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Split-Award Auctions: Insights from Theory and Experiments: Electronic Companion

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Appendix A: Another Example for Optimality of Non-Greedy Allocation

Consider $n = 4$ suppliers and the following sourcing rules: $A = 55\%$, $B = 0$ and $M = 0$. The suppliers' costs are distributed according to the following non-regular density function:

$$f(x) = \begin{cases} 0.35 \text{ if } 0 \leq x \leq 1 \\ 0.35 + 199650(x-1) \text{ if } 1 \leq x \leq 1.001 \\ 200 \text{ if } 1.001 \leq x \leq 1.0037 \\ 0 \text{ otherwise.} \end{cases} \tag{A1}$$

In this scenario the optimal splits will be $Q_1 = 0.55, Q_2 = 0.45/3$, $Q_3 = 0.45/3$ and $Q_4 = 0.45/3$. Compared to the optimal splits the non-optimal (greedy) splits of $Q_1 = 0.55, Q_2 = 0.45$, $Q_3 = 0$ and $Q_4 = 0$ would result in an increase of about 2% in the expected buyer cost.

Appendix B: Sensitivity of Buyer's Cost

Define the buyer's expected cost given the sourcing constraints (1b) as

$$C^*_{buyer}(A, B, M) = \begin{cases} \min_{\mathbf{Q}} C_{buyer}(\mathbf{Q}) \\ \text{s.t. the sourcing constraints (1b) are satisfied.} \end{cases} \tag{A2}$$

The next result characterizes the sensitivity of the buyer's expected cost with respect to the sourcing rules.

Proposition A1. *For a regular cost distribution the buyer's expected cost, $C^*_{buyer}(A, B, M)$ is convex decreasing in A, and convex increasing in B and M.*

Note that with respect to M we are using discrete convexity since M takes integer values.

Proof of Proposition A1: We show three different proofs of this proposition for A, B and C respectively. In all these proofs, we denote by z the number of suppliers that get a strictly positive allocation. If the allocations are greedy then z can be characterized as: $z = \max(M, \lceil 1/A \rceil)$.

Constraint A: Let $z_A \geq 0$ denote the number of suppliers that get allocated A under the optimal allocation. Since the greedy allocation is optimal for regular cost distributions, z_A can be characterized as: $z_A = \left\lfloor \frac{1-B\cdot z}{A-B} \right\rfloor$. Indeed, $dz_A/dA \leq 0$. We next consider cases for which $dz_A/dA = 0$, i.e., changing A to $A+\epsilon$ for an $\epsilon > 0$ does not change z_A. For $z_A \geq 1$, changing A to $A+\epsilon$ (such that $dz_A/dA = 0$) would transfer ϵ amount to each of the allocations $Q_1, \ldots, Q_{z_A}$ from the allocation Q_{z_A+1} (since allocating greedily is optimal). From Proposition (2) (buyer's expected cost in sealed-bid auction) we see that this would result in a change in C^*_{buyer} of $z_A\epsilon\mu_{z_A+1} - z_A(z_A+1)\epsilon\mu_{z_A+2} + z_A^2\epsilon\mu_{z_A+1} = -z_A(z_A+1)\epsilon(\mu_{z_A+2} - \mu_{z_A+1})$. Indeed, for $z_A = 0$ and $dz_A/dA = 0$, the constraint A is non-binding and therefore there would be no change in C^*_{buyer}. Hence,

$$dC^*_{buyer}/dA = -z_A(z_A+1)(\mu_{z_A+2} - \mu_{z_A+1}) \leq 0 \text{ for } dz_A/dA = 0. \tag{A3}$$

This implies that C^*_{buyer} is piecewise linear and decreasing in A. Moreover, from Theorem (1) we know that for regularly distributed costs $(z_A+1)\mu_{z_A+2} - z_A\mu_{z_A+1} \geq z_A\mu_{z_A+1} - (z_A-1)\mu_{z_A}$ for all $z_A \geq 1$. Multiplying both sides of the inequality by z_A and then subtracting $z_A\mu_{z_A+1}$ from both sides of this inequality gives $z_A(z_A+1)(\mu_{z_A+2} - \mu_{z_A+1}) \geq (z_A-1)z_A(\mu_{z_A+1} - \mu_{z_A})$. Therefore, dC^*_{buyer}/dA is non-decreasing in A.

Constraint B: Let z_B denote the number of suppliers that get allocated B. Since the greedy allocation is optimal for regular distributions, z_B can be characterized as: $z_B = z - \left\lceil \frac{1-B\cdot z}{A-B} \right\rceil$. Indeed, $dz_B/dB \geq 0$. Since greedy is optimal, for $z_B \geq 1$, the allocations $Q_{z-z_B+1} = \ldots = Q_z = B$. For $z_B \geq 1$, changing B to $B+\epsilon$ such that $dz_B/dB = 0$ for an $\epsilon > 0$ would result in a transfer of ϵ to each $Q_{z-z_B+1}, \ldots, Q_{z_B}$. Since the greedy allocation is optimal, $z_B \cdot \epsilon$ (the total allocation transferred) would be transferred from Q_{z-z_B}. For $dz_B/dB = 0$, from Proposition (2) we can write

$$\frac{dC^*_{buyer}(A,B,M)}{dB} = \frac{C^*_{buyer}(A,B+\epsilon,M) - C^*_{buyer}(A,B,M)}{\epsilon} = \left\{ \begin{array}{l} -\langle (z-z_B)\mu_{z-z_B+1} - (z-z_B-1)\mu_{z-z_B} \rangle \cdot z_B \\ +\langle (z-z_B+1)\mu_{z-z_B+2} - (z-z_B)\mu_{z-z_B+1} \rangle + \ldots \\ +\langle (z\mu_{z+1} - (z-1)\mu_z \rangle \end{array} \right\}.$$

For regular distributions we know from Theorem (1) that $(z-z_B+m+1)\mu_{z-z_B+m+2} - (z-z_B+m)\mu_{z-z_B+m+1} \geq (z-z_B)\mu_{z-z_B+1} - (z-z_B-1)\mu_{z-z_B}$ for any $m \geq -1$. Therefore $dC^*_{buyer}/dB \geq 0$. Indeed, for $z_B = 0$, there would be no change in C^*_{buyer} if B is changed, as long as $dz_B/dB = 0$. Next, we show how dC^*_{buyer}/dB changes as one changes z_B. For this we increase B to B' such that $z'_B = z_B + 1$. We can then write

$$\begin{array}{l} dC^*_{buyer}/dB = -z_B(\mu_{z-z_B+1} - \mu_{z-z_B})(z-z_B-1) + z(\mu_{z+1} - \mu_{z-z_B+1}) \\ \text{for } dz_B/dB = 0 \text{ and for } z_B \geq 1 \text{ and } dC^*_{buyer}/dB = 0 \text{ for } z_B = 0. \end{array} \tag{A4}$$

Note that z remains unaffected from changes in B. Again, for regular distributions we know that $(z-z_B-1)\mu_{z-z_B}-(z-z_B-2)\mu_{z-z_B-1}\leq(z-z_B)\mu_{z-z_B+1}-(z-z_B-1)\mu_{z-z_B}$. Hence $dC^*_{buyer}(A,B^{'},M)/dB\geq dC^*_{buyer}(A,B,M)/dB$. Therefore, dC^*_{buyer}/dB is non-decreasing in B.

Constraint M: Note that changing M would effect the parameters z and z_A defined above. We denote these parameters as functions of M, i.e., as $z(M)$ and $z_A(M)$. Increasing M to $M+1$ would increase C^*_{buyer} only if an additional bidder gets an allocation of B. Thus $C^*_{buyer}(M+1)-C^*_{buyer}(M)\neq 0$ only if $z(M+1)-z(M)>0$, i.e., if $z(M)=M$. For $z(M)=M$, increasing M to $M+1$ implies that the $(M+1)^{th}$ lowest cost supplier now gets allocated B instead of 0. Since allocations are greedy, the allocation B to $(M+1)^{th}$ ranked supplier is transferred from the allocation of $(z_A(M)+1)^{th}$ and higher ranked (lower cost) suppliers (such that the sourcing constraints are not violated). A maximum allocation of $Q_{z_A(M)+1}-B$ can be transferred from the $(z_A(M)+1)^{th}$ ranked supplier. The remaining $2B-Q_{z_A(M)+1}$ has to be transferred from higher ranked (lower cost) suppliers. A maximum of $A-B$ can be transferred from each of the higher ranked suppliers, therefore $A-B$ is transferred from $z_M=\max(\lfloor\frac{2B-Q_{z_A(M)+1}}{A-B}\rfloor,0)$ suppliers. To sum up, a total of B is transferred to Q_{M+1} as follows: $Q_{z_A(M)+1}-B$ from $Q_{z_A(M)+1}$, $A-B$ each from $Q_{z_A(M)},\ldots,Q_{z_A(M)-z_M+1}$ and $(2B-(A-B)z_M-Q_{z_A(M)+1})^+$ from $Q_{z_A(M)-z_M}$. Thus the net change in C^*_{buyer} can be found from Proposition (2) as

$$\begin{aligned}&C^*_{buyer}(A,B,M+1)-C^*_{buyer}(A,B,M)=B((M+1)\mu_{M+2}-M\mu_{M+1})\\&-\langle(z_A(M)+1)\mu_{z_A(M)+2}-z_A(M)\mu_{z_A(M)+1}\rangle(Q_{z_A(M)+1}-B)\\&-z_A(M)\mu_{z_A(M)+1}(A-B)+(z_A(M)-z_M)\mu_{z_A(M)-z_M+1}(A-B)\\&-\langle(z_A(M)-z_M)\mu_{z_A(M)-z_M+1}-(z_A(M)-z_M-1)\mu_{z_A(M)-z_M}\rangle\cdot(2B-(A-B)z_M+Q_{z_A(M)+1})^+,\\&\text{ for } z(M)=M\\&\text{ and } C^*_{buyer}(A,B,M+1)-C^*_{buyer}(A,B,M)=0 \text{ otherwise,}\end{aligned}\tag{A5}$$

where $(x-a)^+\equiv\max(x-a,0)$. Since $M\geq z_A(M)$ and since for regular distributions conditions of Lemma (2) hold, therefore $C^*_{buyer}(M+1)-C^*_{buyer}(M)\geq 0$. Also $z_A(M+1)\leq z_A(M)-z_M$, since $A-B$ was transferred from $Q_{z_A(M)},\ldots,Q_{z_A(M)-z_M+1}$ ranked suppliers. Hence increasing $M+1$ to $M+2$ would transfer B to the $(M+2)^{th}$ ranked supplier from suppliers ranked z_A-z_M and higher (i.e. lower cost). Hence $C^*_{buyer}(M+2)-C^*_{buyer}(M+1)\geq C^*_{buyer}(M+1)-C^*_{buyer}(M)$.■

Appendix C: Equilibrium bid function in sealed-bid first-price auction with uniform costs and $n=4$ suppliers

For $n=4$ suppliers invited to bid, the equilibrium expected allocation in equation (4) (of the paper) can be written as

$$H(x)=\begin{pmatrix}(1-F(x))^3\cdot Q_1+\binom{3}{1}\cdot F(x)\cdot(1-F(x))^2\cdot Q_2+\\ \binom{3}{2}\cdot F^2(x)\cdot(1-F(x))\cdot Q_3+\\ \binom{3}{3}\cdot F^3(x)\cdot Q_4\end{pmatrix}.$$

For uniformly distributed cost, substituting the values $F(x) = \frac{x - \underline{c}}{\bar{c} - \underline{c}}$ and $1 - F(x) = \frac{\bar{c} - x}{\bar{c} - \underline{c}}$ in the above equation, we get

$$H(x) = \begin{pmatrix} (\bar{c} - x)^3 \cdot Q_1 + 3 \cdot (x - \underline{c}) \cdot (\bar{c} - x)^2 \cdot Q_2 + \\ 3 \cdot (x - \underline{c})^2 \cdot (\bar{c} - x) \cdot Q_3 + (x - \underline{c})^3 \cdot Q_4 \end{pmatrix} \cdot \left(\frac{1}{\bar{c} - \underline{c}} \right)^3 .$$

Therefore,

$$- \int_{x=c}^{\bar{c}} x dH(x) =$$

$$\frac{3}{(\bar{c} - \underline{c})^3} \cdot \int_{x=c}^{\bar{c}} \left(x(\bar{c} - x)^2 (Q_1 - Q_2) + 2x(\bar{c} - x)(x - \underline{c})(Q_2 - Q_3) + x(x - \underline{c})^2 (Q_3 - Q_4) \right) dx.$$

Evaluating the above integral, we get

$$- \int_{x=c}^{\bar{c}} x dH(x) = \frac{1}{(\bar{c} - \underline{c})^3} \cdot \left(\begin{array}{r} \left((3c + \bar{c}) \cdot (\bar{c} - c)^3 \right) \cdot \dfrac{Q_1 - Q_2}{4} \\ + \left(6 \cdot (c^2 - cs) \cdot (\bar{c} - c)^2 + (3c - 2\underline{c} + \bar{c}) \cdot (\bar{c} - c)^3 \right) \cdot \dfrac{Q_2 - Q_3}{2} \\ + \left((3\bar{c} + \underline{c}) \cdot (\bar{c} - \underline{c})^3 - (\underline{c} + 3c)(c - \underline{c})^3 \right) \cdot \dfrac{Q_3 - Q_4}{4} \end{array} \right).$$

Substituting these values in equation (6) gives the equilibrium bidding function of a supplier with cost c.

$$\beta(c) = \frac{\left(\begin{array}{r} \dfrac{(\bar{c} - c)^3}{4} \cdot (3c + \bar{c}) \cdot Q_1 \\ + (\bar{c} - c)^2 \cdot \left(2c^2 - 2c\underline{c} + c\bar{c} - \bar{c}\underline{c} + \dfrac{(\bar{c} - c)^2}{4} \right) \cdot Q_2 \\ + \left((c - \underline{c})^2 \cdot \left(2c\bar{c} - 2c^2 - c\underline{c} + \bar{c}\underline{c} - \dfrac{(c - \underline{c})^2}{4} \right) + \dfrac{(\bar{c} - \underline{c})^4}{4} \right) Q_3 \\ + \left(c(c - \underline{c})^3 + \dfrac{(\bar{c} - \underline{c})^4}{4} - \dfrac{(c - \underline{c})^4}{4} \right) Q_4 \end{array} \right)}{\left((\bar{c} - c)^3 \cdot Q_1 + 3 \cdot (c - \underline{c}) \cdot (\bar{c} - c)^2 \cdot Q_2 + 3 \cdot (c - \underline{c})^2 \cdot (\bar{c} - c) \cdot Q_3 + (c - \underline{c})^3 \cdot Q_4 \right)}.$$

The above expression can be cross-checked by taking $\underline{c} = 0$, $\bar{c} = 1$, $Q_1 = 1$ and $Q_2 = Q_3 = Q_4 = 0$. We then get $\beta(c) = (3c + 1)/4$ which is precisely the expected cost of the second-lowest cost supplier conditional upon the cost of the lowest-cost supplier being c. ■

Appendix D: Compendium of Instructions to Participants

Sections in black are for sealed-bid treatments with regular cost distribution. Sections marked in red are for non-regular distributions. Sections marked in blue are for the open-bid treatment.

You are about to participate in an experiment in the economics of decision-making. If you follow these instructions carefully and make good decisions, you will earn money that will be paid to you in cash at the end of the session. If you have a question at any time, please raise your hand and the experimenter will come to your station and answer it. We ask that you not talk with one another for the duration of the experiment.

In this experiment, you will be in the role of a **Supplier** who participates in a Reverse Auction, competing against **three** competitors in this room.

How you earn money

You will bid in 40 auctions for the right to provide 100 units of a commodity to a computerized buyer. In each auction the number of units awarded to each bidder are determined as follows: [In

Sealed-Bid Treatments:

- The bidder with the lowest bid will provide 50 units,
- The bidder with the second lowest bid will provide 35 units,
- The bidder with the third lowest bid will provide 15 units,
- The bidder with the highest bid will provide 0 units.]

[In the open-bid treatment:

- The bidder who drops out first wins 0 units
- The bidder who drops out second wins 25 units
- The bidder who drops out third wins 35 units
- The bidder who does not drop out wins 40 units.]

Your per-unit cost, as well as the costs of the three competitors in each auction is an integer from 0 to 100, with each integer in that range being equally likely. [In non-regular treatments: The chart below shows the probability of having a particular cost.
For example, there is almost a 25% chance of having a cost between 0 and 5, and there is about a 45% chance of having a cost between 96 and 100.]

There is no relationship between your cost and any of the other bidders' costs, or between costs in different auctions (all costs are independent).

You will see your own cost at the start of each auction. Each bidder, including you, only knows their own cost but not the cost of any other bidders.

[In the sealed-bid treatments: After you observe your cost you will place a bid in the bid box on your computer screen and click the "Submit" button. Your bid can be any number, with at most two decimal places, from 0 to 100.
You earn money by being awarded units at a good price:
Your Earnings = (Your Bid - Your Cost) x (Number of Units Awarded to You),

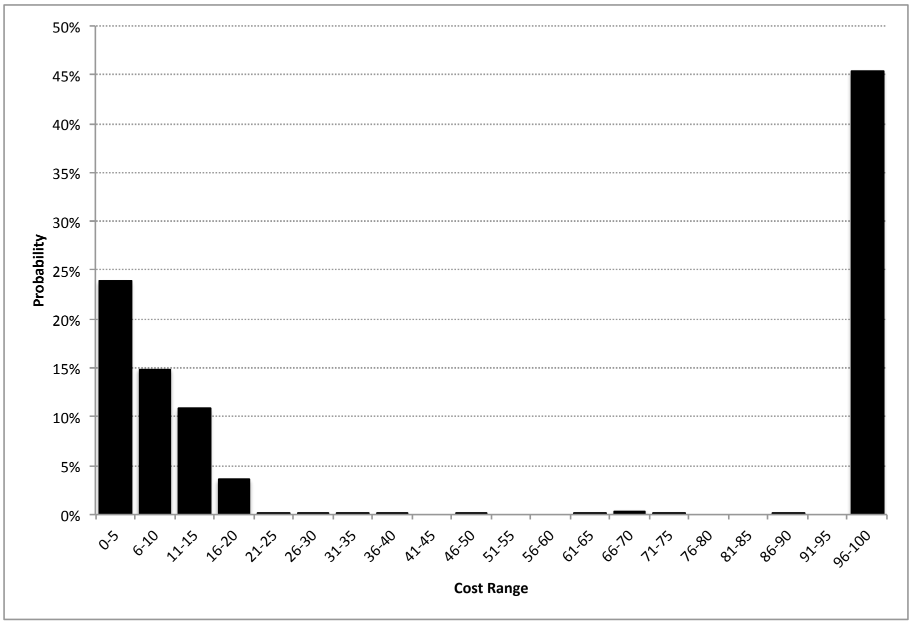

Figure 1 This figure is included in non-regular treatments only

where the number of units awarded to you is determined in the manner stated above.

[In the open-bid treatment:

How you bid

The Bidding Screen will display a Bid that will start at 100 and will automatically decrease by 1 every second. Every bidder will see the same Bid. Every bidder will have a Drop Out button on the screen. When you decide the Bid is too low, click the Drop Out button.

Dropping out of the auction does not affect either the number of units you win or the price you win them at. The number of units you win and how much you are paid for them are based on the drop out bids of the bidders who dropped out before you.

The first bidder to drop out wins 0 units and the remaining three bidders win 25 units at the drop out bid of the first bidder dropping out.

The second bidder to drop out wins only the 25 units at the price the first bidder dropped out, and the two remaining bidders win an additional 10 units at the drop out bid of the second bidder.

Finally, the third bidder to drop out wins 35 (25 at the first drop out price and 10 at the second drop out price) and the remaining bidder wins an additional 5 units at the drop out bid of the third bidder.

Your earnings are as follows:

- If you dropped out first, you earn 0.
- If you dropped out second, you earn 25 units x (Price at which the first bidder dropped out your per unit cost)
- If you dropped out third, you earn 25 units x (Price at which the first bidder dropped out your per unit cost) + 10 units x (Price at which the second bidder dropped out your per unit cost)
- If you did not drop out, you earn 25 units x (Price at which the first bidder dropped out your per unit cost) + 10 units x (Price at which the second bidder dropped out your per unit cost) + 5 units x (Price at which the third bidder dropped out your per unit cost)]

Please note that if your bid is below your cost, and you are awarded any units, you will lose money on each unit, so bid carefully.

Information you will see at the end of each auction

At the end of each auction you will see the following information:

- Your own cost and bid in this auction
- The bids placed by competitors
- The number of units each of the bidders were awarded
- Your earnings from the auction.

You will also have access to this information for all past auctions.

How you will be paid

At the end of the session you will see a final screen summarizing your earnings for the session. This screen will calculate your net profits from 40 auctions, convert them to US dollars at the rate of 3200 ECU per $1, and add them to your $5 participation fee.

Please use this information to fill out your check-out form and wait quietly until the monitor calls you to come to the front of the room and be paid your earnings in private and in cash. After you have been paid, you will be free to leave the laboratory.

Appendix E: Tobit Regression

To further understand overly-aggressive bidding observed for regular distribution treatments we analyze a tobit regression for each split, with the dependent variable bid and independent variable cost.[1] We then compare coefficients for the cost and the constant term obtained from the regression

[1] We use the tobit model with the upper limit at 100 and the lower limit at cost, because the bids are clearly censored at those levels. If we estimate uncensored model, results are virtually unchanged. We use random effects for individuals.

Table 1 Estimates of linear approximation of bid functions.

	Split							
Independent Variable	100–0–0	80–15–5	50–50–0	50–35–15	50–25–25	40–35–25	34–34–34	34–34–17–17
Constant	15.34* (0.455) [25.00]	22.46* (0.508) [32.83]	29.38* (0.978) [43.78]	36.30* (0.907) [49.30]	41.39* (1.125) [53.76]	42.00* (0.6645) [60.08]	61.44 (0.873) [61.60]	68.43 (0.981) [70.12]
Cost	0.840* (0.005) [0.750]	0.777* (0.005) [0.699]	0.660* (0.010) [0.536]	0.599* (0.008) [0.506]	0.552* (0.008) [0.468]	0.542* (0.006) [0.378]	0.325 (0.008) [0.317]	0.348* (0.010) [0.283]

Note: Standard errors are in parenthesis. Estimates of linear approximations of RNNE bid functions are in square brackets; * $p < 0.001$ for comparing the coefficient to the corresponding RNNE linear approximation coefficient in square brackets.

analysis to corresponding coefficients for the linear approximation of the RNNE bid function, for the entire cost range for regular distribution treatments. [2] We summarize regression results in Table 1.

This regression analysis shows that overly-aggressive bidding in regular cost distribution treatments is independent of bidders' cost. For treatments with regular cost distributions, most of the estimated constant terms are significantly lower than the corresponding constant term should be for a linear approximation of the RNNE bid function (all p-values are below 0.001). In those treatments, bidders make up some ground because cost coefficients are significantly above the RNNE linear approximation, but on balance the bids are too low due to the too-low constant term.

Appendix F: Characterization of $R_l(\cdot)$ and $R_w(\cdot)$:

Denoting by f_m the density function of the m^{th} smallest order statistic out of a sample of $n-1$ we get

$$R_l(\cdot) = L \cdot \max \begin{pmatrix} \int_{x=\beta^{-1}_{regret}(c_i)}^{\beta^{-1}_{regret}(b_i)} Q_1 \cdot (\beta_{regret}(x) - c_i) f_1(x) dx; \\ \int_{x=\beta^{-1}_{regret}(c_i)}^{\beta^{-1}_{regret}(b_i)} Q_2 \cdot (\beta_{regret}(x) - c_i) f_2(x) dx; \ldots; \\ \int_{x=\beta^{-1}_{regret}(c_i)}^{\beta^{-1}_{regret}(b_i)} Q_{n-1} \cdot (\beta_{regret}(x) - c_i) f_{n-1}(x) dx \end{pmatrix} \quad \text{(A6)}$$

$$R_w(\cdot) = W \cdot \max \begin{pmatrix} \int_{y=\beta^{-1}_{regret}(b_i)}^{\beta^{-1}_{regret}(\bar{c})} Q_1 \cdot (\beta_{regret}(y) - b_i) f_1(y) dy; \\ \int_{y=\beta^{-1}_{regret}(b_i)}^{\beta^{-1}_{regret}(\bar{c})} Q_2 \cdot (\beta_{regret}(y) - b_i) f_2(y) dy; \ldots; \\ \int_{y=\beta^{-1}_{regret}(b_i)}^{\beta^{-1}_{regret}(\bar{c})} Q_{n-1} \cdot (\beta_{regret}(y) - b_i) f_{n-1}(y) dy; Q_n(\bar{c} - b_i) \end{pmatrix} \quad \text{(A7)}$$

■

[2] Although optimal functions are non-linear (with the exception of the 100-0-0 split) non-linearities are not very strong, and the linear approximation captures the effect of the intercept and the cost, providing a fair comparison for the estimates.

Appendix G: Example with Asymmetric Suppliers

Here we discuss an example where supplier costs are asymmetrically distributed. Specifically, consider $n = 4$ suppliers with the cost of two "low-cost" suppliers distributed uniformly in the interval $[0, 0.95]$ and the cost of two "high-cost" suppliers distributed uniformly in the interval $[0.9, 1]$. The buyer organizes an open-descending price-clock auction which is exactly the same as the one described in §3.2. The dominant strategy of each supplier would be to drop out of the auction when the price-clock reaches their per-unit cost. Thus, the buyer's expected cost, C_{buyer}, can be characterized by equations (7) and (8). For the asymmetric distribution described above we find the first to fourth-lowest order statistics respectively are $\mu_1 = 0.3167$, $\mu_2 = 0.6326$, $\mu_3 = 0.9340$ and $\mu_4 = 0.9668$. For sourcing constraints $A = 1/3$, $B = 0$ and $M = 0$ (i.e., the maximum business given to any supplier is 1/3) the buyer's optimization program (1) gives optimal splits as $1/3, 2/9, 2/9, 2/9$. This is indeed different from greedy splits of $1/3, 1/3, 1/3$.■