E-companion to:

Strategic Open Routing in Service Networks

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Appendix A: Proofs of Results in Sections 4 and 5

Proof of Proposition 1. The fact that priorities are drawn uniformly at random implies that the game is symmetric, and we thus consider player (or customer) i, where i is an arbitrary player index. Suppose that all customers take route AB and that $N \geq 2\mu_A/\mu_B + 1$. The system time experienced by player i depends on which priority she is assigned. Because all customers are taking route AB, by Property 1, customer i will always find station B idle when she finishes service at station A. When assigned priority j, customer i will wait for $j-1$ players to be served at station A, be served herself, and then immediately be served at station B. Thus, as a function of her priority j, customer i's total system time $S^A(j)$ is given by

$$S^A(j) = \frac{j}{\mu_A} + \frac{1}{\mu_B}, \quad j = 1, \dots, N.$$

Let $T(1,m)$ denote customer i's expected system time if she chooses route AB and m other customers also choose route AB, and let $T(0,m)$ denote customer i's expected system time if she chooses route BA and m other customers choose route AB. As priorities are drawn uniformly at random, customer i's expected system time when following the candidate equilibrium strategy is given by

$$T(1,N-1) = \sum_{j=1}^{N} \frac{1}{N} S^A(j) = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right) = \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}.$$

If she takes route BA, then player i will be behind all $N-1$ other players when she gets to station A. So, her total system time $T(0,N-1)$ is deterministic and is given by

$$T(0,N-1) = \frac{1}{\mu_B} + \left(\frac{N}{\mu_A} - \frac{1}{\mu_B}\right) = \frac{N}{\mu_A}.$$

Now, our assumption that $N \geq 2\mu_A/\mu_B + 1$ implies that

$$\frac{1}{\mu_B} + \frac{1}{2\mu_A} \leq \frac{N}{2\mu_A} \implies \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \leq \frac{N}{\mu_A}.$$

Therefore, customer i has no incentive to deviate as $T(1,N-1) \leq T(0,N-1)$, and we have a Nash equilibrium.

With the herding equilibrium at station A established, assume now that $\mu_B < 2\mu_A$ and $N \geq \max\{\mu_B/\mu_A + 1, 2\mu_A + \mu_B/(2\mu_A - \mu_B)\}$, and suppose that all customers take route BA. We will evaluate whether any customer has incentive to deviate. The condition $N \geq (\mu_B/\mu_A) + 1$ ensures that, if player i deviates and takes route AB, then station B will not finish serving all $N-1$ other players before player i finishes at station A. Applying the same notation as the previous case, customer i's total system time $T(1,0)$ from taking route AB is deterministic and is given by

$$T(1,0) = \frac{N}{\mu_B}.$$

If customer i takes route BA, then her priority at station B is drawn uniformly at random, and with probability $1/N$ she will be in position j, for $j = 1, \ldots, N$. Suppose that she draws priority j; she will wait for $j-1$ customers to be served at station B, be served herself there, and then wait in a queue at station A. Because all customers take route BA, station A will idle for the first $1/\mu_B$ units of time, and then will work continuously until it has processed all N customers. Player i's system time corresponds to the time when station A finishes with the j-th job. Thus, we have that customer i's system time $S^B(j)$ is

$$S^B(j) = \frac{j}{\mu_A} + \frac{1}{\mu_B}.$$

We can now calculate her expected total system time $T(0,0)$ as

$$T(0,0) = \sum_{j=1}^{N} \frac{1}{N} S^B(j) = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right) = \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}.$$

By our assumption that $\mu_B < 2\mu_A$ and $N \geq (2\mu_A + \mu_B)/(2\mu_A - \mu_B)$, we have

$$2\mu_A + \mu_B \leq (2\mu_A - \mu_B)N \implies \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \leq \frac{N}{\mu_B}.$$

Therefore we have $T(0,0) \leq T(1,0)$, implying that no customer has incentive to deviate, and we have a Nash equilibrium. $\square$

Proof of Proposition 2. As discussed, submodularity is equivalent to the decreasing differences condition (3). For our proof, we divide $T(1,m)$, the expected system time for a fixed player choosing route AB when m other players are also choosing the same route, into two components. We first define $\widetilde{T}(1,m)$ to be the expected system time for the player choosing route AB, not counting any waiting time that she may experience in the queue at station B. Because priorities are drawn uniformly at random, we have

$$\widetilde{T}(1,m) := \sum_{j=1}^{m+1} \frac{1}{m+1}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right) = \frac{1}{\mu_B} + \frac{m+2}{2\mu_A}. \tag{EC.1}$$

Next, we define δ_m as the difference between the total expected system time and the expression in equation (EC.1), i.e., $\delta_m := T(1,m) - \widetilde{T}(1,m)$. By definition, then, δ_m is equal to the expected time spent waiting in the queue at station B.

Station A never idles until it finishes as long as at least one player chooses route AB. Therefore, for $m \geq 1$, the expected system time for a player choosing route BA when m other players choose route AB is given by

$$T(0,m) = \sum_{\ell=1}^{N-m} \frac{1}{N-m}\left(\frac{m+\ell}{\mu_A}\right) = \frac{N+m+1}{2\mu_A}. \tag{EC.2}$$

By equations (EC.1) and (EC.2), we have that for all $m \geq 1$,

$$d_m := T(1,m) - T(0,m) = \delta_m + \widetilde{T}(1,m) - T(0,m) = \delta_m + \frac{1}{\mu_B} - \frac{N-1}{2\mu_A}, \tag{EC.3}$$

which implies that for all $1 \le \widetilde{m} \le m \le N-1$,

$$d_m \le d_{\widetilde{m}} \text{ if and only if } \delta_m \le \delta_{\widetilde{m}}. \tag{EC.4}$$

For $m \ge 1$, define $\delta_m^{(j)}$ to be the wait time in queue at station B experienced by the fixed player when m other players choose route AB, given that she chooses route AB and receives priority j at station A. Then we have the representations

$$\delta_m^{(1)} = \left(\frac{N-m-1}{\mu_B} - \frac{1}{\mu_A}\right)^+, \text{ and } \delta_m^{(j+1)} = \left(\delta_m^{(j)} + \frac{1}{\mu_B} - \frac{1}{\mu_A}\right)^+ \text{ for all } j = 1, \dots m.$$

Because $1/\mu_B - 1/\mu_A < 0$, we have that

$$\delta_m^{(j+1)} \le \delta_m^{(j)} \text{ for all } j = 1, \dots, m. \tag{EC.5}$$

Moreover, because the function $f(x) := (x - 1/\mu_A)^+$ is non-decreasing, we also have

$$\delta_{m+1}^{(1)} \le \delta_m^{(1)},$$

and therefore, for $j = 2, \dots, m+1$,

$$\delta_{m+1}^{(j)} = \left(\delta_{m+1}^{(j-1)} + \frac{1}{\mu_B} - \frac{1}{\mu_A}\right)^+ \le \left(\delta_m^{(j-1)} + \frac{1}{\mu_B} - \frac{1}{\mu_A}\right)^+ = \delta_m^{(j)}. \tag{EC.6}$$

We have thus established that $\delta_m^{(j)}$ is monotonically decreasing both in j for any given m and in m for any given j. Next, because priorities are drawn uniformly at random, we have that

$$\delta_m = \frac{1}{m+1} \sum_{j=1}^{m+1} \delta_m^{(j)} \quad \text{for all } m \ge 1. \tag{EC.7}$$

Equation (EC.7) expresses δ_m as the average of the terms $\delta_m^{(1)}, \dots, \delta_m^{(m+1)}$. By equation (EC.5), $\delta_m^{(j)}$ is non-increasing in j, and therefore the average of $\delta_m^{(1)}, \dots, \delta_m^{(m+1)}$ is bounded above by the average of $\delta_m^{(1)}, \dots, \delta_m^{(m)}$. We then have

$$\delta_m = \frac{1}{m+1} \sum_{j=1}^{m+1} \delta_m^{(j)} \le \frac{1}{m} \sum_{j=1}^{m} \delta_m^{(j)} \le \frac{1}{m} \sum_{j=1}^{m} \delta_{m-1}^{(j)} = \delta_{m-1} \text{ for all } 2 \le m \le N-1, \tag{EC.8}$$

where the second inequality comes from the monotonicity in m in equation (EC.6). Equations (EC.4) and (EC.8) then imply that

$$d_m \le d_{m-1} \quad \text{for all } 2 \le m \le N-1, \tag{EC.9}$$

which satisfies the decreasing differences condition. Moreover, when $2 \le m < N - \mu_B/\mu_A$, we have that

$$\delta_m^{(1)} = \left(\frac{N-m-1}{\mu_B} - \frac{1}{\mu_A}\right)^+ < \left(\frac{N-m}{\mu_B} - \frac{1}{\mu_A}\right)^+ = \delta_{m-1}^{(1)}, \tag{EC.10}$$

so we conclude that the inequalities in equations (EC.8) and (EC.9) hold strictly in this range.

Lastly, we directly evaluate and compare d_1 and d_0. Observe that $N > N_{\text{sub}}$ implies that $2/\mu_A < (N-1)/\mu_B$, so when $m = 0$ or $m = 1$, if the fixed player chooses route AB, then she always faces a queue at station B. Therefore,

$$T(1,1) = \frac{1}{2}\left(\frac{N-1}{\mu_B} + \frac{N}{\mu_B}\right) = \frac{2N-1}{2\mu_B} \quad \text{and} \quad T(1,0) = \frac{N}{\mu_B}.$$

When the player chooses route BA, we note that station A will not idle if $m=1$, but it idles for $1/\mu_B$ units of time if $m=0$. Therefore,

$$T(0,1)=\frac{1}{N-1}\sum_{j=1}^{N-1}\frac{j+1}{\mu_A}=\frac{N+2}{2\mu_A} \quad \text{and} \quad T(0,0)=\sum_{j=1}^{N}\frac{1}{N}\left(\frac{j}{\mu_A}+\frac{1}{\mu_B}\right)=\frac{1}{\mu_B}+\frac{N+1}{2\mu_A}.$$

Applying the equations above, we have the relation

$$d_1=\frac{N-1}{\mu_B}-\frac{N+1}{2\mu_A}+\left(\frac{1}{2\mu_B}-\frac{1}{2\mu_A}\right)<\frac{N-1}{\mu_B}-\frac{N+1}{2\mu_A}=T(1,0)-T(0,0)=d_0. \tag{EC.11}$$

Equations (EC.9) and (EC.11) imply that the expected system time has decreasing differences and is therefore submodular, and equations (EC.10) and (EC.11) imply equation (4). □

Proof of Corollary 1. If $m<N-\mu_B/\mu_A$, then the statement is immediately verified by Proposition 2. If $m\geq N-\mu_B/\mu_A$, we then must have

$$\frac{N-m}{\mu_B}\leq\frac{1}{\mu_A},$$

which implies that the queue at station B clears before the first AB customer arrives. Recall that $\delta_m=T(1,m)-\widetilde{T}(1,m)$ is equal to the expected time spent waiting in the queue at station B. Because the queue at station B clears before the first AB customer arrives, we have $\delta_m=0$, and equation (EC.3) implies that

$$d_m=\frac{1}{\mu_B}-\frac{N-1}{2\mu_A}<\frac{1}{\mu_B}-\frac{1}{\mu_A}<0,$$

where the first inequality follows because $N>N_{\text{sub}}>3$. □

Proof of Proposition 3.

Proof. Let $\langle\cdot,\cdot\rangle$ denote the standard vector product, and let $\boldsymbol{d}$ be the vector of differences $d_m=T(1,m)-T(0,m)$ of waiting times for choosing routes AB and BA, i.e., $\boldsymbol{d}:=(d_0,d_1,\ldots,d_{N-1})$. Define $\pi_i^{(t)}$ as the expected difference in system times for player i, given her belief at time t, which is given by

$$\pi_i^{(t)}:=\langle\boldsymbol{\beta}_i^{(t)},\boldsymbol{d}\rangle.$$

We proceed by cases.

Case 1: suppose that $x^{(\ell)}\geq m^*+1$. Then for any player i who chose route AB in period ℓ, we must have $x_{-i}^{(\ell)}\geq m^*$, and also $\pi_i^{(\ell)}\leq 0$ because otherwise player i would have chosen route BA in period ℓ. This implies that

$$\pi_i^{(\ell+1)}=(1-\alpha_\ell)\pi_i^{(\ell)}+\alpha_\ell\langle\boldsymbol{e}(x_{-i}^{(\ell)}),\boldsymbol{d}\rangle\leq(1-\alpha_\ell)\pi_i^{(\ell)}+\alpha_\ell d_{m^*}\leq 0, \tag{EC.12}$$

where the first inequality follows from the submodularity of the expected system time, and the second inequality follows by the definition of m^*. Equation (EC.12) implies that all of the customers who chose route AB in period ℓ will do so again in periods $\ell+1$, $\ell+2$, ... Therefore, for any $t\geq\ell$, we have that $x_i^{(t)}=1$ for each player i that chose route AB in period ℓ, and this implies that

$$m^*+1\leq x^{(\ell)}\leq x^{(t)} \qquad \text{for all } t\geq\ell.$$

For $\bar{t} \geq \ell$ and for any customer i who chose route BA in period ℓ, we have that

$$\begin{aligned}\pi_i^{(\bar{t}+1)} &= \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\pi_i^{(\ell)} + \sum_{t=\ell}^{\bar{t}} \alpha_t \langle \boldsymbol{e}(x_{-i}^{(t)}), \boldsymbol{d}\rangle \prod_{s=t+1}^{\bar{t}}(1-\alpha_s) \\ &\leq \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\pi_i^{(\ell)} + \sum_{t=\ell}^{\bar{t}} \alpha_t d_{m^*+1} \prod_{s=t+1}^{\bar{t}}(1-\alpha_s) \\ &= \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\right) d_{m^*+1},\end{aligned}$$

where the inequality follows by the submodularity of the expected system time. Note that we always have $d_{m^*+1} < 0$, because if $d_{m^*} < 0$, then $d_{m^*+1} \leq d_{m^*} < 0$; and if $d_{m^*} = 0$, then by Corollary 1, we have $d_{m^*+1} < 0$. Combining this with the assumption in equation (6), there must exist t_0 such that for all $\bar{t} \geq t_0$,

$$\pi_i^{(\bar{t}+1)} \leq \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\right) d_{m^*+1} < 0.$$

This implies that for all $\bar{t} \geq t_0$, we have $x_i^{(\bar{t}+1)} = 1$ for every customer i that chose route BA in period ℓ.

Case 2: suppose that $x^{(\ell)} \leq m^* - 1$. Then for any customer i who chose route BA in period ℓ, we must have $\pi_i^{(\ell)} > 0$ and $x_{-i}^{(\ell)} \leq m^* - 1$. This implies that

$$\pi_i^{(\ell+1)} = (1-\alpha_\ell)\pi_i^{(\ell)} + \alpha_\ell \langle \boldsymbol{e}(x_{-i}^{(\ell)}), \boldsymbol{d}\rangle \geq (1-\alpha_\ell)\pi_i^{(\ell)} + \alpha_\ell d_{m^*-1} > 0,$$

where the first inequality follows from the submodularity of the expected system time, and the second inequality follows from the definition of m^*. Therefore, for any $t \geq \ell$, we have that $x_i^{(t)} = 0$ for every customer i that chose route BA in period ℓ. Finally, for $\bar{t} \geq \ell$ and for any customer i that chose route AB in period ℓ, we have

$$\pi_i^{(\bar{t}+1)} \geq \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\right) d_{m^*-2}.$$

The assumption in equation (6) and the fact that $d_{m^*-2} > 0$ together imply that there must exist t_0 such that for all $\bar{t} \geq t_0$,

$$\pi_i^{(\bar{t}+1)} \geq \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}}(1-\alpha_t)\right) d_{m^*-2} > 0.$$

This implies that for all $\bar{t} \geq t_0$, we have $x_i^{(\bar{t}+1)} = 0$ for each customer i that chose route AB in period ℓ. □

Proof of Corollary 2. By Proposition 3, we know that if there exists some $\ell \geq 0$ such that $x^{(\ell)} \neq m^*$, then Cournot best-response will converge to herding because it is a special case of $\{\alpha_t\}$-learning. We proceed to show that with Cournot best-response, there always exists some such ℓ. Consider an arbitrary path of play in which customers play Cournot best-response, that is, $\{\alpha_t\}$-learning with $\alpha_t = 1$ for all $t \geq 1$. In period t, $x^{(t)}$ players choose route AB. If $x^{(t)} \neq m^*$, then Proposition 3 implies that play will converge to herding in finitely many periods. If instead $x^{(t)} = m^*$, then we have $d_{x^{(t)}} \leq 0$ and $d_{x^{(t)}-1} > 0$ by the definition of m^*. Clearly, if N is odd, then $m^* \neq N/2$, and next, we show that $m^* \neq N/2$ when N is even. Define $Q_{\frac{N}{2}}$ by

$$Q_{\frac{N}{2}} := \min\{\frac{N}{2}, \lfloor \frac{\mu_A(\frac{N}{2}-1)}{\mu_B - \mu_A} \rfloor\}.$$

The quantity $Q_{\frac{N}{2}}$ represents the number of AB customers who face a queue at station B when they depart station A, given that a total of $N/2$ customers chose route AB. We now have

$$T(1,\frac{N}{2}-1)=\sum_{k=1}^{Q_{\frac{N}{2}}}\frac{1}{(\frac{N}{2})}\left(\frac{k+\frac{N}{2}}{\mu_B}\right)+\sum_{k=Q_{\frac{N}{2}}+1}^{\frac{N}{2}}\frac{1}{(\frac{N}{2})}\left(\frac{k}{\mu_A}+\frac{1}{\mu_B}\right)$$

and

$$\begin{aligned}T(0,\frac{N}{2}-1)&=\sum_{j=1}^{\frac{N}{2}+1}\left(\frac{1}{\frac{N}{2}+1}\right)\left(\frac{j+\frac{N}{2}-1}{\mu_A}\right)\\&\geq\sum_{k=1}^{\frac{N}{2}}\frac{1}{(\frac{N}{2})}\left(\frac{j+\frac{N}{2}-1}{\mu_A}\right),\end{aligned}\tag{EC.13}$$

where the inequality in equation (EC.13) comes from the fact that the average of a set of $(N/2)+1$ real numbers is larger than the average of the smallest $N/2$ numbers in the set. We can then write

$$\begin{aligned}d_{\frac{N}{2}-1}&=T(1,\frac{N}{2}-1)-T(0,\frac{N}{2}-1)\\&\leq\sum_{k=1}^{Q_{\frac{N}{2}}}\frac{1}{(\frac{N}{2})}\left(\frac{k+\frac{N}{2}}{\mu_B}-\frac{k+\frac{N}{2}-1}{\mu_A}\right)+\sum_{k=Q_{\frac{N}{2}}+1}^{\frac{N}{2}}\frac{1}{(\frac{N}{2})}\left(\frac{k}{\mu_A}+\frac{1}{\mu_B}-\frac{k+\frac{N}{2}-1}{\mu_A}\right).\end{aligned}$$

From the assumption that $N>2\mu_A/(\mu_B-\mu_A)$, we get

$$\frac{k+\frac{N}{2}}{\mu_B}-\frac{k+\frac{N}{2}-1}{\mu_A}=-\frac{N(\mu_B-\mu_A)}{2\mu_A\mu_B}+\frac{k}{\mu_B}-\frac{k-1}{\mu_A}<\frac{k-1}{\mu_B}-\frac{k-1}{\mu_A}\leq 0;$$

which implies that every term in the first summation above is strictly negative. Also, by the assumption that $N\geq N_{\text{sub}}+1=2\mu_B/\mu_A+2$, we have

$$\frac{1}{\mu_B}-\frac{N}{2\mu_A}+\frac{1}{\mu_A}\leq\frac{1}{\mu_B}-\frac{\mu_B}{\mu_A^2}<\frac{1}{\mu_B}-\frac{1}{\mu_A}<0;$$

which implies that every term in the second summation is strictly negative. By this reasoning and the decreasing differences condition, we conclude that

$$d_{\frac{N}{2}}\leq d_{\frac{N}{2}-1}<0,$$

contradicting the definition of m^*, which requires that $d_{m^*-1}>0$. Therefore $m^*\neq N/2$. If $x^{(t)}=m^*$, then in period $t+1$ the customers who played route AB in period t will switch to route BA, and the customers who played route BA will switch to route AB. The fact that $m^*\neq N/2$ implies that $x^{(t+1)}\neq m^*$, and therefore play will converge to herding by Proposition 3. ☐

Proof of Corollary 3. Assume by contradiction that there exists a pure-strategy Nash equilibrium in which $0<N_{AB}<N$ players choose route AB, and $N_{BA}=N-N_{AB}$ players choose route BA. These assumptions imply that $d_{N_{AB}}\geq 0$ and $d_{N_{AB}-1}\leq 0$. But by Proposition 2, we must also have $d_{N_{AB}}\leq d_{N_{AB}-1}$. This implies that $d_{N_{AB}}=d_{N_{AB}-1}=0$, which contradicts Corollary 1. Therefore, there exist no pure-strategy Nash equilibria besides the herding equilibria of Proposition 1. ☐

Proof of Corollary 4. Proposition 2 tells us that if $N > N_{\text{sub}}$, then the game has decreasing differences, i.e., $d_{N-1} \le d_{N-2} \le \cdots \le d_1 \le d_0$. The relation $2\mu_A \le \mu_B$ then implies that

$$\frac{\mu_B}{2\mu_A} > \frac{N-1}{N+1} \implies \frac{N-1}{\mu_B} < \frac{N+1}{2\mu_A} \implies d_0 = T(1,0) - T(0,0) = \frac{N-1}{\mu_B} - \frac{N+1}{2\mu_A} < 0.$$

Decreasing differences and the fact that $d_0 < 0$ gives us that $d_m = T(1,m) - T(0,m) < 0$ for all $m \in \{0,1,\ldots,N-1\}$. Therefore a player's expected system time is always smaller for route AB than route BA, no matter how many other players choose route AB. $\square$

Proof of Proposition 4. For players $i = 1,2,\ldots,N$, let $s_i \in [0,1]$ denote player i's strategy—specifically, the probability that player i chooses route AB. Assume by way of contradiction that there exists a Nash equilibrium with some players adopting mixed strategies and other players adopting pure strategies. Let N_{AB} be the number of customers playing the pure strategy of choosing route AB, N_{BA} be the number of players playing the pure strategy of choosing route BA, and $N_M := N - N_{AB} - N_{BA}$ be the number of customers playing "properly" mixed strategies, i.e., placing strictly positive probability on both routes. Let the index i be defined such that $s_i = 1$ for $i = 1,\ldots,N_{AB}$; $s_i = 0$ for $i = N_{AB}+1,\ldots,N_{AB}+N_{BA}$; and $0 < s_i < 1$ for $i = N_{AB}+N_{BA}+1,\ldots,N$. By our assumption, $N_{AB}+N_{BA} < N$ and either $N_{AB} \ge 1$ or $N_{BA} \ge 1$.

Let Γ_i denote the difference between player i's expected system time from choosing route AB and the expected system time from choosing route BA. Let η_k be the probability, given their strategies, that exactly k players choose route AB among players in the set $\{N_{AB}+N_{BA}+2,\ldots,N\}$. Player $N_{AB}+N_{BA}+1$'s difference between her expected system times from taking routes AB and BA can be expressed as

$$0 = \Gamma_{N_{AB}+N_{BA}+1} = \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k}, \tag{EC.14}$$

where the fact that $\Gamma_{N_{AB}+N_{BA}+1} = 0$ is true by assumption; if player $N_{AB}+N_{BA}+1$ is employing a mixed strategy in this Nash equilibrium, then she must be indifferent between route AB and route BA.

Assume first that $N_{AB} \ge 1$. Then player 1 is choosing route AB. In this case, the difference between her expected system times from taking routes AB and BA can be expressed as

$$\begin{aligned}\Gamma_1 &= s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k} + (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1} \\ &= (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1},\end{aligned}$$

where the last equality holds by equation (EC.14). Moreover, because player 1 is choosing route AB, that route must be weakly better for her, implying that

$$\sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1} \le 0. \tag{EC.15}$$

Now, combining equations (EC.14), (EC.15), and the decreasing differences property, we get

$$\sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1} = \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k} = 0 \tag{EC.16}$$

If the decreasing differences condition holds strictly (that is, if $d_{N_{AB}+k} < d_{N_{AB}+k-1}$) for some $k \in \{0, 1, \ldots N_M - 1\}$, then equation (EC.16) cannot hold. Therefore we must have

$$d_{N_{AB}-1} = d_{N_{AB}} = \ldots = d_{N_{AB}+N_M-1} = 0. \tag{EC.17}$$

Equation (EC.17) contradicts Corollary 1, and thus N_{AB} must be equal to zero and therefore $N_{BA} \geq 1$. In that case, player 1 is choosing route BA, and the difference between her expected times from choosing routes AB and BA can be expressed as

$$\Gamma_1 = s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_M-1} \eta_k d_{1+k} + (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_M-1} \eta_k d_k = s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_M-1} \eta_k d_{1+k},$$

where the last equality holds by equation (EC.14). Moreover, because player 1 is choosing route BA, that route must be weakly better for her, implying that

$$\sum_{k=0}^{N_M-1} \eta_k d_{1+k} \geq 0. \tag{EC.18}$$

Now, applying a similar argument as in the case with $N_{AB} \geq 1$, combining equations (EC.14), (EC.18), and the decreasing differences property gives us

$$d_0 = d_1 = \ldots = d_{N_M} = 0. \tag{EC.19}$$

Equation (EC.19) also contradicts Corollary 1. Therefore, there are no Nash equilibria of the open routing game in which some players adopt mixed strategies and other players adopt pure strategies.

Consider now an equilibrium in which all N players mix, that is, $N_M = N$ and $N_{AB} = N_{BA} = 0$. Let ν_k be the probability, given their strategies, that exactly k players choose route AB among players in the set $\{3, \ldots, N\}$. Player 1's expected savings from taking route BA instead of route AB is given by

$$\Gamma_1 = s_2 \sum_{k=0}^{N-2} \nu_k d_{1+k} + (1 - s_2) \sum_{k=0}^{N-2} \nu_k d_k = 0,$$

by assumption since player 1 must be indifferent if she plays a mixed strategy in equilibrium. Denote by $\widetilde{\Gamma}_1^\epsilon$ the perturbed expected savings for player 1 if player 2 increases her probability of route AB to $s_2 + \epsilon$. We have

$$\begin{aligned}\widetilde{\Gamma}_1^\epsilon &= (s_2 + \epsilon) \sum_{k=0}^{N-2} d_{1+k} \nu_k + (1 - s_2 - \epsilon) \sum_{k=0}^{N-2} d_k \nu_k \\ &= \epsilon \left(\sum_{k=0}^{N-2} \nu_k (d_{1+k} - d_k) \right) \\ &< 0,\end{aligned}$$

where the strict inequality follows from decreasing differences and the fact that $d_1 < d_0$, from the proof of Proposition 2. Therefore, for any $\epsilon > 0$, if player 2 perturbs her strategy by increasing her probability of route AB to $s_2 + \epsilon$, then s_1 is no longer a best response for player 1, and thus this mixed-strategy Nash equilibrium is not "ϵ-stable." The same argument applies to the perturbation of any one player's strategy. It should be observed that this notion of ϵ-stability is related to but much stronger than the notion of an "evolutionarily stable strategy" as defined in Hassin and Haviv (2003). □

Proof of Corollary 5. Similar to the argument of Corollary 3, suppose that there is a pure-strategy Nash equilibrium besides the herding equilibria. Then there must exist some $N_{AB} < m < N_{AB} + N_S - 1$ such that $d_m \geq 0$ and $d_{m-1} \leq 0$. Combining this with Proposition 2, we must have $d_m = d_{m-1} = 0$, but this contradicts Corollary 1. Therefore, there cannot be non-herding pure-strategy Nash equilibria. Next, we identify three regimes corresponding to the three conclusions in the statement of the proposition.

Case 1: $d_{N_{AB}} < 0$.

If $d_{N_{AB}} < 0$, then even if only non-strategic customers take route AB, it is in the interest of each strategic customer to deviate to route AB. Decreasing differences implies that it is a dominant strategy for all of the strategic players to choose route AB.

Case 2: $d_{N_{AB}+N_S-1} > 0$.

If $d_{N_{AB}+N_S-1} > 0$, then even if all of the other strategic customers are taking route AB, a strategic customer would rather take route BA. Decreasing differences gives that it is a dominant strategy for all of the strategic players to choose route BA.

Case 3: $d_{N_{AB}} \geq 0$ and $d_{N_{AB}+N_S-1} \leq 0$.

In this case, both herding profiles are Nash equilibria. □

Proof of Proposition 5. Assume that all N players visit station A first, so station B is initially empty until the first departure from station A. Observe that if all N customers visit station A first, then station B behaves like a $GI/GI/1$ queueing system with arrival rate μ_A and service rate μ_B (recall that $\mu_A < \mu_B$, so such a system would be stable). Let $F_{W_0^B}$ be the distribution function for a random variable which is independent of the arrival and service processes and which may modify the initial state of the queueing system. Let W_k^B, $k \geq 1$, be the waiting time that the k-th departure from station A experiences at station B, and let $F_{W_k^B}$ be the distribution function of W_k^B. Note that with probability 1 we have $W_0^B = 0$ and $W_1^B = 0$ because station B is initially empty. Therefore, $F_{W_1^B}$ stochastically dominates $F_{W_0^B}$, which we denote as

$$F_{W_0^B} \leq_{st} F_{W_1^B}.$$

Define $F_{W_\infty^B}$ as the stationary waiting-time distribution function for a $GI/GI/1$ queueing system with arrival rate μ_A and service rate μ_B. By Theorem 6.2.1 in Müller and Stoyan (2002), we then have that

$$F_{W_k^B} \leq_{st} F_{W_\infty^B} \quad \text{for all } k = 1, 2, \ldots. \tag{EC.20}$$

Armed with the stochastic dominance relation (EC.20), we consider the candidate equilibrium profile in which all customers visit station A first and evaluate the prospect of deviating to route BA. If the customer follows her current strategy, she will receive priority k at station A, for $k = 1, \ldots N$, with probability $1/N$. Conditional on her priority k, her expected total system time $\mathbb{E}[S^A|k]$ is given by

$$\mathbb{E}[S^A|k] = \frac{k}{\mu_A} + \mathbb{E}[W_k^B] + \frac{1}{\mu_B}.$$

Equation (EC.20) implies that $\mathbb{E}[W_k^B] \leq \mathbb{E}[W_\infty^B]$, where $\mathbb{E}[W_\infty^B]$ is the steady-state expected waiting time from the distribution function $F_{W_\infty^B}$. We then have the bound

$$\mathbb{E}[S^A|k] \leq \frac{k}{\mu_A} + \mathbb{E}[W_\infty^B] + \frac{1}{\mu_B} \leq \frac{k}{\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} + \frac{1}{\mu_B}, \tag{EC.21}$$

where the last inequality follows from bounds for the steady-state expected waiting time in queue for a $GI/GI/1$ queue found in Kingman (1962). Taking expectation over the priority in equation (EC.21), we have

$$\begin{aligned}\mathbb{E}[S^A] &\leq \frac{1}{N}\sum_{k=1}^{N}\left(\frac{k}{\mu_A}+\frac{\mu_A(\sigma_A^2+\sigma_B^2)}{2(1-\mu_A/\mu_B)}+\frac{1}{\mu_B}\right)\\ &= \frac{N+1}{2\mu_A}+\frac{\mu_A(\sigma_A^2+\sigma_B^2)}{2(1-\mu_A/\mu_B)}+\frac{1}{\mu_B}.\end{aligned}$$

If a customer deviates and visits station B first, then she will be the last to be served at station A, so her expected system time $\mathbb{E}[S^B]$ satisfies $\mathbb{E}[S^B] \geq N/\mu_A$. Finally, by equation (7), we have

$$\begin{aligned}&\frac{2\mu_A}{\mu_B}+\frac{\mu_A^2(\sigma_A^2+\sigma_B^2)}{1-\mu_A/\mu_B} < N-1\\ &\Rightarrow \frac{\mu_A(\sigma_A^2+\sigma_B^2)}{2(1-\mu_A/\mu_B)}+\frac{1}{\mu_B} < \frac{N-1}{2\mu_A}\\ \Longrightarrow \mathbb{E}[S^A] \leq \frac{N+1}{2\mu_A}&+\frac{\mu_A(\sigma_A^2+\sigma_B^2)}{2(1-\mu_A/\mu_B)}+\frac{1}{\mu_B} < \frac{N}{\mu_A} \leq \mathbb{E}[S^B].\end{aligned}$$

As a customer's expected total system time is shorter if she follows the candidate profile and visits station A first, she has no incentive to deviate. We conclude that it is a Nash equilibrium for all customers to visit station A first. □

Proof of Proposition 6. Let X_k^A and X_k^B be the service time experienced by the k-th customer to be served at stations A and B, respectively. Suppose that all players visit station B first, and let W_k^A be the waiting time at station A experienced by the k-th departure from station B. Define U_k by

$$U_k := X_k^A - X_{k+1}^B.$$

By Lindley's equation, then, we have

$$\begin{aligned}W_1^A &= 0\\ W_2^A &= \max\{0, W_1^A + U_1\} = \max\{0, U_1\}\\ &\cdots\\ W_{k+1}^A &= \max\{0, U_k, U_k + U_{k-1}, \ldots, U_k + U_{k-1} + \cdots + U_1\}.\end{aligned}$$

Now, define I_k^A to be the cumulative idle time experienced by station A before the arrival of the k-th customer, when all customers visit station A first. We can relate the waiting time W_k^A of the k-th customer to the idle time and the excess workload by

$$\begin{aligned}I_k^A &= W_k^A - \left(\sum_{i=1}^{k-1} X_i^A - \sum_{j=1}^{k} X_j^B\right)\\ &= W_k^A - \sum_{i=1}^{k-1} U_i + X_1^B\\ &= X_1^B + \max\left\{-\sum_{i=1}^{k-1} U_i, -\sum_{i=1}^{k-2} U_i, \ldots, -U_1, 0\right\}.\end{aligned}$$

Note then that $-U_i$ has the same distribution as $X^B_{k-i} - X^A_{k-i+1}$, and therefore, $I^A_k - X^B_1$ has the same distribution as

$$W^B_k := \max\left\{\sum_{i=1}^{k-1}(X^B_i - X^A_{i+1}), \sum_{i=2}^{k-1}(X^B_i - X^A_{i+1}), \ldots, (X^B_{k-1} - X^A_k), 0\right\}, \tag{EC.22}$$

which is the "dual process" of W^A_k. Moreover, by Lindley's equation, we can view the equation (EC.22) as the wait time of a single server queue with interarrival time distribution corresponding to station A's service time distribution, and service time distribution corresponding to station A's interarrival time distribution. From equation (EC.20) in the proof of Proposition 5 and Kingman (1962), we have that

$$\mathbb{E}[W^B_k] \le \mathbb{E}[W^B_\infty] \le \frac{\mu_A(\sigma^2_A + \sigma^2_B)}{2(1-\mu_A/\mu_B)}. \tag{EC.23}$$

Let S^A be the system time associated with choosing route AB, and S^B be the system time associated with choosing route BA. Now, given a priority k at station B, we have

$$\begin{aligned}\mathbb{E}[S^B|k] &= \frac{k}{\mu_A} + \mathbb{E}[I^A_k] \\ &= \frac{k}{\mu_A} + \frac{1}{\mu_B} + \mathbb{E}[W^B_k] \\ &\le \frac{k}{\mu_A} + \frac{1}{\mu_B} + \mathbb{E}[W^B_\infty],\end{aligned}$$

where the equality follows from the fact that $I^A_k - X^B_1$ has the same distribution as W^A_k, while the inequality follows from equation (EC.23). Applying this inequality, we get that

$$\begin{aligned}\mathbb{E}[S^B] &\le \mathbb{E}[W^B_\infty] + \frac{1}{\mu_B} + \sum_{k=1}^N \frac{1}{N}\left(\frac{k}{\mu_A}\right) \\ &= \mathbb{E}[W^B_\infty] + \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \\ &\le \frac{\mu_A(\sigma^2_A + \sigma^2_B)}{2(1-\mu_A/\mu_B)} + \frac{1}{\mu_B} + \frac{N+1}{2\mu_A},\end{aligned}$$

with the last inequality follows from equation (EC.23). Finally, because a player deviating to route AB will be the last customer served at station B, we clearly have

$$\mathbb{E}[S^A] \ge \frac{N}{\mu_B}.$$

Equation (8) then gives

$$\begin{aligned}&\frac{1}{2\mu_A} + \frac{1}{\mu_B} + \frac{\mu_A\mu_B(\sigma^2_A + \sigma^2_B)}{2(\mu_B - \mu_A)} \le N\left(\frac{2\mu_A - \mu_B}{2\mu_A\mu_B}\right) \\ \Longrightarrow\ &\mathbb{E}[S^B] \le \frac{N+1}{2\mu_A} + \frac{1}{\mu_B} + \frac{\mu_A(\sigma^2_A + \sigma^2_B)}{2(1-\mu_A/\mu_B)} \le \frac{N}{\mu_B} \le \mathbb{E}[S^A].\end{aligned}$$

As it is, the expected system time from choosing route BA is less than that from choosing route AB. Therefore, no customer has incentive to deviate, and we have that it is a Nash equilibrium for every player to choose route BA. □

Proof of Proposition 7. We establish the subgame perfect equilibrium using backward induction. First, equations (9) and (10) tell us that the unique optimal strategy for customer N is to take route BA if $y^A_{N-1} = N-1$ and route AB otherwise. Note that this strategy ensures that in equilibrium always at least one customer will take route AB. Consequently, being the slower station, in equilibrium station A will never idle until it has processed all N customers. Next, we use induction to identify the optimal strategy for all other players. For the induction hypothesis, assume for some integer $2 \le n \le N-1$ that the strategy of subsequent customers $N-n'$, where $1 \le n' < n$, is to always take route AB, and that the final customer to move, customer N, follows the optimal strategy derived above.

To serve as the base case, we first verify the induction hypothesis for $n = 2$ by deriving the equilibrium strategy for customer $N-1$. Player N, the last to move, is the only customer that follows player $N-1$, and in equilibrium customer N will follow the strategy of visiting station B first if $y^A_{N-1} = N-1$, and station A otherwise. So, if customer $N-1$ visits station B first, then customer N will visit station A first, and customer $N-1$ will be the last person served at station A. She will then experience system time given by

$$S^B_{N-1} = \frac{N}{\mu_A} \tag{EC.24}$$

because, as noted, station A never idles given player N's equilibrium strategy. If customer $N-1$ takes route AB, then her system time will depend on how many players before her made the same choice. If a small enough number of them chose route AB that when player $N-1$ departs from station A she will find station B busy, then her system time S^A_{N-1} is given by

$$S^A_{N-1} = \frac{N-1}{\mu_B} < \frac{N-1}{\mu_A} < \frac{N-1}{\mu_A} + \frac{1}{\mu_A} = S^B_{N-1}. \tag{EC.25}$$

On the other hand, if enough customers before customer $N-1$ chose route AB that she will find station B idle when she departs from station A, then her system time S^A_{N-1} is given by

$$S^A_{N-1} = \frac{y^A_{N-2}+1}{\mu_A} + \frac{1}{\mu_B} < \frac{N-1}{\mu_A} + \frac{1}{\mu_A} = S^B_{N-1} \tag{EC.26}$$

Thus, the equilibrium strategy for customer $N-1$ is to visit station A first in every subhistory. Combined with the equilibrium strategy for customer N, equations (EC.24)-(EC.26) verify the induction hypothesis for $n = 2$.

Now, assume that the induction hypothesis holds for some integer $2 \le n \le N-1$. Let S^B_{N-n} be the system time that customer $N-n$ experiences if she chooses to join station B first. Similarly, let S^A_{N-n} be the system time that customer $N-n$ experiences if she chooses to join station A first. If customer $N-n$ chooses route AB, then everyone after her will join station A first, and customer $N-n$ will be the last to be served at station A. Because station A never idles, we then must have $S^B_{N-n} = N/\mu_A$. To study S^A_{N-n}, we need to consider the following two cases.

Case 1: The number of AB customers before customer $N-n$, denoted by y^A_{N-n-1}, is small enough that customer $N-n$ finds station B busy when she finishes at station A. Then by Property 2, station B has never idled since starting service and we must have $S^A_{N-n} = (N-n)/\mu_B$. Moreover,

$$S^A_{N-n} = \frac{N-n}{\mu_B} < \frac{N-n}{\mu_A} < \frac{N-1}{\mu_A} + \frac{1}{\mu_A} = S^B_{N-n}. \tag{EC.27}$$

Case 2: The number of AB customers before customer $N-n$, denoted by y^A_{N-n-1}, is big enough that customer $N-n$ finds station B idle when she finishes at station A. In this case, we have $S^A_{N-n} = (y^A_{N-n-1} + 1)/\mu_A + (1/\mu_B)$. Therefore, we have that

$$S^A_{N-n} = \frac{y^A_{N-n-1}+1}{\mu_A} + \frac{1}{\mu_B} < \frac{N-1}{\mu_A} + \frac{1}{\mu_A} = S^B_{N-n}.$$

In either case, choosing route AB results in a strictly shorter system time for customer $N-n$. Thus, if the induction hypothesis holds for some $n \geq 2$, we now have that for any $n \leq \widetilde{n} \leq N-1$, the unique optimal strategy for customer $N-\widetilde{n}$ is to take route AB regardless of the subhistory, given that all subsequent customers act optimally. Having verified the induction hypothesis for $n=2$, we obtain the unique subgame perfect equilibrium of our game, comprised of the strategies stated in the proposition. Furthermore, inspection of the strategies reveals that the resulting equilibrium path entails the first $N-1$ players taking route AB, and the final player N taking route BA. $\square$

Appendix B: Additional Discussion and Proofs for Section 6

We now study the *cumulative system time*, that is the sum of the system times of all customers, under different routing assignments. Suppose that x customers are assigned to route AB and the remaining $N-x$ customers are assigned to route BA. Let Q be the number of AB customers who, after their service at station A, find station B busy. We use $D^i_A(x), i \in \{1,\dots,x\}$, to denote the total time spent in the system by the i-th AB customer; similarly, we use $D^j_B(x), j \in \{1,\dots,N-x\}$, to denote the total time spent in the system by the j-th BA customer. Finally, define $D(x)$ to be the cumulative system time, i.e.,

$$D(x) := \sum_{i=1}^{x} D^i_A(x) + \sum_{j=1}^{N-x} D^j_B(x). \tag{EC.28}$$

If the i-th AB customer finds station B busy upon completing her service at station A, then her total system time is given by

$$D^i_A(x) = \frac{N-x}{\mu_B} + \frac{i}{\mu_B}, \quad i = 1,\dots,Q. \tag{EC.29}$$

We may understand this by recalling from Property 1 that if the i-th customer finds station B busy, then station B must necessarily have never been idle since the start of service availability. Thus, the i-th AB customer will enter service at station B exactly when station B has finished processing all $N-x$ of the BA customers plus the $i-1$ customers who visited station A first and are in front of the i-th AB customer. Adding her own service time, she will experience the total system time related in equation (EC.29).

On the other hand, if the i-th AB customer finishes service at station A and finds station B idle, then her total system time is given by

$$D^i_A(x) = \frac{i}{\mu_A} + \frac{1}{\mu_B}, \qquad \text{for } i = Q+1,\dots,x. \tag{EC.30}$$

Finding station B idle is equivalent to not facing a wait at station B, so a customer's total system time is the sum of the time that she spends at station A and her service time at station B. She must wait at station A for the service of the $i-1$ customers in front of her there, and then she must be served. She then moves to station B, where she immediately enters service, resulting in the system time given by equation (EC.30).

Finally, we have that the j-th BA customer has total system time given by

$$D_B^j(x) = \frac{x+j}{\mu_A}, \qquad \text{for } j = 1, \ldots, N-x, \tag{EC.31}$$

if $x > 0$. To build some intuition around this last quantity, we note that the j-th BA customer will enter service at station A exactly when station A finishes processing all x of the AB customers plus all $j-1$ of the BA customers in front of her at station B. We recall from Property 2 that station A never idles as long as $x > 0$, so the time until she enters service at station A is given by the sum of the service times of $x+j-1$ customers at station A. Adding her own service time at station A, we see that her system time is as expressed in equation (EC.31). Note that if $x = 0$, then we have $D_B^j(x) = j/\mu_A + 1/\mu_B$, i.e., the quantity in equation (EC.31) plus $1/\mu_B$, where $1/\mu_B$ is the time that station A is idle before its first customer departs from station B.

Therefore, the cumulative system time, $D(x)$, is simply the sum of all of the terms in equations (EC.29)-(EC.31), and we present this quantity in the following lemma as a function of the number x of customers that visit station A first.

LEMMA EC.1 **(Cumulative System Time)**. *Take $x \in \{0, \ldots, N\}$, and define $\tilde{Q}$ as*

$$\tilde{Q} := \max\left\{k \in \mathbb{N} : \frac{k}{\mu_A} \le \frac{N-x+k-1}{\mu_B}\right\} = \left\lfloor \frac{\mu_A(N-x-1)}{\mu_B - \mu_A} \right\rfloor.$$

The number Q of AB customers who find station B busy upon service completion at station A is given by

$$Q = \min\{x, \tilde{Q}\},$$

and the cumulative system time $D(x)$ is given by

$$D(x) = \begin{cases} \dfrac{N}{\mu_B} + \dfrac{N^2+N}{2\mu_A} & x = 0, \\ \dfrac{x(2N-x+1)}{2\mu_B} + \dfrac{(N-x)(N+x+1)}{2\mu_A} & 1 \le x \le \dfrac{\mu_A(N-1)}{\mu_B}, \\ \dfrac{\tilde{Q}(2N-2x+\tilde{Q}-1)+2x}{2\mu_B} + \dfrac{N+N^2-\tilde{Q}-\tilde{Q}^2}{2\mu_A} & \dfrac{\mu_A(N-1)}{\mu_B} < x \le N - \dfrac{\mu_B}{\mu_A}, \\ \dfrac{x}{\mu_B} + \dfrac{N^2+N}{2\mu_A} & N - \dfrac{\mu_B}{\mu_A} < x \le N. \end{cases} \tag{EC.32}$$

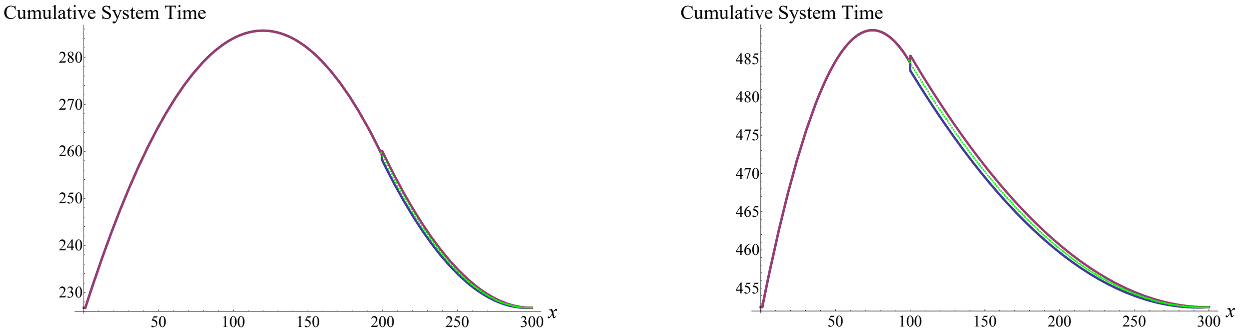
Substituting $x = N$ into the function $D(x)$, we observe that the cumulative system time is the same under either herding profile, i.e., $D(0) = D(N)$.

Proof. First, observe that

$$\begin{aligned} x &> \frac{\mu_A(N-1)}{\mu_B} \\ \mu_B x - \mu_A x &> \mu_A(N-1) - \mu_A x \\ x &> \frac{\mu_A(N-1-x)}{\mu_B - \mu_A} \\ x &> \left\lfloor \frac{\mu_A(N-1-x)}{\mu_B - \mu_A} \right\rfloor = \tilde{Q}, \end{aligned}$$

where the last equivalence condition holds because x is an integer. Therefore we have

$$Q = \tilde{Q} < x \text{ if } x > \frac{\mu_A(N-1)}{\mu_B} \quad \text{ and } \quad Q = x \text{ if } x \le \frac{\mu_A(N-1)}{\mu_B}. \tag{EC.33}$$



(a) $N=300$, $\mu_A=2/3$, $\mu_B=1$. (b) $N=300$, $\mu_A=1/3$, $\mu_B=1$.

Figure EC.1 Cumulative system time as a function of the number of AB customers x.

Next, we will study $D(x)$ under the four cases listed in equation (EC.32).

Case 1: $x=0$. In this case, station A will be idle for the first $1/\mu_B$ units of time. Since all customers visit station B first, each customer must wait for the customers in front of her to finish at station A before she is served there. Hence the customer with priority j at station B will finish at station A exactly when station A completes its j-th service, which because of the idling will occur after $(1/\mu_B)+(j/\mu_A)$ units of time. Thus, the cumulative system time is equal to

$$D(x)=\sum_{j=1}^{N} D_B^j(0)=\sum_{j=1}^{N}\left(\frac{1}{\mu_B}+\frac{j}{\mu_A}\right)=\frac{N}{\mu_B}+\frac{N^2+N}{2\mu_A}.$$

Case 2: $1\le x\le \mu_A(N-1)/\mu_B$. By equation (EC.33), here we have $Q=x$. Equations (EC.28) and (EC.29) give us that

$$D(x)=\sum_{i=1}^{x} D_A^i(x)+\sum_{j=1}^{N-x} D_B^j(x)=\frac{x(2N-x+1)}{2\mu_B}+\frac{(N-x)(N+x+1)}{2\mu_A}.$$

Case 3: $\mu_A(N-1)/\mu_B<x\le N-\mu_B/\mu_A$. By equation (EC.33), in this case $Q=\tilde{Q}$. By equations (EC.28), (EC.29), and (EC.30), we get

$$D(x)=\sum_{i=1}^{\tilde{Q}} D_A^i(x)+\sum_{i=\tilde{Q}+1}^{x} D_A^i(x)+\sum_{j=1}^{N-x} D_B^j(x)=\frac{\tilde{Q}(2N-2x+\tilde{Q}-1)+2x}{2\mu_B}+\frac{N+N^2-\tilde{Q}-\tilde{Q}^2}{2\mu_A}.$$

Case 4: $N-\mu_B/\mu_A<x\le N$. Recall that if station B idles then it will never build up a queue again. In this case, even the first AB customer does not face a queue at station B, and therefore the cumulative system time is equal to

$$D(x)=\sum_{i=1}^{x} D_A^i(x)+\sum_{j=1}^{N-x} D_B^j(x)=\frac{x}{\mu_B}+\frac{N^2+N}{2\mu_A}. \quad \square$$

The four ranges for x that appear in equation (EC.32) correspond to four regimes for the number of AB customers: if $x=0$, then there are no AB customers and station A will be idle during the first $1/\mu_B$ time periods; if $1\le x\le \mu_A(N-1)/\mu_B$, then all of the AB customers will face a queue at station B; if $\mu_A(N-1)/\mu_B<x\le N-\mu_B/\mu_A$, then some of the earlier AB customers will face a queue at station B, while the later AB customers will be served at station B immediately; and finally, if $N-\mu_B/\mu_A<x\le N$, then no AB customers will face a queue at station B.

Lemma EC.1 provides an explicit formula for the cumulative system time $D(x)$, when x customers visit station A first. Because $\tilde{Q}$ is a discrete function of x, we now develop tight upper and lower bounds for $D(x)$ when $x \in \{\lceil \mu_A(N-1)/\mu_B \rceil, \ldots, \lfloor N - (\mu_B/\mu_A) \rfloor\}$. Both the upper and lower bounds are outputs of continuous functions, which are much more amenable to analysis, and the lower bound will be used in the proof of Proposition 8.

Lemma EC.2 **(Bounds on Cumulative System Time)**. *Defining the functions $S_1(\cdot)$ and $S_2(\cdot)$ and the quantity S_3 by*

$$\begin{aligned} S_1(t) &= \frac{t}{2}\left(\frac{N-x+1}{\mu_B} + \frac{N-x+t}{\mu_B}\right), \\ S_2(t) &= \frac{x-t}{2}\left(\frac{t+1}{\mu_A} + \frac{1}{\mu_B} + \frac{x}{\mu_A} + \frac{1}{\mu_B}\right), \\ \text{and } S_3 &= \frac{N-x}{2}\left(\frac{x+1}{\mu_A} + \frac{N}{\mu_A}\right), \end{aligned}$$

we have that if $\mu_A(N-1)/\mu_B < x \le N - (\mu_B/\mu_A)$, then $D(x)$ is bounded by

$$\begin{aligned} D(x) &\ge S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A} - 1\right) + S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right) + S_3, \\ D(x) &\le S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right) + S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A} - 1\right) + S_3. \end{aligned} \tag{EC.34}$$

Proof. Recall from the proof of Lemma EC.1 that if $(\mu_A(N-1)/\mu_B) < x \le N - (\mu_B/\mu_A)$, then

$$D(x) = \sum_{i=1}^{\tilde{Q}} \frac{N-x+i}{\mu_B} + \sum_{i=\tilde{Q}+1}^{x} \left(\frac{i}{\mu_A} + \frac{1}{\mu_B}\right) + \sum_{j=1}^{N-x} \frac{x+j}{\mu_A} = S_1(\tilde{Q}) + S_2(\tilde{Q}) + S_3.$$

The function $S_1(t)$ is non-decreasing on $\mathbb{R}_+$, and the function $S_2(t)$ is non-increasing on $\mathbb{R}_+$. Here we have that $x \le N - \mu_B/\mu_A$, so

$$1 \le \tilde{Q} \le \frac{\mu_A(N-x-1)}{\mu_B-\mu_A},$$

and the monotonicity properties of $S_1(\cdot)$ and $S_2(\cdot)$ give us that

$$\begin{aligned} D(x) &\ge S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A} - 1\right) + S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right) + S_3 \\ \text{and } D(x) &\le S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right) + S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A} - 1\right) + S_3. \quad \square \end{aligned}$$

The lemma follows from the upper and lower bounds on $\tilde{Q}$ given by

$$\frac{\mu_A(N-x-1)}{\mu_B-\mu_A} - 1 < \tilde{Q} = \left\lfloor \frac{\mu_A(N-x-1)}{\mu_B-\mu_A} \right\rfloor \le \frac{\mu_A(N-x-1)}{\mu_B-\mu_A}.$$

In turn, the bounds that we obtain are extremely tight when N is large, as they are essentially the bounds for the discretization error. Figure EC.1 plots the function $D(x)$, along with these upper and lower bounds, for two problem instances.

Proof of Proposition 8. We will establish that $D(1)$ is a lower bound on the value of $D(x)$ everywhere except the region where $\tilde{Q}$ appears in the expression for $D(x)$, and we will bound the difference between $D(1)$ and the minimum possible value of $D(x)$ in that region.

Case 1: $x=0$. Substituting 1 for x in equation (EC.32), we get

$$\begin{aligned} D(1) &= \frac{N}{\mu_B} + \frac{(N-1)(N+2)}{2\mu_A} \\ &= \frac{N}{\mu_B} + \frac{N^2+N}{2\mu_A} - \frac{1}{\mu_A} \\ &= D(0) - \frac{1}{\mu_A}, \end{aligned}$$

and therefore $1/\mu_A \le D_N^H - D_N^*$.

Case 2: $1 \le x \le \mu_A(N-1)/\mu_B$. In this interval, the continuous extension of $D(x)$ is a concave quadratic function. Thus, the minimum of $D(x)$ in this region must be at one of the endpoints. We evaluate $D(\mu_A(N-1)/\mu_B)$ and get

$$D\left(\frac{\mu_A(N-1)}{\mu_B}\right) = \frac{N^2+N}{2\mu_A} + \frac{N^2(\mu_A^2\mu_B - \mu_A^3)}{2\mu_A\mu_B^3} + \frac{N(2\mu_A^3 + \mu_A^2\mu_B - \mu_A\mu_B^2)}{2\mu_A\mu_B^3} + \frac{\mu_B^2 - 2\mu_A\mu_B - \mu_A^2}{2\mu_B^3}.$$

The difference $D(\mu_A(N-1)/\mu_B) - D(1)$ is then given by

$$D\left(\frac{\mu_A(N-1)}{\mu_B}\right) - D(1) = \frac{N^2\mu_A(\mu_B-\mu_A)}{2\mu_B^3} - \frac{N(\mu_B-\mu_A)(2\mu_A+3\mu_B)}{2\mu_B^3} + \frac{(\mu_B-\mu_A)(\mu_A^2+3\mu_A\mu_B+2\mu_B^2)}{2\mu_A\mu_B^3}$$

The larger root of this convex quadratic function of N occurs at $N = 1 + 2\mu_B/\mu_A$, and thus

$$N \ge 1 + \frac{2\mu_B}{\mu_A} \implies D\left(\frac{\mu_A(N-1)}{\mu_B}\right) - D(1) \ge 0.$$

We deduce therefore that $D(1)$ is the minimum value of $D(x)$ for $1 \le x \le \mu_A(N-1)/\mu_B$.

Case 3: $\mu_A(N-1)/\mu_B < x \le N - \mu_B/\mu_A$. In this region we will work relative to the lower bound in equation (EC.34). Evaluating this bound, we have

$$\begin{aligned} D(x) &\ge S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A} - 1\right) + S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right) + S_3 \\ &= x^2\left(\frac{\mu_A}{2\mu_B(\mu_B-\mu_A)}\right) + x\left(\frac{5}{2(\mu_B-\mu_A)} - \frac{\mu_A(2N+1)}{2\mu_B(\mu_B-\mu_A)}\right) \\ &\quad + N^2\left(\frac{\mu_B-\mu_A}{2\mu_A\mu_B} + \frac{1}{2(\mu_B-\mu_A)}\right) + N\left(\frac{\mu_B+\mu_A}{2\mu_A\mu_B} - \frac{2}{\mu_B-\mu_A}\right) + \frac{2\mu_A+\mu_B}{2\mu_B(\mu_B-\mu_A)} \\ &=: \underline{D}(x). \end{aligned}$$

We note that $\underline{D}(x)$ is a convex quadratic function in x. Differentiating with respect to x, we get

$$\frac{\partial \underline{D}}{\partial x} = \frac{5\mu_B - \mu_A(1+2N-2x)}{2\mu_B(\mu_B-\mu_A)}.$$

Because the first-order condition is sufficient for a minimum, we set the derivative $\partial\underline{D}/\partial x$ equal to zero and solve for the root x^*, obtaining

$$\begin{aligned} 5\mu_B - \mu_A(1+2N-2x^*) &= 0 \\ 2\mu_A x^* &= \mu_A(2N+1) - 5\mu_B \\ x^* &= N + \frac{1}{2} - \frac{5\mu_B}{2\mu_A}. \end{aligned}$$

We then have that $\underaccent{\bar}{D}(x^*)$ is a lower bound on the value of $D(x)$ for any $\mu_A(N-1)/\mu_B < x \le N - \mu_B/\mu_A$. The bound holds whether or not x^* falls within the relevant interval, as $\underaccent{\bar}{D}(x^*)$ is the global minimum of the quadratic function. Evaluating $\underaccent{\bar}{D}(x^*)$, we get

$$\begin{aligned}\underaccent{\bar}{D}(x^*) &= \frac{N^2+N}{2\mu_A} + \frac{N}{\mu_B} - \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)}\\ &= D(0) - \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)}\\ &= D(1) + \frac{1}{\mu_A} - \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)}.\end{aligned}$$

Because $\mu_A < \mu_B$, we have

$$\begin{aligned}0 &< 17\mu_B^2 - 6\mu_A\mu_B - 7\mu_A^2\\ 8\mu_B^2 - 8\mu_A\mu_B &< 25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2\\ \frac{1}{\mu_A} &< \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)},\end{aligned}$$

which implies that $\underaccent{\bar}{D}(x^*) < D(1)$.

Case 4: $N - \mu_B/\mu_A < x \le N$. The cumulative system time $D(x)$ is an increasing linear function on the half-open interval $(N - \mu_B/\mu_A, N]$. Therefore, the infimum of $D(x)$ on this interval is given by the output of the linear function evaluated at $N - \mu_B/\mu_A$, i.e.,

$$\begin{aligned}\inf_{x\in(N-\mu_B/\mu_A,N]} D(x) &= \frac{N - \mu_B/\mu_A}{\mu_B} + \frac{N^2+N}{2\mu_A}\\ &= \frac{N}{\mu_B} + \frac{N^2+N}{2\mu_A} - \frac{1}{\mu_A}\\ &= D(1).\end{aligned}$$

Collecting the results from the four cases, we conclude that

$$\frac{1}{\mu_A} \le D_N^H - D_N^* \le \frac{1}{\mu_A} + \frac{17\mu_B^2 - 6\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)}. \quad \square$$

Appendix C: Results from Section 7

Customers Who Visit Only One Station

PROPOSITION EC.1 **(Nash Equilibrium with Route *AB*—Dedicated Customers)**. *If*

$$N_S \ge (N_B+1)\Big(\frac{2\mu_A}{\mu_B}\Big) + 1 - N_A, \tag{EC.35}$$

then it is a Nash equilibrium for all N_S strategic players to visit station A first.

Proof. Suppose that all of the strategic players choose route AB. If so, then a player who deviates to take route BA will be served at station A after all N_A of the "A-only" players and after the other $N_S - 1$ strategic players. We therefore have the following lower bound on the expected system time $\mathbb{E}[S^B]$ from deviation given by

$$\mathbb{E}[S^B] \ge \frac{N_A + N_S}{\mu_A}. \tag{EC.36}$$

Next we calculate an upper bound on the expected system time from following the profile and taking route AB. Here we note that Property 1 still applies to this system. Namely, if station B ever idles, then it will never build up a queue again. First, consider the case with no B-only players (that is, $N_B = 0$). In this case, there will never be a queue at station B, and strategic customers departing from station A will immediately enter service at station B. Therefore, with no B-only customers we have an exact expression for $\mathbb{E}[S^A]$ given by

$$\mathbb{E}[S^A] = \sum_{j=1}^{N_A+N_S} \frac{1}{N_A+N_S}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right).$$

Now, for a given priority j, if N_B is greater than 0, then the maximum increase in the system time S^A from the case with no B-only players is equal to N_B/μ_B, the increased initial workload at station B. Thus, for any value of N_B we have the bound

$$\mathbb{E}[S^A] \le \sum_{j=1}^{N_A+N_S} \frac{1}{N_A+N_S}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B} + \frac{N_B}{\mu_B}\right) = \frac{N_A+N_S+1}{2\mu_A} + \frac{N_B+1}{\mu_B}. \tag{EC.37}$$

Equations (EC.35), (EC.36), and (EC.37) give

$$\begin{aligned} &(N_B+1)(\frac{2\mu_A}{\mu_B}) + 1 \le N_A + N_S \\ \implies \mathbb{E}[S^A] \le &\frac{N_B+1}{\mu_B} + \frac{N_A+N_S+1}{2\mu_A} \le \frac{N_A+N_S}{\mu_A} \le \mathbb{E}[S^B]. \end{aligned}$$

We conclude that strategic players have no incentive to deviate, and therefore it is a symmetric Nash equilibrium for all strategic players to visit station A first. □

PROPOSITION EC.2 **(Nash Equilibrium with Route BA—Dedicated Customers)**. *If* $\mu_B < 2\mu_A$ *and*

$$N_S \ge \frac{\mu_B(2N_A+1) - \mu_A(N_B-2)}{2\mu_A - \mu_B}, \tag{EC.38}$$

then it is a Nash equilibrium for all N_S *strategic players to visit station* B *first.*

Proof. Suppose that all of the strategic customers are following the profile of visiting station B first, and consider a player who contemplates deviating and visiting station A first. If she deviates, then she will certainly not enter service at station B until after station B processes all N_B B-only players as well as the other $N_S - 1$ strategic customers. Therefore, we can bound her expected system time $\mathbb{E}[S^A]$ from deviating by

$$\mathbb{E}[S^A] \ge \frac{N_B + N_S}{\mu_B}. \tag{EC.39}$$

Next, suppose that she follows the profile and visits station B first, and further suppose that she receives priority k at station B. Let Z_k be the random variable representing the number of strategic customers among the first $k-1$ players at station B. Given that there are $N_B + N_S - 1$ other players at station B, $N_S - 1$ of which are strategic, the random variable Z_k has a hypergeometric distribution with $k-1$ trials from a population of size $N_B + N_S - 1$ containing $N_S - 1$ successes. Therefore, its mean is

$$\mathbb{E}[Z_k] = (k-1)\frac{N_S - 1}{N_B + N_S - 1}. \tag{EC.40}$$

Table EC.1 Percent Deviation from Submodularity with Dedicated Customers ($N_S = 50, \mu_B = 1$).

	μ_A				
N_A, N_B	1.1	1.3	1.5	1.7	1.9
0,5	2	0	0	0	0
0,10	0	2	0	0	0
5,0	0	2	2	4	4
5,10	0	0	0	0	0
10,0	2	0	4	4	6
10,5	0	0	0	2	4
10,10	0	0	0	0	0

With a large number of B-only players it is possible that station A will become idle while some strategic customers have not yet been served at station B. Consider a strategic customer who is assigned priority k at station B. The greatest amount of idle time that could possibly be introduced at station A is the sum of the service times at station B of the B-only customers with a higher priority. There are exactly $k-1-Z_k$ such players. Define I_k as the amount of time that station A spends idle before the customer with priority k at station B finishes her service at station B. If $N_A = 0$, then station A would also idle for $1/\mu_B$ before it receives its first customer. An upper bound on the mean of $\mathbb{E}[I_k]$ is then

$$\mathbb{E}[I_k] \le \frac{1}{\mu_B} + \mathbb{E}\left[\frac{k-1-Z_k}{\mu_B}\right] = \frac{1}{\mu_B} + \frac{k-1-\mathbb{E}[Z_k]}{\mu_B}. \tag{EC.41}$$

We can now express the expected system time from choosing route BA by

$$\begin{aligned}\mathbb{E}[S^B] &= \sum_{k=1}^{N_B+N_S} \frac{1}{N_B+N_S}\left(\frac{N_A+1+\mathbb{E}[Z_k]}{\mu_A} + \mathbb{E}[I_k]\right)\\ &\le \sum_{k=1}^{N_B+N_S} \frac{1}{N_B+N_S}\left(\frac{N_A+1+\mathbb{E}[Z_k]}{\mu_A} + \frac{k-1-\mathbb{E}[Z_k]}{\mu_B} + \frac{1}{\mu_B}\right)\\ &= \frac{1}{\mu_B} + \frac{N_A+1}{\mu_A} + \frac{N_S-1}{2\mu_A} + \frac{N_B}{2\mu_B},\end{aligned} \tag{EC.42}$$

where the inequality follows from equation (EC.41) and the last equality follows from substituting the expression in equation (EC.40) for $\mathbb{E}[Z_k]$ and then evaluating the summation. Because $\mu_B < 2\mu_A$, equations (EC.38), (EC.39), and (EC.42) then give

$$\begin{aligned}&\frac{\mu_B(2N_A+1)-\mu_A(N_B-2)}{2\mu_A-\mu_B} \le N_S\\ &\implies \frac{2N_A+1}{2\mu_A} - \frac{N_B-2}{2\mu_B} \le N_S\left(\frac{1}{\mu_B} - \frac{1}{2\mu_A}\right)\\ &\implies \mathbb{E}[S^B] \le \frac{1}{\mu_B} + \frac{N_A+1}{\mu_A} + \frac{N_S-1}{2\mu_A} + \frac{N_B}{2\mu_B} \le \frac{N_B+N_S}{\mu_B} \le \mathbb{E}[S^A].\end{aligned}$$

Because $\mathbb{E}[S^B] \le \mathbb{E}[S^A]$, we conclude that no player has incentive to deviate, and thus it is a Nash equilibrium for all N_S strategic players to visit station B first. $\square$

We also observe in a series of numerical experiments that, if there is not too much imbalance in the number of dedicated customers at each station (i.e., only station A vs. only station B), then the expected system time for the strategic customers is often submodular (or close to submodular) even in the presence of dedicated customers. The results of the numerical study are summarized in Table EC.1. We fix $N_S = 50$

and $\mu_B = 1$. The number of customers who are dedicated to each station varies; there are always 50 strategic customers, and any non-strategic customers increase the total population size. For each routing profile of the strategic customers (i.e., for $x = 0, 1, \ldots, 50$ customers on route AB, with the remainder on route BA), we simulate the system 1,000 times. Service times are deterministic, so the only randomness in the system is the arrangement of customers of different types in the queues. Using the sample average system time for AB and BA customers for each routing profile, we compute the sample differences vector. Table EC.1 depicts the percentage of difference comparisons (i.e., d_0 vs. d_1, d_1 vs. d_2, etc.) which do *not* satisfy the decreasing differences condition. We note that all instances satisfy the condition within a .5% tolerance. These numerical experiments provide evidence that the expected system time continues to be submodular or almost submodular in the presence of dedicated customers, if most of the customers are strategic and there is not too much imbalance in the number of customers dedicated to each station.

$\mathcal{S}$-Station Open Routing Game

Proposition EC.3 **(Nash Equilibrium for Unobservable $\mathcal{S}$-Station System)**. *If we have*

$$N \geq 1 + 2\mu_1 \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi}, \tag{EC.43}$$

then the unobservable $\mathcal{S}$-station open routing game has a Nash equilibrium in which all players choose the routing vector $(1, 2, \ldots, \mathcal{S})$.

Proof. Suppose that all players are following the routing vector $(1, 2, \ldots, \mathcal{S})$, and consider a player who contemplates deviation. On path (that is, if everyone follows the prescribed profile), the only time any customer will face a queue is at station 1. All of the other stations start empty and then receive arrivals only when the station immediately before them completes a service. Because the routing vector is in order of increasing service rate, a queue will never build up at any station other than station 1. Thus, if the customer follows the prescribed profile, then her expected system time $\mathbb{E}[S^{\mathrm{EQ}}]$ is given by

$$\mathbb{E}[S^{\mathrm{EQ}}] = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu_1}\right) + \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} = \frac{N+1}{2\mu_1} + \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi}.$$

Because the player will not face a queue at any station besides station 1, she cannot possibly improve her system time by changing the order of stations that she visits after station 1. Thus the only deviations that we must consider are those which involve a vector that starts at a station other than station 1. If the player starts at station $\xi \geq 2$, then she will necessarily be the last customer to be served at station 1. Thus, we easily have a lower bound on her expected system time $\mathbb{E}[S^{\mathrm{D}}]$ from deviating given by

$$\mathbb{E}[S^{\mathrm{D}}] \geq \frac{N}{\mu_1}.$$

Equation (EC.43) then gives

$$\begin{aligned} N &\geq 1 + 2\mu_1 \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} \\ \Longrightarrow \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} &\leq \frac{N-1}{2\mu_1} \\ \Longrightarrow \mathbb{E}[S^{\mathrm{EQ}}] = \frac{N+1}{2\mu_1} + \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} &\leq \frac{N}{\mu_1} \leq \mathbb{E}[S^{\mathrm{D}}]. \end{aligned}$$

The player's expected system time is less if she follows the prescribed profile, and we therefore have a symmetric Nash equilibrium where all customers herd at station 1 and follow the routing vector $(1, 2, \ldots, \mathcal{S})$. □

Appendix D: Proof of Result from Section 8

Proof of Corollary 6. Let $\mathbb{E}[S^A]$ be the expected system time for a customer who takes route AB, and similarly let $\mathbb{E}[S^B]$ be the expected system time for a customer who takes route BA. Also, take $L_A(\mathbf{p}, \mathbf{z})$ and $L_B(\mathbf{p}, \mathbf{z})$ to be the steady-state expected number of customers at stations A and B, respectively, when the strategy profile specified by $\mathbf{p}$ and $\mathbf{z}$ is played. By Lemma 1, we have

$$\mathbb{E}[S^A] = \frac{L_A(\mathbf{p}, \mathbf{z})}{\mu_A} + \frac{L_B(\mathbf{p}, \mathbf{z})}{\mu_B} = \mathbb{E}[S^B]. \quad \square$$