Supplementary Appendix A: Proofs

Proof of Proposition 1: First, consider the following optimization problem:

$$\begin{aligned} &\max_{0\leq\lambda\leq\mu_1\leq\frac{r\alpha}{c_1},0\leq\lambda\leq\mu_2\leq\frac{r\alpha}{c_2}} r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda = \alpha - \beta\left(w_1(\mu_1,\lambda) + w_2(\mu_2,\lambda)\right) \end{aligned} \tag{A.1}$$

Since the constraint set is compact and the objective function is continuous, by Weierstass Theorem, there is a maximum to (A.1). Let's denote the optimal solution to (A.1) as $\lambda^*, \mu_1^*, \mu_2^*$, then the corresponding optimal objective value $\pi^* = r\lambda^* - c_1\mu_1^* - c_2\mu_2^*$.

Lemma A.1. *If $\pi^* > 0$, then $(\lambda^*, \mu_1^*, \mu_2^*)$ must be the optimal solution to the optimization problem (1); if $\pi^* \leq 0$, then there is no solution to (1) that yields positive profit.*

Proof of Lemma A.1: If $\pi^* > 0$, then we must have $\lambda^* > 0$. Thus, $(\lambda^*, \mu_1^*, \mu_2^*)$ is also a feasible solution to (1). Suppose the optimal solution to (1) is $(\lambda^b, \mu_1^b, \mu_2^b) \neq (\lambda^*, \mu_1^*, \mu_2^*)$. Then, it means that $\pi^b = r\lambda^b - c_1\mu_1^b - c_2\mu_2^b > \pi^* > 0$. Then, it is easy to check that $(\lambda^b, \mu_1^b, \mu_2^b)$ is a feasible solution to (A.1) as well. But it leads to a higher objective value π^b for (A.1), which contradicts to the fact that $(\lambda^*, \mu_1^*, \mu_2^*)$ is the optimal solution to (A.1). Thus, we must have $(\lambda^b, \mu_1^b, \mu_2^b) = (\lambda^*, \mu_1^*, \mu_2^*)$.

If $\pi^* \leq 0$, suppose there exists a solution $(\lambda^b, \mu_1^b, \mu_2^b)$ to (1) such that $\pi^b = r\lambda^b - c_1\mu_1^b - c_2\mu_2^b > 0$. Then, it is easy to check that $(\lambda^b, \mu_1^b, \mu_2^b)$ should also be a feasible solution to (A.1), which leads to a positive objective value π^b for (A.1). This contradicts to the fact that $\pi^* \leq 0$. Thus, there is no solution to (1) that yields positive profit. □

By Lemma A.1, finding the optimal solution to (1) that yields positive profit is equivalent to finding the optimal solution to (A.1) that yields positive profit.

The Lagrangian of (A.1) is defined as follows:

$$L(\lambda, \mu_1, \mu_2, \rho) = r\lambda - c_1\mu_1 - c_2\mu_2 + \rho\left[\lambda - \alpha + \beta\left(\frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}\right)\right]$$

where $\rho \in \mathbb{R}$ is the Lagrange multiplier. To find the critical points of $L(\lambda, \mu_1, \mu_2, \rho)$, we solve the following equation set:

$$\begin{aligned} &\tfrac{\partial L}{\partial \lambda} = r + \rho + \rho\beta\tfrac{1}{(\mu_1-\lambda)^2} + \rho\beta\tfrac{1}{(\mu_2-\lambda)^2} = 0 \\ &\tfrac{\partial L}{\partial \mu_i} = -c_i - \rho\beta\tfrac{1}{(\mu_i-\lambda)^2} = 0, i = 1, 2 \\ &\lambda - \alpha + \beta\left(\tfrac{1}{\mu_1-\lambda} + \tfrac{1}{\mu_2-\lambda}\right) = 0 \\ &0 \leq \lambda \leq \mu_1 \leq \tfrac{r\alpha}{c_1}, 0 \leq \lambda \leq \mu_2 \leq \tfrac{r\alpha}{c_2} \end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.1) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the boundary conditions that $0 \leq \lambda$ and $\mu_i \leq \frac{r\alpha}{c_i}$, and solve the equation set above, which gives us a unique solution:

$$\begin{aligned} &\lambda^* = \alpha - \beta\sqrt{\tfrac{c_1}{\beta(r-c_1-c_2)}} - \beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}} \\ &\mu_1^* = \lambda^* + \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_1}} \\ &\mu_2^* = \lambda^* + \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}} \end{aligned} \tag{A.2}$$

So, if $\lambda^* \geq 0$ and $\mu_i^* \leq \frac{r\alpha}{c_i}$, $i = 1, 2$, then $(\lambda^*, \mu_1^*, \mu_2^*)$ must be the optimal solution to (A.1); otherwise, the optimal solution to (A.1) must be corner solution (i.e., at least one of the follow holds, $\lambda^* = 0$ and $\mu_i^* = \frac{r\alpha}{c_i}$, $i = 1, 2$), in which case the optimal value is clearly nonpositive. Note, with $(\lambda^*, \mu_1^*, \mu_2^*)$ defined in (A.2), we have $\pi^* = r\lambda^* - c_1\mu_1^* - c_2\mu_2^* = (r - c_1 - c_2)\alpha - 2\sqrt{\beta(r - c_1 - c_2)c_1} - 2\sqrt{\beta(r - c_1 - c_2)c_2}$, which is positive if $\alpha > \frac{2\sqrt{\beta}(\sqrt{c_1}+\sqrt{c_2})}{\sqrt{r-c_1-c_2}}$. It is easy to check that if $\alpha > \frac{2\sqrt{\beta}(\sqrt{c_1}+\sqrt{c_2})}{\sqrt{r-c_1-c_2}}$, we have $\lambda^* \geq 0$ and $\mu_i^* \leq \frac{r\alpha}{c_i}$, $i = 1, 2$. This implies that there is an optimal solution $(\lambda^*, \mu_1^*, \mu_2^*)$ that yields positive profit for (A.1) if and only if $\alpha > \bar{\alpha} = \frac{2\sqrt{\beta}(\sqrt{c_1}+\sqrt{c_2})}{\sqrt{r-c_1-c_2}}$. Then, by Lemma A.1, we can conclude that same result holds for the optimization problem (1). □

Proof of Proposition 2: Suppose we set the two capacity levels at μ_1^b, μ_2^b (i.e., the optimal solution in the base model). Then, the demand rates are

$$\begin{aligned}
\lambda_o^\star &= \left[\alpha - \xi\beta w_2(\mu_2^b, (1-\theta)\lambda_s^\star + \theta\lambda_o^\star)\right]^+ \\
\lambda_s^\star &= \left[\alpha - \beta\left(w_1(\mu_1^b, (1-\theta)\lambda_s^\star) + w_2(\mu_2^b, (1-\theta)\lambda_s^\star + \theta\lambda_o^\star)\right)\right]^+
\end{aligned}$$

It is easy to check $(1-\theta)\lambda_s^\star + \theta\lambda_o^\star > \lambda^b$. This implies that the profit with μ_1^b, μ_2^b and online self-order technology is greater than π^b. Therefore, under optimal solution, there must be customers coming to store after the implementation of online self-order technology, i.e., at least one of λ_s^o and λ_o^o is positive. Note, if $\lambda_s^o > 0$, we must have $\lambda_o^o > 0$. Thus, we must have $\lambda_o^o > 0$ in the optimal solution. So the optimization problem (2) is equivalent to the following problem

$$\begin{aligned}
\max_{0\leq\lambda_o, (1-\theta)\lambda_s\leq\mu_1, (1-\theta)\lambda_s+\theta\lambda_o\leq\mu_2} \quad & r\left((1-\theta)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\
s.t. \quad \lambda_o &= \alpha - \xi\beta w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o) \\
\lambda_s &= \left[\alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s) + w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)\right]^+
\end{aligned} \tag{A.3}$$

Then, the firm has two choices:

i. Shut down stage 1 (i.e., $\mu_1 = 0$) and sell only to online customers. In this case, the optimal solution is obtained by solving the following optimization problem:

$$\begin{aligned}
\max_{0\leq\lambda_o, \theta\lambda_o\leq\mu_2\leq\frac{r\alpha}{c_2}} \quad & r\theta\lambda_o - c_2\mu_2 \\
s.t. \quad & \lambda_o = \alpha - \xi\beta w_2(\mu_2, \theta\lambda_o)
\end{aligned} \tag{A.4}$$

The Lagrangian of (A.4) is defined as follows:

$$L\left(\lambda_o, \mu_2, \rho\right) = r\theta\lambda_o - c_2\mu_2 + \rho\left[\lambda_o - \alpha + \xi\beta\frac{1}{\mu_2 - \theta\lambda_o}\right]$$

where $\rho \in \mathbb{R}$ is the Lagrange multiplier. To find the critical points of $L\left(\lambda_o, \mu_2, \rho\right)$, we solve the following equation set:

$$\begin{aligned}
&\tfrac{\partial L}{\partial\lambda_o} = r\theta + \rho + \rho\xi\beta\theta\tfrac{1}{(\mu_2-\theta\lambda_o)^2} = 0 \\
&\tfrac{\partial L}{\partial\mu_2} = -c_2 - \rho\xi\beta\tfrac{1}{(\mu_2-\theta\lambda_o)^2} = 0 \\
&\lambda_o - \alpha + \xi\beta\tfrac{1}{\mu_2-\theta\lambda_o} = 0 \\
&0 \leq \lambda_o, \theta\lambda_o \leq \mu_2 \leq \tfrac{r\alpha}{c_2}
\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.4) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the conditions that $0 \le \lambda_o$ and $\mu_2 \le \frac{r\alpha}{c_2}$, and solve the equation set above, which gives us a unique solution:

$$\begin{aligned} \lambda_o^* &= \alpha - \xi\beta\sqrt{\tfrac{c_2}{\theta\xi\beta(r-c_2)}} \\ \mu_2^* &= \lambda_o^* + \sqrt{\tfrac{\theta\xi\beta(r-c_2)}{c_2}} \end{aligned} \tag{A.5}$$

So, if $\lambda^* \ge 0$ and $\mu_2^* \le \frac{r\alpha}{c_2}$, then (λ_o^*, μ_2^*) together with $\lambda_1^* = 0, \mu_1^* = 0$ must be the optimal solution to (A.4); otherwise, the optimal solution to (A.4) must be corner solution (i.e., at least one of the follow holds, $\lambda_o^* = 0$ and $\mu_2^* = \frac{r\alpha}{c_2}$), in which case the optimal value is clearly nonpositive, and thus the firm won't choose this option at all.

ii. Sell to both types of customers. In this case, the optimal solution is obtained by solving the following optimization problem:

$$\begin{aligned} &\max_{0\le\lambda_o, 0\le(1-\theta)\lambda_s\le\mu_1\le\frac{r\alpha}{c_1}, (1-\theta)\lambda_s+\theta\lambda_o\le\mu_2\le\frac{r\alpha}{c_2}} r\left((1-\theta)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda_o = \alpha - \xi\beta w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o) \\ &\qquad \lambda_s = \alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s) + w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right) \end{aligned} \tag{A.6}$$

The Lagrangian of (A.6) is defined as follows:

$$\begin{aligned} \tilde{L}\left(\lambda_s, \lambda_o, \mu_1, \mu_2, \rho_1, \rho_2\right) =& r\left((1-\theta)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\ &+ \rho_1\left[\lambda_o - \alpha + \xi\beta\frac{1}{\mu_2 - (1-\theta)\lambda_s - \theta\lambda_o}\right] \\ &+ \rho_2\left[\lambda_s - \alpha + \beta\frac{1}{\mu_1 - (1-\theta)\lambda_s} + \beta\frac{1}{\mu_2 - (1-\theta)\lambda_s - \theta\lambda_o}\right] \end{aligned}$$

where $\rho_1, \rho_2 \in \mathbb{R}$ are the Lagrange multipliers. To find the critical points of $\tilde{L}\left(\lambda_s, \lambda_o, \mu_1, \mu_2, \rho_1, \rho_2\right)$, we solve the following equation set:

$$\begin{aligned} &\tfrac{\partial\tilde{L}}{\partial\lambda_s} = r(1-\theta) + \rho_2 + \rho_1\xi\beta(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} + \rho_2\beta(1-\theta)\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^2} \\ &\qquad + \rho_2\beta(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} = 0 \\ &\tfrac{\partial\tilde{L}}{\partial\lambda_o} = r\theta + \rho_1 + \rho_1\xi\beta\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} + \rho_2\beta\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} = 0 \\ &\tfrac{\partial\tilde{L}}{\partial\mu_1} = -c_1 - \rho_2\beta\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^2} = 0 \\ &\tfrac{\partial\tilde{L}}{\partial\mu_2} = -c_2 - \rho_1\xi\beta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} - \rho_2\beta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} = 0 \\ &\lambda_o - \alpha + \xi\beta\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o} = 0 \\ &\lambda_s - \alpha + \beta\tfrac{1}{\mu_1-(1-\theta)\lambda_s} + \beta\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o} = 0 \\ &0 \le \lambda_o, 0 \le (1-\theta)\lambda_s \le \mu_1 \le \tfrac{r\alpha}{c_1}, (1-\theta)\lambda_s + \theta\lambda_o \le \mu_2 \le \tfrac{r\alpha}{c_2} \end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.6) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the boundary conditions (i.e., $0 \le \lambda_o$, $0 \le \lambda_s$ and $\mu_i \le \frac{r\alpha}{c_i}$ $i = 1, 2$), and solve the equation set above, which gives us a unique solution:

$$\begin{aligned}
\tilde{\lambda}_s^* &= \alpha - \beta\sqrt{\tfrac{c_1}{(1-\theta)\beta(r-c_1-c_2)}} - \beta\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}\\
\tilde{\lambda}_o^* &= \alpha - \xi\beta\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}\\
\tilde{\mu}_1^* &= (1-\theta)\tilde{\lambda}_s^o + \sqrt{\tfrac{(1-\theta)\beta(r-c_1-c_2)}{c_1}}\\
\tilde{\mu}_2^* &= (1-\theta)\tilde{\lambda}_s^o + \theta\tilde{\lambda}_o^o + \sqrt{\tfrac{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}{c_2}}
\end{aligned}$$

So, if $(\tilde{\lambda}_o^*, \tilde{\lambda}_s^*, \tilde{\mu}_1^*, \tilde{\mu}_2^*)$ is an interior solution, then it must be the optimal solution to (A.6); otherwise, the solution to (A.6) must be corner solution, in which case we can check the optimal value is either nonpositive or strictly less than that under option (i), and thus the firm never chooses this option.

Therefore, the optimal solution to (A.3) (and thus to (2)) must be chosen from the interior solutions under the two options, whichever gives higher optimal value. For option (i), under the interior solution $(\lambda_o^*, \lambda_s^*, \mu_1^*, \mu_2^*)$, the optimal value is $\pi^* = r((1-\theta)\lambda_s^* + \theta\lambda_o^*) - c_1\mu_1^* - c_2\mu_2^* = (r-c_2)\theta\alpha - 2\sqrt{\theta\xi\beta(r-c_2)c_2}$; for option (ii), under the interior solution $(\tilde{\lambda}_o^*, \tilde{\lambda}_s^*, \tilde{\mu}_1^*, \tilde{\mu}_2^*)$, the optimal value is $\tilde{\pi}^* = r((1-\theta)\tilde{\lambda}_s^* + \theta\tilde{\lambda}_o^*) - c_1\tilde{\mu}_1^* - c_2\tilde{\mu}_2^* = (r-c_2-c_1(1-\theta))\alpha - 2\sqrt{(1-\theta)\beta(r-c_1-c_2)c_1} - 2\sqrt{[(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)]c_2}$. Thus, $\pi^* < \tilde{\pi}^*$ if and only if $\alpha > \bar{\alpha}'' = \frac{2\sqrt{\beta c_1}}{\sqrt{(1-\theta)(r-c_1-c_2)}} + \frac{2\sqrt{[(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)]c_2}}{(r-c_1-c_2)(1-\theta)} - \frac{2\sqrt{\theta\xi\beta(r-c_2)c_2}}{(r-c_1-c_2)(1-\theta)}$. Then, we conclude the proof by setting $\bar{\alpha}' = \max(\bar{\alpha}, \bar{\alpha}'')$. $\square$

Proof of Proposition 3: From Propositions 1 and 2, we get $w_1^b = \sqrt{\frac{c_1}{\beta(r-c_1-c_2)}}$, $w_2^b = \sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}$, $w_1^o = \sqrt{\frac{c_1}{(1-\theta)\beta(r-c_1-c_2)}}$, and $w_2^o = \sqrt{\frac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}$. Since $\theta > 0$, we have $w_1^o > w_1^b$. Also, note that $\frac{\partial(w_2^o - w_2^b)}{\partial r} > 0$ and $\lim_{r\to c_1+c_2}\left(w_2^o - w_2^b\right) < 0$. Thus, given c_1 and c_2, $\exists \bar{r} > c_1 + c_2$ such that $w_2^o < w_2^b$ if and only if $\frac{c_1+c_2}{r} > \frac{c_1+c_2}{\bar{r}} = m_w$.

$\square$

Proof of Proposition 4: $\lambda_o^o = \alpha - \sqrt{\frac{\xi^2\beta^2 c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}} \geq \alpha - \sqrt{\frac{\beta c_2}{(1-\theta)(r-c_1-c_2)+\theta(r-c_2)}} > \alpha - \sqrt{\frac{\beta c_2}{r-c_1-c_2}} > \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{r-c_1-c_2}} = \lambda^b$ where the first inequality is because of $\xi \leq 1$.

Next, let's prove the second bullet point in Proposition 4. Note,

$$\left(\lambda_s^o - \lambda_s^b\right)\frac{\sqrt{r-c_1-c_2}}{\beta} = \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{(1-\theta)\beta}} - \sqrt{\frac{c_2}{(1-\theta)\beta + \theta\xi\beta\frac{r-c_2}{r-c_1-c_2}}}$$

which is decreasing in r for $r \geq c_1 + c_2$. When $r \to c_1 + c_2$, $\left(\lambda_s^o - \lambda_s^b\right)\frac{\sqrt{r-c_1-c_2}}{\beta} \to \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{(1-\theta)\beta}} > 0$ if $\theta < \psi_s = 1 - \left(\frac{\sqrt{c_1}}{\sqrt{c_1}+\sqrt{c_2}}\right)^2$. Then, we can conclude the result.

By Proposition 3, $\lambda_s^o > \lambda^b$ must happen when $w_2^o < w_2^b$, and thus we have $m_\lambda > m_w$.

Finally, let's prove the third point in Proposition 4. Note $(1-\theta)\lambda_s^o + \theta\lambda_o^o = \alpha - \sqrt{\frac{(1-\theta)\beta c_1}{r-c_1-c_2}} - \frac{\sqrt{c_2}\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\sqrt{r-\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}}$. Also, $\frac{\partial\sqrt{(1-\theta)\beta c_1}}{\partial\theta} < 0$, $\frac{\partial\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta} \leq 0$, and $\frac{\partial\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta} < 0$. Thus,

$(1-\theta)\,\lambda_s^o + \theta\lambda_o^o$ is increasing in θ. Note when $\theta = 0$, we have $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o = \lambda^b$. Thus, $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o > \lambda^b$ for all $\theta > 0$. $\square$

Proof of Proposition 5: Note $\mu_1^o = \sqrt{1-\theta}\left(\sqrt{1-\theta}\alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{(r-c_1-c_2)+\theta/(1-\theta)\xi(r-c_2)}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}\right)$, which is decreasing in θ. Since when $\theta = 0$ we have $\mu_1^o = \mu_1^b$, we can conclude $\mu_1^o < \mu_1^b$ for any $\theta > 0$.

Next, let's prove part (ii). Note that

$$\frac{\mu_2^o - \mu_2^b}{\sqrt{\beta\,(r-c_1-c_2)}} = \frac{\sqrt{c_1} - \sqrt{(1-\theta)\,c_1}}{r-c_1-c_2} + \frac{\sqrt{c_2} - (1-\theta+\theta\xi)\sqrt{\frac{c_2}{1-\theta+\theta\xi\frac{r-c_2}{r-c_1-c_2}}}}{r-c_1-c_2} + \sqrt{\frac{1-\theta+\theta\xi\frac{r-c_2}{r-c_1-c_2}}{c_2}} - \sqrt{\frac{1}{c_2}} \tag{A.7}$$

It is easy to find the 1st and 3rd terms in (A.7) are decreasing in r. Also, since $\xi \leq 1$, we have $\sqrt{c_2} - (1-\theta+\theta\xi)\sqrt{\frac{c_2}{1-\theta+\theta\xi\frac{r-c_2}{r-c_1-c_2}}} > 0$. Thus, we find the 2nd term in (A.7) is also decreasing in r. In sum, we find $\frac{\mu_2^o-\mu_2^b}{\sqrt{\beta(r-c_1-c_2)}}$ is decreasing in r. Note, when $-\beta\,(r-c_1-c_2) + \xi\beta\,(r-c_2) = 0$ (i.e., $r = \frac{c_1+c_2-\xi c_2}{1-\xi} > c_1+c_2$). then $\mu_2^o - \mu_2^b = \left(1-\sqrt{1-\theta}\right)\sqrt{\frac{\beta c_1}{r-c_1-c_2}} + (1-\xi)\,\theta\sqrt{\frac{\beta c_2}{r-c_1-c_2}} > 0$. Thus, there exists $\bar{r} > \frac{c_1+c_2-\xi c_2}{1-\xi}$ such that $\mu_2^o > \mu_2^b$ if and only if $r < \bar{r}$. Then, we can define $m_\mu = \frac{c_1+c_2}{\bar{r}}$ and conclude the result. Because $(1-\theta)\lambda_s^o + \theta\lambda_o^o > \lambda^b$, $w_2^o < w_2^b$ implies $\mu_2^o > \mu_2^b$. Thus, we must have $m_\mu < m_w$. $\square$

Proof of Proposition 6:

$$\begin{aligned}
k_1^o + k_2^o - k_1^b - k_2^b = {} & -\theta\alpha\tau_1 + \sqrt{\tfrac{\beta c_1}{r-c_1-c_2}}\,(\tau_1+\tau_2)\left(1-\sqrt{1-\theta}\right) \\
& - (\tau_1+\tau_2)\left[\left((1-\theta)\,\beta + \theta\xi\beta\right)\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}} - \beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}}\right] \\
& - \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_1}}\tau_1\left(1-\sqrt{1-\theta}\right) + \tau_2\left(\sqrt{\tfrac{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}{c_2}} - \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}}\right) \\
& + \tau_1\theta\xi\beta\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}
\end{aligned} \tag{A.8}$$

Let's first show the 3rd term in (A.8), i.e., $-(\tau_1+\tau_2)\left[\left((1-\theta)\,\beta+\theta\xi\beta\right)\sqrt{\frac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}} - \beta\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}\right]$ (denoted as $f_3(r)$) is decreasing in r:

$$\frac{\partial f_3}{\partial r} = -(\tau_1+\tau_2)\left[-\frac{\sqrt{c_2}}{2}\frac{\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\left(r-c_2-\frac{(1-\theta)\beta c_1}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{3}{2}}} + \frac{\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}}\right]$$

Since $\frac{\partial[(1-\theta)\beta+\theta\xi\beta]}{\partial\theta} \leq 0$ and $\frac{\partial\left[r-c_2-\frac{(1-\theta)\beta c_1}{(1-\theta)\beta+\theta\xi\beta}\right]}{\partial\theta} > 0$, we can find that $\frac{\partial f_3}{\partial r}$ is decreasing in θ. Note when $\theta = 0$, $\frac{\partial f_3}{\partial r} = 0$. Thus, $\frac{\partial f_3}{\partial r} < 0$ for all $\theta > 0$.

Next, let's show the 5th term in (A.8), i.e., $\tau_2\left(\sqrt{\frac{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}{c_2}} - \sqrt{\frac{\beta(r-c_1-c_2)}{c_2}}\right)$

(denoted as $f_5(r)$) is decreasing in r: Note, $\frac{\partial\sqrt{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}{\partial r} = \frac{\sqrt{(1-\theta)\beta+\theta\xi\beta}}{2\sqrt{r-\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}}$. Since $\frac{\partial\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta} \leq 0$ and $\frac{\partial\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta} < 0$, we find $\frac{\partial\sqrt{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}{\partial r}$ is decreasing in θ, and thus $\frac{\partial f_5}{\partial r}$ is decreasing in θ. Note when $\theta = 0$, we have $\frac{\partial f_5}{\partial r} = 0$. Thus, $\frac{\partial f_5}{\partial r} < 0$ for all $\theta > 0$.

Therefore, we can conclude that $\frac{\partial(k_1^o+k_2^o-k_1^b-k_2^b)}{\partial r} < 0$. Note, if $r \to c_1 + c_2$, we have $(k_1^o + k_2^o - k_1^b - k_2^b)\sqrt{r-c_1-c_2} \to (\tau_1+\tau_2)\sqrt{\beta c_2} + \sqrt{\beta c_1}(\tau_1+\tau_2)(1-\sqrt{(1-\theta)}) > 0$, which implies $k_1^o + k_2^o - k_1^b - k_2^b > 0$ if r is very close to $c_1 + c_2$. Thus, there exists $\bar{r} > c_1 + c_2$ such that $k_1^o + k_2^o - k_1^b - k_2^b > 0$ if and only if $r < \bar{r}$. Then, we can define $m_k = \frac{c_1+c_2}{\bar{r}}$. Since to have $k_1^o + k_2^o - k_1^b - k_2^b > 0$, we must have $\mu_2^o > \mu_2^b$, this implies $m_k > m_\mu$. □

Proof of Proposition 7: The retailer has three options to choose from:

1. Serve only tech savvy customers (i.e., by setting $\mu_{1h} = 0$), and the optimization problem is

$$\begin{aligned}&\max_{\substack{0\leq\eta\lambda_m\leq\mu_{1m}\leq\frac{r\alpha}{c_{1m}}\\0\leq\eta\lambda_m\leq\mu_2\leq\frac{r\alpha}{c_2}}} r\eta\lambda_m - c_{1m}\mu_{1m} - c_2\mu_2\\&s.t. \lambda_m = \alpha - \beta\frac{1}{\mu_{1m}-\eta\lambda_m} - \beta\frac{1}{\mu_2-\eta\lambda_m}\end{aligned}$$

   if we ignore the boundary conditions, we can derive the unique optimal solution as

$$\begin{aligned}\tilde{\lambda}_m^* &= \alpha - \beta\sqrt{\frac{c_{1m}}{\eta\beta(r-c_{1m}-c_2)}} - \beta\sqrt{\frac{c_2}{\eta\beta(r-c_{1m}-c_2)}}\\\tilde{\mu}_{1m}^* &= \eta\tilde{\lambda}_m^* + \sqrt{\frac{\eta\beta(r-c_{1m}-c_2)}{c_{1m}}}\\\tilde{\mu}_2^* &= \eta\tilde{\lambda}_m^* + \sqrt{\frac{\eta\beta(r-c_{1m}-c_2)}{c_2}}\end{aligned} \tag{A.9}$$

   and the corresponding optimal value is $\tilde{\pi}^* = (r-c_{1m}-c_2)\eta\alpha - 2\sqrt{\eta\beta(r-c_{1m}-c_2)c_{1m}} - 2\sqrt{\eta\beta(r-c_{1m}-c_2)c_2}$. So, if $(\tilde{\lambda}_m^*, \tilde{\mu}_{1m}^*, \tilde{\mu}_2^*)$ is an interior solution, then it must be the optimal solution to (A.9); otherwise, the solution to (A.9) must be corner solution, in which case we can check the optimal value is clearly nonpositive, and thus the firm never chooses this option.

2. Serve only traditional customers (i.e., by setting $\mu_{1m} = 0$), and the optimization problem is

$$\begin{aligned}&\max_{\substack{0\leq(1-\eta)\lambda_h\leq\mu_{1h}\leq\frac{r\alpha}{c_1}\\0\leq(1-\eta)\lambda_h\leq\mu_2\leq\frac{r\alpha}{c_2}}} r(1-\eta)\lambda_h - c_{1h}\mu_{1h} - c_2\mu_2\\&s.t. \lambda_h = \alpha - \beta\frac{1}{\mu_{1h}-(1-\eta)\lambda_h} - \beta\frac{1}{\mu_2-(1-\eta)\lambda_h}\end{aligned} \tag{A.10}$$

   if we ignore the boundary conditions, we can derive the unique optimal solution as

$$\begin{aligned}\hat{\lambda}_h^* &= \alpha - \beta\sqrt{\frac{c_1}{(1-\eta)\beta(r-c_1-c_2)}} - \beta\sqrt{\frac{c_2}{(1-\eta)\beta(r-c_1-c_2)}}\\\hat{\mu}_1^* &= (1-\eta)\hat{\lambda}_m^* + \sqrt{\frac{(1-\eta)\beta(r-c_1-c_2)}{c_1}}\\\hat{\mu}_2^* &= (1-\eta)\hat{\lambda}_m^* + \sqrt{\frac{(1-\eta)\beta(r-c_1-c_2)}{c_2}}\end{aligned}$$

   and the corresponding optimal value is $\hat{\pi}^* = (r-c_1-c_2)(1-\eta)\alpha - 2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_1} - 2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_2}$. So, if $(\hat{\lambda}_h^*, \hat{\mu}_1^*, \hat{\mu}_2^*)$ is an interior solution, then it must be the optimal solution to (A.10); otherwise, the solution to (A.10) must be corner solution, in which case we can check the optimal value is clearly nonpositive, and thus the firm never chooses this option.

3. Serve both types of customers. In this case, the optimal solution is obtained by solving the following optimization problem:

$$\begin{aligned} \max_{\substack{0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}},\\ 0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1},\\ \eta\lambda_m+(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}}} \quad & r\left(\eta\lambda_m+(1-\eta)\lambda_h\right)-c_{1m}\mu_{1m}-c_1\mu_{1h}-c_2\mu_2 \\ s.t. \quad \lambda_m &= \alpha-\beta w_{1m}\left(\mu_{1m},\eta\lambda_m\right)-\beta w_2\left(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h\right) \\ \lambda_h &= \alpha-\beta w_{1h}\left(\mu_{1h},(1-\eta)\lambda_h\right)-\beta w_2\left(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h\right) \end{aligned} \tag{A.11}$$

Since the constraint set is compact and the objective function is continuous, by Weierstrass Theorem, there is a maximum to (A.11).

The Lagrangian of (A.11) is defined as follows:

$$\begin{aligned} L(\lambda_m,\lambda_h,\mu_{1m},\mu_{1h},\mu_2,\rho_m,\rho_h) = {} & r\left(\eta\lambda_m+(1-\eta)\lambda_h\right)-c_{1m}\mu_{1m}-c_1\mu_{1h}-c_2\mu_2 \\ & -\rho_m\left(\lambda_m-\alpha+\beta\frac{1}{\mu_{1m}-\eta\lambda_m}+\beta\frac{1}{\mu_2-\eta\lambda_m-(1-\eta)\lambda_h}\right) \\ & -\rho_h\left(\lambda_h-\alpha+\beta\frac{1}{\mu_{1h}-(1-\eta)\lambda_h}+\beta\frac{1}{\mu_2-\eta\lambda_m-(1-\eta)\lambda_h}\right) \end{aligned}$$

where $\rho_m,\rho_h\in\mathbb{R}$ are the Lagrange multiplier. To find the critical points of $L(\lambda_m,\lambda_h,\mu_{1m},\mu_{1h},\mu_2,\rho_m,\rho_h)$, we solve the following equation set:

$$\begin{aligned} &\frac{\partial L}{\partial\lambda_m}=r\eta-\rho_m-\frac{\rho_m\beta\eta}{(\mu_{1m}-\eta\lambda_m)^2}-\frac{(\rho_m+\rho_h)\beta\eta}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0 \\ &\frac{\partial L}{\partial\lambda_h}=r(1-\eta)-\rho_h-\frac{\rho_h\beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^2}-\frac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0 \\ &\frac{\partial L}{\partial\mu_{1m}}=-c_{1m}+\frac{\rho_m\beta}{(\mu_{1m}-\eta\lambda_m)^2}=0 \\ &\frac{\partial L}{\partial\mu_{1h}}=-c_1+\frac{\rho_h\beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^2}=0 \\ &\frac{\partial L}{\partial\mu_2}=-c_2+\frac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0 \\ &0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}},\ 0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1},\ \eta\lambda_m+(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2} \end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.11) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the boundary conditions, and solve the equation set above, which gives us a unique solution:

$$\begin{aligned} \lambda_m^* &= \alpha-\beta\sqrt{\frac{c_{1m}}{\beta\eta(r-c_{1m}-c_2)}}-\beta\sqrt{\frac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}} \\ \lambda_h^* &= \alpha-\beta\sqrt{\frac{c_1}{\beta(1-\eta)(r-c_1-c_2)}}-\beta\sqrt{\frac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}} \\ \mu_{1m}^* &= \eta\lambda_m^*+\sqrt{\frac{\beta\eta(r-c_{1m}-c_2)}{c_{1m}}} \\ \mu_{1h}^* &= (1-\eta)\lambda_h^*+\sqrt{\frac{\beta(1-\eta)(r-c_1-c_2)}{c_1}} \\ \mu_2^* &= \eta\lambda_m^*+(1-\eta)\lambda_h^*+\sqrt{\frac{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}{c_2}} \end{aligned}$$

Note it is easy to check that if $\lambda_m^*\ge 0$ and $\lambda_h^*\ge 0$, then $\mu_i\le\frac{r\alpha}{c_i}$. Thus, if $\lambda_m^*\ge 0$ and $\lambda_h^*\ge 0$ (or α is large), then $(\lambda_m^*,\lambda_h^*,\mu_{1m}^*,\mu_{1h}^*,\mu_2^*)$ is the optimal solution to (A.11), and the corresponding optimal value is $\pi^*=(r-c_{1m}-c_2)\eta\alpha+(r-c_1-c_2)(1-\eta)\alpha-2\sqrt{\eta\beta c_{1m}(r-c_{1m}-c_2)}-2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_1}-2\sqrt{\beta c_2[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}$.

So, if $(\lambda_m^*,\lambda_h^*,\mu_{1m}^*,\mu_{1h}^*,\mu_2^*)$ is an interior solution, then it must be the optimal solution to (A.11); otherwise, the solution to (A.11) must be corner solution, in which case we can check the optimal value is either nonpositive or strictly less than that under the first two options, and thus the firm never chooses this option.

Note $\pi^* - \tilde{\pi}^*$ and $\pi^* - \hat{\pi}^*$ are linearly increasing in α, and thus if and only if α is large enough, we have $\pi^* > \max(\tilde{\pi}^*, \hat{\pi}^*)$, and thus it is optimal for the firm to serve both types of customers.

$\square$

Proof of Lemma 1: From the optimal solution given in Proposition 7, we know $w_{1h}^s = \sqrt{\frac{c_1}{\beta(1-\eta)(r-c_1-c_2)}}$, $w_{1m}^s = \sqrt{\frac{c_{1m}}{\beta\eta(r-c_{1m}-c_2)}}$, and $w_2^s = \sqrt{\frac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}}$.

Then, $\eta w_{1m}^s + (1-\eta) w_{1h}^s = \sqrt{\frac{\eta c_{1m}}{\beta(r-c_{1m}-c_2)}} + \sqrt{\frac{(1-\eta)c_1}{\beta(r-c_1-c_2)}}$ which is increasing in c_{1m}. Also, when $c_{1m} = 0$, $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$. Thus, there exists $\bar{c}_{1m} > 0$ such that $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$ if and only if $c_{1m} < \bar{c}_{1m}$. $\square$

Proof of Proposition 8: Because $\eta < 1$, we can find $w_{1h}^s > w_1^b$. Since $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$, we must have $w_{1m}^s < w_1^b$. Also, since $c_{1m} < c_1$, we have $w_2^s < w_2^b$. $\square$

Proof of Proposition 9: First, because $w_{1m}^s < w_1^b$ and $w_2^s < w_2^b$, we find $\lambda_m^s = \alpha - \beta w_{1m}^s - \beta w_2^s > \alpha - \beta w_1^b - \beta w_2^b = \lambda^b$.

Second, note

$$\frac{\left(\lambda_h^s - \lambda^b\right)\sqrt{r-c_1-c_2}}{\beta} = \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{\beta\left(1-\eta\right)}} - \sqrt{\frac{c_2}{\beta\eta\frac{r-c_{1m}-c_2}{r-c_1-c_2} + \beta\left(1-\eta\right)}}$$

which is decreasing in r. Then, if $\lim\limits_{r\to c_1+c_2} \frac{\left(\lambda_h^s - \lambda^b\right)\sqrt{r-c_1-c_2}}{\beta} = \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{\beta(1-\eta)}} > 0$ (i.e., η is small enough), then there exists $\bar{r} > c_1 + c_2$ such that $\lambda_h^s > \lambda^b$ if and only if $r < \bar{r}$ (or $\frac{c_1+c_2}{r} < \frac{c_1+c_2}{\bar{r}} = m'_\lambda$).

Finally, note

$$\begin{aligned}
\eta\lambda_m^s + \left(1-\eta\right)\lambda_h^s =& \alpha - \sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\frac{\left(1-\eta\right)\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{\left(r-c_{1m}-c_2\right)\eta + \left(r-c_1-c_2\right)\left(1-\eta\right)}} \\
&> \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{\left(r-c_1-c_2\right)}} \\
&= \lambda^b
\end{aligned}$$

where the inequality is due to $c_{1m} < c_1$ and Assumption 1, which implies $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$ (by Lemma 1) or $\sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$). $\square$

Proof of Proposition 10: Note $\mu_{1h}^s = \sqrt{1-\eta}\left(\sqrt{1-\eta}\alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{(r-c_1-c_2)+\eta/(1-\eta)(r-c_{1m}-c_2)}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}\right)$, which is decreasing in η. Since when $\eta = 0$ we have $\mu_{1h}^s = \mu_1^b$, we can conclude

$\mu_{1h}^s < \mu_1^b$ for any $\eta > 0$. Also, note

$$\begin{aligned}\mu_{1m}^s + \mu_{1h}^s =& \alpha - \sqrt{\frac{\eta\beta c_{1m}}{r - c_{1m} - c_2}} - \sqrt{\frac{(1-\eta)\,\beta c_1}{r - c_1 - c_2}} \\ & - \sqrt{\frac{\beta c_2}{(r - c_{1m} - c_2)\,\eta + (r - c_1 - c_2)\,(1-\eta)}} \\ & + \sqrt{\frac{\beta\eta\,(r - c_{1m} - c_2)}{c_{1m}}} + \sqrt{\frac{\beta\,(1-\eta)\,(r - c_1 - c_2)}{c_1}}\end{aligned}$$

Because of Assumption 1 and Lemma 1, we have $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, and thus $\sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$. Because $c_{1m} < c_1$, we have $\sqrt{\frac{\beta c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}} < \sqrt{\frac{\beta c_2}{r-c_1-c_2}}$. Also, define function $f(\eta) = \sqrt{\frac{\beta\eta(r-c_{1m}-c_2)}{c_{1m}}} + \sqrt{\frac{\beta(1-\eta)(r-c_1-c_2)}{c_1}}$. We can easily check that f is a concave function. Note $f(0) = \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}$ and $f(1) > \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}$. This implies that $f(\eta) > \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}$ for any $\eta \in (0,1]$. Based on the results above, we can conclude that $\mu_{1m}^s + \mu_{1h}^s > \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{r-c_1-c_2}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}} = \mu_1^s$.

Second, note

$$\begin{aligned}\mu_2^s =& \alpha - \sqrt{\frac{\eta\beta c_{1m}}{r - c_{1m} - c_2}} - \sqrt{\frac{(1-\eta)\,\beta c_1}{r - c_1 - c_2}} \\ & - \sqrt{\frac{\beta c_2}{(r - c_{1m} - c_2)\,\eta + (r - c_1 - c_2)\,(1-\eta)}} + \sqrt{\frac{\beta\,[(r - c_{1m} - c_2)\,\eta + (r - c_1 - c_2)\,(1-\eta)]}{c_2}}\end{aligned}$$

Because of Assumption 1 and Lemma 1, we have $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, and thus $\sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$. Also, define function $g(\eta) = -\sqrt{\frac{\beta c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}} + \sqrt{\frac{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}{c_2}}$. We can easily check that g is an increasing function. Thus, for any $\eta \in (0,1]$, $g(\eta) > g(0) = \sqrt{\frac{\beta c_2}{r-c_1-c_2}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_2}}$. Based on the results above, we can conclude that $\mu_2^s > \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{r-c_1-c_2}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_2}} = \mu_2^s$. □

Proof of Proposition 11: Note

$$\frac{\partial k_1^s}{\partial r} = \frac{\tau_1\sqrt{\beta\,(1-\eta)\,c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\tau_1\,(1-\eta)\,\sqrt{\beta c_2}}{2[(r - c_{1m} - c_2)\,\eta + (r - c_1 - c_2)\,(1-\eta)]^{\frac{3}{2}}} + \frac{\tau_1\sqrt{\beta\,(1-\eta)}}{2\sqrt{c_1\,(r-c_1-c_2)}}$$

which is decreasing in η. Since $k_1^s = k_1^b$ when $\eta = 0$, we can conclude that $\frac{\partial k_1^s}{\partial r} < \frac{\partial k_1^b}{\partial r}$ as $\eta > 0$.

Note

$$\begin{aligned}\frac{\partial k_2^s}{\partial r} =& \frac{\sqrt{\eta\beta c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta\,(1-\eta)\,c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} \\ & + \frac{\sqrt{\beta c_2}}{2[(r - c_{1m} - c_2)\,\eta + (r - c_1 - c_2)\,(1-\eta)]^{\frac{3}{2}}} + \frac{\sqrt{\beta}}{2\sqrt{c_2\,[(r - c_{1m} - c_2)\,\eta + (r - c_1 - c_2)\,(1-\eta)]}}\end{aligned}$$

Because of $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$ (by Assumption 1 and Lemma 1) and $c_{1m} < c_1$, we have $\frac{\sqrt{\eta\beta c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} < \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}}$. Also, define function $f(\eta) = \frac{\sqrt{\beta c_2}}{2[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}} + \frac{\sqrt{\beta}}{2\sqrt{c_2[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}}$, which is a decreasing function. Thus, for any $\eta > 0$, $f(\eta) < f(0)$. Based on the results above, we can conclude that $\frac{\partial k_2^s}{\partial r} < \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta}}{2\sqrt{c_2(r-c_1-c_2)}} = \frac{\partial k_2^b}{\partial r}$.

Because $\frac{\partial k_2^s}{\partial r} < \frac{\partial k_2^b}{\partial r}$ and $\frac{\partial k_1^s}{\partial r} < \frac{\partial k_1^b}{\partial r}$, we can conclude that $\frac{\partial(k_1^s+k_2^s)}{\partial r} < \frac{\partial(k_1^b+k_2^b)}{\partial r}$. Note when $r \to c_1+c_2$, then $k_1^s+k_2^s-k_1^b-k_2^b > 0$. Thus, there exists $\bar{r} > c_1+c_2$ such that $k_1^s+k_2^s-k_1^b-k_2^b > 0$ if and only if $r < \bar{r}$. Then, define $m_k' = (c_1+c_2)/\bar{r}$ and we get the result. □

Proof of Proposition 12: Based on the optimal solutions, we have

- $\pi^b = r\lambda^b - c_1\mu_1^b - c_2\mu_2^b = (r-c_1-c_2)\alpha - 2\sqrt{\beta(r-c_1-c_2)c_1} - 2\sqrt{\beta(r-c_1-c_2)c_2}$
- $\pi^o = r((1-\theta)\lambda_1^o + \theta\lambda_2^o) - c_1\mu_1^o - c_2\mu_2^o = [r-c_2-c_1(1-\theta)]\alpha - 2\sqrt{(1-\theta)\beta(r-c_1-c_2)c_1} - 2\sqrt{[(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)]c_2}$
- $\pi^s = r\left((1-\eta)\lambda_h^s + \eta\lambda_m^s\right) - c_{1m}\mu_{1m}^s - c_1\mu_1^s - c_2\mu_2^s = (r-c_{1m}-c_2)\eta\alpha + (r-c_1-c_2)(1-\eta)\alpha - 2\sqrt{\eta\beta c_{1m}(r-c_{1m}-c_2)} - 2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_1} - 2\sqrt{\beta c_2[(r-c_{1m}-c_2)\eta + (r-c_1-c_2)(1-\eta)]}$

Let's first compare π^o and π^b.

Suppose we keep using μ_1^b and μ_2^b in the case of online self-order technology. Then, we want to show that total demand rate $\lambda > \lambda^b$. Suppose not, i.e., $\lambda \leq \lambda^b$. Then, we have $w_2 = 1/(\mu_2^b - \lambda) \leq w_2^b$. Because $\xi \leq 1$, we have $\lambda_o = \alpha - \xi\beta w_2 > \lambda_o^b$. This means that $\lambda_s < \lambda_s^b$ (otherwise we cannot have $\lambda = (1-\theta)\lambda_s + \theta\lambda_o \leq \lambda^b$). Then, $w_1 = 1/(\mu_1^b - (1-\theta)\lambda_s) < w_1^b$. This implies that $\lambda_s = \alpha - \beta w_1 - \beta w_2 > \lambda_s^b$. This contradicts the result $\lambda_s < \lambda_s^b$ that we just obtained. So we must have $\lambda > \lambda^b$. Then, with this feasible solution μ_1^b and μ_2^b with online self-order technology, the total profit is $r\lambda - c_1\mu_1^b - c_2\mu_2^b > \pi^b$. Therefore, the optimal profit $\pi^o > \pi^b$.

Next, we look at how $\pi^o - \pi^b$ changes as different model parameters change.

(1) $\frac{\partial(\pi^o-\pi^b)}{\partial r} = \sqrt{\frac{c_1\beta}{r-c_1-c_2}} + \sqrt{\frac{c_2\beta}{r-c_1-c_2}} - \sqrt{\frac{(1-\theta)c_1\beta}{r-c_1-c_2}} - \sqrt{\frac{(1-\theta+\theta\xi)c_2\beta}{r-\frac{1-\theta}{1-\theta+\theta\xi}c_1-c_2}} > 0$, because $\theta \leq 1$ and $\xi \leq 1$.

(2) $\frac{\partial(\pi^o-\pi^b)}{\partial c_2} = \mu_2^b - \mu_2^o$. Since $\frac{\partial\mu_2^o}{\partial\xi} > 0$, we have $\frac{\partial^2(\pi^o-\pi^b)}{\partial c_2\partial\xi} < 0$. Also, note that if $\xi = 1$, then $\mu_2^b < \mu_2^o$, and thus $\frac{\partial(\pi^o-\pi^b)}{\partial c_2} < 0$.

(3) Let's first prove the following lemma:

Lemma A.2. *π^o is increasing in θ.*

Proof of Lemma A.2: For $\theta_1, \theta_2 \in [0,1]$, suppose $\theta_1 < \theta_2$. Denote the optimal solution when $\theta = \theta_1$ as $\mu_1^\triangle$, $\mu_2^\triangle$, $\lambda_s^\triangle$, $\lambda_o^\triangle$. The corresponding total demand and profit are $\lambda^\triangle = (1-\theta_1)\lambda_s^\triangle + \theta_1\lambda_o^\triangle$ and $\pi^\triangle$. It is easy to check that $\lambda_s^\triangle < \lambda_o^\triangle$.

When $\theta=\theta_2$, suppose we keep using $\mu_1^{\triangle}$ and $\mu_2^{\triangle}$. Then, under this feasible solution, we want to show that total demand rate $\lambda>\lambda^{\triangle}$. Suppose not, i.e., $\lambda\le\lambda^{\triangle}$. Then, $w_2=\frac{1}{\mu_2^{\triangle}-\lambda}\le\frac{1}{\mu_2^{\triangle}-\lambda^{\triangle}}=w_2^{\triangle}$, which implies that $\lambda_o\ge\lambda_o^{\triangle}$. Since $\lambda\le\lambda^{\triangle}$, $\theta_1<\theta_2$ and $\lambda_o\ge\lambda_o^{\triangle}$, we can find $\lambda_s\le\lambda_s^{\triangle}$. Then, $w_1=\frac{1}{\mu_1^{\triangle}-(1-\theta_2)\lambda_s}<\frac{1}{\mu_1^{\triangle}-(1-\theta_1)\lambda_s^{\triangle}}=w_1^{\triangle}$. However, if $w_2<w_2^{\triangle}$ and $w_1\le w_1^{\triangle}$, then we must have $\lambda_s>\lambda_s^{\triangle}$, which contradicts to what we just found (i.e., $\lambda_s\le\lambda_s^{\triangle}$). Thus, we must have $\lambda>\lambda^{\triangle}$. This implies that when $\theta=\theta_2$, with the feasible solution $\mu_1^{\triangle}$ and $\mu_2^{\triangle}$, we have more demand and thus more profit. Therefore, the optimal profit when $\theta=\theta_2$, denoted as $\pi^{\triangle\triangle}$, must be greater than $\pi^{\triangle}$. This completes the proof. $\square$

According to Lemma A.2, we have $\frac{\partial\pi^o}{\partial\theta}>0$. Therefore, $\frac{\partial(\pi^o-\pi^s)}{\partial\theta}>0$.

(4) It is easy to find that $\frac{\partial\pi^o}{\partial\xi}<0$. Thus, $\frac{\partial(\pi^o-\pi^b)}{\partial\xi}<0$.

Nest, let's compare π^s and π^b.

Note

$$\begin{aligned}
\frac{\partial\left(\pi^s-\pi^b\right)}{\partial\eta} &= \left(c_1-c_{1m}\right)\alpha-\sqrt{\frac{\beta c_{1m}\left(r-c_{1m}-c_2\right)}{\eta}}+\sqrt{\frac{\beta c_1\left(r-c_1-c_2\right)}{1-\eta}}-\frac{\sqrt{\beta c_2}\left(c_1-c_{1m}\right)}{\sqrt{\left(r-c_{1m}-c_2\right)\eta+\left(r-c_1-c_2\right)\left(1-\eta\right)}}\\
&> \left(c_1-c_{1m}\right)\sqrt{\frac{\beta c_{1m}}{\eta\left(r-c_{1m}-c_2\right)}}-\sqrt{\frac{\beta c_{1m}\left(r-c_{1m}-c_2\right)}{\eta}}+\sqrt{\frac{\beta c_1\left(r-c_1-c_2\right)}{1-\eta}}\\
&= \sqrt{\frac{r-c_1-c_2}{r-c_{1m}-c_2}}\sqrt{c_1c_{1m}\beta}\left(\sqrt{\frac{r-c_{1m}-c_2}{\left(1-\eta\right)c_{1m}}}-\sqrt{\frac{r-c_1-c_2}{\eta c_1}}\right)\\
&> 0
\end{aligned}$$

where the first inequality is because of $\lambda_m^s>0$ (and thus $\alpha>\beta\sqrt{\frac{c_{1m}}{\beta\eta(r-c_{1m}-c_2)}}+\beta\sqrt{\frac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}}$), and the second inequality is because $w_{1m}^s<w_{1h}^s$ (and thus $\sqrt{\frac{r-c_{1m}-c_2}{(1-\eta)c_{1m}}}-\sqrt{\frac{r-c_1-c_2}{\eta c_1}}>0$).

Next, we look at how $\pi^s-\pi^b$ changes as different model parameters change.

(1) Note

$$\begin{aligned}
\frac{\partial\left(\pi^s-\pi^b\right)}{\partial r} = &\sqrt{\frac{c_1}{r-c_1-c_2}}+\sqrt{\frac{c_2}{r-c_1-c_2}}-\sqrt{\frac{\eta c_{1m}}{r-c_{1m}-c_2}}-\sqrt{\frac{(1-\eta)c_1}{r-c_1-c_2}}\\
&-\sqrt{\frac{c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}}
\end{aligned}$$

Because of Assumption 1 and Lemma 1, we have $\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, and thus $\sqrt{\frac{\eta c_{1m}}{r-c_{1m}-c_2}}+\sqrt{\frac{(1-\eta)c_1}{r-c_1-c_2}}<\sqrt{\frac{c_1}{r-c_1-c_2}}$. Also, because $c_{1m}<c_1$, we must have $\sqrt{\frac{c_2}{r-c_1-c_2}}>\sqrt{\frac{c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}}$. Thus, $\frac{\partial\left(\pi^s-\pi^b\right)}{\partial r}>0$.

(2) $\frac{\partial(\pi^s-\pi^b)}{\partial c_2}=\mu_2^b-\mu_2^s<0$.

(3) Let's first prove the following lemma:

Lemma A.3. π^s *is increasing in* η.

Proof of Lemma A.3: For $\eta_1, \eta_2 \in [0,1]$, suppose $\eta_1 < \eta_2$. Denote the optimal solution when $\eta = \eta_1$ as μ_{1m}^*, μ_{1h}^*, μ_2^*, λ_s^*, λ_o^*. The corresponding total demand and profit are $\lambda^* = (1-\eta_1)\lambda_h^* + \eta_1\lambda_m^*$ and π^*. Since $w_{1m}^* < w_{1h}^*$, we have $\lambda_m^* > \lambda_h^*$.

When $\eta = \eta_2$, consider the following feasible solution: $\lambda_m^{\triangle} = \lambda_m^*$, $\lambda_h^{\triangle} = \lambda_h^*$, $\mu_{1m}^{\triangle} = \mu_{1m}^* + (\eta_2-\eta_1)\lambda_m^*$, $\mu_{1h}^{\triangle} = \mu_{1h}^* - (\eta_2-\eta_1)\lambda_h^*$, $\mu_2^{\triangle} = \mu_2^* + (\eta_2-\eta_1)(\lambda_m^*-\lambda_h^*)$. Then, the corresponding profit $\pi^{\triangle} = \pi^* + r(\lambda_m^* - \lambda_h^*) - c_{1m}(\eta_2-\eta_1)\lambda_m^* + c_1(\eta_2-\eta_1)\lambda_h^* - c_2(\eta_2-\eta_1)(\lambda_m^* - \lambda_h^*) > \pi^* + (r-c_1-c_2)(\eta_2-\eta_1)(\lambda_m^*-\lambda_h^*) > 0$, where the first inequality is due to $c_{1m} < c_1$. Therefore, the optimal profit with $\eta = \eta_2$ must be greater than π^*. This concludes the proof. $\square$

By Lemma A.3, we find $\frac{\partial(\pi^s-\pi^b)}{\partial\eta} > 0$.

(4) It is easy to find that $\frac{\partial\pi^s}{\partial c_{1m}} < 0$. Thus, $\frac{\partial(\pi^s-\pi^b)}{\partial c_{1m}} < 0$.

$\square$

Proof of Proposition 13:

$$\begin{aligned}\pi^o - \pi^s = &- c_1(1-\theta)\alpha + [c_{1m}\eta + c_1(1-\eta)]\alpha \\ &+ 2\sqrt{\eta\beta(r-c_{1m}-c_2)c_{1m}} + 2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_1} + 2\sqrt{\beta c_2[(r-c_{1m}-c_2)\eta + (r-c_1-c_2)(1-\eta)]} \\ &- 2\sqrt{(1-\theta)\beta(r-c_1-c_2)c_1} - 2\sqrt{[(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)]c_2}\end{aligned}$$

Note that $\frac{\partial^2(\pi^o-\pi^s)}{\partial\eta^2} < 0$. When $\eta = 0$, $\pi^s = \pi^b < \pi^o$. Define $\bar{\eta}$ such that $c_1(1-\theta) = c_{1m}\bar{\eta} + c_1(1-\bar{\eta})$. When $\eta = \bar{\eta}$, then we can check that $\pi^o - \pi^s > 0$. Thus, for $\eta \in (0, \bar{\eta}]$, we have $\pi^o - \pi^s > 0$, or $\bar{\beta} = 0$. [It is easy to check that $\bar{\eta} > \theta$. Thus, if $\eta \in (0, \theta]$, we must have $\pi^o > \pi^s$ or $\bar{\beta} = 0$.]

If $\eta > \bar{\eta}$, we have $-c_1(1-\theta)\alpha + [c_{1m}\eta + c_1(1-\eta)]\alpha < 0$. Then, $\frac{\partial \frac{\pi^o-\pi^s}{\sqrt{\beta}}}{\partial\beta} = \frac{\partial \frac{-c_1(1-\theta)\alpha + [c_{1m}\eta + c_1(1-\eta)]\alpha}{\sqrt{\beta}}}{\partial\beta} > 0$. Thus, there exists $\bar{\beta} \geq 0$ such that $\pi^o - \pi^s > 0$ if and only if $\beta > \bar{\beta}$.

$\square$

References

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Supplementary Appendix B: Extensions

We have three main insights in the paper:

1. [Demand] With self-order technologies, total demand increases; and those who don't use the technology may also benefit from their implementation and choose to visit store more often. (I.e., Propositions 4 and 9)
2. [Workforce Level] With self-order technologies, total workforce level may increase, especially for firms with high cost-revenue ratio. (I.e., Propositions 6 and 11)
3. [Profit] Online self-order technology generates more profit than offline self-order technology if and only if customer wait sensitivity is large. (I.e., Proposition 13)

Below, we extend our basic model in three different ways. The goal is to show the robustness of the results listed above. All proofs for the analytical results in this appendix are presented in Section B4.

B1 Customer Heterogeneity

Suppose there are two types of customers. A fraction η are tech-savvy customers, and the rest $1-\eta$ are traditional customers. Tech-savvy customers and traditional customers have different base shopping rate and wait sensitivity, denoted as α_m and α_h, β_m and β_h, respectively. This general demand model can capture customer heterogeneity in terms of two aspects: (1) Customers may have different loyalty levels towards the firm (which is reflected by different base shopping rates), and (2) customers may have different sensitivity levels towards wait in store. The model presented in the paper is a special case of this general model with $\alpha_h = \alpha_m$ and $\beta_h = \beta_m$. We assume tech-savvy customers are more wait sensitive than traditional customers (i.e., $\beta_m \geq \beta_h$); this is natural since customers adopt self-order technology primarily because of their impatience with waiting lines (eMarketer, 2014).

Similar to what we did in the paper, in the following analysis, we focus on the case where the firm serves both types of customers, which is a valid assumption when α_m and α_h are large.

B1.1 Base Model

Given the demand rate function of each type of customers, i.e., $\lambda_m = [\alpha_m - \beta_m(w_1 + w_2)]^+$ and $\lambda_h = [\alpha_h - \beta_h(w_1 + w_2)]^+$, the firm chooses the capacity level at each stage, μ_1 and μ_2, to maximize the profit rate, i.e.,

$$\begin{aligned} &\max_{(1-\eta)\lambda_h+\eta\lambda_m\leqslant\mu_1,(1-\eta)\lambda_h+\eta\lambda_m\leqslant\mu_2} r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda_m = [\alpha_m - \beta_m\left(w_1(\mu_1,(1-\eta)\lambda_h+\eta\lambda_m) + w_2(\mu_2,(1-\eta)\lambda_h+\eta\lambda_m)\right)]^+ \\ &\qquad \lambda_h = [\alpha_h - \beta_h\left(w_1(\mu_1,(1-\eta)\lambda_h+\eta\lambda_m) + w_2(\mu_2,(1-\eta)\lambda_h+\eta\lambda_m)\right)]^+ \end{aligned} \tag{B1.1}$$

Proposition B1.1. *The firm's optimal solution is given as follows:*

- $\mu_1^b = \lambda^b + \sqrt{\frac{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}{c_1}}$;
- $\mu_2^b = \lambda^b + \sqrt{\frac{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}{c_2}}$,

where $\lambda^b = (1-\eta)\lambda_h^b + \eta\lambda_m^b$, *where* $\lambda_h^b = \alpha_h - \beta_h\sqrt{\frac{c_1}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}}$ *and* $\lambda_m^b = \alpha_m - \beta_m\sqrt{\frac{c_1}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}} - \beta_m\sqrt{\frac{c_2}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}}$

B1.2 Online Self-Order Technology

Suppose there is online self-order technology, then some tech-savvy customers will use the technology and become online customers. Suppose a fraction $\theta \leq \eta$ of all customers are online customers. Because of the instant-order and advance-order effects, their shopping rate is given by $\lambda_{mo} = [\alpha_m - \xi\beta_m w_2]^+$, where $\xi \in (0,1]$. As for store customers, there are two types: traditional customers (with demand function $\lambda_h = [\alpha_h - \beta_h(w_1+w_2)]^+$) and the remaining tech-savvy customers who don't have access to the online technology (with demand function $\lambda_{ms} = [\alpha_m - \beta_m(w_1+w_2)]^+$).

The firm chooses the capacity level at each stage, i.e., μ_1 and μ_2, to maximize profit rate:

$$\begin{aligned}
\max_{\substack{(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\leq\mu_1,\\ \theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\leq\mu_2}} \quad & r\left(\theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}\right) - c_1\mu_1 - c_2\mu_2 \\
s.t. \quad & \lambda_{mo} = [\alpha_m - \xi\beta_m w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms})]^+ \\
& \lambda_h = [\alpha_h - \beta_h\left(w_1(\mu_1, (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}) + w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms})\right)]^+ \\
& \lambda_{ms} = [\alpha_m - \beta_m(w_1(\mu_1, (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}) + w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}))]^+
\end{aligned} \tag{B1.2}$$

Proposition B1.2. *With online self-order technology, the firm's optimal solution is given as follows:*

- $\mu_1^o = (1-\eta)\lambda_h^o + (\eta-\theta)\lambda_{ms}^o + \sqrt{\frac{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}{c_1}}$;
- $\mu_2^o = (1-\eta)\lambda_h^o + (\eta-\theta)\lambda_{ms}^o + \theta\lambda_{mo}^o + \sqrt{\frac{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}}$,

where

- $\lambda_{mo}^o = \alpha_m - \xi\beta_m\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$;
- $\lambda_h^o = \alpha_h - \beta_h\sqrt{\frac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$;
- $\lambda_{ms}^o = \alpha_m - \beta_m\sqrt{\frac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}} - \beta_m\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$.

Proposition B1.3. *With online self-order technology,*

- *online customers come more often than before, i.e.,* $\lambda_{mo}^o > \lambda_m^b$;
- *given* c_1 *and* c_2*, store customers come to store more often (i.e.,* $\lambda_h^o > \lambda_h^b$ *and* $\lambda_{ms}^o > \lambda_m^b$*) if and only if* $\frac{c_1+c_2}{r} > m_\lambda$ *and* $\theta \in (0, \psi_s)$ *for some* $m_\lambda < 1$ *and* $\psi_s > 0$;

- *total demand increases, i.e.,* $\theta\lambda_{mo}^o + (1-\eta)\lambda_h^o + (\eta-\theta)\lambda_{ms}^o > (1-\eta)\lambda_h^b + \eta\lambda_m^b$.

This shows that our original Proposition 4 still holds in this case.

Proposition B1.4. *Given* c_1 *and* c_2*, then there exists a threshold* $m_k < 1$ *such that the firm increases total workforce level after implementing online self-order technology (i.e.,* $k_1^o + k_2^o > k_1^b + k_2^b$*) if and only if* $\frac{c_1+c_2}{r} > m_k$.

This shows that our original Proposition 6 still holds in this case.

B1.3 Offline Self-Order Technology

With offline self-order technology, same as the base model, tech-savvy customers always prefer to use the self-order machines, while traditional customers place an order only with human servers. Then, the firm's optimization problem is given as follows:

$$\begin{aligned} \max_{\substack{\eta\lambda_m \le \mu_{1m}, \\ (1-\eta)\lambda_h \le \mu_{1h}, \\ \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2}} \quad & r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\ s.t. \quad & \lambda_m = \left[\alpha_m - \beta_m w_{1m}\left(\mu_{1m}, \eta\lambda_m\right) - \beta_m w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right]^+ \\ & \lambda_h = \left[\alpha_h - \beta_h w_{1h}\left(\mu_{1h}, (1-\eta)\lambda_h\right) - \beta_h w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right]^+ \end{aligned} \tag{B1.3}$$

Proposition B1.5. *With offline self-order technology, the firm's optimal solution is given as follows:*

- $\mu_{1m}^s = \eta\lambda_m^s + \sqrt{\frac{\beta_m\eta(r-c_{1m}-c_2)}{c_{1m}}}$;
- $\mu_{1h}^s = (1-\eta)\lambda_h^s + \sqrt{\frac{\beta_h(1-\eta)(r-c_1-c_2)}{c_1}}$;
- $\mu_2^s = \eta\lambda_m^s + (1-\eta)\lambda_h^s + \sqrt{\frac{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}{c_2}}$,

where

- $\lambda_m^s = \alpha_m - \beta_m\sqrt{\frac{c_{1m}}{\beta_m\eta(r-c_{1m}-c_2)}} - \beta_m\sqrt{\frac{c_2}{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}}$;
- $\lambda_h^s = \alpha_h - \beta_h\sqrt{\frac{c_1}{\beta_h(1-\eta)(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}}$.

Similar to what we did in the paper, in the following analysis, we assume the machine capacity cost c_{1m} is small enough such that the average wait time at stage 1 is shorter with self-order technology compared to the base case, i.e., $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$.

Proposition B1.6. *With offline self-order technology,*

- *tech-savvy customers come more often than before, i.e.,* $\lambda_m^s > \lambda_m^b$*;*
- *given* c_{1m}, c_1, c_2*, traditional customers come to store more often (i.e.,* $\lambda_h^s > \lambda_h^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_\lambda$ *and* $\eta \in (0, \psi'_s)$ *for some* $m'_\lambda < 1$ *and* $\psi'_s > 0$*;*

- *total demand increases, i.e.,* $\eta\lambda_m^s + (1-\eta)\lambda_h^s > (1-\eta)\lambda_h^b + \eta\lambda_m^b$.

This shows that our original Proposition 9 still holds in this case.

Proposition B1.7. *Given* c_{1m}, c_1 *and* c_2*, then there exists a threshold* $m'_k < 1$ *such that the firm increases total workforce level after implementing offline self-order technology (i.e.,* $k_1^s + k_2^s > k_1^b + k_2^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_k$.

This shows that our original Proposition 11 still holds in this case.

B1.4 Profit Implications

Suppose $\beta_h = bb_h$ and $\beta_m = bb_m$, i.e., b measures the base wait sensitivity in the market.

Proposition B1.8. *There exists* $\bar{b} \geq 0$ *such that online self-order technology generates more profit than offline self-order technology (i.e.,* $\pi^o > \pi^s$*) if and only if* $b > \bar{b}$.

This shows that our original Proposition 13 still holds in this case.

B2 Convex Impact of Wait Time

In the paper, we assumed demand is a linear function of wait time. In this extension, we relax this assumption. Specifically, we consider the following demand function: $\lambda = \alpha - \beta w_1^\phi - \beta w_2^\phi$, where w_1 and w_2 represent the wait times at stages 1 and 2, and parameter $\phi \in (0, 1]$. Since $\phi \in (0, 1]$, demand λ is convex with respect to wait time at each stage. Note, the linear demand model presented in the paper is a special case of this general demand model with $\phi = 1$.

Similar to what we did in the paper, in the following analysis, we focus on the case where the firm serves all types of customers (including online, store, tech-savvy, traditional), which is a valid assumption when α is large.

B2.1 Base Model

Given the demand function, the firm chooses the capacity level at each stage, μ_1 and μ_2, to maximize the profit rate, i.e.,

$$\begin{aligned} &\max_{\lambda \leqslant \mu_1, \lambda \leqslant \mu_2} \; r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda = \left[\alpha - \beta\left(w_1(\mu_1, \lambda)\right)^\phi - \beta\left(w_2(\mu_2, \lambda)\right)^\phi\right]^+ \end{aligned} \tag{B2.1}$$

Proposition B2.1. *The firm's optimal solution is given as follows:*

- $\mu_1^b = \lambda^b + \left(\frac{\phi\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}$;
- $\mu_2^b = \lambda^b + \left(\frac{\phi\beta(r-c_1-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}$,

where $\lambda^b = \alpha - \beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}$.

B2.2 Online Self-Order Technology

With online self-order technology, because of the instant-order and advance-order effects, online customer's demand function is given as $\lambda_o = \alpha - \xi\beta w_2^{\phi}$. The store customer's demand function remains the same as before, i.e., $\lambda_s = \alpha - \beta w_1^{\phi} - \beta w_2^{\phi}$. Therefore, the firm's optimization problem is as follows:

$$\begin{aligned}
&\max_{(1-\theta)\lambda_s \le \mu_1, (1-\theta)\lambda_s + \theta\lambda_o \le \mu_2} \quad r\left((1-\theta)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\
&s.t. \quad \lambda_o = \left[\alpha - \xi\beta\left(w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)^{\phi}\right]^{+} \\
&\qquad \lambda_s = \left[\alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s)\right)^{\phi} - \beta\left(w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)^{\phi}\right]^{+}
\end{aligned} \tag{B2.2}$$

Proposition B2.2. *With online self-order technology, the firm's optimal solution is given as follows:*

- $\mu_1^o = (1-\theta)\lambda_s^o + \left(\frac{\phi(1-\theta)\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}$;
- $\mu_2^o = (1-\theta)\lambda_s^o + \theta\lambda_o^o + \left(\frac{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}$,

where

- $\lambda_s^o = \alpha - \beta\left(\frac{c_1}{\phi(1-\theta)\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}}$,
- $\lambda_o^o = \alpha - \xi\beta\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}}$.

Proposition B2.3. *With online self-order technology,*

- *online customers come more often than before, i.e.,* $\lambda_o^o > \lambda^b$*;*
- *given* c_1 *and* c_2*, store customers come to store more often (i.e.,* $\lambda_s^o > \lambda^b$*) if and only if* $\frac{c_1+c_2}{r} > m_\lambda$ *and* $\theta \in (0, \psi_s)$ *for some* $m_\lambda < 1$ *and* $\psi_s > 0$*;*
- *total demand increases, i.e.,* $(1-\theta)\lambda_s^o + \theta\lambda_o^o > \lambda^b$*.*

This shows that our original Proposition 4 still holds in this case.

Proposition B2.4. *Given* c_1 *and* c_2*, then there exists a threshold* $m_k < 1$ *such that the firm increases total workforce level after implementing online self-order technology (i.e.,* $k_1^o + k_2^o > k_1^b + k_2^b$*) if and only if* $\frac{c_1+c_2}{r} > m_k$*.*

This shows that our original Proposition 6 still holds in this case.

B2.3 Offline Self-Order Technology

With self-order technology, the firm's optimization problem is given as follows:

$$\begin{aligned}
\max_{\substack{\eta\lambda_m \le \mu_{1m},\\ (1-\eta)\lambda_h \le \mu_{1h}\\ \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2}} \quad & r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\
s.t. \quad \lambda_m = & \left[\alpha - \beta(w_{1m}\left(\mu_{1m}, \eta\lambda_m\right))^{\phi} - \beta(w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right))^{\phi}\right]^+ \\
\lambda_h = & \left[\alpha - \beta(w_{1h}\left(\mu_{1h}, (1-\eta)\lambda_h\right))^{\phi} - \beta(w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right))^{\phi}\right]^+
\end{aligned} \tag{B2.3}$$

Proposition B2.5. *With offline self-order technology, the firm's optimal solution is given as follows:*

- $\mu_{1m}^s = \eta\lambda_m^s + \left(\frac{\phi\beta\eta(r-c_{1m}-c_2)}{c_{1m}}\right)^{\frac{1}{1+\phi}}$;
- $\mu_{1h}^s = (1-\eta)\lambda_h^s + \left(\frac{\phi\beta(1-\eta)(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}$;
- $\mu_2^s = \eta\lambda_m^s + (1-\eta)\lambda_h^s + \left(\frac{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}$,

where

- $\lambda_m^s = \alpha - \beta\left(\frac{c_{1m}}{\phi\beta\eta(r-c_{1m}-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}$;
- $\lambda_h^s = \alpha - \beta\left(\frac{c_1}{\phi\beta(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}$.

Similar to what we did in the paper, in the following analysis, we assume the machine capacity cost c_{1m} is small enough such that the average wait time at stage 1 is shorter with self-order technology compared to the base case, i.e., $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$.

Proposition B2.6. *With offline self-order technology,*

- *tech-savvy customers come more often than before, i.e.,* $\lambda_m^s > \lambda^b$*;*
- *given* c_{1m}, c_1, c_2*, traditional customers come to store more often (i.e.,* $\lambda_h^s > \lambda^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_\lambda$ *and* $\eta \in (0, \psi'_s)$ *for some* $m'_\lambda < 1$*) and* $\psi'_s > 0$*;*
- *total demand increases, i.e.,* $\eta\lambda_m^s + (1-\eta)\lambda_h^s > \lambda^b$*.*

This shows that our original Proposition 9 still holds in this case.

Proposition B2.7. *Given* c_{1m}*,* c_1 *and* c_2*, then there exists a threshold* $m'_k < 1$ *such that the firm increases total workforce level after implementing offline self-order technology (i.e.,* $k_1^s + k_2^s > k_1^b + k_2^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_k$*.*

This shows that our original Proposition 11 still holds in this case.

B2.4 Profit Implications

Proposition B2.8. *There exists $\bar{\beta} \geq 0$ such that online self-order technology generates more profit than offline self-order technology (i.e., $\pi^o > \pi^s$) if and only if $\beta > \bar{\beta}$.*

This shows that our original Proposition 13 still holds in this case.

B3 Alternative Wait Time Function

In the paper, we formulated firm's optimization problem with respect to capacity μ. The three optimization problems (1, 2, 3) can also be reformulated with respect to the number of servers k as follows:

- Basic model:

$$\begin{aligned} &\max_{\lambda \leqslant k_1/\tau_1, \lambda \leqslant k_2/\tau_2} \quad r\lambda - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda = \left[\alpha - \beta\left(w_1(k_1, \lambda) + w_2(k_2, \lambda)\right)\right]^+ \end{aligned} \tag{1'}$$

- Online self-order technology:

$$\begin{aligned} &\max_{(1-\theta)\lambda_s \leq k_1/\tau_1, \theta\lambda_o + (1-\theta)\lambda_s \leq k_2/\tau_2} \quad r\left(\theta\lambda_o + (1-\theta)\lambda_s\right) - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda_o = \left[\alpha - \xi\beta w_2(k_2, \theta\lambda_o + (1-\theta)\lambda_s)\right]^+ \\ &\qquad \lambda_s = \left[\alpha - \beta\left(w_1(k_1, (1-\theta)\lambda_s) + w_2(k_2, \theta\lambda_o + (1-\theta)\lambda_s)\right)\right]^+ \end{aligned} \tag{2'}$$

- Offline self-order technology:

$$\begin{aligned} &\max_{\substack{\eta\lambda_m \leq k_{1m}/\tau_1, \\ (1-\eta)\lambda_h \leq k_1/\tau_1, \\ \eta\lambda_m + (1-\eta)\lambda_h \leq k_2/\tau_2}} \quad r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - l_{1m} k_{1m} - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda_m = \left[\alpha - \beta\left(w_1\left(k_{1m}, \eta\lambda_m\right) + w_2\left(k_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right)\right]^+ \\ &\qquad \lambda_h = \left[\alpha - \beta\left(w_1\left(k_1, (1-\eta)\lambda_h\right) + w_2\left(k_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right)\right]^+ \end{aligned} \tag{3'}$$

where $l_1 = c_1/\tau_1$, $l_{1m} = c_{1m}/\tau_1$, $l_2 = c_2/\tau_2$, $w_i(k, \lambda) = \frac{1}{k/\tau_i - \lambda}$, and τ_i is the average service time at stage $i = 1, 2$. Here, l_1 and l_2 can be interpreted as the labor cost per unit of time at stages 1 and 2, l_{1m} is the corresponding cost for machines. The number of machine servers is denoted by k_{1m}.

In the paper, we have assumed wait time function takes the following form: $w_i(k, \lambda) = \frac{1}{k/\tau_i - \lambda}$. In this section, we numerically test the robustness of our main insights with a different wait time function, i.e.,

$$w_i(k, \lambda) = \left(\frac{\tau_i}{k}\right)\left(\frac{\rho_i^{\sqrt{2(k+1)}-1}}{1-\rho_i}\right) + \tau_i \tag{B3.1}$$

where $\rho_i = \frac{\lambda\tau_i}{k}$. This corresponds to the approximated average wait time in a M/M/k queue (Cachon and Terwiesch, 2009).

In the numerical study, we consider the following parameter values:

- $\tau_1 = \{\frac{0.5}{60}, \frac{1}{60}\}$. We assume the average service requirement is 0.5/60 or 1/60 hour (i.e., 30 seconds or 1 min) to place an order.
- $\tau_2 = t\tau_1$, where $t = \{3, 5, 7\}$. Here, we only look at the case where $\tau_2 > \tau_1$ because it generally takes longer cooking food at stage 2 than processing an order at stage 1.
- $\alpha = 100$. Here, we assume the maximum traffic (i.e., if there is no wait) in a store is 100 people per hour.
- $\beta = \{300, 400, 500, 600\}$. This implies that the longest amount of wait time people can tolerate (after which their shopping rate is 0) is $\alpha/\beta = \{1/3, 1/4, 1/5, 1/6\}$ hours.
- $\eta = \{0.6, 0.7, 0.8, 0.9\}$
- $\theta = \zeta\eta$, where $\zeta = \{0.2, 0.4, 0.6, 0.8\}$.
- $\xi = \{0.2, 0.4, 0.6, 0.8\}$
- $l_1 = l_2 = \{8, 9, 10, 11\}$ The range of the hourly wage is consistent with the data provided by the Bureau of Labor Statistics `http://www.bls.gov/oes/current/oes353021.htm`.
- $l_{1m} = xl_1$, where $x = \{0.01, 0.1\}$.
- $\frac{l_1\tau_1 + l_2\tau_2}{r} = \{0.1, 0.3, 0.5, 0.7\}$. The range of the cost-revenue ratio is selected based on the following fact: According to National Restaurant Association (2010), for a restaurant, the median cost of food and beverage sales is 31.9%, and the median cost of salaries and wages is 29.4%. Note, $l_1\tau_1 + l_2\tau_2$ is the cost to serve one customer; r is the sales revenue from each customer net of cost of food and beverage. Then, the data above implies that the median of the cost-revenue ratio $\frac{l_1\tau_1 + l_2\tau_2}{r}$ should be around $29.4\%/(1 - 31.9\%) = 43.2\%$.

There are 49.152 cases in total. After checking with the assumptions we made in the paper, we end up having 35,620 cases. For each case, we solve the three optimization problems above with the wait time function (B3.1). To simplify calculation, we assume $k_1, k_2, k_{1m} \in \mathbb{R}_+$.

First, we check the impact of self-order technology on demand. Here are the results:

- With online technology, compared to the base scenario:
  - total demand increases (i.e., $\theta\lambda_o^o + (1 - \theta)\lambda_s^o > \lambda^b$) in all cases;
  - online customers come more often (i.e., $\lambda_o^o > \lambda^b$) in all cases;
  - store customers come more often (i.e., $\lambda_s^o > \lambda^b$) in about 21.7% of cases.
- With offline technology, compared to the base scenario:
  - total demand increases (i.e., $\eta\lambda_m^s + (1 - \eta)\lambda_h^s > \lambda^b$) in all cases;
  - tech-savvy customers come more often (i.e., $\lambda_m^s > \lambda^b$) in all cases;
  - traditional customers come more often (i.e., $\lambda_h^s > \lambda^b$) in about 5.4% of cases.

These results are consistent with Propositions 4 and 9 in the paper.

Next, we check the impact of self-order technology on workforce level:

- Figure B3.1 shows the proportion of instances that $k_1^o + k_2^o > k_1^b + k_2^b$ given the cost-revenue ratio $(l_1\tau_1 + l_2\tau_2)/r$. It shows that a firm with higher cost-revenue ratio will be more likely to increase workforce level after the implementation of online self-order technology, which is consistent with Proposition 6.
- Figure B3.2 shows the proportion of instances that $k_1^s + k_2^s > k_1^b + k_2^b$ given the cost-revenue ratio $(l_1\tau_1 + l_2\tau_2)/r$. It shows that a firm with higher cost-revenue ratio will be more likely to increase workforce level after the implementation of offline self-order technology, which is consistent with Proposition 11.

Finally, we check the optimal choice between online and offline self-order technologies: Figure B3.3 shows the proportion of instances that $\pi^o > \pi^s$ given β. It implies that online self-order technology is more profitable if β is large, which is consistent with Proposition 13.

Figure B3.1: Proportion of instances that total workforce level increases after the implementation of online self-order technology (i.e., $k_1^o + k_2^o > k_1^b + k_2^b$)

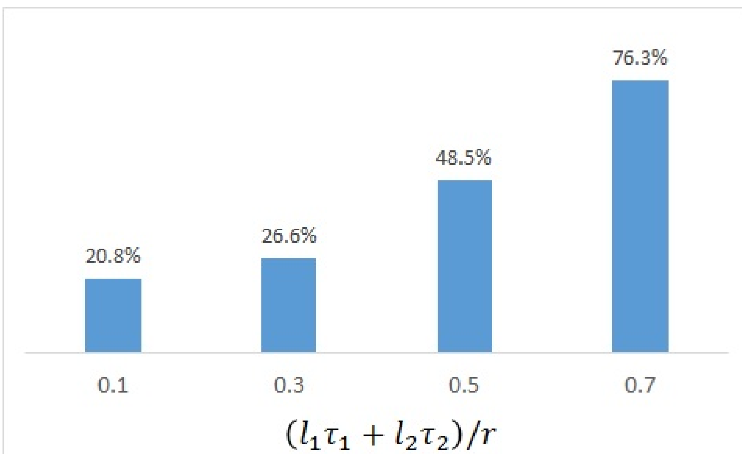


Figure B3.2: Proportion of instances that total workforce level increases after the implementation of offline self-order technology (i.e., $k_1^s + k_2^s > k_1^b + k_2^b$)

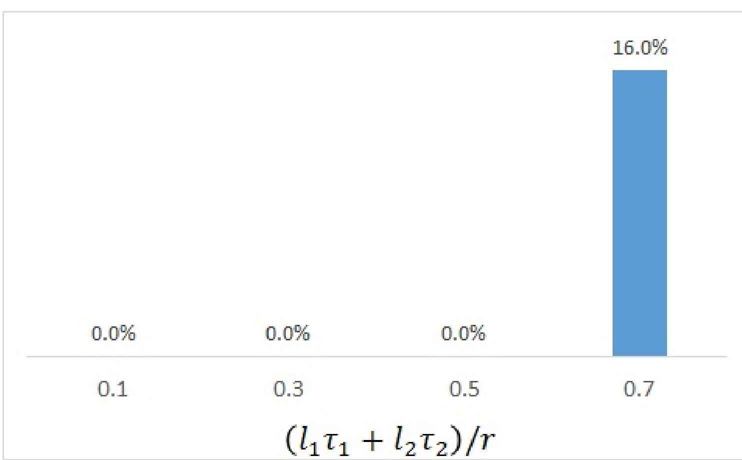
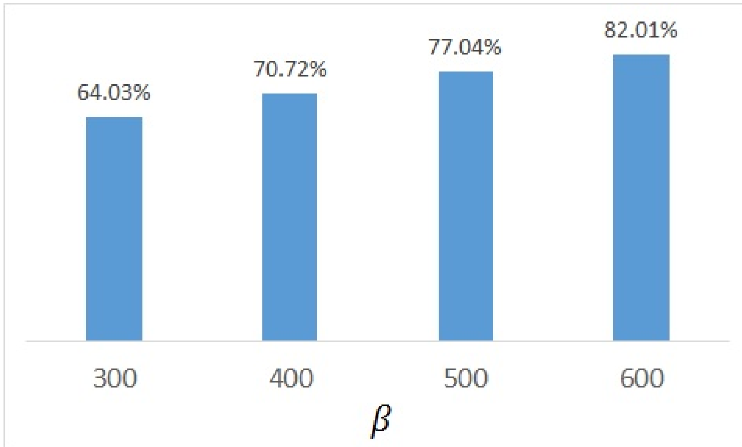


Figure B3.3: Proportion of instances that online self-order technology generates more profit than offline self-order technology (i.e., $\pi^o > \pi^s$)



B4 Proofs of Results in Extensions

Proof of Proposition B1.1: Let's define $\alpha = (1-\eta)\alpha_h + \eta\alpha_m$ and $\beta = (1-\eta)\beta_h + \eta\beta_m$.

Since we only focus on the case where the retailer serves both types, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned}
\max_{\substack{\lambda_m, \lambda_h \geq 0, \\ (1-\eta)\lambda_h + \eta\lambda_m \leqslant \mu_1 \leqslant \frac{r\alpha}{c_1}, \\ (1-\eta)\lambda_h + \eta\lambda_m \leqslant \mu_2 \leqslant \frac{r\alpha}{c_2}}} \quad & r((1-\eta)\lambda_h + \eta\lambda_m) - c_1\mu_1 - c_2\mu_2 \\
s.t. \quad & \lambda_m = \alpha_m - \beta_m \left(w_1(\mu_1, (1-\eta)\lambda_h + \eta\lambda_m) + w_2(\mu_2, (1-\eta)\lambda_h + \eta\lambda_m)\right) \\
& \lambda_h = \alpha_h - \beta_h \left(w_1(\mu_1, (1-\eta)\lambda_h + \eta\lambda_m) + w_2(\mu_2, (1-\eta)\lambda_h + \eta\lambda_m)\right)
\end{aligned}$$

the optimal solution of which can be obtained by the following optimization problem:

$$\begin{aligned}
\max_{0 \leq \lambda \leq \mu_i \leq \frac{r\alpha}{c_i}} \quad & r\lambda - c_1\mu_1 - c_2\mu_2 \\
s.t. \quad & \lambda = \alpha - \beta\left(w_1(\mu_1, \lambda) + w_2(\mu_2, \lambda)\right)
\end{aligned} \tag{B4.1}$$

where $\lambda = (1-\eta)\lambda_h + \eta\lambda_m$. The Lagrangian of (B4.1) is defined as follows:

$$L(\lambda, \mu_1, \mu_2, \rho) = r\lambda - c_1\mu_1 - c_2\mu_2 + \rho\left[\lambda - \alpha + \beta\left(\frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}\right)\right]$$

where $\rho \in \mathbb{R}$ is the Lagrange multiplier. To find the critical points of $L(\lambda, \mu_1, \mu_2, \rho)$, we solve the following equation set:

$$\begin{aligned}
&\frac{\partial L}{\partial \lambda} = r + \rho + \rho\beta\frac{1}{(\mu_1-\lambda)^2} + \rho\beta\frac{1}{(\mu_2-\lambda)^2} = 0 \\
&\frac{\partial L}{\partial \mu_i} = -c_i - \rho\beta\frac{1}{(\mu_i-\lambda)^2} = 0, i = 1, 2 \\
&\lambda - \alpha + \beta\left(\frac{1}{\mu_1-\lambda} + \frac{1}{\mu_2-\lambda}\right) = 0 \\
&0 \leq \lambda \leq \mu_1 \leq \frac{r\alpha}{c_1}, 0 \leq \lambda \leq \mu_2 \leq \frac{r\alpha}{c_2}
\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (B4.1) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$\begin{aligned}
\lambda^b &= \alpha - \beta\sqrt{\tfrac{c_1}{\beta(r-c_1-c_2)}} - \beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}} \\
\mu_1^b &= \lambda^b + \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_1}} \\
\mu_2^b &= \lambda^b + \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}}
\end{aligned}$$

Then, we can find $\lambda_h^b = \alpha_h - \beta_h w_1(\mu_1^b, \lambda^b) - \beta_h w_2(\mu_2^b, \lambda^b) = \alpha_h - \beta_h\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}$ and $\lambda_m^b = \alpha_m - \beta_m w_1(\mu_1^b, \lambda^b) - \beta_m w_2(\mu_2^b, \lambda^b) = \alpha_m - \beta_m\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}} - \beta_m\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}$. □

Proof of Proposition B1.2: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned}
&\max_{\substack{\lambda_{mo},\lambda_h,\lambda_{ms}\geq 0,\\ (1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\leq\mu_1\leq\frac{r\alpha}{c_1},\\ \theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\leq\mu_2\leq\frac{r\alpha}{c_2}}} \quad r\left(\theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}\right) - c_1\mu_1 - c_2\mu_2 \\
&s.t.\quad \lambda_{mo} = \alpha_m - \xi\beta_m w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}) \\
&\qquad \lambda_h = \alpha_h - \beta_h\left(w_1(\mu_1, (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}) + w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms})\right) \\
&\qquad \lambda_{ms} = \alpha_m - \beta_m(w_1(\mu_1, (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}) + w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}))
\end{aligned}$$

the optimal solution of which can be obtained by solving the following optimization problem:

$$\begin{aligned}
&\max_{\substack{0\leq(1-\theta)\lambda_s\leq\mu_1\leq\frac{r\alpha}{c_1},\\ \theta\lambda_{mo}+(1-\theta)\lambda_s\leq\mu_2}} \quad r\left(\theta\lambda_{mo} + (1-\theta)\lambda_s\right) - c_1\mu_1 - c_2\mu_2 \\
&s.t.\quad \lambda_{mo} = \alpha_m - \xi\beta_m w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s) \\
&\qquad \lambda_s = \alpha_s - \beta_s\left(w_1(\mu_1, \lambda_s) + w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s)\right)
\end{aligned} \tag{B4.2}$$

where $\lambda_s = \frac{(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}}{1-\theta}$, $\alpha_s = \frac{(1-\eta)\alpha_h+(\eta-\theta)\alpha_m}{1-\theta}$ and $\beta_s = \frac{(1-\eta)\beta_h+(\eta-\theta)\beta_m}{1-\theta}$. The Lagrangian of (B4.2) is

$$\begin{aligned}
L\left(\lambda_{mo}, \lambda_s, \mu_1, \mu_2, \rho_1, \rho_2\right) =& r\left(\theta\lambda_{mo} + (1-\theta)\lambda_s\right) - c_1\mu_1 - c_2\mu_2 \\
&+ \rho_1\left(\lambda_{mo} - \alpha_m + \xi\beta_m w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s)\right) \\
&+ \rho_2\left(\lambda_s - \alpha_s + \beta_s\left(w_1(\mu_1, \lambda_s) + w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s)\right)\right)
\end{aligned}$$

To find the critical points of $L\left(\lambda_{mo}, \lambda_s, \mu_1, \mu_2, \rho_1, \rho_2\right)$, we solve the following equation set:

$$\begin{aligned}
&\tfrac{\partial L}{\partial\lambda_s} = r(1-\theta) + \rho_2 + \rho_1\xi\beta_m(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} + \rho_2\beta_s(1-\theta)\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^2} \\
&\qquad +\rho_2\beta_s(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} = 0 \\
&\tfrac{\partial L}{\partial\lambda_{mo}} = r\theta + \rho_1 + \rho_1\xi\beta_m\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} + \rho_2\beta_s\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} = 0 \\
&\tfrac{\partial L}{\partial\mu_1} = -c_1 - \rho_2\beta_s\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^2} = 0 \\
&\tfrac{\partial L}{\partial\mu_2} = -c_2 - \rho_1\xi\beta_m\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} - \rho_2\beta_s\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} = 0 \\
&\lambda_{mo} - \alpha_m + \xi\beta_m\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo}} = 0 \\
&\lambda_s - \alpha_s + \beta_s\tfrac{1}{\mu_1-(1-\theta)\lambda_s} + \beta_s\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo}} = 0 \\
&0 \leq (1-\theta)\lambda_s \leq \mu_1 \leq \tfrac{r\alpha}{c_1}, 0 \leq \lambda_{mo}, (1-\theta)\lambda_s + \theta\lambda_{mo} \leq \mu_2 \leq \tfrac{r\alpha}{c_2}
\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (B4.2) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve all types of customers, the solution must be interior, which gives us a unique solution:

$$\begin{aligned}
\mu_1^o &= (1-\theta)\,\lambda_s^o + \sqrt{\tfrac{\beta_s(r-c_1-c_2)}{c_1}} \\
\mu_2^o &= (1-\theta)\,\lambda_s^o + \theta\lambda_{mo}^o + \sqrt{\tfrac{\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}} \\
\lambda_{mo}^o &= \alpha_m - \xi\beta_m\sqrt{\tfrac{c_2}{\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}} \\
\lambda_s^o &= \alpha_s - \beta_s\sqrt{\tfrac{c_1}{\beta_s(r-c_1-c_2)}} - \beta_h\sqrt{\tfrac{c_2}{\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}
\end{aligned}$$

Then, $\lambda_s^o = \alpha_h - \beta_h w_1(\mu_1^o,(1-\theta)\lambda_s^o) - \beta_h w_2(\mu_2^o,\theta\lambda_{mo}^o+(1-\theta)\lambda_s^o) = \alpha_h - \beta_h\sqrt{\frac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$, and $\lambda_{ms}^o = \alpha_m - \beta_m w_1(\mu_1^o,(1-\theta)\lambda_s^o) - \beta_m w_2(\mu_2^o,\theta\lambda_{mo}^o + (1-\theta)\lambda_s^o) = \alpha_m - \beta_m\sqrt{\frac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}} - \beta_m\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$. $\square$

Proof of Proposition B1.3: $\lambda_{mo}^o = \alpha_m - \xi\beta_m\sqrt{\frac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}} \geq \alpha_m - \beta_m\sqrt{\frac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\beta_m(r-c_2)}} > \alpha_m - \beta_m\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}} > \alpha_m - \beta_m\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}} - \beta_m\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}} = \lambda_m^b$, where the first inequality is because of $\xi \leq 1$, and the second inequality is because $\beta = (1-\eta)\beta_h + \eta\beta_m = (1-\theta)\beta_s + \theta\beta_m$.

Note $\left(\lambda_h^o - \lambda_h^b\right)\frac{\sqrt{r-c_1-c_2}}{\beta_h} = \left(\lambda_{ms}^o - \lambda_{ms}^b\right)\frac{\sqrt{r-c_1-c_2}}{\beta_m} = \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{(1-\theta)\beta_s}} - \sqrt{\frac{c_2}{(1-\theta)\beta_s+\theta\beta_m\frac{r-c_2}{r-c_1-c_2}}}$, which is decreasing in r. Then if $\sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{(1-\theta)\beta_s}} > 0$ (or θ is small enough), then given c_1 and c_2, there exists $\bar{r} > c_1 + c_2$ such that $\lambda_h^o - \lambda_h^b > 0$ and $\lambda_{ms}^o - \lambda_m^b > 0$ if and only if $\frac{c_1+c_2}{r} > \frac{c_1+c_2}{\bar{r}}$. $\square$

Proof of Proposition B1.4:

$$\begin{aligned}
k_1^o + k_2^o - k_1^b - k_2^b = {} & -\theta\alpha\tau_1 + \sqrt{\tfrac{\beta c_1}{r-c_1-c_2}}\,(\tau_1+\tau_2)\left(1-\sqrt{1-\theta}\right) \\
& - (\tau_1+\tau_2)\left[\left((1-\theta)\,\beta_s + \theta\xi\beta_m\right)\sqrt{\tfrac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}} - \beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}}\right] \\
& - \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_1}}\tau_1\left(1-\sqrt{1-\theta}\right) + \tau_2\left(\sqrt{\tfrac{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}} - \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}}\right) \\
& + \tau_1\theta\xi\beta_m\sqrt{\tfrac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}
\end{aligned} \tag{B4.3}$$

Let's first show the 3rd term in (B4.3), i.e., $-(\tau_1+\tau_2)\left[\left((1-\theta)\,\beta_s+\theta\xi\beta_m\right)\sqrt{\frac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}} - \beta\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}\right]$ (denoted as $f_3(r)$) is decreasing in r:

$$\frac{\partial f_3}{\partial r} = -(\tau_1+\tau_2)\left[-\frac{\sqrt{c_2}}{2}\frac{\sqrt{(1-\theta)\beta_s+\theta\xi\beta_m}}{\left(r-c_2-\frac{(1-\theta)\beta_s c_1}{(1-\theta)\beta_s+\theta\xi\beta_m}\right)^{\frac{3}{2}}} + \frac{\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}}\right]$$

Since $\frac{\partial[(1-\theta)\beta_s+\theta\xi\beta_m]}{\partial\theta} \leq 0$ and $\frac{\partial\left[r-c_2-\frac{(1-\theta)\beta_s c_1}{(1-\theta)\beta_s+\theta\xi\beta_m}\right]}{\partial\theta} > 0$ and $\frac{\partial\beta}{\partial\theta} = \beta_m - \beta_s \geq 0$ (because $\beta_m \geq \beta_h$), we can find that $\frac{\partial f_3}{\partial r}$ is decreasing in θ. Note when $\theta = 0$, $\frac{\partial f_3}{\partial r} = 0$. Thus, $\frac{\partial f_3}{\partial r} < 0$ for all $\theta > 0$.

Next, let's show the 5th term in (B4.3), i.e., $\tau_2\left(\sqrt{\frac{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}}-\sqrt{\frac{\beta(r-c_1-c_2)}{c_2}}\right)$ (denoted as $f_5(r)$) is decreasing in r:

$$\frac{\partial f_5}{\partial r}=\frac{\tau_2}{2\sqrt{c_2}}\left(\frac{\sqrt{(1-\theta)\,\beta_s+\theta\xi\beta_m}}{\sqrt{r-\frac{(1-\theta)\beta_s(c_1+c_2)+\theta\xi\beta_m c_2}{(1-\theta)\beta_s+\theta\xi\beta_m}}}-\frac{\sqrt{\beta}}{\sqrt{r-c_1-c_2}}\right)<0$$

Therefore, we can conclude that $\frac{\partial(k_1^o+k_2^o-k_1^b-k_2^b)}{\partial r}<0$. Note, if $r\to c_1+c_2$, we have $(k_1^o+k_2^o-k_1^b-k_2^b)\sqrt{r-c_1-c_2}\to(\tau_1+\tau_2)\sqrt{\beta c_2}+\sqrt{\beta c_1}(\tau_1+\tau_2)(1-\sqrt{1-\theta})>0$, which implies $k_1^o+k_2^o-k_1^b-k_2^b>0$ if r is very close to c_1+c_2. Thus, there exists $\bar{r}>c_1+c_2$ such that $k_1^o+k_2^o-k_1^b-k_2^b>0$ if and only if $r<\bar{r}$. Then, we can define $m_k=\frac{c_1+c_2}{\bar{r}}$.

$\square$

Proof of Proposition B1.5: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned}\max_{\substack{0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}},\\ 0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1},\\ \eta\lambda_m+(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}}}\quad & r\left(\eta\lambda_m+(1-\eta)\,\lambda_h\right)-c_{1m}\mu_{1m}-c_1\mu_{1h}-c_2\mu_2\\ s.t.\quad \lambda_m&=\alpha_m-\beta_m w_{1m}\left(\mu_{1m},\eta\lambda_m\right)-\beta_m w_2\left(\mu_2,\eta\lambda_m+(1-\eta)\,\lambda_h\right)\\ \lambda_h&=\alpha_h-\beta_h w_{1h}\left(\mu_{1h},(1-\eta)\,\lambda_h\right)-\beta_h w_2\left(\mu_2,\eta\lambda_m+(1-\eta)\,\lambda_h\right)\end{aligned}\tag{B4.4}$$

The Lagrangian of (B4.4) is

$$\begin{aligned}L\left(\lambda_m,\lambda_h,\mu_{1h},\mu_{1m},\mu_2,\rho_h,\rho_m\right)=&r\left(\eta\lambda_m+(1-\eta)\,\lambda_h\right)-c_1\mu_{1h}-c_{1m}\mu_{1m}-c_2\mu_2\\ &-\rho_h\left(\lambda_h-\alpha_h+\beta_h\frac{1}{\mu_{1h}-(1-\eta)\,\lambda_h}+\beta_h\frac{1}{\mu_2-(1-\eta)\,\lambda_h-\eta\lambda_m}\right)\\ &-\rho_m\left(\lambda_m-\alpha_m+\beta_m\frac{1}{\mu_{1m}-\eta\lambda_m}+\beta_m\frac{1}{\mu_2-(1-\eta)\,\lambda_h-\eta\lambda_m}\right)\end{aligned}$$

To find the critical points, we solve the following equation set:

$$\begin{aligned}&\tfrac{\partial L}{\partial\lambda_m}=r\eta-\rho_m-\tfrac{\rho_m\beta_m\eta}{(\mu_{1m}-\eta\lambda_m)^2}-\tfrac{(\rho_m\beta_m+\rho_h\beta_h)\eta}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0\\ &\tfrac{\partial L}{\partial\lambda_h}=r\left(1-\eta\right)-\rho_h-\tfrac{\rho_h\beta_h(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^2}-\tfrac{(\rho_m\beta_m+\rho_h\beta_h)(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0\\ &\tfrac{\partial L}{\partial\mu_{1m}}=-c_{1m}+\tfrac{\rho_m\beta_m}{(\mu_{1m}-\eta\lambda_m)^2}=0\\ &\tfrac{\partial L}{\partial\mu_{1h}}=-c_1+\tfrac{\rho_h\beta_h(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^2}=0\\ &\tfrac{\partial L}{\partial\mu_2}=-c_2+\tfrac{(\rho_m\beta_m+\rho_h\beta_h)(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0\\ &0\le\eta\lambda_m\le\mu_{1m}\le\tfrac{r\alpha}{c_{1m}},0\le(1-\eta)\,\lambda_h\le\mu_{1h}\le\tfrac{r\alpha}{c_1},\eta\lambda_m+(1-\eta)\,\lambda_h\le\mu_2\le\tfrac{r\alpha}{c_2}\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (B4.2) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve all types of customers, the solution must be interior, which gives us a unique solution:

$$\begin{aligned}
\lambda_m^s &= \alpha_m - \beta_m\sqrt{\tfrac{c_{1m}}{\beta_m\eta(r-c_{1m}-c_2)}} - \beta_m\sqrt{\tfrac{c_2}{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}}\\
\lambda_h^s &= \alpha_h - \beta_h\sqrt{\tfrac{c_1}{\beta_h(1-\eta)(r-c_1-c_2)}} - \beta_h\sqrt{\tfrac{c_2}{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}}\\
\mu_{1m}^s &= \eta\lambda_m^s + \sqrt{\tfrac{\beta_m\eta(r-c_{1m}-c_2)}{c_{1m}}}\\
\mu_{1h}^s &= (1-\eta)\,\lambda_h^s + \sqrt{\tfrac{\beta_h(1-\eta)(r-c_1-c_2)}{c_1}}\\
\mu_2^s &= \eta\lambda_m^s + (1-\eta)\,\lambda_h^s + \sqrt{\tfrac{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}{c_2}}
\end{aligned}$$

□

Proof of Proposition B1.6: First, note

$$\frac{\left(\lambda_m^s-\lambda_m^b\right)\sqrt{\beta}}{\beta_m} = \sqrt{\frac{c_1}{r-c_1-c_2}}+\sqrt{\frac{c_2}{r-c_1-c_2}}-\sqrt{\frac{c_{1m}\beta}{\beta_m\eta\left(r-c_{1m}-c_2\right)}}-\sqrt{\frac{c_2\beta}{\beta_m\eta\left(r-c_{1m}-c_2\right)+\beta_h\left(1-\eta\right)\left(r-c_1-c_2\right)}}$$

Because $c_{1m} < c_1$, we have $\sqrt{\frac{c_2}{r-c_1-c_2}} - \sqrt{\frac{c_2\beta}{\beta_m\eta(r-c_{1m}-c_2)+\beta_h(1-\eta)(r-c_1-c_2)}} > 0$. Also, we can easily check that $w_{1h}^s > w_1^b$. Then, since $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, we must have $w_{1m}^s < w_1^b$, which implies $\sqrt{\frac{c_1}{r-c_1-c_2}} - \sqrt{\frac{c_{1m}\beta}{\beta_m\eta(r-c_{1m}-c_2)}} > 0$. Thus, we have $\frac{\left(\lambda_m^s-\lambda_m^b\right)\sqrt{\beta}}{\beta_m} > 0$ or $\lambda_m^s > \lambda_m^b$.

Second, note

$$\frac{\left(\lambda_h^s-\lambda_h^b\right)\sqrt{\beta\left(r-c_1-c_2\right)}}{\beta_h} = \sqrt{c_1}+\sqrt{c_2}-\sqrt{\frac{c_1\beta}{\beta_h\left(1-\eta\right)}}-\sqrt{\frac{c_2\beta}{\beta_m\eta\frac{r-c_{1m}-c_2}{r-c_1-c_2}+\beta_h\left(1-\eta\right)}}$$

which is decreasing in r. Then if $\sqrt{c_1}+\sqrt{c_2}-\sqrt{\frac{c_1\beta}{(1-\eta)\beta_h}} > 0$ (or η is small enough), then there exists $\bar{r} > c_1+c_2$ such that $\lambda_h^s - \lambda_h^b > 0$ if and only if $\frac{c_1+c_2}{r} > \frac{c_1+c_2}{\bar{r}}$.

Finally, let's look at total demand rate. We first prove the following lemma:

Lemma B4.1. *If $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, then $\sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$.*

Proof of Lemma B4.1: Because $\eta w_{1m}^s+(1-\eta)w_{1h}^s < w_1^b$, we have $\sqrt{\frac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}} + \sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$. So all we need show is $\sqrt{\frac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}} + \sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}} \geq \sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}$. Note

$$\begin{aligned}
&\sqrt{\tfrac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}} + \sqrt{\tfrac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}} - \sqrt{\tfrac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\tfrac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}\\
&\geq \sqrt{\tfrac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}} - \sqrt{\tfrac{\beta_h(1-\eta)c_1}{r-c_1-c_2}} - \left[\sqrt{\beta_m^2}-\sqrt{\beta^2}\right]\left[\sqrt{\tfrac{c_1}{\beta(r-c_1-c_2)}} - \sqrt{\tfrac{(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}\right]\\
&= \sqrt{\tfrac{(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}\left(\beta_m-\beta_h\right)(1-\eta)\left[\tfrac{1}{1-\eta} - \sqrt{\tfrac{\beta_h}{(1-\eta)\beta}}\right]\\
&\geq 0
\end{aligned}$$

where the first inequality is because $\sqrt{\frac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}} + \sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}} - \sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}$ is decreasing in c_{1m} and $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, and the second inequality is because $\beta_m \geq \beta_h$ and $\beta > (1-\eta)\beta_h$. This completes the proof. □

Because $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$, Lemma B4.1 and $c_{1m} < c_1$, we can check that $\eta\lambda_m^s + (1-\eta)\lambda_h^s = \alpha - \sqrt{\frac{\beta_m \eta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}} - \beta\sqrt{\frac{c_2}{\beta_m\eta(r-c_{1m}-c_2)+\beta_h(1-\eta)(r-c_1-c_2)}} > \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \beta\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}} = \lambda^b$ □

Proof of Proposition B1.7: Note $\frac{\partial k_1^s}{\partial r} = \frac{\tau_1\sqrt{\beta_h(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\tau_1(1-\eta)\beta_h\beta\sqrt{c_2}}{2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}} + \frac{\tau_1\sqrt{\beta_h(1-\eta)}}{2\sqrt{c_1(r-c_1-c_2)}} < \frac{\tau_1\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\tau_1\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\tau_1\sqrt{\beta}}{2\sqrt{c_1(r-c_1-c_2)}} = \frac{\partial k_1^b}{\partial r}$, where the inequality is because $c_{1m} < c_1$ and $\beta > (1-\eta)\beta_h$.

Also, note

$$\begin{aligned}\frac{\partial k_2^s}{\partial r} &= \frac{\sqrt{\eta\beta_m c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta_h(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} \\ &\quad + \frac{\beta^2\sqrt{c_2}}{2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}} + \frac{\beta}{2\sqrt{c_2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]}}\end{aligned}$$

By Lemma B4.1 and $c_{1m} < c_1$, we have $\frac{\sqrt{\eta\beta_m c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta_h(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} < \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}}$. Also, define function $f(c_{1m}) = \frac{\beta^2\sqrt{c_2}}{2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}} + \frac{\beta}{2\sqrt{c_2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]}}$, which is an increasing function. Thus, $f(c_{1m}) < f(c_1)$. Since $\frac{\partial k_2^b}{\partial r} = \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + f(c_1)$, we have $\frac{\partial k_2^s}{\partial r} < \frac{\partial k_2^b}{\partial r}$.

Therefore, we can conclude that $\frac{\partial(k_1^s+k_2^s-k_1^b-k_2^b)}{\partial r} < 0$. Note, if $r \to c_1+c_2$, we have $(k_1^s+k_2^s-k_1^b-k_2^b)\sqrt{r-c_1-c_2} \to (\tau_1+\tau_2)(\sqrt{\beta c_1}+\sqrt{\beta c_2}-\sqrt{\beta_h(1-\eta)c_1}) > 0$, which implies $k_1^s+k_2^s-k_1^b-k_2^b > 0$ if r is very close to c_1+c_2. Thus, there exists $\bar{r} > c_1+c_2$ such that $k_1^s+k_2^s-k_1^b-k_2^b > 0$ if and only if $r < \bar{r}$. Then, we can define $m_k' = \frac{c_1+c_2}{\bar{r}}$. □

Proof of Proposition B1.8:

$$\begin{aligned}\pi^o - \pi^s = &-c_1(1-\theta)\alpha + [c_{1m}\eta + c_1(1-\eta)]\alpha \\ &+ 2\sqrt{\eta\beta_m(r-c_{1m}-c_2)c_{1m}} + 2\sqrt{(1-\eta)\beta_h(r-c_1-c_2)c_1} \\ &+ 2\sqrt{c_2[\beta_m(r-c_{1m}-c_2)\eta + \beta_h(r-c_1-c_2)(1-\eta)]} \\ &- 2\sqrt{(1-\theta)\beta_s(r-c_1-c_2)c_1} - 2\sqrt{[(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_o(r-c_2)]c_2}\end{aligned}$$

Note that $\frac{\partial^2(\pi^o-\pi^s)}{\partial\theta^2} < 0$. Also note that when $\theta = \eta$, we must have the $\pi^o > \pi^s$, the proof of which is as follows: Suppose the optimal solution for the offline model is $\mu_{1m}^s, \mu_{1h}^s, \mu_2^s, \lambda_m^s, \lambda_h^s$. Consider the following feasible solution for the online model: $\mu_1^\triangle, \mu_2^\triangle, \lambda_m^\triangle, \lambda_h^\triangle$, where $\mu_1^\triangle = \mu_{1h}^s$ and $\mu_2^\triangle = \mu_2^s$. Then, suppose $\eta\lambda_m^\triangle + (1-\eta)\lambda_h^\triangle \le \eta\lambda_m^s + (1-\eta)\lambda_h^s$. Then, we must have $w_2^\triangle \le w_2^s$. Then, because online customers don't wait at stage 1, we have $\lambda_m^\triangle > \lambda_m^s$. Then, $\lambda_h^\triangle < \lambda_h^s$. Then, $w_1^\triangle < w_1^s$. However, if $w_1^\triangle < w_1^s$ and $w_2^\triangle < w_2^s$, then this means that $\lambda_h^\triangle > \lambda_h^s$. We get a contradiction. Thus, we must have $\eta\lambda_m^\triangle + (1-\eta)\lambda_h^\triangle > \eta\lambda_m^s + (1-\eta)\lambda_h^s$. This implies that $\pi^\triangle > \pi^s$. Thus, we must have $\pi^o \ge \pi^\triangle > \pi^s$. The results above implies that $\exists\bar{\theta} \le \eta$ such that $\pi^o > \pi^s$ if and only if $\theta > \bar{\theta}$.

Note when $c_{1m}\eta = c_1(\eta-\theta)$ (or $\theta = \frac{(c_1-c_{1m})\eta}{c_1}$), we have $\pi^o - \pi^s > 0$ for any $b > 0$. This implies that $\bar{\theta} \le \frac{(c_1-c_{1m})\eta}{c_1}$. Thus, for any $b > 0$, if $\theta \ge \frac{(c_1-c_{1m})\eta}{c_1}$ (or $c_{1m}\eta - c_1(\eta-\theta) \ge 0$), we must have $\pi^o - \pi^s > 0$, i.e., $\bar{b} = 0$. If $c_{1m}\eta - c_1(\eta - \theta) < 0$, then $\frac{\partial \frac{\pi^o - \pi^s}{\sqrt{b}}}{\partial b} = \frac{\partial \frac{(c_{1m}\eta - c_1(\eta-\theta))\alpha}{\sqrt{b}}}{\partial b} > 0$. Thus, there exists $\bar{b} \geq 0$ such that $\pi^o - \pi^s > 0$ if and only if $b > \bar{b}$. □

Proof of Proposition B2.1: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned} &\max_{0 \leqslant \lambda \leqslant \mu_1 \leqslant \frac{r\alpha}{c_1}, \lambda \leqslant \mu_2 \leqslant \frac{r\alpha}{c_2}} r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda = \alpha - \beta\left(w_1(\mu_1, \lambda)\right)^{\phi} - \beta\left(w_2(\mu_2, \lambda)\right)^{\phi} \end{aligned} \tag{B4.5}$$

The Lagrangian of (B4.5) is

$$L(\lambda, \mu_1, \mu_2, \rho) = r\lambda - c_1\mu_1 - c_2\mu_2 + \rho\left(\lambda - \alpha + \beta\frac{1}{(\mu_1 - \lambda)^{\phi}} + \beta\frac{1}{(\mu_2 - \lambda)^{\phi}}\right)$$

To find the critical points, we solve the following equation set:

$$\begin{aligned} &\tfrac{\partial L}{\partial \lambda} = r + \rho + \tfrac{\phi\rho\beta}{(\mu_1 - \lambda)^{1+\phi}} + \tfrac{\phi\rho\beta}{(\mu_2-\lambda)^{1+\phi}\underline{\partial}} = 0 \\ &\tfrac{\partial L}{\partial \mu_1} = -c_1 - \tfrac{\phi\rho\beta}{(\mu_1-\lambda)^{1+\phi}} = 0 \\ &\tfrac{\partial L}{\partial \mu_2} = -c_2 - \tfrac{\phi\rho\beta}{(\mu_2-\lambda)^{1+\phi}} = 0 \\ &\lambda - \alpha + \beta\tfrac{1}{(\mu_1-\lambda)^{\phi}} + \beta\tfrac{1}{(\mu_2-\lambda)^{\phi}} \\ &\lambda \leq \mu_1, \lambda \leq \mu_2 \end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (B4.1) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$\begin{aligned} &\lambda^b = \alpha - \beta\left(\tfrac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\tfrac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} \\ &\mu_1^b = \lambda^b + \left(\tfrac{\phi\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}} \\ &\mu_2^b = \lambda^b + \left(\tfrac{\phi\beta(r-c_1-c_2)}{c_2}\right)^{\frac{1}{1+\phi}} \end{aligned}$$

□

Proof of Proposition B2.2: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned} &\max_{\substack{0 \leq (1-\theta)\lambda_s \leq \mu_1 \leq \frac{r\alpha}{c_1}, \\ 0 \leq \lambda_o, \\ (1-\theta)\lambda_s + \theta\lambda_o \leq \mu_2 \leq \frac{r\alpha}{c_2}}} r\left((1-\theta)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda_o = \alpha - \xi\beta\left(w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)^{\phi} \\ &\qquad\;\; \lambda_s = \alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s)\right)^{\phi} - \beta\left(w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)^{\phi} \end{aligned} \tag{B4.6}$$

The Lagrangian of (B4.6) is given as follows:

$$\begin{aligned} L\left(\lambda_o,\lambda_s,\mu_1,\mu_2,\rho_o,\rho_s\right) = & r\left(\theta\lambda_o+(1-\theta)\lambda_s\right)-c_1\mu_1-c_2\mu_2 \\ & -\rho_1\left(\lambda_o-\alpha+\xi\beta(w_2(\mu_2,\theta\lambda_o+(1-\theta)\lambda_s))^{\phi}\right) \\ & -\rho_2\left(\lambda_s-\alpha+\beta(w_1(\mu_1,\lambda_s))^{\phi}+\beta(w_2(\mu_2,\theta\lambda_{mo}+(1-\theta)\lambda_s))^{\phi}\right) \end{aligned}$$

To find the critical points, we solve the following equation set:

$$\begin{aligned} & \tfrac{\partial L}{\partial\lambda_s}=r(1-\theta)+\rho_2+\phi\rho_1\xi\beta(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^{1+\phi}}+\phi\rho_2\beta(1-\theta)\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^{1+\phi}} \\ & \qquad +\phi\rho_2\beta(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^{1+\phi}}=0 \\ & \tfrac{\partial L}{\partial\lambda_o}=r\theta+\rho_1+\phi\rho_1\xi\beta\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^{1+\phi}}+\phi\rho_2\beta\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^{1+\phi}}=0 \\ & \tfrac{\partial L}{\partial\mu_1}=-c_1-\phi\rho_2\beta\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^{1+\phi}}=0 \\ & \tfrac{\partial L}{\partial\mu_2}=-c_2-\phi\rho_1\xi\beta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^{1+\phi}}-\rho_2\beta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^{1+\phi}}=0 \\ & \lambda_o-\alpha+\xi\beta\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o}=0 \\ & \lambda_s-\alpha+\beta\tfrac{1}{\mu_1-(1-\theta)\lambda_s}+\beta\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o}=0 \\ & 0\le(1-\theta)\lambda_s\le\mu_1\le\tfrac{r\alpha}{c_1},0\le\lambda_o,(1-\theta)\lambda_s+\theta\lambda_o\le\mu_2\le\tfrac{r\alpha}{c_2} \end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (B4.1) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$\begin{aligned} & \lambda_s^o=\alpha-\beta\left(\tfrac{c_1}{\phi(1-\theta)\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}-\beta\left(\tfrac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}} \\ & \lambda_o^o=\alpha-\beta_o\left(\tfrac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}} \\ & \mu_1^o=(1-\theta)\lambda_s^o+\left(\tfrac{\phi(1-\theta)\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}} \\ & \mu_2^o=(1-\theta)\lambda_s^o+\theta\lambda_o^o+\left(\tfrac{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}{c_2}\right)^{\frac{1}{1+\phi}} \end{aligned}$$

□

Proof of Proposition B2.3: $\lambda_o^o=\alpha-\left[\frac{c_2(\xi\beta)^{1+\phi}}{\phi\xi\beta\theta(r-c_2)+\phi\beta(r-c_1-c_2)(1-\theta)}\right]^{\frac{\phi}{1+\phi}}\ge\alpha-\left[\frac{c_2\beta^{\phi}}{\phi\theta(r-c_2)+\phi(r-c_1-c_2)(1-\theta)}\right]^{\frac{\phi}{1+\phi}}>\alpha-\left[\frac{c_2\beta^{\phi}}{\phi(r-c_1-c_2)}\right]^{\frac{\phi}{1+\phi}}>\alpha-\beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}-\beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}=\lambda^b$ where the first inequality is because of $\xi\le1$.

Next, let's prove the second bullet point. Note

$$\left(\lambda_s^o-\lambda^b\right)\frac{(r-c_1-c_2)^{\frac{\phi}{1+\phi}}}{\beta}=\left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}}+\left(\frac{c_2}{\phi\beta}\right)^{\frac{\phi}{1+\phi}}-\left(\frac{c_1}{\phi\beta(1-\theta)}\right)^{\frac{\phi}{1+\phi}}-\left(\frac{c_2}{\phi\beta(1-\theta)+\phi\xi\beta\theta\frac{r-c_2}{r-c_1-c_2}}\right)^{\frac{\phi}{1+\phi}}$$

which is decreasing in r. When $r\to c_1+c_2$, $\left(\lambda_s^o-\lambda^b\right)\frac{(r-c_1-c_2)^{\frac{\phi}{1+\phi}}}{\beta}\to\left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}}+\left(\frac{c_2}{\phi\beta}\right)^{\frac{\phi}{1+\phi}}-\left(\frac{c_1}{\phi\beta(1-\theta)}\right)^{\frac{\phi}{1+\phi}}>0$ if θ is small enough. Then, we can conclude the result.

Finally, let's prove the third point. Note $(1-\theta)\lambda_s^o+\theta\lambda_o^o=\alpha-((1-\theta)\beta)^{\frac{1}{1+\phi}}\left(\frac{c_1}{\phi(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}-\frac{c_2^{\frac{\phi}{1+\phi}}((1-\theta)\beta+\theta\xi\beta)^{\frac{1}{1+\phi}}}{\left(r-\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{\phi}{1+\phi}}}$, which is increasing in θ. Note when $\theta=0$, we have $(1-\theta)\lambda_s^o+\theta\lambda_o^o=\lambda^b$. Thus, $(1-\theta)\lambda_s^o+\theta\lambda_o^o>\lambda^b$ for all $\theta>0$. □

Proof of Proposition B2.4:

$$
\begin{aligned}
k_1^o+k_2^o-k_1^b-k_2^b= & -\theta\alpha\tau_1+(\tau_1+\tau_2)\left(1-(1-\theta)^{\frac{\phi}{1+\phi}}\right)\beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{1}{1+\phi}} \\
& -(\tau_1+\tau_2)\left[((1-\theta)\beta+\theta\xi\beta)\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{1}{1+\phi}}-\beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{1}{1+\phi}}\right] \\
& -\left(\frac{\phi\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}\tau_1\left(1-(1-\theta)^{\frac{1}{1+\theta}}\right)+\tau_2\left(\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{-\frac{1}{1+\phi}}-\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{-\frac{1}{1+\phi}}\right) \\
& +\tau_1\theta\xi\beta\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{1}{1+\phi}}
\end{aligned}
\tag{B4.7}
$$

Let's first show the 3rd term in (B4.7), i.e., $-(\tau_1+\tau_2)\left[((1-\theta)\beta+\theta\xi\beta)\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{1}{1+\phi}}-\beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{1}{1+\phi}}\right]$ (denoted as $f_3(r)$) is decreasing in r:

$$
\frac{\partial f_3}{\partial r}=-(\tau_1+\tau_2)\left[-\frac{c_2^{\frac{1}{1+\phi}}}{(1+\phi)\,\phi^{\frac{1}{1+\phi}}}\frac{((1-\theta)\beta+\theta\xi\beta)^{\frac{\phi}{1+\phi}}}{\left(r-c_2-\frac{(1-\theta)\beta c_1}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{1}{1+\phi}+1}}+\frac{\beta^{\frac{\phi}{1+\phi}}c_2^{\frac{1}{1+\phi}}}{(1+\phi)\,\phi^{\frac{1}{1+\phi}}(r-c_1-c_2)^{\frac{1}{1+\phi}+1}}\right]
$$

which is decreasing in θ. Note when $\theta=0$, $\frac{\partial f_3}{\partial r}=0$. Thus, $\frac{\partial f_3}{\partial r}<0$ for all $\theta>0$.

Next, let's show the 5th term in (B4.7), i.e., $\tau_2\left(\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{-\frac{1}{1+\phi}}-\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{-\frac{1}{1+\phi}}\right)$ (denoted as $f_5(r)$) is decreasing in r:

$$
\frac{\partial f_5}{\partial r}=\frac{c_2^{-\frac{1}{1+\phi}}}{(1+\phi)\,\phi^{-\frac{1}{1+\phi}}}\left(\frac{((1-\theta)\beta+\theta\xi\beta)^{\frac{1}{1+\phi}}}{\left(r-\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{\phi}{1+\phi}}}-\frac{\beta^{\frac{1}{1+\phi}}}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}}}\right)<0
$$

Therefore, we can conclude that $\frac{\partial(k_1^o+k_2^o-k_1^b-k_2^b)}{\partial r}<0$. Note, if $r\to c_1+c_2$, we have $(k_1^o+k_2^o-k_1^b-k_2^b)(r-c_1-c_2)^{\frac{1}{1+\phi}}\to(\tau_1+\tau_2)\beta\left(\frac{c_2}{\phi\beta}\right)^{\frac{1}{1+\phi}}+\left(1-(1-\theta)^{\frac{\phi}{1+\phi}}\right)\beta\left(\frac{c_1}{\phi\beta}\right)^{\frac{1}{1+\phi}}(\tau_1+\tau_2)(1-\sqrt{(1-\theta)})>0$, which implies $k_1^o+k_2^o-k_1^b-k_2^b>0$ if r is very close to c_1+c_2. Thus, there exists $\bar{r}>c_1+c_2$ such that $k_1^o+k_2^o-k_1^b-k_2^b>0$ if and only if $r<\bar{r}$. Then, we can define $m_k=\frac{c_1+c_2}{\bar{r}}$. □

Proof of Proposition B2.5: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$
\begin{aligned}
\max_{\substack{0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}},\\ 0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1}\\ \eta\lambda_m+(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}}} \quad & r\left(\eta\lambda_m+(1-\eta)\lambda_h\right)-c_{1m}\mu_{1m}-c_1\mu_{1h}-c_2\mu_2 \\
s.t.\quad & \lambda_m=\alpha-\beta(w_{1m}(\mu_{1m},\eta\lambda_m))^{\phi}-\beta(w_2(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h))^{\phi} \\
& \lambda_h=\alpha-\beta(w_{1h}(\mu_{1h},(1-\eta)\lambda_h))^{\phi}-\beta(w_2(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h))^{\phi}
\end{aligned}
\tag{B4.8}
$$

The Lagrangian of (B4.8) is given as follows:

$$
\begin{aligned}
L(\lambda_m, \lambda_h, \mu_{1h}, \mu_{1m}, \mu_2, \rho_m, \rho_h) =& r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\
& - \rho_m \left(\lambda_m - \alpha + \beta\Big(\frac{1}{\mu_{1m} - \eta\lambda_m}\Big)^{\phi} + \beta\Big(\frac{1}{\mu_2 - \eta\lambda_m - (1-\eta)\lambda_h}\Big)^{\phi}\right) \\
& - \rho_h \left(\lambda_h - \alpha + \beta\Big(\frac{1}{\mu_{1h} - (1-\eta)\lambda_h}\Big)^{\phi} + \beta\Big(\frac{1}{\mu_2 - \eta\lambda_m - (1-\eta)\lambda_h}\Big)^{\phi}\right)
\end{aligned}
$$

To find the critical points, we solve the following equation set:

$$
\begin{aligned}
&\tfrac{\partial L}{\partial \lambda_m} = r\eta - \rho_m - \tfrac{\rho_m \beta\eta}{(\mu_{1m}-\eta\lambda_m)^{\phi+1}} - \tfrac{(\rho_m+\rho_h)\beta\eta}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \lambda_h} = r(1-\eta) - \rho_h - \tfrac{\rho_h \beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^{\phi+1}} - \tfrac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \mu_{1m}} = -c_{1m} + \tfrac{\rho_m \beta}{(\mu_{1m}-\eta\lambda_m)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \mu_{1h}} = -c_1 + \tfrac{\rho_h \beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \mu_2} = -c_2 + \tfrac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\eta\lambda_m \le \mu_{1m}, (1-\eta)\lambda_h \le \mu_{1h}, \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2
\end{aligned}
$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (B4.1) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$
\begin{aligned}
&\lambda_m^s = \alpha - \beta\Big(\tfrac{c_{1m}}{\phi\beta\eta(r-c_{1m}-c_2)}\Big)^{\frac{\phi}{1+\phi}} - \beta\Big(\tfrac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}} \\
&\lambda_h^s = \alpha - \beta\Big(\tfrac{c_1}{\phi\beta(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}} - \beta\Big(\tfrac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}} \\
&\mu_{1m}^s = \eta\lambda_m^s + \Big(\tfrac{\phi\beta\eta(r-c_{1m}-c_2)}{c_{1m}}\Big)^{\frac{1}{1+\phi}} \\
&\mu_{1h}^s = (1-\eta)\lambda_h^s + \Big(\tfrac{\phi\beta(1-\eta)(r-c_1-c_2)}{c_1}\Big)^{\frac{1}{1+\phi}} \\
&\mu_2^s = \eta\lambda_m^s + (1-\eta)\lambda_h^s + \Big(\tfrac{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}{c_2}\Big)^{\frac{1}{1+\phi}}
\end{aligned}
$$

□

Proof of Proposition B2.6: First note

$$
\begin{aligned}
\frac{\left(\lambda_m^s - \lambda^b\right)\phi^{\frac{\phi}{1+\phi}}}{\beta^{\frac{1}{1+\phi}}} =& \Big(\frac{c_1}{r-c_1-c_2}\Big)^{\frac{\phi}{1+\phi}} + \Big(\frac{c_2}{r-c_1-c_2}\Big)^{\frac{\phi}{1+\phi}} \\
& - \Big(\frac{c_{1m}}{\eta(r-c_{1m}-c_2)}\Big)^{\frac{\phi}{1+\phi}} - \Big(\frac{c_2}{\eta(r-c_{1m}-c_2)+(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}}
\end{aligned}
$$

Because $c_{1m} < c_1$, we have $\Big(\frac{c_2}{r-c_1-c_2}\Big)^{\frac{\phi}{1+\phi}} - \Big(\frac{c_2}{\eta(r-c_{1m}-c_2)+(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}} > 0$. Also, we can easily check that $w_{1h}^s > w_1^b$. Then, since $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, we must have $w_{1m}^s < w_1^b$, which implies $\Big(\frac{c_1}{r-c_1-c_2}\Big)^{\frac{\phi}{1+\phi}} - \Big(\frac{c_{1m}}{\eta(r-c_{1m}-c_2)}\Big)^{\frac{\phi}{1+\phi}} > 0$. Thus, we have $\frac{(\lambda_m^s-\lambda^b)\phi^{\frac{\phi}{1+\phi}}}{\beta^{\frac{1}{1+\phi}}} > 0$, or $\lambda_m^s > \lambda^b$.

Second, note

$$\frac{\left(\lambda_h^s-\lambda_h^b\right)(r-c_1-c_2)^{\frac{\phi}{1+\phi}}\phi^{\frac{\phi}{1+\phi}}}{\beta^{\frac{1}{1+\phi}}}=c_1{}^{\frac{\phi}{1+\phi}}+c_2{}^{\frac{\phi}{1+\phi}}-\left(\frac{c_1}{1-\eta}\right)^{\frac{\phi}{1+\phi}}-\left(\frac{c_2}{\eta\frac{r-c_{1m}-c_2}{r-c_1-c_2}+(1-\eta)}\right)^{\frac{\phi}{1+\phi}}$$

which is decreasing in r. Then if $c_1{}^{\frac{\phi}{1+\phi}}+c_2{}^{\frac{\phi}{1+\phi}}-\left(\frac{c_1}{1-\eta}\right)^{\frac{\phi}{1+\phi}}>0$ (or η is small enough), then there exists $\bar{r}>c_1+c_2$ such that $\lambda_h^s-\lambda_h^b>0$ if and only if $\frac{c_1+c_2}{r}>\frac{c_1+c_2}{\bar{r}}$.

Finally, let's look at total demand rate. We first prove the following lemma:

Lemma B4.2. *If $\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, then $\eta(w_{1m}^s)^\phi+(1-\eta)(w_{1h}^s)^\phi<(w_1^b)^\phi$.*

Proof of Lemma B4.2: Because $0<\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, we have $(\eta w_{1m}^s+(1-\eta)w_{1h}^s)^\phi<(w_1^b)^\phi$. Note $(\eta w_{1m}^s+(1-\eta)w_{1h}^s)^\phi\geq\eta^\phi(w_{1m}^s)^\phi+(1-\eta)^\phi(w_{1h})^\phi\geq\eta(w_{1m}^s)^\phi+(1-\eta)(w_{1h})^\phi$, where the second inequality is because of $\eta\in(0,1)$ and $\phi\in(0,1]$. Thus, we have $\eta(w_{1m}^s)^\phi+(1-\eta)(w_{1h}^s)^\phi<(w_1^b)^\phi$. □

Because $\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, Lemma B4.1 and $c_{1m}<c_1$, we can check that $\eta\lambda_m^s+(1-\eta)\lambda_h^s=\alpha-\eta\beta\left(\frac{c_{1m}}{\phi\eta\beta(r-c_{1m}-c_2)}\right)^{\frac{\phi}{1+\phi}}-(1-\eta)\beta\left(\frac{c_1}{\phi(1-\eta)\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}-\beta\left(\frac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}>\alpha-\beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}-\beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}=\lambda^b$ □

Proof of Proposition B2.7: Note $\frac{\partial k_1^s}{\partial r}=\frac{\phi^{\frac{1}{1+\phi}}\beta^{\frac{1}{1+\phi}}(1-\eta)^{\frac{1}{1+\phi}}c_1^{\frac{\phi}{1+\phi}}\tau_1}{(1+\phi)(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}}+\frac{\tau_1(1-\eta)\beta^2\phi^2c_2^{\frac{\phi}{1+\phi}}}{(1+\phi)[\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta)]^{\frac{\phi}{1+\phi}+1}}+\frac{\tau_1(\beta\phi(1-\eta))^{\frac{1}{1+\phi}}}{(1+\phi)c_1^{\frac{1}{1+\phi}}(r-c_1-c_2)^{\frac{\phi}{1+\phi}}}<\frac{\phi^{\frac{1}{1+\phi}}\beta^{\frac{1}{1+\phi}}c_1^{\frac{\phi}{1+\phi}}\tau_1}{(1+\phi)(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}}+\frac{\tau_1\beta^2\phi^2c_2^{\frac{\phi}{1+\phi}}}{(1+\phi)[\beta(r-c_1-c_2)]^{\frac{\phi}{1+\phi}+1}}+\frac{\tau_1(\beta\phi)^{\frac{1}{1+\phi}}}{(1+\phi)c_1^{\frac{1}{1+\phi}}(r-c_1-c_2)^{\frac{\phi}{1+\phi}}}=\frac{\partial k_1^b}{\partial r}$.

Also, note

$$\begin{aligned}\frac{\partial k_2^s}{\partial r}&=\eta\beta\left(\frac{c_{1m}}{\phi\beta\eta}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_{1m}-c_2)^{\frac{\phi}{1+\phi}+1}}+(1-\eta)\beta\left(\frac{c_1}{\phi\beta(1-\eta)}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}}\\&\quad+\phi\beta^2c_2^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta))^{\frac{\phi}{1+\phi}+1}}+\frac{\phi\beta}{(1+\phi)c_2^{\frac{1}{1+\phi}}}\frac{1}{(\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta))^{\frac{\phi}{1+\phi}}}\end{aligned}$$

By Lemma B4.2 and $c_{1m}<c_1$, we have

$$\eta\beta\left(\frac{c_{1m}}{\phi\beta\eta}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_{1m}-c_2)^{\frac{\phi}{1+\phi}+1}}+(1-\eta)\beta\left(\frac{c_1}{\phi\beta(1-\eta)}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}}<\beta\left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}}.$$

Also, define function

$$\begin{aligned}f(c_{1m})=&\phi\beta^2c_2^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta))^{\frac{\phi}{1+\phi}+1}}\\&+\frac{\phi\beta}{(1+\phi)c_2^{\frac{1}{1+\phi}}}\frac{1}{(\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta))^{\frac{\phi}{1+\phi}}},\end{aligned}$$

which is an increasing function. Thus, $f(c_{1m}) < f(c_1)$. Since $\frac{\partial k_2^b}{\partial r} = \beta\left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}} + f\left(c_1\right)$, we have $\frac{\partial k_2^s}{\partial r} < \frac{\partial k_2^b}{\partial r}$.

Therefore, we can conclude that $\frac{\partial(k_1^s+k_2^s-k_1^b-k_2^b)}{\partial r} < 0$. Note, if $r \to c_1 + c_2$, we have $(k_1^s + k_2^s - k_1^b - k_2^b)(r - c_1 - c_2)^{\frac{\phi}{1+\phi}} \to (\tau_1 + \tau_2)\beta^{\frac{1}{1+\phi}}\phi^{\frac{-\phi}{1+\phi}}(c_1^{\frac{\phi}{1+\phi}} + c_2^{\frac{\phi}{1+\phi}} - c_1^{\frac{\phi}{1+\phi}}(1-\eta)^{\frac{1}{1+\phi}}) > 0$, which implies $k_1^s + k_2^s - k_1^b - k_2^b > 0$ if r is very close to $c_1 + c_2$. Thus, there exists $\bar{r} > c_1 + c_2$ such that $k_1^s + k_2^s - k_1^b - k_2^b > 0$ if and only if $r < \bar{r}$. Then, we can define $m_k' = \frac{c_1+c_2}{\bar{r}}$. □

Proof of Proposition B2.8: Note that

$$\pi^o = \left[r - c_1\left(1-\theta\right) - c_2\right]\alpha - \left(\frac{c_1}{\phi}\right)^{\frac{\phi}{1+\phi}}\left[\beta\left(r-c_1-c_2\right)\left(1-\theta\right)\right]^{\frac{1}{1+\phi}} - \phi^{\frac{1}{1+\phi}}c_1^{\frac{\phi}{1+\phi}}\left[\beta\left(r-c_1-c_2\right)\left(1-\theta\right)\right]^{\frac{1}{1+\phi}}$$
$$-\left(\frac{c_2}{\phi}\right)^{\frac{\phi}{1+\phi}}\left[\left(1-\theta\right)\beta\left(r-c_1-c_2\right)+\theta\xi\beta\left(r-c_2\right)\right]^{\frac{1}{1+\phi}} - \phi^{\frac{1}{1+\phi}}c_2^{\frac{\phi}{1+\phi}}\left[\left(1-\theta\right)\beta\left(r-c_1-c_2\right)+\theta\xi\beta\left(r-c_2\right)\right]^{\frac{1}{1+\phi}}$$

and

$$\begin{aligned}\pi^s = &\left(r-c_2\right)\alpha - c_{1m}\eta\alpha - c_1\left(1-\eta\right)\alpha \\ &-\left[\phi^{-\frac{\phi}{1+\phi}} + \phi^{\frac{1}{1+\phi}}\right]c_{1m}^{\frac{\phi}{1+\phi}}\left[\eta\beta\left(r-c_{1m}-c_2\right)\right]^{\frac{1}{1+\phi}} - \left[\phi^{-\frac{\phi}{1+\phi}} + \phi^{\frac{1}{1+\phi}}\right]c_1^{\frac{\phi}{1+\phi}}\left[\left(1-\eta\right)\beta\left(r-c_1-c_2\right)\right]^{\frac{1}{1+\phi}} \\ &-\left[\phi^{-\frac{\phi}{1+\phi}} + \phi^{\frac{1}{1+\phi}}\right]c_2^{\frac{\phi}{1+\phi}}\left[\eta\beta\left(r-c_{1m}-c_2\right)+\beta\left(1-\eta\right)\left(r-c_1-c_2\right)\right]^{\frac{1}{1+\phi}}\end{aligned}$$

Note that $\frac{\partial^2(\pi^o-\pi^s)}{\partial\theta^2} < 0$. Also note that when $\theta = \eta$, we must have the $\pi^o > \pi^s$, the proof of which is as follows: Suppose the optimal solution for the offline model is $\mu_{1m}^s, \mu_{1h}^s, \mu_2^s, \lambda_m^s, \lambda_h^s$. Consider the following feasible solution for the online model: $\mu_1^\triangle, \mu_2^\triangle, \lambda_m^\triangle, \lambda_h^\triangle$, where $\mu_1^\triangle = \mu_{1h}^s$ and $\mu_2^\triangle = \mu_2^s$. Then, suppose $\eta\lambda_m^\triangle + (1-\eta)\lambda_h^\triangle \le \eta\lambda_m^s + (1-\eta)\lambda_h^s$. Then, we must have $w_2^\triangle \le w_2^s$. Then, because online customers don't wait at stage 1, we have $\lambda_m^\triangle > \lambda_m^s$. Then, $\lambda_h^\triangle < \lambda_h^s$. Then, $w_1^\triangle < w_1^s$. However, if $w_1^\triangle < w_1^s$ and $w_2^\triangle < w_2^s$, then this means that $\lambda_h^\triangle > \lambda_h^s$. We get a contradiction. Thus, we must have $\eta\lambda_m^\triangle + (1-\eta)\lambda_h^\triangle > \eta\lambda_m^s + (1-\eta)\lambda_h^s$. This implies that $\pi^\triangle > \pi^s$. Thus, we must have $\pi^o \ge \pi^\triangle > \pi^s$. The results above implies that $\exists\bar{\theta} \le \eta$ such that $\pi^o > \pi^s$ if and only if $\theta > \bar{\theta}$.

Note when $c_{1m}\eta = c_1(\eta-\theta)$ (or $\theta = \frac{(c_1-c_{1m})\eta}{c_1}$), we have $\pi^o - \pi^s > 0$ for any $\beta > 0$. This implies that $\bar{\theta} \le \frac{(c_1-c_{1m})\eta}{c_1}$. Thus, for any $\beta > 0$, if $\theta \ge \frac{(c_1-c_{1m})\eta}{c_1}$ (or $c_{1m}\eta - c_1(\eta-\theta) \ge 0$), we must have $\pi^o - \pi^s > 0$, i.e., $\bar{b} = 0$. If $c_{1m}\eta - c_1(\eta-\theta) < 0$, then $\frac{\partial\frac{\pi^o-\pi^s}{\beta^{\frac{1}{1+\phi}}}}{\partial\beta} = \frac{\partial\frac{(c_{1m}\eta-c_1(\eta-\theta))\alpha}{\beta^{\frac{1}{1+\phi}}}}{\partial\beta} > 0$. Thus, there exists $\bar{\beta} \ge 0$ such that $\pi^o - \pi^s > 0$ if and only if $\beta > \bar{\beta}$. □

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