## Appendix I: Proofs of Technical Results

**Proof of Theorem 1:** Consider any policy $\pi$ which is feasible to $\mathcal{P}^E_{static}(\vec{\beta})$. Define the vector $(\tilde{x},\tilde{y})$ as follows:

$$\tilde{x}_{s,i,l} = \frac{\sum_{t\in\mathcal{T}_{s,i}} \mathbb{E}_{\vec{U}_{t-1}}\left[x^\pi_{t,l}\big(\vec{j}^\pi_t(\vec{U}_{t-1})\big)\right]}{\Delta}, \tag{22}$$

$$\tilde{y}_{s,i,l,c} = \frac{\sum_{t\in\mathcal{T}_{s,i}} \mathbb{E}_{\vec{U}_{t-1}}\left[y^\pi_{t,l,c}\big(\vec{j}^\pi_t(\vec{U}_{t-1})\big)\right]}{\Delta}. \tag{23}$$

Our proof involves demonstrating the following claims:

(a) $(\tilde{x},\tilde{y})$ is feasible to $\hat{\mathcal{P}}(\vec{\beta})$.

(b) $\pi(\vec{\beta})$ is feasible to $\mathcal{P}^E_{static}(\vec{\beta})$.

(c) $f^\pi \geq \hat{f}(\tilde{x})$.

(d) $\hat{f}(\tilde{x}) \geq \hat{f}\big(\hat{x}^*(\vec{\beta})\big)$.

(e) $\hat{f}\big(\hat{x}^*(\vec{\beta})\big) = f^{\pi(\vec{\beta})}$.

Combining (a) – (e) we obtain the desired result that $f^{\pi(\vec{\beta})} \leq f^\pi$ for any policy $\pi$ which is feasible for $\mathcal{P}^E_{static}(\vec{\beta})$. Thus, $\pi(\vec{\beta})$ is optimal for $\mathcal{P}^E_{static}(\vec{\beta})$. It only remains to show (a) – (e).

Proof of (a): The feasibility of $(\tilde{x},\tilde{y})$ for $\hat{\mathcal{P}}(\vec{\beta})$ follows from its definition and the feasibility of policy $\pi$ for problem $\mathcal{P}^E_{static}(\vec{\beta})$.

Proof of (b): The feasibility of $\pi(\vec{\beta})$ for problem $\mathcal{P}^E_{static}(\vec{\beta})$ follows from its definition and the feasibility of $\big(\hat{x}^*(\vec{\beta}),\ \hat{y}^*(\vec{\beta})\big)$ for problem $\hat{\mathcal{P}}(\vec{\beta})$.

Proof of (c): The following two inequalities follow from Jensen's inequality:

$$\mathbb{E}_{\vec{U}_{t-1}}\left[f_{i,l}\Big(x^\pi_{t,l}\big(\vec{j}^\pi_t(\vec{U}_{t-1})\big)\Big)\right] \geq f_{i,l}\Big(\mathbb{E}_{\vec{U}_{t-1}}\left[x^\pi_{t,l}\big(\vec{j}^\pi_t(\vec{U}_{t-1})\big)\right]\Big),\ \forall\ t\in\mathcal{T}_{s,i},\ s\in\mathcal{S},\ i\in\mathcal{I},\ l\in\mathcal{L},$$

and

$$\frac{\sum\limits_{t\in\mathcal{T}_{s,i}} \mathbb{E}_{\vec{U}_{t-1}}\left[f_{i,l}\Big(x^\pi_{t,l}\big(\vec{j}^\pi_t(\vec{U}_{t-1})\big)\Big)\right]}{\Delta} \geq f_{i,l}\left(\frac{\sum_{t\in\mathcal{T}_{s,i}} \mathbb{E}_{\vec{U}_{t-1}}\left[x^\pi_{t,l}\big(\vec{j}^\pi_t(\vec{U}_{t-1})\big)\right]}{\Delta}\right) = f_{i,l}(\tilde{x}_{i,l}),\ \forall\ s\in\mathcal{S},\ i\in\mathcal{I},\ l\in\mathcal{L}.$$

Multiplying both sides by $\Delta q_{i,l}$ and summing over $l\in\mathcal{L}$, we obtain

$\sum\limits_{t\in\mathcal{T}_{s,i}}\sum\limits_{l\in\mathcal{L}} q_{i,l}\mathbb{E}_{\vec{U}_{t-1}}\left[f_{i,l}\Big(x^\pi_{t,l}\big(\vec{j}^\pi_t(\vec{U}_{t-1})\big)\Big)\right] \geq \Delta\sum\limits_{l\in\mathcal{L}} q_{i,l}f_{i,l}(\tilde{x}_{i,l})$. Summing over $s\in\mathcal{S},\ i\in\mathcal{I}$, we have

$$f^\pi \geq \hat{f}(\tilde{x}).$$

Proof of (d): This claim follows immediately from claim (a) and the optimality of $\hat{x}^*(\vec{\beta})$ for problem $\hat{\mathcal{P}}(\vec{\beta})$.

Proof of (e): Since $\pi(\vec{\beta})$ is a state independent policy, directly substituting its definition in the expression $f^{\pi}$ gives $f^{\pi(\vec{\beta})} = \hat{f}\big(\hat{x}^*(\vec{\beta})\big)$. ■

**Proof of Theorem 2:** We want to find a bound for the ratio

$$\frac{f^{\pi(\vec{\beta}^*)}}{\mathrm{Opt}(\vec{M},\alpha)}\,.$$

Recall that problem $\mathcal{P}^E_{static}(\alpha\vec{M})$ is a relaxation of problem $\mathcal{P}_{static}(\vec{M},\alpha)$ and that $\hat{\mathcal{P}}(\alpha\vec{M})$ is equivalent to $\mathcal{P}^E_{static}(\alpha\vec{M})$. Let $\pi(\alpha\vec{M}) = \big(\hat{x}^*(\alpha\vec{M}),\ \hat{y}^*(\alpha\vec{M})\big)$ be the solution of problem $\hat{\mathcal{P}}(\alpha\vec{M})$, where $\hat{x}^*(\alpha\vec{M}) = \big(\hat{x}^*_{s,i,l}(\alpha\vec{M}):\ s\in\mathcal{S},\ i\in\mathcal{I},\ l\in\mathcal{L}\big)$ and $\hat{y}^*(\alpha\vec{M}) = \big(\hat{y}^*_{s,i,l,c}(\alpha\vec{M}):\ s\in\mathcal{S},\ i\in\mathcal{I},\ l\in\mathcal{L},\ c\in\mathcal{C}\big)$. Therefore,

$$\mathrm{Opt}(\vec{M},\alpha) \ge \hat{f}\big(\hat{x}^*(\alpha\vec{M})\big) = \sum_{s\in\mathcal{S}}\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}} \Delta\ q_{i,l}\ f_{i,l}\big(\hat{x}^*_{s,i,l}(\alpha\vec{M})\big). \tag{24}$$

We can obtain a feasible solution for problem $\hat{\mathcal{P}}(\vec{M}+z_\alpha\vec{M}_r)$, where $\vec{M}_r\in\mathbb{R}^{|\mathcal{C}|}_+$ is a vector with elements $\sqrt{M_c}, c\in\mathcal{C}$, by simply multiplying the solution $\big(\hat{x}^*(\alpha\vec{M}),\ \hat{y}^*(\alpha\vec{M})\big)$ of problem $\hat{\mathcal{P}}(\alpha\vec{M})$ by the factor

$$\gamma = \max_{c\in\mathcal{C}}\left[\frac{M_c + z_\alpha\sqrt{M_c}}{\alpha M_c}\right] = \max_{c\in\mathcal{C}}\left[\frac{1}{\alpha}\left(1+\frac{z_\alpha}{\sqrt{M_c}}\right)\right].$$

Thus, $\pi^\gamma(\alpha\vec{M}) := \big(\gamma\hat{x}^*(\alpha\vec{M}),\ \gamma\hat{y}^*(\alpha\vec{M})\big)$ is a feasible solution for problem $\hat{\mathcal{P}}(\vec{M}+z_\alpha\vec{M}_r)$.

We know that the cost associated with policy $\pi^\gamma(\alpha\vec{M})$ is $f^{\pi^\gamma(\alpha\vec{M})}$. Since $\pi^\gamma(\alpha\vec{M})$ is only a feasible policy for problem $\hat{\mathcal{P}}(\vec{M}+z_\alpha\vec{M}_r)$, the cost corresponding to $\pi^\gamma(\alpha\vec{M})$, i.e., $f^{\pi^\gamma(\alpha\vec{M})}$, is higher than the optimal cost for problem $\hat{\mathcal{P}}(\vec{M}+z_\alpha\vec{M}_r)$, i.e., $f^{\pi(\vec{\beta}^*)}$. Thus, $f^{\pi(\vec{\beta}^*)} \le f^{\pi^\gamma(\alpha\vec{M})}$. We also know that $f^{\pi^\gamma(\alpha\vec{M})} = \sum_{s\in\mathcal{S}}\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\Delta\ q_{i,l}\ f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)$. Therefore,

$$f^{\pi(\vec{\beta}^*)} \le \sum_{s\in\mathcal{S}}\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\Delta\ q_{i,l}\ f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big). \tag{25}$$

From (24) and (25) we can write,

$$\frac{f^{\pi(\vec{\beta}^*)}}{\mathrm{Opt}(\vec{M},\alpha)} \le \frac{\sum_{s\in\mathcal{S}}\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\Delta\ q_{i,l}\ f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}{\sum_{s\in\mathcal{S}}\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\Delta\ q_{i,l}\ f_{i,l}\big(\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}. \tag{26}$$

Now, consider the ratio,

$$\frac{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}{f_{i,l}\big(\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}.$$

Since $\gamma \geq 1$, $\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M}) \geq \hat{x}^*_{s,i,l}(\alpha\vec{M})$. Further, we know that $f_{i,l}(\cdot)$ is convex. Thus, we can write,

$$\begin{aligned}\frac{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}{f_{i,l}\big(\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)} &\leq \frac{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big) - \big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M}) - \hat{x}^*_{s,i,l}(\alpha\vec{M})\big)f'_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)},\\ &= \frac{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big) - (\gamma-1)\hat{x}^*_{s,i,l}(\alpha\vec{M})f'_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)},\\ &= \frac{1}{1 - \frac{(\gamma-1)}{\gamma}\frac{\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})f'_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}}.\end{aligned}$$

Now, the above inequality can be rewritten as

$$\begin{aligned}\frac{f_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)}{f_{i,l}\big(\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)} &\leq \frac{1}{1 - \frac{(\gamma-1)}{\gamma}\psi_{i,l}\big(\gamma\hat{x}^*_{s,i,l}(\alpha\vec{M})\big)},\\ &\leq \frac{1}{1 - \frac{(\gamma-1)}{\gamma}\overline{\psi}}.\end{aligned} \tag{27}$$

Using (26) and (27), we obtain the desired result that

$$\frac{f^{\pi(\vec{\beta}^*)}}{\mathrm{Opt}(\vec{M},\alpha)} \leq \frac{1}{1 - \frac{(\gamma-1)}{\gamma}\overline{\psi}}.$$

This completes the proof of Theorem 2. ∎

**Proof of Proposition 1:** The simple intuition behind the proof is as follows: Consider the set of active campaigns at location $l$ at any point in time. As time progresses, some campaigns end, resulting in fewer campaigns being active in the later time blocks. Thus, a higher number of active campaigns in the earlier time blocks leads to higher target win-probabilities than those in the later time blocks.

Let $\hat{x}(\vec{\beta}) = \{\hat{x}_{s,i,l}(\vec{\beta}) :\ s \in \mathcal{S},\ i \in \mathcal{I},\ l \in \mathcal{L}\}$ and $\hat{y}(\vec{\beta}) = \{\hat{y}_{s,i,l,c}(\vec{\beta}) :\ s \in \mathcal{S},\ i \in \mathcal{I},\ l \in \mathcal{L},\ c \in \mathcal{C}\}$ be an optimal solution of problem $\hat{\mathcal{P}}(\vec{\beta})$. Consider an arbitrary time block $i \in \mathcal{I}$ and location $l \in \mathcal{L}$, and assume, for a contradiction, that there exist two consecutive time periods $d$ and $d+1$ with $\hat{x}_{d,i,l}(\vec{\beta}) < \hat{x}_{d+1,i,l}(\vec{\beta})$. That is, in time block $i$ at location $l$, the target win-probability in the $d$-th period is strictly less than the target win-probability in the $(d+1)$-th period. We will construct another feasible solution $\big(\hat{x}'(\vec{\beta}),\ \hat{y}'(\vec{\beta})\big)$ such that (i) $\hat{x}'_{d,i,l}(\vec{\beta}) = \hat{x}'_{d+1,i,l}(\vec{\beta})$, (ii) $\hat{x}'_{s,i,l}(\vec{\beta}) = \hat{x}_{s,i,l}(\vec{\beta}),\ s \in \mathcal{S} \setminus \{d, d+1\}$, (iii) $\hat{x}'_{s',i',l'}(\vec{\beta}) = \hat{x}_{s',i',l'}(\vec{\beta}),\ (s',i',l') \in \mathcal{S} \times \mathcal{I} \times \mathcal{L} \setminus \big\{(d,i,l),\ (d+1,i,l)\big\}$, and (iv) the objective function value of the solution $\big(\hat{x}'(\vec{\beta}), \hat{y}'(\vec{\beta})\big)$ is strictly less than that of the solution $\big(\hat{x}(\vec{\beta}), \hat{y}(\vec{\beta})\big)$.

Let

$$\bar{x}_{d,d+1,i,l}(\vec{\beta}) = \frac{\hat{x}_{d,i,l}(\vec{\beta}) + \hat{x}_{d+1,i,l}(\vec{\beta})}{2}\ .$$

For location $l$ and time block $i$, define

$$\hat{x}'_{d,i,l}(\vec{\beta}) = \hat{x}'_{d+1,i,l}(\vec{\beta}) = \bar{x}_{d,d+1,i,l}(\vec{\beta}), \qquad (28)$$
$$\hat{x}'_{s,i,l}(\vec{\beta}) = \hat{x}_{s,i,l}(\vec{\beta}),\ \forall\ s \in \mathcal{S} \setminus \{d,\ d+1\}.$$

For all other time periods, time blocks, and locations define

$$\hat{x}'_{s',i',l'}(\vec{\beta}) = \hat{x}_{s',i',l'}(\vec{\beta}),\ (s',i',l') \in \mathcal{S} \times \mathcal{I} \times \mathcal{L} \setminus \big\{(d,i,l),\ (d+1,i,l)\big\}.$$

We now define the allocation probabilities $\hat{y}'(\vec{\beta})$ by adjusting the original allocation probabilities $\hat{y}(\vec{\beta})$. We do this by advancing the procurment for some campaigns from time period $d+1$ to time period $d$. Let

$$c^* = \min\left\{\tilde{c} : \sum_{c=1}^{\tilde{c}} \hat{y}_{d+1,i,l,c}(\vec{\beta}) \geq \hat{x}_{d+1,i,l} - \bar{x}_{d,d+1,i,l}(\vec{\beta})\right\}.$$

For location $l$, define

$$\begin{aligned}
\hat{y}'_{d,i,l,c}(\vec{\beta}) &= \hat{y}_{d,i,l,c}(\vec{\beta}) + \hat{y}_{d+1,i,l,c}(\vec{\beta}), && \forall\ c \in \{1,2,\ldots,c^*-1\},\\
\hat{y}'_{d+1,i,l,c}(\vec{\beta}) &= 0, && \forall\ c \in \{1,2,\ldots,c^*-1\},\\
\hat{y}'_{d,i,l,c^*}(\vec{\beta}) &= \hat{y}_{d,i,l,c^*}(\vec{\beta}) + \left[\big(\hat{x}_{d+1,i,l}(\vec{\beta}) - \bar{x}_{d,d+1,i,l}(\vec{\beta})\big) - \sum_{c=1}^{c^*-1} \hat{y}_{d+1,i,l,c}(\vec{\beta})\right],\\
\hat{y}'_{d+1,i,l,c^*}(\vec{\beta}) &= \hat{y}_{d+1,i,l,c^*}(\vec{\beta}) - \left[\big(\hat{x}_{d+1,i,l}(\vec{\beta}) - \bar{x}_{d,d+1,i,l}(\vec{\beta})\big) - \sum_{c=1}^{c^*-1} \hat{y}_{d+1,i,l,c}(\vec{\beta})\right],\\
\hat{y}'_{d,i,l,c}(\vec{\beta}) &= \hat{y}_{d,i,l,c}(\vec{\beta}) && \forall\ c \in \{c^*+1, c^*+2, \ldots, |\mathcal{C}|\},\\
\hat{y}'_{d+1,i,l,c}(\vec{\beta}) &= \hat{y}_{d+1,i,l,c}(\vec{\beta}) && \forall\ c \in \{c^*+1, c^*+2, \ldots, |\mathcal{C}|\}.
\end{aligned}$$

For all other time periods, time blocks, and locations define

$$\hat{y}'_{s',i',l',c}(\vec{\beta}) = \hat{y}_{s',i',l',c}(\vec{\beta}),\ (s',i',l') \in \mathcal{S} \times \mathcal{I} \times \mathcal{L} \setminus \big\{(d,i,l),\ (d+1,i,l)\big\},\ c \in \mathcal{C}.$$

It is easy to verify that $\big(\hat{x}'(\vec{\beta}),\ \hat{y}'(\vec{\beta})\big)$ is a feasible solution of problem $\hat{\mathcal{P}}(\vec{\beta})$. The fact that the cost of $\big(\hat{x}'(\vec{\beta}),\ \hat{y}'(\vec{\beta})\big)$ is strictly less than that of $\big(\hat{x}(\vec{\beta}),\ \hat{y}(\vec{\beta})\big)$ is a direct consequence of the strict convexity of $f_{i,l}(\cdot)$. This completes the proof. ■

**Proof of Proposition 2:** Define $\hat{x}^*(\vec{\beta}) = \{\hat{x}^*_{s,i,l}(\vec{\beta}) :\ s \in \mathcal{S},\ i \in \mathcal{I},\ l \in \mathcal{L}\}$ and $\hat{y}^*(\vec{\beta}) = \{\hat{y}^*_{s,i,l,c}(\vec{\beta}) :\ s \in \mathcal{S},\ i \in \mathcal{I},\ l \in \mathcal{L},\ c \in \mathcal{C}\}$. Let $\big(\hat{x}^*(\vec{\beta}),\ \hat{y}^*(\vec{\beta})\big)$ be an optimal solution of problem $\hat{\mathcal{P}}(\vec{\beta})$. We want to prove that the optimal win-probability in location $l$ during time block $i$ of time period $s$ decreases with an increase in the impression-arrival probability $q_{i,l}$. That is, we want to show that $\hat{x}^*_{s,i,l}(\vec{\beta})$ decreases with $q_{i,l}$. Without loss of generality, we prove this for $\hat{x}^*_{1,1,1}(\vec{\beta})$, i.e., we prove that $\hat{x}^*_{1,1,1}(\vec{\beta})$ decreases with an increase in $q_{1,1}$. Define $\mathcal{B}_{-1} = \mathcal{S} \times \mathcal{I} \times \mathcal{L} \setminus (1,1,1)$. Let $\hat{x}_{-1}(\vec{\beta}) = \big\{\hat{x}_{s,i,l}(\vec{\beta}) :\ (s,i,l) \in \mathcal{B}_{-1}\big\}$. We now define a new problem in which for a given value of win-probability in location $l=1$ for time block $i=1$ and time period $s=1$, we find all other win-probabilities. For a given value of $\hat{x}_{1,1,1}$, we define $z = \Delta q_{1,1}\hat{x}_{1,1,1}$.

Thus, we solve the following optimization problem to obtain the value of objective function at the optimal values of the other win-probabilities, given $\hat{x}_{1,1,1}$:

$$\hat{\mathcal{P}}_{-1}(\vec{\beta}) : \begin{cases} \mathcal{F}(z) = \min\limits_{\hat{x}_{-1},\hat{y}} \sum\limits_{(s,i,l)\in\mathcal{B}_{-1}} \Delta\ q_{i,l}\ f_{i,l}(\hat{x}_{s,i,l}) \\ \text{subject to:} \\ \sum\limits_{c\in\mathcal{C}} \hat{y}_{s,i,l,c}\ =\ \hat{x}_{s,i,l},\ \forall\ (s,i,l)\in\mathcal{B}_{-1}, & (29) \\ \Delta q_{1,1} \sum\limits_{c\in\mathcal{C}} \hat{y}_{1,1,1,c}\ =\ z, \\ \sum\limits_{(s,i,l)\in\mathcal{B}_{-1}} \Delta\ q_{i,l}\hat{y}_{s,i,l,c} + \Delta q_{1,1}\hat{y}_{1,1,1,c} \geq \beta_c,\ \forall\ c\in\mathcal{C},\ K_c > 1, & (30) \\ \hat{y}_{s,i,l,c} = 0,\ \forall\ c\in\mathcal{C},\ l\in\mathcal{L}_c,\ i\in\mathcal{I},\ s > K_c, & (31) \\ \hat{x}_{s,i,l} \in [0,1],\ \forall\ (s,i,l)\in\mathcal{B}_{-1}, \\ \hat{y}_{s,i,l,c} \in [0,1],\ \forall\ (s,i,l)\in\mathcal{S}\times\mathcal{I}\times\mathcal{L},\ c\in\mathcal{C}. \end{cases}$$

Since $f_{i,l}(\cdot)$ is a convex function and minimization preserves convexity, function $\mathcal{F}(z)$ is a convex function of $z$. For a given value of the $z$, we can obtain the value of $\mathcal{F}(z)$ by solving the above optimization problem. Thus, we can find the optimal value of $\hat{x}_{1,1,1}$ by solving the following problem:

$$\min_{0\leq \hat{x}_{1,1,1}\leq 1} \quad \Delta q_{1,1} f_{1,1}(\hat{x}_{1,1,1}) + \mathcal{F}(\Delta q_{1,1}\hat{x}_{1,1,1}).$$

Let $\hat{x}^*_{1,1,1}(\vec{\beta})$ be an optimal solution of the above problem. Then, the first-order condition for the above optimization problem results in

$$f'_{1,1}\big(\hat{x}^*_{1,1,1}(\vec{\beta})\big) + \mathcal{F}'\big(\Delta q_{1,1}\hat{x}^*_{1,1,1}(\vec{\beta})\big) = 0. \tag{32}$$

Since $f_{1,1}(\cdot)$ and $\mathcal{F}(\cdot)$ are convex functions, $f'_{1,1}(\cdot)$ and $\mathcal{F}'(\cdot)$ are increasing functions. It now immediately follows that $\hat{x}^*_{1,1,1}(\vec{\beta})$ decreases with an increase in $q_{1,1}$. ■

**Proof of Corollary 3:** We want to find a bound for the ratio

$$\frac{\mathbb{E}[G(\bar{\mathbf{M}})]}{G(\alpha\mathbf{M})}\ . \tag{33}$$

Recall that $G(\bar{\mathbf{M}})$ is the objective function of problem $\mathcal{P}_1(\bar{\mathbf{M}})$ and $G(\alpha\mathbf{M})$ is the objective function of problem $\mathcal{P}_1(\alpha\mathbf{M})$. Let $\pi(\alpha\mathbf{M}) = \big(\hat{x}^*(\alpha\mathbf{M}),\ \hat{y}^*(\alpha\mathbf{M})\big)$ be the solution of problem $\mathcal{P}_1(\alpha\mathbf{M})$, where $\hat{x}^*(\alpha\mathbf{M}) = \big(\hat{x}^*_{i,l}(\alpha\mathbf{M}):\ i\in\mathcal{I},\ l\in\mathcal{L}\big)$ and $\hat{y}^*(\alpha\mathbf{M}) = \big(\hat{y}^*_{i,l,c}(\alpha\mathbf{M}):\ i\in\mathcal{I},\ l\in\mathcal{L},\ c\in\mathcal{C}\big)$.

We can obtain a feasible solution for problem $\mathcal{P}_1(\bar{\mathbf{M}})$ by simply multiplying the solution $\big(\hat{x}^*(\alpha\mathbf{M}),\ \hat{y}^*(\alpha\mathbf{M})\big)$ of problem $\mathcal{P}_1(\alpha\mathbf{M})$ by a factor

$$\gamma^r = \max_{c\in\mathcal{C}} \left[\frac{\bar{M}_c}{\alpha M_c}\right],$$

where $r$ stands for our rolling-horizon policy. Since $\bar{M}_c$ is a random variable, $\gamma^r$ is also a random variable. Therefore, we have

$$\frac{\mathbb{E}[G(\bar{\mathbf{M}})]}{G(\alpha\mathbf{M})} \le \frac{\Delta\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\mathbb{E}_{\gamma^r}\Big[q_{i,l}f_{i,l}\big(\gamma^r\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)\Big]}{\Delta\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\Big[q_{i,l}f_{i,l}\big(\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)\Big]}\ . \tag{34}$$

We know that

$$\bar{M}_{s,c} = \sum_{\hat{s}=s-K+1}^{s}\frac{\tilde{M}_{\hat{s},c}}{K} + z_\alpha\sqrt{\frac{\tilde{M}_{\hat{s},c}}{K}}.$$

We know that the support of the random variable $\tilde{M}_{\hat{s},c}$ is $[0,\ M_c^{max}]$. Thus, we have $\bar{M}_{s,c} \ \le\ M_c^{max} + z_\alpha\sqrt{KM_c^{max}}$ .

We now define

$$\gamma^r_{max} = \max_{c\in\mathcal{C}}\left[\frac{M_c^{max} + z_\alpha\sqrt{KM_c^{max}}}{\alpha M_c}\right].$$

It is easy to see that $\gamma^r \le \gamma^r_{max}$ with probability 1. Thus, from (34) we have

$$\frac{\mathbb{E}[G(\bar{\mathbf{M}})]}{G(\alpha\mathbf{M})} \le \frac{\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\Big[q_{i,l}f_{i,l}\big(\gamma^r_{max}\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)\Big]}{\sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\Big[q_{i,l}f_{i,l}\big(\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)\Big]}\ . \tag{35}$$

Now, consider the ratio,

$$\frac{f_{i,l}\big(\gamma^r_{max}\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)}{f_{i,l}\big(\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)}.$$

By following an analysis identical to that in the proof of Theorem 2, we obtain

$$\frac{f_{i,l}\big(\gamma^r_{max}\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)}{f_{i,l}\big(\hat{x}^*_{i,l}(\alpha\mathbf{M})\big)} \le \frac{1}{1-\frac{(\gamma^r_{max}-1)}{\gamma^r_{max}}\overline{\psi}_r}. \tag{36}$$

Using (33) and (36), we obtain the desired result that

$$\frac{\mathbb{E}[G(\bar{\mathbf{M}})]}{G(\alpha\mathbf{M})} \le \frac{1}{1-\frac{(\gamma^r_{max}-1)}{\gamma^r_{max}}\overline{\psi}_r}.$$

This completes the proof of Corollary 3. ■

## Appendix II: Details of the Regression for Estimating the Win-Curves Using Real Data (Section 4.2)

Here we provide the details of our logistic regression to estimate the win-curves. A few fields in our dataset are: (a) winning status (binary variable taking value 1 when the bid is won, 0 otherwise), (b) click status (binary variable taking value 1 if the impression is clicked, 0 otherwise), (c) bid amount, (d) zip-code, (e) device maker (Apple, Samsung, etc.), (f) operating system (iOS, android, etc.). In the data set, we have 52,460 records corresponding to 5 different zip-codes in the Boston area. We ran five different regressions, corresponding to each of the five zip-codes, to estimate the win-curves in the respective locations. The bid amount is the independent variable and the winning status is the dependent variable. After estimating the regression coefficients, we also applied the chi-square test to check the goodness-of-fit of the regression model. For location $l \in \{1,\ 2,\ 3,\ 4,\ 5\}$, (locations 1, 2, 3, 4, and 5, corresponding to zip-codes 02110, 02114, 02116, 02118, and 02119, respectively) we estimate the win-probability for a bid $b$ as $x_l(b) = \frac{e^{\beta_0^l + \beta_1^l b}}{1+e^{\beta_0^l + \beta_1^l b}}$. Then, we obtain the expected-cost function for location $l$ using the relation $f_l(x) = x_l b_l(x)$, where $b_l(\cdot)$ is the inverse function of $x_l(\cdot)$. The estimated function $f_l(\cdot)$ is convex at each location, as is assumed in our theoretical analysis (see Figure 5). The results of the regression and the goodness-of-fit test for each zip-code are in Table 1.

**Table 1 Results of the Regression**

| Zip Code | Number of Records | Regression Constant ($\beta_0^l$) | Regression Coefficient for Bid ($\beta_1^l$) | Chi-Square Test Statistic |
|---|---|---|---|---|
| 02110 | 1086 | $-2.281^{***}$ (std. error: 0.2141) | $0.7051^{***}$ (std. error: 0.1290) | $31.23302^{***}$ |
| 02114 | 27693 | $-2.291^{***}$ (std. error: 0.03693) | $1.04294^{***}$ (std. error: 0.02224) | $2415.644^{***}$ |
| 02116 | 6794 | $-1.905^{***}$ (std. error: 0.07184) | $0.87651^{***}$ (std. error: 0.04388) | $426.4403^{***}$ |
| 02118 | 2833 | $-1.936^{***}$ (std. error: 0.11432) | $0.76762^{***}$ (std. error: 0.06958) | $128.1244^{***}$ |
| 02119 | 14054 | $-2.193^{***}$ (std. error: 0.05212) | $0.97615^{***}$ (std. error: 0.03133) | $1053.156^{***}$ |

Significance codes: '***' $p < 0.001$, '**' $p < 0.01$, '*' $p < 0.05$, '.' $p < 0.1$. The chi-square test with p-value of less than 0.001 tells us that our model as a whole fits significantly better than an empty model.