Online Appendix
Multichannel Distribution Strategy: Selling to a Competing Buyer with Limited Supplier Capacity

Zhibin (Ben) Yang, Xinxin Hu, Haresh Gurnani, and Huiqi Guan

A Definitions and Technical Results

Definition A.1. *$K^o(w)$ is such that:*

If $S/2 \le B \le S$, then

$$K^o(w) = \begin{cases} \frac{B^2}{4w} - \left(\frac{2B-S}{3w}\right)\left(\frac{2B-S}{3} - w\right) & \text{if } w \le \frac{2B-S}{6} \\ \frac{B^2}{4w} - \frac{1}{2w}\left(\frac{2B-S}{2} - w\right)^2 & \text{if } \frac{2B-S}{6} \le w \le \frac{2B-S}{2} \\ \frac{B^2}{4w} & \text{if } \frac{2B-S}{2} \le w \le \frac{B}{2} \\ (B-w)^+ & \text{if } \frac{B}{2} \le w \end{cases}; \tag{A.1}$$

If $S < B \le 2S$, then

$$K^o(w) = \begin{cases} \frac{B^2}{4w} - \left(\frac{2B-S}{3w}\right)\left(\frac{2B-S}{3} - w\right) & \text{if } w \le \frac{2B-S}{6} \\ \frac{B^2}{4w} - \frac{1}{2w}\left(\frac{2B-S}{2} - w\right)^2 & \text{if } \frac{2B-S}{6} \le w \le \frac{(B-S)+\sqrt{B(2S-B)}}{2} \\ \frac{B-w}{2} + \sqrt{\frac{(B-w)^2}{4} - \frac{1}{2}(\frac{2B-S}{2} - w)^2} & \text{if } \frac{(B-S)+\sqrt{B(2S-B)}}{2} \le w \le \frac{2B-S}{2} \\ (B-w)^+ & \text{if } \frac{2B-S}{2} \le w \end{cases}. \tag{A.2}$$

$K^o(w)$ is illustrated in Figure A.1.

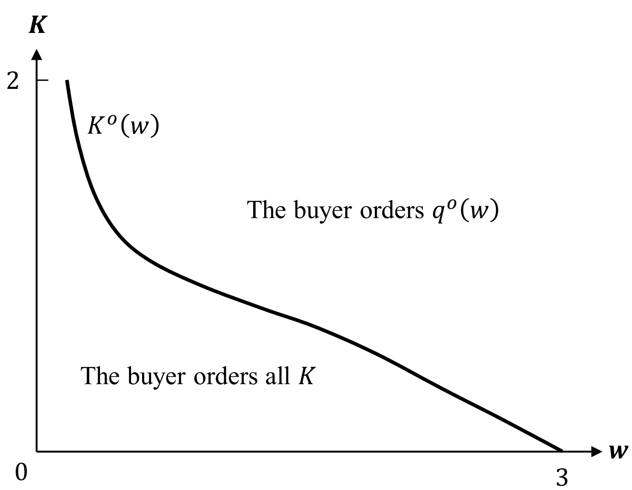

Figure A.1: An example of $K^o(w)$, the buyer's critical capacity in relation to w, under $S < B \le 2S$. In this example, $S = 2$ and $B = 3$.

Lemma A.1. *$K^o(w)$ is continuous and monotone decreasing in w. If $S \le B \le 2S$ and $\frac{3}{2}z_b^o \ge w \ge \frac{1}{2}z_b^o$, then $K^o(w) \ge \frac{1}{2}(\frac{3}{2}z_b^o + S - w)$. Its inverse function $w^o(K)$ is:*

If $S/2 \le B \le S$, then

$$w^o(K) = \begin{cases} (B-K)^+ & \text{if } K \le \frac{B}{2} \\ \frac{B^2}{4K} & \text{if } \frac{B}{2} < K \le \frac{B^2}{6z_b^o} \\ \frac{3}{2}z_b^o - K + \sqrt{(S-K)K + 2(K-\frac{B}{2})^2} & \text{if } \frac{B^2}{6z_b^o} < K \le \frac{B^2}{2z_b^o} - z_b^o \\ \frac{(2z_b^o - B)(2z_b^o + B)}{4(z_b^o - K)} & \text{if } K > \frac{B^2}{2z_b^o} - z_b^o \end{cases}; \tag{A.3}$$

If $S < B \le 2S$, then

$$w^o(K) = \begin{cases} (B-K)^+ & \text{if } K \le \frac{S}{2} \\ \frac{3}{2}z_b^o - K + \sqrt{SK - K^2} & \text{if } \frac{S}{2} < K \le \frac{B}{2} \\ \frac{3}{2}z_b^o - K + \sqrt{(S-K)K + 2(K-\frac{B}{2})^2} & \text{if } \frac{B}{2} < K \le \frac{B^2}{2z_b^o} - z_b^o \\ \frac{(2z_b^o - B)(2z_b^o + B)}{4(z_b^o - K)} & \text{if } K > \frac{B^2}{2z_b^o} - z_b^o \end{cases}. \tag{A.4}$$

Definition A.2. *Define*

$$K^c \overset{def}{=} \begin{cases} \frac{3(7S^2 - 8BS + 4B^2)}{25S^2 - 40BS + 19B^2} z_b^o & \frac{16-\sqrt{3}}{11}S < B \le \frac{16+\sqrt{3}}{11}S \\ \frac{7S^2 - 8BS + 4B^2}{18z_b^o - 2\sqrt{6(B-S)(7S-B)}} & S < B \le \frac{16-\sqrt{3}}{11}S \text{ or } \frac{16+\sqrt{3}}{11}S < B \le t_0 S \\ \hat{K}^c & t_0 S < B \le 2S \end{cases}, \tag{A.5}$$

where $\hat{K}^c$ is defined in Definition A.3 and t_0 is provided in equation (A.6):

$$t_0 \overset{def}{=} \frac{3}{5} + \sqrt[3]{\frac{509}{250} + \frac{7\sqrt{854}}{100}} + \sqrt[3]{\frac{509}{250} - \frac{7\sqrt{854}}{100}} \approx 1.985. \tag{A.6}$$

The expression for $w^o(K)$ in cases (C) and (D) of Proposition 1 can be derived from equation (A.4)*. Specifically,*

$$w^o(K) = \begin{cases} (B-K)^+ & \text{condition (D) \& } K \le S/2 \\ \frac{3}{2}z_b^o - K + \sqrt{(S-K)K} & \text{condition (D) \& } K > S/2 \\ \frac{3}{2}z_b^o - K + \sqrt{(S-K)K + 2(K-\frac{B}{2})^2} & \text{condition (C) \& } K \le \frac{B^2}{2z_b^o} - z_b^o \\ \frac{(2z_b^o - B)(2z_b^o + B)}{4(z_b^o - K)} & \text{condition (C) \& } K > \frac{B^2}{2z_b^o} - z_b^o \end{cases}. \tag{A.7}$$

Definition A.3. *Define $\hat{K}^c$ as the value of $K \in [S/2, S]$ that solves*

$$\frac{S^2}{4} + \frac{(B-S)^2}{3} = \frac{3}{2}z_b^o K - K^2 + K\sqrt{(S-K)K}. \tag{A.8}$$

The buyer's break-even price for given K in Lemma 3 is

$$\hat{w}^o(K) = \begin{cases} \frac{3}{2}z_b^o - K + \sqrt{(S-K)K} & \text{cases (C2), (D) \& } K > S/2 \\ B - K & \text{case (D) \& } K \le S/2 \end{cases}. \tag{A.9}$$

Lemma A.2. *Let $w^a(K)$ and $\pi_s^a(K)$ be the optimal w and objective value for subprogram (9a), and $w^b(K)$ and $\pi_s^b(K)$ for subprogram (9b), respectively. They have the following properties:*

1. *For subprogram (9a), we have* $w^a(K) = w^o(K)$ *and* $\pi_s^a(K) = w^o(K)K$*, and* $\pi_s^a(K)$ *is continuous in* K*. There exists* K^a *such that* $\pi_s^a(K)$ *increases in* $K \le K^a$ *and decreases in* $K \ge K^a$*, where* $K^a = B/2$ *if* $S/2 \le B \le S$*, and* $S/2 < K^a < B/2$ *if* $S < B \le 2S$*.*

2. *For subprogram (9b),* $\pi_s^b(K)$ *is continuous in* K*. There exists* K^b *such that* $\pi_s^b(K)$ *increases in* $K \le K^b$ *and is constant for* $K \ge K^b$*. Moreover,*

   (a) *Under* $S/2 \le B \le S$*, we have* $K^b = S/2$*,* $w^b(K) \ge \max\left(w^o(K), w^o(K^b)\right)$ *for all* K*, and* $\pi_s^b(K) = \frac{S^2}{4}$ *for* $K \ge K^b$*;*

   (b) *Under* $S < B \le 2S$*, we have* $K^b = K^o(\frac{2B+S}{6}) \ge S/2$*,* $w^b(K) = \max\left\{w^o(K), w^o(K^b)\right\}$ *for all* K*, and* $\pi_s^b(K) = \frac{S^2}{4} + \frac{(B-S)^2}{3}$ *for* $K \ge K^b$*.*

We compare $\pi_s^a(K)$ *with* $\pi_s^b(K)$ *to derive the following properties:*

3. *If* $S/2 \le B \le S$*, then* $\pi_s^b(K) \ge \pi_s^a(K)$ *for all* K*.*

4. *If* $S < B \le 2S$*, there exists* K^c *such that* $\pi_s^a(K) \ge \pi_s^b(K)$ *for* $K \le K^c$ *and* $\pi_s^b(K) \ge \pi_s^a(K)$ *for* $K \ge K^c$*, where* K^c *(see Definition A.2) is the solution to* $\frac{S^2}{4} + \frac{(B-S)^2}{3} = w^o(K)K$*. Furthermore, we have* $K^c > \max(K^a, K^b)$ *and* K^c *has the following properties:*

   (a) *If* $\frac{16-\sqrt{3}}{11}S \le B \le \frac{16+\sqrt{3}}{11}S$*, then* $K^c \ge \frac{B^2}{2z_b^o} - z_b^o$*;*

   (b) *If* $S \le B \le \frac{16-\sqrt{3}}{11}S$ *or* $\frac{16+\sqrt{3}}{11}S < B \le t_0 S$*, then* $\frac{B}{2} \le K^c \le \frac{B^2}{2z_b^o} - z_b^o$*;*

   (c) *If* $t_0 S \le B \le 2S$*, then* $\frac{S}{2} \le K^c \le \frac{B}{2}$*, where* t_0 *is defined in (A.6).*

B Proofs

Proof of Lemma 1. If $K - q \ge z_s^o$ and $q \ge z_b^o$, case (I) follows. If $K - q \ge z_s^o$ and $q \le z_b^o$, then $z_b^* = q$, and from (4) $z_s^* = \min\left\{\frac{S-q}{2}, K-q\right\}$. Note that $q \le z_b^o$ implies that $\frac{S-q}{2} \ge \frac{S-z_b^o}{2} = z_s^o$. Hence, with $K - q \ge \frac{S-q}{2}$ and $q \le z_b^o$, we get case (II) proved. The conditions of $\frac{S-q}{2} \ge K - q \ge z_s^o$ and $q \le z_b^o$ lead to $z_s^* = K - q$ and $z_b^* = q$.

If $0 \le K - q \le z_s^o$, then $z_s^* = K - q$, and from (4) $z_b^* = \min\left\{\frac{B-(K-q)}{2}, q\right\}$. Note that $K - q \le z_s^o$ implies that $\frac{B-(K-q)}{2} \ge \frac{B-z_s^o}{2} = z_b^o$. Hence, if $q \ge \frac{B-(K-q)}{2}$ and $0 \le K - q \le z_s^o$, case (III) is established. Some algebra simplifies the corresponding conditions to $0 \le K - q \le z_s^o$ and $q \ge B - K$. Instead, with $q \le B - K$ and $0 \le K - q \le z_s^o$, we have $z_s^* = K - q$ and $z_b^* = q$ again. Combining the conditions $\frac{S-q}{2} \ge K - q \ge z_s^o$ and $q \le z_b^o$ with those $q \le B - K$ and $0 \le K - q \le z_s^o$, we conclude case (IV).

Finally, the corresponding buyer's profit follows by evaluating $\pi_b(z_s^*, z_b^*)$. Also notice that with Assumption 1, both $(B - z_s^* - z_b^*)z_b^*$ in (2) and $(S - z_s^* - z_b^*)z_s^*$ in (3) are non-negative. ■

Proofs of Lemmas 2 and A.1. We first show that the buyer's profit function $\pi_b(q, w)$ has at most two local maximal points: $q^a \stackrel{\text{def}}{=} K$ or $q^b \stackrel{\text{def}}{=} \min\left\{\left(\frac{3}{2}z_b^o - w\right)^+, z_b^o, (2K-S)^+\right\}$, and thus the optimal q^* can only take one of these two values. According to the buyer's profit defined in Lemma 1, we notice the following facts: (a) the border between region (I) and region (III) can never be optimal since the buyer's profit in region (I) is decreasing in q; (b) q^b is a possible optimal, since the buyer's profit in region (II) is concave with the unrestricted maximal point equal to $\frac{3}{2}z_b^o - w$ and that in region (IV) is decreasing when $B - K - w < 0$; and (c) q^a is a possible optimal, since the buyer's profit in region (III) is convex and in region (IV) is increasing when $B - K - w > 0$. Hence, it suffices to prove that the border between regions (III) and (IV) (i.e., $q = B - K$ where $\frac{B}{2} \le K \le \frac{B+S}{3}$) cannot be a local optimal. Note that $q = B - K$ is the regional optimal point in (IV) only when $K < B - w$. In region (III), $\pi_b(z_s^*, z_b^*)$ is convex, and is symmetric with respect to its global minimal point which falls on the line $K = q + B - 2w$. Hence, the regional maximal point must be on the border which is farther away from the global minimal point. Also the middle line between the borders $K = B - q$ and $K = q$ is $q = B/2$. Due to these properties, consider $q = B/2$, by the global minimal line, we have $K = \frac{3B}{2} - 2w$. It implies that $q = B - K$ is the regional optimal point in (III) only when $\frac{B+S}{3} \ge K \ge \max\left\{\frac{3B}{2} - 2w, \frac{B}{2}\right\}$. It is easy to check that the following conditions $\frac{B}{2} \le K \le \frac{B+S}{3}$, $K < B - w$, and $K \ge \frac{3B}{2} - 2w$ conflict with each other. Hence, the conclusion follows.

We shall derive the buyer's profits at q^a and q^b, denoted as $\pi_b^a(K, w)$ and $\pi_b^b(K, w)$, and then compare them.

First derive the buyer's profit function at $q = q^a = K$: $\pi_b^a(K, w) = \left[B - z_s^*(K) - z_b^*(K)\right]z_b^*(K) - wK$, which can be evaluated based on Lemma 1. (i) If $K \le B/2$, the (q, K) pair falls in region (IV); the selling quantities are $(z_s^*, z_b^*) = (0, K)$; the buyer's profit is

$$\pi_1^a(K, w) \stackrel{\text{def}}{=} (B - K - w)K. \tag{B.10}$$

(ii) If $K > B/2$, the (q, K) pair falls in region (III); the selling quantities are $(z_s^*, z_b^*) = (0, B/2)$; the buyer's profit is

$$\pi_2^a(K, w) \stackrel{\text{def}}{=} \frac{B^2}{4} - wK. \tag{B.11}$$

Overall, the buyer's profit at $q = q^a = K$ can be concisely expressed as

$$\pi_b^a(K, w) = \begin{cases} \pi_1^a & K \le B/2 \\ \pi_2^a & K > B/2. \end{cases} \tag{B.12}$$

One can verify that π_b^a is concave in K and is linearly decreasing in $K \geq B/2$; moreover, π_b^a is linearly decreasing in w.

Next establish the switching curve $K^o(w)$ and its inverse function $w^o(K)$. In order to specify the equation $\pi_b^a(K,w) = \pi_b^b(K,w)$, which determines $K^o(w)$ and $w^o(K)$, we consider the definition of $\pi_b^b(K,w) = (B - z_s^* - z_b^*)z_b^* - wq^b$ by the following three cases,

(1) If $w \geq \frac{3}{2}z_b^o$, then $q^b = 0$ and $\pi_b^b = 0$. Since $\pi_1^a = 0$ if and only if $K = 0$ or $K = B - w$, corresponding to whether $K = B - w \leq B/2$ or not, we can determine $K^o(w)$ (or $w^o(K)$) by either $\pi_1^a = 0$ or $\pi_2^a = 0$. Note that $w \geq \max\left\{\frac{3}{2}z_b^o, \frac{B}{2}\right\} \Rightarrow K = B - w \leq \frac{B}{2}$, we have $K^o(w) = B - w$ (or $w^o(K) = B - K$). Furthermore, if $\frac{3}{2}z_b^o < \frac{B}{2}$ (i.e., $B < S$), then for $\frac{3}{2}z_b^o \leq w \leq \frac{B}{2}$, we have $K = B - w \geq \frac{B}{2}$, and $K^o(w) = \frac{B^2}{4w}$ (or $w^o(K) = \frac{B^2}{4K}$). Clearly, K^o is continuous and decreasing in the current range, and the corresponding $w^o(K)$ has the similar properties.

(2) If $\frac{3}{2}z_b^o \geq w \geq \frac{1}{2}z_b^o$, then $q^b = \min\left\{\frac{3}{2}z_b^o - w, (2K-S)^+\right\}$. Define $K_1^b = \frac{1}{2}\left(\frac{3}{2}z_b^o + S - w\right)$ and $w_1^b = \frac{3}{2}z_b^o + S - 2K$ by equating $\frac{3}{2}z_b^o - w = 2K - S$. Let $\pi_1^a(K,w) = (B-K-w)(2K-S)$ and $\pi_2^b(w) = \frac{1}{2}\left(\frac{3}{2}z_b^o - w\right)^2$, then

$$\pi_b^b(K,w) = \begin{cases} 0 & K \leq S/2 \\ \pi_1^b & S/2 < K \leq K_1^b \\ \pi_2^b & K > K_1^b \end{cases}. \tag{B.13}$$

One can verify that $\pi_b^b(K,w)$ is continuous in K, π_1^b is concavely increasing in $K \in (S/2, K_1^b]$, and π_2^b is constant in K. Therefore, the maximum of $\pi_b^b(K,w) = \pi_2^b$ and is obtained by any $K \geq K_1^b$.

Recall the property of π_b^a with respect to K, we conclude that $\pi_b^a \leq \pi_b^b$ when K is large enough. In term of w, π_b^b is linearly decreasing when $w \leq w_1^b$ and becomes convex afterward. Since π_b^a linearly decreases in w, we know that $\pi_b^a \leq \pi_b^b$ when w is big enough.

On the other hand, $\pi_b^a(K_1^b, w) = \pi_1^a(K_1^b, w)$ if $K_1^b \leq \frac{B}{2}$ (i.e., $w \geq \frac{S}{2}$) and $\pi_b^a(K_1^b, w) = \pi_2^a(K_1^b, w)$ if otherwise. Due to the facts of $\frac{B}{2} \leq \frac{B+S}{3}$ and $\frac{3}{2}z_b^o \geq w \geq \frac{1}{2}z_b^o \Leftrightarrow \frac{S}{2} \leq K_1^b \leq \frac{B+S}{3}$, we can show $\pi_b^a(K_1^b, w) \geq \pi_2^b$ for both the cases where $S/2 \leq B \leq S$ and $S \leq B \leq 2S$.

Consequently, the equation is either $\pi_1^a(K,w) = \pi_2^b$ or $\pi_2^a(K,w) = \pi_2^b$, and the corresponding solutions are $K_1^o(w) = \frac{B-w}{2} + \sqrt{\frac{(B-w)^2}{4} - \frac{1}{2}\left(\frac{3}{2}z_b^o - w\right)^2}$ and $K_2^o(w) = \frac{B^2}{4w} - \frac{1}{2w}\left(\frac{3}{2}z_b^o - w\right)^2$. Both of them can be shown decreasing in w. We can also identify $w_b = \frac{(z_b^o - z_s^o) + \sqrt{3Bz_s^o}}{2}$ by the equation $\pi_b^a(\frac{B}{2}, w) = \pi_2^b$. Hence, if $S/2 \leq B \leq S$, then $K^o(w) = K_2^o(w)$, and if $S \leq B \leq 2B$, $K^o(w) = \mathbf{1}\left(\frac{3}{2}z_b^o \geq w \geq w_b\right) K_1^o(w) + \mathbf{1}\left(w_b \geq w \geq \frac{1}{2}z_b^o\right) K_2^o(w)$. Due to $K^o(w) \geq K_1^b$, we know that if $K \geq K^o(w)$, $q^* = \frac{3}{2}z_b^o - w$.

If we solve the equations with respect to w, we have the equation of $\pi_2^a = \pi_2^b$ for $\frac{B^2}{6z_b^o} \leq K \leq \frac{B^2}{2z_b^o} - z_b^o$ in the case of $S/2 \leq B \leq S$. Notice that $\pi_2^b(\frac{3}{2}z_b^o) = 0$ and $\pi_2^a(\frac{3}{2}z_b^o) \leq 0$. Hence, we should choose the quadratic solution which is closer to $\frac{3}{2}z_b^o$. That is, $w_2^o(K) = \frac{3}{2}z_b^o - K + \sqrt{K^2 + \frac{B^2}{2} - 3z_b^o K}$. Similarly, for the case of $S \le B \le 2S$, the equation is $\mathbf{1}\left(\frac{S}{2} \le K \le \frac{B}{2}\right)\pi_1^a + \mathbf{1}\left(\frac{B}{2} \le K \le \frac{B^2}{2z_b^o} - z_b^o\right)\pi_2^a = \pi_2^b$, and we can show that the solution is $w^o(K) = \mathbf{1}\left(\frac{S}{2} \le K \le \frac{B}{2}\right)w_1^o(K) + \mathbf{1}\left(\frac{B}{2} \le K \le \frac{B^2}{2z_b^o} - z_b^o\right)w_2^o(K)$, where $w_1^o(K) = \frac{3}{2}z_b^o - K + \sqrt{SK - K^2}$.

(3) If $\frac{1}{2}z_b^o \ge w \ge 0$, then $q^b = \min\{z_b^o, (2K-S)^+\}$. Define $K_2^b = \frac{B+S}{3}$ by equating $z_b^o = 2K - S$. Let $\pi_3^b = z_b^o(z_b^o - w)$, then $\pi_b^b(K) = \mathbf{1}\left(\frac{S}{2} \le K \le K_2^b\right)\pi_1^b + \mathbf{1}\left(K_2^b \le K\right)\pi_3^b$. Follow the similar logic as above, by showing that $\pi_b^a(\frac{B+S}{3}) \ge \pi_3^b$, we specify the equation as $\pi_2^a(K) = \pi_3^b$, which determines $K^o(w) = \frac{B^2}{4w} - \frac{z_b^o}{w}(z_b^o - w)$, and $w^o(K) = \frac{7B^2 - 16BS + 4S^2}{36(z_b^o - K)}$. Due to $K^o(w) \ge K_2^b$, we have if $K \ge K^o(w)$, $q^* = z_b^o$.

In summary, $K^o(w)$ is continuously decreasing in w. If $K \le K^o(w)$, $q^* = q^a = K$, so that according to Lemma 1, $z_s^* = 0$, $z_b^* = \min\{K, B/2\}$, and $\pi_s(z_s^*, z_b^*, q^*, w) = wK$. If $K \ge K^o(w)$, $q^* = q^b = \min\left\{\left(\frac{3}{2}z_b^o - w\right)^+, z_b^o\right\}$. More specifically, when $w \ge \frac{3}{2}z_b^o$, then $q^* = 0$, $z_s^* = \min\left\{\frac{S}{2}, K\right\}$, and $\pi_s(z_s^*, z_b^*, q^*, w) = (S - z_s^*)z_s^*$. When $w \le \frac{3}{2}z_b^o$, $q^* = \min\left\{\frac{3}{2}z_b^o - w, z_b^o\right\}$, $z_s^* = \frac{S-q^*}{2}$, and $\pi_s(z_s^*, z_b^*, q^*, w) = \left(\frac{S-q^*}{2}\right)^2 + wq^*$. ■

Proof of Lemma A.2. We shall prove the four properties in order.

Property 1. Note that $\pi_s^a = w^o(K)K = wK^o(w)$. Hence, we can equivalently check its property with respect to w. We start with $S/2 \le B \le S$ to make the discussion as follows:

(1) if $w \le \frac{1}{2}z_b^o$, then $\pi_s^a = \frac{B^2}{4} - (z_b^o)^2 + z_b^o w$ increases in w.

(2) if $\frac{1}{2}z_b^o \le w \le \frac{3}{2}z_b^o$, then $\pi_s^a = \frac{B^2}{4} - \frac{1}{2}\left(\frac{3}{2}z_b^o - w\right)^2$ increases in w.

(3) if $\frac{3}{2}z_b^o \le w \le \frac{B}{2}$, then $\pi_s^a = \frac{B^2}{4}$, constant in w.

(4) if $\frac{B}{2} \le w$, then $\pi_s^a = (B - w)w$ decreases in w.

Hence, by letting $K^a = K^o(\frac{B}{2}) = \frac{B}{2}$, π_s^a increases when $K \le K^a$, constant as $\frac{B^2}{4}$ when $K^a \le K \le K^o(\frac{3}{2}z_b^o)$, decreases when $K \ge K^o(\frac{3}{2}z_b^o)$.

We next focus on $S \le B \le 2S$, especially the case of $w_b \le w \le \frac{3}{2}z_b^o$, where $w_b = \frac{(z_b^o - z_s^o) + \sqrt{3Bz_s^o}}{2}$. Note that $K^o(w)$ is the solution to the equation $\frac{1}{2}\left(\frac{3}{2}z_b^o - w\right)^2 = (B - K - w)K$, hence, $\pi_s^a = (B - K^o)K^o - \frac{1}{2}\left(\frac{3}{2}z_b^o - w\right)^2$. Straightforward algebra tells us that $\frac{\partial^2 K^o}{\partial w^2} \le 0$, and consequently $\frac{\partial^2 \pi_s^a}{\partial w^2} \le 0$. Note also that $K^o(w_b) = \frac{B}{2}$, which implies $\frac{\partial \pi_s^a}{\partial w}(w_b) \ge 0$. Combining with $\frac{\partial \pi_s^a}{\partial w}(\frac{3}{2}z_b^o) \le 0$, we conclude that π_s^a is concave in the range with its peak contained in the range. Suppose $w_a \in [w_b, \frac{3}{2}z_b^o]$ maximizes π_s^a. Let $K^a \overset{\text{def}}{=} K^o(w_a)$, we know that $\frac{S}{2} < K^a < \frac{B}{2}$ and that property 2 also holds for the case $S \le B \le 2S$.

Property 2. Clearly, when $w \ge w^o$, $q^* = q^o$. Denote the objective function in the subprogram (9b) as π^b. Based on the definition of q^o, without loss of generality, consider $w^o \le \frac{1}{2}z_b^o$. We have the following results:

(1) For $w^o \le w \le \frac{1}{2}z_b^o$, $q^* = z_b^o$ and $\pi^b = (z_s^o)^2 + wz_b^o$, increasing in w.

(2) For $\frac{1}{2}z_b^o \le w \le \frac{3}{2}z_b^o$, $q^* = \frac{3}{2}z_b^o - w$ and $\pi^b = \frac{1}{4}\left(S - \frac{3}{2}z_b^o + w\right)^2 + w\left(\frac{3}{2}z_b^o - w\right)$, concave in w, with the unrestricted global maximum of $\frac{S^2}{4} + \frac{(B-S)^2}{3}$, achieved at $w_s = \frac{2B+S}{6}$. It is clear that $w_s \ge \frac{1}{2}z_b^o$ and $w_s - \frac{3}{2}z_b^o = \frac{2}{3}(S-B) \ge 0 \Leftrightarrow S \ge B$. Hence, if $S/2 \le B \le S$, π^b increases in w; if $S \le B \le 2S$, it increases first then decreases.

(3) For $\frac{3}{2}z_b^o \le w$, $q^* = 0$, $z_s^* = \min(S/2, K)$, and $z_b^* = 0$. Hence, $\pi^b = \mathbf{1}(K \le S/2)\,(S - K)K + \mathbf{1}(K \ge S/2)\,\frac{S^2}{4}$, constant with respect to w, but increases in K. Notice that the maximum of $(S-K)K$ is achieved at $K = S/2$, equal to $S^2/4$.

Based on the above results, we know that π^b is unimodal in w. More specific,

(a) if $S/2 \le B \le S$, let $K^b = \frac{S}{2}$. (i) When $K \ge K^b$, $w^o(K) \le w^o(K^b)$, π^b increases when $w^o(K) \le w \le w^o(K^b)$ and is constant $\frac{S^2}{4}$ when $w \ge w^o(K^b)$. Hence, $w^b(K) \ge w^o(K^b)$, and $\pi_s^b(K) = \frac{S^2}{4}$. (ii) When $K \le K^b$, $w^o(K) \ge w^o(K^b)$, $\pi^b = (S-K)K$ for all $w \ge w^o(K)$. Hence, $w^b(K) \ge w^o(K)$ and $\pi_s^b = (S-K)K$, increasing in K.

(b) if $S \le B \le 2S$, then $K^b = K^o(w_s) \ge K^o(\frac{3}{2}z_b^o) = \frac{S}{2}$. (i) When $K \ge K^b$, we know π^b increases when $w^o \le w \le w_s$, decreases when $w_s \le w \le \frac{3}{2}z_b^o$, and keeps constant when $w \ge \frac{3}{2}z_b^o$. Hence, $w^b(K) = w^o(K^b)$ and $\pi_s^b = \frac{S^2}{4} + \frac{(B-S)^2}{3}$. (ii) When $K^b \ge K \ge S/2$, $w^o \ge w_s$, π^b decreases when $w^o \le w \le \frac{3}{2}z_b^o$ and keeps constant afterward. Hence, $w^b(K) = w^o(K)$, and $\pi_s^b = \frac{1}{2}\left(S - \frac{3}{2}z_b^o + w^o\right)^2 + w^o\left(\frac{3}{2}z_b^o - w^o\right)$ increases in K, since w^o decreases in K. (iii) When $K \le S/2$, $\pi_s^b = (S-K)K$ increases in K, and $w^b(K) = w^o(K)$.

Property 3. Based on the above analysis, when $S/2 \le B \le S$, it is $K^b = \frac{S}{2} \ge K^a = \frac{B}{2}$. For $K \le K^a$, $\pi_s^b = (S-K)K \ge (B-K)K = \pi_s^a$; for $K \ge K^a$, $\pi_s^b > \frac{B^2}{4} \ge \pi_s^a$. Hence, conclusion follows.

Property 4. First, we show the existence and the uniqueness of K^c. If $K \le \frac{S}{2}$, $\pi_s^b = (S-K)K \le (B-K)K = \pi_s^a$. If $\frac{S}{2} \le K \le K^a$, we know $K^a < \frac{B}{2}$, $\frac{3}{2}z_b^o \ge w^o > w_b$, and at the price w^o, the buyer is indifferent of ordering K or $q^o = \frac{3}{2}z_b^o - w^o$. Hence, the comparison of the supplier's profits is equivalent to the comparison of the supply chain's profits. When the buyer orders K, the supply chain profit is $\Pi^a = (B-K)K$; when q^o, the profit is $\Pi^a = (S - \frac{S+q^o}{2})\frac{S-q^o}{2} + (B - \frac{S+q^o}{2})q^o = (S-B)\frac{S-q^o}{2} + (B - \frac{S+q^o}{2})\frac{S+q^o}{2}$. Note that $S \le B$ and $\frac{S+q^o}{2} \le K \le \frac{B}{2}$, we have $\Pi^b \le (B - \frac{S+q^o}{2})\frac{S+q^o}{2} \le (B-K)K = \Pi^a$, and thus, $\pi_s^a \ge \pi_s^b$. If $K \ge \frac{B}{2}$, then $w^o \le w_b$, π_s^a decreases in K while π_s^b does not decrease in K. Also, $\lim_{w^o \to 0} \pi_s^a \to \frac{B^2}{4} - \frac{(2B-S)^2}{9} \le \frac{S^2}{4} + \frac{(B-S)^2}{3} = \lim_{w^o \to 0} \pi_s^b$. Hence, there exists $K^c \ge K^a$ such that $K \le K^c \Rightarrow \pi_s^a \ge \pi_s^b$ and $K \ge K^c \Rightarrow \pi_s^a \le \pi_s^b$.

Next, we characterize K^c in two steps: demonstrate that $K^c \geq K^o(w_s)$ and identify the expression of K^c.

Demonstrate $K^c \geq K^o(w_s) = K^b$: Note that $\pi_s^b(K^b) = \frac{S^2}{4} + \frac{(B-S)^2}{3}$. Focus on $\pi_s^a(K^o(w_s))$. If $S \leq B \leq 1.6S$, then $\frac{1}{2}z_b^o \leq w_s \leq w_b$, $\frac{B}{2} \leq K^o(w_s) \leq \frac{B^2}{2z_b^o} - z_b^o$, and $\pi_s^a(w_s) = \frac{B^2}{4} - \frac{1}{2}\left(\frac{2(B-S)}{3}\right)^2$. We can calculate that $\pi_s^b(K^o(w_s)) \leq \pi_s^a(K^o(w_s)) = \frac{(11B-29S)(B-S)}{36} \leq 0$. If $1.6S \leq B \leq 2S$, then $w_b \leq w_s \leq \frac{3}{2}z_b^o$, $\frac{S}{2} \leq K^o(w_s) \leq \frac{B}{2}$, and $\pi_s^a(K^o(w_s)) = (B - K^o(w_s))K^o(w_s) - \frac{1}{2}\left(\frac{3}{2}z_b^o - w_s\right)^2$. Evaluating $\pi_s^b(K^o(w_s)) - \pi_s^a(K^o(w_s)) = \frac{(11B-29S)(B-S)}{36} + \left(\frac{2B+S}{12} - \sqrt{\frac{-16B^2+56BS-31S^2}{144}}\right)^2$, we can show that $\pi_s^b(K^o(w_s)) \leq \pi_s^a(K^o(w_s))$ for $S \leq B \leq 2S$.

Identify the expression of K^c: From the above analysis, the equation to identify K^c must be $\pi_s^b(K^b) = \pi_s^a(K)$. According to the magnitude of K, π_s^a has three possibilities:

(1) If $K \geq \frac{B^2}{2z_b^o} - z_b^o$, i.e., $w^o(K) \leq \frac{1}{2}z_b^o$, then $\pi_s^a(K) = \frac{B^2}{4} - z_b^o(z_b^o - w^o(K)) \leq \frac{B^2}{4} - \frac{1}{2}(z_b^o)^2$. Comparing $\pi_s^b(K^b)$ and $\frac{B^2}{4} - \frac{1}{2}(z_b^o)^2$, we have $\pi_s^b \leq \pi_s^a$ if and only if $\frac{16-\sqrt{3}}{11}S \leq B \leq \frac{16+\sqrt{3}}{11}S$. Hence, for $\frac{16-\sqrt{3}}{11}S \leq B \leq \frac{16+\sqrt{3}}{11}S$, we should solve the equation $\pi_s^b(K^b) = \frac{B^2}{4} - z_b^o(z_b^o - w^o(K))$ with respect to K and determine $K^c = \frac{21S^2-24BS+12B^2}{25S^2-40BS+19B^2}z_b^o$.

(2) If $\frac{B^2}{2z_b^o} - z_b^o \geq K \geq \frac{B}{2}$, i.e., $\frac{1}{2}z_b^o \leq w^o(K) \leq w_b$, then $\pi_s^a(K) = \frac{B^2}{4} - \frac{1}{2}(\frac{3}{2}z_b^o - w^o(K))^2$. Comparing $\pi_s^b(K^b)$ and $\pi_s^a(\frac{B}{2})$, we have $\pi_s^b \geq \pi_s^a$ if and only if $t_0 S \leq B \leq 2S$, where t_0 is given in equation (A.6). Hence, for $S \leq B \leq \frac{16-\sqrt{3}}{11}S$ or $\frac{16+\sqrt{3}}{11}S \leq B \leq t_0 S$, we should the equation $\pi_s^b(K^b) = \frac{B^2}{4} - \frac{1}{2}(\frac{3}{2}z_b^o - w^o(K))^2$ with respect to K and determine $K^c = \frac{7S^2-8BS+4B^2}{18z_b^o - 2\sqrt{6(B-S)(7S-B)}}$.

(3) Based on the results in (1) and (2), we know that when $t_0 S \leq B \leq 2S$, we need to use $\pi_s^a(K) = (B-K)K - \frac{1}{2}(\frac{3}{2}z_b^o - w^o)^2$ in the equation to determine K^c. That is, K^c must be the solution to the equation $\frac{S^2}{4} + \frac{(B-S)^2}{3} = \frac{3}{2}z_b^o K - K^2 + K\sqrt{SK - K^2}$. ■

Proof of Proposition 1. (A): Based on Lemma A.2, when $S/2 \leq B \leq S$, $\pi_s^b(K) \geq \pi_s^a(K)$ for any $K \geq 0$. Hence, $w^* = w^b(K) \geq \max\{w^o(K), w^o(S/2)\}$, i.e., $w^* = \infty$ is optimal to the supplier. According to Definition A.1, $K^o(w^*) = 0$, and by Lemma 2, we know the remaining conclusions hold.

(B): Based on Lemma A.2, with $S \leq B \leq 2S$ and $K \geq K^c$, $\pi_s^b(K) \geq \pi_s^a(K)$. Hence, $w^* = w^b(K) = \max\{w^o(K), w^o(K^b)\}$. Since $K^c \geq K^b = K^o(\frac{2B+S}{6})$ and w^o decreases in K, we get $w^* = w^o(K^b) = \frac{2B+S}{6} = \frac{1}{2}z_b^o + \frac{S}{3}$. It also means that $K \geq K^o(w^*)$. By noting that $\frac{1}{2}z_b^o \leq w^* \leq \frac{3}{2}z_b^o$, from Lemma 2, we can verify the value of q^*, z_b^*, z_s^*, and π_s^*. The calculation of π_b^* is straightforward based on Lemma 1.

(C) and (D) can be similarly argued. ■

Proof of Corollary 1. Given $S \geq B \leq 2S$, according to Proposition 1, $K \geq K^c \Rightarrow z_s^* + z_b^* =$

$z_b^o + z_s^o - \frac{S}{6} = \frac{2B+S}{6}$. On the otherhand, let $w_s = \frac{2B+S}{6}$. It is easy to check, from Lemma A.1, $\frac{3}{2}z_b^o \geq w_s \geq \frac{1}{2}z_b^o \Rightarrow K^o(w_s) \geq \frac{2B+S}{6}$. Hence, the conclusion follows from Lemma A.2, which states that $K^c \geq \max(K^a, K^b)$, where $K^b = K^o(w_s)$. ■

Proof of Proposition 2. We show first proof for the supplier's profit and then for the buyer's profit.

The break-even K for the supplier. When $S/2 \leq B \leq S$ (i.e., in region (A)), the supplier's profit is $\pi_s^* = \frac{S^2}{4}$ for $K > S/2$ and $\pi_s^* = (S-K)K$ for $K \leq S/2$. It is always true $(S-K)K \leq \frac{S^2}{4}$. The result follow.

We now consider the case of $S < B \leq 2S$. We shall compare the supplier's profit in region (B) to that in regions (C) and (D). From Proposition 1, the supplier's profit in region (B) is $\pi_s^* = \frac{S^2}{4} + \frac{(B-S)^2}{3}$ and is $\pi_s^* = \pi_s^a = w^o(K)K$ in regions (C) and (D). We compare these two values to identify the break-even K. From the definition of K^c, the supplier is indifferent between the two values at $K = K^c$. We shall identify another break-even K that is less than K^c. Recall property 1 of Lemma A.2: under $S < B \leq 2S$, there exist $K^a \in (S/2, B/2)$ such that π_s^a is continuous in K and π_s^a increases in $K \leq K^a$ and decreases in $K > K^a$. Because $K^a < B/2 < K^c$, if there exists a break-even $K < K^c$, such K must satisfy $K < B/2$, that is, region (D). Using the expressions for $w^o(K)$ in Definition A.2 in Appendix A, we derive π_s^a in region (D) to be:

$$\pi_s^a = \begin{cases} \left[B - K - S/2 + \sqrt{(S-K)K}\right] K & K > S/2 \\ (B-K)K & K \leq S/2 \end{cases} \tag{B.14}$$

It takes some algebra to show that at $K = S/2$, $\pi_s^a - \left[\frac{S^2}{4} + \frac{(B-S)^2}{3}\right] = -\frac{1}{3}\left(B - \frac{5}{2}S\right)(B-S) > 0$. Because π_s^a is increasing in $K \leq K^a$, the break-even value of K must be some $K < S/2$, under which $\pi_s^a = (B-K)K$. We solve

$$(B-K)K - \left[\frac{S^2}{4} + \frac{(B-S)^2}{3}\right] = 0 \tag{B.15}$$

and use the solution that is less than $B/2$. The result in equation (10) follows.

The break-even K for the buyer. The result for $S/2 \leq B \leq S$ (i.e., in region (A)) is trivial. We shall focus on the case of $S < B \leq 2S$. It takes two steps.

Step 1. We establish that the buyer's profit π_b^* increases in $K \leq K^c$ (i.e., regions (C) and (D)). When $K \leq S/2$ (i.e., lower region (D)), the buyer's profit is $\pi_b^* = (B-K)K - w^o(K)K$ (from Proposition 1). From Lemma A.1, under $S < B \leq 2S$ and $K \leq S/2$ the expression for $w^o(K) = B - K$. Therefore, $\pi_b^* = 0$.

We now show that π_b^* is increasing in $K \in (S/2, K^c]$ (i.e., upper region (D) and region (C)), where the buyer orders $q^* \equiv K$. Note that, to induce the buyer to order $q = K$, the supplier sets the wholesale price to be $w^o(K)$, which breaks even between the buyer's profit from ordering $q = K$ and its profit from order $q = q^o\big(w^o(K)\big) < K$ (i.e., the (q, K) pair falls in region (II) of Lemma 1). Therefore, it suffices to obtain the buyer's profit at $q = q^o\big(w^o(K)\big)$. From Lemma A.1 and its proof, under $\frac{S}{2} < K \le \frac{B}{2}$, $w^o(K)$ is the solution to $\frac{1}{2}(\frac{3}{2}z_b^o - w)^2 = (B - K - w)K$; under $\frac{B}{2} < K \le \frac{B^2}{2z_b^o} - z_b^o$, $w^o(K)$ is the solution to $\frac{1}{2}(\frac{3}{2}z_b^o - w)^2 = \frac{B^2}{4} - wK$; under $K > \frac{B^2}{2z_b^o} - z_b^o$, $w^o(K)$ is the solution to $z_b^o(z_b^o - w) = \frac{B^2}{4} - wK$. The left-hand-sides of these equations are the buyer's profits at $q = q^o\big(w^o(K)\big)$ at different capacity levels. From these expressions, we obtain the buyer's profit under $K \le K^c$:

$$\pi_b^* = \begin{cases} \frac{1}{2}\left[\frac{3}{2}z_b^o - w^o(K)\right]^2 & \frac{S}{2} < K \le \min\left\{K^c, \frac{B^2}{2z_b^o} - z_b^o\right\} \\ z_b^o\left[z_b^o - w^o(K)\right] & \min\left\{K^c, \frac{B^2}{2z_b^o} - z_b^o\right\} < K \le K^c. \end{cases} \tag{B.16}$$

where $w^o(K) < \frac{3}{2}z_b^o$ because $K > S/2$.

When $K \in \left(S/2, \min\left\{K^c, \frac{B^2}{2z_b^o} - z_b^o\right\}\right]$, the buyer's profit π_b^* is continuous, increasing in K because $w^o(K)$ is continuous, decreasing in K (see Lemma A.1) and $w^o(K) < \frac{3}{2}z_b^o$. When $K \in \left(\min\left\{K^c, \frac{B^2}{2z_b^o} - z_b^o\right\}, K^c\right]$, π_b^* is also continuous, increasing in K because $w^o(K)$ is continuous, decreasing in K. Furthermore, if $K^c > \frac{B^2}{2z_b^o} - z_b^o$, π_b^* is continuous at $K = \frac{B^2}{2z_b^o} - z_b^o$ because $\frac{1}{2}\left[\frac{3}{2}z_b^o - w^o(K)\right]^2 = z_b^o\left[z_b^o - w^o(K)\right] = \frac{B^2}{4} - w^o(K)K$.

Therefore, π_b^* is continuously increasing in $K \le K^c$. This concludes step 1.

Step 2. We drive the expressions for K_b, which breaks even between the buyer's profit under $K > K^c$ (i.e., in region (B)) and the profit under $K \le K^c$ where the buyer orders $q = K$. Under $K > K^c$, from Proposition 1, the supplier sets the wholesale price to be $w_s \stackrel{\text{def}}{=} \frac{1}{2}z_b^o + \frac{S}{3} = \frac{2B+S}{6}$, so that the buyer orders $q = q^o(w_s) < K$. From the definition of $K^o(w)$, the break-even K_b coincides with $K^o(w)$ at $w = w_s$, that is, $K_b = K^o(w_s)$.

We derive the expressions for $K^o(w_s)$. One can verify that $\frac{1}{2}z_b^o < w_s \le \frac{3}{2}z_b^o$ and that $w_s > \frac{(z_b^o - z_s^o) + \sqrt{3Bz_s^o}}{2}$ if and only if $B > \frac{8}{5}S$. From Definition A.1, we obtain the expression of $K^o(w_s)$ under $\frac{1}{2}z_b^o < w_s \le \frac{(z_b^o - z_s^o) + \sqrt{3Bz_s^o}}{2}$ (under $B \le \frac{8}{5}S$) and $\frac{(z_b^o - z_s^o) + \sqrt{3Bz_s^o}}{2} < w_s \le \frac{3}{2}z_b^o$ (under $B > \frac{8}{5}S$). The result in equation (11) follows.

For a proof of $K^c > K_b > K_s$, please see Lemma B.3. ■

Proof of Proposition 3. Based on Table 1, the conclusion follows by noticing that $K \ge \frac{B}{2}$ or not differentiates cases C and D and that $S \le B \le 2S \Rightarrow \frac{2B+S}{6} \le \frac{B}{2}$. For a proof of $K^c > K_b > K_u > K_s$, please see Lemma B.3. ■

Lemma B.3. *For $B \in (S, 2S]$, we have $K^c > K_b > K_u > K_s$.*

Proof of Lemma B.3. From Lemma A.1, it is straightforward to check $K_u \leq K^o(\frac{2B+S}{6})$. From Lemma A.2, we know $K^c \geq K^o(\frac{2B+S}{6}) \geq K_u$. The remaining inequality can be verified based on (13), (11), and (10). ■

Proof of Lemma 3 and Proposition 5 (for the supplier and buyer). We shall analyze the problem backwardly from Stage 3 to Stage 2 and to Stage 1.

Stage 3. The buyer's selling quantity has to be his order quantity in stage 2. Hence, the optimization in stage 3 is on the supplier's profit $\pi_s(z_s) = (S - z_s - q)z_s + wq$ subject to $0 \leq z_s \leq K - q$. Applying the first order condition, we can conclude the supplier's optimal selling quantity $\hat{z}_s(q)$ is

$$\hat{z}_s(q) = \begin{cases} K - q & \text{if } K \leq \frac{S}{2} \\ \min\left(\frac{S-q}{2}, K - q\right) & \text{if } \frac{S}{2} \leq K \leq S \\ \left[\frac{S-q}{2}\right]^+ & \text{if } S \leq K \end{cases},$$

with $q \in [0, K]$.

Stage 2. In order to charaterize the buyer's optimal quantity $\hat{q}(w)$, we try to establish the capacity threshold $\hat{K}^o(w)$ similar that to Lemma 2.

Given $\hat{z}_s(q)$, we now specify the buyer's profit $\pi_b(q)$ according to K:

(1) If $K \leq \frac{S}{2}$, then $\pi_b(q) = (B - K - w)q$, which is linear in q and its monotonicity depends on if $K \leq B - w$ or not. The corresponding maximal point is either K or 0.

(2) If $\frac{S}{2} \leq K \leq S$, then $\pi_b(q) = \left(B - \frac{S+q}{2}\right) q - wq$ (which is concave in q with the unrestricted global maximal point satisfying $q = \frac{3}{2}z_b^o - w$) when $0 \leq q \leq 2K - S$ and $\pi_b(q) = (B - K - w)q$ when $2K - S \leq q \leq K$. The corresponding maximal point is either K or $\min\left\{\left(\frac{3}{2}z_b^o - w\right)^+, 2K - S\right\}$.

(3) If $S \leq K$, then $\pi_b(q) = \left(B - \frac{S+q}{2}\right) q - wq$ when $0 \leq q \leq S$ and $\pi_b(q) = (B - q - w)q$ (which is concave in q with the unrestricted global maximal point satisfying $q = \frac{B-w}{2}$) when $S \leq q \leq K$. Notice, under the assumption $\frac{S}{2} \leq B \leq 2S$, that $\frac{B-w}{2} \leq \frac{B}{2} \leq S$. Hence, $\pi_b(q)$ is unimodal in q when $S \leq K$. The corresponding maximal point is $\min\left\{\left(\frac{3}{2}z_b^o - w\right)^+, S\right\}$.

In summary, we can define $\hat{q}^a = K$ and $\hat{q}^b = \min\left\{\left(\frac{3}{2}z_b^o - w\right)^+, S, (2K - S)^+\right\}$, so that $\hat{q}$ can takes either $\hat{q}^a$ or $\hat{q}^b$. Through comparing $\pi_b(\hat{q}^a)$ and $\pi_b(\hat{q}^b)$, we can determine $\hat{K}^o$. Note that $\pi_b(\hat{q}^a) = (B - K - w)K$, and $\pi_b(\hat{q}^b)$ is discussed as below:

- If $\frac{3}{2}z_b^o \leq w \leq B$, then $\hat{q}^b = 0$ and $\pi_b(\hat{q}^b) = 0$. Since $\pi_b(\hat{q}^b) \leq \pi_b(\hat{q}^a) \Leftrightarrow 0 \leq K \leq B - w$, we can define $K^o(w) = B - w$ for $\frac{3}{2}z_b^o \leq w \leq B$.

• If $\left(\frac{3}{2}z_b^o - S\right)^+ \leq w \leq \frac{3}{2}z_b^o$, it is easy to get that $K \leq \frac{1}{2}(\frac{3}{2}z_b^o + S - w) \Rightarrow \hat{q}^b = (2K-S)^+$ and $K \geq \frac{1}{2}(\frac{3}{2}z_b^o + S - w) \Rightarrow \hat{q}^b = \frac{3}{2}z_b^o - w$. Also notice that $\frac{S}{2} \leq \frac{1}{2}(\frac{3}{2}z_b^o + S - w) \leq \max(B-w, S)$. Combining this fact with (1) and (2), we conclude that when $K \leq \frac{1}{2}(\frac{3}{2}z_b^o + S - w)$, $\pi_b(q)$ is monotonically increases in q, implying that $\pi_b(\hat{q}^b) < \pi_b(\hat{q}^a)$. Hence, by recalling (3), we can focus on $S \geq K \geq \frac{1}{2}(\frac{3}{2}z_b^o + S - w)$ to derive $\hat{K}^o$, where $\hat{q}^b = \frac{3}{2}z_b^o - w$ and $\pi_b(\hat{q}^b) = \frac{1}{2}(\frac{3}{2}z_b^o - w)^2$. By algebra calculation, we find that $\hat{K}^o(w) = \frac{B-w}{2} + \sqrt{\frac{(B-w)^2}{4} - \frac{1}{2}(\frac{3}{2}z_b^o - w)^2}$ is the value of K that solves $\pi_b(\hat{q}^a) = \pi_b(\hat{q}^b)$.

• If $0 \leq w \leq \left(\frac{3}{2}z_b^o - S\right)^+$, it is easy to check that $K \leq S \Rightarrow \hat{q}^b = (2K-S)^+$ and $K \geq S \Rightarrow \hat{q}^b = S$. Since $B - w \geq B - \left(\frac{3}{2}z_b^o - S\right)^+ \geq S$, from (1) and (2), we know $K \leq S \Rightarrow \pi_b(\hat{q}^b) \leq \pi_b(\hat{q}^a)$. Again with (3), we can conclude that $\hat{K}^o(w) = S$ if $\frac{3}{2}z_b^o - S = B - \frac{3}{2}S > 0$.

It is easy to check that $\hat{K}^o(w)$ is continuous in w. Furthermore, we can show that $\hat{K}^o(w) \leq K^o(w)$ by comparing $w\hat{K}^o(w)$ and $wK^o(w)$. Since $(B-w)w \leq \frac{B^2}{4}$, from (A.1) and (A.2), it is clear that $w \geq \frac{3}{2}z_b^o \Rightarrow w\hat{K}^o(w) \leq wK^o(w)$. Notice that $\left(\frac{3}{2}z_b^o - S\right)^+ \leq \frac{1}{2}z_b^o$ and $\left(\frac{3}{2}z_b^o - S\right)^+ \leq w \leq \frac{3}{2}z_b^o \Rightarrow w\hat{K}^o = (B-K)K - \frac{1}{2}(\frac{3}{2}z_b^o - w)^2$. Since $(B-K)K \leq \frac{B^2}{4}$ and $\frac{1}{2}(\frac{3}{2}z_b^o - w)^2 \geq z_b^o(z_b^o - w)$, from (A.1) and (A.2) again, we know that $\left(\frac{3}{2}z_b^o - S\right)^+ \leq w \leq \frac{3}{2}z_b^o \Rightarrow w\hat{K}^o \leq wK^o$. Since $K^o(w)$ decreases in w and $\hat{K}^o(w) = S$ if $0 \leq w \leq \left(\frac{3}{2}z_b^o - S\right)^+$ is not singleton, we know the same conclusion also holds if $0 \leq w \leq \left(\frac{3}{2}z_b^o - S\right)^+$.

The final conclusion for stage 2 is that, if $K \leq \hat{K}^o(w)$, then $\hat{q} = \hat{q}^a = K$, and otherwise, $\hat{q} = \hat{q}^b = \min\left\{\left(\frac{3}{2}z_b^o - w\right)^+, S\right\}$.

Stage 1. Denote $\hat{K} = \hat{K}^o(\left(\frac{3}{2}z_b^o - S\right)^+)$. Clearly, $S/2 \leq \hat{K} \leq S$. For $K \geq \hat{K}$, $\hat{q}(w) = \hat{q}^b$ and the supplier's profit can be derived as

$$\pi_s(\hat{q}(w), w) = \begin{cases} wS, & \text{if } 0 \leq w \leq (\frac{3}{2}z_b^o - S)^+ \\ \frac{1}{4}\left(S - \frac{3}{2}z_b^o + w\right)^2 + w\left(\frac{3}{2}z_b^o - w\right), & \text{if } (\frac{3}{2}z_b^o - S)^+ \leq w \leq \frac{3}{2}z_b^o \\ \frac{S^2}{4}, & \text{if } \frac{3}{2}z_b^o \leq w \end{cases} \tag{B.17}$$

Applying the similar logic for arguing Lemma A.2 (Property 1), we conclude that (a) if $S \leq B \leq 2S$, $\pi_s(\hat{q}(w), w)$ is maximized at $\hat{w} = w_s = \frac{2B+S}{6}$ and $(\frac{3}{2}z_b^o - S)^+ \leq w_s \leq \frac{3}{2}z_b^o$. The resulting supplier's profit is $\hat{\pi}_s = \frac{S^2}{4} + \frac{(B-S)^2}{3}$, and that (b) if $S/2 \leq B \leq S$, $w_s \geq \frac{3}{2}z_b^o$, and thus $\pi_s(\hat{q}(w), w)$ is non-decreasing in w. We conclude $\hat{w} = \infty$, and $\hat{\pi}_s = \frac{S^2}{4}$.

For $K < \hat{K}$, we use again the divide-and-conquer method to analyze the supplier's wholesale price decision. From the $\hat{K}^o(w)$ derived above, we can calculate its inverse function

$$\hat{w}^o(K) = \begin{cases} \frac{3}{2}z_b^o - K + \sqrt{SK - K^2} & \text{if } S/2 \leq K \leq \hat{K} \\ B - K & \text{if } 0 \leq K \leq S/2 \end{cases}.$$

Denote $\hat{\pi}_s^b(K) = \max_{w \geq \hat{w}^o(K)} \pi_s(\hat{q}(w), w)$. Adopting the similar logic for arguing A.2 (Property 2), we know the optimal point is $\hat{w}^a = \hat{w}^o$, and

$$\hat{\pi}_s^a(K) = \begin{cases} (B-K)K - \frac{1}{2}(\frac{3}{2}z_b^o - w^o(K))^2 = \frac{3}{2}z_b^o K - K^2 + K\sqrt{SK-K^2} & \text{if } S/2 \leq K \leq \hat{K} \\ (B-K)K & \text{if } 0 \leq K \leq S/2 \end{cases}.$$

Checking its first and second derivatives with respect to K, we conclude that $\hat{\pi}_s^a$ is unimodal in K. More specifically, if $S/2 \leq B \leq S$, then it is maximized at $K = B/2$ and $\hat{\pi}_s^a(B/2) = \frac{B^2}{4}$; if $S \leq B \leq 2S$, then its maximal point falls in $[S/2, S]$.

Denote $\hat{\pi}_s^a(K) = \max_{w \leq \hat{w}^o(K)} \pi_s(\hat{q}(w), w)$. The discussion is similar to that for the case where $K \geq \hat{K}$. If $S/2 \leq B \leq S$, then $w_s \geq \frac{3}{2}z_b^o$, implying that $\hat{w}^b = \infty$ and

$$\hat{\pi}_s^b(K) = \begin{cases} \frac{S^2}{4} & \text{if } S/2 \leq K \leq \hat{K} \\ (S-K)K & \text{if } 0 \leq K \leq S/2 \end{cases}$$

If $S \leq B \leq 2S$, then $w_s \leq \frac{3}{2}z_b^o$. Hence,

$$\hat{\pi}_s^b(K) = \begin{cases} \frac{S^2}{4} + \frac{(B-S)^2}{3} & \text{if } \hat{K}^o(w_s) \leq K \leq \hat{K}, \text{ with } \hat{w}^b = w_s \\ \frac{1}{2}\left(S - \frac{3}{2}z_b^o + w^o\right)^2 + w\left(\frac{3}{2}z_b^o - w^o\right) & \text{if } S/2 \leq K \leq \hat{K}^o(w_s), \text{ with } \hat{w}^b = \hat{w}^o \\ (S-K)K & \text{if } 0 \leq K \leq S/2, \text{ with } \hat{w}^b = \infty \end{cases}$$

We next compare $\hat{\pi}_s^b(K)$ and $\hat{\pi}_s^a(K)$ for both $S \leq B \leq 2S$ and $S/2 \leq B \leq S$.

(a) If $S \leq B \leq 2S$, following the logic to prove A.2 (Property 4), we know that $\hat{\pi}_s^b(\hat{K}^o(w_s)) \leq \hat{\pi}_s^a(\hat{K}^o(w_s))$. We can also show that $\hat{\pi}_s^b(\hat{K}^o(w_s)) \geq \hat{\pi}_s^a(\hat{K}^o((\frac{3}{2}z_b^o - S)^+))$. Hence, we can identify $\hat{K}^c \in [\hat{K}^o(w_s), \hat{K}^o((\frac{3}{2}z_b^o - S)^+)]$, by solving the equation $\frac{S^2}{4} + \frac{(B-S)^2}{3} = \hat{w}^o(K)K$, as defined in Definition A.3 such that $K \leq \hat{K}^c \Leftrightarrow \hat{\pi}_s^a(K) \geq \hat{\pi}_s^b(K)$. Notice that we have shown $\hat{K}^o(w) \leq K^o(w)$, and consequently $\hat{w}^o(K) \leq w^o(K)$. Since K^c is determined by solving the equation $\frac{S^2}{4} + \frac{(B-S)^2}{3} = w^o(K)K$, we can argue that $\hat{K}^c \leq K^c$. The remaining conclusions for the case $S \leq B \leq 2S$ in Lemma 3 follows.

(b) If $S/2 \leq B \leq S$, it is easy to know that $\hat{\pi}_s^a \leq \hat{\pi}_s^b$. Hence, all the desired conclusions in Lemma 3 follows.

Now, we are ready to compare $\hat{\pi}_s$ and π_s^* as well as $\hat{\pi}_b$ and π_b^*. Notice that $K \leq \hat{K}^c \Rightarrow \hat{w}^o(K)K \leq w^o(K)K$. Then, based on Lemma A.2, it is easy to see that $\hat{\pi}_s \leq \pi_s^*$. Focus on comparing $\hat{\pi}_b$ and π_b^* for the case where $S \leq B \leq 2S$. Since $\hat{w}^o(K)$ is the indifferent wholesale price for the buyer to order full capacity and to order partial capacity, we know $\frac{B}{2} \leq K \leq \hat{K}^c \Rightarrow \hat{\pi}_b = (B-K)K - \hat{w}^o(K)K = \frac{1}{2}(\frac{3}{2}z_b^o - \hat{w}^o(K))^2$. The similar property also holds for $w^o(K)$. Hence, due to $\hat{w}^o(K) \leq w^o(K)$, it is easy to check that $\frac{1}{2}(\frac{3}{2}z_b^o - \hat{w}^o(K))^2 \geq \frac{1}{2}(\frac{3}{2}z_b^o - w^o(K))^2$. Recall Proposition 2, it suffices to show that $\hat{K}^c \geq K_b$. According to Definition A.3, recall that

$\hat{K}^c \in [\hat{K}^o(w_s), \hat{K}^o((\frac{3}{2}z_b^o - S)^+)]$ and that $\hat{\pi}_s^a$ is maximized at some $K \in [S/2, S]$, we can claim that $\hat{K}^c \geq K_b \Leftrightarrow \frac{S^2}{4} + \frac{(B-S)^2}{3} \leq \frac{3}{2}z_b^o K_b - K_b^2 + K_b\sqrt{SK_b - K_b^2}$. This statement can be verified, based on the expression of K_b in (11), through straightforward and tedious algebra. Hence, we can confirm the results about the supplier and buyer in Proposition 5. ■

Proof of Proposition 4. The key is to show that the border between regions (B) and (C), K^c, is non-monotone in B. K^c is defined for $B \in (S, 2S]$, and its expression is presented in equation (A.5). Specifically, we shall prove the following properties of K^c in relation to B:

- K^c increases in $B \in (S, t^*S]$ and decreases in $B \in (t^*S, t_*S]$, where $t^* \approx 1.337$ and $t_* \approx 1.878$.
- $K^c \approx 1.054S$ when $B = t^*S$ and $K^c \approx 0.989S$ at $B = t_*S$.

Give that S is a constant, we divide both side of equation (A.5) by S and define $t = B/S$ and $\kappa^c(t) \stackrel{\text{def}}{=} K^c/S$, obtaining the following expression of $\kappa^c(t)$ for $t \in (1, t_*]$ (i.e., $S < B \leq t_*S$):

$$\kappa^c(t) \stackrel{\text{def}}{=} \begin{cases} \frac{(2t-1)(7-8t+4t^2)}{25-40t+19t^2} & \frac{16-\sqrt{3}}{11} < t \leq \frac{16+\sqrt{3}}{11} \\ \frac{7-8t+4t^2}{(12t-6)-2\sqrt{6(t-1)(7-t)}} & 1 < t \leq \frac{16-\sqrt{3}}{11} \text{ or } \frac{16+\sqrt{3}}{11} < t \leq t_* \end{cases}. \tag{B.18}$$

To prove the properties of K^c defined above, without loss of generality, we can prove the following properties of $\kappa^c(t)$:

- $\kappa^c(t)$ increases in $t \in (1, t^*]$ and decreases in $t \in (t^*, t_*]$.
- $\kappa^c(t^*) \approx 1.054$ and $\kappa^c(t_*) \approx 0.989$.

It follows from the continuity of K^c that $\kappa^c(t)$ is continuous. t^*, t_*, $\kappa^c(t^*)$ and $\kappa^c(t_*)$ are illustrated in Figure B.2.

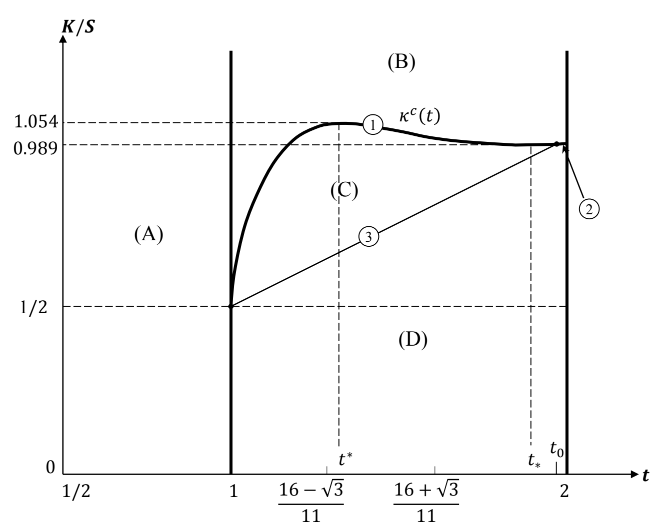

Figure B.1: $\kappa^c(t)$.

We take derivative of $\kappa^c(t)$, obtaining: if $\frac{16-\sqrt{3}}{11} < t \le \frac{16+\sqrt{3}}{11}$, then

$$(\kappa^c)'(t) = \frac{270 + 2t\{-367 + t[491 + 4t(-80 + 19t)]\}}{[25 + t(-40 + 19t)]^2} \tag{B.19}$$

and, if $1 < t \le \frac{16-\sqrt{3}}{11}$ or $\frac{16+\sqrt{3}}{11} < t \le t_*$, then

$$(\kappa^c)'(t) = \frac{4 - 4t}{3 + \sqrt{6}\sqrt{-(-7+t)(-1+t)} - 6t} - \frac{\left[12 + \frac{2\sqrt{6}(-4+t)}{\sqrt{-(-7+t)(-1+t)}}\right][7 + 4(-2+t)t]}{4\left[3 + \sqrt{6}\sqrt{-(-7+t)(-1+t)} - 6t\right]^2}. \tag{B.20}$$

One can verify that $(\kappa^c)'(t)$ is continuous at $t = \frac{16-\sqrt{3}}{11}$ and $t = \frac{16+\sqrt{3}}{11}$. Therefore, $\kappa^c(t)$ is differentiable for all $t \in (1, t_*]$.

Over the interval of $(1, t_*]$, $(\kappa^c)'(t) = 0$ is obtained at two values of t: $t = t^*$, which satisfies $\frac{16-\sqrt{3}}{11} < t^* < \frac{16+\sqrt{3}}{11}$; and $t = t_*$, which satisfies $t_* > \frac{16+\sqrt{3}}{11}$. Furthermore, one can verify that $(\kappa^c)''(t^*) < 0$ and $(\kappa^c)''(t_*) > 0$. Therefore, over $(1, t_*]$, $\kappa^c(t)$ is maximized by t^* and is increasing for $t \in (1, t^*]$ and is decreasing for $t \in (t^*, t_*]$. Finally, from equation (B.18) it is straightforward to verify that $\kappa^c(t^*) \approx 1.054$ and $\kappa^c(t_*) \approx 0.989$. ■

Proof of Proposition 5 (for the channel and consumer). The conclusions about the supplier and buyer come as a part of the proof for Lemma 3. We now focus on the channel and the consumer.

The channel profit. Combining the supplier and the buyer together, with $S \le B \le 2S$, it is easy to know the corresponding system profit is $(B - z)z$, where z is the total quantity sold to market. The system profit is subject to $z \le K$. Clearly, the first best quantity of the system is $\min(K, \frac{B}{2})$. Hence, the desired conclusions follow.

The consumer surplus. The values of the consumer surplus in the model with and without buyer withholding can be derived by substituting $(z_s, z_b) = (z_s^*, z_b^*)$ and $(z_s, z_b) = (\hat{z}^s, \hat{z}^b)$, respectively, in the expression of (12). We omit the details of the derivation. ■

Proof of Lemma 4. Since the supplier does not sell to the market, the buyer will order as many as he wants to sell. The corresponding buyer's profit at given wholesale price w is $\pi_b(q, w) = (B - q - w)q$, where q is the buyer's order quantity. Clearly, the buyer's optimal quantity is contingent to w and the supplier's capacity K: $\tilde{q}(w, K) = \min\left(\frac{B-w}{2}, K\right)$.

Based on the buyer's response to w and K, we can formulate the supplier's profit as

$$\pi_s(w, K) = w\tilde{q}(w, K) = \begin{cases} \frac{(B-w)w}{2} & \text{if } w \ge B - 2K \\ wK & \text{if } w \le B - 2K \end{cases}$$

Notice that the global maximizer to $\frac{(B-w)w}{2}$ is $w = \frac{B}{2}$, and $\frac{B}{2} \ge B - 2K \Leftrightarrow K \ge \frac{B}{4}$. We can make the following statements:

• if $K \geq \frac{B}{4}$, the supplier's optimal profit is $\max\left((B-2K)K, \frac{B^2}{8}\right)$. Since $(B-2K)K$ is concave in K and maximized with $K = \frac{B}{4}$, which results exactly $\frac{B^2}{8}$. Hence, The supplier's optimal profit is $\tilde{\pi}_s = \frac{B^2}{8}$, and the optimal wholesale price $\tilde{w} = \frac{B}{2}$, causing the buyer's optimal order quantity equal to $\tilde{q} = \frac{B}{4}$.

• if $K \leq \frac{B}{4}$, the supplier's profit function is clearly maximized at $w = B - 2K$. Hence, the desired results follow. ■

Proof of Proposition 6. Given $S \leq B \leq 2S$, from Lemma A.2 and Proposition 1, we know that π_s^* is continuous in K, with maximum being attained at some $K \in [S/2, \min(B/2, K^c)]$.

On the other hand, $\tilde{\pi}_s$ is continuous increases when $K \leq B/4$ and then keeps constant when $K \geq B/4$. Hence, the continuity of $\pi_s^* - \tilde{\pi}_s$ is obvious. Check Definition A.2, we know that when $K \leq B/4$, $\pi_s^* - \tilde{\pi}_s = (B-K)K - (B-2K)K = K^2$ increasing in K. Since $S \leq B \leq 2S \Rightarrow B/4 \leq S/2$, we can finally conclude that $\pi_s^* - \tilde{\pi}_s$ is unimodal and its maximum must between $S/2$ and $\min(B/2, K^c)$. ■

Proofs of Propositions 7 and 8. Using Definition A.2 and Proposition 2, we illustrate $K = z_b^o + z_s^o$ versus $K = K^c$ and $K = K_b$ in Figure B.3. Recall that the area between line 1 and line 4 is the region limited capacity has "win-win-win" effect. We focus on the un-shaded area shown in Figure B.3, that is, $K > z_b^o + z_s^o$ and $S < B \leq 2S$. We prove the results backwards, starting from Stage 3.

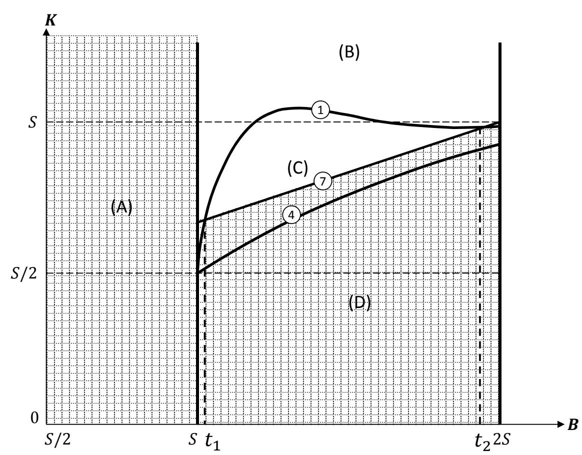

Figure B.2: The unshaded region represents the set of situations considered in this proof. Line 1: $K = K^c$; line 4; $K = K_b$; line 7: $K = z_s^o + z_b^o = \frac{B+S}{3}$. Lines 1 and 4 are identical to those defined in Figure 4.

Stage 3. Given that the supplier's wholesale price is w, the spot market price is p, the buyer orders q and q_p respectively from the supplier and the spot market, we determine the market equilibrium $(\check{z}_s, \check{z}_b)$ for the supplier and the buyer from their profit maximization programs:

$$\max_{0\le z_s \le K-q} \pi_s(z_s, z_b) = (S - z_s - z_b)z_s + wq \tag{B.21}$$

$$\max_{0\le z_b \le q+q_p} \pi_b(z_s, z_b) = (B - z_s - z_b)z_b - wq - pq_p. \tag{B.22}$$

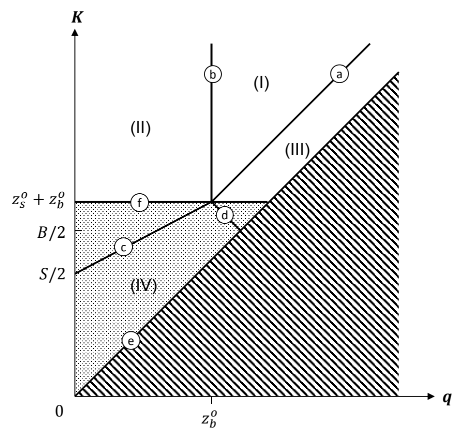

Figure B.3: Regions to be considered for the analysis of stage 3.

Following the same logic for proving Lemma 1, we can derive the conclusions and illustrate them in Figure B.4. As you can notice, Figure B.4 differs from Figure 1 by having line (f), which is defined by $K = z_s^o + z_b^o$, and removing the area below line (f). We end up with region (I) and upper regions (II) and (III) with $K \ge z_s^o + z_b^o$. In this proof, we slightly abuse the notation by referring to them as regions (I), (II) and (III). For these three regions in Figure B.4, we conclude:

(I) The spot market quantity q_p does not impact on the market equilibrium, which is still $\check{z}_s = z_s^o$ and $\check{z}_b = z_b^o$. We denote the buyer's profit as

$$\check{\pi}_b^I = (z_b^o)^2 - wq - pq_p.$$

(II) The market equilibrium depends on the buyer's total order from the supplier and the spot market $q + q_p$: $\check{z}_s = \max\left(\frac{S-(q+q_p)}{2}, z_s^o\right)$ and $\check{z}_b = \min(q + q_p, z_b^o)$. Note that when $q + q_p \ge z_b^o$, the equilibrium converges to $\check{z}_s = z_s^o$ and $\check{z}_b = z_b^o$. Hence, we focus on $q + q_p < z_b^o$ and denote the buyer's resulted profit as

$$\check{\pi}_b^{II} = \left[B - \frac{S + (q + q_p)}{2}\right](q + q_p) - wq - pq_p.$$

(III) Like (I), the order from the spot market does not change the market equilibrium: $\check{z}_s = K - q$ and $\check{z}_b = \frac{B-(K-q)}{2}$. The buyer's profit is denoted as

$$\check{\pi}_b^{III} = \left[\frac{B - (K - q)}{2}\right]^2 - wq - pq_p.$$

Stage 2. Based on the conclusion on Stage 3, we now explore the buyer's best ordering quantities from both the supplier and the spot market, given to the supplier's wholesale price w and the spot market price p.

Notice that both $\breve{\pi}_b^I$ and $\breve{\pi}_b^{III}$ are decreasing in q_p. It indicates the following two facts: (1) If $q \geq z_b^o$, then $\check{q}_p = 0$. Hence, the areas (I) and (III) in Figure B.4 become identical to the no spot market case, and the reginal best action is to buy all the supplier's capacity K but nothing from the spot market. (2) If $q \leq z_b^o$, then $0 \leq \check{q}_p \leq z_b^o - q$, and the optimization in area (II) must be based on $\breve{\pi}_b^{II}$. Hence, the reginal optimal purchasing strategy is that the buyer orders only from one source, the one with the lower price, and the order quantity is $\min\left(\left(\frac{3}{2}z_b^o - \min(w,p)\right)^+, z_b^o\right)$.

In conclusion, the buyer has only two possible optimal order quantities in response to (w,p):

(i) Only buy out the supplier: $q = K$ and $q_p = 0$. The corresponding buyer's profit is

$$\breve{\pi}_b^i = \frac{B^2}{4} - wK.$$

(ii) Only order from the cheaper source: If $w \leq q$, then order nothing from the spot market so that $q = \min\left(\left(\frac{3}{2}z_b^o - w\right)^+, z_b^o\right)$ and $q_p = 0$. The corresponding profit is

$$\breve{\pi}_b^{ii} = \left(B - \frac{S + \min\left(\left(\frac{3}{2}z_b^o - w\right)^+, z_b^o\right)}{2} - w\right)\left(\min\left(\left(\frac{3}{2}z_b^o - w\right)^+, z_b^o\right)\right).$$

When $w \geq q$, order nothing from the supplier so that $q = 0$ and $q_p = \min\left(\left(\frac{3}{2}z_b^o - p\right)^+, z_b^o\right)$. Correspondingly, the buyer's profit becomes independent of w, and

$$\breve{\pi}_b^{ii} = \left(B - \frac{S + \min\left(\left(\frac{3}{2}z_b^o - p\right)^+, z_b^o\right)}{2} - p\right)\left(\min\left(\left(\frac{3}{2}z_b^o - p\right)^+, z_b^o\right)\right).$$

Notice that both $\breve{\pi}_b^i$ and $\breve{\pi}_b^{ii}$ are non-increasing in w.

Stage 1. We discuss two cases of the capacity: $z_s^o + z_b^o < K \leq K^c$ and $K > K^c$.

First, given that $z_s^o + z_b^o < K \leq K^c$, without spot market, Proposition 1 shows that $w^* = w^o(K)$ and $q^* = K$. According to Lemma A.1 and Definition A.1, we know that at $w^* = w^o(K)$ the buyer is indifferent between $q^* = K$ and $q^* = q^o(w^o(K)) = \min\left(\frac{3}{2}z_b^o - w^o(K), z_b^o\right)$, which results in market competition in Stage 3. However, the supplier's profit is higher if the buyer chooses $q^* = K$. We assume that when two actions are indifferent to the buyer, she chooses the one which benefits the supplier.

With spot market, it is either $p > w^o(K)$ or $p \leq w^o(K)$.

Suppose $p > w^o(K)$, then supplier may choose either $w \geq p$ or $w \leq p$. Since $w^o(K)$ is the inverse function of $K^o(w)$, we can revise the conclusion in Stage 2 as follows: (a) If $w \leq w^o(K)$, the comparison between $\breve{\pi}_b^i$ and $\breve{\pi}_b^{ii}$ is identical to the no spot market cases. According to Lemma 2(body), we must have $\breve{\pi}_b^i \geq \breve{\pi}_b^{ii}$, suggesting that the buyer's decision is $\check{q} = K$ and $\check{q}_p = 0$; (b) If $w^o(K) \leq w \leq p$, spot market is not used. Still by Lemma 2(body), we know $\breve{\pi}_b^{ii} \geq \breve{\pi}_b^i$. Hence, the buyer's decision is $\check{q} = \min\left(\left(\frac{3}{2}z_b^o - w\right)^+, z_b^o\right)$ and $\check{q}_p = 0$; and (c) If $p \leq w$, $\breve{\pi}_b^i$ decreases in w, while $\breve{\pi}_b^{ii}$ remains at a constant larger than $\breve{\pi}_b^i(p)$. Hence, the buyer's decision is $\check{q} = 0$ and $\check{q}_p = \min\left(\left(\frac{3}{2}z_b^o - p\right)^+, z_b^o\right)$.

Hence, the buyer's decision is identical to the no spot market case when $w \leq p$. Notice that $p \geq w^o(K)$ and $w^o(K)$ is the optimal price that the supplier will charge without spot market. We know $w^o(K)$ is also the best price to the supplier given that $w \leq p$. For $w > p$, the supplier's profit is constant and must be lower than what supplier can get when the buyer orders from the supplier at the price $w = p$. As the result, we have the following result:

Conclusion (2.b) Given $z_s^o + z_b^o < K \leq K^c$ and $p > w^o(K)$, the supplier offers wholesale price $\breve{w} = w^o(K)$, the buyers order $\check{q} = K$ from the supplier and $\check{q}_p = 0$ from the spot market. The buyer sells $\breve{z}_b = \frac{B}{2}$ and the suppliers sells $\breve{z}_s = 0$.

Obviously, the spot market does not have any impact on the supplier and the buyer's decisions as well as their profits. We can also expect that the consumer surplus must be the same as the no spot market case.

Suppose $p \leq w^o(K)$, the supplier still can choose either $w \geq p$ or $w \leq p$. Similar to the logic used above, when $w \leq p$, we know that the spot market is not used and that $\breve{\pi}_b^i \geq \breve{\pi}_b^{ii}$. When $w \geq p$, $\breve{\pi}_b^i$ decreases in w, while $\breve{\pi}_b^{ii}$ is fixed to the number smaller than $\breve{\pi}_b^i(p)$ and bigger than $\breve{\pi}_b^i(w^o(K))$. Hence, we can conclude that there exists a wholesale price threshold $p^o(K,p) \in (p, w^o(K))$, determined by the following rules:

$$\begin{cases} \frac{B^2}{4} - p^o K = (z_b^o)^2 - z_b^o p, & \text{if } p \leq \frac{1}{2}z_b^o \\ \frac{B^2}{4} - p^o K = \frac{1}{2}\left(\frac{3}{2}z_b^o - p\right)^2, & \text{if } \frac{1}{2}z_b^o \leq p \leq w^o(K) \end{cases}, \tag{B.23}$$

such that (a) if $w \leq p^o(K,p)$, the buyer decides to order $\check{q} = K$ and $\check{q}_p = 0$; and (b) if $w \geq p^o(K,p)$, the buyer orders $\check{q} = 0$ and $\check{q}_p = \min\left(\left(\frac{3}{2}z_b^o - p\right)^+, z_b^o\right)$. It is also easy to check that $w^o(K) \leq \frac{3}{2}z_b^o$ and that $p^o(K,p)$ decreases in K and increases in p.

Based on the buyer's response, if $w \leq p^o(K,p)$, then the supplier's profit is wK; otherwise, it is

$$\breve{\pi}_s^b = \left(\frac{S - \min\left(\frac{3}{2}z_b^o - p, z_b^o\right)}{2}\right)^2. \tag{B.24}$$

Clearly, supplier's final decision depends on the order between $\breve{\pi}_s^a = p^o(K,p)K$ and $\breve{\pi}_s^b$. Note that, $\breve{\pi}_s^a$ can be derived from (B.23), and we can also formulate

$$\breve{\pi}_s^b = \begin{cases} (z_s^o)^2 & \text{if } p \leq \frac{1}{2}z_b^o \\ \frac{\left(S - \frac{3}{2}z_b^o + p\right)^2}{4} & \text{if } \frac{1}{2}z_b^o \leq p \leq w^o(K) \end{cases}.$$

Straightforward algebra shows that $\breve{\pi}_s^a \geq \breve{\pi}_s^b$. That means the following result:

Conclusion (2.a) Given $z_s^o + z_b^o \leq K \leq K^c$ and $p \leq w^o(K)$, the supplier offers wholesale price $\breve{w} = p^o(K,p)$, which is increasing in p and $p < \breve{w} \leq w^o(K)$. The buyers order $\breve{q} = K$ from the supplier and $\breve{q}_p = 0$ from the spot market. The buyer sells $\breve{z}_b = \frac{B}{2}$ and the supplier sells $\breve{z}_s = 0$.

A lower wholesale price offered by the supplier is the only difference to the no spot market results. It is easy to see that the supplier's profit is reduced due to the spot market, while the buyer's profit is increased. The consumer surplus is not changed by the existence of the spot market.

Next, we explore the case where $K > K^c$. Without the spot market, the supplier sets $w^* = \frac{1}{2}z_b^o + \frac{S}{3}$, buyer orders $q^* = z_b^o - \frac{S}{3}$. From Lemma A.1 and Definition A.2, we can show that $w^* > w^o(K)$. With the spot market, we consider three different cases: $p > w^*$, $w^* \geq p > w^o(K)$, and $w^o(K) \geq p$. It suffices to focus on the case where $w^* \geq p > w^o(K)$, since the other two can be argued by exactly the same logic as before. We can conclude the following results:

Conclusion (1.d) Given $K > K^c$ and $p > \frac{1}{2}z_b^o + \frac{S}{3}$, the supplier offers wholesale price $\breve{w} = \frac{1}{2}z_b^o + \frac{S}{3}$. The buyer orders $\breve{q} = z_b^o - \frac{S}{3}$ from the supplier and $\breve{q}_p = 0$ from the spot market. The buyer sells $\breve{z}_b = z_b^o - \frac{S}{3}$ and the supplier sells $\breve{z}_s = z_s^o + \frac{S}{6}$.

Again, all the results coincide with the case without the spot market. Hence, the existence of the spot market does not change the supplier's profit, the buyer's profit, and consumer surplus.

It is also easy to see that the supplier's equilibrium profit $\breve{\pi}_s^*$ is constant to K.

Conclusion (1.a) Given $K > K^c$ and $p \leq w^o(K)$, the supplier offers wholesale price $\breve{w} = p^o(K,p)$, which is increasing in p and $p < \breve{w} \leq w^o(K)$. The buyers order $\breve{q} = K$ from the supplier and $\breve{q}_p = 0$ from the spot market. The buyer sells $\breve{z}_b = \frac{B}{2}$ and the supplier sells $\breve{z}_s = 0$.

Compared to the no spot market case, the supplier gives up competing with the buyer in the end market and offers a much lower wholesale price to ensure the full capacity purchase from the buyer. Notice that, without the spot market, if the supplier offers $\breve{w}$ as the wholesale price, the buyer's ordering decision is also K. It suggests that the supplier's profit at price $\breve{w}$ cannot be better than that at price w^*. Hence, the spot market reduces the supplier's profit. On the other hand, the spot market benefits the buyer, since he becomes the monopolist in the market by paying a wholesale price even lower than $w^o(K)$. Based on Table 1, we can easily check that the consumer surplus is improved due to the spot market.

The analysis for the case of $w^* \geq p > w^o(K)$ is very similar to the one for $z_s^o + z_b^o < K \leq K^c$ and $p > w^o(K)$, especially for final characterization of the buyer's decisions.

The supplier's profit is composed by three parts. When $w \leq w^o(K)$, it is wK, which is maximized at $w = w^o(K)$, giving the profit of $w^o(K)K$. When $w^o(K) \leq w \leq p$, it is

$$\left(\frac{S - \min\left(\frac{3}{2}z_b^o - w, z_b^o\right)}{2}\right)^2 + w\left(\min\left(\frac{3}{2}z_b^o - w, z_b^o\right)\right),$$

concave in w. Without constraint, it is maximized at $w = \frac{1}{2}z_b^o + \frac{S}{3}$. Hence, the regional highest profit is achieved at $w = p$, equal to

$$\left(\frac{S - \min\left(\frac{3}{2}z_b^o - p, z_b^o\right)}{2}\right)^2 + p\left(\min\left(\frac{3}{2}z_b^o - p, z_b^o\right)\right).$$

When $p \leq w$, it is no longer the function of w and formulated same to $\breve{\pi}_s^b$ presented in (B.24). Clearly, when compared to the previous ranges, $\breve{\pi}_s^b$ cannot be the global highest profit.

We can establish a spot price threshold $\breve{p}^c(K)$ such that when $p = \breve{p}^c(K)$, we have

$$w^o(K)K = \left(\frac{S - \min\left(\frac{3}{2}z_b^o - p, z_b^o\right)}{2}\right)^2 + p\left(\min\left(\frac{3}{2}z_b^o - p, z_b^o\right)\right).$$

From Lemma A.2(body), we find that $\breve{p}^c(K)$ decreases in K, and $\breve{p}^c(K^c) = \frac{1}{2}z_b^o + \frac{S}{3}$. Hence, we can make the following conclusions:

<u>Conclusion (1.c)</u> Given $K > K^c$ and $\breve{p}^c(K) < p \leq \frac{z_b^o}{2} + \frac{S}{3}$, the supplier matches the spot market price $\breve{w} = p$. The buyer responses by ordering $\breve{q} = \min\left(\left(\frac{3}{2}z_b^o - p\right)^+, z_b^o\right)$ from supplier and $\breve{q}_p = 0$ from the spot market. The buyer sells everything on the market $\breve{z}_b = \breve{q}$ and the suppliers sells $\breve{z}_s = \frac{S - \breve{q}}{2}$.

The spot market forces the supplier to reduce the wholesale price, but does not change the channel structure. The supplier still sells and competes with the buyer at the same time. Note that, with wholesale price equal to $\breve{w}$, the buyer's response as well as the followed market equilibrium remain the same no matter if there is a spot market or not. Hence, we can claim that the spot market makes the supplier worse off. Analyzing the buyer's profit shows that the buyer is better off, since he can buy more at a lower price to compete with the supplier in the end market. By plugging $\breve{z}_s$ and $\breve{z}_b$ in (12) and comparing it to case (B) of $\pi_u(z_s^*, z_b^*)$ in Table 1, we find that the spot market increases consumer surplus.

It is also easy to see that the supplier's equilibrium profit $\breve{\pi}_s^*$ is constant to K.

Conclusion (1.b) Given $K > K^c$ and $w^o(K) < p \leq \breve{p}^c(K)$, the supplier offers wholesale price $\breve{w} = w^o(K)$, lower than the spot market price. The buyer orders $\breve{q} = K$ from supplier and $\breve{q}_p = 0$ from the spot market. The buyer sells everything on the market $\breve{z}_b = \frac{B}{2}$ and the suppliers sells nothing $\breve{z}_s = 0$.

Similar to the discussion on Conclusion (1.a), we conclude that the spot market benefits the buyer and the consumers but not the supplier.

Now, we are ready to compare the supplier and buyer profits and the consumer surplus in cases (1.c) and (1.d), which are equivalent to the infinite capacity case, to those in the other cases, in order to prove the conclusions in Proposition 9 (body).

First, if $p > \frac{z_b^o}{2} + \frac{S}{3}$, we are comparing case (1.d) and (2.b). In both cases, the spot market does not have any impact on either supplier or buyer's decisions, indicating that all the results for no spot market scenario still hold. Notice the relationship between $z_b^o + z_s^o$ and K_b, we can easily derive the conclusion.

Next, we consider $p \leq \frac{z_b^o}{2} + \frac{S}{3}$. Note that $\breve{K}^c(p)$ is the inverse function of $\breve{p}^c(K)$ and decreases in P. Based on the definition of $\breve{p}^c(K)$, we know that supplier breaks even at $K = \breve{K}^c(p)$ between offering wholesale price p and $w^o(K)$. Since the total supply chain profit reaches the highest level of $\frac{B^2}{4}$ when the wholesale price is $w^o(K)$, it is easy to that at $K = \breve{K}^c(p)$, the supplier should offer wholesale price $w^o(K)$ to ensure a higher profit to the buyer. The continuity of the buyer's profit in case (1.b) guarantees that there exists $K \in [K^c, \breve{K}^c(p))$ such that the buyer's profit is higher than that in case (1.c).

For consumer surplus, according to its definition in (12) (body), we can see that it is $\frac{B^2}{8}$ when in cases (1.a), (1.b), (2.a), and (2.b). In case (1.c), it is $\frac{(B+S)^2}{18}$ if $p \leq \frac{z_b^o}{2}$ and $\frac{1}{2}\left(\frac{S}{2} + B - p\right)^2$ if otherwise. Straightforward algebra shows that if and only if $p > \frac{S}{2}$, consumer surplus in case (1.c) is less than $\frac{B^2}{8}$.

For supplier's profit, we can derive the case dependent profit functions. In cases (1.a) and (2.a), it is $\breve{\pi}_s^*(a) = p^o(K,p)K$. In cases (1.b) and (2.b), it is $\breve{\pi}_s^*(b) = w^o(K)K$. In case (1.c), if $p \leq \frac{z_b^o}{2}$, it is $\breve{\pi}_s^{i*}(c) = pz_b^o + \frac{(2S-B)^2}{9}$, and it is $\breve{\pi}_s^{ii*}(c) = p\left(B - \frac{S}{2} - p\right) + \frac{1}{4}\left(\frac{3S}{2} - B + p\right)^2$. Note that the supplier's profit is continuous in K. Based on Lemma A.2 and the definition of $\breve{p}^c(K)$, we know $\breve{p}^c(K) \geq K^c \geq \max(K^a, K^b)$. According to Lemma A.2, it means that there exists $K \in [K^c, \breve{K}^c(p))$ such that the supplier's profit is higher than that in case (1.c). ■

Proof of Proposition 9. We prove the results for the model with withholding and the similar logic can be applied for the argument of the results for the model without withholding.

When withholding is allowed, the solution to Stage 3 is the same, no matter if the supplier contract on Q or not. Hence, Lemma 1 can be referred directly. The only difference on Stage 2 is that the constraint is no longer $q \leq K$ but $q \leq Q$. For the convenience of further discussion, if the supplier has capacity of K, we denote $\pi_b(q, w, Q; K)$ as the buyer's profit at ordering q in Stage 2 when the supplier offers (w, Q) and the market equilibrium will be applied in Stage 3. Recall that we have shown in the proofs of Lemmas 2 and A.1 that $\pi_b(q, w, K; K)$ has at most two local optimal points: $q^a \stackrel{\text{def}}{=} K$ or $q^b \stackrel{\text{def}}{=} \min\left\{\left(\frac{3}{2}z_b^o - w\right)^+, z_b^o, (2K-S)^+\right\}$. It is easy to see that with $q \leq Q$, the buyer's optimal order quantity, denoted as $\bar{q}$, must be

$$\bar{q} = \begin{cases} \min\{q^a, Q\} & \text{or} \\ \min\{q^b, Q\} \end{cases}$$

Now we solve the Stage 1 problem for the supplier's optimal (w, Q) contract, denoted as $(\bar{w}, \bar{Q})$. We focus on the case where $S < B \leq 2S$, which has the most representative analysis. We consider four cases of K: (1) $K \leq \frac{S}{2}$, (2) $\frac{S}{2} < K \leq \frac{B}{2}$, (3) $\frac{B}{2} < K \leq \frac{B+S}{3}$, and (4) $K > \frac{B+S}{3}$.

(1) $\boldsymbol{K \leq \frac{S}{2}}$. By referring to Figure 1 and Lemma 1, we know the buyer's profit function is $\pi_b(q, w, Q; K) = (B - K - w)q$ subject to $q \leq Q$. Hence, we have $w \leq B - K \Rightarrow \bar{q} = Q$ and $w \geq B - K \Rightarrow \bar{q} = 0$. By Figure B.5, we show why $\bar{Q} = K$ and $\bar{w} = B - K$.

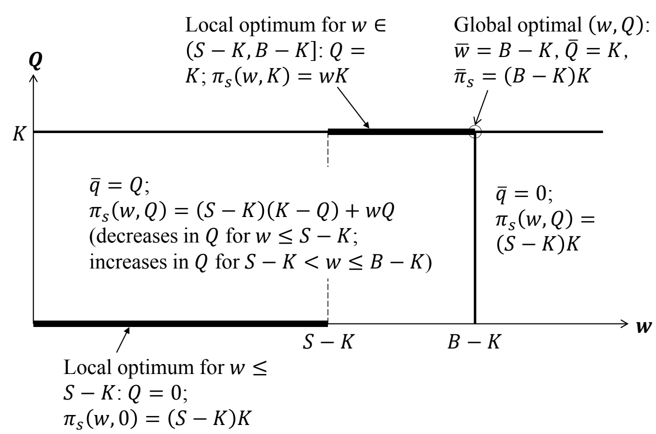

Figure B.4: Illustration of the proof when $K \leq S/2$.

(2) $\boldsymbol{\frac{S}{2} < K \leq \frac{B}{2}}$. We refer to Figure 1 and Lemma 1 again and construct Figure B.6 by the following steps: (Denote $w_o(K) = \frac{3}{2}z_b^o - (2K - S)$, and notice that $2K - S \leq z_b^o$ and $2K - S \leq K$.)

(2.a) If $w < w_o(K)$, then $2K - S \leq \frac{3}{2}z_b^o - w$. It implies that $q^b = 2K - S$ and $\pi_b(q, w, K; K)$ is increasing in q when $q \leq q^b$. Since $w < w_o \Rightarrow w < B - K$, we know that $\pi_b(q, w, K; K)$ is also increasing in q when $q^b \leq q \leq K$. Hence, q^b is not a local maximum point, and we must have $\bar{q} = Q$. The supplier's profit can be specified as (1) if $Q \le 2K - S$, then $\pi_s(w, Q) = \frac{1}{4}(S - Q)^2 + wQ$; and (2) if otherwise, $\pi_s(w, Q) = (S - K)(K - Q) + wQ$.

(2.b) If $w_o(K) \le w < w^o(K)$, then $q^b = \frac{3}{2}z_b^o - w$ is a local maximum point, and $\pi_b(q, w, K; K)$ is decreasing in q when $q^b \le q \le 2K - S$. We consequently establish a threshold

$$Q_1(w) = q^b.$$

From the definition of $w^o(K)$, we know q^a is the global maximum point. Hence, we must be able to find a break even threshold $Q_2(w)$, satisfying $2K - S \le Q_2(w) \le K$, such that $\pi_b(q^b, w, K; K) = \pi_b(Q_2(w), w, K; K)$. Based on Lemma 1, we can calculate $\pi_b(q^b, w, K; K) = \frac{1}{2}(\frac{3}{2}z_b^o - w)^2$ and $\pi_b(Q_2(w), w, K; K) = (B - K - w)Q_2(w)$. Hence, we have

$$Q_2(w) = \frac{(\frac{3}{2}z_b^o - w)^2}{2(B - K - w)}.$$

Notice that $Q_1(w) \le 2K - S \le Q_2(w)$. Now, we are ready to characterize $\bar{q}$ based on the thresholds $Q_1(w)$ and $Q_2(w)$. (1) If $Q \le Q_1(w)$, then $\bar{q} = Q$ and $\pi_s(w, Q) = (\frac{S-Q}{2})^2 + wQ$. (2) If $Q_1(w) \le Q < Q_2(w)$, then $\bar{q} = q^b$ and $\pi_s(w, Q) = \frac{1}{4}\left(S - \frac{3}{2}z_b^o + w\right)^2 + w\left(\frac{3}{2}z_b^o - w\right)$. (3) If $Q \ge Q_2(w)$, then $\bar{q} = Q$ and $\pi_s(w, Q) = (S - K)(K - Q) + wQ$.

(2.c) if $w \ge w^o(K)$, then $q^b = \left(\frac{3}{2}z_b^o - w\right)^+$ is the global maximum point. Hence, for the same defined $Q_1(w)$ as in (2.b), we know that (1) if $Q \le Q_1(w)$, then $\bar{q} = Q$ and $\pi_s(w, Q) = (\frac{S-Q}{2})^2 + wQ$; (2) if otherwise, $\bar{q} = q^b$ and $\pi_s(w, Q) = \frac{1}{4}\left(S - (\frac{3}{2}z_b^o - w)^+\right)^2 + w\left(\frac{3}{2}z_b^o - w\right)^+$.

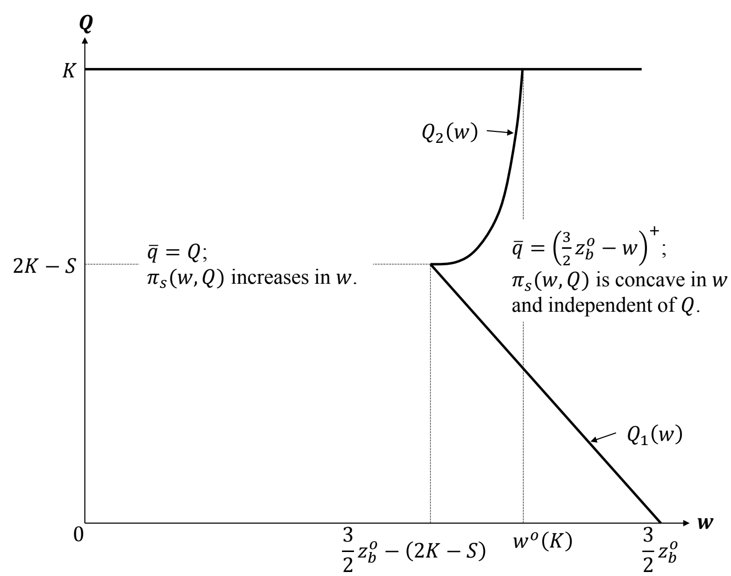

Figure B.5: Illustration of the proof when $S/2 < K \le B/2$.

From the above analysis, we find that $\pi_s(w, Q)$ is continuous in w when $Q \le 2K - S$. Hence, the determination of $(\bar{w}, \bar{Q})$ can be simplified in three steps:

(i). Solve the program $\max_{w_o(K)\le w\le w^o(K)} \pi_s^i(w)$, where $\pi_s^i(w) = (S-K)(K-Q_2(w)) + wQ_2(w)$. Note that $\pi_s^i(w^o(K)) = w^o(K)$. The corresponding buyer's optimal order in Stage 2 is K and the profit is $\pi_b(K, w^o(K), K; K) = (B-K)K - w^o(K)K$. The supply chain total profit is $\Pi(w^o(K), K) = (B-K)K$, which is the supply chain first best profit. Hence, we have $\Pi(w^o(K), K) \ge \Pi(w, Q_2(w))$, where $w_o(K) \le w \le w^o(K)$ and $\Pi(w, Q_2(w)) = \pi_s^i(w) + \pi_b(Q_2(w), w, Q_2(w); K)$. Recall that $Q_2(w)$ is the buyer's break even threshold where $\frac{1}{2}(\frac{3}{2}z_b^o - w)^2 = \pi_b(Q_2(w), w, K; K)$, and notice that $\pi_b(Q_2(w), w, K; K) = \pi_b(Q_2(w), w, Q_2(w); K)$ and $\frac{1}{2}(\frac{3}{2}z_b^o - w)^2$ is decreasing in w for $w_o(K) \le w \le w^o(K)$. Hence, we can conclude that $\pi_s^i(w^o(K)) \ge \pi_s^i(w)$, and $w^o(K)$ is the optimal solution.

(ii). Solve the program $\max_{w_o(K)\le w} \pi_s^{ii}(w)$, where $\pi_s^{ii}(w) = \frac{1}{4}\left(S - (\frac{3}{2}z_b^o - w)^+\right)^2 + w\left(\frac{3}{2}z_b^o - w\right)^+$. It is straightforward to check that the objective function is unimodal in w, with the global maximum obtained at $w_s = \frac{2B+S}{6}$. Hence, we have two possible cases: (1) if $w_o(K) \ge w_s$, (i.e., if $\frac{S}{2} \le K \le \frac{2B+S}{6}$), then the optimal solution is $w_o(K)$; and (2) if otherwise, (i.e., if $\frac{2B+S}{6} \le K \le \frac{B}{2}$), then the optimal solution is w_s. The supplier's respective profit is $\pi_s(w_o(K), \frac{3}{2}z_b^o - w_o(K)) = (S-K)^2 + w_o(K)(2K-S)$ and $\pi_s(w_s, \frac{3}{2}z_b^o - w_s) = \frac{S^2}{4} + \frac{(B-S)^2}{3}$. The buyer's optimal order in Stage 2 is respectively $\frac{3}{2}z_b^o - w_o(K) = 2K - S$ and $\frac{3}{2}z_b^o - w_s = \frac{2}{3}(B-S)$. The corresponding profit is respectively $\pi_b(2K-S, w_o(K), 2K-S; K) = \frac{1}{2}(\frac{3}{2}z_b^o - w_o(K))^2$ and $\pi_b(\frac{2}{3}(B-S), w_s, \frac{2}{3}(B-S); K) = \frac{1}{2}(\frac{3}{2}z_b^o - w_s)^2$. We denote the respective supply chain total profit as $\Pi(w_o(K), \frac{3}{2}z_b^o - w_o(K))$ and $\Pi(w_s, \frac{3}{2}z_b^o - w_s)$.

(iii). Comparing the optimal solutions from (i) and (ii) to determine which one is the global optimal solution. We start with the comparison on the supply chain's total profit. Notice that $\Pi(w^o(K), K)$ is the supply chain's first best profit, implying that $\Pi(w^o(K), K) \ge \Pi(w_o(K), \frac{3}{2}z_b^o - w_o(K))$ and $\Pi(w_s, K) \ge \Pi(w_s, \frac{3}{2}z_b^o - w_s)$. Notice also that $w^o(K)$ is the break even wholesale price for the buyer to order K and to order $\frac{3}{2}z_b^o - w^o(K)$, implying that $\pi_b(K, w^o(K), K; K) = \pi_b(\frac{3}{2}z_b^o - w^o(K), w^o(K), \frac{3}{2}z_b^o - w^o(K); K) = \frac{1}{2}(\frac{3}{2}z_b^o - w^o(K))^2$. We next focus on the two cases specified in (ii). (1) If $\frac{S}{2} \le K \le \frac{2B+S}{6}$, $w_s \le w_o(K) \le w^o(K) \le \frac{3}{2}z_b^o$ implies that $\frac{1}{2}(\frac{3}{2}z_b^o - w_o(K))^2 \ge \pi_b(K, w^o(K), K; K)$. Hence, $\pi_s(w^o(K), K) \ge \pi_s(w_o(K), \frac{3}{2}z_b^o - w_o(K))$, and $(\bar{w}, \bar{Q}) = (w^o(K), K)$. It is easy to see from Lemma A.2, $w^o(K)$ is also the supplier's optimal price when there is no contract on Q. (2) If $\frac{2B+S}{6} \le K \le \frac{B}{2}$, then $w_s \le \frac{3}{2}z_b^o$ but it is not always less than $w^o(K)$. If $w_s \le w^o(K)$, by the similar logic as above, we have $\pi_s(w^o(K), K) \ge \pi_s(w_s, \frac{3}{2}z_b^o - w_s)$ and $(\bar{w}, \bar{Q}) = (w^o(K), K)$, consistent to the optimal policy under wholesale price only contract. If $w^o(K) \le w_s$, the comparison becomes an application of Lemma A.2.

In summary, we conclude that $\bar{Q} = K$.

(3) $\boldsymbol{\frac{B}{2} < K \leq \frac{B+S}{3}}$. By the similar logic as in (2), we can construct Figure B.7 and make the analysis based on it. Notice here that, by letting $w_t = K - \frac{S}{2} + \sqrt{(B-K)(B+S-3K)}$,

$$Q_2(w) = \begin{cases} \frac{(\frac{3}{2}z_b^o - w)^2}{2(B-K-w)} & \text{if } \frac{3}{2}z_b^o - (2K-S) \leq w \leq w_t \\ 2w - (B-K) + \sqrt{4w^2 - 4w(B-K) + 2(\frac{3}{2}z_b^o - w)^2} & \text{if } w_t \leq w \leq w^o(K) \end{cases}$$

and

$$\pi_s^i(w) = \begin{cases} (S-K)(K - Q_2(w)) + wQ_2(w) & \text{if } \frac{3}{2}z_b^o - (2K-S) \leq w \leq w_t \\ (S - \frac{B+(K-Q_2(w))}{2})(K - Q_2(w)) + wQ_2(w) & \text{if } w_t \leq w \leq w^o(K) \end{cases}$$

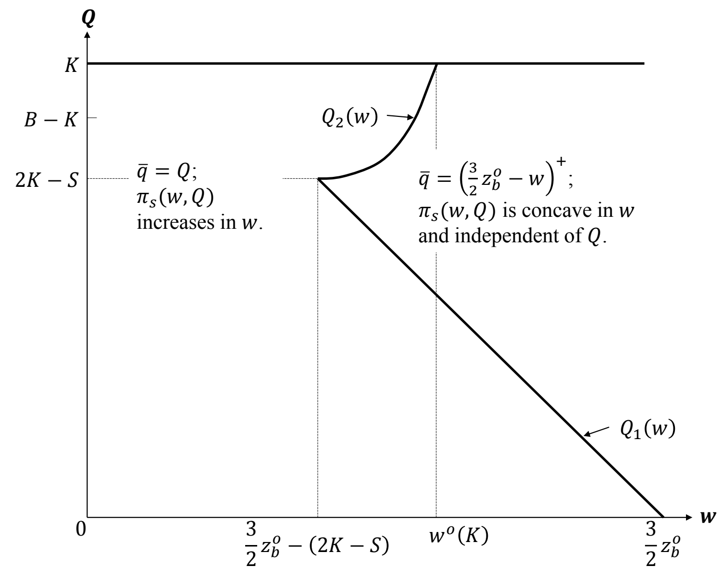

Figure B.6: Illustration of the proof when $B/2 < K \leq \frac{B+S}{3}$.

(4) $\boldsymbol{K > \frac{B+S}{3}}$. We do the similar analysis based on Figure B.8. Now, definitions for both $Q_1(w)$ and $Q_2(w)$ have to be updated. We have $Q_1(w) = \frac{3}{2}z_b^o - \max(w, \frac{1}{2}z_b^o)$,

$$Q_2(w) = \begin{cases} 2w - (B-K) + \sqrt{4w^2 - 4w(B-K) + 4z_b^o(z_b^o - w)} & \text{if } 0 \leq w \leq \min(w^o(K), \frac{1}{2}z_b^o) \\ 2w - (B-K) + \sqrt{4w^2 - 4w(B-K) + 2(\frac{3}{2}z_b^o - w)^2} & \text{if } \min(w^o(K), \frac{1}{2}z_b^o) \leq w \leq w^o(K) \end{cases}$$

and

$$\pi_s^i(w) = (S - \frac{B + (K - Q_2(w))}{2})(K - Q_2(w)) + wQ_2(w).$$

■

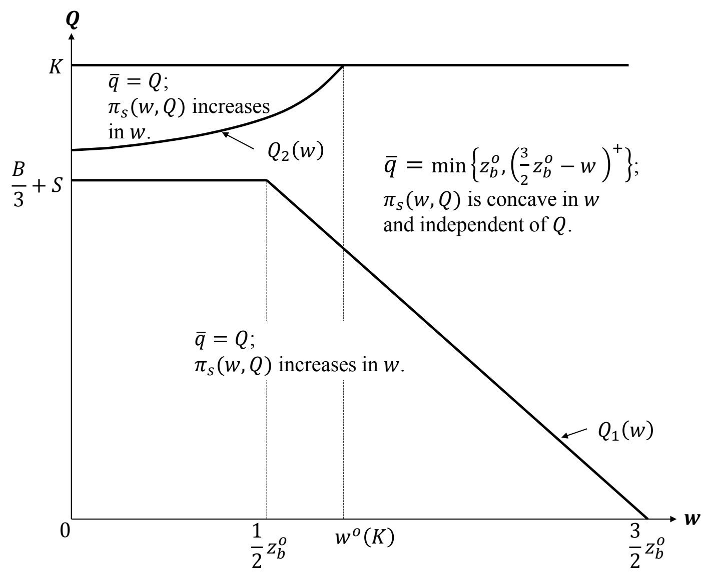

Figure B.7: Illustration of the proof when $K > \frac{B+S}{3}$.