# Appendix A

**The Carr Madan Implied Volatility (CM IMV)**

The following is the CBOE formula for the VIX index, denoted as CM IMV.

$\sigma^2_{CM} = \frac{2}{\tau}\sum_i \frac{\Delta K_i}{K_i^2}e^{r\tau}Q(\tau, K_i) - \frac{1}{\tau}[\frac{F}{K_0} - 1]^2$

$CM\ IMV = \sigma_{CM} * 100$

| | |
|---|---|
| $\tau$ | : Time to expiration |
| $F$ | : Forward index level derived from put-call parity |
| $K_0$ | : First strike price below the forward index level, $F$ |
| $K_i$ | : Strike price of the $i^{th}$ out-of-the-money option; a call if $K_i > K_0$, and a put if $K_i < K_0$; both put and call if $K_i = K_0$ |
| $\Delta K_i$ | $= \frac{K_{i+1} - K_{i-1}}{2}$ |
| $r$ | : Risk-free spot rate of interest |
| $Q(\tau, K_i)$ | : The ask price/bid price for the option with strike $K_i$ |

In the VIX index calculation, $Q(K_i)$ is the midpoint of the bid-ask spread for each option with strike price $K_i$. The forward index level is:

$$F = strike\ price + e^{r\tau}(\ Call\ price - Put\ price\ )$$

where the strike price selected is that for which the absolute difference between the call and put prices is the smallest.

In our paper, we use the strike price that is closest to the spot price to calculate the forward index level, sometimes called the effective forward price. The original formula proposed by Carr and Madan doesn't include the term involving $[\frac{F}{K_0} - 1]$.

# Appendix B

**Appendix B-1 (Expected Profit and Variance)**

**Portfolio Expected Profit Given $\sigma_Y$:**

Assume $y = \ln Y \sim N(\mu_Y, \sigma_Y^2)$ and $Z \sim N(0, 1)$, then

$$X = \sigma_Y Z + \mu_Y, Y = exp\,(\sigma_Y Z + \mu_Y)$$

The calculation of $E\,[Y \cdot I(Y \geq y)]$ and $Var\,[Y \cdot I(Y \geq y)]$ is detailed as follows:

$$\begin{aligned}
E\,[Y \cdot I(Y \geq y)] &= E\,[\exp(\sigma_Y Z + \mu_Y) \cdot I(\exp(\sigma_Y Z + \mu_Y) \geq y)] \\
&= E\left[\exp(\sigma_Y Z + \mu_Y) \cdot I(Z \geq \frac{\ln(y) - \mu_Y}{\sigma_Y})\right] \\
&= \int_z^\infty \exp(\sigma_Y u + \mu_Y) \cdot \frac{1}{\sqrt{2\pi}}\exp(-\frac{u^2}{2})du; Let\ z = \frac{\ln(y) - \mu_Y}{\sigma_Y} \\
&= \exp(\frac{\sigma_Y^2}{2} + \mu_Y)\int_z^\infty \frac{1}{\sqrt{2\pi}}\exp(-\frac{(u - \sigma_Y)^2}{2})du \\
&= \exp(\frac{\sigma_Y^2}{2} + \mu_Y) \cdot \mathbf{N}(\sigma_Y - z)
\end{aligned}$$

$$
\begin{aligned}
E\left[Y^2 \cdot I(Y \geq y)\right] =& E\left[\exp(2\sigma_Y Z + 2\mu_Y) \cdot I(\exp(\sigma_Y Z + \mu_Y) \geq y)\right] \\
=& \int_z^{\infty} \exp(2\sigma_Y u + 2\mu_Y) \cdot \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du; Let\ z = \frac{\ln(y) - \mu_Y}{\sigma_Y} \\
=& \exp(2\sigma_Y^2 + 2\mu_Y) \int_z^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{(u - 2\sigma_Y)^2}{2}) du \\
=& \exp(2\sigma_Y^2 + 2\mu_Y) \cdot \mathbf{N}(2\sigma_Y - z)
\end{aligned}
$$

$$
\begin{aligned}
Var\left[Y \cdot I(Y \geq y)\right] =& E\left[Y^2 \cdot I(Y \geq y)\right] - \{E\left[Y \cdot I(Y \geq y)\right]\}^2 \\
=& \exp(\sigma_Y^2 + 2\mu_Y) \left\{\exp(\sigma_Y^2) \cdot \mathbf{N}(2\sigma_Y - z) - [\mathbf{N}(\sigma_Y - z)]^2\right\}
\end{aligned}
$$

Let $y \to 0$, that is $z \to -\infty$, we have

$$
\begin{aligned}
&E\left[Y\right] = \exp(\frac{\sigma_Y^2}{2} + \mu_Y) \\
&Var\left[Y\right] = \exp(\sigma_Y^2 + 2\mu_Y) \left[\exp(\sigma_Y^2) - 1\right]
\end{aligned}
$$

Let $C_\tau = (P_0 Y - P_0 k)^+$. The expectation and variance of the portfolio value can be explicitly calculated below. let $z = \frac{\ln(k) - \mu_Y}{\sigma_Y}$, $\Delta$ is the holded stocks for hedging, $P_0 k$ is the strike price, and $B = p_c - \Delta_a P_0$ is the amount of money market account at very beginning.

$$
\begin{aligned}
E\left[W_a\right] =& E[\Delta_a \cdot P_\tau - C_\tau + Be^{r\tau}] \\
=& E[\Delta_a \cdot P_0 Y - (P_0 Y - P_0 k)^+ + Be^{r\tau}] \\
=& \Delta_a \cdot P_0 \cdot \exp(\frac{\sigma_Y^2}{2} + \mu_Y) - P_0 \left[\exp(\frac{\sigma_Y^2}{2} + \mu_Y) \cdot \mathbf{N}(\sigma_Y - z) - k \cdot \mathbf{N}(-z)\right] + Be^{r\tau}.
\end{aligned}
$$

If we express the mean and volatility in annualized term, we can show the formula for $C_\tau$ is the same as Black-Scholes-Merton pricing model. Let $\mu_Y = \mu\tau$, $\sigma_Y^2 = \sigma^2\tau$ and $K = P_0 k$. Given the expectation operator is the risk neutral probability, we have $e^{(\frac{\sigma^2\tau}{2} + \mu\tau)} = e^{r\tau}$. Then the following equation for $C_\tau$ is exactly the same as Black-Scholes-Merton Formula after discounted by risk-free rate.

$$
\begin{aligned}
&P_0 \left[\exp(\tfrac{\sigma_Y^2}{2} + \mu_Y) \cdot \mathbf{N}(\sigma_Y - z) - k \cdot \mathbf{N}(-z)\right] \\
&= P_0 e^{r\tau} \cdot \mathbf{N}\left(\tfrac{\sigma_Y^2 - \ln(k) + \mu_Y}{\sigma_Y}\right) - P_0 k \cdot \mathbf{N}(\tfrac{-\ln(k) + \mu_Y}{\sigma_Y}) \\
&= P_0 e^{r\tau} \cdot \mathbf{N}\left(\tfrac{\sigma_Y^2/2 - \ln(k) + \ln(e^{\mu_Y + \sigma_Y^2/2})}{\sigma_Y}\right) - P_0 k \cdot \mathbf{N}(\tfrac{-\sigma_Y^2/2 - \ln(k) + \ln(e^{\mu_Y + \sigma_Y^2/2})}{\sigma_Y}) \\
&= P_0 e^{r\tau} \cdot \mathbf{N}\left(\tfrac{\ln(\frac{P_0 e^{\mu_Y + \sigma_Y^2/2}}{P_0 k}) + \sigma_Y^2/2}{\sigma_Y}\right) - K \cdot \mathbf{N}\left(\tfrac{\ln(\frac{P_0 e^{\mu_Y + \sigma_Y^2/2}}{P_0 k}) - \sigma_Y^2/2}{\sigma_Y}\right) \\
&= P_0 e^{r\tau} \cdot \mathbf{N}\left(\tfrac{\ln(\frac{P_0}{K}) + r\tau + \sigma_Y^2\tau/2}{\sigma_Y\sqrt{\tau}}\right) - K \cdot \mathbf{N}(\tfrac{\ln(\frac{P_0}{K}) + r\tau - \sigma_Y^2\tau/2}{\sigma_Y\sqrt{\tau}})
\end{aligned}
$$

The variance of the portfolio is

$$
\begin{aligned}
Var\left[W_a\right] =& Var[\Delta_a \cdot P_\tau - C_\tau + B_a \cdot e^{r\tau}] \text{ and } B_a = C_0^a - \Delta_a P_0 \\
=& Var[\Delta_a \cdot P_0 Y] + Var\left[(P_0 Y - P_0 k)^+\right] - 2\Delta_a \times cov(P_0 Y, (P_0 Y - P_0 k)^+) \\
=& \left[\Delta_a \cdot P_0\right]^2 Var[Y] + P_0^2 Var\left[(Y-k)^+\right] - 2\Delta_a P_0^2 \times cov(Y, (Y-k)^+)
\end{aligned}
$$

$$
\begin{aligned}
Var\left[W_b\right] =& Var[\Delta_b \cdot P_\tau + C_\tau + B_b \cdot e^{r\tau}] \text{ and } B_b = -C_0^b - \Delta_b P_0 \\
=& Var[\Delta_b \cdot P_0 Y] + Var\left[(P_0 Y - P_0 k)^+\right] + 2\Delta_b \times cov(P_0 Y, (P_0 Y - P_0 k)^+) \\
=& \left[\Delta_b \cdot P_0\right]^2 Var[Y] + P_0^2 Var\left[(Y-k)^+\right] + 2\Delta_b P_0^2 \times cov(Y, (Y-k)^+)
\end{aligned}
$$

We need to calculate $Var\left[(Y-k)^+\right]$ and $cov(Y,(Y-k)^+)$ .

$$
\begin{aligned}
Var\left[(Y-K)^+\right] =& Var\left[(Y-k)\cdot I(Y \geqslant k)\right] \\
=& Var\left[Y \cdot I(Y \geqslant k) - k \cdot I(Y \geqslant k)\right] \\
=& Var\left[Y \cdot I(Y \geqslant k)\right] + k^2 Var\left[I(Y \geqslant k)\right] - 2k \cdot cov\left[Y \cdot I(Y \geqslant k), I(Y \geqslant k)\right] \\
=& \exp(\sigma_Y^2 + 2\mu_Y)\left[\exp(\sigma_Y^2)\mathbf{N}(2\sigma_Y - z) - [\mathbf{N}(\sigma_Y - z)]^2\right] \\
&+ k^2 \mathbf{N}(z)\mathbf{N}(-z) - 2k \cdot \exp(\frac{\sigma_Y^2}{2} + \mu_Y)\mathbf{N}(\sigma_Y - z)\mathbf{N}(z)
\end{aligned}
$$

$$
\begin{aligned}
& Cov\left[Y, (Y-k)^+\right] \\
&= Cov\left[Y, (Y-k)\cdot I(Y \geqslant k)\right] \\
&= E\left[Y(Y-k)\cdot I(Y \geqslant k)\right] - \exp\left(\frac{\sigma_Y^2}{2} + \mu_Y\right)\left[\exp\left(\frac{\sigma_Y^2}{2} + \mu_Y\right)\mathbf{N}(\sigma_Y - z) - k \cdot \mathbf{N}(-z)\right] \\
&= E\left[Y^2 \cdot I(Y \geqslant k)\right] - k \cdot E[Y \cdot I(Y \geqslant k)] - \exp\left(\frac{\sigma_Y^2}{2} + \mu_Y\right)\cdot\left[\exp\left(\frac{\sigma_Y^2}{2} + \mu_Y\right)\mathbf{N}(\sigma_Y - z) - k \cdot \mathbf{N}(-z)\right] \\
&= \exp\left(2\sigma_Y^2 + 2\mu_Y\right)\mathbf{N}(2\sigma_Y - z) - \left(k + \exp\left(\frac{\sigma_Y^2}{2} + \mu_Y\right)\right)\cdot \exp\left(\frac{\sigma_Y^2}{2} + \mu_Y\right)\mathbf{N}(\sigma_Y - z) \\
&+ k \cdot \exp\left(\frac{\sigma_Y^2}{2} + \mu_Y\right)\mathbf{N}(-z)
\end{aligned}
$$

Plugging $Var\left[(Y-k)^+\right]$ and $cov(Y,(Y-k)^+)$ into $Var\left[W_a\right]$, and rearranging the terms, we obtain

$$
Var\left[W_a\right] = Var\left[W_b\right] = P_0^2\left[C + A_1 \cdot \mathbf{N}(2\sigma_Y - z) + A_2 \cdot \mathbf{N}(\sigma_Y - z) + A_3 \cdot \mathbf{N}(-z)\right]
$$

where

$$
\begin{aligned}
C &= \Delta_a^2 \exp\left(\sigma_Y^2 + 2\mu_Y\right)\left[\exp\left(\sigma_Y^2\right) - 1\right] \\
A_1 &= \exp\left(2\sigma_Y^2 + 2\mu_Y\right)(1 - 2\Delta_a) \\
A_2 &= -\exp(\frac{\sigma_Y^2}{2} + \mu_Y)\left[\exp(\frac{\sigma_Y^2}{2} + \mu_Y)\mathbf{N}(\sigma_Y - z) + 2k \cdot \mathbf{N}(z) - 2\Delta_a\left(k + \exp(\frac{\sigma_Y^2}{2} + \mu_Y)\right)\right] \\
A_3 &= k\left[k \cdot \mathbf{N}(z) - 2\Delta_a \exp(\frac{\sigma_Y^2}{2} + \mu_Y)\right]
\end{aligned}
$$

**Portfolio Mean and Variance given a Bernoulli type Volatility.**

- We assume a Bernoulli type random volatility, which is a probability with independence assumption (4) for different annual volatility levels, $\sigma_L$ and $\sigma_H$. In our math derivation, we let $\sigma_L$=$\sigma \cdot L$ and $\sigma_H$=$\sigma \cdot H$. We also use $(L, H)$ in subscripts to denote the volatilities $(\sigma_L, \sigma_H)$.

- Traders assign the subjective probability $(1-\phi)$ and $\phi$ to volatility $\sigma L$ and $\sigma H$, respectively.

**Expected Profit given a Bernoulli type Volatility**

For a "written" and a "long" call option position, after traders apply delta hedging, the final wealth of $W_a$ and $W_b$ are

$$W_a = \Delta_a \cdot P_T - C_T + (C_a^0 - \Delta_a \cdot P_0) \cdot e^{r\tau}$$

$$W_b = \Delta_b \cdot P_T + C_T + (-C_b^0 - \Delta_b \cdot P_0) \cdot e^{r\tau}$$

where $C_a^0$ and $C_b^0$ are selling price and buying price respectively, $C_T$ is final option payoff ($C_T = Max[P_T - K, 0]$), and $\Delta$ are the hedging positions. $C_T \geqslant 0$ and is cash-outflow for a written call, while $C_a^0 \geqslant 0$ and is cash inflow for a written call. Conversely, $C_T$ and $C_b^0$ are cash-inflow and cash-outflow respectively for a long position. A positive (negative) $\Delta$ means a long (short) position of the stock; therefore, if $\Delta_a$ is positive , $\Delta_a \cdot P_T$ is the amount of cash-inflow from selling stocks at $P_T$ and$-\Delta_a \cdot P_0$ is the cash-outflow to buy stocks at $P_0$. Given assumption (4),

$$E(P_T) = P_0 E(\frac{P_T}{P_0}) = P_0 \left\{(1-\phi)E_L(\frac{P_T}{P_0}) + \phi E_H(\frac{P_T}{P_0})\right\} = P_0 \left\{(1-\phi)e^{r\tau} + \phi e^{r\tau}\right\} = P_0 e^{r\tau}.$$

And the expected final wealth for selling a call option and purchasing a call options are,

$$\begin{aligned} E(W_a) =& \phi E_H \left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\} + (1-\phi)E_L \left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\} \\ =& C_a^0 e^{r\tau} - \phi E_H \left\{C_T\right\} - (1-\phi)E_L \left\{C_T\right\} = C_a^0 e^{r\tau} - E\left\{C_T\right\} \end{aligned}$$

$$\begin{aligned} E(W_b) =& \phi E_H \left\{-C_b^0 e^{r\tau} + C_T + \Delta_b[P_T - P_0 e^{r\tau}]\right\} + (1-\phi)E_L \left\{-C_b^0 e^{r\tau} + C_T + \Delta_b[P_T - P_0 e^{r\tau}]\right\} \\ =& E\left\{C_T\right\} - C_b^0 e^{r\tau} \end{aligned}$$

Using Appendix B-1 formula, we can derive the option price formula which is the same as Black-Schole-Merton Model. The expected prices under different volatility levels are,

$$\begin{aligned} E_H\left[C_\tau\right] =& e^{r\tau}\left[P_0 \cdot \mathbf{N}(z_H + \sigma H\sqrt{\tau}) - Ke^{-r\tau} \cdot \mathbf{N}(z_H)\right] \quad and \quad z_H = \frac{\ln\frac{P_0}{K} + (r - \frac{1}{2}\sigma^2 H^2)\tau}{\sigma H\sqrt{\tau}} \\ E_L\left[C_\tau\right] =& e^{r\tau}\left[P_0 \cdot \mathbf{N}(z_L + \sigma L\sqrt{\tau}) - Ke^{-r\tau} \cdot \mathbf{N}(z_L)\right] \quad and \quad z_L = \frac{\ln\frac{P_0}{K} + (r - \frac{1}{2}\sigma^2 L^2)\tau}{\sigma L\sqrt{\tau}} \end{aligned}$$

where $z_H = \frac{\ln(K) - E_H(\frac{P_T}{P_0})}{\sigma_J}$.Therefore, the expected final wealth is

$$E(W_a) = C_a^0 e^{r\tau} - e^{r\tau}\left\{P_0\left[\phi \cdot \mathbf{N}(z_H + \sigma H\sqrt{\tau}) + (1-\phi) \cdot \mathbf{N}(z_L + \sigma L\sqrt{\tau})\right] - Ke^{-r\tau}\left[\phi \cdot \mathbf{N}(z_H) + (1-\phi) \cdot \mathbf{N}(z_L)\right]\right\}$$

**Variance given a Bernoulli type Volatility**

$$\begin{aligned} Var(W_a) = & \phi Var_H \left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\} + (1-\phi)Var_L \left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\} \\ & + \phi(1-\phi)(E_L - E_H)^2 \end{aligned}$$

where $E_L = E_L\left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\}$ and $E_H = \left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\}$.

To calculate $Var_H\left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\}$, we can rewrite $Var_H\left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\}$ as

$$Var_H\left\{C_a^0 e^{r\tau} - C_T + \Delta_a[P_T - P_0 e^{r\tau}]\right\} = \Delta_a^2 Var_H\{P_T\} + Var_H\{C_T\} - 2\cdot\Delta_a\cdot Cov_H(P_T, C_T)$$

$\ln(\frac{P_\tau}{P_0})$ follows $N\left((r - \frac{1}{2}\sigma^2 H^2)\tau, \sigma^2 H^2 \tau\right)$. Then, the variance of $P_\tau$ is [22]

$$Var_H\{P_T\} = E_H\left\{P_T^2\right\} - [E_H(P_T)]^2 = P_0^2 e^{2r\tau}\left[e^{\sigma^2 H^2 \tau} - 1\right].$$

The variance of $C_\tau$ is

$$\begin{aligned}
Var_H(C_\tau) =& Var_H\left[(P_\tau - K)\cdot I(P_\tau \geq K)\right] \\
=& Var_H\left[P_\tau \cdot I(P_\tau \geq K) - K\cdot I(P_\tau \geq K)\right] \\
=& Var_H[P_\tau \cdot I(P_\tau \geq K)] + K^2 \cdot Var_H\left[I(P_\tau \geq K)\right] - 2K\cdot Cov_H[P_\tau\cdot I(P_\tau \geq K), I(P_\tau \geq K)] \\
=& P_0^2 e^{2r\tau}\left\{e^{\sigma^2 H^2 \tau}\mathbf{N}(2\sigma\cdot H\sqrt{\tau} + z_H) - \left[\mathbf{N}(\sigma\cdot H\sqrt{\tau} + z_H)\right]^2\right\} + K^2\mathbf{N}(z_H)\mathbf{N}(-z_H) \\
&- 2P_0 K e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau} + z_H)\mathbf{N}(-z_H)
\end{aligned}$$

The co-variance of $P_\tau$ and $C_\tau$ is

$$\begin{aligned}
Cov_H(P_\tau, C_\tau) =& Cov_H\left(P_T, [P_T - K]\cdot I(P_T \geq K)\right) \\
=& E_H\left\{{P_T}^2 \cdot I(P_T \geq K)\right\} - K\cdot E_H\left\{P_T \cdot I(P_T \geq K)\right\} - E_H(P_T)\cdot E_H\left[(P_T - K)^+\right] \\
=& P_0^2 e^{2r\tau + \sigma^2 H^2 \tau}\mathbf{N}\left(2\sigma H\sqrt{\tau} + z_H\right) - \left(P_0 K e^{r\tau} + P_0^2 e^{2r\tau}\right)\mathbf{N}\left(\sigma H\sqrt{\tau} + z_H\right) + P_0 K e^{r\tau}\mathbf{N}(z_H)
\end{aligned}$$

Therefore,

$$Var_H\left\{C_a^0 e^{r\tau} - C_T + \Delta_a\left[P_T - P_0 e^{r\tau}\right]\right\} = C + A_1\mathbf{N}(2\sigma H\sqrt{\tau} + z_H) + A_2\mathbf{N}(\sigma H\sqrt{\tau} + z_H) + A_3\mathbf{N}(z_H)$$

$$\begin{cases}
C = P_0^2 \Delta_a^2 e^{2r\tau}\left[e^{\sigma^2 H^2 \tau} - 1\right] \\
A_1 = P_0^2 e^{2r\tau + \sigma^2 H^2 \tau}(1 - 2\Delta_a) \\
A_2 = -P_0^2 e^{2r\tau}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) - 2P_0 K e^{r\tau}\mathbf{N}(-z_H) + 2\Delta_a P_0 e^{r\tau}(K + P_0 e^{r\tau}) \\
A_3 = K^2\mathbf{N}(-z_H) - 2\Delta_a P_0 K e^{r\tau}
\end{cases}$$

The derivation for $Var_L$ is the same. And the unconditional Variance for a bernoulli type random volatility is:

$$\begin{aligned}
Var(W_a) =& \phi Var_H\left\{-C_T + \Delta_a \cdot P_T\right\} + (1-\phi)Var_L\left\{-C_T + \Delta_a\cdot P_T\right\} \\
&+ \phi(1-\phi)(E_L\left[C_T\right] - E_H\left[C_T\right])^2
\end{aligned}$$

[22] For the general log-normal random variable $\ln(X) \sim N(\mu, \sigma^2)$, we have the following general results:
(1)The First and Second Moment of $X$.

$$E(X) = e^{\mu + \frac{1}{2}\sigma^2} \quad and \quad E(X^2) = e^{2\mu + 2\sigma^2}.$$

(2)The Restricted First and Second Moment of $X$.

$$E\left[X\cdot I(X \geq x)\right] = E(X)\cdot\Phi\left(\sigma + \frac{-\ln(x) + \mu}{\sigma}\right) \quad and \quad E\left[X^2\cdot I(X \geq x)\right] = E(X^2)\cdot\Phi\left(2\sigma + \frac{-\ln(x) + \mu}{\sigma}\right).$$

$$
\begin{aligned}
Var(W_b) = &\phi Var_H \{C_T + \Delta_b \cdot P_T\} + (1-\phi) Var_L \{C_T + \Delta_b \cdot P_T\} \\
&+ \phi(1-\phi)(E_L[C_T] - E_H[C_T])^2
\end{aligned}
$$

The derivation of $Var(W_b)$ is the same as we did for $Var(W_a)$. Variance are the same, because $\Delta_a = -\Delta_b$, the proof of which is shown next.

**Appendix B-2 (The Optimal Delta)**

The portfolio value at the expiration date $W_a$ and $W_b$ can have two volatility realizations $(\sigma L, \sigma H)$ with subjective probability $1-\phi$ and $\phi$ respectively. The variance of $W_a$ and $W_b$ can be written as

$$
\begin{aligned}
Var(W_a) = &(1-\phi) Var_L[W_a] + \phi Var_H[W_a] + (1-\phi)\phi \left[E_L(W_a) - E_H(W_a)\right]^2 \\
= &(1-\phi)\left\{\Delta_a^2 Var_L(P_T) + Var_L[C_T] - 2\Delta_a Cov_L(P_T, C_T)\right\} \\
&+ \phi\left\{\Delta_a^2 Var_H(P_T) + Var_H[C_T] - 2\Delta_a Cov_H(P_T, C_T)\right\} + (1-\phi)\phi\left[E_L(W_a) - E_H(W_a)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
Var(W_b) = &(1-\phi) Var_L[W_b] + \phi Var_H[W_b] + (1-\phi)\phi \left[E_L(W_b) - E_H(W_b)\right]^2 \\
= &(1-\phi)\left\{\Delta_b^2 Var_L(P_T) + Var_L[C_T] + 2\Delta_b Cov_L(P_T, C_T)\right\} \\
&+ \phi\left\{\Delta_b^2 Var_H(P_T) + Var_H[C_T] + 2\Delta_b Cov_H(P_T, C_T)\right\} + (1-\phi)\phi\left[E_L(W_b) - E_H(W_b)\right]^2
\end{aligned}
$$

The necessary conditions to minimize $Var[W_a]$ and $Var[W_b]$ are

$$
\begin{aligned}
&0 = \tfrac{\partial Var(W_a)}{\partial \Delta} \\
&= (1-\phi)\left\{\Delta_a Var_L(P_T) - Cov_L(P_T, (P_T-K)^+)\right\} + \phi\left\{\Delta_a Var_H(P_T) - Cov_H(P_T, (P_T-K)^+)\right\} \\
&+ (1-\phi)\phi\left\{\Delta_a E_L(P_T) - E_L\left[(P_T-K)^+\right] - \Delta_a E_H(P_T) + E_H\left[(P_T-K)^+\right]\right\}\left[E_L(P_T) - E_H(P_T)\right]
\end{aligned}
$$

$$
\begin{aligned}
&0 = \tfrac{\partial Var(W_b)}{\partial \Delta} \\
&= (1-\phi)\left\{\Delta_b Var_L(P_T) + Cov_L(P_T, (P_T-K)^+)\right\} + \phi\left\{\Delta_b Var_H(P_T) + Cov_H(P_T, (P_T-K)^+)\right\} \\
&+ (1-\phi)\phi\left\{\Delta_b E_L(P_T) + E_L\left[(P_T-K)^+\right] - \Delta_b E_H(P_T) - E_H\left[(P_T-K)^+\right]\right\}\left[E_L(P_T) - E_H(P_T)\right]
\end{aligned}
$$

Therefore, the optimal $\Delta_a^*$ and $\Delta_b^*$ are

$$
\begin{aligned}
&\Delta_a^* = \tfrac{E(Cov([P_T-K]^+, P_T|V))}{E[Var(P_T)|V]} \\
&= \tfrac{(1-\phi)Cov_L(P_T,(P_T-K)^+) + \phi Cov_H\left(P_T,(P_T-K)^+\right) + (1-\phi)\phi[E_L(P_T)-E_H(P_T)]\left\{E_L\left[(P_T-K)^+\right] - E_H\left[(P_T-K)^+\right]\right\}}{(1-\phi)Var_L(P_T) + \phi Var_H(P_T) + (1-\phi)\phi[E_L(P_T)-E_H(P_T)]^2}
\end{aligned}
$$

$$
\begin{aligned}
&\Delta_b^* = -\tfrac{E(Cov([P_T-K]^+, P_T|V))}{E[Var(P_T)|V]} \\
&= -\tfrac{(1-\phi)Cov_L(P_T,(P_T-K)^+) + \phi Cov_H\left(P_T,(P_T-K)^+\right) + (1-\phi)\phi[E_L(P_T)-E_H(P_T)]\left\{E_L\left[(P_T-K)^+\right] - E_H\left[(P_T-K)^+\right]\right\}}{(1-\phi)Var_L(P_T) + \phi Var_H(P_T) + (1-\phi)\phi[E_L(P_T)-E_H(P_T)]^2}
\end{aligned}
$$

**Appendix B-3 (Comparative Statics for $\tau$)**

Let $f(A, \sigma, L, H, \tau; \phi) = E(W_a) - \frac{Q}{\gamma} Var(W_a) - c = 0$. We show the following propositions.

When time to maturity decreases, the ask volatility increases for ATM options, i.e.

$$\frac{\partial A}{\partial \tau} = -\frac{\frac{\partial f}{\partial \tau}}{\frac{\partial f}{\partial A}} < 0$$

Assume the ask volatility corresponds to the ask price $C_a^0$ in the risk neutral probability measure, i.e.

$$C_a^0 = P_0 \cdot \mathbf{N}(\sigma A\sqrt{\tau} + z_A) - Ke^{-r\tau} \cdot \mathbf{N}(z_A) \quad and \quad z_A = \frac{\ln \frac{P_0}{K} + (r - \frac{1}{2}A^2\sigma^2)\tau}{\sigma A\sqrt{\tau}}$$

We have

$$\frac{\partial f}{\partial A} = e^{r\tau} \times \frac{\partial C_a^0}{\partial A} > 0.$$

Now, we derive the equations for $\frac{\partial f}{\partial \tau}$ and $\frac{\partial A}{\partial \tau}$.

$$\frac{\partial f}{\partial \tau} = \frac{\partial E(W_a)}{\partial \tau} - \frac{Q}{\gamma}\frac{\partial Var(W_a)}{\partial \tau}$$

To calculate $\frac{\partial E(W_a)}{\partial \tau}$, we first calculate $\frac{\partial\left(e^{r\tau}C_0^a\right)}{\partial \tau}$.[23]

$$\frac{\partial\left(e^{r\tau}C_0^a\right)}{\partial \tau} = \frac{K \cdot \mathbf{n}(z_A) \cdot \sigma A}{2\sqrt{\tau}} + rP_0e^{r\tau} \cdot \mathbf{N}(z_A + \sigma \cdot A\sqrt{\tau})$$

Since $\frac{\partial C_H}{\partial \tau}$ and $\frac{\partial C_L}{\partial \tau}$ are similar to $\frac{\partial\left(e^{r\tau}C_0^a\right)}{\partial \tau}$ with $H, L$ replacing $a$ respectively, we can finally write $\frac{\partial E(Wa)}{\partial \tau}$ as

$$\begin{aligned}\frac{\partial E(W_a)}{\partial \tau} = \quad & \frac{K\sigma}{2\sqrt{\tau}}\left[\mathbf{n}(z_A)A - \phi \cdot \mathbf{n}(z_H)H - (1-\phi)\cdot \mathbf{n}(z_L)L\right] \\ & + rP_0e^{r\tau}\left[\mathbf{N}(z_A + \sigma A\sqrt{\tau}) - \phi \cdot \mathbf{N}(z_H + \sigma H\sqrt{\tau}) - (1-\phi)\cdot \mathbf{N}(z_L + \sigma L\sqrt{\tau})\right]\end{aligned}$$

To calculate $\frac{\partial Var(W_a)}{\partial \tau}$, we first calculate $\frac{\partial Var_H}{\partial \tau}$ and the details follow.

$$\begin{aligned}\frac{\partial Var_H}{\partial \tau} = \quad & \frac{\partial C}{\partial \tau} + \frac{\partial A_1}{\partial \tau}\mathbf{N}(2\sigma H\sqrt{\tau} + z_H) + \frac{\partial A_2}{\partial \tau}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) + \frac{\partial A_3}{\partial \tau}\mathbf{N}(z_H) \\ & + A_1\frac{\partial \mathbf{N}(2\sigma H\sqrt{\tau}+z_H)}{\partial \tau} + A_2\frac{\partial \mathbf{N}(\sigma H\sqrt{\tau}+z_H)}{\partial \tau} + A_3\frac{\partial \mathbf{N}(z_H)}{\partial \tau}\end{aligned}$$

We have

[23] We have the basic results regarding the normal density function.

$$\mathbf{n}(z_A + \sigma A\sqrt{\tau}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\sigma A\sqrt{\tau}+z_A)^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\sigma^2A^2\tau - \frac{1}{2}z_A^2 - \sigma A\sqrt{\tau}z_A} = \frac{K}{P_0}e^{-r\tau}\mathbf{n}(z_A)$$

$$\mathbf{n}(z_A + 2\sigma A\sqrt{\tau}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(2\sigma A\sqrt{\tau}+z_A)^2} = \frac{1}{\sqrt{2\pi}}e^{-2\sigma^2A^2\tau - \frac{1}{2}z_A^2 - 2\sigma A\sqrt{\tau}z_A} = \frac{K^2}{P_0^2}e^{-2r\tau-\sigma^2A^2\tau}\mathbf{n}(z_A)$$

$\frac{\partial C}{\partial \tau} = P_0^2\Delta^2\left[(2r+\sigma^2H^2)e^{2r\tau+\sigma^2H^2\tau} - 2re^{2r\tau}\right]$

$\frac{\partial A_1}{\partial \tau}\mathbf{N}(2\sigma H\sqrt{\tau}+z_H) = P_0^2(2r+\sigma^2H^2)(1-2\Delta)e^{2r\tau+\sigma^2H^2\tau}\mathbf{N}(2\sigma H\sqrt{\tau}+z_H)$

$A_1\frac{\partial \mathbf{N}(2\sigma H\sqrt{\tau}+z_H)}{\partial \tau} = K^2(1-2\Delta)\left(\frac{\sigma H}{\sqrt{\tau}} + \frac{\partial z_H}{\partial \tau}\right)\mathbf{n}(z_H)$

$$
\begin{aligned}
&\frac{\partial A_2}{\partial \tau}\mathbf{N}(\sigma H\sqrt{\tau}+z_H)\\
&= -KP_0e^{r\tau}\left(\frac{\sigma H}{2\sqrt{\tau}} + \frac{\partial z_H}{\partial \tau}\right)\mathbf{n}(z_H)\mathbf{N}(\sigma H\sqrt{\tau}+z_H) - 2rP_0^2e^{2r\tau}\left[\mathbf{N}(\sigma H\sqrt{\tau}+z_H)\right]^2\\
&+2KP_0e^{r\tau}\frac{\partial z_H}{\partial \tau}\mathbf{n}(z_H)\mathbf{N}(\sigma H\sqrt{\tau}+z_H) - 2rKP_0e^{r\tau}\mathbf{N}(-z_H)\mathbf{N}(\sigma H\sqrt{\tau}+z_H)\\
&+2r\Delta KP_0e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau}+z_H) + 4r\Delta P_0^2e^{2r\tau}\mathbf{N}(\sigma H\sqrt{\tau}+z_H)
\end{aligned}
$$

$$
\begin{aligned}
&A_2\frac{\partial \mathbf{N}(\sigma H\sqrt{\tau}+z_H)}{\partial \tau}\\
&= \left\{-KP_0e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau}+z_H) - 2K^2\mathbf{N}(-z_H) + 2\Delta K^2 + 2\Delta KP_0e^{r\tau}\right\}\left(\frac{\sigma H}{2\sqrt{\tau}} + \frac{\partial z_H}{\partial \tau}\right)\mathbf{n}(z_H)
\end{aligned}
$$

$\frac{\partial A_3}{\partial \tau}\mathbf{N}(z_H) = \left(-K^2\frac{\partial z_H}{\partial \tau}\mathbf{n}(z_H) - 2r\Delta KP_0e^{r\tau}\right)\mathbf{N}(z_H)$

$A_3\frac{\partial \mathbf{N}(z_H)}{\partial \tau} = \left(K^2\mathbf{N}(-z_H) - 2\Delta P_0Ke^{r\tau}\right)\frac{\partial z_H}{\partial \tau}\mathbf{n}(z_H)$

Adding up all the parts and rearranging, we finally have[24]

$$
\begin{aligned}
\frac{\partial Var_H}{\partial \tau} = \quad & D + B_1\mathbf{N}(2\sigma H\sqrt{\tau}+z_H) + B_2\mathbf{N}(\sigma H\sqrt{\tau}+z_H) + B_3\mathbf{N}(z_H)\\
&+\left[K^2\mathbf{N}(z_H) - KP_0e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau}+z_H) - \Delta K^2 + \Delta KP_0e^{r\tau}\right]\frac{\sigma H}{\sqrt{\tau}}\mathbf{n}(z_H)
\end{aligned}
$$

$$
\begin{cases}
D = P_0^2\Delta^2\left[(2r+\sigma^2H^2)e^{2r\tau+\sigma^2H^2\tau} - 2re^{2r\tau}\right]\\
B_1 = P_0^2(2r+\sigma^2H^2)(1-2\Delta)e^{2r\tau+\sigma^2H^2\tau}\\
B_2 = -2rP_0^2e^{2r\tau}\mathbf{N}(\sigma H\sqrt{\tau}+z_H) - 2rKP_0e^{r\tau}\mathbf{N}(-z_H) + 2r\Delta KP_0e^{r\tau} + 4r\Delta P_0^2e^{2r\tau}\\
B_3 = -2r\Delta KP_0e^{r\tau}
\end{cases}
$$

To calculate $\frac{\partial Var(W_a)}{\partial \tau}$, we also need to calculate $\frac{\partial (E_H-E_L)^2}{\partial \tau}$.

$$
\begin{aligned}
\frac{\partial (E_H-E_L)^2}{\partial \tau} \quad &= [E_H\{C_T\} - E_L\{C_T\}] \times \frac{\partial (E_H\{C_T\}-E_L\{C_T\})}{\partial \tau}\\
&= [E_H\{C_T\} - E_L\{C_T\}] \times \left[\frac{K\sigma}{2\sqrt{\tau}}(\mathbf{n}(Z_H)H - \mathbf{n}(Z_L)L) + rP_0e^{r\tau}(\mathbf{N}(z_H+\sigma\cdot H\sqrt{\tau}) - \mathbf{N}(z_L+\sigma\cdot L\sqrt{\tau}))\right]
\end{aligned}
$$

We discuss the limiting behavior of $\frac{\partial f}{\partial \tau}$ for ATM, OTM and ITM call options in turn.

(1) For ATM call option, i.e. $K = P_0e^{r\tau}$, we have

$$\mathbf{n}(z_H) = \mathbf{n}(z_L) = \mathbf{n}(z_A) \to \frac{1}{\sqrt{2\pi}} \quad and \quad \mathbf{N}(z_H) = \mathbf{N}(z_L) = \mathbf{N}(z_A) \to \frac{1}{2}, \quad when \quad \tau \to 0$$

24 $\frac{\partial Var_L}{\partial \tau}$ can be written in the similar format with $L$ replacing $H$ .

Then we have the limiting behavior $\frac{\partial E(W_a)}{\partial \tau}$ and $\frac{\partial Var(W_a)}{\partial \tau}$ [25]as follows,

$$\frac{\partial E(W_a)}{\partial \tau} = \frac{K\sigma}{2\sqrt{2\pi\tau}}(A - \phi H - (1-\phi)L) + o(\frac{1}{\sqrt{\tau}})$$
$$\frac{\partial Var(W_a)}{\partial \tau} = \phi P_0^2\sigma^2 H^2(\Delta_a^2 - \Delta_a + \frac{1}{2} - \frac{1}{2\pi}) + (1-\phi)P_0^2\sigma^2 L^2(\Delta_a^2 - \Delta_a + \frac{1}{2} - \frac{1}{2\pi}) + \phi(1-\phi)\frac{K^2\sigma^2(H-L)^2}{4\pi} + o(1)$$

As $\tau \to 0$, $\frac{\partial E(W_a)}{\partial \tau}$approaches positive infinity [26] , while $\frac{\partial Var(W_a)}{\partial \tau}$ is a constant. Therefore, $\frac{\partial f}{\partial \tau} > 0$ as time approaches maturity date, which implies that $\frac{\partial A}{\partial \tau} < 0$.

(2) For OTM call option, i.e. $K > P_0e^{r\tau}$, we have

$$\frac{\mathbf{n}(z_H)}{\sqrt{\tau}} = \frac{\mathbf{n}(z_L)}{\sqrt{\tau}} = \frac{\mathbf{n}(z_A)}{\sqrt{\tau}} \to 0 \quad and \quad \mathbf{N}(z_H) = \mathbf{N}(z_L) = \mathbf{N}(z_A) \to 1, \quad when \quad \tau \to 0$$

Then, the limiting behavior of $\frac{\partial E(W_a)}{\partial \tau}$ and $\frac{\partial Var(W_a)}{\partial \tau}$ is as follows,

$$\begin{aligned}\frac{\partial E(W_a)}{\partial \tau} &= o(1) \\ \frac{\partial Var(W_a)}{\partial \tau} &= \phi P_0^2\Delta_a^2\sigma^2 H^2 + (1-\phi)P_0^2\Delta_a^2\sigma^2 L^2 + o(1)\end{aligned}$$

---

[25]The last term of $\frac{\vartheta Var_H}{\vartheta \tau}$ is messy, and needs special attention. The detailed derivation, assuming $\tau \to 0$, follows.

$$\left[K^2\mathbf{N}(z_H) - KP_0e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) - \Delta K^2 + \Delta KP_0e^{r\tau}\right]\frac{\sigma H}{\sqrt{\tau}}\mathbf{n}(z_H) = \left[K^2\mathbf{N}(-\frac{1}{2}\sigma H\sqrt{\tau}) - K^2\mathbf{N}(\frac{1}{2}\sigma H\sqrt{\tau})\right]\frac{\sigma H}{\sqrt{\tau}}\mathbf{n}(z_H)$$

$$= -\sigma^2H^2K^2 \times \frac{\mathbf{N}(\frac{1}{2}\sigma H\sqrt{\tau}) - \mathbf{N}(-\frac{1}{2}\sigma H\sqrt{\tau})}{\sigma H\sqrt{\tau}} \times \mathbf{n}(-\frac{1}{2}\sigma H\sqrt{\tau})$$

As time $\tau$ goes to 0, we have

$$\lim_{\tau\to 0}\frac{\mathbf{N}(\frac{1}{2}\sigma H\sqrt{\tau}) - \mathbf{N}(-\frac{1}{2}\sigma H\sqrt{\tau})}{\sigma H\sqrt{\tau}} = \mathbf{n}(0) = \frac{1}{\sqrt{2\pi}} \quad and \quad \lim_{\tau\to 0}\mathbf{n}(-\frac{1}{2}\sigma H\sqrt{\tau}) = \mathbf{n}(0) = \frac{1}{\sqrt{2\pi}}$$

Therefore,

$$\lim_{\tau\to 0}\left[K^2\mathbf{N}(z_H) - KP_0e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) - \Delta K^2 + \Delta KP_0e^{r\tau}\right]\frac{\sigma H}{\sqrt{\tau}}\mathbf{n}(z_H) = -\frac{1}{2\pi}\sigma^2H^2K^2$$

The limiting behavior of $\frac{\vartheta(E_H - E_L)^2}{\vartheta \tau}$ is also tricky, and the derivation is

$$\begin{aligned}\frac{\vartheta(E_H - E_L)^2}{\vartheta \tau} &= [E_H\{C_\tau\} - E_L\{C_\tau\}] \times \frac{K\sigma}{2\sqrt{\tau}}(\mathbf{n}(Z_H)H - \mathbf{n}(Z_L)L) + o(1) \\ &= 2P_0e^{r\tau}\left[\mathbf{N}(\frac{1}{2}\sigma H\sqrt{\tau}) - \mathbf{N}(\frac{1}{2}\sigma L\sqrt{\tau})\right] \times \frac{K\sigma}{2\sqrt{\tau}}(\mathbf{n}(Z_H)H - \mathbf{n}(Z_L)L) + o(1) \\ &= \frac{1}{2}KP_0e^{r\tau}\sigma^2(H-L)^2 \times \frac{\mathbf{N}(\frac{1}{2}\sigma H\sqrt{\tau}) - \mathbf{N}(\frac{1}{2}\sigma L\sqrt{\tau})}{\frac{1}{2}\sigma(H-L)\sqrt{\tau}} \times \mathbf{n}(0) + o(1)\end{aligned}$$

As time $\tau$ goes to 0, we have

$$\lim_{\tau\to 0}\frac{\vartheta(E_H - E_L)^2}{\vartheta \tau} = \frac{K^2\sigma^2(H-L)^2}{4\pi} + o(1)$$

[26]To compensate for the hedging uncertainty and transaction cost, we have $E(W_a) > 0$. For ATM call option,

$$\begin{aligned}&E(W_a) = K[2\cdot\mathbf{N}(\frac{1}{2}\sigma A\sqrt{\tau}) - 1] - \phi\cdot K[2\cdot\mathbf{N}(\frac{1}{2}\sigma H\sqrt{\tau}) - 1] - (1-\phi)K[2\cdot\mathbf{N}(\frac{1}{2}\sigma L\sqrt{\tau}) - 1] > 0 \\ &\Rightarrow \mathbf{N}(\frac{1}{2}\sigma A\sqrt{\tau}) - \phi\cdot\mathbf{N}(\frac{1}{2}\sigma H\sqrt{\tau}) - (1-\phi)\cdot\mathbf{N}(\frac{1}{2}\sigma L\sqrt{\tau}) > 0 \\ &\Rightarrow \left(\mathbf{N}(0) + \frac{1}{2}\sigma A\sqrt{\tau}\mathbf{n}(0)\right) - \phi\cdot\left(\mathbf{N}(0) + \frac{1}{2}\sigma H\sqrt{\tau}\mathbf{n}(0)\right) - (1-\phi)\cdot\left(\mathbf{N}(0) + \frac{1}{2}\sigma L\sqrt{\tau}\mathbf{n}(0)\right) + o(\sqrt{\tau}) > 0 \\ &\Rightarrow A - \phi H - (1-\phi)L + o(1) > 0 \Rightarrow A - (1-\lambda)H - \lambda L > 0 \quad as \quad \tau \to 0\end{aligned}$$

As $\tau \to 0$, $\frac{\partial E(W_a)}{\partial \tau}$ becomes 0, while $\frac{\partial Var(W_a)}{\partial \tau}$ becomes a constant positive number. Therefore, $\frac{\partial f}{\partial \tau} < 0$ as time approaches maturity date, which implies that $\frac{\partial A}{\partial \tau} > 0$.

(3) For ITM call option, i.e. $K < P_0 e^{r\tau}$, we have

$$\frac{\mathbf{n}(z_H)}{\sqrt{\tau}} = \frac{\mathbf{n}(z_L)}{\sqrt{\tau}} = \frac{\mathbf{n}(z_A)}{\sqrt{\tau}} \to 0 \quad and \quad \mathbf{N}(z_H) = \mathbf{N}(z_L) = \mathbf{N}(z_A) \to 1, \quad when \quad \tau \to 0$$

Then, the limiting behavior $\frac{\partial E(W_a)}{\partial \tau}$ and $\frac{\partial Var(W_a)}{\partial \tau}$ is

$$\begin{aligned}\frac{\partial E(W_a)}{\partial \tau} =& o(1)\\ \frac{\partial Var(W_a)}{\partial \tau} =& \phi P_0^2(1-\Delta_a)^2\sigma^2 H^2 + (1-\phi)P_0^2(1-\Delta_a)^2\sigma^2 L^2 + o(1)\end{aligned}$$

As $\tau \to 0$, $\frac{\partial E(W_a)}{\partial \tau}$ becomes 0, while $\frac{\partial Var(W_a)}{\partial \tau}$ becomes a positive constant. Therefore, $\frac{\partial f}{\partial \tau} < 0$ as time approaches maturity date, which implies that $\frac{\partial A}{\partial \tau} > 0$.

**Appendix B-4 (Second Derivative of $\tau$)**

$$\frac{\partial^2 A}{\partial \tau^2} = -\frac{\partial\left[\frac{\partial f}{\partial \tau}/\frac{\partial f}{\partial A}\right]}{\partial \tau} = \frac{\frac{\partial f}{\partial \tau}\frac{\partial^2 f}{\partial A \partial \tau} - \frac{\partial^2 f}{\partial \tau^2}\frac{\partial f}{\partial A}}{\left(\frac{\partial f}{\partial A}\right)^2}$$

Based on B-S model, we have

$$\begin{aligned}\frac{\partial f}{\partial A} =& K\sigma\sqrt{\tau}\mathbf{n}(z_A) > 0\\ \frac{\partial^2 f}{\partial A \partial \tau} =& K\sigma\sqrt{\tau}\mathbf{n}(z_A)\left[\frac{1}{2\tau} - z_A \times \frac{\partial z_A}{\partial \tau}\right]\end{aligned}$$

In Proposition 1, we have already discussed the limiting behavior of $\frac{\partial f}{\partial \tau}$ for ATM ,OTM and ITM call options, and now we continue on discussing the limiting behavior of $\frac{\partial^2 f}{\partial \tau^2}$ for the three different cases.

(1) For ATM call option, we have

$$\begin{aligned}\tfrac{\partial^2 E(W_a)}{\partial \tau^2} &= -\tfrac{K\sigma}{4\cdot\sqrt{2\pi\tau^3}}(A - \phi H - (1-\phi)L) + o(\tfrac{1}{\sqrt{\tau^3}})\\ \tfrac{\partial Var(W_a)}{\partial \tau} &= O(1)\\ \tfrac{\partial^2 f}{\partial A \partial \tau} &= \tfrac{K\sigma\mathbf{n}(z_A)}{2\sqrt{\tau}} + o(\tfrac{1}{\sqrt{\tau}})\end{aligned}$$

As $\tau \to 0$, $\frac{\partial^2 E(W_a)}{\partial \tau^2}$ becomes negative, $\frac{\partial Var(W_a)}{\partial \tau}$ tends to a positive constant, so $\frac{\partial^2 f}{\partial \tau^2}$ is negative. Also, as $\tau \to 0$, $\frac{\partial^2 f}{\partial A \partial \tau}$ becomes positive. Therefore, we can conclude that as time approaches maturity date, the ask volatility will increase at an increasing rate.

(2) For OTM call option, we have

$$\frac{\partial^2 E(W_a)}{\partial \tau^2} = O(\frac{\mathbf{n}(z_A)}{\sqrt{\tau^5}}) \quad \frac{\partial^2 Var(W_a)}{\partial \tau^2} = O(1) \quad and \quad \frac{\partial^2 f}{\partial A \partial \tau} = O(\frac{\mathbf{n}(z_A)}{\sqrt{\tau^3}})$$

As $\tau \to 0$, $\frac{\partial^2 E(W_a)}{\partial \tau^2}$ becomes 0, $\frac{\partial^2 Var(W_a)}{\partial \tau^2}$ tends to a positive constant, so $\frac{\partial^2 f}{\partial \tau^2}$ is a negative constant . The order of $\frac{\partial f}{\partial A}$ is $O(\mathbf{n}(z_A)\sqrt{\tau})$, while the order of $\frac{\partial^2 f}{\partial A \partial \tau}$ is $O(\frac{\mathbf{n}(z_A)}{\sqrt{\tau^3}})$, which implies that $\frac{\partial^2 f}{\partial A \partial \tau}$ dominates $\frac{\partial f}{\partial A}$. Therefore, as time approaches the maturity date, the ask volatility will decrease at a decreasing rate.

(3) For ITM call option, we have the same conclusion as for the OTM call option. The insight lies in noting that for OTM and ITM call option, the normal density function of $z$ is an infinitesimal in any order of $\tau$, i.e. $\frac{\mathbf{n}(z)}{\tau^m} \to 0$ as $\tau \to 0$, for all $m \in R$.

**Appendix B-5 (The Volatility Level Effect)**

Consider

$$\frac{\partial A}{\partial \sigma} = -\frac{\frac{\partial f}{\partial \sigma}}{\frac{\partial f}{\partial A}}$$

In Proposition 2, we proved that $\frac{\partial f}{\partial A} = K\sigma\sqrt{\tau}\mathbf{n}(z_A) > 0$. Hence, we only need to determine the sign of $\frac{\partial f}{\partial \sigma} = \frac{\partial E(W_a)}{\partial \sigma} - \frac{Q}{\gamma}\frac{\partial Var(W_a)}{\partial \sigma}$.

For $\frac{\partial E(W_a)}{\partial \sigma}$, we have

$$\frac{\partial E(W_a)}{\partial \sigma} = K\left[\mathbf{n}(z_A)A\sqrt{\tau} - \phi \cdot \mathbf{n}(z_H)H\sqrt{\tau} - (1-\phi) \cdot \mathbf{n}(z_L)L\sqrt{\tau}\right]$$

For $\frac{\partial Var(W_a)}{\partial \sigma}$, we first calculate $\frac{\partial Var_H}{\partial \sigma}$ using the Chain Rule.

$$\begin{aligned}\frac{\partial Var_H}{\partial \sigma} =& \frac{\partial C}{\partial \sigma} + \frac{\partial A_1}{\partial \sigma}\mathbf{N}(2\sigma H\sqrt{\tau} + z_H) + \frac{\partial A_2}{\partial \sigma}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) + \frac{\partial A_3}{\partial \sigma}\mathbf{N}(z_H) \\ &+ A_1\frac{\partial \mathbf{N}(2\sigma H\sqrt{\tau} + z_H)}{\partial \sigma} + A_2\frac{\partial \mathbf{N}(\sigma H\sqrt{\tau} + z_H)}{\partial \sigma} + A_3\frac{\partial \mathbf{N}(z_H)}{\partial \sigma}\end{aligned}$$

For every part of $\frac{\partial Var_H}{\partial \sigma}$, we have

$\frac{\partial C}{\partial \sigma} = 2P_0^2\Delta_a^2 H^2\sigma\tau e^{2r\tau+\sigma^2H^2\tau}$

$\frac{\partial A_1}{\partial \sigma}\mathbf{N}(2\sigma \cdot H\sqrt{\tau} + z_H) = 2(1-2\Delta_a)P_0^2H^2\tau\sigma e^{2r\tau+\sigma^2H^2\tau}\mathbf{N}(2\sigma H\sqrt{\tau} + z_H)$

$A_1\frac{\partial \mathbf{N}(2\sigma H\sqrt{\tau}+z_H)}{\partial \sigma} = K^2(1-2\Delta_a)\left(2H\sqrt{\tau} + \frac{\partial z_H}{\partial \sigma}\right)\mathbf{n}(z_H)$

$\frac{\partial A_2}{\partial \sigma}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) = P_0Ke^{r\tau}\left(\frac{\partial z_H}{\partial \sigma} - H\sqrt{\tau}\right)\mathbf{n}(z_H)\mathbf{N}(\sigma H\sqrt{\tau} + z_H)$

$A_2\frac{\partial \mathbf{N}(\sigma H\sqrt{\tau}+z_H)}{\partial \sigma} = \left\{-KP_0e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) - 2K^2\mathbf{N}(-z_H) + 2\Delta_aK^2 + 2\Delta_aKP_0e^{r\tau}\right\}\left(H\sqrt{\tau} + \frac{\partial z_H}{\partial \sigma}\right)\mathbf{n}(z_H)$

$\frac{\partial A_3}{\partial \sigma}\mathbf{N}(z_H) = -K^2\frac{\partial z_H}{\partial \sigma}\mathbf{n}(z_H)\mathbf{N}(z_H)$

$A_3\frac{\partial \mathbf{N}(z_H)}{\partial \sigma} = \left(K^2\mathbf{N}(-z_H) - 2\Delta_aP_0Ke^{r\tau}\right)\frac{\partial z_H}{\partial \sigma}\mathbf{n}(z_H)$

Adding up all the parts and rearranging, we finally have[27]

$$\begin{aligned}\frac{\partial Var_H}{\partial \sigma} &= 2P_0^2\Delta^2H^2\tau\sigma e^{2r\tau+\sigma^2H^2\tau} + 2(1-2\Delta_a)P_0^2H^2\tau\sigma e^{2r\tau+\sigma^2H^2\tau}\mathbf{N}(2\sigma H\sqrt{\tau} + z_H) \\ &+2\left[K^2\mathbf{N}(z_H) - KP_0e^{r\tau}\mathbf{N}(\sigma H\sqrt{\tau} + z_H) - \Delta_aK^2 + \Delta_aKP_0e^{r\tau}\right]H\sqrt{\tau}\mathbf{n}(z_H)\end{aligned}$$

To calculate $\frac{\partial Var(W_a)}{\partial \sigma}$, we also need to calculate $\frac{\partial (E_H - E_L)^2}{\partial \sigma}$ and the details follow.

$$\begin{aligned}\frac{\partial (E_H - E_L)^2}{\partial \sigma} &= [E_H\{C_\tau\} - E_L\{C_\tau\}] \times \frac{\partial (E_H\{C_\sigma\} - E_L\{C_\tau\})}{\partial \sigma} \\ &= [E_H\{C_\tau\} - E_L\{C_\tau\}] \times K\left[\mathbf{n}(z_H)H\sqrt{\tau} - \mathbf{n}(z_L)L\sqrt{\tau}\right]\end{aligned}$$

[27] $\frac{\vartheta Var_L}{\vartheta \sigma}$ can be written in the similar format with $L$ replacing $H$ .

We will discuss the limiting behavior of $\frac{\partial f}{\partial \sigma}$ for ATM, OTM and ITM call options in turn.

(1) For ATM call option, i.e. $K = P_0 e^{r\tau}$, we have

$$\mathbf{n}(z_H) = \mathbf{n}(z_L) = \mathbf{n}(z_A) \to \frac{1}{\sqrt{2\pi}} \quad and \quad \mathbf{N}(z_H) = \mathbf{N}(z_L) = \mathbf{N}(z_A) \to \frac{1}{2}, \quad when \quad \tau \to 0$$

Then we have the limiting behavior $\frac{\partial E(W_a)}{\partial \tau}$ and $\frac{\partial Var(W_a)}{\partial \tau}$ as follows,

$$\frac{\partial E(W_a)}{\partial \sigma} = \frac{K\sqrt{\tau}}{\sqrt{2\pi}}(A - \phi H - (1-\phi)L) + o(\sqrt{\tau})\tilde{}O(\sqrt{\tau}) \quad and \quad \frac{\partial Var(W_a)}{\partial \sigma} = O(\tau)$$

As $\tau \to 0$, $\frac{\partial E(W_a)}{\partial \sigma}$ dominates $\frac{\partial Var(W_a)}{\partial \sigma}$ . Therefore, $\frac{\partial f}{\partial \sigma} > 0$ as time approaches maturity date, which implies that $\frac{\partial A}{\partial \sigma} < 0$.

(2) For OTM call option, i.e. $K > P_0 e^{r\tau}$, we have

$$\frac{\partial E(W_a)}{\partial \sigma} = O(\mathbf{n}(z)\sqrt{\tau}) \quad and \quad \frac{\partial Var(W_a)}{\partial \sigma} = O(\tau)$$

Since $\mathbf{n}(z)$ is an infinitesimal of $o(\tau^n)$ for any $n$ when $z \to \propto$, $\frac{\partial Var(W_a)}{\partial \sigma}$ dominates $\frac{\partial E(W_a)}{\partial \sigma}$ , which implies that $\frac{\partial f}{\partial \sigma} < 0$ as $\tau \to 0$. Therefore, $\frac{\partial A}{\partial \sigma} > 0$ as time approaches maturity date.

(3) For ITM call option, i.e. $K > P_0 e^{r\tau}$, we use the similar reasoning as in OTM, and we reach the same conclusion.

# Appendix C

**Appendix C-1 (The optimal dynamic hedging strategy)**

We first compute the variance over two periods. According to the law of total variance, the total variance for a two-periods model is

$$\begin{aligned}
& Var_0\left(W_{a,2}\right) \\
& = E_0[Var_1(W_{a,2})] + Var_0\left[E_1(W_{a,2})\right] \\
& \quad \because Var_1(W_{a,2}) = E_1[Var_2(W_{a,2})] + Var_1\left[E_2(W_{a,2})\right] \;\; and \; Var_2(W_{a,2}) = 0 \\
& = E_0[Var_0[E_1(W_{a,2})] + Var_1\left[E_2(W_{a,2})\right]] \\
& = E_0\left[Var_0\left(-C_{1,2} + \Delta_0 P_1\right) + Var_1\left(-C_{2,2} + \Delta_1 P_2\right)\right]
\end{aligned}$$

where

$$\begin{aligned}
& Var_0\left\{E_1(W_{a,2})\right\} \\
& = Var_0\left\{\left[-e^{-r\Delta\tau} \cdot E_1(P_2 - K)^+ + \Delta_0 e^{-r\Delta\tau} \cdot E_1(P_2)\right]\right\} \\
& = Var_0\left\{\left[-C_{1,2} + \Delta_0 P_1\right]\right\}
\end{aligned}$$

$$Var_1\left\{E_2(W_{a,2})\right\} = Var_1[-E_2(P_2 - K)^+ + \Delta_1 P_2] = Var_1\left[-C_{2,2} + \Delta_1 P_2\right]$$

Therefore,

$$\begin{aligned}
& Var_0\left(W_{a,2}\right) \\
& = E_0\left\{Var_0[-C_{1,2} + \Delta_0 P_1] + Var_1\left[-C_{2,2} + \Delta_1 P_2\right]\right\}
\end{aligned}$$

We let $C_{1,2}$ denote the equilibrium call price at time 1 where $C_{1,2}e^{r\Delta\tau} = E_1(P_2 - K)^+$. In addition, $C_{2,2} = (P_2 - K)^+$.

Now we derive the optimal hedging strategy. We follow the methodology in Basak and Chabakauri (2010) and apply dynamic programming to the value function $J_t$ , which is defined as

$$J = Var_t(W_{a,2}).$$

The law of total variance yields a recursive representation for the value function.

$$J_t = \min_{\Delta_t} \left\{ E_t(J_{t+\Delta\tau}) + Var_t[-E_{t+\Delta\tau}([P_T - K]^+ + \Delta_t P_{t+\Delta\tau})] \right\}.$$

where $\Delta_t$ is the stock holding and $\Delta_\tau$is time interval. We first check optimization for period 1.

$$J_0 = \min_{\Delta_0} \left\{ E_0(J_1) + Var_0[-E_1([P_2 - K]^+ + \Delta_0 P_1)] \right\} = \min_{\Delta_0} \{E_0(J_1) + Var_0[-C_{1,2} + \Delta_0 P_1]\}.$$

By F.O.C, we get optimal $\Delta_0$as,

$$\Delta_0^* = \tfrac{Cov_0(C_{1,2},P_1)}{Var_0[P_1]}.$$

We continue to get optimal $\Delta_1$ for period 2.

$$J_1 = \min_{\Delta_1} E_1(J_2) + Var_1[-E_2([P_2 - K]^+ + \Delta_1 P_2)] = \min_{\Delta_0} E_0(J_2) + Var_1[-C_{2,2} + \Delta_1 P_2].$$

The solution is

$$\Delta_1^* = \tfrac{Cov_1(C_{2,2},P_2)}{Var_1[P_2]}.$$

The general solution for multiple-periods model is also provided by Basak and Chabakauri (2012). To get analytical solution for $\Delta^*$, we advance to compute covariance. Given in each period we have two possible realizations $(\mu_H\Delta\tau, \sigma_H^2\Delta\tau)$ and $(\mu_L\Delta\tau, \sigma_L^2\Delta\tau)$ with a bernoulli random arrival rate, law of total covariance yields

$$\begin{aligned}
&Cov_1(C_{2,2}, P_1) \\
&= (1-\phi)Cov_{1,L}\left[C_{2,2}, P_2\right] + \phi Cov_{1,H}\left[C_{2,2}, P_2\right] \\
&+ Cov_1\left[(1-\phi)E_{1,L}\left(C_{2,2}\right) + \phi E_{1,H}\left(C_{2,2}\right), (1-\phi)E_{1,L}\left(P_2\right) + \phi E_{1,H}\left(P_2\right)\right].
\end{aligned}$$

Given $E_1(P_2|V) = P_1 e^{r\Delta\tau}$ is constant by assumption (4), we simplify the covariance as

$$\begin{aligned}
Cov_1(C_{2,2}, P_2) &= (1-\phi)Cov_{1,L}\left[C_{2,2}, P_2\right] + \phi Cov_{1,H}\left[C_{2,2}, P_2\right], \\
Cov_0(C_{1,2}, P_1) &= (1-\phi)Cov_{0,L}\left[C_{1,2}, P_1\right] + \phi Cov_{0,H}\left[C_{1,2}, P_1\right].
\end{aligned}$$

**Appendix C-2 (The optimal dynamic hedging strategy)**

Here we compute $C_{1,2}$ and $C_{0,2}$.

$$\begin{aligned}
&C_{1,2} = E_1[(P_2 - K)^+] \\
&= e^{-r\Delta\tau}\left\{\phi E_1\left[(P_2-K)^+ | X_1 = x_1, V_2 = \sigma_H\right] + (1-\phi)E_1\left[(P_2-K)^+ | X_1 = x_1, V_2 = \sigma_L\right]\right\} \\
&= \phi BSM_{x_1,H} + (1-\phi)BSM_{x_1,L},
\end{aligned}$$

where $x_1$, $\sigma_H$, $\sigma_L$ are the realizations of $X_1$ and $V_2$.

$$BSM_{x_1,H} = P_1 \cdot \mathbf{N}\left(\frac{\ln(\frac{P_1}{K}) + [r\Delta\tau + \frac{\sigma_H^2\Delta\tau}{2}]}{\sigma_H\sqrt{\Delta\tau}}\right) - e^{-r\triangle\tau}K \cdot \mathbf{N}\left(\frac{\ln(\frac{P_1}{K}) + [r\Delta\tau - \frac{\sigma_H^2\Delta\tau}{2}]}{\sigma_H\sqrt{\Delta\tau}}\right)$$

$$BSM_{x_1,L} = P_1 \cdot \mathbf{N}\left(\frac{\ln(\frac{P_1}{K}) + [r\Delta\tau + \frac{\sigma_L^2\Delta\tau}{2}]}{\sigma_L\sqrt{\Delta\tau}}\right) - e^{-r\triangle\tau}K \cdot \mathbf{N}\left(\frac{\ln(\frac{P_1}{K}) + [r\Delta\tau - \frac{\sigma_L^2\Delta\tau}{2}]}{\sigma_L\sqrt{\Delta\tau}}\right).$$

$$
\begin{aligned}
&C_{0,2} = e^{-2r\Delta\tau} E_0[(P_2-K)^+] \\
&= e^{-2r\Delta\tau} \left\{ \phi^2 E_0 \left( E_1 \left[ (P_2-K)^+ \,|X_1, V_2=\sigma_H \right] |V_1=\sigma_H \right) + (1-\phi)\phi E_0 \left( E_1 \left[ (P_2-K)^+ \,|X_1, V_2=\sigma_H \right] |V_1=\sigma_L \right) \right\} \\
&+ e^{-2r\triangle\tau} \left\{ (\phi(1-\phi) E_0 \left( E_1 \left[ (P_2-K)^+ \,|X_1, V_2=\sigma_L \right] |V_1=\sigma_H \right) + (1-\phi)^2 E_0 \left( E_1 \left[ (P_2-K)^+ \,|X_1, V_2=\sigma_L \right] |V_1=\sigma_L \right) \right\} \\
&= \phi^2 BSM_{H,H} + 2\phi(1-\phi) BSM_{H,L} + (1-\phi)^2 BSM_{L,L}
\end{aligned}
$$

The following is the derivation for $BSME_{H,H}$, $BSME_{H,L}$and $BSM_{L,L}$. Here in Appendix C2, for the notation convenience, we let $\Delta\tau = 1$.

$$
\begin{aligned}
&E_0 \left( E_1 \left[ (P_2-K)^+ \,|X_1, V_2=\sigma_H \right] |V_1=\sigma_H \right) \\
&= E_0(E_1[P_2 \cdot I(P_2>K)|X_1, V_2=\sigma_H]|V_1=\sigma_H) - E_0(E_1[K \cdot I(P_2>K)|X_1, V_2=\sigma_H]|V_1=\sigma_H)
\end{aligned}
$$

Let $f_{X_1,H}$ the normal density function. The first term can be derived as

$$
\begin{aligned}
&E_0(E_1[P_2 \cdot I(P_2>K)|X_1, V_2=\sigma_H]|V_1=\sigma_H) \\
&= \int_{X_{1,H}} P_0 e^{X_{1,H}+X_{2,H}} \Pr(X_{2,H} > \ln(\tfrac{K}{P_0}) - X_{1,H}|X_{1,H}) f_{X_{1,H}} dX_{1,H} \\
&\quad \because X_{2,H} = u_H + \sigma \cdot H\varepsilon_2 \\
&= \int_{X_{1,H}} P_0 e^{X_{1,H}+u_H+\sigma_H\varepsilon_2} \Pr(\varepsilon_2 > \frac{\ln(\frac{K}{P_0})-X_{1,H}-u_H}{\sigma_H}|X_{1,H}) f_{X_{1,H}} dX_{1,H}. \\
&\quad\quad let\; Z = \frac{\ln(\frac{K}{P_0})-X_{1,H}-u_H}{\sigma_H} \\
&= \int_{X_{1,H}} P_0 e^{X_{1,H}+u_H} \left[ \int_{\varepsilon_2>Z} e^{\sigma_H\varepsilon_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{\varepsilon_2^2}{2}} d\varepsilon_2 \right] f_{X_{1,H}} dX_{1,H} \\
&= \int_{X_{1,H}} P_0 e^{X_{1,H}+u_H+\frac{\sigma^2 H^2}{2}} \left[ \int_Z \frac{1}{\sqrt{2\pi}} e^{-\frac{(\varepsilon_2-\sigma_H)^2}{2}} d\varepsilon_2 \right] f_{X_{1,H}} dX_{1,H} \\
&= \int_{X_{1,H}} P_0 e^{X_{1,H}+u_H+\frac{\sigma_H^2}{2}} \left[ 1 - \mathbf{N}(\frac{\ln(\frac{K}{P_0})-X_{1,H}-u_H}{\sigma_H} - \sigma_H) \right] f_{X_{1,H}} dX_{1,H} \\
&= \int_{X_{1,H}} P_0 e^{u_H+\sigma\varepsilon_1+u_H+\frac{\sigma_H^2}{2}} \left[ \mathbf{N}(\frac{X_{1,H}-\left[\ln(\frac{K}{P_0})-u_H-\sigma_H^2\right]}{\sigma_H}) \right] \frac{1}{\sqrt{2\pi}\sigma_H} e^{-\frac{(X_{1,H}-u_H)^2}{2\sigma_H^2}} d[\sigma_H\varepsilon_1] \\
&\quad Note\, \mathbf{N}(\frac{X_{1,H}-[\ln(\frac{K}{P_0})-u_H-\sigma_H^2}{\sigma_H}) = \Pr(\varepsilon_2 > \frac{\ln(\frac{K}{P_0})-X_{1,H}-u_H-\sigma_H^2}{\sigma_H}). \\
&= P_0 e^{2[u+\frac{\sigma_H^2}{2}]} \int_{\varepsilon_1} \left[ \mathbf{N}(\frac{\ln(\frac{P_0}{K})+2u_H+\sigma_H^2}{\sigma_H} + \varepsilon_1) \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{(\varepsilon_1-\sigma_H)^2}{2}} d\varepsilon_1 \\
&\quad apply\; theorem: \int \mathbf{N}(m+s\varepsilon_1) \frac{1}{\sqrt{2\pi}} e^{-\frac{(\varepsilon_1-g)^2}{2}} d\varepsilon_1 = \mathbf{N}\left( \frac{\frac{m}{s}+g}{\sqrt{(\frac{1}{s})^2+1}} \right). \\
&= P_0 e^{2[u_H+\frac{\sigma_H^2}{2}]} \mathbf{N}\left( \frac{\ln(\frac{P_0}{K})+2u_H+2\sigma_H^2}{\sqrt{2}\sigma_H} \right).
\end{aligned}
$$

Then we derive the second term.

$$
\begin{aligned}
&E_0(E_1[K \cdot I(P_2>K|X_1, V_2=\sigma_H)|V_1=\sigma_H]) \\
&= K \int_{X_{1,H}} \Pr(X_{2,H} > \ln(\tfrac{K}{P_0}) - X_{1,H}|X_{1,H}) f_{X_{1,H}} dX_{1,H} \\
&= K \int_{X_{1,H}} \Pr(\varepsilon_2 > \frac{\ln(\frac{K}{P_0})-X_{1,H}-u_H}{\sigma_H}|X_{1,H}) f_{X_{1,H}} dX_{1,H} \\
&= \int_{X_{1,H}} K \left[ \mathbf{N}(\frac{-\ln(\frac{K}{P_0})+X_{1,H}+u_H}{\sigma_H}) \right] \frac{1}{\sqrt{2\pi}\sigma_H} e^{-\frac{(X_1-u_H)^2}{2\sigma_H^2}} dX_{1,H} \\
&= \int_{\varepsilon_1} K \left[ \mathbf{N}(\frac{\ln(\frac{P_0}{K})+2u_H}{\sigma_H} + \varepsilon_1) \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{(\varepsilon_1)^2}{2}} d\varepsilon_1 \\
&= K \cdot \mathbf{N}\left( \frac{\ln(\frac{P_0}{K})+2u_H}{\sqrt{2}\sigma_H} \right)
\end{aligned}
$$

Given $u_H + \frac{(\sigma_H)^2}{2} = r$,

$$\begin{aligned}
&e^{-2r}\left\{E_0\left(E_1\left[(P_2-K)^+\,|X_1, V_2=\sigma_H\right]|V_1=\sigma_H\right)\right\}\\
&= P_0\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r+\frac{\sigma_H^2}{2}]*2}{\sigma_H\sqrt{2}}\right) - e^{-2r}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r-\frac{\sigma_H^2}{2}]*2}{\sigma_H\sqrt{2}}\right)\\
&= BSM_{H,H}
\end{aligned}$$

Using the same procedure, we can derive

$$\begin{aligned}
&e^{-2r}\left\{E_0\left(E_1\left[(P_2-K)^+\,|X_1, V_2=\sigma_L\right]|V_1=\sigma_L\right)\right\}\\
&= P_0\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r+\frac{\sigma_L^2}{2}]*2}{\sigma_L\sqrt{2}}\right) - e^{-2r}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r-\frac{\sigma_L^2}{2}]*2}{\sigma_L\sqrt{2}}\right)\\
&= BSM_{L,L}
\end{aligned}$$

and

$$\begin{aligned}
&e^{-2r}\left\{E_0\left(E_1\left[(P_2-K)^+\,|X_1, V_2=\sigma_H\right]|V_1=\sigma_L\right)\right\}\\
&= P_0\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r+\frac{\sigma_H^2}{2}+r+\frac{\sigma_L^2}{2}]}{\sqrt{\sigma_H^2+\sigma_L^2}}\right) - e^{-2r}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r-\frac{\sigma_H^2}{2}+r-\frac{\sigma_L^2}{2}]}{\sqrt{\sigma_H^2+\sigma_L^2}}\right)\\
&= BSM_{L,H}
\end{aligned}$$

and

$$\begin{aligned}
&e^{-2r}\left\{E_0\left(E_1\left[(P_2-K)^+\,|X_1, V_2=\sigma L\right]|V_1=\sigma H\right)\right\}\\
&= P_0\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r+\frac{\sigma_H^2}{2}+r+\frac{\sigma_L^2}{2}]}{\sqrt{\sigma_H^2+\sigma_L^2}}\right) - e^{-2r}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_0}{K})+[r-\frac{\sigma_H^2}{2}+r-\frac{\sigma_L^2}{2}]}{\sqrt{\sigma_H^2+\sigma_L^2}}\right)\\
&= BSM_{H,L}
\end{aligned}$$

**Appendix C-3 (Proof of Proposition 5)**

Let $Y = \sum_{j=t+\Delta\tau}^{T} X_j$, $P_T = P_0 e^Y$ with each $X_j$ a Bernoulli-type random normal distribution.
Therefore, we know $C_{t,T} = \sum_{i=0}^{n}\binom{n}{i}\phi^i(1-\phi)^{n-i}E_t\left[(P_T-K)^+|Var(Y) = i\cdot(\sigma_H)^2\Delta\tau + (n-i)\cdot(\sigma_L)^2\Delta\tau\right]$

$$\begin{aligned}
&C_{t,T} = \sum_{i=0}^{n}\binom{n}{i}\phi^i(1-\phi)^{n-i}\left\{P_t\cdot\mathbf{N}(d_1) - e^{-nr\Delta\tau}K\cdot\mathbf{N}(d_2)\right\}\\
&d_1 = \frac{\ln(\frac{P_t}{K})+i\cdot[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}]+(n-i)\cdot[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n-i)\sigma_L^2\Delta\tau}} \qquad ,\\
&d_2 = \frac{\ln(\frac{P_t}{K})+i\cdot[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}]+(n-i)\cdot[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n-i)\sigma_L^2\Delta\tau}}, n = \frac{T}{\Delta\tau}
\end{aligned}$$

Next,

$$\begin{aligned}
&Cov_t(C_{t+1,T}\ , P_{t+1})\\
&= E_t\left[Cov_t(C_{t+1,T}, P_{t+1}|V_{t+1})\right] + Cov_t\left[E_t(C_{t+1,T}|V_{t+1}), E_t(P_{t+1}|V_{t+1})\right]\\
&\quad \because E_t(P_{t+1}|V_{t+1}) \text{ is constant by our assumption.}\\
&= E_t\left[Cov_t(C_{t+1,T}, P_{t+1}|V_{t+1})\right]\\
&= \phi\left\{E_t(C_{t+1,T}P_{t+1}|V_{t+1}=\sigma_H) - E_t(C_{t+1,T}|V_{t+1}=\sigma_H)E(P_{t+1}|V_{t+1}=\sigma_H)\right\}\\
&\quad +(1-\phi)\left\{E_t(C_{t+1,T}P_{t+1}|V_{t+1}=\sigma_L) - E_t(C_{t+1,T}|V_{t+1}=\sigma_L)E(P_{t+1}|V_{t+1}=\sigma_L)\right\}
\end{aligned}$$

The detail of first term of covariance is

$$E_t(C_{t+1,T}P_{t+1}|V_{t+1}=\sigma_H)$$
$$=E_t\left\{\sum_{i=0}^{n^*}\binom{n^*}{i}\phi^i(1-\phi)^{n^*-i}\left\{P_{t+1}^2\mathbf{N}(d_1)-P_{t+1}e^{-n^*r\Delta\tau}K\mathbf{N}(d_2)\right\}|V_{t+1}=\sigma_H\right\},$$

where $n^*=\frac{T-(t+\Delta\tau)}{\Delta\tau}$. And we advance to express the terms as

$$E_t\left\{P_{t+1}^2\mathbf{N}(d_1)-P_{t+1}e^{-n^*r\Delta\tau}K\mathbf{N}(d_2)|V_{t+1}=\sigma_H\right\}$$
$$=E_t\left[\begin{array}{c} P_t^2e^{2u_H\Delta\tau+2\sigma_H\sqrt{\Delta\tau}\varepsilon_{t+1}}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}\right]+u_H\Delta\tau}{\sqrt{i\cdot(\sigma_H)^2\Delta\tau+(n^*-i)(\sigma_L)^2\Delta\tau}}+\varepsilon_{t+1}\right) \\ -P_te^{-n^*r\Delta\tau}e^{u_H\Delta\tau+\sigma_H\sqrt{\Delta\tau}\varepsilon_{t+1}}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}\right]+u_H\Delta\tau}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}+\varepsilon_{t+1}\right)\end{array}|V_{t+1}=\sigma_H\right]$$

$$\because\int\mathbf{N}(m+s\varepsilon)\frac{1}{\sqrt{2\pi}}e^{-\frac{(\varepsilon_1-g)^2}{2}}d\varepsilon_1=\mathbf{N}\left(\frac{\frac{m}{s}+g}{\sqrt{(\frac{1}{s})^2+1}}\right)$$
$$=P_t^2e^{2r\Delta\tau+\sigma_H^2\Delta\tau}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}\right]+[r\Delta\tau+\frac{3\sigma_H^2\Delta\tau}{2}]}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right)$$
$$-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}\right]+[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}]}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right).$$

Let $r_H^+=r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}$, $r_L^+=r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}$, $r_H^-=r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}$ $and\, r_L^-=r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}$ .

$$E_t\left\{P_{t+1}^2\mathbf{N}(d_1)-P_{t+1}e^{-n^*r\Delta\tau}K\mathbf{N}(d_2)|V_{t+1}=\sigma_H\right\}$$
$$=P_t^2e^{2\cdot r_H^+}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot r_H^++(n^*-i)\cdot r_L^++\left[r_H^++\sigma_H^2\Delta\tau\right]}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right)$$
$$-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot r_H^-+(n^*-i)\cdot r_L^-+\left[r_H^+\right]}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right).$$

Similarly, details inside $E_t(C_{t+1,T}P_{t+1}|V_{t+1}=\sigma_L)$ are

$$E_t\left\{P_{t+1}^2\cdot\mathbf{N}\left(d_1\right)-P_{t+1}e^{-n^*r\Delta\tau}K\cdot\mathbf{N}\left(d_2\right)|V_{t+1}=\sigma_L\right\}$$

$$=E_t\left[\begin{array}{c}P_t^2e^{2u_L\Delta\tau+2\sigma_L\sqrt{\Delta\tau}\varepsilon_{t+1}}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}\right]+u_L\Delta\tau}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}+\varepsilon_{t+1}\right)\\ -P_te^{-n^*r\Delta\tau}e^{u_L\Delta\tau+\sigma_L\sqrt{\Delta\tau}\varepsilon_{t+1}}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}]\right]+u_L\Delta\tau}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n-i)\sigma_L^2\Delta\tau}}+\varepsilon_{t+1}\right)\end{array}|V_{t+1}=\sigma_L\right]$$

$$=P_t^2e^{2r\Delta\tau+\sigma_L^2\Delta\tau}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}\right]+[r\Delta\tau+\frac{3\sigma_L^2\Delta\tau}{2}]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right)$$
$$-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}\right]+[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right)$$

$$=P_t^2e^{2\cdot r_L^+}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot r_H^++(n^*-i)\cdot r_L^++\left[r_L^++\sigma_L^2\Delta\tau\right]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right)$$
$$-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot r_H^-+(n^*-i)\cdot r_L^-+\left[r_L^+\right]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right).$$

The second term $E_t(C_{t+1,T}|V_{t+1}=\sigma_H)E(P_{t+1}|V_{t+1}=\sigma_H)$ and $E_t(C_{t+1,T}|V_1=\sigma_H)E(P_{t+1}|V_{t+1}=\sigma_L)$ can be expressed separately as

$$E_t(C_{t+1,T}|V_{t+1}=\sigma_H)E(P_{t+1}|V_{t+1}=\sigma_H)$$
$$=\sum_{i=0}^{n^*}\binom{n^*}{i}\phi^i(1-\phi)^{n^*-i}E_t\left\{\left[P_{t+1}\cdot\mathbf{N}\left(d_1\right)-e^{-n^*r\Delta\tau}K\cdot\mathbf{N}\left(d_2\right)\right]|V_{t+1}=\sigma_H\right\}P_te^{u_H\Delta\tau+\frac{(\sigma_H)^2\Delta\tau}{2}}$$

and

$$E_t\left\{\left\{P_{t+1}\cdot\mathbf{N}\left(d_1\right)-e^{-n^*r\Delta\tau}K\cdot\mathbf{N}\left(d_2\right)\right\}|V_{t+1}=\sigma_H\right\}\cdot P_te^{u_H\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}}$$

$$=P_te^{r\Delta\tau}E_t\left[\begin{array}{c}P_te^{u_H\Delta\tau+\sigma_H\sqrt{\Delta\tau}\varepsilon_{t+1}}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]]+(n^*-i)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}\right]+u_H\Delta\tau}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}+\varepsilon_{t+1}\right)\\ -e^{-n^*r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}]+u_H\Delta\tau\right]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}+\varepsilon_{t+1}\right)\end{array}|V_{t+1}=\sigma_H\right]$$

$$=P_t^2e^{2r\Delta\tau}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+(i+1)\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}\right]}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right)$$
$$-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+(i+1)\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}\right]}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right)$$

$$=P_t^2e^{2r\Delta\tau}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+(i+1)\cdot r_H^++(n^*-i)\cdot r_L^+}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right)$$
$$-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+(i+1)\cdot r_H^-+(n^*-i)\cdot r_L^-}{\sqrt{(i+1)\cdot\sigma_H^2\Delta\tau+(n^*-i)\sigma_L^2\Delta\tau}}\right)$$

Similarly

$$
\begin{aligned}
&E_t(C_{t+1,T}|V_1=\sigma_L)E(P_{t+1}|V_1=\sigma_L)\\
&=\sum_{i=0}^{n^*}\binom{n^*}{i}\phi^i(1-\phi)^{n^*-i}E_t\left\{\left[P_{t+1}\cdot\mathbf{N}\left(d_1\right)-e^{-n^*r\Delta\tau}K\cdot\mathbf{N}\left(d_2\right)\right]|V_{t+1}=\sigma_L\right\}\cdot P_te^{u_L\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&E_t\left\{\left[P_{t+1}\cdot\mathbf{N}\left(d_1\right)-e^{-n^*r\Delta\tau}K\cdot\mathbf{N}\left(d_2\right)\right]|V_{t+1}=\sigma_L\right\}\cdot P_te^{u_L\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}}\\
&=P_te^{r\Delta\tau}E_t\left[\begin{array}{c}P_te^{u_L\Delta\tau+\sigma_L\sqrt{\Delta\tau}\varepsilon_{t+1}}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}\right]+u_L\Delta\tau}{\sqrt{i\cdot\sigma H^2\Delta\tau+(n^*-i)\sigma L^2\Delta\tau}}+\varepsilon_{t+1}\right)\\-e^{-n^*r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}\right]+u_L\Delta\tau}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n-i)\sigma_L^2\Delta\tau}}+\varepsilon_{t+1}\right)\end{array}|V_{t+1}=\sigma_L\right]
\end{aligned}
$$

$$
\begin{aligned}
&=P_t^2e^{2r\Delta\tau}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau+\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i+1)\cdot\left[r\Delta\tau+\frac{\sigma_L^2\Delta\tau}{2}]\right]}{\sqrt{i\cdot(\sigma_H)^2\Delta\tau+(n^*-i+1)(\sigma_L)^2\Delta\tau}}\right)\\
&\quad-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot\left[r\Delta\tau-\frac{\sigma_H^2\Delta\tau}{2}\right]+(n^*-i+1)\cdot\left[r\Delta\tau-\frac{\sigma_L^2\Delta\tau}{2}\right]}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=P_t^2e^{2r\Delta\tau}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot r_H^++(n^*-i+1)\cdot r_L^+}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right)\\
&\quad-P_te^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+i\cdot r_H^-+(n^*-i+1)\cdot r_L^-}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right).
\end{aligned}
$$

Combining,

$$
\begin{aligned}
&\phi E_t(C_{t+1,T}|V_1=\sigma_H)E(P_{t+1}|V_1=\sigma_H)+(1-\phi)E_t(C_{t+1,T}|V_1=\sigma_L)E(P_{t+1}|V_1=\sigma_L)\\
&=\sum_{i=0}^{n^*+1}\binom{n^*+1}{i}\phi^i(1-\phi)^{n^*+1-i}\cdot P_t\left[\begin{array}{c}P_te^{2r\Delta\tau}\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+(i+1)\cdot r_H^++(n^*-i+1)\cdot r_L^+}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right)\\-e^{-(n^*-1)r\Delta\tau}K\cdot\mathbf{N}\left(\frac{\ln(\frac{P_t}{K})+(i+1)\cdot\cdot r_H^-+(n^*-i+1)\cdot r_L^-}{\sqrt{i\cdot\sigma_H^2\Delta\tau+(n^*-i+1)\sigma_L^2\Delta\tau}}\right)\end{array}\right]
\end{aligned}
$$

The analytical solution for covariance is

$$
\begin{aligned}
&Cov_t(C_{t+1,T}\ ,P_{t+1}) \\
&= \phi \sum \binom{n^*}{i} \phi^i (1-\phi)^{n^*-i} \left\{ P_t^2 e^{2r_H^+} \mathbf{N}\left(d_{1,H}\right) - P_t e^{-(n^*-1)r\Delta\tau} K \cdot \mathbf{N}\left(d_{2,H}\right) \right\} \\
&\quad + (1-\phi) \sum \binom{n^*}{i} \phi^i (1-\phi)^{n^*-i} \left\{ P_t^2 e^{2r_L^+} \mathbf{N}\left(d_{1,L}\right) - P_t e^{-(n^*-1)r\Delta\tau} K \cdot \mathbf{N}\left(d_{2,L}\right) \right\} \\
&\quad - \textstyle\sum_{i=0}^{n^*+1} \binom{n^*+1}{i} \phi^i (1-\phi)^{n^*+1-i} \cdot \left\{ P_t^2 e^{2r\Delta\tau} \mathbf{N}\left(d_1^*\right) - P_t e^{-(n^*-1)r\Delta\tau} K \cdot \mathbf{N}\left(d_2^*\right) \right\}.
\end{aligned}
$$

*and*

$$d_{1,H} = \frac{\ln(\frac{P_t}{K}) + i \cdot r_H^+ + (n^*-i) \cdot r_L^+ + \left[ r_H^+ + \sigma_H^2 \Delta\tau \right]}{\sqrt{(i+1) \cdot \sigma_H^2 \Delta\tau + (n^*-i) \sigma_L^2 \Delta\tau}}$$

$$d_{2,H} = \frac{\ln(\frac{P_t}{K}) + i \cdot r_H^- + (n^*-i) \cdot r_L^- + \left[ r_H^+ \right]}{\sqrt{(i+1) \cdot \sigma_H^2 \Delta\tau + (n^*-i) \sigma_L^2 \Delta\tau}}$$

$$d_{1,L} = \frac{\ln(\frac{P_t}{K}) + i \cdot r_H^+ + (n^*-i) \cdot r_L^+ + \left[ r_L^+ + \sigma_L^2 \Delta\tau \right]}{\sqrt{i \cdot \sigma_H^2 \Delta\tau + (n^*-i+1) \sigma_L^2 \Delta\tau}}$$

$$d_{2,L} = \frac{\ln(\frac{P_t}{K}) + i \cdot r_H^- + (n^*-i) \cdot r_{\sigma_L}^- + \left[ r_L^+ \right]}{\sqrt{i \cdot \sigma_H^2 \Delta\tau + (n^*-i+1) \sigma_L^2 \Delta\tau}}$$

$$d_1^* = \frac{\ln(\frac{P_t}{K}) + (i+1) \cdot r_H^+ + (n^*-i+1) \cdot r_L^+}{\sqrt{i \cdot \sigma_H^2 \Delta\tau + (n^*-i+1) \sigma_L^2 \Delta\tau}}$$

$$d_2^* = \frac{\ln(\frac{P_t}{K}) + (i+1) \cdot r_H^- + (n^*-i+1) \cdot r_L^-}{\sqrt{i \cdot \sigma_H^2 \Delta\tau + (n^*-i+1) \sigma_L^2 \Delta\tau}}$$