# Additional Material

## EC.1. Reduction to Convex Risk Measures

In order to show that hypotheses 1 to 4 are equivalent to imposing that the risk measure should be a convex risk measure, we start by demonstrating how each hypothesis about the risk preference relation implies that a certain constraint must be imposed on the risk measure that will be used. We then make sure that the risk measures that are circumscribed by these constraints represent risk preference relations that satisfy the spelled out hypotheses.

One can easily verify, following the Debreu's representation theorem, that the monotonicity property as presented in hypothesis 1 implies the monotonicity axiom of convex risk measures:

$$Z_1(\omega) \geq Z_2(\omega)\,,\, \forall\, \omega \in \Omega \;\Rightarrow\; Z_1 \succeq Z_2 \;\Rightarrow\; \rho(\vec{Z}_1) \leq \rho(\vec{Z}_2)\,.$$

Secondly, strict monotonicity for certain payoffs, together with the continuity and monotonicity of $\rho(\cdot)$, ensures that every random payoff has a well-defined and unique certainty equivalent, i.e. $\mathbb{CE}(Z)$ is the cash amount that is considered as risky as $Z$.[12] Hence, without loss of generality we can normalize the risk measure such that $\rho(\vec{Z}) = -\mathbb{CE}(Z)$ simply by considering $\rho'(\vec{Z}) = f(\rho(\vec{Z}))$, with $f(y) = \sup\{z \in \mathbb{R} | \rho(-z) \leq y\}$. Indeed, we have that $\rho'(\vec{Z}) = f(\rho(\vec{Z})) = -\mathbb{CE}(Z)$, and $\rho'(\vec{Z})$ always captures the same preferences since $f(\cdot)$ is strictly increasing by the strict monotonicity of $\rho(\cdot)$ for certain payoffs.

Using the notion that $\rho(\vec{Z})$ is the negative of the certainty equivalent of $Z$, one can show that translation invariance as presented in hypothesis 4 implies the cash-invariance axiom of convex risk measures. Since we have that

$$\rho(\vec{Z}) = \rho(\mathbb{CE}(Z)) \Rightarrow \rho(\vec{Z}) = \rho(-\rho(\vec{Z})) \tag{EC.1}$$

it must be that

$$\rho(\vec{Z}) \leq \rho(-\rho(\vec{Z})) \;\Rightarrow\; \rho(\vec{Z}+c) \leq \rho(-\rho(\vec{Z})+c) = \rho(\vec{Z}) - c\,,$$

and similarly that

$$\rho(\vec{Z}) \geq \rho(-\rho(\vec{Z})) \;\Rightarrow\; \rho(\vec{Z}+c) \geq \rho(-\rho(\vec{Z})+c) = \rho(\vec{Z}) - c\,,$$

where the last equalities of the above two equations are due to $\rho(\vec{Z}) = -\mathbb{CE}(Z)$ again. Hence, we must have that $\rho(\vec{Z}+c) = \rho(\vec{Z}) - c$.

[12] Indeed, for any $Z$, the set $\{y \in \mathbb{R}\,|\,\rho(y) = \rho(Z)\}$ is non-empty since the monotonicity property guarantees that $\rho(\min_{\omega\in\Omega} Z(\omega)) \leq \rho(Z) \leq \rho(\max_{\omega\in\Omega} Z(\omega))$, and the continuity of $\rho(\cdot)$ implies that there must be a value $y$ between the two bounds for which $\rho(y) = \rho(Z)$. Furthermore, strict monotonicity for certain payoffs ensures that such a value of $y$ is unique, otherwise there would be $y_1 < y_2$ for which $\rho(y_1) = \rho(y_2)$, meaning that $y_1 \sim y_2$.

Together with translation invariance, quasiconvexity is actually equivalent to the convexity axiom of convex risk measures. Specifically, since we have that

$$\rho(\vec{Z}_i + \rho(\vec{Z}_i)) = \rho(\vec{Z}_i) - \rho(\vec{Z}_i) = 0 \leq \rho(\mathbf{0}),\ i = 1, 2,$$

i.e. $\vec{Z}_1 + \rho(\vec{Z}_1)$ and $\vec{Z}_2 + \rho(\vec{Z}_2)$ are considered less risky than the zero payoff, we must have by quasiconvexity that

$$\rho(\theta(\vec{Z}_1 + \rho(\vec{Z}_1)) + (1-\theta)(\vec{Z}_2 + \rho(\vec{Z}_2))) \ \leq\ \rho(\mathbf{0}) = 0\,.$$

Hence, it must be the case that

$$\rho(\theta\vec{Z}_1 + (1-\theta)\vec{Z}_2 + \theta\rho(\vec{Z}_1) + (1-\theta)\rho(\vec{Z}_2)) = \rho(\theta\vec{Z}_1 + (1-\theta)\vec{Z}_2) - \theta\rho(\vec{Z}_1) - (1-\theta)\rho(\vec{Z}_2) \leq 0\,,$$

so that $\rho(\cdot)$ must satisfy convexity.

For completeness, we verify the other direction, in other words that the constraints we impose in the definition of $\mathfrak{R}$ do imply that the risk measures $\rho \in \mathfrak{R}$ can only represent preference relations $Z_1 \succeq_\rho Z_2 \Leftrightarrow \rho(\vec{Z}_1) \leq \rho(\vec{Z}_2)$ that agree with hypotheses 1 to 4. First, given that $\Omega$ is a finite outcome space, the monotonicity and translation invariance axioms ensure that $\rho$ is finite. Together with the convexity axiom, it is therefore necessarily the case that $\rho$ is continuous. Based on its construction, the risk preference relation $\succeq_\rho$ is therefore necessarily complete, transitive, and continuous. Furthermore, one easily verifies that the monotonicity axiom guarantees that if $\vec{Z}_1 \geq \vec{Z}_2$ then $\rho(\vec{Z}_1) \leq \rho(\vec{Z}_2)$ and thus $\vec{Z}_1$ is considered not riskier than $\vec{Z}_2$. The cash invariance axiom ensures that both the strict monotonicity for certain payoffs and our translation invariance hypothesis 4 are satisfied. Finally, convexity of risk measures ensures that the preferences that it captures satisfy the quasiconvexity hypothesis since:

$$\rho(\theta\vec{Z}_1 + (1-\theta)\vec{Z}_2) \leq \theta\rho(\vec{Z}_1) + (1-\theta)\rho(\vec{Z}_2) \leq \max(\rho(\vec{Z}_1), \rho(\vec{Z}_2)) \leq \rho(\vec{Z}_3)\,.$$

## EC.2. Reduction to Coherent and Law-invariant Risk Measures

Regarding the scale invariance hypothesis, a similar argument as for translation invariance can be made. Since we have that $\rho(\vec{Z}) = -\mathbb{CE}(Z)$, it must be that $\rho(\vec{Z}) = \rho(-\rho(\vec{Z}))$. Hypothesis 5 is therefore equivalent to saying that $\rho(\lambda\vec{Z}) \leq \rho(-\lambda\rho(\vec{Z})) = \lambda\rho(\vec{Z})$ and that $\rho(\lambda\vec{Z}) \geq \rho(-\lambda\rho(\vec{Z})) = \lambda\rho(\vec{Z})$. The risk measure must therefore be a member of $\mathcal{R}_{Coh}$. Conversely, any risk measure $\rho(\cdot) \in \mathcal{R}_{Coh}$ necessarily satisfies the scale invariance hypothesis since for $\lambda \geq 0$

$$Z_1 \succeq Z_2 \Rightarrow \rho(\vec{Z}_1) \leq \rho(\vec{Z}_2) \Rightarrow \lambda\rho(\vec{Z}_1) \leq \lambda\rho(\vec{Z}_2) \Rightarrow \rho(\lambda\vec{Z}_1) \leq \rho(\lambda\vec{Z}_2) \Rightarrow \lambda Z_1 \succeq \lambda Z_2\,.$$

Regarding law invariance, the equivalence simply follows from the definition of $\rho(\cdot)$ which captures the preference relation, i.e. $Z_1 \sim Z_2 \Leftrightarrow \rho(\vec{Z}_1) = \rho(\vec{Z}_2)$. Hence, if $Z_1$ and $Z_2$ have the same distribution then they must be equivalent in terms of risk so that $\rho(\vec{Z}_1) = \rho(\vec{Z}_2)$. The other direction follows as easily.

## EC.3. Proof of Proposition 3

LEMMA EC.1. *Given a set $\mathcal{E}$ of $K$ comparisons, let the set $\{X_j\}_{j=1}^J := 0 \cup \bigcup_{k=1}^K \{W_k, Y_k\}$ be the support set of all random payoffs involved in one of the elicited comparisons and the zero payoff which we identify as $X_1$. Given any random payoff $Z$, the worst-case risk measure of $Z$ under $\mathcal{R}_{El}(\mathcal{E})$ is the optimal value of the following optimization problem:*

$$\varrho_{\mathcal{R}_{El}(\mathcal{E})}(\vec{Z}) \;=\; \max_{\boldsymbol{\delta}\in\Delta}\; \rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{Z})$$

*where $\mathcal{A}(\mathbb{X},\boldsymbol{\delta})$ denotes the monotone convex hull of the set $\{\vec{X}_j + \delta_j\}_{j=1}^J$ which admits the following representation*

$$\mathcal{A}(\mathbb{X},\boldsymbol{\delta}) = \{\vec{Z}\in\mathbb{R}^M \,|\, \exists\boldsymbol{\theta}\in\mathbb{R}^J,\;\; \vec{Z} \geq \mathbb{X}\boldsymbol{\theta} + \boldsymbol{\delta}^\top\boldsymbol{\theta},\, \mathbf{1}^\top\boldsymbol{\theta} = 1,\, \boldsymbol{\theta}\geq 0\},$$

*where the matrix $\mathbb{X}\in\mathbb{R}^{M\times J}$ is composed from a set of reference random payoffs $\{\vec{X}_j\}_{j=1,\ldots,J}$ as its column vectors, i.e. $\mathbb{X} = [\vec{X}_1, \vec{X}_2 \cdots \vec{X}_J]$, and where the set $\Delta$ is the set represented by*

$$\Delta := \{\boldsymbol{\delta}\in\mathbb{R}^J \,|\, \rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_i + \delta_i) \geq 0,\, i = 1,\ldots,J,\, \delta_1 = 0,\, \delta_i \leq \delta_j,\, \forall\,(i,j)\in\bar{\mathcal{E}}\}\,.$$

The intuition behind the lemma above is fairly straightforward. For a given random payoff, the search for the acceptance set that leads to the worst-case risk reduces to searching among the feasible risk values $\boldsymbol{\delta}\in\Delta$ for the reference set of random payoffs $\{X_j\}_{j=1}^J$. Once this is done, the worst-case risk is obtained by considering the monotone convex hull of the random payoffs $\vec{X}_j + \delta_j$. Note that at this point, it is not clear whether the worst-case acceptance set is independent of the random payoff $Z$ that is analyzed or not.

*Proof of Lemma EC.1:* First, we decompose the worst-case analysis into two steps: a search for the worst-case $\boldsymbol{\delta}$ followed by a search for the worst-case risk measure with $\rho(\vec{X}_j) = \delta_j$. Mathematically, we consider

$$\varrho_{\mathcal{R}_{El}(\mathcal{E})}(\vec{Z}) \;=\; \max_{\boldsymbol{\delta}} \quad \sup_{\rho\in\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)} \quad \rho(\vec{Z}) \tag{EC.2a}$$

$$\text{subject to} \quad \delta_i \leq \delta_j\,\forall\,(i.j)\in\bar{\mathcal{E}} \tag{EC.2b}$$

$$\delta_1 = 0 \tag{EC.2c}$$

$$\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J) \neq \emptyset\,, \tag{EC.2d}$$

where $\boldsymbol{\delta}\in\mathbb{R}^J$ and where $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J) \subset \mathfrak{R}$ is the set of convex risk measures that consider the risk of each $\vec{X}_j$ to be respectively $\delta_j$, i.e. that $\rho(\cdot)\in\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$ if and only if it is a convex risk measure that evaluates $\rho(\vec{X}_j) = \delta_j$. Hence if $\rho\in\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$, then it necessarily satisfies the elicited comparison.

In order to show that constraints (EC.2b) to (EC.2d) are equivalent to the set $\Delta$, one needs to establish that constraint (EC.2d) is equivalent to $\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_j+\delta_j)\geq 0\,,\,\forall j$. The statement that for all $\boldsymbol{\delta}\in\mathbb{R}^J$,

$$\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_j+\delta_j)\geq 0\,,\,\forall j\;\Rightarrow\;\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)\neq\emptyset$$

is somewhat trivial since $\mathcal{A}(\mathbb{X},\boldsymbol{\delta})$ is constructed to make each $\vec{X}_j+\delta_j$ acceptable and therefore

$$0\geq\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_j+\delta_j)\geq 0\Rightarrow\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_j+\delta_j)=0\Rightarrow\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_j)=\delta_j\,,$$

so that $\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}\in\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)\neq\emptyset$.

In the other direction, if $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)\neq\emptyset$. Let $\mathcal{A}'$ denote an acceptance set such that $\rho_{\mathcal{A}'}\in\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$. Since $\rho(\vec{X}_j)=\delta_j$ implies that $\rho(\vec{X}_j+\delta_j)=0$, each random payoff $X_j+\delta_j$ must therefore be members of $\mathcal{A}'$. Provided that $\mathcal{A}(\mathbb{X},\boldsymbol{\delta})$ is the smallest monotone convex set containing all $\vec{X}_j+\delta_j$ (see proof of Proposition 2), it is necessarily the case that $\mathcal{A}(\mathbb{X},\boldsymbol{\delta})\subseteq\mathcal{A}'$ and that

$$\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_j+\delta_j)\geq\rho_{\mathcal{A}'}(\vec{X}_j+\delta_j)=\rho_{\mathcal{A}'}(\vec{X}_j)-\delta_j=0\,,$$

where the first equality comes from translation invariance and the second from the definition of $\mathcal{A}'$.

We are left with showing that for any $\boldsymbol{\delta}$ such that $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)\neq 0$, and any $Z$, we have that

$$\sup_{\rho\in\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)}\rho(\vec{Z})=\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{Z})\,.$$

Yet, in this regard, we just showed that $\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\cdot)$ is always a member of $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$ when the latter is non-empty, and that $\mathcal{A}(\mathbb{X},\boldsymbol{\delta})$ is a subset of the acceptance set associated to any member of $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$. Hence, it must be that

$$\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{Z})\geq\sup_{\rho\in\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)}\rho(\vec{Z})\geq\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{Z})\,.$$

This completes the proof. □

In order to establish that the worst-case $\boldsymbol{\delta}\in\Delta$ does not depend on the random payoff $\vec{Z}$ that is being evaluated, we will show through the following two lemmas that $\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{Z})$ is non-decreasing in $\boldsymbol{\delta}$ and that problem (6) returns a vector $\bar{\boldsymbol{\delta}}$ for which each entry is at the maximum value that it can achieve. Together with Lemma EC.1, this completes the proof of Proposition 3 since we can conclude that

$$\max_{\boldsymbol{\delta}\in\Delta}\;\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{Z})=\rho_{\mathcal{A}(\mathbb{X},\bar{\boldsymbol{\delta}})}(\vec{Z})\,.$$

LEMMA EC.2. *Given any random payoff $Z$, the risk measure $\rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{Z})$ is non-decreasing in $\boldsymbol{\delta}$.*

*Proof of Lemma EC.2:* Let $\boldsymbol{\delta}_1 \geq \boldsymbol{\delta}_2$, and let $(t_1, \boldsymbol{\theta}_1)$ and $(t_2, \boldsymbol{\theta}_2)$ be the optimal solutions of

$$\begin{aligned} \min_{t,\boldsymbol{\theta}} \quad & t \\ \text{subject to} \quad & \vec{Z} + t \geq \mathbb{X}\boldsymbol{\theta} + \boldsymbol{\delta}^\top \boldsymbol{\theta} \\ & \mathbf{1}^\top \boldsymbol{\theta} = 1,\ \boldsymbol{\theta} \geq 0 \end{aligned}$$

when $\boldsymbol{\delta} = \boldsymbol{\delta}_1$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_2$ respectively. Since $\boldsymbol{\theta}_1 \geq 0$, we have that

$$\vec{Z} + t_1 \geq \mathbb{X}\boldsymbol{\theta}_1 + \boldsymbol{\delta}_1^\top \boldsymbol{\theta}_1 \geq \mathbb{X}\boldsymbol{\theta}_1 + \boldsymbol{\delta}_2^\top \boldsymbol{\theta}_1\,,$$

hence $(t_1, \boldsymbol{\theta}_1)$ is a feasible solution when $\boldsymbol{\delta} = \boldsymbol{\delta}_2$. Since $t_2$ is the optimal solution when $\boldsymbol{\delta} = \boldsymbol{\delta}_2$, we have that $t_2 \leq t_1$. □

Lemma EC.3. *Let $\bar{\boldsymbol{\delta}}$ be the optimal solution of $\max_{\boldsymbol{\delta}\in\Delta} \sum_j \delta_j$, then each $\bar{\delta}_i$ is the optimal value of $\max_{\boldsymbol{\delta}\in\Delta}\ \delta_i$. Furthermore, the problem $\max_{\boldsymbol{\delta}\in\Delta} \sum_j \delta_j$ is equivalent to problem (6).*

*Proof of Lemma EC.3:* We first present the details of $\max_{\boldsymbol{\delta}\in\Delta} \sum_j \delta_j$ to simplify the discussion:

$$\begin{aligned} \max_{\boldsymbol{\delta}} \quad & \sum_{j=1}^{J} \delta_j, && \text{(EC.3a)} \\ \text{subject to} \quad & \delta_i \leq \delta_j\,,\ \forall (i,j) \in \bar{\mathcal{E}} && \text{(EC.3b)} \\ & \rho_{\mathcal{A}(\mathbb{X},\boldsymbol{\delta})}(\vec{X}_j + \delta_j) \geq 0\,,\ \forall j = 1, \ldots, J && \text{(EC.3c)} \\ & \delta_1 = 0\,. && \text{(EC.3d)} \end{aligned}$$

Next, we let $\hat{\boldsymbol{\delta}}^{(i)}$ be the optimal solution of problem (EC.3) when the objective is replaced with the objective of maximizing $\delta_i$. One can actually show that the solution $\bar{\boldsymbol{\delta}}$ composed such that $\bar{\delta}_i = \hat{\delta}_i^{(i)}$ is feasible for problem (EC.3) and therefore an optimal solution of this problem.

In the case of constraint (EC.3b), by construction of $\bar{\boldsymbol{\delta}}$ we have that $\bar{\boldsymbol{\delta}} \geq \hat{\boldsymbol{\delta}}^{(i)}$ for all $i$. Therefore, one can confirm that for any pair $(i,j) \in \bar{\mathcal{E}}$

$$\bar{\delta}_i = \hat{\delta}_i^{(i)} \leq \hat{\delta}_j^{(i)} \leq \hat{\delta}_j^{(j)} = \bar{\delta}_j\,.$$

As for constraint (EC.3c), for any fixed $j$, the representation of this constraint which is based on optimization states that

$$\begin{aligned} \min_{t,\boldsymbol{\theta}} \quad & t && \geq 0\,. \\ \text{subject to} \quad & \vec{X}_j + \delta_j + t \geq \mathbb{X}\boldsymbol{\theta} + \boldsymbol{\delta}^\top \boldsymbol{\theta} \\ & \mathbf{1}^\top \boldsymbol{\theta} = 1\,,\ \boldsymbol{\theta} \geq 0 \end{aligned}$$

After replacing $t' := t + \delta_j - \boldsymbol{\delta}^\top \boldsymbol{\theta}$, one obtains the equivalent constraint:

$$\begin{aligned} \min_{t',\boldsymbol{\theta}} \ & t' - \delta_j + \boldsymbol{\delta}^\top \boldsymbol{\theta} & \geq 0\,. \\ \text{subject to } & \vec{X}_j + t' \geq \mathbb{X}\boldsymbol{\theta} & \\ & \mathbf{1}^\top \boldsymbol{\theta} = 1\,,\ \boldsymbol{\theta} \geq 0 & \end{aligned} \quad \text{(EC.4)}$$

One can verify that $\bar{\boldsymbol{\delta}}$ satisfies this constraint since for all feasible $t'$ and $\boldsymbol{\theta}$, one has that

$$t' - \bar{\delta}_j + \bar{\boldsymbol{\delta}}^\top \boldsymbol{\theta} = t' - \hat{\delta}_j^{(j)} + \sum_i \hat{\delta}_i^{(i)} \theta_i \geq t' - \hat{\delta}_j^{(j)} + \sum_i \hat{\delta}_i^{(j)} \theta_i \geq 0\,.$$

In order to show that the optimization problem $\max_{\boldsymbol{\delta}\in\Delta} \sum_j \delta_j$ is equivalent to problem (6), we make use of duality to reformulate each of the constraints in (EC.3c). In particular, for a fixed $j$ the constraint takes the explicit form presented in (EC.4). One can verify that a feasible solution always exists for this minimization problem with $t' = 0$ and $\boldsymbol{\theta} = \boldsymbol{e}_j$ where $\boldsymbol{e}_j$ is the vector of all zeros except for a one at the $j$-th entry. Linear programming duality theory therefore states that the following constraint is equivalent to constraint (EC.4):

$$\begin{aligned} \max_{\boldsymbol{y},\phi} \ & -\vec{X}_j^\intercal \boldsymbol{y} - \delta_j + \phi & \geq 0 \\ \text{subject to } & \vec{X}_i^\intercal \boldsymbol{y} + \delta_i - \phi \geq 0\,, \forall i = 1, \ldots, J & \\ & \mathbf{1}^\top \boldsymbol{y} = 1,\ \boldsymbol{y} \geq 0\,, & \end{aligned}$$

where $\boldsymbol{y} \in \mathbb{R}^M$ and $\phi \in \mathbb{R}$ are the respective dual variables of the two constraints. Necessarily, the maximization operation in this constraint can be merged with the overall maximization of problem (EC.3) as long as $\boldsymbol{y}$ and $\phi$ are properly indexed with $j$. In other words, for each $j$, one should add $\boldsymbol{y}_j \in \mathbb{R}^M$ and $\phi_j \in \mathbb{R}$ as decision variables and replace constraint (EC.3c) with:

$$\begin{aligned} & -\vec{X}_j^\intercal \boldsymbol{y}_j - \delta_j + \phi_j \geq 0 \\ & \vec{X}_i^\intercal \boldsymbol{y}_j + \delta_i - \phi_j \geq 0\,, \forall i = 1, \ldots, J \\ & \mathbf{1}^\top \boldsymbol{y}_j = 1,\ \boldsymbol{y}_j \geq 0\,. \end{aligned}$$

Yet, one can even see that the constraints reduce to

$$\begin{aligned} & \vec{X}_j^\intercal \boldsymbol{y}_j + \delta_j - \phi_j = 0 \\ & \vec{X}_i^\intercal \boldsymbol{y}_j + \delta_i - \phi_j \geq 0\,, \forall i \neq j \\ & \mathbf{1}^\top \boldsymbol{y}_j = 1,\ \boldsymbol{y}_j \geq 0\,, \end{aligned}$$

so that the decision variable $\phi_j$ can be replaced to get

$$\begin{aligned} & (\vec{X}_i - \vec{X}_j)^\intercal \boldsymbol{y}_j + \delta_i - \delta_j \geq 0\,, \forall i \neq j \\ & \mathbf{1}^\top \boldsymbol{y}_j = 1,\ \boldsymbol{y}_j \geq 0\,, \end{aligned}$$

This completes the proof as we are left with problem (6). $\square$

## EC.4. Proof of Proposition 5

We present the proof of this proposition in two steps. First, we show how the worst-case law-invariant risk measure $\varrho_{\mathcal{R}_{LE}(\mathcal{E})}(\vec{Z})$ can be represented as a finite dimensional convex optimization problem when the probability space involves a uniform measure. Next, we consider how a general preference robust risk minimization problem (9) with $\mathcal{R}_{LE}(\{(F_k^W, F_k^Y)\}_{k=1}^K)$ might be equivalently represented in this space as long as the probabilities are rational, and show how to reduce the associated optimization problems to forms of more reasonable sizes that are presented in (10) and (11).

**Step 1: The case of uniform probability measure**

In this first step, we focus our attention on probability spaces with a uniform measure.

Assumption EC.1. *The probability measure $F$ associated to the discrete probability space is the uniform measure, i.e. $P(\{\omega_i\}) = 1/M$ for any $\omega_i \in \Omega$. Recall that $|\Omega| = M$.*

In this kind of probability space, law invariance is intimately related to the notion of permutation, which we make more precise in the following definition.

Definition EC.1. A permutation over $M$ elements is a bijective function $\bar{\sigma} : \{1,...,M\} \to \{1,...,M\}$. We call the random variable permutation operator $\sigma$, the operator that permutes the values associated to each outcome of a random variable according to the bijection $\bar{\sigma}$. Mathematically speaking, we have that $(\sigma(X))(\omega_i) = X(\omega_{\sigma^{-1}(i)})$. For simplicity of exposure, we also overload this notation such that $(\sigma(\vec{X}))_i = \vec{X}(\omega_{\bar{\sigma}^{-1}(i)})$. Finally, we will denote with $\Sigma$ the set of all random variable permutation operators in the $\Omega$ outcome space.

Since $\sigma(Z) =_F Z$ (i.e. in distribution) when $F$ is the uniform distribution, we must have for any preference of the type $W \succeq Y$ that $\sigma(W) \succeq W \succeq Y \succeq \sigma'(Y)$ for any $\sigma, \sigma' \in \Sigma$. This leads us to consider an augmented set of elicited comparison

$$\Sigma(\mathcal{E}) := \{(W,Y) | \exists \sigma \in \Sigma, \sigma' \in \Sigma, (\sigma(W), \sigma'(Y)) \in \mathcal{E}\}.$$

Risk measures that are known to comply with the elicited comparisons $\mathcal{E}$ and to be law invariant should also respect preference orderings described in the augmented set $\Sigma(\mathcal{E})$. What is more interesting is actually the reverse statement, which we prove in the following lemma that robust risk measures if constructed directly based on the augmented set $\Sigma(\mathcal{E})$ would coincide with the law invariant measures based on $\mathcal{E}$. This result shifts the complexity of incorporating the hypothesis of law invariance from the set of law-invariant risk measures $\mathcal{R}_{LE}(\mathcal{E})$ to the set of convex risk measures $\mathcal{R}_{El}(\Sigma(\mathcal{E}))$ (or from $\mathcal{R}_{CLE}(\mathcal{E})$ to $\mathcal{R}_{CE}(\Sigma(\mathcal{E}))$).

LEMMA EC.4. *The preference robust risk measure* $\varrho_{\mathcal{R}_{LE}(\mathcal{E})}(\vec{Z}) := \sup_{\rho\in\mathcal{R}_{LE}(\mathcal{E})}\rho(\vec{Z})$ *is equivalent to* $\varrho_{\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}(\vec{Z})$ *and* $\varrho_{\mathcal{R}_{El}(\Sigma(\mathcal{E}))}(\vec{Z})$*. Similarly, the preference robust risk measure* $\varrho_{\mathcal{R}_{CLE}(\mathcal{E})}(\vec{Z})$ *is equivalent to* $\varrho_{\mathcal{R}_{CLE}(\Sigma(\mathcal{E}))}(\vec{Z})$ *and* $\varrho_{\mathcal{R}_{CE}(\Sigma(\mathcal{E}))}(\vec{Z})$*.*

*Proof of Lemma EC.4:* In the case of robust convex risk measures, one can first establish that $\varrho_{\mathcal{R}_{LE}(\mathcal{E})}(\vec{Z}) = \varrho_{\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}(\vec{Z})$ if it can be verified that $\mathcal{R}_{LE}(\mathcal{E}) = \mathcal{R}_{LE}(\Sigma(\mathcal{E}))$. This can be done with the following argument. Given that the later set imposes more constraints on the risk measure, it must be that $\mathcal{R}_{LE}(\mathcal{E}) \supseteq \mathcal{R}_{LE}(\Sigma(\mathcal{E}))$. In fact, given any $\rho \in \mathcal{R}_{LE}(\mathcal{E})$, one can confirm that it is a member of $\mathcal{R}_{LE}(\Sigma(\mathcal{E}))$ since for any pair $(W', Y') \in \Sigma(\mathcal{E})$, there must be a comparison $(W, Y) \in \mathcal{E}$ and a pair of permutation operators $(\sigma_W, \sigma_Y)$ such that $W' = \sigma_W(W)$ and $Y' = \sigma_Y(Y)$, hence that

$$\rho(\vec{W}') = \rho(\sigma_W(\vec{W})) = \rho(\vec{W}) \geq \rho(\vec{Y}) = \rho(\sigma_Y(\vec{Y})) = \rho(\vec{Y}'),$$

where the second and third equalities are due to the fact that $\rho$ is law invariant.

Next, we prove that $\varrho_{\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}(\vec{Z})$ is equivalent to $\varrho_{\mathcal{R}_{El}(\Sigma(\mathcal{E}))}(\vec{Z})$. Since $\mathcal{R}_{LE}(\Sigma(\mathcal{E})) \subseteq \mathcal{R}_{El}(\Sigma(\mathcal{E}))$, we must have that $\varrho_{\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}(\vec{Z}) \leq \varrho_{\mathcal{R}_{El}(\Sigma(\mathcal{E}))}(\vec{Z})$. To get the reverse, we can consider that given any $\rho \in \mathcal{R}_{El}(\Sigma(\mathcal{E}))$, one can construct a new risk measure $\rho_\Sigma(\vec{Z}) := \max_{\sigma\in\Sigma}\rho(\sigma(\vec{Z}))$. It is clear that $\rho_\Sigma \in \mathcal{R}_{LE}(\Sigma(\mathcal{E}))$ and that $\rho_\Sigma(\vec{Z}) \geq \rho(\vec{Z})$. [13] We can conclude that

$$\varrho_{\mathcal{R}_{El}(\Sigma(\mathcal{E}))}(\vec{Z}) = \sup_{\rho\in\mathcal{R}_{El}(\Sigma(\mathcal{E}))}\rho(\vec{Z}) \leq \sup_{\rho\in\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}\rho(\vec{Z}) = \varrho_{\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}(\vec{Z}).$$

Hence, we know that

$$\varrho_{\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}(\vec{Z}) \leq \varrho_{\mathcal{R}_{El}(\Sigma(\mathcal{E}))}(\vec{Z}) \leq \varrho_{\mathcal{R}_{LE}(\Sigma(\mathcal{E}))}(\vec{Z}).$$

The proof for the case of robust coherent risk measures is similar. □

The complexity of constructing risk measures $\varrho_{\mathcal{R}_{El}(\Sigma(\mathcal{E}))}(\vec{Z})$ (resp. $\varrho_{\mathcal{R}_{CE}(\Sigma(\mathcal{E}))}(\vec{Z})$) lies in the size of the augmented set $\Sigma(\mathcal{E})$, which grows exponentially with respect to the number of elicited comparisons. Our next result is to show that the convex optimization problems formulated based on Proposition 3 (resp. Proposition 4) for $\varrho_{\mathcal{R}_{El}(\Sigma(\mathcal{E}))}(\vec{Z})$ (resp. $\varrho_{\mathcal{R}_{CE}(\Sigma(\mathcal{E}))}(\vec{Z})$) can be reduced to problems whose size no longer depend on the size of the set of permutations and grow polynomially with the number of elicited comparisons. In the following lemma, we present first the reduced formulation for the offline optimization problem (6) in Proposition 3 with the augmented set $\Sigma(\mathcal{E})$.

[13] We use the fact $W =_F Y \Leftrightarrow \exists\sigma : \vec{Y} = \sigma(\vec{W})$ here. The direction $\Leftarrow$ is clear as discussed. To see $\Rightarrow$, note that following Assumption EC.1, any law $F^X$ can be expressed by $P(X = x_k) = |\{\omega\in\Omega \mid X(\omega) = x_k\}|/M, \forall k$. $W =_F Y$ implies that $|\{\omega\in\Omega \mid W(\omega) = x_k\}| = |\{\omega\in\Omega \mid Y(\omega) = x_k\}|, \forall k$. The fact that $\vec{Y} = \sigma(\vec{W})$ easily follows.

LEMMA EC.5. *The optimization problem*

$$\max_{\boldsymbol{\delta},\{\boldsymbol{y}_{j,\sigma}\}_{j=1,\sigma\in\Sigma}^{J}} \quad \sum_{j=1}^{J}\sum_{\sigma\in\Sigma}\delta_{j,\sigma}, \tag{EC.5a}$$

$$\text{subject to} \quad \delta_{i,\sigma}\leq\delta_{j,\sigma'}\,,\,\forall(i,j)\in\bar{\mathcal{E}},\,\forall\sigma\in\Sigma,\,\forall\sigma'\in\Sigma \tag{EC.5b}$$

$$(\sigma(\vec{X}_i)-\sigma'(\vec{X}_j))^\intercal\boldsymbol{y}_{j,\sigma'}+\delta_{i,\sigma}-\delta_{j,\sigma'}\geq 0\,,\,\forall i\neq j,\,\forall\sigma,\,\forall\sigma' \tag{EC.5c}$$

$$\mathbf{1}^\top\boldsymbol{y}_{j,\sigma}=1,\;\boldsymbol{y}_{j,\sigma}\geq 0,\,\forall j,\,\forall\sigma\in\Sigma \tag{EC.5d}$$

$$\delta_{1,\sigma}=0,\,\forall\sigma\in\Sigma\,, \tag{EC.5e}$$

*where* $\boldsymbol{\delta}\in\mathbb{R}^{J\times|\Sigma|}$, $\boldsymbol{y}_{j,\sigma}\in\mathbb{R}^M$, *and where* $\bar{\mathcal{E}}$ *is the set of edges in the partial ordering of* $\{X_j\}_{j=1}^J$ *described by the elicited comparisons: i.e.*

$$\bar{\mathcal{E}}:=\{(i,j)\in\{1,2,\dots,J\}^2\,|\,(X_i,X_j)\in\mathcal{E}\}\,,$$

*has an optimal solution for which* $\delta_{j,\sigma}=\delta_{j,\sigma'}$ *for all* $j$*, all* $\sigma\in\Sigma$*, and all* $\sigma'\in\Sigma$*. Furthermore, it reduces to solving the following linear programming problem*

$$\max_{\boldsymbol{\delta},\{\boldsymbol{y}_j\}_{j=1}^J,\{\boldsymbol{v}_{i,j},\boldsymbol{w}_{i,j}\}_{i=1,j=1}^{i=J,j=J}} \quad |\Sigma|\sum_{j=1}^{J}\delta_j, \tag{EC.6a}$$

$$\text{subject to} \quad \delta_i\leq\delta_j\,,\,\forall(i,j)\in\bar{\mathcal{E}} \tag{EC.6b}$$

$$\mathbf{1}^\top\boldsymbol{v}_{i,j}+\mathbf{1}^\top\boldsymbol{w}_{i,j}-\vec{X}_j^\top\boldsymbol{y}_j+\delta_i-\delta_j\geq 0\,,\,\forall i\neq j \tag{EC.6c}$$

$$\vec{X}_i\boldsymbol{y}_j^\top-\boldsymbol{v}_{i,j}\mathbf{1}^\top-\mathbf{1}\boldsymbol{w}_{i,j}\geq 0\,,\,\forall i\neq j \tag{EC.6d}$$

$$\mathbf{1}^\top\boldsymbol{y}_j=1,\;\boldsymbol{y}_j\geq 0\,,\,\forall j \tag{EC.6e}$$

$$\delta_1=0\,, \tag{EC.6f}$$

*where each* $\boldsymbol{y}_j\in\mathbb{R}^M$*, each* $\boldsymbol{v}_{i,j}\in\mathbb{R}^M$*, and each* $\boldsymbol{w}_{i,j}\in\mathbb{R}^M$*.*

*Proof of Lemma EC.5:* To demonstrate this lemma, we will prove that given any feasible solution $(\boldsymbol{\delta},\{\boldsymbol{y}_{j,\sigma}\}_{j=1,\sigma\in\Sigma}^J)$, one can construct a feasible solution

$$\bar{\delta}_{i,\sigma}:=\frac{1}{|\Sigma|}\sum_{\sigma'\in\Sigma}\delta_{i,\sigma'},\;\forall\sigma\in\Sigma,\;\;\forall i=1,...,J$$

$$\bar{\boldsymbol{y}}_{i,\sigma}:=\frac{1}{|\Sigma|}\sigma\left(\sum_{\sigma'\in\Sigma}\sigma'^{-1}(\boldsymbol{y}_{i,\sigma'})\right)\,,\,\forall\sigma\in\Sigma,\;\;\forall i=1,...,J$$

that achieves the same objective value and has the added property that the value of $\bar{\delta}_{i,\sigma}$ is the same for all permutations.

First, the objective is necessarily the same since

$$\begin{aligned}\sum_{j=1}^{J}\sum_{\sigma\in\Sigma}\bar{\delta}_{j,\sigma} &= \sum_{j=1}^{J}\sum_{\sigma\in\Sigma}\frac{1}{|\Sigma|}\sum_{\sigma'\in\Sigma}\delta_{j,\sigma'} \\ &= \sum_{j=1}^{J}\sum_{\sigma'\in\Sigma}\sum_{\sigma\in\Sigma}\frac{1}{|\Sigma|}\delta_{j,\sigma'} \\ &= \sum_{j=1}^{J}\sum_{\sigma'\in\Sigma}\delta_{j,\sigma'}\,.\end{aligned}$$

Next, we confirm one constraint at a time that each of constraints (EC.5b) to (EC.5e) are satisfied. For constraint (EC.5b), we have that:

$$\bar{\delta}_{i,\sigma} = \frac{1}{|\Sigma|}\sum_{\sigma''\in\Sigma}\delta_{i,\sigma''} \leq \frac{1}{|\Sigma|}\sum_{\sigma'''\in\Sigma}\delta_{j,\sigma'''} = \bar{\delta}_{j,\sigma'}\,.$$

In the case of constraint (EC.5c), the work is a bit more tedious

$$\begin{aligned}&(\sigma(\vec{X}_i)-\sigma'(\vec{X}_j))^{\intercal}\bar{\boldsymbol{y}}_{j,\sigma'}+\bar{\delta}_{i,\sigma}-\bar{\delta}_{j,\sigma'} \\ &\quad= (\sigma(\vec{X}_i)-\sigma'(\vec{X}_j))^{\intercal}\frac{1}{|\Sigma|}\sigma'\left(\sum_{\sigma''\in\Sigma}\sigma''^{-1}(\boldsymbol{y}_{j,\sigma''})\right)+\bar{\delta}_{i,\sigma}-\bar{\delta}_{j,\sigma'} \\ &\quad= \frac{1}{|\Sigma|}(\sigma'^{-1}(\sigma(\vec{X}_i)-\sigma'(\vec{X}_j)))^{\intercal}\left(\sum_{\sigma''\in\Sigma}\sigma''^{-1}(\boldsymbol{y}_{j,\sigma''})\right)+\bar{\delta}_{i,\sigma}-\bar{\delta}_{j,\sigma'} \\ &\quad= \frac{1}{|\Sigma|}\sum_{\sigma''\in\Sigma}(\sigma'^{-1}(\sigma(\vec{X}_i))-\vec{X}_j)^{\intercal}\sigma''^{-1}(\boldsymbol{y}_{j,\sigma''})+\bar{\delta}_{i,\sigma}-\bar{\delta}_{j,\sigma'} \\ &\quad= \frac{1}{|\Sigma|}\sum_{\sigma''\in\Sigma}(\sigma''(\sigma'^{-1}(\sigma(\vec{X}_i)))-\sigma''(\vec{X}_j))^{\intercal}(\boldsymbol{y}_{j,\sigma''})+\bar{\delta}_{i,\sigma}-\bar{\delta}_{j,\sigma'} \\ &\quad\geq \frac{1}{|\Sigma|}\sum_{\sigma''\in\Sigma}\delta_{j,\sigma''}-\delta_{i,\sigma''\circ\sigma'^{-1}\circ\sigma}+\bar{\delta}_{i,\sigma}-\bar{\delta}_{j,\sigma'} = 0\,,\end{aligned}$$

where we used the fact that

$$(\sigma''(\sigma'^{-1}(\sigma(\vec{X}_i)))-\sigma''(\vec{X}_j))^{\intercal}(\boldsymbol{y}_{j,\sigma''}) \geq \delta_{j,\sigma''}-\delta_{i,\sigma''\circ\sigma'^{-1}\circ\sigma}$$

and in the last step we used the fact the sum is over all possible permutations. Finally, constraints (EC.5d) and (EC.5e) can easily be verified.

$$\mathbf{1}^{\top}\bar{\boldsymbol{y}}_{j,\sigma} = \frac{1}{|\Sigma|}\mathbf{1}^{\top}\sigma\left(\sum_{\sigma'\in\Sigma}\sigma'^{-1}(\boldsymbol{y}_{j,\sigma'})\right) = \frac{1}{|\Sigma|}\sum_{\sigma'\in\Sigma}\mathbf{1}^{\top}\sigma'^{-1}(\boldsymbol{y}_{j,\sigma'}) = \frac{1}{|\Sigma|}\sum_{\sigma'\in\Sigma}\mathbf{1}^{\top}\boldsymbol{y}_{j,\sigma'} = 1$$

$$\bar{\delta}_{1,\sigma} = \frac{1}{|\Sigma|}\sum_{\sigma'\in\Sigma}\delta_{1,\sigma'} = 0\,.$$

This completes the proof that problem (EC.5) has an optimal solution for which $\delta_{j,\sigma} = \delta_{j,\sigma'}$ for all $j$, all $\sigma\in\Sigma$, and all $\sigma'\in\Sigma$.

Given the existence of an optimal solution with such structure it is possible to simplify the problem by optimizing only over all $\delta_{j,\sigma} = \delta_j$ and $\boldsymbol{y}_{i,\sigma} := \sigma(\boldsymbol{y}_i)$. This gives rise to the following problem:

$$\max_{\boldsymbol{\delta}, \{\boldsymbol{y}_j\}_{j=1}^J} \quad \sum_{j=1}^{J} \sum_{\sigma \in \Sigma} \delta_j, \tag{EC.7a}$$

$$\text{subject to} \quad \delta_i \le \delta_j \,,\, \forall (i,j) \in \bar{\mathcal{E}} \tag{EC.7b}$$

$$(\sigma(\vec{X}_i) - \vec{X}_j)^\intercal \boldsymbol{y}_j + \delta_i - \delta_j \ge 0 \,,\, \forall i \ne j, \forall \sigma \in \Sigma \tag{EC.7c}$$

$$\mathbf{1}^\top \boldsymbol{y}_j = 1,\ \boldsymbol{y}_j \ge 0, \forall j \tag{EC.7d}$$

$$\delta_1 = 0 \,, \tag{EC.7e}$$

where we used the fact that

$$\mathbf{1}^\top \sigma(\boldsymbol{y}_j) = \mathbf{1}^\top \boldsymbol{y}_j$$

and the fact that

$$(\sigma(\vec{X}_i) - \sigma'(\vec{X}_j))^\intercal \sigma'(\boldsymbol{y}_j) + \delta_i - \delta_j = ({\sigma'}^{-1}(\sigma(\vec{X}_i)) - \vec{X}_j)^\intercal \boldsymbol{y}_j + \delta_i - \delta_j \,.$$

We are left with constraint (EC.7c) which can be stated as

$$\min_{\sigma \in \Sigma} \sigma(\vec{X}_i)^\intercal \boldsymbol{y}_j - \vec{X}_j^\intercal \boldsymbol{y}_j + \delta_i - \delta_j \ge 0 \,,\, \forall i \ne j \,, \tag{EC.8}$$

and that we will reduce using duality theory.

Consider that $\min_{\sigma \in \Sigma} \sigma(\vec{X}_i)^\intercal \boldsymbol{y}_j$ is equal to the optimal value of the optimization problem

$$\begin{aligned} \min_{Q} \quad & \boldsymbol{y}^\top Q \vec{X} \\ \text{subject to} \quad & Q^\top \mathbf{1} = \mathbf{1} \\ & Q\mathbf{1} = \mathbf{1} \\ & Q_{k,l} \in \{0,1\}^{M \times M} \,. \end{aligned}$$

Due to the result of Birkhoff (1946), the above problem, also known as linear assignment problem, can be solved exactly by relaxing the binary constraints to the constraint that each variable is a real value between 0 and 1. Since the relaxed form of this problem is feasible, we have that strong linear programming duality holds thus that the optimal value can also be obtained through the following dual problem :

$$\begin{aligned} \max_{\boldsymbol{v}, \boldsymbol{w}} \quad & \mathbf{1}^\top \boldsymbol{v} + \mathbf{1}^\top \boldsymbol{w} \\ \text{subject to} \quad & \vec{X} \boldsymbol{y}^\top - \boldsymbol{v} \mathbf{1}^\top - \mathbf{1} \boldsymbol{w}^\top \ge 0 \,, \end{aligned}$$

where $\boldsymbol{v} \in \mathbb{R}^M$ and $\boldsymbol{w} \in \mathbb{R}^M$. When replacing the first term of the constraint stated in equation (EC.8), we get that constraint (EC.7c) is satisfied as long as there exists a set of values for $\boldsymbol{v}_{i,j}$ and $\boldsymbol{w}_{i,j}$ that satisfy constraints (EC.6c) and (EC.6d). This completes our proof. □

We are left with showing that the problem (5) in Proposition 3 can as well be reduced a problem independent of the size of the set of permutations $\Sigma$. We prove this in the following proposition, which also wraps up our result for the case of law-invariant risk measures based on a uniform probability measure.

PROPOSITION EC.1. *Given a set $\mathcal{E}$ of $K$ comparisons, let the set $\{X_j\}_{j=1}^J := 0 \cup \bigcup_{k=1}^K \{W_k, Y_k\}$ be the support set of all random payoffs involved in one of the elicited comparisons and the zero payoff which we identify as $X_1$. When the distribution on the discrete probability space is uniform, the preference robust risk minimization problem (2) with $\mathcal{R}_{LE}(\mathcal{E})$ is equivalent to the optimization problem:*

$$\min_{\boldsymbol{x}\in\mathcal{X},t,\boldsymbol{\theta},\{Q_j\}} \quad t, \tag{EC.9a}$$

$$\text{subject to} \quad \vec{Z}(\boldsymbol{x}) + t \geq \sum_j Q_j \vec{X}_j + \bar{\boldsymbol{\delta}}^\top \boldsymbol{\theta}, \tag{EC.9b}$$

$$Q_j \mathbf{1} = \theta_j, \, \forall j \tag{EC.9c}$$

$$Q_j^\top \mathbf{1} = \theta_j, \, \forall j \tag{EC.9d}$$

$$\mathbf{1}^\top \boldsymbol{\theta} = 1 \tag{EC.9e}$$

$$\boldsymbol{\theta} \geq 0, \, Q_j \geq 0 \, \forall j \tag{EC.9f}$$

*where $t \in \mathbb{R}$, $\boldsymbol{\theta} \in \mathbb{R}^J$, and each $Q_j \in \mathbb{R}^{M\times M}$, and where $\bar{\boldsymbol{\delta}} \in \mathbb{R}^J$ is the optimal solution of the linear program (EC.6). Moreover, problem (EC.9) is a convex optimization problem when each term $(\vec{Z}(\boldsymbol{x}))_i$ is a concave function of $\boldsymbol{x}$, and problem (EC.6) is feasible if and only if $\mathcal{R}_{LE}(\mathcal{E})$ is non-empty.*

*Proof of Proposition EC.1:* Based on Lemma EC.4, we have that the preference robust risk minimization problem (2) with $\mathcal{R}_{LE}(\mathcal{E})$ is equivalent to using the set of risk measure $\mathcal{R}_{El}(\Sigma(\mathcal{E}))$. According to Proposition 3, minimizing this preference robust risk measure can be achieved by solving:

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathcal{X},t,\boldsymbol{\lambda}} \quad & t \\ \text{subject to} \quad & \vec{Z}(\boldsymbol{x}) + t \geq \sum_{j,\sigma} \lambda_{j,\sigma}(\sigma(\vec{X}_j) + \bar{\delta}_{j,\sigma}) \\ & \sum_{j,\sigma} \lambda_{j,\sigma} = 1, \, \boldsymbol{\lambda} \geq 0 \,, \end{aligned}$$

where $t \in \mathbb{R}$, $\boldsymbol{\lambda} \in \mathbb{R}^{J\times|\Sigma|}$, and where $\bar{\delta}_{j,\sigma}$ is an optimal solution to problem (EC.5). Yet, according to Lemma EC.5, problem (EC.5) reduces to problem (EC.6) which always identifies solution for which $\bar{\delta}_{j,\sigma}$ is constant over $\sigma \in \Sigma$. This means that the optimization problem described above reduces to:

$$\min_{\boldsymbol{x}\in\mathcal{X},t,\boldsymbol{\lambda},\boldsymbol{\theta}} \quad t$$

$$\begin{aligned} \text{subject to} \quad & \vec{Z}(\boldsymbol{x}) + t \geq \sum_{j,\sigma} \lambda_{j,\sigma} \sigma(\vec{X}_j) + \sum_j \theta_j \bar{\delta}_j \\ & \theta_j = \sum_{\sigma \in \Sigma} \lambda_{j,\sigma} \, , \, \forall j \\ & \sum_{j,\sigma} \lambda_{j,\sigma} = 1, \, \boldsymbol{\lambda} \geq 0 \, , \end{aligned}$$

with $\boldsymbol{\theta} \in \mathbb{R}^J$, which can be shown equivalent to

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathcal{X}, t, \boldsymbol{\lambda}', \boldsymbol{\theta}} \quad & t \\ \text{subject to} \quad & \vec{Z}(\boldsymbol{x}) + t \geq \sum_j \theta_j (\sum_\sigma \lambda'_{j,\sigma} Q_\sigma) \vec{X}_j + \sum_j \theta_j \bar{\delta}_j \\ & \sum_{\sigma \in \Sigma} \lambda'_{j,\sigma} = 1 \, , \, \forall j, \, \boldsymbol{\lambda}' \geq 0 \, , \\ & \sum_j \theta_j = 1, \, \boldsymbol{\theta} \geq 0 \, , \end{aligned}$$

where $Q_\sigma \in \mathbb{R}^{M \times M}$ is the permutation matrix associated to the $\sigma$ permutation operator. This equivalence follows from the fact that a feasible solution $(\boldsymbol{x}, t, \boldsymbol{\lambda}', \boldsymbol{\theta})$ to the second model can be used to construct a feasible solution to the first one simply by assigning $\lambda_{j,\sigma} := \theta_j \lambda'_{j,\sigma}$ while the reverse is also true when using:

$$\lambda'_{j,\sigma} := \begin{cases} 1/|\Sigma| & \text{if } \theta_j = 0 \\ \lambda_{j,\sigma}/\theta_j & \text{otherwise.} \end{cases} .$$

One can finally realize that for each $j$, the set captured by $\{\sum_\sigma \lambda'_{j,\sigma} Q_\sigma \mid \sum_{\sigma \in \Sigma} \lambda'_{j,\sigma} = 1, \, \lambda'_{j,\sigma} \geq 0 \, \forall \sigma\}$ describes the convex hull of all permutation matrices which is known given the result of Birkhoff (1946) to be equivalently represented by $\{Q_j \in \mathbb{R}^{M \times M} \mid Q_j \mathbf{1} = \mathbf{1}, Q_j^\top \mathbf{1} = \mathbf{1}, Q_j \geq 0\}$. Using this simpler representation we obtain the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathcal{X}, t, \boldsymbol{\theta}, \{Q_j\}} \quad & t \\ \text{subject to} \quad & \vec{Z}(\boldsymbol{x}) + t \geq \sum_j \theta_j Q_j \vec{X}_j + \sum_j \theta_j \bar{\delta}_j \\ & Q_j \mathbf{1} = \mathbf{1} \, , \, Q_j^\top \mathbf{1} = \mathbf{1} \\ & \sum_j \theta_j = 1 \\ & Q_j \geq 0, \, \theta_j \geq 0 \, , \, \forall j \, . \end{aligned}$$

This last version of the problem reduces to problem (EC.9) when replacing $Q'_j := \theta_j Q_j$. □

**Step 2: The case of general measures**

We are ready to consider the general setup of the preference robust risk minimization problem (9) with $\mathcal{R}_{LE}(\{(F_k^W, F_k^Y)\}_{k=1}^K)$. Given the set of distributions $\{F_j\}_{j=1,\ldots,J}$ involved in $\mathcal{E}$ and the distribution of $\boldsymbol{\xi}$, we can convert each distribution to a random payoff in an outcome space endowed with a uniform probability measure. We can achieve this by first representing each probability value using a common denominator $M$. This can be done since, according to Assumption 2, each probability value is a rational number and there are a finite number of them. Without loss of generality, we can then assume that all distributions are induced from an outcome space $\Omega' := \{\tilde{\omega}_d\}_{d=1,...,M}$ endowed with a uniform probability measure $P$, i.e. $P(\{\tilde{\omega}_d\}) = 1/M$, $d = 1, ..., M$. In this space, a distribution $F_j$ given in the form of $P(X_j = x_k) = p_k^j$, $k = 1, ..., M_j$ can be expressed as a random payoff in $\Omega'$ that takes the form

$$\vec{X}_j' = h(F_j) = [\underbrace{(\vec{X}_j)_1 \cdots (\vec{X}_j)_1}_{\pi_1^{(j)}} \underbrace{(\vec{X}_j)_2 \cdots (\vec{X}_j)_2}_{\pi_2^{(j)}} \cdots \underbrace{(\vec{X}_j)_{M_j} \cdots (\vec{X}_j)_{M_j}}_{\pi_{M_j}^{(j)}}]^\top, \tag{EC.10}$$

where $\pi_k^{(j)} = p_k^j \cdot M$ for $k = 1, ..., M_j$, and we use $h(\cdot)$ to stand for an operator that maps a given distribution to a random payoff as described above. Each entry of this $M$-dimensional vector corresponds to the mapping from an outcome $\tilde{\omega} \in \Omega'$ to a real value. A similar mapping can be used for $\vec{Z}(\boldsymbol{x})$ with

$$\vec{Z}'(\boldsymbol{x}) = [\underbrace{r(\boldsymbol{x}, \boldsymbol{\xi}_1) \cdots r(\boldsymbol{x}, \boldsymbol{\xi}_1)}_{\pi_1^{\xi}} \underbrace{r(\boldsymbol{x}, \boldsymbol{\xi}_2) \cdots r(\boldsymbol{x}, \boldsymbol{\xi}_2)}_{\pi_2^{\xi}} \cdots \underbrace{r(\boldsymbol{x}, \boldsymbol{\xi}_{M_\xi}) \cdots r(\boldsymbol{x}, \boldsymbol{\xi}_{M_\xi})}_{\pi_{M_\xi}^{\xi}}]^\top,$$

To facilitate the exposition of the proof below, we use $l_k^j$ to denote the set of indexes $l_k^j := \{k' \in \mathbb{N} | (\vec{X}_j')_{k'} = (\vec{X}_j)_k\}$. Using this notation, we can say that $(\vec{X}_j')_d = (\vec{X}_j)_k$ for all $d \in l_k^j$. Similarly, we will refer to $l_k^\xi := \{k' \in \mathbb{N} | (\vec{Z}'(\boldsymbol{x}))_{k'} = r(\boldsymbol{x}, \boldsymbol{\xi}_k)\}$. By such conversions, we can reformulate the problem (9) into

$$\min_{\boldsymbol{x} \in \mathcal{X}} \sup_{\rho \in \mathcal{R}_{LE}(\{h(F_k^W), h(F_k^Y)\}_{k=1}^K)} \rho(\vec{Z}'(\boldsymbol{x})),$$

which admits a convex optimization formulation following Lemma EC.5 and Proposition EC.1. In the next two corollaries, we show how the convex reformulation can be further reduced.

Corollary EC.1. *The optimization problem (EC.6) in Lemma EC.5 with $\vec{X}_i' = h(F_i)$ and $\vec{X}_j' = h(F_j)$ can be reduced to the problem of (11).*

*Proof of Corollary EC.1:* Consider first the constraints (EC.6c) and (EC.6d)

$$\mathbf{1}^\top \boldsymbol{v}_{i,j} + \mathbf{1}^\top \boldsymbol{w}_{i,j} - \vec{X}_j'^\top \boldsymbol{y}_j + \delta_i - \delta_j \geq 0 \qquad \forall i \neq j \tag{EC.11}$$

$$\vec{X}_i' \boldsymbol{y}_j^\top - \boldsymbol{v}_{i,j} \mathbf{1}^\top - \mathbf{1} \boldsymbol{w}_{i,j}^\top \geq 0 \qquad \forall i \neq j \tag{EC.12}$$

For any fixed $i$ and $k \in \{1,...,M_i\}$, we can re-write the constraint (EC.12) as

$$(\boldsymbol{v}_{i,j})_d \leq (\vec{X}_i)_k(\boldsymbol{y}_j)_m - (\boldsymbol{w}_{i,j})_m,\ \forall d \in l^i_k\,, \forall m \in \{1,...,M\}\,.$$

Observe that each entry of $(\boldsymbol{v}_{i,j})_d$ in this range is bounded above by the same value for any fixed $\boldsymbol{y}^*_j$ and $\boldsymbol{w}^*_{i,j}$. Observe also that increasing $(\boldsymbol{v}_{i,j})_d$ will not violate any other constraint in (EC.6). Thus, for any given optimal solution $\boldsymbol{v}^*_{i,j}$ with $(\boldsymbol{v}^*_{i,j})_{d_1} \neq (\boldsymbol{v}^*_{i,j})_{d_2}$ for some $d_1, d_2 \in l^i_k$, we can always increase each entry of $(\boldsymbol{v}^*_{i,j})_d$ up to the same value (the upper bound) and obtain a new optimal solution $v^{**}$ that satisfies $(\boldsymbol{v}^*_{i,j})_{d_1} = (\boldsymbol{v}^*_{i,j})_{d_2}$ for any $d_1, d_2 \in l^i_k$.

Therefore, we can impose $(\boldsymbol{v}_{i,j})_d = (\tilde{\boldsymbol{v}}_{i,j})_k, \forall d \in l^i_k$, where $\tilde{\boldsymbol{v}}_{i,j} \in \mathbb{R}^{M_i}$ and reformulate the constraints into

$$\mathbf{1}^\top(\pi^{(i)} \circ \tilde{\boldsymbol{v}}_{i,j}) + \mathbf{1}^\top \boldsymbol{w}_{i,j} - \vec{X}'^\top_j \boldsymbol{y}_j + \delta_j - \delta_i \geq 0 \qquad \forall i \neq j \tag{EC.13}$$

$$\vec{X}_i \boldsymbol{y}_j^\top - \tilde{\boldsymbol{v}}_{i,j}\mathbf{1}^\top - \mathbf{1}\boldsymbol{w}^\top_{i,j} \geq 0 \qquad \forall i \neq j. \tag{EC.14}$$

Next, suppose that $\tilde{\boldsymbol{v}}^*_{i,j}, \boldsymbol{w}^*_{i,j}, \boldsymbol{y}^*_j, \delta^*_i, \delta^*_j$ are an optimal solution of the problem (EC.6) with (EC.6c) and (EC.6d) replaced by the above two sets of constraints (EC.13) and (EC.14). We claim that the solution $\tilde{\boldsymbol{v}}^*_{i,j}, \delta^*_i, \delta^*_j$ together with the newly constructed $\boldsymbol{y}^{**}_j$ and $\boldsymbol{w}^{**}_{i,j}$:

$$(\boldsymbol{y}^{**}_j)_{\hat{d}} := (1/\pi^{(j)}_{k(j,\hat{d})}) \sum_{d \in l^j_{k(j,\hat{d})}} (\boldsymbol{y}^*_j)_d$$

$$(\boldsymbol{w}^{**}_{i,j})_{\hat{d}} = (1/\pi^{(j)}_{k(j,\hat{d})}) \sum_{d \in l^j_{k(j,\hat{d})}} (\boldsymbol{w}^*_{i,j})_d\,,$$

where $k(j,\hat{d})$ refers to the only index such that $\hat{d} \in l^j_k$, will also be optimal. Substituting this new solution into the constraint (EC.13), we have

$$\begin{aligned}
&\mathbf{1}^\top(\pi^{(i)} \circ \tilde{\boldsymbol{v}}^*_{i,j}) + \mathbf{1}^\top \boldsymbol{w}^{**}_{i,j} - \vec{X}'^\top_j \boldsymbol{y}^{**}_j + \delta^*_i - \delta^*_j \\
&\qquad = \mathbf{1}^\top(\pi^{(i)} \circ \tilde{\boldsymbol{v}}^*_{i,j}) + \sum_{k=1}^{M_j} \pi^{(j)}_k (1/\pi^{(j)}_k) \sum_{d \in l^j_k} (\boldsymbol{w}^*_{i,j})_d - \sum_{k=1}^{M_j} (\vec{X}_j)_k \pi^{(j)}_k (1/\pi^{(j)}_k) \sum_{d \in l^j_k} (\boldsymbol{y}^*_j)_d + \delta^*_i - \delta^*_j \\
&\qquad = \mathbf{1}^\top(\pi^{(i)} \circ \tilde{\boldsymbol{v}}^*_{i,j}) + \mathbf{1}^\top \boldsymbol{w}^*_{i,j} - \vec{X}'^\top_j \boldsymbol{y}^*_j + \delta^*_i - \delta^*_j \ \geq\ 0
\end{aligned}$$

To verify the feasibility of the second constraint (EC.14), let us consider the following derivations for any fixed $j$ and fixed $k \in \{1, 2, \ldots, M_j\}$:

$$\begin{aligned}
\vec{X}_i \boldsymbol{y}^{*\top}_j - \tilde{\boldsymbol{v}}^*_{i,j}\mathbf{1}^\top - \mathbf{1}\boldsymbol{w}^{*\ \top}_{i,j} \geq 0 &\Rightarrow \vec{X}_i (\sum_{d \in l^j_k} (\boldsymbol{y}^*_j)_d) - \tilde{\boldsymbol{v}}^*_{i,j}\pi^{(j)}_k - \mathbf{1}(\sum_{d \in l^j_k} (\boldsymbol{w}^*_{i,j})_d) \geq 0 \\
&\Rightarrow \vec{X}_i (1/\pi^{(j)}_k)(\sum_{d \in l^j_k} (\boldsymbol{y}^*_j)_d) - \tilde{\boldsymbol{v}}^*_{i,j} - \mathbf{1}(1/\pi^{(j)}_k)(\sum_{d \in l^j_k} (\boldsymbol{w}^*_{i,j})_d) \geq 0 \\
&\Rightarrow \vec{X}_i \boldsymbol{y}^{**\top}_j - \tilde{\boldsymbol{v}}^*_{i,j}\mathbf{1}^\top - \mathbf{1}\boldsymbol{w}^{**\top}_{i,j} \geq 0.
\end{aligned}$$

where the first step is obtained by summing the columns in the range $l_k^j$ for the matrix on the lefthand side of the inequality.

It is straightforward to see $\mathbf{1}^\top \boldsymbol{y}_j^{**} = \mathbf{1}^\top \boldsymbol{y}_j^* = 1$, which verifies the feasibility of the constraint (EC.6e).

Thus, we can impose that $(\boldsymbol{y}_j)_d = (\tilde{\boldsymbol{y}}_j)_{k(j,d)}$ for all $j$ and $d$, where $\tilde{\boldsymbol{y}}_j \in \mathbb{R}^{M_j}$, and that $(\boldsymbol{w}_{i,j})_d = (\tilde{\boldsymbol{w}}_{i,j})_k \; d \in l_k^j$, where $\tilde{\boldsymbol{w}}_{i,j} \in \mathbb{R}^{M_j}$, and reformulate the constraints (EC.13), (EC.14) and (EC.6e) into

$$\begin{aligned}
&\mathbf{1}^\top(\pi^{(i)} \circ \tilde{\boldsymbol{v}}_{i,j}) + \mathbf{1}^\top(\pi^{(j)} \circ \tilde{\boldsymbol{w}}_{i,j}) - \vec{X}_j^\top(\pi^{(j)} \circ \tilde{\boldsymbol{y}}_j) + \delta_j - \delta_i \geq 0 && \forall i \neq j\\
&\vec{X}_i \tilde{\boldsymbol{y}}_j^\top - \tilde{\boldsymbol{v}}_{i,j}\mathbf{1}^\top - \mathbf{1}\tilde{\boldsymbol{w}}_{i,j}^\top \geq 0 && \forall i \neq j\\
&\mathbf{1}^\top(\pi^{(j)} \circ \tilde{\boldsymbol{y}}_j) = 1 && \forall j.
\end{aligned}$$

Let $\hat{\boldsymbol{v}}_{i,j} = \pi^{(i)} \circ \tilde{\boldsymbol{v}}_{i,j}$ , $\hat{\boldsymbol{w}}_{i,j} = \pi^{(j)} \circ \tilde{\boldsymbol{w}}_{i,j}$, and $\hat{\boldsymbol{y}}_j = \pi^{(j)} \circ \tilde{\boldsymbol{y}}_j$. The above second constraint becomes

$$\vec{X}_i((\pi^{(j)})^{-1} \circ \hat{\boldsymbol{y}}_j)^\top - ((\pi^{(i)})^{-1} \circ \hat{\boldsymbol{v}}_{i,j})\mathbf{1}^\top - \mathbf{1}((\pi^{(j)})^{-1} \circ \tilde{\boldsymbol{w}}_{i,j})^\top \geq 0\,,$$

where $(\pi^{(i)})^{-1}$ satisfies $(\pi^{(i)})^{-1} \circ (\pi^{(i)}) = \mathbf{1}$. Finally, multiplying $(\pi^{(i)}\mathbf{1}^\top)$ to the inequality we have

$$\begin{aligned}
&(\pi^{(i)}\mathbf{1}^\top)\circ\left(\vec{X}_i((\pi^{(j)})^{-1} \circ \hat{\boldsymbol{y}}_j)^\top - ((\pi^{(i)})^{-1}\hat{\boldsymbol{v}}_{i,j})\mathbf{1}^\top - \mathbf{1}((\pi^{(j)})^{-1} \circ \tilde{\boldsymbol{w}}_{i,j})^\top\right) \geq 0\\
&\Rightarrow \Pi_{i,j} \circ \vec{X}_i \hat{\boldsymbol{y}}_j{}^\top - \hat{\boldsymbol{v}}_{i,j}\mathbf{1}^\top - \Pi_{i,j} \circ \mathbf{1}\tilde{\boldsymbol{w}}_{i,j}^\top \geq 0,
\end{aligned}$$

where $\Pi \in \mathbb{R}^{M_i \times M_j}$ and

$$\Pi_{i,j} = (\pi^{(i)})((\pi^{(j)})^{-1})^\top,$$

as described in the corollary. This completes the proof. □

Note that in the proof below, for simplicity we will use the notation

$$V_{(a_1:a_2,b_1:b_2)} = \begin{bmatrix} V_{a_1,b_1} & \cdots & V_{a_1,b_2} \\ \vdots & \ddots & \vdots \\ V_{a_2,b_1} & \cdots & V_{a_2,b_2} \end{bmatrix}$$

to describe a submatrix of a matrix $V$. The notation $V_{(:,k)}$ (respectively $V_{(k,:)}$) will refer to the $k$th-column (respectively $k$th-row) of the matrix $V$.

Corollary EC.2. *The problem (EC.9) in Proposition EC.1 with $\vec{Z}'(\boldsymbol{x})$ and $\vec{X}_j' = h(F_j)$ can be reduced to the problem of (10)*

*Proof of Corollary EC.2:* Suppose that $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*, \{Q_j^*\}$ are an optimal solution of (EC.9). We claim that the solution $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*$ together with the newly constructed $Q_j^{**}$ such that for all $k \in \{1, 2, \ldots, M_j\}$:

$$Q_j^{**}(:, l_k^j) := (1/\pi_k^{(j)}) \sum_{d \in l_k^j} Q_j^*(:, d)\mathbf{1}^\top$$

is also optimal.

Substituting into the constraint (EC.9b), we have that for each $j \in \{1, ..., J\}$ ,

$$Q_j^{**}\vec{X}_j' = \sum_{k=1}^{M_j} \pi_k^{(j)} \cdot ((\vec{X}_j)_k/\pi_k^{(j)}) \left(\sum_{d\in l_k^j} (Q_j^*)_{(:,d)}\right) = \sum_{k=1}^{M_j} (\vec{X}_j)_k \left(\sum_{d\in l_k^j} (Q_j^*)_{(:,d)}\right) = Q_j^*\vec{X}_j'.$$

Substituting into the constraint (EC.9c), we have for each $j \in \{1, ..., J\}$

$$Q_j^{**}\mathbf{1} = \sum_{k=1}^{M_j} \pi_k^{(j)} \cdot (1/\pi_k^{(j)}) \left(\sum_{d\in l_k^j} (Q_j^*)_{(:,d)}\right) = Q_j^*\mathbf{1} = \theta_j^*.$$

Substituting into the constraint (EC.9d), we have for each $j \in \{1, ..., J\}$ and for each $\tilde{d} \in \{1, ..., M\}$ we have

$$\begin{aligned}
(Q_j^{**\top}\mathbf{1})_{\tilde{d}} &= (1/\pi_{k(j,\tilde{d})}^{(j)}) \sum_{m=1}^{M} \sum_{d\in l_{k(j,\hat{d})}^j} (Q_j^*)_{m,d} \\
&= (1/\pi_{k(j,\tilde{d})}^{(j)}) \sum_{d\in d\in l_{k(j,\hat{d})}^j} \sum_{m=1}^{M} (Q_j^*)_{m,d} \\
&= (1/\pi_{k(j,\tilde{d})}^{(j)}) \sum_{d\in d\in l_{k(j,\hat{d})}^j} \theta_j^* = \theta_j^*.
\end{aligned}$$

As $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*, \{Q_j^{**}\}$ satisify all constraints and $t^*$ remains the same minimum value, $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*, \{Q_j^{**}\}$ are also an optimal solution. Therefore, we can impose that for all $j$ and all $k \in \{1, 2, \ldots, M_j\}$ we have that $(Q_j)_{(:,l_k^j)} = (\tilde{Q}_j)_{(:,k)}\mathbf{1}^\top$, where $\tilde{Q}_j \in \mathbb{R}^{M\times M_j}$, and reformulate (EC.9) into

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathcal{X}, t, \boldsymbol{\theta}, \{\tilde{Q}_j\}} \quad & t && \text{(EC.15a)} \\
\text{subject to} \quad & \vec{Z}'(\boldsymbol{x}) + t \geq \sum_j ((\mathbf{1}\pi^{(j)\top}) \circ \tilde{Q}_j)\vec{X}_j + \bar{\boldsymbol{\delta}}^\top \boldsymbol{\theta} && \text{(EC.15b)} \\
& ((\mathbf{1}\pi^{(j)\top}) \circ \tilde{Q}_j)\mathbf{1} = \theta_j, \forall j && \text{(EC.15c)} \\
& \tilde{Q}_j^\top \mathbf{1} = \theta_j, \forall j && \text{(EC.15d)} \\
& \mathbf{1}^\top \boldsymbol{\theta} = 1 && \text{(EC.15e)} \\
& \boldsymbol{\theta} \geq 0, \tilde{Q}_j \geq 0, \forall j. && \text{(EC.15f)}
\end{aligned}$$

Next, suppose that $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*, \{\tilde{Q}_j^*\}$ is the optimal solution for the above problem. We claim that that the solution $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*$ together with the newly constructed $\tilde{Q}_j^{**}$ such that for all $k = 1, ..., M_\xi$ and all $\tilde{d} \in l_k^\xi$ we have that

$$(\tilde{Q}_j^{**})_{(\tilde{d},:)} = (1/\pi_k^\xi) \sum_{d\in l_k^\xi} (\tilde{Q}_j^*)_{(d,:)}$$

is also optimal.

Substituting into the constraint (EC.15b), we have for all $\tilde{d} \in \{1, ..., M\}$ we have that

$$\left(\sum_j ((\mathbf{1}\pi^{(j)\top}) \circ \tilde{Q}_j^{**})\vec{X}_j\right)_{\tilde{d}} = (1/\pi^{\xi}_{k(j,\tilde{d})})\left(\sum_{d\in l^{\xi}_{k(j,\tilde{d})}} \sum_j (\pi^{(j)\top} \circ \tilde{Q}_j^*(d,:))\vec{X}_j\right)$$
$$\leq (1/\pi^{\xi}_{k(j,\tilde{d})})(\sum_{d\in l^{\xi}_{k(j,\tilde{d})}} (\vec{Z}'(\boldsymbol{x}))_d + t^* - \bar{\boldsymbol{\delta}}^\top \boldsymbol{\theta}^*)$$
$$= (1/\pi^{\xi}_{k(j,\tilde{d})})(\pi^{\xi}_{k(j,\tilde{d})}((\vec{Z}'(\boldsymbol{x}))_{\tilde{d}} + t^* - \bar{\boldsymbol{\delta}}^\top \boldsymbol{\theta}^*)$$
$$= (\vec{Z}'(\boldsymbol{x}))_{\tilde{d}} + t^* - \bar{\boldsymbol{\delta}}^\top \boldsymbol{\theta}^*.$$

Substituting into the constraint (EC.15c), we have for each $j$ and for all $\tilde{d} \in \{1, ..., M\}$ that

$$\left(((\mathbf{1}\pi^{(j)\top}) \circ \tilde{Q}_j^{**})\mathbf{1}\right)_{\tilde{d}} = (1/\pi^{\xi}_{k(j,\tilde{d})}) \sum_{m=1}^{M_j} \pi^{(j)}_m \sum_{d\in l^{\xi}_{k(j,\tilde{d})}} (\tilde{Q}_j^*)_{(d,m)}$$
$$= (1/\pi^{\xi}_{k(j,\tilde{d})}) \sum_{d\in l^{\xi}_{k(j,\tilde{d})}} \sum_{m=1}^{M_j} (\pi^{(j)}_m)(\tilde{Q}_j^*)_{(d,m)}$$
$$= (1/\pi^{\xi}_{k(j,\tilde{d})}) \sum_{d\in l^{\xi}_{k(j,\tilde{d})}} \theta_j^* = \theta_j^*.$$

Substituting into the constraint (EC.15d), we have

$$\tilde{Q}_j^{**\top}\mathbf{1} = \sum_{k=1}^{M_\xi} \pi^{\xi}_k \cdot (1/\pi^{\xi}_k) \sum_{d\in l^{\xi}_k} (\tilde{Q}_j^*)_{(d,:)} = \tilde{Q}_j^{*\top}\mathbf{1} = \theta_j^*.$$

As $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*, \{\tilde{Q}_j^{**}\}$ satisfy all constraints and $t^*$ remains the same minimum value, $\boldsymbol{x}^*, t^*, \boldsymbol{\theta}^*, \tilde{Q}_j^{**}$ are also an optimal solution. We can now impose that for all $k \in \{1, 2, \ldots, M_\xi\}$ we have that $(\tilde{Q}_j)_{(l^{\xi}_k,:)} = \mathbf{1}(\hat{Q}_j)_{(k,:)}$, where $\hat{Q}_j \in \mathbb{R}^{M_\xi \times M_j}$, and reformulate the problem (EC.15) into

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathcal{X}, t, \boldsymbol{\theta}, \{\hat{Q}_j\}} \quad & t \\
\text{subject to} \quad & \vec{Z}_\xi(\boldsymbol{x}) + t \geq \sum_j ((\mathbf{1}\pi^{(j)\top}) \circ \hat{Q}_j)\vec{X}_j + \bar{\boldsymbol{\delta}}^\top \boldsymbol{\theta}, \\
& ((\mathbf{1}\pi^{(j)\top}) \circ \hat{Q}_j)\mathbf{1} = \theta_j, \forall j \\
& ((\pi^{\xi}\mathbf{1}^\top) \circ \hat{Q}_j)^\top \mathbf{1} = \theta_j, \forall j \\
& \mathbf{1}^\top \boldsymbol{\theta} = 1 \\
& \boldsymbol{\theta} \geq 0, \hat{Q}_j \geq 0, \forall j.
\end{aligned}$$

Finally, Let $\hat{Q}'_j = (\mathbf{1}\pi^{(j)\top}) \circ \hat{Q}_j$. The left-hand-side of the above third constraint can be written as $((\pi^{\xi}\mathbf{1}^\top) \circ ((\mathbf{1}(\pi^{(j)})^{-1\top}) \circ \hat{Q}'_j) = (\pi^{\xi}((\pi^{(j)})^{-1})^\top) \circ \hat{Q}'_j$. Having $\Pi_j = \pi^{\xi}((\pi^{(j)})^{-1})^\top$, we arrive at the formulation. □

## EC.5. Alternate proofs for Propositions 2, 3 and 4

*Alternate proof of Proposition 2:* It is well known that all risk measures in $\mathfrak{R}$ can be represented as

$$\rho(\vec{Z}) := \sup_{\boldsymbol{p}\in\mathcal{P}} \boldsymbol{p}^\top(-\vec{Z}) - \psi(\boldsymbol{p}) \,,$$

where $\boldsymbol{p}\in\mathbb{R}^M$, for some $\mathcal{P}\subseteq\{\boldsymbol{p}\in\mathbb{R}^M_+ \,|\, \mathbf{1}^\top\boldsymbol{p}=1\}$, and some convex function $\psi:\mathbb{R}^M\to\mathbb{R}$. This can be equivalently represented as

$$\rho(\vec{Z}) := \sup_{(\boldsymbol{p},s)\in\mathcal{U}} \boldsymbol{p}^\top(-\vec{Z}) - s \,, \tag{EC.16}$$

with $\mathcal{U} := \{(\boldsymbol{p},s)\in\mathcal{P}\times\mathbb{R} \,|\, s\geq\psi(\boldsymbol{p})\}$. Additionally, the constraint that $\rho(\mathbf{0})=0$ implies that

$$\rho(\mathbf{0}) = \sup_{(\boldsymbol{p},s)\in\mathcal{U}} -s = 0$$

which further implies that $\mathcal{U}\subseteq\mathcal{P}\times\mathbb{R}_+$. Finally, if $\rho\in\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)$, then it must be that for all $k$

$$\rho(\vec{W}_k)\leq 0 \Leftrightarrow \sup_{(\boldsymbol{p},s)\in\mathcal{U}} \boldsymbol{p}^\top(-\vec{W}_k) - s\leq 0 \Leftrightarrow \mathcal{U}\subset\{(\boldsymbol{p},s)\in\mathbb{R}^M\times\mathbb{R} \,|\, -\vec{W}_k^\top\boldsymbol{p}-s\leq 0\} \,.$$

Now, let us construct a risk measure $\bar{\rho}(\vec{Z}) := \sup_{(\boldsymbol{p},s)\in\bar{\mathcal{U}}} \boldsymbol{p}^\top(-\vec{Z}) - s$ such that

$$\bar{\mathcal{U}} := \{(\boldsymbol{p},s)\in\mathcal{P}\times\mathbb{R}_+ \,|\, -\vec{W}_k^\top\boldsymbol{p}-s\leq 0, \forall\, k=1,\ldots,K\} \,.$$

We can show that $\varrho_{\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)}(\vec{Z}) = \bar{\rho}(\vec{Z})$. First, it is clear that $\varrho_{\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)}(\vec{Z}) \leq \bar{\rho}(\vec{Z})$ since for each $\rho\in\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)$, the set $\mathcal{U}$ that corresponds to the measure in representation (EC.16) is a subset of $\bar{\mathcal{U}}$. Now, to verify the reverse direction, i.e. $\varrho_{\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)}(\vec{Z}) \geq \bar{\rho}(\vec{Z})$, it suffices to show that $\bar{\rho}(\vec{Z})$ is a member of $\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)$. This can be easily verified if $\bar{\rho}(\mathbf{0})=0$, since for all $k$

$$\bar{\rho}(\vec{W}_k) = \sup_{(\boldsymbol{p},s)\in\bar{\mathcal{U}}} -\boldsymbol{p}^\top\vec{W}_k - s\leq 0 = \bar{\rho}(\mathbf{0}) \,,$$

and by construction $\bar{\rho}$ is a convex risk measure. Hence, if $\bar{\rho}(\mathbf{0})=0$, we have

$$\begin{aligned}
\varrho_{\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)}(\vec{Z}) = \max_{\boldsymbol{p},s} \quad & -\boldsymbol{p}^\top\vec{Z} - s, \\
\text{subject to} \quad & -\vec{W}_k^\top\boldsymbol{p} - s\leq 0, \forall\, k=1,\ldots,K \\
& \mathbf{1}^\top\boldsymbol{p}=1,\ \boldsymbol{p}\geq 0 \\
& s\geq 0,
\end{aligned}$$

which is equivalent to the minimization form in equation (4) since the latter is the dual formulation of the linear program presented here. Note that strong duality holds here since problem (4) is always feasible.

In the case that $\bar{\rho}(\mathbf{0}) \neq 0$, then $\bar{\rho}(\mathbf{0})$ is necessarily strictly smaller than zero, and we show that in this case $\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)$ must be empty. The strict inequality $\bar{\rho}(\mathbf{0}) < 0$ implies that the following system of linear inequalities is infeasible:

$$-\vec{W}_k^\top \boldsymbol{p} \leq 0, \,\forall\, k = 1,\ldots,K$$
$$\mathbf{1}^\top \boldsymbol{p} = 1, \,\boldsymbol{p} \geq 0\,.$$

By Farkas lemma, one can show that this implies that there exists a convex combination characterized by $\vec{W}_\theta := \sum_k \theta_k \vec{W}_k$ such that $\vec{W}_\theta < 0$. Yet, given any risk measure $\rho(\cdot)$ that satisfies all conditions expressed in $\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)$, we should have $\rho(\vec{W}_k) \leq 0$ and also that

$$0 \geq \sum_k \theta_k \rho(\vec{W}_k) \geq \rho(\vec{W}_\theta) > \rho(\mathbf{0}) = 0\,,$$

which indicates a contradiction, and hence $\mathcal{R}_{El}(\{(W_k,0)\}_{k=1}^K)$ must be empty. □

*Alternate proof of Proposition 3:* We present the proof of this proposition in three steps. First, we show how problem (5) corresponds to the problem of minimizing the worst-case risk when the risk measure is constrained to be equal to $\delta_1,\delta_2,\ldots,\delta_J$ for the random payoffs $\vec{X}_1,\vec{X}_2,\ldots,\vec{X}_J$ respectively, i.e. the risk is evaluated based on the inner maximization problem of (EC.2). We then explain how for the set of elicited comparisons, the constraints described in problem (6) characterize all feasible risk assignments for the random payoffs $\vec{X}_1,\vec{X}_2,\ldots,\vec{X}_J$ such that these respect the $K$ comparisons described in $\mathcal{E}$, i.e. the set of $\delta_1,\delta_2,\ldots,\delta_J$ that satisfies (6) is equivalent to the one that satisfies (EC.2d). We conclude the proof with the same arguments as used in the proof of Proposition 3 that indicate that the worst-case risk assignments can be found by employing the objective function of problem (6).

*Step 1:* Similarly as was done for the alternate proof of Proposition 2, we can argue that any risk measure $\rho \in \mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$ (see the definition in Section EC.3) can be represented as $\rho(\vec{Z}) := \sup_{(\boldsymbol{p},s)\in\mathcal{U}} \boldsymbol{p}^\top(-\vec{Z}) - s$ for some $\mathcal{U} \subseteq \mathcal{P}\times\mathbb{R}_+$. Yet, in order to respect the fact that $\rho(\vec{X}_j) = \delta_j$ for all $j$, it must also be the case that $\rho(\vec{X}_j + \delta_j) \leq 0$ so that $\mathcal{U} \subset \{(\boldsymbol{p},s)\in\mathbb{R}^M\times\mathbb{R} \,|\, -(\vec{X}_j+\delta_j)^\top\boldsymbol{p} - s \leq 0,\,\forall\, j = 1,2,\ldots,J\}$. This leads to constructing the risk measure $\bar{\rho}$ whose associated $\bar{\mathcal{U}}$ is described as:

$$\bar{\mathcal{U}} := \{(\boldsymbol{p},s)\in\mathcal{P}\times\mathbb{R}_+ \,|\, -(\vec{X}_j+\delta_j)^\top\boldsymbol{p} - s \leq 0,\,\forall\, j = 1,\ldots,J\}\,.$$

We can show that if $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$ is not empty, then $\varrho_{\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)}(\vec{Z}) = \bar{\rho}(\vec{Z})$. Similarly as in the alternate proof of Proposition 2, by construction of $\bar{\mathcal{U}}$ it is clear that $\varrho_{\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)}(\vec{Z}) \leq \bar{\rho}(\vec{Z})$. The reverse is also true since once again either $\bar{\rho}(\vec{Z}) \in \mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$ or the latter set is empty. To confirm this, one needs to study whether for all $j$, the value $\bar{\rho}(\vec{X}_j+\delta_j) = 0$. Taking the case of

$j=J$ as an example, this can be expressed as whether there is a $\boldsymbol{p}\in\mathbb{R}^M$ and a $s\in\mathbb{R}$ that satisfy the following linear constraints:

$$\begin{aligned}
&-\boldsymbol{p}^\top(\vec{X}_J+\delta_J)-s=0\\
&-(\vec{X}_j+\delta_j)^\top\boldsymbol{p}-s\leq 0,\,\forall j=1,\ldots,J-1\\
&\mathbf{1}^\top\boldsymbol{p}=1,\,\boldsymbol{p}\geq 0\\
&s\geq 0\,.
\end{aligned}$$

According to Farkas lemma, this system can only be infeasible if there exists a $\boldsymbol{\theta}\in\mathbb{R}^J$ such that

$$\begin{aligned}
&\sum_{j\neq J}\theta_j\leq\theta_J\\
&\sum_{j\neq J}(\vec{X}_j+\delta_j)\theta_j<\theta_J(\vec{X}_J+\delta_J)\\
&\theta_j\geq 0,\,\forall j=1,\ldots,J\,.
\end{aligned}$$

The latter can only happen under two circumstances. First, it might be the case that $\sum_{j\neq J}\theta_j=0$ which would indicate that $\theta_J(\vec{X}_J+\delta_J)>0$. Yet, this could only occur if $\vec{X}_J+\delta_J>0$ but this contradicts the fact that $\rho(\vec{X}_J+\delta_J)=0$ because $\rho(\vec{X}_J+\delta_J)\leq\rho(\min_i(\vec{X}_J)_i+\delta_J)=-(\min_i(\vec{X}_J)_i+\delta_J)<0$. The alternative would be that $\sum_{j\neq J}\theta_j>0$ which implicates that

$$\vec{X}_J+\delta_J>\sum_{j\neq J}\frac{\theta_j}{\theta_J}(\vec{X}_j+\delta_j)+\frac{\theta_J-\sum_{j\neq J}\theta_j}{\theta_J}\mathbf{0}\,.$$

Looking more closely at the right hand side of this inequality, we can recognize a convex combination of $J$ random variables that are supposed to have a risk equal to zero for all risk measures that are in $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$. This leads to a contradiction since by monotonicity arguments it would imply that the risk of $\vec{X}_J+\delta_J$ is strictly negative following

$$\rho(\vec{X}_J-\delta_J)<\rho\left(\sum_{j\neq J}\frac{\theta_j}{\theta_J}(\vec{X}_j+\delta_j)+\frac{\theta_J-\sum_{j\neq J}\theta_j}{\theta_J}\mathbf{0}\right)\leq\sum_{j\neq J}\frac{\theta_j}{\theta_J}\rho(\vec{X}_j+\delta_j)=0\,,$$

This also cannot happen for risk measures in $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$.

To summarize, we have shown that if $\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)$ is non-empty, then

$$\begin{aligned}
\varrho_{\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)}(\vec{Z})=\quad&\max_{\boldsymbol{p},s}\quad -\boldsymbol{p}^\top\vec{Z}-s &\text{(EC.17a)}\\
&\text{subject to}\quad -(\vec{X}_j+\delta_j)^\top\boldsymbol{p}-s\leq 0,\,\forall j=1,\ldots,J &\text{(EC.17b)}\\
&\qquad\mathbf{1}^\top\boldsymbol{p}=1,\,\boldsymbol{p}\geq 0 &\text{(EC.17c)}\\
&\qquad s\geq 0\,, &\text{(EC.17d)}
\end{aligned}$$

which by duality theory can be shown equivalent to the minimum in $t$ and $\boldsymbol{\theta}$ expressed in problem (5).

*Step 2:* We start by describing a number of conditions that must be satisfied by a joint assignment $(\delta_1, \delta_2, \ldots, \delta_J)$ in order for this assignment to capture plausible measurements of the risk associated to each of the random payoffs $(\vec{X}_1, \ldots, \vec{X}_J)$. First, based on $\mathcal{E}$:

$$\forall (i,j) \in \bar{\mathcal{E}}, \rho(\vec{X}_i) \leq \rho(\vec{X}_j) \Rightarrow \delta_i \leq \delta_j \, .$$

Next, by convexity of $\rho(\cdot)$, for each $j$, there must exist a hyperplane that supports the $\rho(\cdot)$ function at $\vec{X}_j$. Specifically, there must exist a vector $\boldsymbol{y}_j \in \mathbb{R}^M$ such that $\rho(\vec{Z}) \geq \rho(\vec{X}_j) - \boldsymbol{y}_j^\top(\vec{Z} - \vec{X}_j)$ for all random variables $\vec{Z}$. In particular, it must be the case that for all $i = 1, 2, \ldots, J$, we have that $\rho(\vec{X}_i) \geq \rho(\vec{X}_j) - \boldsymbol{y}_j^\top(\vec{X}_i - \vec{X}_j)$. This leads to the condition that

$$\forall i \neq j, \, \rho(\vec{X}_i) \geq \rho(\vec{X}_j) - \boldsymbol{y}_j^\top(\vec{X}_i - \vec{X}_j) \Rightarrow \delta_i \geq \delta_j - \boldsymbol{y}_j^\top(\vec{X}_i - \vec{X}_j) \, .$$

Furthermore, the fact that $\rho(\cdot)$ is monotone and translation invariant implies additional properties about $\boldsymbol{y}_j$. Namely, monotonicity implies that for all $i = 1, \ldots, M$,

$$\rho(\vec{X}_j + e_i) \leq \rho(\vec{X}_j) \Rightarrow \rho(\vec{X}_j) \geq \rho(\vec{X}_j + e_i) \geq \rho(\vec{X}_j) - \boldsymbol{y}_j^\top e_i \Rightarrow (\boldsymbol{y}_j)_i \geq 0 \, ,$$

where $e_i$ refers to the $i$-th column of the identity matrix and which can be summarized as $\boldsymbol{y}_j \geq 0$. Regarding translation invariance, one can discover that

$$\rho(\vec{X}_j + \mathbf{1}) = \rho(\vec{X}_j) - 1 \Rightarrow \rho(\vec{X}_j) - 1 = \rho(\vec{X}_j + 1) \geq \rho(\vec{X}_j) - \boldsymbol{y}_j^\top \mathbf{1} \Rightarrow \mathbf{1}^\top \boldsymbol{y}_j \geq 1 \, ,$$

and similarly that

$$\rho(\vec{X}_j - \mathbf{1}) = \rho(\vec{X}_j) + 1 \Rightarrow \rho(\vec{X}_j) + 1 \geq \rho(\vec{X}_j) - \boldsymbol{y}_j^\top(-\mathbf{1}) \Rightarrow \mathbf{1}^\top \boldsymbol{y}_j \leq 1 \, .$$

Together with the fact that $\delta_1 = \rho(\mathbf{0}) = 0$, this means that for any feasible assignment there must exist a set of $\{\boldsymbol{y}_j\}_{j=1}^J$ that satisfies constraints (6b) to (6d).

Now to demonstrate that these conditions are also sufficient, we can simply verify that the risk measure constructed as

$$\bar{\rho}(\vec{Z}) := \max_{j=1,\ldots,J} \delta_j - \boldsymbol{y}_j^\top(\vec{Z} - \vec{X}_j)$$

based on a feasible solution of $\{(\delta_j, \boldsymbol{y}_j)\}_{j=1}^J$ is necessarily a member of $\mathcal{R}_\delta(\{(\vec{X}_j, \delta_j)\}_{j=1}^J)$. One should start by confirming that $\bar{\rho}(\vec{X}_j) = \delta_j$ as expected :

$$\bar{\rho}(\vec{X}_j) = \max_{i=1,\ldots,J} \delta_i - \boldsymbol{y}_i^\top(\vec{X}_j - \vec{X}_i) = \delta_j \, ,$$

since $\delta_j - \boldsymbol{y}_j^\top(\vec{X}_j - \vec{X}_j) = \delta_j \geq \delta_i - \boldsymbol{y}_i^\top(\vec{X}_j - \vec{X}_i)$ for all $i \neq j$. Next, one can confirm that

$$\bar{\rho}(\vec{Z}) = \max_{j=1,\ldots,J} \delta_j - \boldsymbol{y}_j^\top(\vec{Z} - \vec{X}_j) = \sup_{(\boldsymbol{p},s) \in \bar{\mathcal{U}}} -\boldsymbol{p}^\top \vec{Z} - s \, ,$$

for the set $\bar{\mathcal{U}} = \text{ConvexHull}(\{(\boldsymbol{y}_j, -\delta_j - \boldsymbol{y}_j^\top \vec{X}_j)\}_{j=1}^J)$. One can straightforwardly confirm that $\bar{\mathcal{U}}$ is a subset of $\mathcal{P} \times \mathbb{R}$ so that $\bar{\rho}(\cdot)$ is a convex risk measure.

*Step 3:* Similarly as in Section EC.3, one can rely on Lemma EC.2 to confirm that $\varrho_{\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)}(\vec{Z})$ is an increasing function of $\boldsymbol{\delta}$, and rely on Lemma EC.3 to obtain that the optimal solution $\bar{\boldsymbol{\delta}}$ of problem (6) returns the largest value achievable for each $\delta_j$. This provides the guarantee that $\varrho_{\mathcal{R}_{El}(\mathcal{E})}(\vec{Z}) = \varrho_{\mathcal{R}_\delta(\{(\vec{X}_j,\bar{\delta}_j)\}_{j=1}^J)}(\vec{Z})$ which by duality theory is equivalent to what is optimized in problem (5). □

*Alternate proof of Proposition 4:* The alternate proof for this proposition is very similar to the proof of Proposition 3. Namely, in step 1 one should exploit positive homogeneity to further impose that $s=0$ as is known to be the case for coherent risk measures. This leads to the following formulation:

$$\begin{aligned} \varrho_{\mathcal{R}_{Coh}\cap\mathcal{R}_\delta(\{(\vec{X}_j,\delta_j)\}_{j=1}^J)}(\vec{Z}) = \quad & \max_{\boldsymbol{p},s} \; -\boldsymbol{p}^\top \vec{Z} \\ & \text{subject to} \; -(\vec{X}_j+\delta_j)^\top \boldsymbol{p} \le 0, \, \forall j=1,\ldots,J \\ & \mathbf{1}^\top \boldsymbol{p} = 1, \, \boldsymbol{p} \ge 0\,, \end{aligned}$$

which can be shown equivalent to what is optimized in problem (7) using duality theory.

As for step 2, one should further impose on the assignment $\{(\delta_j,\boldsymbol{y}_j)\}_{j=1}^J$ that it must satisfy a property implied by positive homogeneity. Namely,

$$\rho(2\vec{X}_j) = 2\rho(\vec{X}_j) \Rightarrow 2\rho(\vec{X}_j) = \rho(2\vec{X}_j) \ge \rho(\vec{X}_j) - \boldsymbol{y}_j^\top \vec{X}_j \Rightarrow \delta_j \ge -\boldsymbol{y}_j^\top \vec{X}_j\,,$$

and

$$\rho(0.5\vec{X}_j) = 0.5\rho(\vec{X}_j) \Rightarrow 0.5\rho(\vec{X}_j) \ge \rho(\vec{X}_j) + 0.5\boldsymbol{y}_j^\top \vec{X}_j \Rightarrow \delta_j \le -\boldsymbol{y}_j^\top \vec{X}_j\,,$$

in other words $\delta_j = -\boldsymbol{y}_j^\top \vec{X}_j$. If one makes the proper replacement of variables in problem (6), he straightforwardly obtains problem (8). □

## EC.6. Preference Robust Aspiration Measures

We discuss in this section what one may encounter when following ideas from this article to seek a preference robust measure for the case of aspiration measures (AM). In particular, we consider the case when a decision maker is known to have aspirational preferences but favours only diversification. Moreover, we assume without loss of generality that the aspiration measure $\mathfrak{a}$ that captures the preferences is normalized such that $\mathfrak{a}(\mathbf{0}) = 0$. In this case, given pairs of elicited comparisons we can write down the following set of aspiration measures $\mathfrak{A}$ following the definition given in Brown et al. (2012):

$$\mathfrak{A} := \left\{ \mathfrak{a} : \mathbb{R}^M \to \bar{\mathbb{R}} \;\middle|\; \begin{array}{c} \vec{Z}_1 \ge \vec{Z}_2 \Rightarrow \mathfrak{a}(\vec{Z}_1) \ge \mathfrak{a}(\vec{Z}_2) \\ \min\{\mathfrak{a}(\vec{Z}_1), \mathfrak{a}(\vec{Z}_2)\} \le \mathfrak{a}(\theta\vec{Z}_1 + (1-\theta)\vec{Z}_2)\,, \forall \vec{Z}_1, \vec{Z}_2, 0 \le \theta \le 1 \\ \mathfrak{a}(\mathbf{0}) = 0 \\ \mathfrak{a}(\vec{W}_k) \le \mathfrak{a}(\vec{Y}_k)\,, \forall k \in \{1,2,\ldots,K\} \end{array} \right\}^{14}.$$

[14] Rigorously speaking, it should be also imposed that every function in this set has to be upper semi-continuous given its definition in Brown et al. (2012). However, as seen in Lemma EC.6, the worst-case measure is always upper semi-continuous, and for simplicity we leave this technicality detail in the footnote.

It is not difficult to see that by substituting $\mathfrak{a}(\cdot) = -\rho(\cdot)$ into the above constraints, the set becomes equivalent to the set of convex risk measures $\mathcal{R}_{El}(\mathcal{E})$ without the condition of translation invariance $\rho(\vec{Z}+c) = \rho(\vec{Z}) - c$. Similarly as was done in this paper for the case of convex risk measures, one might consider comparing random payoffs using a worst-case aspiration level perspective, namely by seeking decisions that are optimal with respect to $\max_{\boldsymbol{x}\in\mathcal{X}} \varrho_{\mathfrak{A}}(\vec{Z})$ where $\varrho_{\mathfrak{A}}(\vec{Z}) := \inf_{\mathfrak{a}\in\mathfrak{A}} \mathfrak{a}(\vec{Z})$. Unfortunately, one can show such a scheme ends up exploiting a rather overly simplistic way of comparing payoffs.

Lemma EC.6. *Given any random variable $\vec{Z}$, the preference robust aspiration level of $\vec{Z}$ according to $\varrho_{\mathfrak{A}}$ is either zero or minus infinite. Namely,*

$$\varrho_{\mathfrak{A}}(\vec{Z}) = \begin{cases} 0 & \text{if } \mathfrak{a}(\vec{Z}) \geq 0\,,\ \forall\,\mathfrak{a}\in\mathfrak{A} \\ -\infty & \text{otherwise} \end{cases}.$$

*Proof of Lemma EC.6:* Given any $\mathfrak{a}\in\mathfrak{A}$, one can construct the following indicator function

$$\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Z}) := \begin{cases} 0 & \text{if } \vec{Z}\in\mathcal{A}_{\mathfrak{a}} \\ -\infty & \text{otherwise} \end{cases},$$

where $\mathcal{A}_{\mathfrak{a}} := \{\vec{Z} \mid \mathfrak{a}(\vec{Z}) \geq 0\}$.

We show that $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}} \in \mathfrak{A}$. Firstly, given that $\mathfrak{a}\in\mathfrak{A}$, the acceptance set $\mathcal{A}_{\mathfrak{a}}$ must be monotone, convex, and must contain the zero payoff. Given any $\vec{Z}_1, \vec{Z}_2$ such that $\vec{Z}_1 \geq \vec{Z}_2$, if $\vec{Z}_2 \notin \mathcal{A}_{\mathfrak{a}}$, $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Z}_2) = -\infty \leq \bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Z}_1)$, and if $\vec{Z}_2 \in \mathcal{A}_{\mathfrak{a}}$, by the monotonicity of $\mathcal{A}_{\mathfrak{a}}$ we have $\vec{Z}_1 \in \mathcal{A}_{\mathfrak{a}}$ and therefore $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Z}_1) = 0 \geq \bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Z}_2)$. This verifies the monotonicity of $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}$. Given that $\mathcal{A}_{\mathfrak{a}}$ is convex and contains the zero payoff, its indicator function must be concave and satisfies $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\mathbf{0}) = 0$. Finally, we verify the preference constraints $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{W}_k) \leq \bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Y}_k)$. For any $\vec{W}_k \in \mathcal{A}_{\mathfrak{a}}$, we have $\vec{Y}_k \in \mathcal{A}_{\mathfrak{a}}$ due to $0 \leq \mathfrak{a}(\vec{W}_k) \leq \mathfrak{a}(\vec{Y}_k)$, and thus $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Y}_k) = 0 \geq \bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{W}_k)$. For any $\vec{W}_k \notin \mathcal{A}_{\mathfrak{a}}$, $\bar{\rho}_{\mathcal{A}_{\mathfrak{a}}}(\vec{W}_k) = -\infty \leq \bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Y}_k)$ follows easily.

Hence, since it is clear that $\bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Z}) \leq \mathfrak{a}(\vec{Z})$, then it is necessarily the case that

$$\varrho_{\mathfrak{A}}(\vec{Z}) = \inf_{\mathfrak{a}\in\mathfrak{A}} \mathfrak{a}(\vec{Z}) = \inf_{\mathfrak{a}\in\mathfrak{A}} \bar{\mathfrak{a}}_{\mathcal{A}_{\mathfrak{a}}}(\vec{Z}) = \begin{cases} 0 & \text{if } \mathfrak{a}(\vec{Z}) \geq 0\,,\ \forall\,\mathfrak{a}\in\mathfrak{A} \\ -\infty & \text{otherwise} \end{cases}.$$

□

There are a few flaws in this definition of preference robust aspiration measure. First, we see that it always reduces to classifying random payoffs as either acceptable or intolerable which is a somewhat extreme way of handling uncertainty. Second, it lacks the natural property that the way random payoffs are compared should converge to how the comparison would be made according to the true aspiration measure as more elicited results are obtained, unless of course the true aspiration measure actually takes the form of an indicator function. These flaws are directly related to the fact that aspiration measures do not need to satisfy translation invariance which appears to be the key behind the descriptive power of pairwise comparisons for the case of convex risk measures.

It might be possible to correct for this issue by refining the uncertainty set of aspiration measures. For example, additional constraints such as $\mathfrak{a}(z_{\max}) = 1$ or $\mathfrak{a}(z_{\min}) = -1$ might be considered if further normalization can be well justified, which enforces the resulting measure to take values other than zero or minus infinity. A richer form of questions might also be considered in the elicitation process that can help acquire quantitative information about the aspiration measure. The question can be for example whether the decision maker's aspiration is affected more by improving $\vec{W}$ with a positive amount $\Delta_1$ or by improving $\vec{Y}$ with a positive amount $\Delta_2$, thus possibly leading to the condition $\mathfrak{a}(\vec{W} + \Delta_1) - \mathfrak{a}(\vec{W}) \geq \mathfrak{a}(\vec{Y} + \Delta_2) - \mathfrak{a}(\vec{Y})$ which helps prevent the measure to have an infinite slope. Finally, we suspect that it might be possible to draw more meaningful conclusions with our set of aspiration measures $\mathfrak{A}$ through a framework that compares random payoffs using their respective worst-case certainty equivalents as was done in Armbruster and Delage (2015). Mathematically, this leads to considering $\varrho_{\mathfrak{A}}(\vec{Z}) := \inf_{\mathfrak{a}\in\mathfrak{A}} \sup\{s : \mathfrak{a}(s) \leq \mathfrak{a}(\vec{Z}(x))\}$ which reduces to $\varrho_{\mathfrak{A}}(\vec{Z}) := \inf_{\mathfrak{a}\in\mathfrak{A}} \mathfrak{a}(\vec{Z}(x))$ when translation invariance is imposed.[15] Given that the right correction scheme is not straightforward, we leave the investigation of these different ways of handling aspiration risk measures as a subject of future research.

[15] This approach resolves for instance the issue that all preference robust measurement return either zero or minus infinity as values. Take for instance a case where all possible pairwise comparisons have been made and one needs to evaluate $\varrho_{\mathfrak{A}}(\cdot)$ at some $\vec{Z}$ for which the true $\bar{\mathfrak{a}}(\vec{Z}) = \bar{\mathfrak{a}}(\alpha)$ for some finite and unique $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Since all comparisons have been made then it must be that $\forall \mathfrak{a} \in \mathfrak{A}$, we have that $\mathfrak{a}(\alpha) = \mathfrak{a}(\vec{Z})$ so that $\sup\{s : \mathfrak{a}(s) \leq \mathfrak{a}(\vec{Z})\} \geq \alpha$. On the other hand, uniqueness of $\alpha$ ensures that $\bar{\mathfrak{a}}(\alpha + \epsilon) > \bar{\mathfrak{a}}(\vec{Z})$ for any $\epsilon > 0$ hence $\sup\{s : \bar{\mathfrak{a}}(s) \leq \bar{\mathfrak{a}}(\vec{Z})\} = \alpha$. Since $\bar{\mathfrak{a}} \in \mathfrak{A}$, we can conclude that $\varrho_{\mathfrak{A}}(\vec{Z}) := \inf_{\mathfrak{a}\in\mathfrak{A}} \sup\{s : \mathfrak{a}(s) \leq \mathfrak{a}(\vec{Z})\} = \alpha \notin \{0, -\infty\}$. Similar arguments can also be used to argue that this worst-case certainty equivalent measure should converge to the "certainty equivalent" measure $\sup\{s : \bar{\mathfrak{a}}(s) \leq \bar{\mathfrak{a}}(\vec{Z})\}$ when many comparisons are made.