

## Electronic Companion A Proofs

### A.1 Proof of Proposition 2.1

For notational convenience, we define the nonnegative random variables  $\tilde{z}_+ = (\tilde{d} - m)^+$  and  $\tilde{z}_- = (m - \tilde{d})^+$ . From the definition of  $s$ , we have  $\mathbb{E}(\tilde{z}_+^2) = \frac{1}{2}(1+s)\sigma^2$  and  $\mathbb{E}(\tilde{z}_-^2) = \frac{1}{2}(1-s)\sigma^2$ . Since  $\tilde{d}$  is nonnegative,  $\tilde{z}_-$  must not exceed  $m$ . This implies that  $\mathbb{E}(\tilde{z}_-(m - \tilde{z}_-)) \geq 0$ , or equivalently,  $\mathbb{E}(\tilde{z}_-) \geq \mathbb{E}(\tilde{z}_-^2)/m = (1-s)\sigma^2/(2m)$ . Also since  $\mathbb{E}(\tilde{z}_-^2 | \tilde{z}_- > 0) \geq (\mathbb{E}(\tilde{z}_- | \tilde{z}_- > 0))^2$ , then we have

$$\Pr(\tilde{z}_- > 0) \geq \frac{(\mathbb{E}(\tilde{z}_- | \tilde{z}_- > 0) \Pr(\tilde{z}_- > 0) + 0 \cdot \Pr(\tilde{z}_- = 0))^2}{\mathbb{E}(\tilde{z}_-^2 | \tilde{z}_- > 0) \Pr(\tilde{z}_- > 0) + 0^2 \cdot \Pr(\tilde{z}_- = 0)} = \frac{(\mathbb{E}(\tilde{z}_-))^2}{\mathbb{E}(\tilde{z}_-^2)} \geq \frac{(1-s)\sigma^2}{2m^2}. \quad (\text{A.1})$$

By a similar argument, and since  $\mathbb{E}(\tilde{z}_+) = \mathbb{E}(\tilde{z}_-)$ , we also find that

$$\Pr(\tilde{z}_+ > 0) \geq \frac{(\mathbb{E}(\tilde{z}_+ | \tilde{z}_+ > 0) \Pr(\tilde{z}_+ > 0) + 0 \cdot \Pr(\tilde{z}_+ = 0))^2}{\mathbb{E}(\tilde{z}_+^2 | \tilde{z}_+ > 0) \Pr(\tilde{z}_+ > 0) + 0^2 \cdot \Pr(\tilde{z}_+ = 0)} = \frac{(\mathbb{E}(\tilde{z}_+))^2}{\mathbb{E}(\tilde{z}_+^2)} \geq \frac{(1-s)^2 \sigma^2}{(1+s) 2m^2}. \quad (\text{A.2})$$

Finally, note that by our definition,  $\Pr(\tilde{z}_+ > 0) = \Pr(\tilde{d} > m)$  and  $\Pr(\tilde{z}_- > 0) = \Pr(\tilde{d} < m)$ . Thus, by inequalities (A.1)–(A.2), it follows that

$$1 \geq \Pr(\tilde{d} > m) + \Pr(\tilde{d} < m) \geq \frac{\sigma^2}{m^2} \frac{(1-s)}{(1+s)}, \quad (\text{A.3})$$

which gives us the lower bound on  $s$ . Observe that if  $s = \frac{\sigma^2 - m^2}{\sigma^2 + m^2}$ , then inequalities (A.1)–(A.3) tight. This implies that

$$\begin{aligned} \Pr(\tilde{d} > m) &= \frac{m^2}{m^2 + \sigma^2}, \quad \mathbb{E}\left(\left(\tilde{d} - m\right)^+\right) = \frac{m\sigma^2}{m^2 + \sigma^2}, \\ \Pr(\tilde{d} < m) &= \frac{\sigma^2}{m^2 + \sigma^2}, \quad \mathbb{E}\left(\left(m - \tilde{d}\right)^+\right) = \frac{m\sigma^2}{m^2 + \sigma^2}. \end{aligned}$$

Therefore, we have the following conditional moments:

$$\begin{aligned} \mathbb{E}(\tilde{z}_+ | \tilde{d} > m) &= \frac{\sigma^2}{m}, \quad \mathbb{E}(\tilde{z}_+^2 | \tilde{d} > m) = \frac{\sigma^4}{m^2}, \\ \mathbb{E}(\tilde{z}_- | \tilde{d} < m) &= m, \quad \mathbb{E}(\tilde{z}_-^2 | \tilde{d} < m) = m^2, \end{aligned}$$

where the second moments follow from the expression for  $s$ . Thus,  $\tilde{z}_+$  and  $\tilde{z}_-$  can only have one positive mass above and below  $m$  respectively. Thus,  $\tilde{d}$  can only have strictly positive mass on 0 and  $(m^2 + \sigma^2)/m$ , and a nonnegative mass at  $m$ . However, since  $\Pr(\tilde{d} > m) + \Pr(\tilde{d} < m) = 1$ , then the only possible distribution is a two-point distribution.

Now let us verify that we can construct a nonnegative distribution for any triplet  $(m, \sigma, s)$  satisfying

the conditions in Proposition 2.1. Consider the two-point distribution:

$$\tilde{d} = \begin{cases} m - \sigma \sqrt{\frac{1-s}{1+s}}, & \text{w.p. } \frac{1+s}{2}, \\ m + \sigma \sqrt{\frac{1+s}{1-s}}, & \text{w.p. } \frac{1-s}{2}. \end{cases}$$

These support points are nonnegative in the range (2.1). In fact,  $s$  can be arbitrarily close but never equal to one since, by definition,

$$s = \frac{\mathbb{E} \left( (\tilde{d} - m)^+ \cdot (\tilde{d} - m)^+ \right) - \mathbb{E} \left( (m - \tilde{d})^+ \cdot (m - \tilde{d})^+ \right)}{\sigma^2} = 1 - \frac{2}{\sigma^2} \mathbb{E} \left( (m - \tilde{d})^+ \cdot (m - \tilde{d})^+ \right).$$

Since we assume that  $\sigma > 0$ , then there exists  $\tilde{d} < m$  with nonzero probability.  $\square$

## A.2 Proof of Theorem 2.1

Consider the primal problem

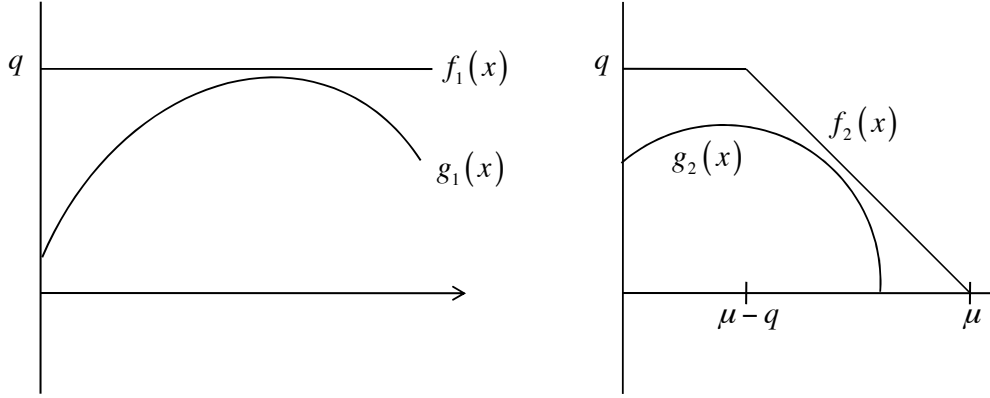
$$\begin{aligned} \inf_f \quad & \mathbb{E}_f \left( \min \{ \tilde{d}, q \} \right) \\ \text{s.t.} \quad & \mathbb{E}_f \left( (\tilde{d} - m)^+ \right) - \mathbb{E} \left( (m - \tilde{d})^+ \right) = 0, \\ & \mathbb{E}_f \left( (\tilde{d} - m)^{+2} \right) = \frac{(1+s)}{2} \sigma^2, \\ & \mathbb{E}_f \left( (m - \tilde{d})^{+2} \right) = \frac{(1-s)}{2} \sigma^2, \\ & \mathbb{E}_f(1) = 1, \quad f(d) \geq 0, \quad \forall d \geq 0. \end{aligned}$$

The constraints are equivalent to having  $\tilde{d}$  be a nonnegative random variable with moments  $\mathbb{E}(\tilde{d}) = m$ ,  $\text{var}(\tilde{d}) = \sigma^2$ , and  $\text{sv}(\tilde{d}) = s\sigma^2$ . Define  $m_1 = \frac{(1+s)}{2}\sigma^2$ ,  $m_2 = \frac{(1-s)}{2}\sigma^2$ . Proposition 2.1 is equivalent to the following:  $\frac{m_2}{m_1}(m_1 + m_2) \leq m^2$  and  $m_1, m_2 > 0$ . Then we can write the dual as

$$\begin{aligned} \sup_{t,r,y_1,y_2} \quad & t + m_1 y_1 + m_2 y_2 \\ \text{s.t.} \quad & t + r(x - m) + y_1(x - m)^2 \leq \min\{x, q\}, \quad \forall x \geq m, \\ & t - r(m - x) + y_2(m - x)^2 \leq \min\{x, q\}, \quad \forall 0 \leq x \leq m. \end{aligned}$$

By a simple transformation, the dual problem can be rewritten as

$$\begin{aligned} \sup_{t,r,y_1,y_2} \quad & t + m_1 y_1 + m_2 y_2 \\ \text{s.t.} \quad & t + r x + y_1 x^2 \leq \min\{x + m, q\}, \quad \forall x \geq 0, \\ & t - r x + y_2 x^2 \leq \min\{m - x, q\}, \quad \forall 0 \leq x \leq m. \end{aligned}$$



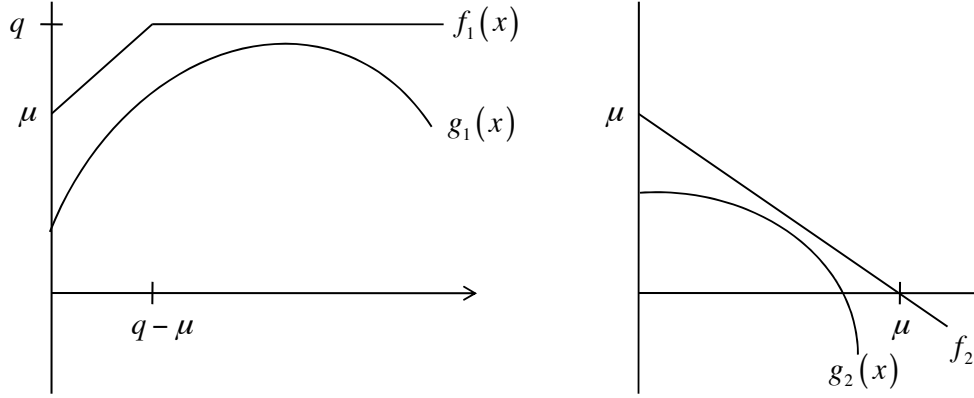
**Figure 12:** Graphical illustration of functions satisfying feasibility conditions of the dual problem under the case when  $q \leq m$ .

Throughout this section, we will refer to this last formulation as the dual problem. Given a set of variables  $t, r, y_1, y_2, m, q$ , we define the following quadratic functions:  $g_1(x) = t + rx + y_1x^2$ ,  $g_2(x) = t - rx + y_2x^2$ , and the following piecewise linear functions:  $f_1(x) = \min\{x + m, q\}$ ,  $f_2(x) = \min\{m - x, q\}$ . Checking dual feasibility of a set of parameters  $t, r, y_1, y_2$  is equivalent to checking if  $g_1(x) \leq f_1(x)$  for  $x \geq 0$ , and  $g_2(x) \leq f_2(x)$  for  $0 \leq x \leq m$ . Our strategy in finding a closed-form solution for finding an optimal solution to the primal problem is by constructing primal and dual feasible solutions which achieve the same objective cost. By weak duality, the two solutions are primal and dual optimal respectively.

**Case 1:  $q \leq m$ .** Under this case, we find that  $f_1(x) = q$  for all  $x \geq 0$ . Figure 12 illustrates a dual feasible solution. Table 5 shows the primal and dual optimal solutions under three subcases, which we discuss briefly.

**Case 1(a):  $0 \leq q \leq \frac{m}{2}$ .** By Proposition 2.1, the support points of the primal distribution are well-defined and the probabilities are valid. Moreover, the range of  $\pi$  is nonempty. This range guarantees that the nonzero support points are at least  $m$ . Then we can verify that this is a primal feasible distribution. Note that  $g_1(x) = f_1(x)$ , for all  $x \geq 0$ , while  $g_2(x)$  is concave and intersects  $f_2(x)$  at points 0 and  $m$ . Moreover,  $g_2(x) \leq q$  for all  $x$ . If  $q \leq \frac{m}{2}$ , then  $g_2'(m) \geq -1$  and  $g_2(x) \leq m - x$  for all  $0 \leq x \leq m$ . Thus, it is dual feasible.

**Case 1(b):  $\frac{m}{2} \leq q \leq m - \frac{m_2}{2} \sqrt{\frac{m_1 + m_2}{m_1 m_2}}$ .** Note that for the range of  $q$  under this case, the support points are well-defined and the probabilities are valid. Moreover, the range of  $\pi$  is nonempty and guarantees that the last two support points are at least  $m$ . We can then verify that the distribution is primal feasible. Note that  $g_1(x) = f_1(x)$  for all  $x \geq 0$ , while  $g_2(x) \leq f_2(x)$  for all  $x$ . Thus, the dual solution is always feasible.



**Figure 13:** Graphical illustration of functions satisfying feasibility conditions of the dual problem under the case when  $q \geq m$ .

**Case 1(c):**  $m - \frac{m_2}{2} \sqrt{\frac{m_1+m_2}{m_1 m_2}} \leq q \leq m$ . Under Proposition 2.1, the support points are always non-negative. It is easy to check that it is a primal feasible distribution. Moreover,  $g_1(x)$  intersects  $q$  and  $g_2(x)$  intersects  $m - x$  at exactly one nonnegative point each. Note that the dual solution is feasible if  $g'_1(0) \geq 0$ , or  $q \geq m - \frac{m_2}{2} \sqrt{\frac{m_1+m_2}{m_1 m_2}}$ .

**Case 2:**  $q \geq m$ . Under this case, it is not difficult to verify that  $f_2(x) = m - x$  for all  $x \geq 0$ . Figure 13 shows the functions corresponding to a dual feasible solution. Table 6 shows the primal and dual optimal solutions under three subcases.

**Case 2(a):**  $m \leq q \leq m + \frac{m_1}{2} \sqrt{\frac{m_1+m_2}{m_1 m_2}}$ . Note that the primal and dual optimal solutions are exactly the same as that of Case 1(c). We can verify that  $g_1(x)$  intersects  $q$  and  $g_2(x)$  intersects  $m - x$  at exactly one nonnegative point each. Dual feasibility is equivalent to the condition that  $g'_2(0) \geq -1$ , or  $q \leq m + \frac{m_1}{2} \sqrt{\frac{m_1+m_2}{m_1 m_2}}$ .

**Case 2(b):**  $m + \frac{m_1}{2} \sqrt{\frac{m_1+m_2}{m_1 m_2}} \leq q \leq m + \frac{m m_1}{2 m_2}$ . Unlike the previous cases, it is not obvious that the range of  $\pi$  is nonempty. To prove that the range is valid, consider the following random variable

$$\tilde{x} = \begin{cases} \frac{2(q-m)}{4(q-m)^2 - m_1} \left( m_1 - \sqrt{4m_2(q-m)^2 - m_1(m_1+m_2)} \sqrt{\frac{1-\hat{\pi}}{\hat{\pi}}} \right), & \text{w.p. } \hat{\pi}, \\ \frac{2(q-m)}{4(q-m)^2 + m_1} \left( m_1 - \sqrt{4m_2(q-m)^2 - m_1(m_1+m_2)} \sqrt{\frac{\hat{\pi}}{1-\hat{\pi}}} \right), & \text{w.p. } 1 - \hat{\pi}, \end{cases}$$

for any  $\hat{\pi} \in (0, 1)$ . Note that under the range of  $q$ ,  $E(\tilde{x}^2) \leq mE(\tilde{x})$ , or equivalently,  $\text{var}(\tilde{x}) \leq E(\tilde{x})(m - E(\tilde{x}))$ . Taking the square of both sides and adding  $\text{var}(\tilde{x})(m - E(\tilde{x}))^2$ , we find that

$$\text{var}(\tilde{x}) \left( \text{var}(\tilde{x}) + (m - E(\tilde{x}))^2 \right) \leq (m - E(\tilde{x}))^2 \left( \text{var}(\tilde{x}) + E(\tilde{x})^2 \right).$$

Substituting the values of the mean and variance of  $\tilde{x}$ , we find that this is equivalent to saying that the range of  $\pi$  is nonempty. The range of values for  $\pi$  ensures that the first two support points of  $\tilde{d}$

<p><b>Case 1(a): <math>0 \leq q \leq \frac{m}{2}</math></b></p> <p>PRIMAL OPTIMAL DISTRIBUTION:</p> $\tilde{d} = \begin{cases} 0, & \text{w.p. } \frac{m_2}{m^2}, \\ \frac{m}{m^2 - m_2} \left( m^2 - \sqrt{m_1(m^2 - m_2) - m_2^2} \sqrt{\frac{1-\pi}{\pi}} \right), & \text{w.p. } \pi \left( 1 - \frac{m_2}{m^2} \right), \\ \frac{m}{m^2 - m_2} \left( m^2 + \sqrt{m_1(m^2 - m_2) - m_2^2} \sqrt{\frac{\pi}{1-\pi}} \right), & \text{w.p. } (1 - \pi) \left( 1 - \frac{m_2}{m^2} \right), \end{cases}$ <p>for any <math>\pi \in \left[ 1 - \frac{m_2^2}{m_1(m^2 - m_2)}, 1 \right)</math></p> <p>DUAL OPTIMAL SOLUTION: <math>t = q, r = 0, y_1 = 0, y_2 = -\frac{q}{m^2}</math></p> <p>PRIMAL AND DUAL OPTIMAL COST: <math>q - \frac{m_2}{m^2}q</math></p>
<p><b>Case 1(b): <math>\frac{m}{2} \leq q \leq m - \frac{m_2}{2} \sqrt{\frac{m_1 + m_2}{m_1 m_2}}</math></b></p> <p>PRIMAL OPTIMAL DISTRIBUTION:</p> $\tilde{d} = \begin{cases} 2q - m, & \text{w.p. } \frac{m_2}{4(m-q)^2}, \\ m + \frac{2(m-q)}{4(m-q)^2 - m_2} \left( m_2 - \sqrt{4m_1(m-q)^2 - m_2(m_1 + m_2)} \sqrt{\frac{1-\pi}{\pi}} \right), & \text{w.p. } \pi \left( 1 - \frac{m_2}{4(m-q)^2} \right), \\ m + \frac{2(m-q)}{4(m-q)^2 - m_2} \left( m_2 + \sqrt{4m_1(m-q)^2 - m_2(m_1 + m_2)} \sqrt{\frac{\pi}{1-\pi}} \right), & \text{w.p. } (1 - \pi) \left( 1 - \frac{m_2}{4(m-q)^2} \right), \end{cases}$ <p>for any <math>\pi \in \left[ 1 - \frac{m_2^2}{m_1(4(m-q)^2 - m_2)}, 1 \right)</math></p> <p>DUAL OPTIMAL SOLUTION: <math>t = q, r = 0, y_1 = 0, y_2 = \frac{-1}{4(m-q)}</math></p> <p>PRIMAL AND DUAL OPTIMAL COST: <math>q - \frac{m_2}{4(m-q)}</math></p>
<p><b>Case 1(c): <math>m - \frac{m_2}{2} \sqrt{\frac{m_1 + m_2}{m_1 m_2}} \leq q \leq m</math></b></p> <p>PRIMAL OPTIMAL DISTRIBUTION:</p> $\tilde{d} = \begin{cases} m - m_2 \sqrt{\frac{m_1 + m_2}{m_1 m_2}}, & \text{w.p. } \frac{m_1}{m_1 + m_2}, \\ m + m_1 \sqrt{\frac{m_1 + m_2}{m_1 m_2}}, & \text{w.p. } \frac{m_2}{m_2 + m_2}, \end{cases}$ <p>DUAL OPTIMAL SOLUTION:</p> $t = \frac{qm_2 + mm_1}{m_1 + m_2} - \frac{1}{2} \sqrt{\frac{m_1 m_2}{m_1 + m_2}},$ $r = \frac{2}{m_1} \sqrt{\frac{m_1 m_2}{m_1 + m_2}} \left( \frac{m_1(q-m)}{m_1 + m_2} + \frac{1}{2} \sqrt{\frac{m_1 m_2}{m_1 + m_2}} \right),$ $y_1 = \frac{-m_2}{m_1(m_1 + m_2)} \left( \frac{m_1(q-m)}{m_1 + m_2} + \frac{1}{2} \sqrt{\frac{m_1 m_2}{m_1 + m_2}} \right),$ $y_2 = \frac{m_1}{m_2(m_1 + m_2)} \left( \frac{m_2(q-m)}{m_1 + m_2} - \frac{1}{2} \sqrt{\frac{m_1 m_2}{m_1 + m_2}} \right),$ <p>PRIMAL AND DUAL OPTIMAL COST: <math>\frac{qm_2 + mm_1}{m_1 + m_2} - \sqrt{\frac{m_1 m_2}{m_1 + m_2}}</math></p>

**Table 5:** Primal and dual optimal solutions when  $q \leq m$

are nonnegative and no more than  $m$ . We can then verify that the distribution is primal feasible. Note that  $g_1(x) \leq f_1(x)$  for all  $x$ , while  $g_2(x) = f_2(x)$  for all  $x \geq 0$ . Thus the solution is dual feasible.

**Case 2(c):  $q \geq m + \frac{mm_1}{2m_2}$ .** Define  $a = \frac{m_1}{2(q-m)^2} - \frac{m_2}{m(q-m)}$ , and  $b = 1 - \frac{m_2}{m^2}$ . Note that  $a \leq 0$  for the range of  $q$  in Case 2(c). By Proposition 2.1,  $b \in (0, 1)$  and  $2a + b \geq 0$  for all  $q$ . All the probabilities take values within the range of 0 to 1 by Proposition 2.1. It can be verified that the distribution is primal feasible. Note that  $g_1(x) \leq f_1(x)$  for all  $x$  and they intersect at exactly two points. Moreover,  $g_2(x)$  is convex and intersects  $f_2(x)$  at some negative value and at  $m$ . Thus, the dual solution is feasible.

If we multiply the primal optimal cost by  $p$  and subtract  $cq$  in all cases, then we get  $\underline{\Pi}^{\text{MVS}}(q)$ . Combining all the cases and letting  $m_1 = (1+s)\sigma^2/2$  and  $m_2 = (1-s)\sigma^2/2$ , we get the closed-form expression.  $\square$

**Case 2(a):**  $m \leq q \leq m + \frac{m_1}{2} \sqrt{\frac{m_1+m_2}{m_1m_2}}$

PRIMAL OPTIMAL DISTRIBUTION:

$$\tilde{d} = \begin{cases} m - m_2 \sqrt{\frac{m_1+m_2}{m_1m_2}}, & \text{w.p. } \frac{m_1}{m_1+m_2}, \\ m + m_1 \sqrt{\frac{m_1+m_2}{m_1m_2}}, & \text{w.p. } \frac{m_2}{m_2+m_2}, \end{cases}$$

DUAL OPTIMAL SOLUTION:

$$\begin{aligned} t &= \frac{qm_2+mm_1}{m_1+m_2} - \frac{1}{2} \sqrt{\frac{m_1m_2}{m_1+m_2}}, \\ r &= \frac{2}{m_1} \sqrt{\frac{m_1m_2}{m_1+m_2}} \left( \frac{m_1(q-m)}{m_1+m_2} + \frac{1}{2} \sqrt{\frac{m_1m_2}{m_1+m_2}} \right), \\ y_1 &= \frac{-m_2}{m_1(m_1+m_2)} \left( \frac{m_1(q-m)}{m_1+m_2} + \frac{1}{2} \sqrt{\frac{m_1m_2}{m_1+m_2}} \right), \\ y_2 &= \frac{m_1}{m_2(m_1+m_2)} \left( \frac{m_2(q-m)}{m_1+m_2} - \frac{1}{2} \sqrt{\frac{m_1m_2}{m_1+m_2}} \right), \end{aligned}$$

PRIMAL AND DUAL OPTIMAL COST:  $\frac{qm_2+mm_1}{m_1+m_2} - \sqrt{\frac{m_1m_2}{m_1+m_2}}$

**Case 2(b):**  $m + \frac{m_1}{2} \sqrt{\frac{m_1+m_2}{m_1m_2}} \leq q \leq m + \frac{mm_1}{2m_2}$

PRIMAL OPTIMAL DISTRIBUTION:

$$\tilde{d} = \begin{cases} m - \frac{2(q-m)}{4(q-m)^2-m_1} \left( m_1 + \sqrt{4m_2(q-m)^2 - m_1(m_1+m_2)} \right) \sqrt{\frac{\pi}{1-\pi}}, & \text{w.p. } (1-\pi) \left( 1 - \frac{m_1}{4(q-m)^2} \right), \\ m - \frac{2(q-m)}{4(q-m)^2-m_1} \left( m_1 - \sqrt{4m_2(q-m)^2 - m_1(m_1+m_2)} \right) \sqrt{\frac{1-\pi}{\pi}}, & \text{w.p. } \pi \left( 1 - \frac{m_1}{4(q-m)^2} \right), \\ 2q - m, & \text{w.p. } \frac{m_1}{4(m-q)^2}, \end{cases}$$

for any  $\pi \in \left[ 1 - \frac{m_2^2}{m_1(4(m-q)^2-m_2)}, 1 \right)$

DUAL OPTIMAL SOLUTION:  $t = m, r = 1, y_1 = \frac{-1}{4(q-m)}, y_2 = 0$

PRIMAL AND DUAL OPTIMAL COST:  $m - \frac{m_1}{4(q-m)}$

**Case 2(c):**  $q \geq m + \frac{mm_1}{2m_2}$ . Define  $a = \frac{m_1}{2(q-m)^2} - \frac{m_2}{m(q-m)}$ , and  $b = 1 - \frac{m_2}{m^2}$ .

PRIMAL OPTIMAL DISTRIBUTION:

$$\tilde{d} = \begin{cases} 0, & \text{w.p. } 1-b, \\ m + (q-m) \left( 1 - \sqrt{\frac{2a+b}{b}} \right), & \text{w.p. } \frac{1}{2} \left( b + \left( b - \frac{(1-b)m}{(q-m)} \right) \sqrt{\frac{b}{2a+b}} \right), \\ m + (q-m) \left( 1 + \sqrt{\frac{2a+b}{b}} \right), & \text{w.p. } \frac{1}{2} \left( b - \left( b - \frac{(1-b)m}{(q-m)} \right) \sqrt{\frac{b}{2a+b}} \right). \end{cases}$$

DUAL OPTIMAL SOLUTION:

$$\begin{aligned} t &= \frac{1}{2} \left( q + m - \frac{(q-m)(a+b)}{\sqrt{b(2a+b)}} \right), \\ r &= \frac{1}{2} \left( 1 + \sqrt{\frac{b}{2a+b}} \right), \\ y_1 &= \frac{-1}{4(q-m)} \sqrt{\frac{b}{2a+b}}, \\ y_2 &= \frac{q-m}{2m^2} \left( \frac{(a+b)}{\sqrt{b(2a+b)}} - 1 \right) + \frac{1}{2m} \left( \sqrt{\frac{b}{2a+b}} - 1 \right), \end{aligned}$$

PRIMAL AND DUAL OPTIMAL COST:  $\frac{1}{2} \left( m + bq - (q-m) \sqrt{b(2a+b)} \right)$

**Table 6:** Primal and dual optimal solutions when  $q \geq m$

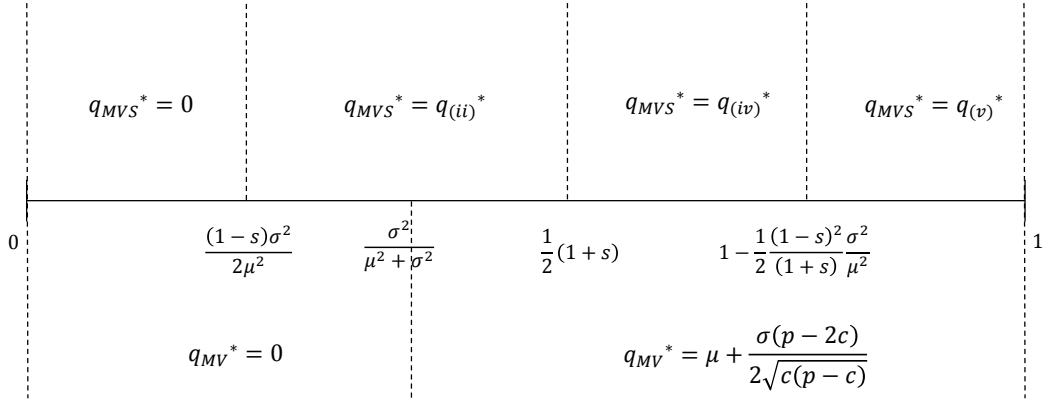
### A.3 Proof of Theorem 2.2

The worst-case profit  $\underline{\Pi}^{\text{MVS}}(q)$  is concave and continuous in  $q$ . Note that unless the profit is maximized by a range of  $q$ , the optimal quantity can only occur at 0 or in Regions (ii), (iv) or (v) (as defined in Theorem 2.1). Let  $q_{\text{MVS}}^*$  be an order quantity that maximizes  $\underline{\Pi}^{\text{MVS}}(q)$ . Define

$$\begin{aligned} q_{(ii)}^* &= m - \frac{\sigma}{2} \sqrt{\frac{(1-s)p}{2(p-c)}}, \\ q_{(iv)}^* &= m + \frac{\sigma}{2} \sqrt{\frac{(1+s)p}{2c}}, \\ q_{(v)}^* &= \frac{m}{b} + \frac{(pb-2c)}{2b} \sqrt{\frac{(1+s)\sigma^2 b - 2(1-b)^2 m^2}{2c(pb-c)}}, \end{aligned}$$

which are the unconstrained maximizers of the aggregate functions in Regions (ii), (iv) and (v), respectively. Suppose  $\frac{c}{p} > 1 - \frac{(1-s)\sigma^2}{2m^2}$ . Then  $\underline{\Pi}^{\text{MVS}}(q)$  is strictly decreasing in Region (i). This implies that the function is strictly decreasing for  $q \geq 0$ . Thus,  $q_{\text{MVS}}^* = 0$ . Note that  $q_{(ii)}^*$  lies in Region (ii) if  $\frac{1}{2}(1-s) \leq \frac{c}{p} \leq 1 - \frac{(1-s)\sigma^2}{2m^2}$ . Under this case,  $q_{\text{MVS}}^* = q_{(ii)}^*$ . Similarly,  $q_{(iv)}^*$  lies in Region (iv) if  $\frac{1}{2} \frac{(1-s)^2 \sigma^2}{(1+s)m^2} \leq \frac{c}{p} \leq \frac{1}{2}(1-s)$ . If this is true, then  $q_{\text{MVS}}^* = q_{(iv)}^*$ . Finally, if  $\frac{c}{p} < \frac{1}{2} \frac{(1-s)^2 \sigma^2}{(1+s)m^2}$ , then  $\underline{\Pi}^{\text{MVS}}(q)$  is increasing in Regions (i) to (iv). Therefore, the maximum is attained in Region (v). Thus,  $q_{\text{MVS}}^* = q_{(v)}^*$ .  $\square$

An illustration of the regions of profit margins  $\frac{p-c}{p}$  for the mean-variance and mean-variance-semivariance distributionally robust newsvendors is shown in Figure 14.



**Figure 14:** Relationship between optimal solutions to the distributionally robust problems (mean-variance and mean-variance-semivariance) under different ranges of the profit margin  $\frac{p-c}{p}$ .

### A.4 Proof of Theorem 3.1

Suppose that the random demand  $\tilde{d}$  has a support set  $\mathbb{R}_+$ . Further, we can partition the support set in the following manner. Let  $\{B_i\}_{i=0}^t$ , where  $B_i = [b_i, b_{i+1})$  for  $i \leq t-1$  and  $B_t = [b_t, \infty)$ , be

non-overlapping intervals whose union is  $\mathbb{R}_+$ . Note that

$$\tilde{d} = \sum_{i=0}^t \tilde{d} \cdot 1_{[\tilde{d} \in B_i]}.$$

Note that, given the  $t$ -partitioned information (3.1), we can rewrite the moment problem as

$$\inf_{f \in \mathcal{F}} E_f \left( p \min \{ \tilde{d}, q \} - cq \right) = pm - cq - \sup_{f \in \mathcal{F}} E_f \left( p \left( \tilde{d} - q \right)^+ \right),$$

where  $\mathcal{F}$  is the set of candidate demand distributions satisfying the known partitioned statistics. Hence, for a fixed  $q$ , the worst-case distribution in moment problem (3.2) can be found by solving

$$\begin{aligned} & \sup_{\lambda} \int_0^{\infty} p(x - q)^+ d\lambda(x) \\ & \text{s.t.} \quad \int_0^{\infty} d\lambda(x) = 1, \\ & \quad \int_0^{\infty} x d\lambda(x) = m, \\ & \quad \int_0^{\infty} x^2 1_{[x \in B_i]} d\lambda(x) = \sigma_i^2, \quad \forall i = 0, \dots, t, \\ & \quad \lambda(x) \geq 0, \quad \forall x \in \mathbb{R}_+. \end{aligned} \tag{A.4}$$

The dual of the above moment problem (A.4) is an upper bound to the optimal value of (A.4). The dual is the following semi-infinite linear program:

$$\begin{aligned} & \inf_{t, r, \mathbf{y}} \quad t + mr + \sum_{i=0}^t \sigma_i^2 y_i \\ & \text{s.t.} \quad t + xr + x^2 y_i \geq 0, \quad \forall x \in B_i, \quad \forall i = 0, \dots, t, \\ & \quad t + xr + x^2 y_i \geq p(x - q), \quad \forall x \in B_i, \quad \forall i = 0, \dots, t. \end{aligned} \tag{A.5}$$

Note that the constraints of (A.5) is equivalent to  $t + xr + x^2 y_i \geq p(x - q)^+$  for all  $x \in B_i$ . We can write the dual problem (A.5) into a second-order cone programming problem by invoking S-lemma (Pólik and Terlaky, 2007), which we state next.

**Proposition A.1** (S-lemma, Pólik and Terlaky, 2007). *Consider two quadratic functions of  $\mathbf{z} \in \mathbb{R}^N$ ,  $q_i(\mathbf{z}) = \mathbf{z}' \mathbf{B}_i \mathbf{z} + 2\mathbf{b}'_i \mathbf{z} + c_i$ ,  $i = 0, 1$ , with  $q_1(\bar{\mathbf{z}}) > 0$  for some  $\bar{\mathbf{z}}$ . Then*

$$q_0(\mathbf{z}) \geq 0 \quad \forall \mathbf{z} \in \{ \mathbf{z} : q_1(\mathbf{z}) \geq 0 \}$$

*if and only if there exists  $\tau \geq 0$  such that*

$$\begin{pmatrix} c_0 & \mathbf{b}'_0 \\ \mathbf{b}_0 & \mathbf{B}_0 \end{pmatrix} - \tau \begin{pmatrix} c_1 & \mathbf{b}'_1 \\ \mathbf{b}_1 & \mathbf{B}_1 \end{pmatrix} \in \mathcal{S}_+^{N+1},$$

where  $\mathcal{S}_+^{N+1}$  denotes the set of all positive-semidefinite matrices of size  $(N+1) \times (N+1)$ .

Note that intervals  $B_i = [b_i, b_{i+1})$  for  $i = 0, \dots, t-1$  are bounded. Hence,

$$B_i = \{x \in \mathbb{R} : (x - b_i)(x - b_{i+1}) \leq 0\}, \quad \forall i = 0, \dots, t-1.$$

By invoking S-lemma, we can write the constraints

$$\begin{aligned} t + xr + x^2y_i &\geq 0, \quad \forall x \in B_i, \quad i = 0, \dots, t-1, \\ t + xr + x^2y_i &\geq p(x - q), \quad \forall x \in B_i, \quad i = 0, \dots, t-1 \end{aligned}$$

as the positive semidefinite constraints

$$\begin{aligned} \begin{pmatrix} t & \frac{1}{2}r \\ \frac{1}{2}r & y_i \end{pmatrix} - \tau_i \begin{pmatrix} -b_i b_{i+1} & \frac{1}{2}(b_i + b_{i+1}) \\ \frac{1}{2}(b_i + b_{i+1}) & -1 \end{pmatrix} &\succeq 0, \quad i = 0, \dots, t-1, \\ \begin{pmatrix} t + pq & \frac{1}{2}(r - p) \\ \frac{1}{2}(r - p) & y_i \end{pmatrix} - v_i \begin{pmatrix} -b_i b_{i+1} & \frac{1}{2}(b_i + b_{i+1}) \\ \frac{1}{2}(b_i + b_{i+1}) & -1 \end{pmatrix} &\succeq 0, \quad i = 0, \dots, t-1, \end{aligned}$$

for some  $\tau_i \geq 0$ ,  $v_i \geq 0$  for  $i = 0, \dots, t-1$ . In fact, for each  $i \leq t-1$ , we can rewrite the pair of positive semidefinite constraints into the following form:

$$t + \tau_i b_i b_{i+1} \geq 0, \quad (\text{A.6})$$

$$(t + \tau_i b_i b_{i+1})(y_i + \tau_i) \geq \frac{1}{4}(r - \tau_i(b_i + b_{i+1}))^2, \quad (\text{A.7})$$

$$t + pq + v_i b_i b_{i+1} \geq 0, \quad (\text{A.8})$$

$$(t + pq + v_i b_i b_{i+1})(y_i + v_i) \geq \frac{1}{4}(r - p - v_i(b_i + b_{i+1}))^2. \quad (\text{A.9})$$

Inequalities (A.7) and (A.9) means that the determinant of the matrix must be nonnegative. Note that these two constraints includes a hyperbolic term. We can further cast constraints (A.6)–(A.9) into second-order constraints (Lobo et al., 1998):

$$t + \tau_i(b_i b_{i+1} + 1) + y_i \geq \sqrt{(t + \tau_i(b_i b_{i+1} - 1) - y_i)^2 + (r - \tau_i(b_i + b_{i+1}))^2}, \quad (\text{A.10})$$

$$t + pq + v_i(b_i b_{i+1} + 1) + y_i \geq \sqrt{(t + pq + v_i(b_i b_{i+1} - 1) - y_i)^2 + (r - p - v_i(b_i + b_{i+1}))^2}, \quad (\text{A.11})$$

for some  $\tau_i \geq 0$ ,  $v_i \geq 0$ ,  $i = 0, \dots, t-1$ . Thus, we have shown that for  $i = 0, \dots, t-1$ , the constraints of the dual problem (A.5) are equivalent to second-order cone constraints.

For the case when  $i = t$ , note that the interval  $B_t = [b_t, \infty)$  is unbounded above. However, we can still invoke S-lemma by recognizing that  $B_t = \{x : x \geq b_t\}$ . Hence, following a similar logic to above,

we can formulate the constraints of (A.5) for  $i = t$  into the following second-order cone constraints:

$$t + \tau_t b_t + y_t \geq \sqrt{(t + \tau_t b_t - y_t)^2 + (r - \tau_t)^2}, \quad (\text{A.12})$$

$$t + pq + v_t b_t + y_t \geq \sqrt{(t + pq + v_t b_t - y_t)^2 + (r - p - v_t)^2}, \quad (\text{A.13})$$

for  $\tau_t \geq 0, v_t \geq 0$ . Hence, we can formulate the dual problem (A.5) as a second-order cone program by replacing the constraints by the second-order cone inequalities (A.10)–(A.13) and the nonnegativity constraints  $\tau \geq 0, v \geq 0$ .  $\square$

## A.5 SOCP best-case bound with $t$ partitions

Suppose that  $\mathcal{F}$  is the set of all distributions satisfying the known  $t$ -partitioned statistics (3.1). For a given  $q$ , the best-case expected profit

$$\sup_{f \in \mathcal{F}} E_f \left( p \min \{ \tilde{d}, q \} - cq \right) = pm - cq - \inf_{f \in \mathcal{F}} E_f \left( p (\tilde{d} - q)^+ \right).$$

Hence, to find the best-case expected profit for a given  $q$ , we are need to solve the following moment problem

$$\begin{aligned} & \inf_{\lambda} \int_0^{\infty} p(x - q)^+ d\lambda(x) \\ \text{s.t.} \quad & \int_0^{\infty} d\lambda(x) = 1, \\ & \int_0^{\infty} x d\lambda(x) = m, \\ & \int_0^{\infty} x^2 1_{[x \in B_i]} d\lambda(x) = \sigma_i^2, \quad \forall i = 0, \dots, t, \\ & \lambda(x) \geq 0, \quad \forall x \in \mathbb{R}_+. \end{aligned} \quad (\text{A.14})$$

Assume that  $q$  lies strictly in the interior of one of the intervals  $B_{k(q)}$  (If  $q$  is on one of the endpoints, the proof can be trivially modified). Define the new intervals  $\hat{B}_i(q) = B_i$  for  $i = 0, \dots, k(q) - 1$  and  $\hat{B}_{k(q)}(q) = [b_{k(q)}, q]$ . We also define  $\bar{B}_i(q) = B_i$  for  $i = k(q) + 1, \dots, m$  and  $\bar{B}_{k(q)}(q) = [q, b_{k(q)+1}]$ . Hence, the dual of the moment problem (A.14) is

$$\begin{aligned} & \sup_{t, r, \mathbf{y}} \quad t + mr + \sum_{i=0}^t \sigma_i^2 y_i \\ \text{s.t.} \quad & t + xr + x^2 y_i \leq p(x - q)^+, \quad \forall x \in B_i, \quad \forall i = 0, \dots, t, \end{aligned}$$



$$\begin{aligned} & \inf_{y_0, \mathbf{y}, \mathbf{Y}} y_0 + \mathbf{y}^\top \mathbf{m} + \mathbf{Y} \cdot \mathbf{Q} \\ \text{s. t. } & y_0 + \mathbf{x}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{Y} \mathbf{x} \geq \max_{A \in 2^{[n]}} \mathbf{p}_A^\top (\mathbf{x} - \mathbf{q}), \quad \forall \mathbf{x}. \end{aligned} \tag{A.16}$$

$$\begin{aligned} & = \inf_{y_0, \mathbf{y}, \mathbf{Y}} y_0 + \mathbf{y}^\top \mathbf{m} + \mathbf{Y} \cdot \mathbf{Q} \\ \text{s.t. } & y_0 + \mathbf{x}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{Y} \mathbf{x} \geq \mathbf{p}_A^\top (\mathbf{x} - \mathbf{q}), \quad \forall \mathbf{x}, \forall A \in 2^{[n]}. \end{aligned} \tag{A.17}$$

Note that for any  $A \in 2^{[n]}$ ,

$$\begin{aligned} & y_0 + \mathbf{x}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{Y} \mathbf{x} \geq \mathbf{p}_A^\top (\mathbf{x} - \mathbf{q}), \quad \forall \mathbf{x}, \\ \Leftrightarrow & \begin{pmatrix} 1 & \mathbf{x}^\top \end{pmatrix} \begin{pmatrix} y_0 + \mathbf{p}_A^\top \mathbf{q} & \frac{1}{2} (\mathbf{y} - \mathbf{p}_A)^\top \\ \frac{1}{2} (\mathbf{y} - \mathbf{p}_A) & \mathbf{Y} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \geq 0, \quad \forall \mathbf{x}, \\ \Leftrightarrow & \begin{pmatrix} y_0 + \mathbf{p}_A^\top \mathbf{q} & \frac{1}{2} (\mathbf{y} - \mathbf{p}_A)^\top \\ \frac{1}{2} (\mathbf{y} - \mathbf{p}_A) & \mathbf{Y} \end{pmatrix} \succeq 0. \end{aligned}$$

Hence, the infinite quadratic inequality constraints in (A.17) is equivalent to  $2^{[n]}$  semidefinite inequality constraints. Thus, the distributionally robust problem (1.1) with mean and covariance information is equivalent to the semidefinite program in Lemma 4.1.  $\square$