# Lower Price or Higher Reward? Measuring The Effect of Consumers' Preferences on Reward Programs

## *Web Appendix*

Federico Rossi*

Krannert School of Management

Purdue University

## A1 Computation of the value functions

In this section we explain in detail how we computed the consumers' value functions. The infinite-horizon dynamic program presented in equation (3) in the paper can be rewritten recursively using the Bellman equation as follows:[1]

$$V(x,\epsilon) = \max_{y(x,\epsilon)} \left\{ u(x,\epsilon,y;\beta) + \delta \int V(x',\epsilon') f(dx' | x,y) g(d\epsilon' | \epsilon) \right\}, \tag{A1}$$

where $f$ and $g$ are the distributions governing the transition between observed states and between unobserved states, respectively. Since we do not observe the stochastic part of the state vector, $\epsilon$, in order to compute the value function we integrate out the shocks. Let $v(x) = \int V(x,\epsilon) g(d\epsilon)$, then we rewrite the Bellman equation as

$$v(x) = \int \max_{y(x,\epsilon)} \left\{ u(x,\epsilon,y;\beta) + \delta \sum_{x'} v(x') f(x' | x,y) \right\} g(d\epsilon). \tag{A2}$$

Given the iid Type I Extreme Value distribution of $\epsilon$, the integral over the current purchase shocks can be solved analytically, while computing the Bellman equation.

*Address: 403 W. State Street, West Lafayette, IN 47907-2056. Email: rossif@purdue.edu.

[1] To simplify notation, we drop the subindex $t$ and use $x'$ and $\epsilon'$ to denote the state variables $x$ and $\epsilon$ in the next period. We also drop the subindex $h$; each value function is specific to consumer $h$.

If the researcher could observe every state variable, from the solution of the value function we could derive an optimal deterministic rule, i.e. the policy function, which maps each state to a fuel brand choice. In the case of dynamic discrete choice models, where some state variables are not observed by the researcher, after solving the value function we instead derive the *conditional choice probability* (CCP henceforth), i.e. the probability that the consumer optimally chooses $y$ given the observed vector of state variables $x$. This probability is derived as follows: If the researcher could observe every state variable, from the solution of the value function we could derive an optimal deterministic rule, i.e. the policy function, which maps each state to a fuel brand choice. In the case of dynamic discrete choice models, where some state variables are not observed by the researcher, after solving the value function we instead derive the *conditional choice probability* (CCP henceforth), i.e. the probability that the consumer optimally chooses $y$ given the observed vector of state variables $x$. This probability is derived as follows:

$$P\left(a\left|x\right.\right)=\int I\left\{a=\underset{j\in\{0,1\}}{\operatorname{argmax}}\left[\omega\left(x,j\right)+\epsilon\left(j\right)\right]\right\}g\left(d\epsilon\right) \tag{A3}$$

where $\omega\left(x,a\right)$ is the value function relative to the fuel brand choice $a\in\{0,1\}$, i.e. $\omega\left(x,a\right)=u\left(x,a\right)+\delta\sum_{x'}\omega\left(x'\right)f\left(x'\left|x,a\right.\right)$.

To solve the function in equation (A2) we apply a policy function iteration algorithm for discrete choice models. The algorithm is based on the theorem of Hotz and Miller (1993), which shows a one-to-one mapping between value functions and choice probabilities. After an initial guess of the CCP (i.e. the policy function in the context of discrete choice models), the first step of the algorithm is in fact to reformulate the value function using the Hotz-Miller representation. In the second step, the CCP is updated using the value function computed in the first step. More specifically, according to the Hotz-Miller representation result,

$$v\left(x\right)=\sum_{a\in\{0,1\}}P\left(a\left|x\right.\right)\left\{u\left(x,a;\beta\right)+E\left[\epsilon\left|x,a\right.\right]+\delta\sum_{x'}f\left(x'\left|x,a\right.\right)v\left(x'\right)\right\} \tag{A4}$$

where $E\left[\epsilon\left|x,a\right.\right]$ is the expectation of the unobservable state variable $\epsilon$ conditional on the optimal choice $a$. Since the shocks are extreme-value distributed, this value can be written in closed form as $e-ln\left(P\left(a\left|x\right.\right)\right)$, where $e$ is the Euler's constant. The logit distribution has mean zero, so we drop the Euler's constant. The value function in (A4) can be rewritten in matrix notation as:

$$v = \left( I - \delta \sum_{a \in \{0,1\}} P(a) * F(a) \right)^{-1} \left\{ \sum_{a \in \{0,1\}} P(a) * [u(a) - ln(P(a))] \right\} \tag{A5}$$

where $*$ is the element-by-element product; $I$ is the identity matrix of size $M$ and $M$ is the total number of observable states; $v$ is a $M \times 1$ vector of value functions $v(x)$; $F(a)$ is the $M \times M$ matrix of conditional transition probabilities, $f(x' | x)$; $P(a)$ and $u(a)$ are $M \times 1$ vectors stacking the corresponding elements in all states conditional on choosing $a$. The equation in (A5) shows how to represent the value function as function of CCPs.

The algorithm proceeds as follows:

**Step 0.** Start with a guess of $P(a|x)^{(0)}$

**Step 1.** Using the representation in (A5), calculate the $M \times 1$ matrix of value functions

$$v^{n+1} = \left( I - \delta \sum_{a \in \{0,1\}} P(a) * F(a) \right)^{-1} \left\{ \sum_{a \in \{0,1\}} P(a) * [u(a) - ln(P(a))] \right\}$$

**Step 2.** Update the CCP:

$$P(a|x)^{(n+1)} = \frac{exp(\omega(x,a))}{\sum_{a \in \{0,1\}} exp(\omega(x,a))}$$

where $\omega(x,a)$ is the value function relative to the choice $a$, defined above.

If $\left\| P(a|x)^{(n+1)} - P(a|x)^{(n)} \right\|$ is smaller than some given tolerance level, then stop. Otherwise, return to Step 1.

After the algorithm stops, the value functions are the elements of the vector $v$. These functions are used to derive the expected future stream of utilities associated with each choice, defined in equation (5).

## A2 Static model

We report the estimation results of the static version of the dynamic model presented in the paper. The estimates are reported in Table A.1.

## References

Hotz, J. V. and R. A. Miller (1993). Conditional choice probabilities and the estimation of dynamic models. *The Review of Economic Studies 60*(3), 497–529.

Table A.1: Estimates of the static model

| | Model 1 | Model 2 | | Model 3 | | | Model 4 | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Seg 1 | Seg 1 | Seg 2 | Seg 1 | Seg 2 | Seg 3 | Seg 1 | Seg 2 | Seg 3 | Seg 4 |
| Intercept | -1.469** | -1.779** | -0.295** | -1.908** | -0.820** | 0.654** | -1.975** | -1.066** | -0.090 | 1.350** |
| | (0.005) | (0.008) | (0.019) | (0.013) | (0.027) | (0.054) | (0.021) | (0.051) | (0.084) | (0.119) |
| Price | -0.155** | -0.202** | -0.183** | -0.208** | -0.250** | -0.162** | -0.215** | -0.219** | -0.253** | -0.191** |
| | (0.005) | (0.007) | (0.011) | (0.009) | (0.015) | (0.019) | (0.011) | (0.024) | (0.036) | (0.025) |
| Segment Size | 1.00 | 0.81** | 0.19** | 0.68** | 0.28** | 0.04** | 0.60** | 0.32** | 0.07** | 0.01** |
| | - | (0.01) | (0.01) | (0.01) | (0.01) | (0.01) | (0.05) | (0.04) | (0.01) | (0.00) |
| Loglik. | -150,763 | -146,932 | | -146,475 | | | -146,412 | | | |
| BIC | 301,550 | 293,926 | | 293,050 | | | 292,963 | | | |

| | Model 5 | | | | | Model 6 | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Seg 1 | Seg 2 | Seg 3 | Seg 4 | Seg 5 | Seg 1 | Seg 2 | Seg 3 | Seg 4 | Seg 5 | Seg 6 |
| Intercept | -2.001** | -1.096** | -0.127** | -1.292** | 1.341** | -2.006** | -1.095** | -1.375** | -0.129 | 1.345** | -0.700** |
| | (0.019) | (0.047) | (0.068) | (0.088) | (0.107) | (0.020) | (0.046) | (0.116) | (0.069) | (0.111) | (0.197) |
| Price | -0.224** | -0.093** | -0.218** | -1.121** | -0.185** | -0.221** | -0.085* | -0.989** | -0.203** | -0.177** | -3.562** |
| | (0.013) | (0.029) | (0.033) | (0.154) | (0.026) | (0.015) | (0.036) | (0.174) | (0.035) | (0.027) | (0.934) |
| Segment Size | 0.56** | 0.28** | 0.08** | 0.07** | 0.01** | 0.55** | 0.27** | 0.08** | 0.08** | 0.01** | 0.01 |
| | (0.05) | (0.04) | (0.01) | (0.02) | (0.00) | (0.07) | (0.07) | (0.01) | (0.02) | (0.00) | (0.00) |
| Loglik. | -146,359 | | | | | -146,341 | | | | | |
| BIC | 292,896 | | | | | 292,897 | | | | | |

In this model consumers are assumed to behave myopically, i.e. while purchasing fuel they do not consider the value of the reward points earned. Each specification uses a panel of 17,860 customers, with a total of 309,178 observations. Parameters are ML estimates. Standard errors are in parenthesis.