Online Supplement to "User-Generated Content and Competing Firms' Product Design"

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Derivation of the Non-Deviation Conditions for and Uniqueness of Equilibrium within the Class of Affine Strategies

We show the equilibrium we derive is unique within the class of affine strategies in next few steps.

(a) Location Choice with Known m: We first establish that for any known m, the firms choose their locations (x_1, x_2) such that $\frac{x_1+x_2}{2} - \frac{3}{2} \leq m \leq \frac{x_1+x_2}{2} + \frac{3}{2}$.

For any location choice (x_1, x_2) and m, a unique point ζ exists where the consumer is indifferent between buying from either firm; that is $U_1(\zeta) = U_2(\zeta)$, which leads to $\zeta = \frac{p_2 - p_1 + t(x_2^2 - x_1^2)}{2t(x_2 - x_1)}$. When $\zeta \in \left[m - \frac{1}{2}, m + \frac{1}{2}\right]$, both firms obtain positive demands, and the demand functions can be formulated as in Equation (1). If ζ is out of this interval, the firm farther away from m obtains zero demand.

We first can rule out the possibility that $x_1 = x_2$ in equilibrium, because the case with $x_1 = x_2$ reduces to the Bertand competition and both firms end up with zero profit. Without loss of generality, we assume $x_1 < x_2$. Assuming $\zeta \in \left[m - \frac{1}{2}, m + \frac{1}{2}\right]$, we can derive firms' equilibrium prices and profits as follows:

$$\begin{aligned} p_1 &= \frac{2t}{3}(x_2 - x_1)\left(\frac{x_1+x_2}{2} + \frac{3}{2} - m\right) + c \\ p_2 &= \frac{2t}{3}(x_1 - x_2)\left(\frac{x_1+x_2}{2} - \frac{3}{2} - m\right) + c \end{aligned} \quad (1)$$

and

$$\begin{aligned} \pi_1 &= \frac{2t}{9}(x_2 - x_1)\left(\frac{x_1+x_2}{2} + \frac{3}{2} - m\right)^2 - f \\ \pi_2 &= \frac{2t}{9}(x_2 - x_1)\left(\frac{x_1+x_2}{2} - \frac{3}{2} - m\right)^2 - f \end{aligned} \quad (2)$$

We can conclude that in equilibrium the location choice must satisfy $\frac{x_1+x_2}{2} - \frac{3}{2} \leq m \leq \frac{x_1+x_2}{2} + \frac{3}{2}$.

This is because if these conditions are not satisfied, one firm must be farther away from m than its competitor, by noticing that $\frac{x_1+x_2}{2} = m$ if the two firms have equal distance to m. Without loss of generality, we assume firm 1 is closer. Under this set of locations, the equilibrium prices of the pricing sub-game is that firm 2 charges price c, and firm 1 charges a price such that it gains all the demand given firm 2 sets a price of c; that is, $p_1 = 2t(x_2 - x_1)\left[\frac{x_2-x_1}{2} - (m + \frac{1}{2})\right] + c$ and $p_2 = c$. In this way, firm 2 cannot benefit by increasing price because increasing price does not increase demand, and it has no incentive to reduce its price either because the price is already at the marginal production cost. As a result, firm 2 obtains zero demand and zero profit. Therefore, firm 2 has incentive to stay not too far away from m such that $\frac{x_1+x_2}{2} - \frac{3}{2} \leq m \leq \frac{x_1+x_2}{2} + \frac{3}{2}$ and it derives positive profit as in Equation (2).

For this reason, for any given x_1 and x_2, we call $\left[\frac{x_1+x_2}{2} - \frac{3}{2}, \frac{x_1+x_2}{2} + \frac{3}{2}\right]$ the competitive interval.

(b) Location Choice with Unknown m: We can show that with unknown m, if the uncertainty over the location of the consumer distribution is relatively small, firms have no incentive to stay far away from the consumer distribution such that they may gain zero demand under some realization. In particular, firms have no incentive to stay far away if

$$\sigma_p \leq \min\left\{\frac{1}{4k}, \frac{1}{4k\left(1-\rho_p+\sqrt{1-\rho_p^2}\right)}\right\} \tag{3}$$

We take firm 2 as an example. We distinguish four cases when firm 2 stays far away from m (from the right-hand side of firm 1): (1) Firm 2 stays a little far away from the consumer distribution such that for some firm 1's realized locations firm 2 always sees positive demand, but for other 1's realized location firm 2 may see zero demand; (2) Firm 2 stays far away from the consumer distribution such that regardless of firm 1's realized location firm 2 may see either zero demand or a positive demand; (3) Firm 2 stays very far away from the consumer distribution such that for some realized locations firm 2 always sees zero demand and for other realized locations, firm 2 may gain zero or positive demand; (4) Firm 2 stays very far away from the consumer distribution such that regardless of firm 1's realized location, firm 2 always sees zero demand. Obviously, Case (4) leads to zero profit and is never as profitable as the equilibrium profit. We next show that firm 2 has no incentive to deviate to the other three cases when σ_p is small.

For a given signal s_2, firm 2 believes m is distributed over interval $[s_2 - k\sigma_p, s_2 + k\sigma_p]$. Given the correlation between s_1 and s_2, we can verify that s_1 is believed to be distributed over interval $\left[s_2 - k\sigma_p\left(1-\rho_p+\sqrt{1-\rho_p^2}\right), s_2 + k\sigma_p\left(1-\rho_p+\sqrt{1-\rho_p^2}\right)\right]$. Notice that when k is large, s_1 and m are in their respective interval with probability 1 approximately. To simplify the notation, we denote $\underline{m} = s_2 - k\sigma$, $\bar{m} = s_2 + k\sigma$, $\underline{s} = s_2 - k\sigma_p\left(1-\rho_p+\sqrt{1-\rho_p^2}\right)$ and $\bar{s} = s_2 + k\sigma_p\left(1-\rho_p+\sqrt{1-\rho_p^2}\right)$, so $[\underline{m}, \bar{m}]$ and $[\underline{s}, \bar{s}]$ represent the intervals for m and s_1, respectively. If the condition in Inequality (3) is satisfied, we have $\bar{s}-\underline{s} \le \frac{1}{2}$ and $\bar{m}-\underline{m} \le \frac{1}{2}$. We consider firm 1 use an affine strategy: $x_1(s_1) = s_1 + b_1$. Notice that $x_1(s_1)$ is increasing in s_1.

Case (1): In this case, there exists some threshold s_1', $s_1' \in [\underline{s}, \bar{s}]$, such that $\frac{x_1(s_1')+x_2}{2} - \frac{3}{2} = \underline{m}$. When $\frac{x_1(s_1)+x_2}{2} - \frac{3}{2} > \underline{m}$ or when $x_1(s_1) > 2\underline{m} + 3 - x_2$, firm 2 may gain zero demand. In this case, firm 2's profit function can be formulated as

$$\begin{aligned}\mathbb{E}(\pi_2^{dev}) &= \int_{\underline{s}}^{x_1^{-1}(2\underline{m}+3-x_2)} \int_{\underline{m}}^{\bar{m}} \frac{2t}{9}\left(x_2 - x_1(s)\right)\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s \\ &+ \int_{x_1^{-1}(2\underline{m}+3-x_2)}^{\bar{s}} \int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \frac{2t}{9}\left(x_2 - x_1(s)\right)\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s - f\end{aligned}$$

By Leibnitz's rule, we have $\frac{\partial \mathbb{E}(\pi_2^{dev})}{\partial x_2}$ as follows:

$$\begin{aligned}&\int_{\underline{m}}^{\bar{m}} \frac{2t}{9}\left(x_2 - (2\underline{m}+3-x_2)\right)\left(\frac{(2\underline{m}+3-x_2)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m, x_1^{-1}(2\underline{m}+3-x_2))\mathrm{d}m \cdot \frac{dx_1^{-1}(2\underline{m}+3-x_2)}{dx_2} \\ +\;& \int_{\underline{s}}^{x_1^{-1}(2\underline{m}+3-x_2)} \int_{\underline{m}}^{\bar{m}} \frac{2t}{9}\left(x_2 - x_1(s)\right)\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right) + \frac{2t}{9}\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s \\ -\;& \int_{\underline{m}}^{\bar{m}} \frac{2t}{9}\left(x_2 - (2\underline{m}+3-x_2)\right)\left(\frac{(2\underline{m}+3-x_2)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m, x_1^{-1}(2\underline{m}+3-x_2))\mathrm{d}m\mathrm{d}s \cdot \frac{dx_1^{-1}(2\underline{m}+3-x_2)}{dx_2} \\ -\;& \int_{x_1^{-1}(2\underline{m}+3-x_2)}^{\bar{s}} \frac{2t}{9}\left(x_2 - x_1(s)\right)\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - (\frac{x_1(s)+x_2}{2} - \frac{3}{2})\right)^2 f(\frac{x_1(s)+x_2}{2} - \frac{3}{2}, s)\mathrm{d}s \cdot \frac{1}{2} \\ +\;& \int_{x_1^{-1}(2\underline{m}+3-x_2)}^{\bar{s}} \int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \frac{2t}{9}\left(x_2 - x_1(s)\right)\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right) + \frac{2t}{9}\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s\end{aligned}$$

Noticing the first and third terms cancel out and the fourth term is zero, we can simplify $\frac{\partial \mathbb{E}(\pi_2^{dev})}{\partial x_2}$ as follows:

$$\begin{aligned}&\int_{\underline{s}}^{x_1^{-1}(2\underline{m}+3-x_2)} \int_{\underline{m}}^{\bar{m}} \frac{2t}{9}\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)\left(x_2 - x_1(s) + \left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)\right) f(m,s)\mathrm{d}m\mathrm{d}s \\ +&\int_{x_1^{-1}(2\underline{m}+3-x_2)}^{\bar{s}} \int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \frac{2t}{9}\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)\left(x_2 - x_1(s) + \left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)\right) f(m,s)\mathrm{d}m\mathrm{d}s\end{aligned} \tag{4}$$

When $x_2 - x_1(s) \ge 1$ for all s (i.e., when $x_2 - x_1(\bar{s}) \ge 1$), for any s and m within their intervals, we have

$$\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m \ge \frac{x_1(\bar{s})-(\bar{s}-\underline{s})+x_2}{2} - \frac{3}{2} - m \ge \underline{m} - \frac{1}{4} - m > -1$$

where the second inequality is because $\frac{x_1(\bar{s})+x_2}{2} - \frac{3}{2} \ge \underline{m}$ and $\bar{s} - \underline{s} \le \frac{1}{2}$ and the last inequality is because $\bar{m} - \underline{m} \leq \frac{1}{2}$. As a result, the integrands in the two integrals in Equation (4) are negative and the whole term in Equation (4) is negative. Therefore, if $\bar{s} - \underline{s} \leq \frac{1}{2}$ and $\bar{m} - \underline{m} \leq \frac{1}{2}$, when $x_2 - x_1 \geq 1$, $\frac{\partial \mathbb{E}(\pi_2^{dev})}{\partial x_2} < 0$ and firm 2 has incentive to move toward the left until it always obtains positive demand regardless of realized location.

When $x_2 - x_1(s) < 1$ for some s (i.e., when $x_2 - x_1(\bar{s}) < 1$), firm 2 can increase its profit by moving to the left of firm 1 and setting $\tilde{x}_2$ at a distance of 1 from $x_1(\underline{s})$; that is, $\tilde{x}_2 = x_1(\underline{s}) - 1$. This is because, by doing so,

$$\frac{x_1(s)+\tilde{x}_2}{2} - \frac{3}{2} \leq \frac{x_1(s)+x_2}{2} - \frac{3}{2} - \frac{1}{2} < \underline{m}$$

Meanwhile, we can verify that $\frac{x_1(s)+\tilde{x}_2}{2} + \frac{3}{2} > \bar{m}$ because

$$\frac{x_1(s)+\tilde{x}_2}{2} + \frac{3}{2} > \frac{x_1(s)+x_2}{2} + \frac{3}{2} - \frac{5}{4} > \frac{x_1(\underline{s})+x_2}{2} + \frac{1}{4} > \frac{x_1(\bar{s})+x_2}{2} > \underline{m} + \frac{3}{2} > \bar{m}$$

Therefore, firm 2's profit on the right side of firm 1 is at most

$$\begin{aligned} \mathbb{E}(\pi_2^{dev}) &= \int_{\underline{s}}^{x_1^{-1}(2\underline{m}+3-x_2)} \int_{\underline{m}}^{\bar{m}} \frac{2t}{9}(1+\tfrac{1}{2}) \left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s \\ &+ \int_{x_1^{-1}(2\underline{m}+3-x_2)}^{\bar{s}} \int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \frac{2t}{9}(1+\tfrac{1}{2}) \left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s - f \end{aligned} \tag{5}$$

and firm 2's profit on the left side of firm 1 is at least

$$\int_{\underline{s}}^{\bar{s}} \int_{\underline{m}}^{\bar{m}} \frac{2t}{9} \left(\frac{x_1(s)+x_2-\frac{5}{2}}{2} + \frac{3}{2} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s = \int_{\underline{s}}^{\bar{s}} \int_{\underline{m}}^{\bar{m}} \frac{2t}{9} \left(\frac{x_1(s)+x_2}{2} + \frac{1}{4} - m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s \tag{6}$$

Notice that for some s_1, we have $\frac{x_1(s_1)+x_2}{2} - \frac{3}{2} > \underline{m}$. We can conclude that, for all s_1, $\frac{x_1(s_1)+x_2}{2} > \underline{s} + \frac{3}{2} - \frac{1}{4} = \underline{s} + \frac{5}{4}$ if $\bar{s} - \underline{s} \leq \frac{1}{2}$. Therefore, for any s_1 and m, we have

$$\frac{3}{2}\left(\frac{x_1(s)+x_2}{2} - \frac{3}{2} - m\right)^2 \leq \frac{3}{2}\left(\bar{m} + \frac{3}{2} - \frac{3}{2} - m\right)^2 < 1 < \left(\frac{x_1(s)+x_2}{2} + \frac{1}{4} - m\right)^2$$

Therefore, the profit in Equation (6) is greater than that in Equation (5), and when $x_2 - x_1(s) < 1$ for some s, firm 2 can increase its profit by moving to the left of firm 1.

All together, we conclude that Case (1) cannot appear in equilibrium, if the uncertainty over the location of the consumer distribution is not too large.

Case (2): In this case, for all s_1, $s_1 \in [\underline{s}, \bar{s}]$, we have $\frac{x_1(s_1)+x_2}{2} - \frac{3}{2} \geq \underline{m}$. Notice when $m \leq$

$\frac{x_1(s_1)+x_2}{2}-\frac{3}{2}$, firm 2 sees zero demand. In this case, firm 2's profit function can be formulated as

$$\mathbb{E}(\pi_2^{dev}) = \int_{\underline{s}}^{\bar{s}} \int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \frac{2t}{9}\left(x_2 - x_1(s)\right)\left(\frac{x_1(s)+x_2}{2}-\frac{3}{2}-m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s - f$$

By Leibnitz's rule, we have $\frac{\partial\mathbb{E}(\pi_2^{dev})}{\partial x_2}$ as follows:

$$-\int_{\underline{s}}^{\bar{s}} \tfrac{2t}{9}\left(x_2-x_1(s)\right)\left(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2}-(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2})\right)^2 f(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2},s)\mathrm{d}s\cdot\tfrac{1}{2}$$

$$+\int_{\underline{s}}^{\bar{s}}\int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \tfrac{2t}{9}\left(x_2-x_1(s)\right)\left(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2}-m\right)+\tfrac{2t}{9}\left(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2}-m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s$$

Similar to Case (1), the first term is zero. Again, we can show that second term is negative when $x_2 - x_1(s) \geq 1$ for all s; When $x_2 - x_1(s) < 1$ for some s, firm 2 can increase its profit by moving to the left of firm 1. Therefore, we conclude that Case (2) cannot appear in equilibrium.

Case (3): In this case, there exists some threshold s_1'', $s_1'' \in [\underline{s},\bar{s}]$, such that $\frac{x_1(s_1'')+x_2}{2}-\frac{3}{2}\geq\bar{m}$. Notice when $\frac{x_1(s_1)+x_2}{2}-\frac{3}{2}>\bar{m}$ or when $x_1(s_1)>2\bar{m}+3-x_2$, firm 2 always sees zero demand. In this case, firm 2's profit function can be formulated as

$$\mathbb{E}(\pi_2^{dev}) = \int_{\underline{s}}^{x_1^{-1}(2\bar{m}+3-x_2)} \int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \frac{2t}{9}\left(x_2-x_1(s)\right)\left(\frac{x_1(s)+x_2}{2}-\frac{3}{2}-m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s - f$$

By Leibnitz's rule, we have $\frac{\partial\mathbb{E}(\pi_2^{dev})}{\partial x_2}$ as follows:

$$\int_{\frac{2\bar{m}+3-x_2+x_2}{2}-\frac{3}{2}}^{\bar{m}} \tfrac{2t}{9}\left(x_2-(2\bar{m}+3-x_2)\right)\left(\tfrac{(2\bar{m}+3-x_2)+x_2}{2}-\tfrac{3}{2}-m\right)^2 f(m,x_1^{-1}(2\bar{m}+3-x_2))\mathrm{d}m\cdot\tfrac{dx_1^{-1}(2\bar{m}+3-x_2)}{dx_2}$$
$$-\int_{\underline{s}}^{x_1^{-1}(2\bar{m}+3-x_2)} \tfrac{2t}{9}\left(x_2-x_1(s)\right)\left(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2}-(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2})\right)^2 f(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2},s)\mathrm{d}s\cdot\tfrac{1}{2}$$
$$+\int_{\underline{s}}^{x_1^{-1}(2\bar{m}+3-x_2)}\int_{\frac{x_1(s)+x_2}{2}-\frac{3}{2}}^{\bar{m}} \tfrac{2t}{9}\left(x_2-x_1(s)\right)\left(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2}-m\right)+\tfrac{2t}{9}\left(\tfrac{x_1(s)+x_2}{2}-\tfrac{3}{2}-m\right)^2 f(m,s)\mathrm{d}m\mathrm{d}s$$

Notice that the first term is zero because $\frac{2\bar{m}+3-x_2+x_2}{2}-\frac{3}{2}=\bar{m}$. Similar to Case (1), the second term is zero. Again, we can show that second term is negative when $x_2-x_1(s)\geq 1$ for all s; When $x_2-x_1(s)<1$ for some s, firm 2 can increase its profit by moving to the left of firm 1. Therefor, we conclude that Case (3) cannot appear in equilibrium.

(c) Boundary Location Choice: Based on the results in part (b), we conclude that, given observed signal s_2, the boundary location that firm 2 may choose is $x_2(s_2)$ such that $\frac{x_1(\bar{s}|s_2)+x_2(s_2)}{2}-\frac{3}{2}=\underline{m}$, which means $x_2(s_2)=3+2\underline{m}-x_1(\bar{s}|s_2)$. In other words, in equilibrium, Firm 2 always chooses

$x_2(s_2) \leq 3 + 2\underline{m} - x_1(\bar{s}|s_2)$.

Notice that $\underline{m} = (s_2 - k\sigma_p)$ and firm 1's equilibrium strategy is $x_1^*(s) = s - b$, where

$$b = \tfrac{1}{12}\left[9 + 2\sigma_p^2(1+\rho_p)\right]$$

We have $x_1^*(\bar{s}|s_2) = s_2 + k\sigma_p\left(1 - \rho_p + \sqrt{1-\rho_p^2}\right) - b$. Therefore, when firm 1 uses equilibrium strategy,

$$\begin{aligned} x_2(s_2) \leq 3 + 2\underline{m} - x_1(\bar{s}|s_2) \quad &= 3 + 2(s_2 - k\sigma_p) - (s_2 + k\sigma_p\left(1 - \rho_p + \sqrt{1-\rho_p^2}\right) - b) \\ &= 3 + s_2 - k\sigma_p\left[3 - \rho_p + \sqrt{1-\rho_p^2}\right] + b \end{aligned}$$

We define

$$\bar{x}_2(s_2) \equiv 3 + s_2 - k\sigma_p\left[3 - \rho_p + \sqrt{1-\rho_p^2}\right] + b = 3 + s_2 - 3k\sigma_p\gamma + b$$

where $\gamma = \frac{3-\rho_p+\sqrt{1-\rho_p^2}}{3}$.

(d) Optimal Location Choice: We verify that firm 2 cannot be better off by deviating to the boundary location. In particular, given any signal s_2, noticing that $x_2^*(s_2) = s_2 + b$, we have firm 2's expected equilibrium profit as follows:

$$\begin{aligned} \mathbb{E}(\pi_2|s_2, x_2^*(s_2)) \quad &= \tfrac{t}{18}\left[x_2^*(s_2) - (s_2 - b)\right]\left[\left[3 + (s_2 - x_2^*(s_2) + b)\right]^2 + 2\sigma_p^2\left(1+\rho_p\right)\right] - f \\ &= \tfrac{t}{18}(2b)\left[9 + 2\sigma_p^2\left(1+\rho_p\right)\right] - f \\ &= \tfrac{t}{18}(2b)(12b) - f \\ &= \tfrac{4t}{3}b^2 - f \end{aligned}$$

When firm 2 chooses the boundary location $\bar{x}_2(s_2) = 3 + s_2 - 3k\sigma_p\gamma + b$, we can derive its expected profit as

$$\begin{aligned} \mathbb{E}(\pi_2|s_2, \bar{x}_2(s_2)) \quad &= \tfrac{t}{18}\left[\bar{x}_2(s_2) - (s_2 - b)\right]\left[\left[3 + (s_2 - \bar{x}_2(s_2) + b)\right]^2 + 2\sigma_p^2\left(1+\rho_p\right)\right] - f \\ &= \tfrac{t}{18}\left(3 - 3k\sigma_p\gamma + 2b\right)\left[\left[3 + b - (3 - 3k\sigma_p\gamma + b)\right]^2 + 2\sigma_p^2\left(1+\rho_p\right)\right] - f \\ &= \tfrac{t}{18}\left(3 - 3k\sigma_p\gamma + 2b\right)\left[(3k\sigma_p\gamma)^2 - 9 + 12b\right] - f \end{aligned}$$

Therefore, the profit difference is as

$$
\begin{aligned}
& \mathbb{E}(\pi_2|s_2, x_2^*(s_2)) - \mathbb{E}(\pi_2|s_2, \bar{x}_2(s_2)) \\
= \ & \tfrac{t}{18}(2b)(12b) - \tfrac{t}{18}\left(3 - 3k\sigma_p\gamma + 2b\right)\left[(3k\sigma_p\gamma)^2 - 9 + 12b\right] \\
= \ & -\tfrac{t}{18}\left(3 - 3k\sigma_p\gamma + 2b\right)\left[(3k\sigma_p\gamma)^2 - 9\right] - \tfrac{t}{18}\cdot 12b\left(3 - 3k\sigma_p\gamma\right) \\
= \ & \tfrac{t}{18}\left(3 - 3k\sigma_p\gamma\right)\left[(3 + 3k\sigma_p\gamma)\left(3 - 3k\sigma_p\gamma + 2b\right) - 12b\right] \\
= \ & \tfrac{t}{6}\left(3 - 3k\sigma_p\gamma\right)\left[3(1 - k^2\sigma_p^2\gamma^2) - (2b - 2bk\sigma_p\gamma)\right] \\
= \ & \tfrac{t}{2}(1 - k\sigma_p\gamma)^2\left[3(1 + k\sigma_p\gamma) - 2b\right]
\end{aligned}
$$

Notice that if $\sigma_p \leq \frac{1}{4k}$, by the definition of b, $b < \frac{1}{12}\left[9 + \frac{2}{16}(1+1)\right] < 1$, and thus we have $3(1 + k\sigma_p\gamma) - 2b > 0$. Therefore, if $\sigma \leq \frac{1}{4k}$, we conclude $\mathbb{E}(\pi_2|s_2, x_2^*(s_2)) - \mathbb{E}(\pi_2|s_2, \bar{x}_2(s_2)) > 0$, and firm 2 has no incentive to deviate to the boundary location.