Online Supplement to:

Online Piracy and the "Longer Arm" of Enforcement

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Appendices

A Model Extensions

In the development of our modeling framework, we made several assumptions, primarily for analytical tractability. In this appendix, we relax three of them and examine the robustness of our results.

A.1 Enforcement and Market Competition

In the paper, we assumed that our manufacturer operates within a monopolistic setting. This assumption is, of course, in line with the prior literature that has typically considered manufacturers of information goods to be monopolies (cf. August and Tunca 2008, Bae and Choi 2006, Lahiri and Dey 2013, among many others). Indeed, information goods intrinsically lack close substitutes. After all, there is only one Microsoft *Windows* as a software product, only one *Life is Beautiful* as a movie, and only one *Thriller* from Michael Jackson as a song. There may be other software products and other great movies or music, but they cannot be considered as perfect substitutes for various reasons, the primary one being consumers' distinct preferences for those particular products. The rationale for a monopolistic setting—the lack of competition—is clearly evidenced in the price commanded by information goods, which would surely have dropped to zero under perfect competition, since information goods have negligible marginal cost. However, since consumers have no incentive to pirate a product that is essentially free, perfect competition would also lead to a complete eradication of piracy.

We do recognize that the very existence of digital piracy in today's marketplace actually hints at a lack of perfect competition. At the same time, we do not accept either that the market is fully monopolistic. For, manufacturers of information products do not have the ability to name any price that they want, without losing some demand to an imperfect substitute. Therefore, we believe that the reality is somewhere in the *middle*—between the two extremes of monopoly and perfect competition—and, though not a monopoly, the manufacturer does retain *some* pricing power. That pricing power, however, is capped because our manufacturer faces some (imperfect) competition. In this section, we wish to examine if our findings extend to this more realistic setting.

To capture the case of imperfect competition, we assume that our manufacturer still has a unique product but now faces a cap on the price it can charge, normalized for the quality of the product. In other words, we now assume that the manufacturer's profit maximization problem, as outlined in Section 3.7, now faces an additional constraint: $\frac{p}{\theta} \leq \delta$, where $0 < \delta < \frac{1}{2}$. Not only does such an abstraction have the advantage of faithfully capturing the reality of the *middle*, but it also consistent with the two extreme cases—when $\delta = \frac{1}{2}$, there is no real cap on the price, and we are back to the monopoly case, whereas when $\delta = 0$, the pricing power disappears completely, as should be the case in a perfect competition.

As before, we only consider the primary piracy region (Cases 1A and 1B) in our analysis. In that region, whenever the interior solution automatically satisfies $\frac{p}{\theta} \leq \delta$—as is the case when δ is sufficiently large—all our earlier results continue to apply without any modification, and there is nothing more to do. Therefore, the only additional case we must examine now is when the interior solution actually violates $\frac{p}{\theta} \leq \delta$ and forces a boundary solution of $\frac{p}{\theta} = \delta$.

Lemma A1 *In Case 1B, the boundary solution of* $\frac{p}{\theta} = \delta$ *is not possible in the long-run equilibrium.*

Lemma A1 is intuitive. Recall that, in Case 1B, the monopolist manufacturer completely ignores the $(1-\lambda)$ segment with access to pirated content and charges a monopoly price to the λ segment. However, when there is a real cap on the price the manufacturer can charge, the profit from just the λ segment decreases. The manufacturer can no longer ignore the $(1-\lambda)$ segment and starts pricing the product to make it attractive to that segment as well, thereby moving the outcome from Case 1B to Case 1A. In other words, the shadow competition from piracy can be ignored by a monopolist in certain situations but, in a competitive market, this shadow competition becomes a relevant issue in the manufacturer's pricing strategy.

Lemma A1 indicates that, faced with a binding price cap due to competition, it is sufficient to consider just Case 1A,[*] the solution for which is our next result.

Proposition A1 *In the long-run equilibrium, the following boundary solution can occur under Case 1A:*

$$\theta^* = \begin{cases} \frac{e\beta\delta\xi}{c} + \frac{\delta(1-\beta-\delta)}{c(1-\beta)}, & \text{if } \beta < 1 \\ \frac{e\delta\xi}{c} - \frac{\beta\delta^2}{c(\beta-1)} & \text{otherwise,} \end{cases} \quad \text{and} \quad p^* = \begin{cases} \frac{e\beta\delta^2\xi}{c} + \frac{\delta^2(1-\beta-\delta)}{c(1-\beta)}, & \text{if } \beta < 1 \\ \frac{e\delta^2\xi}{c} - \frac{\beta\delta^3}{c(\beta-1)} & \text{otherwise.} \end{cases}$$

It is easy to see from Proposition A1 that $\lim_{\delta \to 0} \theta^* = 0$. In other words, as expected, when the market is fiercely competitive, the manufacturer decides against participating in the market. Given the fixed costs of development, the manufacturer stays away from developmental activities unless it can recover those costs by charging a healthy price premium.

Examining the expression for θ^* above, we find that it is independent of r, but is linearly increasing in e. The following result is immediate:

Corollary A1 *The equilibrium quality in Proposition A1 does not change with* r *but increases in* e*.*

Thus, we find that Corollary A1 matches with the finding in Theorem 2 that supply-side enforcement may indeed result in added incentives for innovation and lead to better products. In other words, even when the product faces competition in the marketplace, supply-side enforcement seems to be a better choice as an anti-piracy instrument, from the perspective of innovation.

We conclude this section with comparative statics for private profit and public welfare. We find:

Theorem A1 *For the equilibrium in Proposition A1, the manufacturer's profit is always increasing in both* r *and* e*. The consumer surplus is always decreasing in* r*; however, it is increasing in* e *for* $\beta < 1$ *but not monotonic for* $\beta \geq 1$*. Social welfare is increasing in* r *and* e *for* $\beta < 1$*; for* $\beta \geq 1$*, it is not monotonic in* e *but is monotonically decreasing in* r*.*

In conclusion, we find that Theorem A1 practically echoes the findings in Theorem 3 and Proposition 4. Therefore, our main results derived under a monopoly assumption extend naturally to the competitive setting. When the competition is not very fierce and δ is large, the price cap is not binding. In that case, our earlier results hold verbatim. However, as the competition increases, resulting in a smaller δ, the price cap may become binding. Even in that situation, the basic message that supply-side enforcement is generally a better choice seems to hold. In practice, we expect a market to have many different types of information products, with varying degrees of competition faced by their manufacturers. Since a unique enforcement strategy for every product is not practically viable, in an overall sense, supply-side enforcement is perhaps the path to adopt, till we learn more about the actual consequences of these efforts.

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[*] Although Case 1A was not possible in the long-run equilibrium for $\beta \geq 1$ under monopoly (Lemma 2), curiously, it emerges as a viable equilibrium outcome under imperfect competition.

A.2 Consumer Learning Influenced by e

In Section 3.4, we worked with a fixed threshold T that consumers use in deciding whether to learn and use a new pirate site; this threshold was constant and thus independent of the enforcement level, e. It is, however, conceivable that consumers' learning efforts are actually influenced by e. In other words, the consumer may readjust his learning behavior by using a threshold that is also a function of e:

$$T(e) = T(1 + eB).$$

Clearly, $T(e)$ is increasing (decreasing) in e if the constant B is positive (negative). This flexibility is certainly desirable because there is no clear-cut or well-accepted direction for $T(e)$. For example, consumers may become more dogged in the face of higher enforcement and be ready to incur higher learning costs; if so, $T(e)$ should be increasing in e. In complete contrast, one could also argue that, as e increases and supply dwindles, consumers may actually feel more frustrated and get discouraged faster, making $T(e)$ a decreasing function.

It turns out that all our results in Section 3.6 regarding the equilibrium outcome—Propositions 1 and 2—remain essentially the same, with ξ replaced by $\xi(e) = \mathbf{e}^{\frac{1}{\omega T(1+eB)}} - 1$. Furthermore, since $\xi(e)$ is independent of r, all the comparative statics results with respect to r in Section 4 remain valid as well. Therefore, all we need to do is to focus on the comparative statics with respect to e alone. Fortunately, those results are also valid by and large. In fact, it can be shown that they are all valid as long as $\xi + e\xi' > 0$. Since $e, \xi > 0$ and

$$\xi' = \frac{d\xi}{de} = -\frac{B(\xi + 1)}{\omega T(1 + eB)^2} > 0, \forall B < 0,$$

whenever B is negative, it makes $\xi + e\xi'$ positive, which is sufficient for all our results to carry over. Therefore, to understand the nature of this additional restriction imposed by $\xi + e\xi' > 0$, we only need to consider the case where B is positive. To that end, we define a characteristic function $y = f(x)$ implicitly, as the solution of:

$$1 - \mathbf{e}^{-\frac{1}{y(1+x)}} - \frac{x}{y(1 + x)^2} = 0. \tag{A1}$$

It is easy to verify that the constraint $\xi + e\xi' > 0$ is logically equivalent to $y > f(x)$, with $x = eB$ and $y = \omega T$. We now combine all these observations into our next result.

Theorem A2 *All the comparative statics results stated under Propositions 3 and 4 and Theorems 1, 2, and 3 continue to hold for $B \leq 0$. When $B > 0$, these results hold if $\omega T > f(eB)$.*

Clearly, Theorem A2 tells us that our results do not materially change for $B \leq 0$. However, for $B > 0$, there is an additional condition, $\omega T > f(eB)$, which must now be met. How big a restriction does this condition pose, and can it be satisfied under practical circumstances? Answering these questions is somewhat tricky because $y = f(x)$ is defined only as an implicit function of x in (A1) and, much like a significant portion of this paper, does not have a closed-form solution. Fortunately, though, by substituting $z = \frac{1}{y(1+x)}$ and $a = \frac{x}{1+x}$, (A1) can be rewritten in the fixed-point form:

$$z = F_a(z) = \frac{1 - \mathbf{e}^{-z}}{a}, \tag{A2}$$

with a unique non-zero solution and a guaranteed and fast convergence to it, ensuring the following result:

Corollary A2 *When $B > 0$, all the comparative statics results stated under Propositions 3 and 4 and Theorems 1, 2, and 3 continue to hold as long as $\omega T \geq 0.5$, irrespective of the value of B.*

The result in Corollary A2 is better visualized in Figure A1, which is also the solution of (A1) with $x = eB$ and $y = \omega T$.

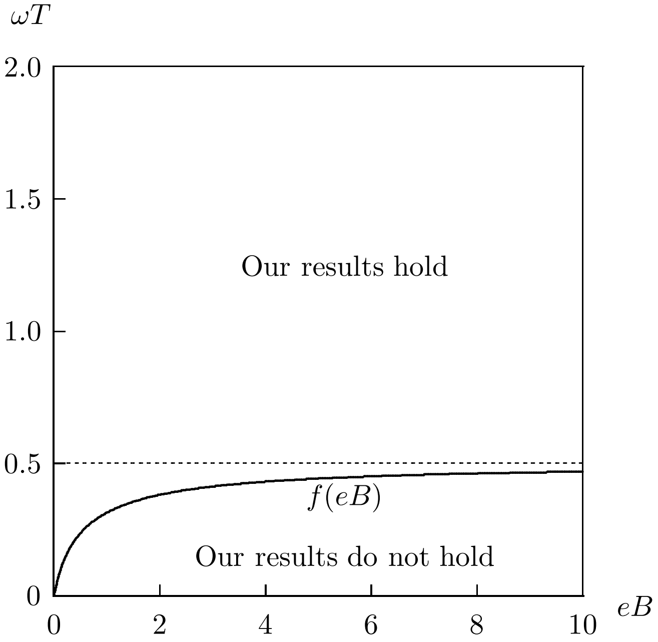


Figure A1: The $(eB, \omega T)$ Space Partitioned by Characteristic Function, $f(eB)$

There are several interesting observations that can be made from Corollary A2 and Figure A1. First, although $f(eB)$ is an increasing function, it is bounded above by the asymptote $\omega T = 0.5$. Therefore, $\omega T \geq 0.5$ is a sufficient condition for $\omega T > f(eB)$ to hold. In other words, as long as ωT is not too small—specifically, as long as $\omega T \geq 0.5$—all our earlier results are good. Clearly, there is a large portion of the parameter space where our results continue to apply. Second, what is really surprising is that this result holds for arbitrarily large values of B. Recall that B represents the relative impact of enforcement on consumers' willingness to learn. Our results indicate that, even when consumers respond in force to a reduction in supply of pirated content (because of higher supply-side enforcement), $\omega T \geq 0.5$ remains a sufficient condition for our results to hold.

How realistic is the restriction of $\omega T \geq 0.5$? Recall that there is a fraction, $g = 1 - \mathbf{e}^{-\frac{1}{\omega T}}$, of consumers who are inherently ethical and do not consider piracy to be an option even when there is no supply-side enforcement (that is, $e = 0$). Also, g is a decreasing function of ωT, approaching one as ωT approaches zero. In particular, $\omega T < 0.5$ is equivalent to $g > 0.865$, implying that, in order to render our analytical results invalid, we would need more than 86.5% of consumers to be inherently ethical, a scenario inconsistent with field observations and quite unlikely in practice. Viewed alternatively, $\omega T \geq 0.5$ requires a meager 13.5% of consumers to be interested in pirated content and is not a wildly restrictive assumption, so our results should extend to most real-world situations.

A.3 Product-Specific Cost for Accessing Pirated Content

In our model in Section 3.4, the learning cost was assumed to be fixed and not specific to any single product (Danaher et al. 2010). While we consider this to be a more realistic representation of the context, one could still ask how robust our results would be if there were also a product-specific cost—similar to the search cost in (Geng and Lee 2013)—for accessing pirated content. Now, if indeed there were a search cost, s, the easiest way to capture it would be to include it in the IR and IC constraints alongside the original piracy cost, r (cf. Duchene and Waelbroeck 2005). In other words, all those constraints would still apply if r is simply replaced by $(r+s)$. Therefore, every expression in Section 3, as well as the entire equilibrium analysis there, would remain valid with $t = r + s$ in place of r.

We now consider the impact of such an extension on the comparative statics in Section 4. Interestingly, if s is independent of e, then replacing r by t simply amounts to a change of origin for r, and the resulting change in the equilibrium outcome is only a lateral shift in the (r, e) space, immediately ensuring that all our comparative statics results are robust to such an extension. In other words, if we were to follow prior literature and include a fixed search cost (Geng and Lee 2013), all our results as stated in Section 4, would continue to hold verbatim.

Therefore, in order to complete our robustness check, we now consider s to be a function of e, with $s'(e) \geq 0$. Now, since $s(e)$ is independent of r and $\frac{dt}{dr} = \frac{dt}{ds} = 1$, we are assured that all the comparative statics with respect to r will remain intact. Hence, we focus on the comparative statics results with respect to e only. Given any metric, M, we can write:

$$\frac{dM}{ds} = \frac{dM}{dt} \times \frac{dt}{ds} = \frac{dM}{dr}.$$

Therefore, we get:

$$\frac{dM}{de} = \left.\frac{dM}{de}\right|_{\text{Old}} + s'(e)\frac{dM}{ds} = \left.\frac{dM}{de}\right|_{\text{Old}} + s'(e)\frac{dM}{dr}, \tag{A3}$$

where the subscript "Old" simply stands for the derivative taken with respect to e for a fixed s and is the same as the one in Section 4, with r replaced by $t = r + s$. It is interesting to note in (A3) that the nature of our results do not depend on the size of $s(e)$, but only on how fast $s(e)$ changes with e. We now consider the relevant metrics.

Social Welfare: We know from Theorem 3 that $\left.\frac{d(SW)}{de}\right|_{\text{Old}} > 0$ and $\frac{d(SW)}{dr}$ could be either positive or negative. If $\frac{d(SW)}{dr}$ is positive, then from (A3), so is $\frac{d(SW)}{de}$. However, if the former is negative, the latter could still be positive, as long as $s'(e)$ is not too large; an estimate of the upper bound on $s'(e)$ can be obtained directly from (A3):

$$s'(e) < -\left.\tfrac{d(SW)}{de}\right|_{\text{Old}} \Big/ \tfrac{d(SW)}{dr}. \tag{A4}$$

Since $\frac{d(SW)}{dr}$ is now negative, the right-hand side of (A4) is always positive, so (A4) would be trivially satisfied if $s'(e) = 0$, exactly what we would expect based on our discussion of a fixed search cost above. In summary, as far as the direction of its impact is concerned, supply-side enforcement cannot be any worse than its demand-side counterpart. This is because, whenever r has a positive impact on social welfare, so does e, but e could have a positive impact even when the impact of r is negative. From a policy angle, this perhaps makes e a slightly better choice over r. A comparison based on only *legal social welfare* also yields similar conclusions.

Quality: If the performance metric is quality, a proxy for innovation in our setup, we arrive at similar conclusions. This is because we know from Theorem 2 that $\frac{d\theta^*}{de}\Big|_{\text{Old}} > 0$, but $\frac{d\theta^*}{dr}$ is always negative. Therefore, from (A3), we find that $\frac{d\theta^*}{de}$ could still be positive if $s'(e)$ is not too large, implying once again that supply-side enforcement cannot be any worse than demand-side. An upper bound similar to (A4) can be found for this case as well.

Manufacturer's Profit: Because the manufacturer's profit increases with either type of enforcement, it can be easily shown using (A3) that the manufacturer's profit is still monotonically increasing in e.

Consumer Surplus: From Proposition 4, we know that $\frac{d(CS)}{dr} < 0$, but $\frac{d(CS)}{de}\Big|_{\text{Old}}$ could be positive or negative. If $\frac{d(CS)}{de}\Big|_{\text{Old}}$ is negative, it is clear from (A3) that so is $\frac{d(CS)}{de}$. However, if $\frac{d(CS)}{de}\Big|_{\text{Old}} > 0$, $\frac{d(CS)}{de}$ can be positive for moderate values of $s'(e)$. In other words, the impact of demand-side enforcement on consumer surplus can never be positive, yet that of supply-side can be, under certain conditions.

All these results are consolidated in Table A1. From this table, along with Table 2 for a comparison, it appears that, even if we were to include a search cost that depends on e, in an overall sense, supply-side enforcement would still be preferable to its demand-side counterpart, although not as strongly as before. Stated differently, our earlier results do get weakened somewhat, but they hold as stated in Section 4 as long as $s'(e)$ is moderate.

Table A1: Long-Run Economic Impacts of Demand- and Supply-Side Enforcement with $s'(e) > 0$

	Demand-Side Impact ($r\uparrow$)	**Supply-Side Impact ($e\uparrow$)**	**Preferable**
Social Welfare	Ambiguous	Ambiguous	$e^\dagger$
Product Quality	Negative	Ambiguous	e
Manufacturer Profit	Positive	Positive	e, r
Consumer Surplus	Negative	Ambiguous	e
Legal Social Welfare	Ambiguous	Ambiguous	$e^\dagger$

Note: †In this case, even though the impact is ambiguous for both e and r, e is mildly preferrable to r because, whenever the demand-side impact is positive, so is the supply-side impact. However, the supply-side impact can be positive even when the demand-side impact is negative.

Another way to interpret our results would be to view $s'(e)$, the spillover to the demand side, as a measure of (un-)desirability of supply-side interventions vis-à-vis the demand-side ones; see Figure A2. When $s'(e)$ is small, a supply-side measure results in better social outcome and is evidently more desirable over a demand-side one. However, as $s'(e)$ increases and supply-side intervention starts manifesting itself more and more on the demand side, the positive impact of the intervention weakens, with its desirability beginning to wane as a result. When $s'(e)$ becomes very large, the two types of interventions become indistinguishable in their impact. This is because, when the spillover is large, every supply-side intervention, in essence, morphs into a

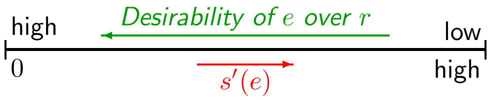


Figure A2: Desirability of Supply-Side Enforcement as Measured by $s'(e)$

demand-side one and the distinction between the two types of enforcement begins to disappear. When that happens, it is doubtful that we would see a positive impact from a supply-side intervention which, now, is largely a demand-side measure in disguise.

A more fundamental question naturally arises at this point. Is there truly a distinction between the two enforcement types, or is this distinction superfluous, cooked up in favor of a convoluted mathematical exercise? If prior research is to be believed, such a distinction does exist (Aguiar et al. 2015, Danaher et al. 2014, 2015, 2016); in fact, the terms “supply-side” and “demand-side” themselves are due to Danaher et al. (2014). Further, if popular press is to be relied upon, such a distinction is already manifest in practice; see Table 1. Therefore, if only to do justice to the nomenclature, a supply-side intervention ought to have its larger or primary impact on the supply side, that is, on pirate suppliers, just like a demand-side intervention ought to, on pirate consumers. A secondary impact larger than the primary impact is, in essence, a *non sequitur*.

There is also ample empirical support that makes a large secondary impact seem unlikely in practice. For example, Aguiar et al. (2015) find that a large majority of consumers had little difficulty—that is, insignificant impact on consumers’ search cost—in finding alternative sites when a widely popular pirate site was taken down. This is also consistent with the observations made by Danaher et al. (2015, 2016) that blocking one or a few site(s) did not increase legal sales significantly, but blocking several of them did. Thus, even if supply-side interventions increased consumers’ search cost for pirated content, such increases were not large enough to deter certain consumers from tracking down alternate pirate sources. This finding seems quite logical. When `MegaUpload.com` was taken down, “many linking websites, custom search engines, and custom streaming scripts that relied on the hosted content became inoperable” (Masnick 2012); as a direct consequence, “some websites were abandoned by their operators, others lost traffic, while still others shifted their business model.” Clearly, the disruptions caused by the shutdown to the supply side was enormous, and it is quite unlikely that consumers alone faced their full brunt. In fact, in our extensive numerical analyses as well, which were done with a wide array of parameter values, we have consistently found the thresholds—the right-hand sides of (A4) and its counterparts for other metrics—to be all quite high; this too makes us surmise that these conditions are rather likely to be satisfied in practice.

B Complete Analysis of the Equilibrium

The purpose of this appendix is to provide all the details related to the overall analysis of the equilibrium and characterize it rigorously. We start by identifying the possible strategies a manufacturer can adopt.

B.1 Manufacturer’s Strategy Space

The manufacturer’s problem is to solve: $\max_{p,\theta} \pi = pq - \frac{c\theta^2}{2}$, where q is the legal demand given by (2). Although conceptually straightforward, solving this problem and analytically characterizing its solution are not simple. This is because the manufacturer’s strategy shifts as we move from one point in the parameter space to another, resulting in singularities with respect to the decision variables at the boundaries of these strategies. Depending on the context parameters, the manufacturer finds it optimal to be in exactly one of the seven cases; please see Figure B1.

Cases 1A, 1B—Limited Supply ($0 < \eta < 1$): Here, a pirated copy has limited availability. As indicated in Figure 2 (a) and (c), the manufacturer can name a price such that it ends up selling to both types of consumers, those who have access to a pirated copy and those who do not (Case 1A). Alternatively,

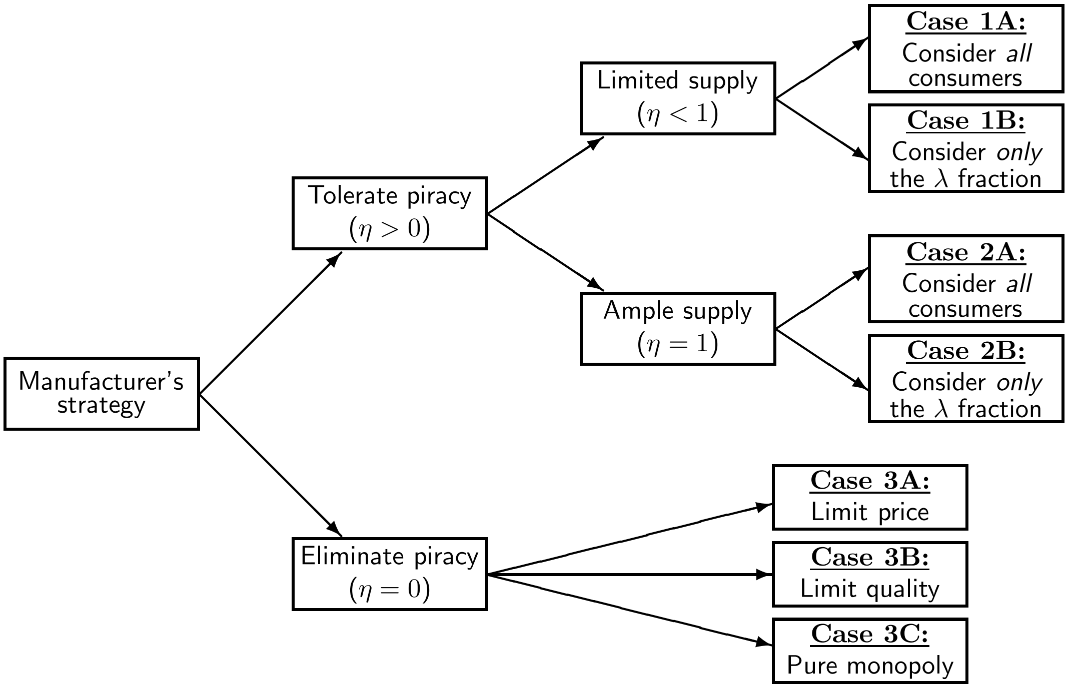


Figure B1: Manufacturer's Strategy Tree

as shown in Figure 2 (b) and (d), it can set p high, effectively shutting out, from the legal product, all consumers who have access to a pirated version (Case 1B).

Cases 2A, 2B—Ample Supply ($\eta = 1$): The supply of pirated content is abundant in this case, and $\eta = 1$ implies $\lambda = \frac{\xi}{\xi+1}$. Similar to Case 1, we again face two possibilities: the manufacturer may serve both groups, those with piracy as an option and those without (Case 2A), or consider just the latter (Case 2B).

Case 3A, 3B, 3C—No Piracy ($\eta = 0$): Piracy ceases to exist when no consumer has the option to use a pirated version, or if the manufacturer chooses the price and/or quality in a way that the pirated product is rendered completely uncompetitive. In equilibrium, they are equivalent, and $\lambda = 1$ in both cases. There are three ways piracy may disappear: In Case 3A, the manufacturer chooses a "limit" price such that the illegal copy becomes barely unattractive when compared to the legal one. In Case 3B, the manufacturer chooses a "limit" quality to the same effect. In both cases, piracy ceases to exist, even though the threat of piracy still remains—unless the manufacturer holds the price or quality at the "limit" level, piracy can resurface. Finally, Case 3C happens when enforcement on either demand- or supply-side, or both, is high enough to suppress all threats completely, resulting in a pure monopoly.

In each of the above seven cases, the manufacturer's objective function as well as the constraints are different. We show all these in Table B1.

B.2 Short-Run Equilibrium

Solving the manufacturer's optimization problem outlined in Table B1, we can easily find the equilibrium outcome in each region. In the short run, θ is exogenous, and the manufacturer only chooses the price for a

Table B1: Manufacturer's Optimization Problem under Different Strategies

	Case	**Objective Function**	**Constraints**
$\beta<1$	Case 1A	$p\left(\lambda\left(1-\frac{p}{\theta}\right)+(1-\lambda)\left(1-\frac{p-r}{(1-\beta)\theta}\right)\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}}$, $\frac{\xi}{\xi+1}<\lambda<1$, and $\frac{p-r}{(1-\beta)\theta}\leq 1$
	Case 1B	$p\lambda\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi}{\theta-\frac{r}{\beta}}$, $\frac{\xi}{\xi+1}<\lambda<1$, and $\frac{p-r}{(1-\beta)\theta}>1$
	Case 2A	$p\left(\frac{\xi}{\xi+1}\left(1-\frac{p}{\theta}\right)+\frac{1}{\xi+1}\left(1-\frac{p-r}{(1-\beta)\theta}\right)\right)-\frac{c\theta^2}{2}$	$\frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}}\leq\frac{\xi}{\xi+1}$, and $\frac{p-r}{(1-\beta)\theta}\leq 1$
	Case 2B	$\frac{p\xi}{\xi+1}\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\frac{e\theta\xi}{\theta-\frac{r}{\beta}}\leq\frac{\xi}{\xi+1}$, and $\frac{p-r}{(1-\beta)\theta}>1$
	Case 3A	$p\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}}=1$, and $\frac{p-r}{(1-\beta)\theta}\leq 1$
	Case 3B	$p\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi}{\theta-\frac{r}{\beta}}=1$, and $\frac{p-r}{(1-\beta)\theta}>1$
	Case 3C	$p\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}}>1$ or $\frac{e\theta\xi}{\theta-\frac{r}{\beta}}>1$
$\beta\geq 1$	Case 1A	$p\left(\lambda\left(1-\frac{p}{\theta}\right)+(1-\lambda)\left(\frac{r-p}{(\beta-1)\theta}-\frac{p}{\theta}\right)\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi}{\theta-\frac{r-p}{\beta-1}}$, $\frac{\xi}{\xi+1}<\lambda<1$, and $p\leq\frac{r}{\beta}$
	Case 1B	$p\lambda\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi}{\theta-\frac{r}{\beta}}$, $\frac{\xi}{\xi+1}<\lambda<1$, and $p>\frac{r}{\beta}$
	Case 2A	$p\left(\frac{\xi}{\xi+1}\left(1-\frac{p}{\theta}\right)+\frac{1}{\xi+1}\left(\frac{r-p}{(\beta-1)\theta}-\frac{p}{\theta}\right)\right)-\frac{c\theta^2}{2}$	$\frac{e\theta\xi}{\theta-\frac{r-p}{\beta-1}}\leq\frac{\xi}{\xi+1}$, and $p\leq\frac{r}{\beta}$
	Case 2B	$\frac{p\xi}{\xi+1}\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\frac{e\theta\xi}{\theta-\frac{r}{\beta}}\leq\frac{\xi}{\xi+1}$, and $p>\frac{r}{\beta}$
	Case 3A	$p\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi}{\theta-\frac{r-p}{\beta-1}}=1$, and $p\leq\frac{r}{\beta}$
	Case 3B	$p\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\lambda=\frac{e\theta\xi}{\theta-\frac{r}{\beta}}=1$, and $p>\frac{r}{\beta}$
	Case 3C	$p\left(1-\frac{p}{\theta}\right)-\frac{c\theta^2}{2}$	$\frac{e\theta\xi}{\theta-\frac{r-p}{\beta-1}}>1$ or $\frac{e\theta\xi}{\theta-\frac{r}{\beta}}>1$

given θ.

$\beta<1$, Case 1A: From the legal demand in (2), the revenue in this case is:

$$R=p\left(\lambda\left(1-\frac{p}{\theta}\right)+(1-\lambda)\left(1-\frac{p-r}{(1-\beta)\theta}\right)\right).$$

We substitute $\lambda=\frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}}$ into this to obtain $R=p+ep\beta\xi-\frac{p(p-r)}{(1-\beta)\theta}$. The first order condition is then given by $\frac{\partial R}{\partial p}=1+e\beta\xi-\frac{2p-r}{(1-\beta)\theta}=0$, which can be easily solved to obtain $p^*(\theta)$. It is easy to see that the second order condition is also satisfied.

$\beta\geq 1$, Case 1A: From (2), the revenue is:

$$R=p\left(\lambda\left(1-\frac{p}{\theta}\right)+(1-\lambda)\left(\frac{r-p}{(\beta-1)\theta}-\frac{p}{\theta}\right)\right).$$

As before, we substitute $\lambda=\frac{e\theta\xi}{\theta-\frac{r-p}{\beta-1}}$ into this to obtain $R=ep\xi+\frac{p(r-p\beta)}{(\beta-1)\theta}$, differentiating which, we obtain the following first order condition $\frac{\partial R}{\partial p}=e\xi+\frac{r-2p\beta}{(\beta-1)\theta}=0$. Since the second order condition is also satisfied, the first order condition yields the desired price.

Case 1B: In Case 1B, λ is independent of p: $\lambda=\frac{e\theta\xi}{\theta-\frac{r}{\beta}}$. Therefore, the revenue, $R=p\lambda\left(1-\frac{p}{\theta}\right)$, is clearly maximized at $p^*(\theta) = \frac{\theta}{2}$.

The derivations for all other cases are analogous. In fact, in all cases, except Case 3A, the optimal price can be obtained by solving the appropriate first order condition. In Case 3A, the limit price is obtained by setting $\lambda = 1$ as indicated in Table B1. Accordingly, we get:

$$p^*(\theta) = \begin{cases} \frac{r}{2} + \frac{\theta(1-\beta)(1+e\beta\xi)}{2}, & \text{Case 1A } (\beta < 1), \\ \frac{r}{2\beta} + \frac{e\theta\xi(\beta-1)}{2\beta}, & \text{Case 1A } (\beta > 1), \\ \frac{r+\theta(1-\beta)(1+\xi)}{2(1+\xi(1-\beta))}, & \text{Case 2A } (\beta < 1), \\ \frac{r+\theta\xi(\beta-1)}{2(\beta+\xi(\beta-1))}, & \text{Case 2A } (\beta > 1), \\ \frac{r}{\beta} + e\theta\xi(1-\beta), & \text{Case 3A } (\beta < 1), \\ r - \theta(\beta-1)(1-e\xi), & \text{Case 3A } (\beta \geq 1), \\ \frac{\theta}{2}, & \text{Cases 1B, 2B, 3B, and 3C.} \end{cases}$$

In addition to satisfying the constraints in Table B1, for the equilibrium to occur within a specific region, its corresponding case must dominate, from the perspective of the manufacturer's profit, all other cases that provide a valid solution. Below, we ensure that all these requirements are met in determining the boundary for each region.

B.2.1 Characterization of Different Equilibrium Regions for $\beta < 1$

First, we determine the boundaries based on profit comparisons between pairs of equilibrium regions. We will subsequently show that the validity conditions in Table B1 are automatically satisfied within each region obtained from such profit comparisons.

Profit Comparisons

Region 1A:
Equating the profits for Cases 1A and 1B, we get:

$$e = h_1(r;\theta) = \frac{(r + (1-\beta)\theta)^2}{\beta\theta\xi(1-\beta)(\beta\theta - r)}.$$

We find that 1A dominates if $e < h_1(r;\theta)$, 1B dominates if $e > h_1(r;\theta)$, and both provide the same profit at $e = h_1(r;\theta)$. Similarly, equating the profits from 1A and 3A, we get the following boundary:

$$e = h_2(r;\theta) = \frac{1}{\xi}\left(\frac{1}{2-\beta} - \frac{r}{\beta\theta(1-\beta)}\right),$$

and 1A is valid only when $e < h_2(r;\theta)$. Now, we compare the profit in 1A with that from 2A and 2B to obtain:

$$e = h_3(r;\theta) = \frac{\sqrt{1+\xi(1-\beta)}(r+\theta(1-\beta)(1+\xi)) - \sqrt{1+\xi}(r+(1-\beta)\theta)(1+\xi(1-\beta))}{\beta\theta\xi\sqrt{1+\xi}(1-\beta)(1+\xi(1-\beta))}, \text{ and}$$
$$e = h_4(r;\theta) = \frac{\theta\left(\sqrt{\xi(1-\beta)} - (1-\beta)\sqrt{1+\xi}\right) - r\sqrt{1+\xi}}{\beta\theta\xi\sqrt{1+\xi}(1-\beta)}.$$

1A dominates to the right of these boundaries, that is, when e is greater then both $h_3(r;\theta)$ and $h_4(r;\theta)$. Region 1A can now be fully characterized as follows:

$$\{(r,e)|\max\{h_3(r;\theta),h_4(r;\theta)\} < e \leq \min\{h_1(r;\theta),h_2(r;\theta)\};e,r \geq 0\}\,. \tag{RGN1Ah}$$

Point to note here is that, in obtaining the boundaries of Region 1A, its profit need not be compared with the profit from either 3B or 3C. This is because, considering the validity conditions, we see that 1A can never have a valid solution when 3B or 3C does.

Region 1B:
The result of profit comparison with 1A is already captured as $e = h_1(r;\theta)$. Next, we compare its profit with that of 3A to obtain the following boundary:

$$e = h_5(r;\theta) = \frac{4r(2r-3\beta\theta)(1-\beta)+\beta^2\theta^2(3-4\beta)+\beta\theta\sqrt{\beta\left(\beta\theta^2(3-4\beta)^2-16r^2(1-\beta)-8r\theta(1-\beta)(1-4\beta)\right)}}{8\beta\theta\xi(1-\beta)^2(\beta\theta-r)},$$

such that Region 1B occurs only above this boundary. Define:

$$h_6(r;\theta) = \begin{cases} h_1(r;\theta), & \text{if } 0 \leq r < \rho_{1h} = \frac{\theta(1-\beta)(3\beta-2)}{4-3\beta}, \\ h_5(r;\theta), & \text{otherwise,} \end{cases}$$

where ρ_{1h} is the solution of $h_1(r;\theta) = h_5(r;\theta)$. Clearly, $e = h_6(r;\theta)$ provides a combined lower boundary for 1B. To find the only other possible lower boundary for this region, we now compare its profit with that of Case 2B to obtain:

$$e = h_7(r;\theta) = \frac{\beta\theta - r}{\beta\theta(1+\xi)}.$$

To find the upper boundary for Region 1B, we equate its profit with that from 3C to obtain:

$$e = \frac{\beta\theta - r}{\beta\theta\xi} = \frac{(1+\xi)h_7(r;\theta)}{\xi}.$$

Therefore, Region 1B can now be fully characterized as follows:

$$\left\{(r,e)|\max\{h_6(r;\theta),h_7(r;\theta)\} \leq e < \tfrac{(1+\xi)h_7(r;\theta)}{\xi}; r \leq \rho_{2h} = \tfrac{\theta(2\beta-1)}{2}; e,r \geq 0\right\}, \tag{RGN1Bh}$$

where ρ_{2h} is the solution of $h_5(r;\theta) = \frac{(1+\xi)h_7(r;\theta)}{\xi}$.

Region 2A:
Comparing the profits from Cases 2A and 2B, we get:

$$r = \rho_{3h} = \theta(1+\xi)\left(\frac{\sqrt{\xi(1-\beta)(1+\xi(1-\beta))}}{1+\xi} - (1-\beta)\right).$$

Region 2A occurs to the right of this boundary and 2B, to the left. The boundary with Region 1A is already derived as $e = h_3(r;\theta)$. We now find the boundary between 2A and 3A by comparing their profits:

$$e = h_8(r;\theta) = \frac{\beta\theta - 2r}{2\beta\theta\xi(1-\beta)} + \frac{1}{2\theta\xi(1-\beta)^2}\sqrt{\frac{(1-\beta)\left(\theta(1-\beta)(1+\xi)(\beta\theta-2r)-r^2\right)}{(1+\xi)(1+\xi(1-\beta))}}.$$

Then, the combined upper boundary can be found as:

$$e = h_9(r;\theta) = \begin{cases} h_3(r;\theta), & \text{if } 0 \le r < \rho_{4h} = \frac{\theta(1-\beta)(1+\xi)}{2-\beta}\left(2\sqrt{1-\frac{\beta\xi}{1+\xi}} - (2-\beta)\right), \\ h_8(r;\theta), & \text{if } \rho_{4h} \le r < \rho_{5h} = \frac{\beta\theta(1-\beta)(1+\xi)}{2-\beta+2\xi(1-\beta)}, \\ 0, & \text{otherwise,} \end{cases}$$

where ρ_{4h} and ρ_{5h} solve $h_3(r;\theta) = h_8(r;\theta)$ and $h_8(r;\theta) = 0$, respectively. Region 2A can now be expressed as:

$$\left\{(r,e) | e \le h_9(r;\theta); \rho_{3h} \le r < \rho_{5h}; e, r \ge 0\right\}. \tag{RGN2Ah}$$

Region 2B:
The possible boundaries with Regions 1A, 1B and 2A have already been found as $e = h_4(r;\theta)$, $e = h_7(r;\theta)$, and $r = \rho_{3h}$, respectively. Of these, $e = h_4(r;\theta)$ and $r = \rho_{3h}$ are valid boundaries for Region 2B only if Case 1A and Case 2A occur for the given set of parameter values. Hence, we modify them in the following manner:

$$e = h_{10}(r;\theta) = \begin{cases} h_4(r;\theta), & \text{if } h_4(r;\theta) < h_2(r;\theta), \\ \infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad r = \rho_{6h} = \begin{cases} \rho_{3h}, & \text{if } \rho_{3h} < \rho_{5h}, \\ \infty, & \text{otherwise.} \end{cases}$$

We now find the possible boundary with Region 3A by comparing the profits:

$$e = h_{11}(r;\theta) = \frac{\beta\theta\left(1 - \frac{1}{\sqrt{1+\xi}}\right) - 2r}{2\beta\theta\xi(1-\beta)}.$$

The complete characterization of Region 2B is, therefore, given by:

$$\left\{(r,e) | e \le \min\{h_7(r;\theta), h_{10}(r;\theta), h_{11}(r;\theta)\}; r < \rho_{6h}; e, r \ge 0\right\}. \tag{RGN2Bh}$$

Region 3A:
The only remaining profit comparison is between Regions 3A and 3C. We do so now to obtain the boundary between them:

$$e = h_{12}(r;\theta) = \frac{\beta\theta - 2r}{2\beta\theta\xi(1-\beta)}.$$

It is now possible to express this region as:

$$\left\{(r,e) | \max\{h_2(r;\theta), h_9(r;\theta), h_{11}(r;\theta)\} \le e \le \min\{h_5(r;\theta), h_{12}(r;\theta)\}; e, r \ge 0\right\}. \tag{RGN3Ah}$$

Region 3B: Since quality θ, is exogenous in the short run, Region 3B (limit quality) cannot occur in the short-run equilibrium.

Region 3C:
When enforcement is very high, on either side, the threat of piracy disappears completely, and we enter this region of pure monopoly. Therefore, there is no upper boundary for this region. It only has a lower boundary, shared with Regions 1B and 3A; these boundaries have been found above as $e = \frac{(1+\xi)h_7(r;\theta)}{\xi}$ and

$e = h_{12}(r;\theta)$, respectively. Therefore, we can characterize this region as:

$$\left\{(r,e)|e > \min\left\{\tfrac{(1+\xi)h_7(r;\theta)}{\xi}, h_{12}(r;\theta)\right\}; e, r \geq 0\right\}. \tag{RGN3Ch}$$

Verifying the Validity Conditions in Table B1

We will first show that the condition $\frac{p-r}{(1-\beta)\theta} \leq 1$ is automatically satisfied in Cases 1A and 2A, which are characterized by (RGN1Ah) and (RGN2Ah), respectively. We prove this by contradiction. Let p^* be the equilibrium solution for a specific (r, e) point satisfying (RGN1Ah), but suppose that $\frac{p^*-r}{(1-\beta)\theta} > 1$. Then, from (5), we should have $\lambda = \frac{e\theta\xi}{\theta - \frac{r}{\beta}}$, the same as the expression for λ in Case 1B. This makes p^* a feasible solution in Case 1B. Furthermore, since $\left(1 - \frac{p^*-r}{(1-\beta)\theta}\right)$ is now negative, comparing the objective functions for Cases 1A and 1B in Table B1, we can immediately infer that the profit from Case 1B is higher. Since the profit from a feasible solution is higher, the optimal profit from Case 1B must also be higher, which is the desired contradiction. A similar argument comparing the profits from Cases 2A and 2B would show that $\frac{p^*-r}{(1-\beta)\theta} \leq 1$ is also satisfied when the equilibrium is obtained from Case 2A.

Let us now show that the condition $\frac{p-r}{(1-\beta)\theta} > 1$ is met in Cases 1B and 2B, which are characterized by (RGN1Bh) and (RGN2Bh), respectively. Let p^* now be the equilibrium solution for a specific (r, e) point satisfying (RGN1Bh), but suppose that $\frac{p^*-r}{(1-\beta)\theta} \leq 1$. Again, (5) tells us that we should have $\lambda = \frac{e\theta\xi(1-\beta)}{p^* - \frac{r}{\beta}}$, which is the same as the expression for λ in Case 1A, making p^* a feasible solution in Case 1A. Now, comparing the objective functions for Cases 1A and 1B, it becomes clear that the profit from Case 1A is higher, which, once again, leads to a contradiction. A similar argument applies to Case 2B as well.

Turning our attention to Case 3A, which must abide by $\frac{p-r}{(1-\beta)\theta} \leq 1$, it is easy to verify that this constraint is also automatically met, because, if not, the solution would become feasible in Case 3B, which is not possible in the short run.

We also observe that, if the equilibrium obtained from Case 1A leads to $\lambda = \frac{e\theta\xi(1-\beta)}{p^* - \frac{r}{\beta}} \leq \frac{\xi}{\xi+1}$, the solution would immediately become a feasible one in Case 2A, and, at the same time, the difference between the Case 1A and Case 2A profits, which is simply $p^*\left(\lambda - \frac{\xi}{\xi+1}\right)\left(\frac{p^*-r}{(1-\beta)\theta} - \frac{p^*}{\theta}\right)$, would be non-positive. Since a feasible solution in Case 2A is now at least as good, Case 2A must dominate Case 1A, again a contradiction to the claim that Case 1A dominates. A similar argument can be used to prove that Case 1B can dominate Case 2B only if $\lambda > \frac{\xi}{\xi+1}$. Clearly, the converse is true as well; unless $\lambda \leq \frac{\xi}{\xi+1}$, Cases 2A and 2B cannot dominate.

Moving on to the $\lambda < 1$ constraint in Case 1A, if this constraint is relaxed to $\lambda \leq 1$, Case 1A would subsume Case 3A, and the optimal solution obtained from either case would be exactly the same whenever $\lambda = 1$. In other words, the point at which the constraint $\lambda \leq 1$ becomes binding would be precisely the one where Case 3A takes over, implying that the constraint $\lambda < 1$ is algebraically equivalent to $e < h_2(r;\theta)$. A similar argument works for Case 1B as well, since violating this constraint would put the equilibrium in Case 3B, which is impossible.

Finally, simple algebra can show that the two constraints associated with Case 3C, namely, $\frac{e\theta\xi(1-\beta)}{p^* - \frac{r}{\beta}} > 1$ and $\frac{e\theta\xi}{\theta - \frac{r}{\beta}} > 1$, are equivalent to $e > h_{12}(r;\theta)$ and $e > \frac{(1+\xi)h_7(r;\theta)}{\xi}$, respectively. So, they are also satisfied when Case 3C dominates.

B.2.2 Characterization of Different Equilibrium Regions for $\beta \geq 1$

Once again, as before, all the regions can be characterized from profit comparisons. As shown later, the validity conditions in Table B1 are automatically satisfied within each region.

Profit Comparisons

Region 1A:
Equating the profits for Cases 1A and 1B, we get:

$$e = k_1(r;\theta) = \frac{r^2}{\theta\xi(\beta-1)(\beta\theta-r)}.$$

Case 1A dominates if $e < k_1(r;\theta)$, 1B dominates if $e > k_1(r;\theta)$, and both provide the same profit at $e = k_1(r;\theta)$. Now, equating the profits from 1A and 3A, we get:

$$e = k_2(r;\theta) = \frac{1}{\xi}\left(\frac{2\beta}{2\beta-1} - \frac{r}{(\beta-1)\theta}\right),$$

and 1A is valid only when $e < k_2(r;\theta)$. Next, we compare the profit in 1A with that from 2A and 2B to obtain:

$$\begin{aligned} e = k_3(r;\theta) &= \frac{\sqrt{\beta}\theta\xi(1-\beta) - r\left(\sqrt{\beta} - \sqrt{(1+\xi)(\beta-\xi(1-\beta))}\right)}{\theta\xi(1-\beta)\sqrt{(1+\xi)(\beta-\xi(1-\beta))}}, \text{ and} \\ e = k_4(r;\theta) &= \frac{\theta\sqrt{\beta\xi(\beta-1)} - r\sqrt{1+\xi}}{\theta\xi\sqrt{1+\xi}(\beta-1)}. \end{aligned}$$

1A dominates to the right of these boundaries, that is, when e is greater then both $k_3(r;\theta)$ and $k_4(r;\theta)$. Region 1A can now be fully characterized as follows:

$$\{(r,e) | \max\{k_3(r;\theta), k_4(r;\theta)\} < e \leq \min\{k_1(r;\theta), k_2(r;\theta)\}; e, r \geq 0\}. \tag{RGN1Ak}$$

As before, Region 1A can never have a valid solution when 3B or 3C does and, in obtaining the boundaries of Region 1A, its profit need not be compared with the profit from either 3B or 3C.

Region 1B:
The result of profit comparison with 1A has already been captured as $e = k_1(r;\theta)$. Next, we compare its profit with that of 3A to obtain the following boundary:

$$e = k_5(r;\theta) = \frac{\theta\sqrt{\beta\theta(24r(\beta-1) + \beta\theta(9-8\beta)) - 16r^2(\beta-1)} - (4r\theta(\beta-1)(4\beta-1) + \beta\theta^2(4\beta(3-2\beta)-3) - 8r^2(\beta-1))}{8\theta\xi(\beta-1)^2(\beta\theta-r)},$$

such that Region 1B occurs only above this boundary. Combining this with $e = k_1(r;\theta)$, we get:

$$e = k_6(r;\theta) = \begin{cases} k_1(r;\theta), & \text{if } 0 \leq r < \rho_{1k} = \frac{2(\beta-1)\beta\theta}{4\beta-3}, \\ k_5(r;\theta), & \text{otherwise,} \end{cases}$$

where ρ_{1k} is the solution of $k_1(r;\theta) = k_5(r;\theta) = k_2(r;\theta)$. Clearly, $e = k_6(r;\theta)$ provides a combined lower boundary for 1B. To find the only other possible lower boundary for this region, we now compare its profit with that of Case 2B to obtain:

$$e = k_7(r;\theta) = \frac{\beta\theta - r}{\beta\theta(1+\xi)}.$$

Finally, to find the upper boundary for Region 1B, we equate its profit with that from 3C to obtain:

$$e = \frac{\beta\theta - r}{\beta\theta\xi} = \frac{(1+\xi)k_7(r;\theta)}{\xi}.$$

Therefore, the full characterization of Region 1B is as follows:

$$\left\{(r,e)|\max\{k_6(r;\theta),k_7(r;\theta)\}\le e<\tfrac{(1+\xi)k_7(r;\theta)}{\xi};r\le\rho_{2k}=\tfrac{\beta\theta}{2};e,r\ge 0\right\}, \tag{RGN1Bk}$$

where ρ_{2k} is the solution of $k_5(r;\theta)=\frac{(1+\xi)k_7(r;\theta)}{\xi}$.

Region 2A:

Comparing the profits from Cases 2A and 2B, we get:

$$r=\rho_{3k}=\theta\left(\sqrt{\xi(\beta-1)(\beta+\xi(\beta-1))}-\xi(\beta-1)\right).$$

Region 2A occurs to the right of this boundary and 2B, to the left. The boundary with Region 1A is already derived as $e=k_3(r;\theta)$. We now find the boundary between 2A and 3A by comparing their profits:

$$e=k_8(r;\theta)=\frac{(2\beta-1)\theta-2r-\sqrt{\frac{\theta^2(\beta-1)(\beta+\xi(2\beta-1))-r^2-2r\theta\xi(\beta-1)}{(\beta-1)(1+\xi)(\beta+\xi(\beta-1))}}}{2\theta\xi(\beta-1)}.$$

Then, the combined upper boundary can be found as:

$$e=k_9(r;\theta)=\begin{cases}k_3(r;\theta), & \text{if } 0\le r<\rho_{4k}=(\beta-1)(1+\xi)\left(\frac{2\theta\sqrt{\beta\left(\beta-\frac{\xi}{1+\xi}\right)}}{2\beta-1}-\frac{\theta\xi}{1+\xi}\right),\\ k_8(r;\theta), & \text{if } \rho_{4k}\le r<\rho_{5k}=\frac{\theta(\beta-1)(2\beta(1+\xi)-\xi)}{2\beta(1+\xi)-1-2\xi},\\ 0, & \text{otherwise,}\end{cases}$$

where ρ_{4k} and ρ_{5k} solve $k_3(r;\theta)=k_8(r;\theta)$ and $k_8(r;\theta)=0$, respectively. Region 2A can now be expressed as:

$$\{(r,e)|e\le k_9(r;\theta);\rho_{3k}\le r<\rho_{5k};e,r\ge 0\}. \tag{RGN2Ak}$$

Region 2B:

The possible boundaries with Regions 1A, 1B, and 2A have already been found as $e=k_4(r;\theta)$, $e=k_7(r;\theta)$, and $r=\rho_{3k}$, respectively. Of these, $e=k_4(r;\theta)$ is a valid boundary for Region 2B only if Case 1A and Case 2A occur for the given set of parameter values. Hence, we modify them:

$$e=k_{10}(r;\theta)=\begin{cases}k_4(r;\theta), & \text{if } k_4(r;\theta)<k_2(r;\theta),\\ \infty, & \text{otherwise,}\end{cases}\quad \text{and}\quad r=\rho_{6k}=\begin{cases}\rho_{3k}, & \text{if } \rho_{3k}<\rho_{5k},\\ \infty, & \text{otherwise.}\end{cases}$$

We now find the possible boundary with Region 3A by comparing profits:

$$e=k_{11}(r;\theta)=\frac{\theta(2\beta-1)-2r-\frac{\theta}{\sqrt{1+\xi}}}{2\theta\xi(\beta-1)}.$$

The complete characterization of Region 2B is, therefore, given by:

$$\{(r,e)|e\le\min\{k_7(r;\theta),k_{10}(r;\theta),k_{11}(r;\theta)\};r<\rho_{6k};e,r\ge 0\}. \tag{RGN2Bk}$$

Region 3A:

The only remaining profit comparison is between Regions 3A and 3C. We do so now to obtain the boundary between them:

$$e = k_{12}(r;\theta) = \frac{\theta(2\beta-1)-2r}{2\theta\xi(\beta-1)}.$$

It is now possible to express this region as:

$$\{(r,e)|\max\{k_2(r;\theta),k_9(r;\theta),k_{11}(r;\theta)\} \le e \le \min\{k_5(r;\theta),k_{12}(r;\theta)\}; e,r \ge 0\}. \tag{RGN3Ak}$$

Region 3B: As before, Region 3B cannot occur in the short-run equilibrium.

Region 3C:
As before, there is no upper boundary for this region. It only has a lower boundary, shared with Regions 1B and 3A. Both these boundaries have been found above. Therefore, we can characterize this region as:

$$\left\{(r,e)|e > \min\left\{\tfrac{(1+\xi)k_7(r;\theta)}{\xi}, k_{12}(r;\theta)\right\}; e,r \ge 0\right\}. \tag{RGN3Ck}$$

Verifying the Validity Conditions in Table B1

The line of argument here is quite similar to the case of $\beta < 1$. We first show that the condition $p \le \frac{r}{\beta}$ is automatically satisfied in Cases 1A and 2A, which are as characterized by (RGN1Ak) and (RGN2Ak), respectively. We prove this by contradiction. Let p^* be the equilibrium solution for a specific (r,e) point satisfying (RGN1Ak), but suppose that $p^* > \frac{r}{\beta}$. Then, from (5), we must have $\lambda = \frac{e\theta\xi}{\theta-\frac{r}{\beta}}$, as in Case 1B. So, p^* is a feasible solution in Case 1B. Furthermore, since $\left(\frac{r-p^*}{(\beta-1)\theta} - \frac{p^*}{\theta}\right)$ is now negative, comparing the objective functions for Cases 1A and 1B in Table B1, we can immediately infer that the profit from Case 1B is higher, which is impossible. A similar argument with Cases 2A and 2B shows that $p \le \frac{r}{\beta}$ is also satisfied when the equilibrium is obtained from Case 2A.

Let us now show that the condition $p > \frac{r}{\beta}$ is met in Cases 1B and 2B, which are characterized by (RGN1Bk) and (RGN2Bk), respectively. Let p^* now be the equilibrium solution for a specific (r,e) point satisfying (RGN1Bk), but suppose that $p^* \le \frac{r}{\beta}$. Again, (5) tells us that we should have $\lambda = \frac{e\theta\xi}{\theta-\frac{r-p^*}{\beta-1}}$, which is the same as the expression for λ in Case 1A, making p^* a feasible solution in Case 1A. Now, comparing the objective functions for Cases 1A and 1B, it becomes clear that the profit from Case 1A is higher, which again leads to a contradiction. A similar argument applies to Case 2B, too.

We now turn our attention to Case 3A, where $p \le \frac{r}{\beta}$ is automatically met, because, if not, the solution would become feasible in Case 3B, which, as before, is not possible in the short run.

Furthermore, if the equilibrium obtained from Case 1A leads to $\lambda \le \frac{\xi}{\xi+1}$, the solution would become a feasible one in Case 2A, and the difference between the Case 1A and Case 2A profits, which is $p^*\left(\lambda - \frac{\xi}{\xi+1}\right)\left(1-\frac{r-p^*}{(\beta-1)\theta}\right)$, would be non-positive, as the existence of piracy in either case ensures that $\frac{r-p^*}{(\beta-1)\theta} < 1$. Since a feasible solution in Case 2A is now at least as good, Case 2A must dominate Case 1A, again a contradiction. A similar argument can be used to prove that Case 1B can dominate Case 2B only if $\lambda > \frac{\xi}{\xi+1}$. Clearly, the converse is true as well; unless $\lambda \le \frac{\xi}{\xi+1}$, Cases 2A and 2B cannot dominate.

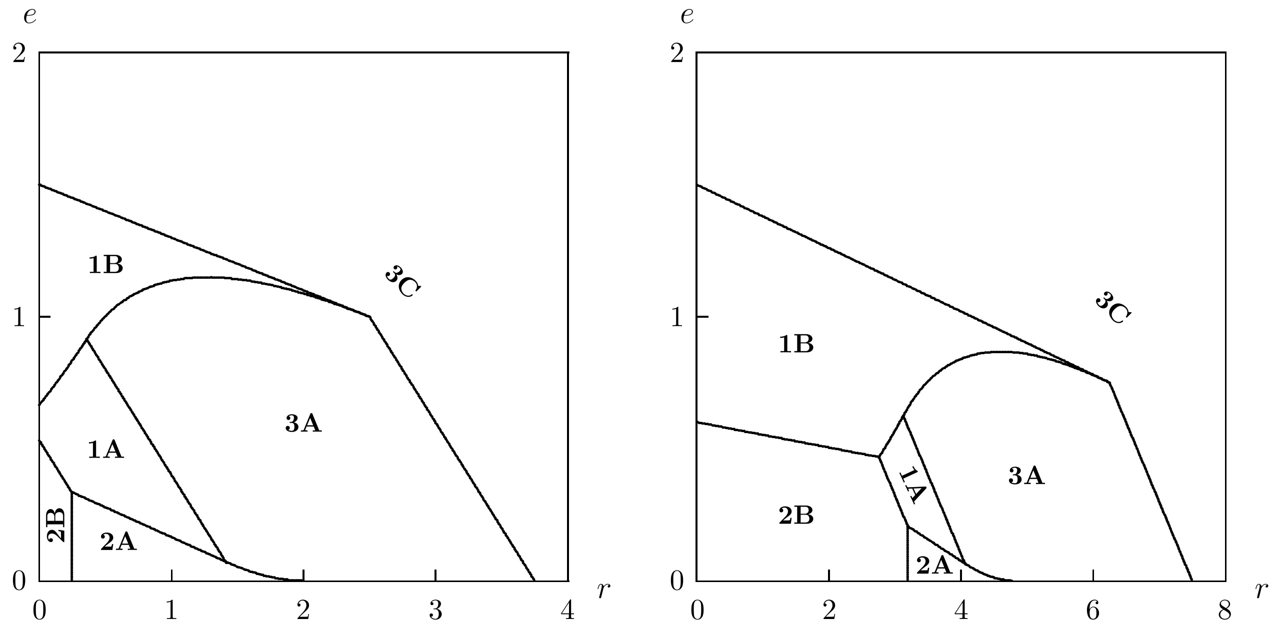
Next, we consider the constraint $\lambda < 1$ in Case 1A; if this constraint is relaxed to $\lambda \le 1$, Case 1A would subsume Case 3A, and the point at which the constraint $\lambda \le 1$ becomes binding would be precisely the one where Case 3A takes over, implying that the constraint $\lambda < 1$ is now algebraically equivalent to $e < k_2(r;\theta)$. A similar argument works for Case 1B as well, as violating this constraint would place the equilibrium in Case 3B, which is impossible.

Finally, simple algebra can show that the two constraints associated with Case 3C, $\frac{e\theta\xi}{\theta-\frac{r-p^*}{\beta-1}} > 1$ and $\frac{e\theta\xi}{\theta-\frac{r}{\beta}} > 1$, are equivalent to $e > k_{12}(r;\theta)$ and $e > \frac{(1+\xi)k_7(r;\theta)}{\xi}$, respectively. So, they are also satisfied when

Case 3C dominates.

B.2.3 Numerical Illustration

We illustrate these regions in Figure B2 for $\xi = \frac{2}{3}$ and $\theta = 10$, with two values of β. In this figure, Region i represents the part of the parameter space where Case i occurs in equilibrium. We conclude by noting that, even though the equilibrium occurs in exactly one of the six possible regions (see Figure B2), not all regions may exist always. Depending on the parameter values, certain regions may actually disappear, implying that the manufacturer would never choose the associated strategy (see Figure B1).



(a) $\beta = \frac{3}{4}$ (b) $\beta = \frac{5}{4}$

Figure B2: Relevant Partitions of the (r, e) Space in the Short Run; $\xi = \frac{2}{3}$, $\theta = 10$

B.3 Long-Run Equilibrium

We now consider the long-run equilibrium, which can be obtained by taking the short-run outcome and simply endogenizing θ. In other words, in each case, we maximize the total profit, $\pi(\theta) = p^*(\theta)q - \frac{c\theta^2}{2}$, to find the optimal quality from the first order condition for that case. As shown in Lemma 2, for $\beta \geq 1$, Cases 1A and 2A cannot occur in the long run.

Case 1A: As per Lemma 2, we only need to consider $\beta < 1$, in which case, the manufacturer's profit can be obtained from (2):

$$\pi = p^*(\theta)\left(\lambda\left(1 - \frac{p^*(\theta)}{\theta}\right) + (1-\lambda)\left(1 - \frac{p^*(\theta) - r}{(1-\beta)\theta}\right)\right) - \frac{c\theta^2}{2},$$

where $\lambda = \frac{e\theta\xi(1-\beta)}{p^*(\theta) - \frac{r}{\beta}}$ and $p^*(\theta) = \frac{r}{2} + \frac{\theta(1-\beta)(1+e\beta\xi)}{2}$. Substituting these, we get:

$$\pi = \frac{(r + \theta(1-\beta)(1+e\beta\xi))^2}{4\theta(1-\beta)} - \frac{c\theta^2}{2}$$

The first order condition can then be derived as:

$$\frac{\partial \pi}{\partial \theta} = \frac{(1-\beta)(1+e\beta\xi)^2}{4} - \frac{r^2}{4\theta^2(1-\beta)} - c\theta = 0. \tag{FOC1A}$$

We now show that (FOC1A) has a unique real positive root that maximizes the profit. First, we observe that (FOC1A) is essentially a cubic equation, and the signs of its coefficients indicate that the product of the three roots is negative but their sum is positive. This immediately implies that there are exactly one negative and at most two positive roots. Of these two positive roots, one is a minimum, and the other a maximum, of the underlying profit. Since $\frac{\partial \pi}{\partial \theta}$ approaches $-\infty$ as θ becomes large, the higher of the two positive roots must be the maximum.

Case 1B: In this case, the profit is: $\pi = p^*(\theta)\lambda\left(1 - \frac{p^*(\theta)}{\theta}\right) - \frac{c\theta^2}{2}$, where $p^*(\theta) = \frac{\theta}{2}$ and $\lambda = \frac{e\theta\xi}{\theta - \frac{r}{\beta}}$. Substituting these, we get :

$$\pi = \frac{e\beta\theta^2\xi}{4(\beta\theta - r)} - \frac{c\theta^2}{2}.$$

The first order condition can then be derived as:

$$\frac{\partial \pi}{\partial \theta} = \frac{e\beta\theta\xi(\beta\theta - 2r)}{4(\beta\theta - r)^2} - c\theta = 0. \tag{FOC1B}$$

Since $\theta \neq 0$, (FOC1B) reduces to a quadratic equation, and the sign of the coefficients indicate that both of its roots are positive. Again, $\frac{\partial \pi}{\partial \theta}$ approaches $-\infty$ as θ becomes very large, implying that the larger of the two roots, $\theta_{1\text{B}} = \frac{r}{\beta} + \frac{e\beta\xi + \sqrt{e\beta\xi(e\beta\xi - 16cr)}}{8c\beta}$, is the unique maximum.

Along similar lines, the first order condition in Case 2A (for $\beta < 1$) can be derived as:

$$\frac{(1-\beta)(1+\xi)}{4(1+\xi(1-\beta))} - \frac{r^2}{4\theta^2(1-\beta)(1+\xi)(1+\xi(1-\beta))} - c\theta = 0, \tag{FOC2A}$$

which has one negative root and at most two positive roots, and the maximum profit is achieved at the largest positive root, $\theta_{2\text{A}}$. In Case 2B, the profit, $\frac{\theta\xi}{4(1+\xi)} - \frac{c\theta^2}{2}$, is convex and is maximized at $\theta_{2\text{B}} = \frac{\xi}{4c(1+\xi)}$.

Further, we can show that, for Case 3A, the first order conditions for $\beta < 1$ and $\beta \geq 1$ are respectively:

$$e\xi(1-\beta)(1 - e\xi(1-\beta)) + \frac{r^2}{\beta^2\theta^2} - c\theta = 0, \text{ and} \tag{FOC3AH}$$

$$(\beta - 1)(1 - e\xi)(e\xi(\beta - 1) - \beta) + \frac{r^2}{\theta^2} - c\theta = 0. \tag{FOC3AK}$$

As the underlying profit is concave, the unique positive roots of these equations, which we name $\theta_{3\text{AH}}$ and $\theta_{3\text{AK}}$, provide the optimal quality levels. For Case 3B, the limit quality can be obtained by setting $\lambda = 1$ as indicated in Table B1. In Case 3C, the profit, $\frac{\theta}{4} - \frac{c\theta^2}{2}$, is convex and is maximized at $\theta_{3\text{C}} = \frac{1}{4c}$.

Taken together, we get the optimal quality as:

$$\theta^* = \begin{cases} \theta_{\text{1A}}, & \text{Case 1A } (\beta < 1), \\ \theta_{\text{1B}} = \frac{r}{\beta} + \frac{e\beta\xi + \sqrt{e\beta\xi(e\beta\xi - 16cr)}}{8c\beta} & \text{Case 1B}, \\ \theta_{\text{2A}}, & \text{Case 2A } (\beta < 1), \\ \theta_{\text{2B}} = \frac{\xi}{4c(1+\xi)}, & \text{Case 2B}, \\ \theta_{\text{3AH}}, & \text{Case 3A } (\beta < 1), \\ \theta_{\text{3AK}}, & \text{Case 3A } (\beta \geq 1), \\ \theta_{\text{3B}} = \frac{r}{\beta(1-e\xi)}, & \text{Case 3B}, \\ \theta_{\text{3C}} = \frac{1}{4c}, & \text{Case 3C}. \end{cases}$$

This result is illustrated in Figure B3, which shows the manufacturer's quality decisions in equilibrium over the entire (r, e) space for $\xi = \frac{2}{3}$ and $c = 0.01$.

From this optimal quality, we can also find the manufacturer's profit in each region. Let π_i be the profit in Region i. We can write the equilibrium profit as:

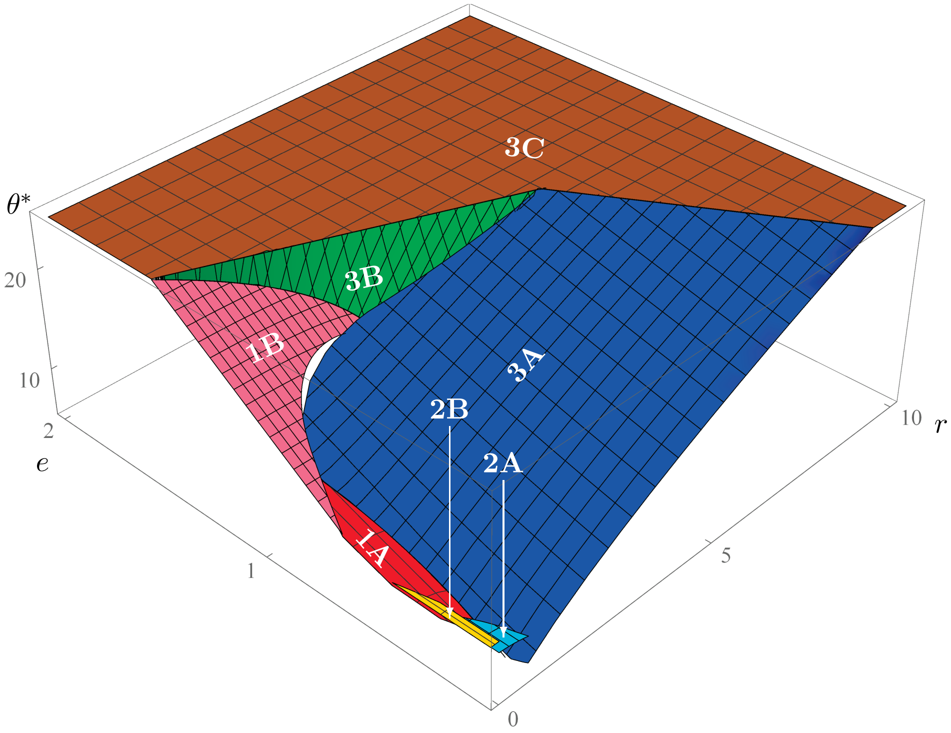
$$\pi^*(\theta^*) = \begin{cases} \pi_{\text{1A}} = \frac{(r+\theta_{\text{1A}}(1-\beta)(1+e\beta\xi))^2}{4\theta_{\text{1A}}(1-\beta)} - \frac{c\theta_{\text{1A}}^2}{2}, & \text{Case 1A } (\beta < 1), \\ \pi_{\text{1B}} = \frac{e\beta\theta_{\text{1B}}^2\xi}{4(\beta\theta_{\text{1B}} - r)} - \frac{c\theta_{\text{1B}}^2}{2}, & \text{Case 1B}, \\ \pi_{\text{2A}} = \frac{(r+\theta_{\text{2A}}(1-\beta)(1+\xi))^2}{4\theta_{\text{2A}}(1-\beta)(1+\xi)(1+\xi(1-\beta))} - \frac{c\theta_{\text{2A}}^2}{2}, & \text{Case 2A } (\beta < 1), \\ \pi_{\text{2B}} = \frac{\xi^2}{32c(1+\xi)^2}, & \text{Case 2B}, \\ \pi_{\text{3AH}} = \frac{(r+e\beta\theta_{\text{3AH}}\xi(1-\beta))(\beta\theta_{\text{3AH}}(1-e\xi(1-\beta))-r)}{\beta^2\theta_{\text{3AH}}} - \frac{c\theta_{\text{3AH}}^2}{2}, & \text{Case 3A } (\beta < 1), \\ \pi_{\text{3AK}} = \frac{(\beta\theta_{\text{3AK}} - r - e\theta\xi(\beta-1))(r-\theta_{\text{3AK}}(\beta-1)(1-e\xi))}{\theta_{\text{3AK}}} - \frac{c\theta_{\text{3AK}}^2}{2}, & \text{Case 3A } (\beta \geq 1), \\ \pi_{\text{3B}} = \frac{r(\beta(1-e\xi)-2cr)}{4\beta^2(1-e\xi)^2}, & \text{Case 3B}, \\ \pi_{\text{3C}} = \frac{1}{32c}, & \text{Case 3C}. \end{cases}$$

B.3.1 Characterization of Different Equilibrium Regions for $\beta < 1$

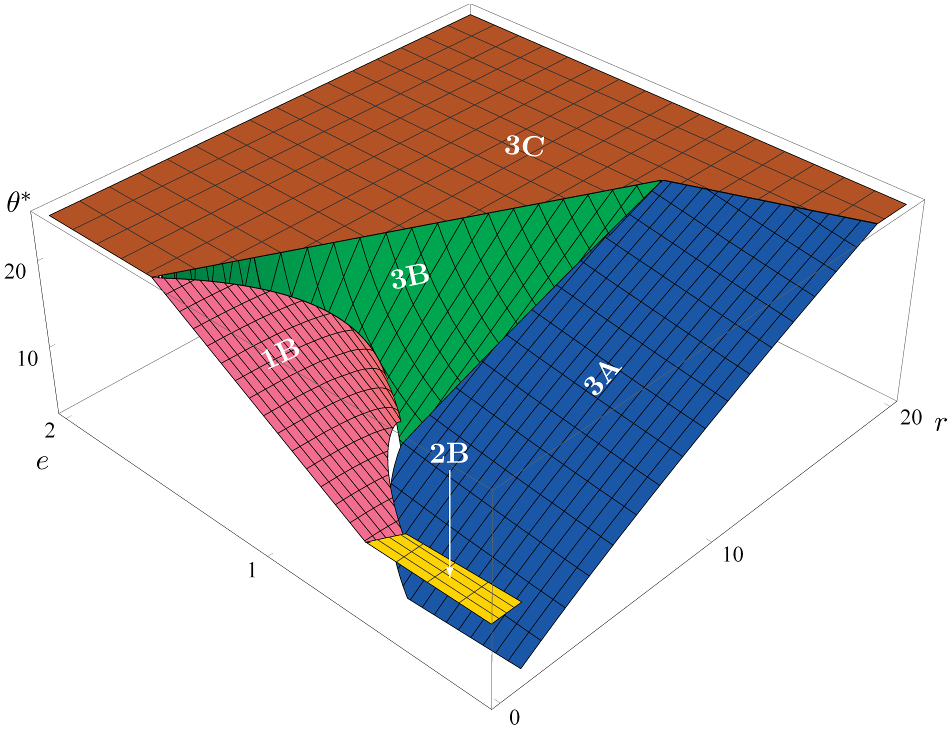
Finding the boundaries of different regions for the long-run equilibrium is now a conceptually straightforward extension of the short-run exercise. To characterize the boundaries, we again compare the profits in these regions, the solutions to which are denoted in Table B2.

Table B2: Boundaries Obtained from Profit Comparisons

Profit Comparison	**Boundary Solution**	**Profit Comparison**	**Boundary Solution**
$\pi_{\text{1A}} = \pi_{\text{1B}}$	$e = H_1(r)$	$\pi_{\text{2A}} = \pi_{\text{2B}}$	$r = \rho_{3H}$
$\pi_{\text{1A}} = \pi_{\text{2A}}$	$e = H_2(r)$	$\pi_{\text{2A}} = \pi_{\text{3AH}}$	$e = H_6(r)$
$\pi_{\text{1A}} = \pi_{\text{2B}}$	$e = H_3(r)$	$\pi_{\text{2B}} = \pi_{\text{3AH}}$	$r = \rho_{4H}(e)$
$\pi_{\text{1A}} = \pi_{\text{3AH}}$	$r = \rho_{1H}(e)$	$\pi_{\text{3AH}} = \pi_{\text{3B}}$	$e = H_7(r)$
$\pi_{\text{1B}} = \pi_{\text{2B}}$	$e = H_4(r)$	$\pi_{\text{3AH}} = \pi_{\text{3C}}$	$e = H_8(r)$
$\pi_{\text{1B}} = \pi_{\text{3AH}}$	$e = H_5(r)$	$\pi_{\text{3B}} = \pi_{\text{3C}}$	$e = H_9(r)$
$\pi_{\text{1B}} = \pi_{\text{3B}}$	$r = \rho_{2H}(e)$		



(a) $\beta = \frac{3}{4}$



(b) $\beta = \frac{5}{4}$

Figure B3: Equilibrium Quality as a Function of e and r; $\xi = \frac{2}{3}$, $c = 0.01$

Region 1A:
Region 1A can be characterized as follows:

$$\{(r,e)|\max\{H_2(r),H_3(r)\}<e\le H_1(r);r<\rho_{1H}(e);e,r\ge 0\}\,. \tag{RGN1AH}$$

Region 1B:
Define:

$$H_{10}(r)=\begin{cases}H_1(r), & \text{if } 0\le r<\rho_{5H},\\ H_5(r), & \text{otherwise,}\end{cases}$$

where ρ_{5H} is the solution of $H_1(r)=H_5(r)$. Clearly, $e=H_{10}(r)$ provides a combined lower boundary for 1B. The only other possible lower boundary for this region is $e=H_4(r)$. The upper/right boundary for Region 1B is obtained by comparing the profits from 1B and 3B. It turns out that $\pi_{1\mathrm{B}}=\pi_{3\mathrm{B}}$ has two roots, only one of which is valid depending on the parameter values:

$$r=\rho_{2H}(e)=\begin{cases}\frac{\beta(1-e\xi)(2e\xi-1)}{4ce\xi}, & \text{if } e\ge\frac{\beta}{24c},\\ \frac{e^2\beta\xi^2(1-e\xi)}{2c(2-e\xi)^2}, & \text{otherwise.}\end{cases}$$

Region 1B can now be fully characterized as follows:

$$\{(r,e)|e>\max\{H_4(r),H_{10}(r)\};r<\min\{\rho_{2H}(e),\rho_{6H}\};e,r\ge 0\}\,, \tag{RGN1BH}$$

where $r=\rho_{6H}=\frac{2\beta-1}{8c}$ solves $H_8(r)=H_9(r)$.

Region 2A:
The combined upper boundary for this region can be found as:

$$e=H_{11}(r)=\begin{cases}H_2(r), & \text{if } 0\le r<\rho_{7H},\\ H_6(r), & \text{if } \rho_{7H}\le r<\rho_{8H},\\ 0, & \text{otherwise,}\end{cases}$$

where ρ_{7H} and ρ_{8H} solve $H_2(r)=H_6(r)$ and $H_6(r)=0$, respectively. Region 2A can now be expressed as:

$$\{(r,e)|e\le H_{11}(r);\rho_{3H}\le r<\rho_{8H};e,r\ge 0\}\,. \tag{RGN2AH}$$

Region 2B:
The boundary between 2A and 2B is relevant only when Case 2A occurs, so we define:

$$\rho_{9H}=\begin{cases}\rho_{3H}, & \text{if } \rho_{3H}<\rho_{8H},\\ \infty, & \text{otherwise.}\end{cases}$$

The complete characterization of Region 2B is then given by:

$$\{(r,e)|e\le\min\{H_3(r),H_4(r)\};r<\min\{\rho_{4H}(e),\rho_{9H}\};e,r\ge 0\}\,. \tag{RGN2BH}$$

Region 3A:

The boundary with 3C is given by $e = H_8(r) = \frac{\beta - 8cr}{2\beta\xi(1-\beta)}$. It is now possible to express this region as:

$$\{(r,e)|H_{11}(r) \leq e \leq \min\{H_5(r), H_7(r), H_8(r)\}; r \geq \max\{\rho_{1H}(e), \rho_{4H}(e)\}; e, r \geq 0\}. \qquad \text{(RGN3AH)}$$

Region 3B:
Solving $\pi_{3\mathrm{B}} = \pi_{3\mathrm{C}}$, we get $e = H_9(r) = \frac{\beta - 4cr}{\beta\xi}$. Then, Region 3B can be characterized as:

$$\{(r,e)|H_7(r) \leq e \leq H_9(r); \rho_{2H}(e) \leq r < \rho_{6H}; e, r \geq 0\}. \qquad \text{(RGN3BH)}$$

Region 3C:
When enforcement is very high, on either side, the threat of piracy disappears completely, and we enter this region of pure monopoly. Therefore, there is no upper boundary for this region. It only has a lower boundary, shared with Regions 3A and 3B. Therefore, we can characterize this region as:

$$\{(r,e)|e > \min\{H_8(r), H_9(r)\}; e, r \geq 0\}. \qquad \text{(RGN3CH)}$$

Verifying the Validity Conditions in Table B1

Verifying these conditions is quite similar to the short run case. We start by showing that the condition $\frac{p-r}{(1-\beta)\theta} \leq 1$ is automatically satisfied in Cases 1A and 2A, which are characterized by (RGN1AH) and (RGN2AH), respectively. We prove this by contradiction. Let (p^*, θ^*) be the equilibrium solution for a specific (r, e) point satisfying (RGN1AH), but suppose that $\frac{p^*-r}{(1-\beta)\theta^*} > 1$. Then, from (5), we should have $\lambda = \frac{e\theta^*\xi}{\theta^* - \frac{r}{\beta}}$, the same as the one in Case 1B. This makes (p^*, θ^*) a feasible solution in Case 1B. Furthermore, since $\left(1 - \frac{p^*-r}{(1-\beta)\theta^*}\right)$ is now negative, comparing the objective functions for Cases 1A and 1B in Table B1, we can immediately infer that the profit from Case 1B is higher, which is a contradiction. A similar argument comparing the profits from Cases 2A and 2B would show that $\frac{p^*-r}{(1-\beta)\theta^*} \leq 1$ is also satisfied when the equilibrium is obtained from Case 2A.

Let us now show that the condition $\frac{p-r}{(1-\beta)\theta} > 1$ is met in Cases 1B and 2B, which are characterized by (RGN1BH) and (RGN2BH), respectively. Let (p^*, θ^*) now be the equilibrium solution for a specific (r, e) point satisfying (RGN1BH), but suppose that $\frac{p^*-r}{(1-\beta)\theta^*} \leq 1$. Again, (5) tells us that we should have $\lambda = \frac{e\theta^*\xi(1-\beta)}{p^* - \frac{r}{\beta}}$, which is the same as the one for λ in Case 1A, making (p^*, θ^*) a feasible solution in Case 1A. Now, comparing the objective functions for Cases 1A and 1B, it becomes clear that the profit from Case 1A is higher, which leads to a contradiction. A similar argument applies to Case 2B as well.

We also observe that, if the equilibrium obtained from Case 1A leads to $\lambda = \frac{e\theta^*\xi(1-\beta)}{p^* - \frac{r}{\beta}} \leq \frac{\xi}{\xi+1}$, the solution would immediately become a feasible one in Case 2A, and, at the same time, the difference between the Case 1A and Case 2A profits, which is simply $p^*\left(\lambda - \frac{\xi}{\xi+1}\right)\left(\frac{p^*-r}{(1-\beta)\theta^*} - \frac{p^*}{\theta^*}\right)$, would be non-positive. Since a feasible solution in Case 2A is now at least as good, Case 2A must dominate Case 1A, again a contradiction to the claim that Case 1A dominates. A similar argument can be used to prove that Case 1B can dominate Case 2B only if $\lambda > \frac{\xi}{\xi+1}$. Clearly, the converse is true as well; unless $\lambda \leq \frac{\xi}{\xi+1}$, Cases 2A and 2B cannot dominate.

Moving on to the $\lambda < 1$ constraint in Case 1A, if this constraint is relaxed to $\lambda \leq 1$, Case 1A would subsume Case 3A, in the sense that the optimal solution obtained from either case would be exactly the same whenever the first order conditions for 1A leads to $\lambda = 1$. In other words, the point at which the constraint $\lambda \leq 1$ becomes binding would be precisely the one where Case 3A takes over, implying that the constraint $\lambda < 1$ is algebraically equivalent to $r < \rho_{1H}(e)$. A similar argument works for Case 1B as well.

Between Cases 3A and 3B, the equilibrium must also satisfy: (i) $\frac{p-r}{(1-\beta)\theta} < 1$ in 3A, and (ii) $\frac{p-r}{(1-\beta)\theta} \geq 1$

in 3B. These must hold since violating one would simply move the equilibrium to the other region.

Finally, simple algebra can show that the two constraints associated with Case 3C, namely, $\frac{e\theta^*\xi(1-\beta)}{p^*-\frac{r}{\beta}} > 1$ and $\frac{e\theta^*\xi}{\theta^*-\frac{r}{\beta}} > 1$, are equivalent to $e > H_8(r)$ and $e > H_9(r)$, respectively.

B.3.2 Characterization of Different Equilibrium Regions for $\beta \geq 1$

When $\beta \geq 1$, Cases 1A and 2A cannot occur. We now compare the profits in the remaining five regions to characterize the boundaries, as shown in Table B3.

Table B3: Boundaries Obtained from Profit Comparisons

Profit Comparison	**Boundary Solution**	**Profit Comparison**	**Boundary Solution**
$\pi_{1\text{B}} = \pi_{2\text{B}}$	$e = K_1(r)$	$\pi_{3\text{AK}} = \pi_{3\text{B}}$	$e = K_3(r)$
$\pi_{1\text{B}} = \pi_{3\text{AK}}$	$e = K_2(r)$	$\pi_{3\text{AK}} = \pi_{3\text{C}}$	$e = K_4(r)$
$\pi_{1\text{B}} = \pi_{3\text{B}}$	$r = \rho_{1K}(e)$	$\pi_{3\text{B}} = \pi_{3\text{C}}$	$e = K_5(r)$
$\pi_{2\text{B}} = \pi_{3\text{AK}}$	$r = \rho_{2K}(e)$		

Region 1B:
As before, it turns out that $\pi_{1\text{B}} = \pi_{3\text{B}}$ has two roots, only one of which is valid:

$$r = \rho_{1K}(e) = \begin{cases} \frac{\beta(1-e\xi)(2e\xi-1)}{4ce\xi}, & \text{if } e \geq \frac{\beta}{24c}, \\ \frac{e^2\beta\xi^2(1-e\xi)}{2c(2-e\xi)^2}, & \text{otherwise.} \end{cases}$$

Region 1B can be characterized as follows:

$$\left\{(r,e)|e > \max\{K_1(r), K_2(r)\}; r < \rho_{1K}(e); e, r \geq 0\right\}. \tag{RGN1BK}$$

Region 2B:
The complete characterization of Region 2B is given by:

$$\left\{(r,e)|e \leq K_1(r); r < \rho_{2K}(e); e, r \geq 0\right\}. \tag{RGN2BK}$$

Region 3A:
The boundary with 3C is given by $e = K_4(r) = \frac{2\beta-1-8cr}{2\xi(\beta-1)}$. It is now possible to express this region as:

$$\left\{(r,e)|e \leq \min\{K_2(r), K_3(r), K_4(r)\}; r \geq \rho_{2K}(e); e, r \geq 0\right\}. \tag{RGN3AK}$$

Region 3B:
Solving $\pi_{3\text{B}} = \pi_{3\text{C}}$, we get $e = K_5(r) = \frac{\beta-4cr}{\beta\xi}$. Then, Region 3B can be characterized as:

$$\left\{(r,e)|K_3(r) \leq e \leq K_5(r); r \geq \rho_{1K}(e); e, r \geq 0\right\}. \tag{RGN3BK}$$

Region 3C:
When enforcement is very high, on either side, the threat of piracy disappears completely, and we enter this region of pure monopoly. Therefore, there is no upper boundary for this region. It only has a lower

boundary, shared with Regions 3A and 3B. Therefore, we can characterize this region as:

$$\{(r,e)|e > \min\{K_4(r), K_5(r)\}\,; e, r \geq 0\}\,. \tag{RGN3CK}$$

Verifying the Validity Conditions in Table B1

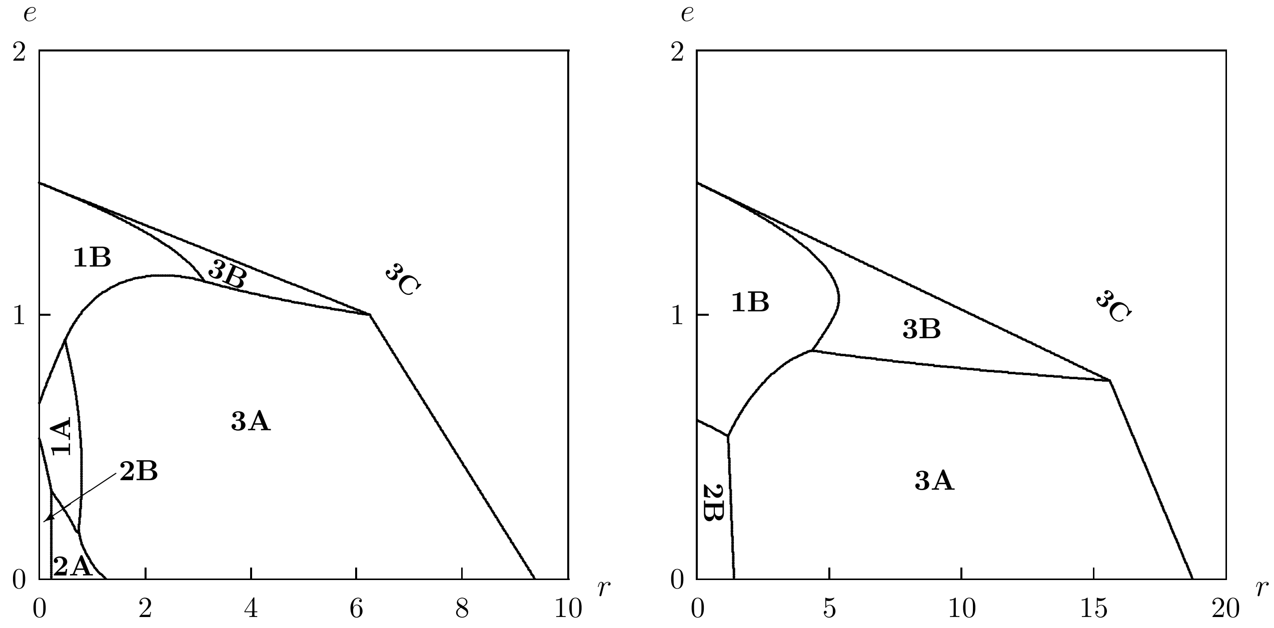
We first show that the condition $p > \frac{r}{\beta}$ is automatically satisfied in Case 1B and 2B, which are characterized by (RGN1BK) and (RGN2BK), respectively. We prove this by contradiction. Let (p^*, θ^*) now be the equilibrium solution for a specific (r, e) point satisfying (RGN1BK), but suppose that $p^* \leq \frac{r}{\beta}$. However, that would make (p^*, θ^*) a feasible solution in Case 1A, which is not possible in the long run. A similar argument applies to Case 2B as well.

Moving on to the $\lambda < 1$ constraint in Case 1B, we note that this constraint is subsumed by the constraint $r < \rho_{1K}(e)$. Also, between Cases 3A and 3B, the equilibrium must also satisfy: (i) $p \leq \frac{r}{\beta}$ in 3A, and (ii) $p > \frac{r}{\beta}$ in 3B. These must also hold since violating one would simply move the equilibrium to the other region.

Finally, simple algebra can show that the two constraints associated with Case 3C, namely, $\frac{e\theta^*\xi}{\theta^* - \frac{r-p^*}{\beta-1}} > 1$ and $\frac{e\theta^*\xi}{\theta^* - \frac{r}{\beta}} > 1$, are equivalent to $e > K_4(r)$ and $e > K_5(r)$, respectively.

B.3.3 Numerical Illustration

We illustrate these regions in Figure B4 for $\xi = \frac{2}{3}$ and $c = 0.01$. Once again, even though the equilibrium occurs in exactly one of the seven possible regions, not all regions may exist always.



(a) $\beta = \frac{3}{4}$ (b) $\beta = \frac{5}{4}$

Figure B4: Relevant Partitions of the (r, e) Space in the Long Run; $\xi = \frac{2}{3}$, $c = 0.01$

C Technical Results and Proofs

C.1 A Few Useful Observations

Before we provide all the proofs, we will state a few results that would be useful later for several of the proofs:

- In Case 1A, if $\beta < 1$, then $\frac{r}{\beta\theta} < \frac{1-\beta}{2-\beta}$. This is because in Case 1A, $e < h_2(r;\theta)$, which can be satisfied only if $h_2(r) > 0$ leading to this result. Of course, this also means that $\frac{r}{\beta\theta} < \frac{1}{2}$.
- In Case 1A, if $\beta \geq 1$, then $\frac{r}{\beta\theta} < \frac{2(\beta-1)}{2\beta-1}$. This is because in Case 1A, $e < k_2(r;\theta)$, which can be satisfied only if $k_2(r) > 0$ leading to this result.
- In Case 1B, if $\beta < 1$, then $\frac{r}{\beta\theta} < \frac{2\beta-1}{2\beta}$. This is because in Case 1B, $r < \rho_{2h} = \frac{\theta(2\beta-1)}{2}$. Since this condition can only be satisfied if $\rho_{2h} > 0$, Case 1B can occur only if $\beta > \frac{1}{2}$.
- In Case 1B, if $\beta \geq 1$, then $\frac{r}{\beta\theta} < \frac{1}{2}$. This is because in Case 1B, $r < \rho_{2k} = \frac{\beta\theta}{2}$.
- In both Cases 1A and 1B, irrespective of the value of β, $e\xi < 1 - \frac{r}{\beta\theta} < 1$. In Case 1A, when $\beta < 1$, $\lambda < 1$ and $\frac{p-r}{(1-\beta)\theta} < 1$ can be combined to obtain the result, and when $\beta \geq 1$, we can get the result from $\lambda < 1$ and $p < \frac{r}{\beta}$. Finally, in case 1B, the result follows directly from $\lambda < 1$.
- In Case 1A, for $\beta < 1$, $(1 - e\xi(1-\beta)) > (1 - 2e\xi(1-\beta)) > 0$. The first inequality is trivially true. To prove the second one, we note that, in this case, $e < h_2(r;\theta)$, which implies that $e\xi < \frac{1}{2-\beta} < \frac{1}{2(1-\beta)}$. The inequality follows.
- All the conditions above, although derived from the short-run equilibrium, must remain valid in the long run as well—a violation would simply mean that the price chosen cannot be the optimal one. Viewed differently, since all these conditions must be satisfied by $p^*(\theta)$ for any given θ, they should clearly be satisfied by $p^*(\theta^*)$.
- In Case 1B, irrespective of the value of β, $\frac{r}{\beta\theta^*} < \frac{1}{3}$. The first order condition with respect to θ leads to $c = \frac{e\beta\xi(\beta\theta^*-2r)}{4(\beta\theta^*-r)^2}$, the second derivative of the profit with respect to θ is simply $\left(-\frac{e\beta^2\theta\xi(\beta\theta-3r)}{4(\beta\theta-r)^3}\right)$. Since this second derivative is negative at an interior maximum, we must have $\beta\theta^* > 3r$.

C.2 Technical Lemmas

We only state the lemmas here; their proofs are available, along with all the other proofs, in Appendix B.3.

Lemma C1 *Let $X(r;\theta)$ be given by:*

$$\begin{aligned}X(r;\theta) = 8r^3 - 3r^2\beta\theta(1-2e\xi(1-\beta)) - 2r\theta^2\left(2-3\beta(1-\beta)+e\beta\xi(4-5\beta)(1-\beta) + 2e^2\beta^2\xi^2(1-\beta)^2\right)\\ + \beta\theta^3(1-2e\xi(1-\beta))(1-\beta)^2(1+e\beta\xi)^2.\end{aligned}$$

Then, the equation $X(r;\theta) = 0$ has two positive roots for $0 < \beta < 1$, the smaller of which is denoted by $r = \sigma_{1\mathrm{A}}(\theta)$. Furthermore, for $r \in \left[0, \frac{\beta\theta(1-\beta)}{2-\beta}\right]$, $X(r;\theta) > 0$ if and only if $r < \sigma_{1\mathrm{A}}(\theta)$.

Lemma C2 *Let $X_1(e;\theta)$ and $X_2(r;\theta)$ be given by:*

$$\begin{aligned}X_1(e;\theta) &= -6r^3 - r^2\theta(4-9\beta+2e\beta\xi(2-3\beta)) + 2r\theta^2(1-\beta)^2(1+e\beta\xi)^2\\ &\quad - \beta\theta^3(1-\beta)(1+e\beta\xi)\left(1-5\beta+7e\beta\xi(1-\beta)+2e^2\beta^2\xi^2(1-\beta)\right), \text{ and}\\ X_2(r;\theta) &= 2r^3\beta - 3r^2\theta(1-\beta)(1+e\beta\xi(1-\beta)) - 2r\theta^2(1-\beta)^2\left(1+\beta+e\beta^2\xi(1+e\xi)\right)\\ &\quad + \theta^3(1-\beta)^3(1+e\beta\xi)^2(1+e\beta\xi(1-\beta)).\end{aligned}$$

i) If $r \in \left[0, \frac{\beta\theta(1-\beta)}{2-\beta}\right]$ and $0 < \beta < 1$, the equation $X_1(e;\theta) = 0$ has three real roots, the largest of which is denoted by $e = \gamma_{1\mathrm{A}}(\theta)$. Then, $X_1(e;\theta) > 0$ at a positive e if and only if $e < \gamma_{1\mathrm{A}}(\theta)$.

ii) The equation $X_2(r;\theta) = 0$ has two positive roots, the smaller of which is denoted by $r = \tau_{1\mathrm{A}}(\theta)$. Then, for $r \in \left[0, \frac{\beta\theta(1-\beta)}{2-\beta}\right]$ and $0 < \beta < 1$, $X_2(r;\theta) > 0$ if and only if $r < \tau_{1\mathrm{A}}(\theta)$.

C.3 Proofs

Proof of Lemma 1

When $\beta = 1$, the pirated version has exactly the same quality as the legal version. Therefore, if $p \leq r$, no consumer will consider the pirated product. That is, however, not allowed in Cases 1A and 2A, where some level of piracy must exist, by definition. On the other hand, if $p > r$, only the ethical consumer segment will purchase the legal product, making the equilibrium outcome fall under either Case 1B or 2B. ■

Proof of Proposition 1

See Section A.2. ■

Proof of Lemma 2

For $\beta \geq 1$, in Case 1A, the manufacturer's profit is given by $\pi = ep\xi + \frac{p(r-p\beta)}{(\beta-1)\theta} - \frac{c\theta^2}{2}$; please see Section A.1. Clearly, this profit can be made arbitrarily large by reducing θ arbitrarily close to 0 (as $p < \frac{r}{\beta}$ in this case). Therefore, an interior solution for Case 1A cannot exist.

For Case 2A, the profit at $p^*(\theta)$ specified in (6) is:

$$\frac{(r + \theta(\beta-1))^2\xi}{4\theta(\beta-1)(2\beta-1)(1+\xi)} - \frac{c\theta^2}{2},$$

which can again be made arbitrarily large by choosing θ arbitrarily small, and an interior solution thus cannot exist. ■

Proof of Proposition 2

See Section A.3. ■

Proof of Theorem 1

In order to prove this, we will show that:

i) If $\beta < 1$, in Case 1A, social welfare is increasing in e if $r < \frac{e\beta^2\theta\xi}{2}$; it is decreasing otherwise. In this case, social welfare is increasing in r if $e < \frac{\beta\theta(1-\beta)-r(4-3\beta)}{2\beta\theta\xi(1-\beta)^2}$, and decreasing otherwise.

ii) If $\beta < 1$, in Case 1B, social welfare is increasing in e if $\beta < \frac{3}{4}$ or if $r > \frac{\theta\sqrt{\beta(4\beta-3)}}{2}$; it is decreasing if neither condition holds. Furthermore, social welfare is increasing in r if $r > \frac{\theta\left(2\beta-\sqrt{3\beta}\right)}{2}$ and $e > \frac{8r(\beta\theta-r)^2}{\beta\theta\xi(4r(2\beta\theta-r)+\beta\theta^2(3-4\beta))}$; otherwise, it is decreasing.

iii) If $\beta \geq 1$, over the entire primary piracy region (Cases 1A and 1B), social welfare is monotonically decreasing in both e and r.

We now make use of (11) to obtain the appropriate expression for social welfare in different cases.

i) $\beta < 1$, Case 1A: Differentiating short-run social welfare with respect to e, we get:

$$\frac{\partial(SW)}{\partial e} = \frac{\xi(1-\beta)\left(e\beta^2\theta\xi - 2r\right)}{4},$$

which would be positive if $r < \frac{e\beta^2\theta\xi}{2}$ and negative otherwise. Similarly, from (11), we can get:

$$\frac{\partial(SW)}{\partial r} = \frac{1}{4}\left(1 - 2e\xi(1-\beta) - \frac{r(4-3\beta)}{\beta\theta(1-\beta)}\right),$$

which would be positive if $e < \frac{\beta\theta(1-\beta)-r(4-3\beta)}{2\beta\theta\xi(1-\beta)^2}$ and negative otherwise.

ii) $\beta < 1$, Case 1B: Again, differentiating the appropriate expression for social welfare with respect to e, we get:

$$\frac{\partial(SW)}{\partial e} = \frac{\xi\left(4r^2 + \beta\theta^2(3-4\beta)\right)}{8(\beta\theta - r)},$$

which is positive if either $\beta \leq \frac{3}{4}$ or $r > \frac{\theta}{2}\sqrt{\beta(4\beta-3)}$, and negative when neither condition holds. Differentiating (11) with respect to r yields:

$$\frac{\partial(SW)}{\partial r} = -\frac{r}{\beta\theta} - \frac{e\xi}{2} + \frac{3e\beta\theta^2\xi}{8(\beta\theta - r)^2},$$

which can be shown to be positive if and only if $r > \frac{\theta\left(2\beta - \sqrt{3\beta}\right)}{2}$ and $e > \frac{8r(\beta\theta-r)^2}{\beta\theta\xi(4r(2\beta\theta-r)+\beta\theta^2(3-4\beta))}$.

iii) $\beta \geq 1$, Case 1A: In this case:

$$\frac{\partial(SW)}{\partial e} = \frac{\theta\xi(\beta-1)\left(\frac{e\xi}{2} - \frac{\beta\theta+r}{\theta}\right)}{2\beta},$$

which is always negative since $\frac{e\xi}{2} < \frac{1}{2} < 1 < \beta < \frac{\beta\theta+r}{\theta}$. Also, we find:

$$\frac{\partial(SW)}{\partial r} = -\frac{2e\xi(\beta-1)^2 + \frac{r(4\beta-3)}{\theta}}{4\beta(\beta-1)},$$

which is clearly negative.

$\beta \geq 1$, Case 1B: Here, we have:

$$\frac{\partial(SW)}{\partial e} = \frac{\xi\left(4r^2 - \beta\theta^2(4\beta-3)\right)}{8(\beta\theta - r)}.$$

Now, since $r < \frac{\beta\theta}{2}$, we find that $4r^2 - \beta\theta^2(4\beta-3) < (\beta\theta)^2 - \beta\theta^2(4\beta-3) = -3\beta\theta^2(\beta-1) < 0$, implying that the numerator is negative. Since the denominator is clearly positive, the partial derivative must be negative overall. Finally, from (11), we get:

$$\frac{\partial(SW)}{\partial r} = -\frac{r}{\beta\theta} - \frac{e\xi}{2} + \frac{3e\beta\theta^2\xi}{8(\beta\theta - r)^2}.$$

Clearly, $\frac{\partial(SW)}{\partial r}$ is a linear function of e and is maximized at an extreme value of e, that is, either at $e = 0$ or at $e = \frac{\beta\theta - r}{\beta\theta\xi}$, the second extreme value being derived from the constraint that $\lambda < 1$ in this case. Now, since $\left.\frac{\partial(SW)}{\partial r}\right|_{e=0} = -\frac{r}{\beta\theta} < 0$, we only need to consider the other extreme value. We find that:

$$\left.\frac{\partial(SW)}{\partial r}\right|_{e\,=\,\frac{\beta\theta-r}{\beta\theta\xi}} = \frac{4r^2-\beta\theta^2(4\beta-3)}{8\beta\theta(\beta\theta-r)},$$

the denominator of which is positive, and the numerator, as shown above, is negative, making the right hand side negative overall.

Since the maximum of $\frac{\partial(SW)}{\partial r}$, taken over the valid parameter range, is negative, $\frac{\partial(SW)}{\partial r}$ must also be negative in this entire range. ■

Proof of Proposition 3

i) We start by showing that, in the primary piracy region, the piracy rate, as given in (8), is monotonically decreasing in both e and r.

$\underline{\beta < 1, \text{ Case 1A:}}$ From (8), the piracy rate in this case is given by:

$$\mu = \frac{r(2-\beta)-\beta\theta(1-\beta)(1-e\xi(2-\beta))}{2(1-\beta)(r-\beta\theta(1-e\xi(1-\beta)))}.$$

Taking partial derivative with respect to e and r, we get:

$$\frac{\partial\mu}{\partial e} = -\frac{\beta^2\theta^2\xi}{2(r-\beta\theta(1-e\xi(1-\beta)))^2} < 0, \quad \text{and} \quad \frac{\partial\mu}{\partial r} = -\frac{\beta\theta}{2(1-\beta)(r-\beta\theta(1-e\xi(1-\beta)))^2} < 0.$$

$\underline{\beta \geq 1, \text{ Case 1A:}}$ From (8):

$$\mu = \frac{r(2\beta-1)-\theta(\beta-1)(e\xi+2\beta(1-e\xi))}{(\beta-1)(r+\theta(e\xi(\beta-1)-2\beta))}.$$

Again, taking partial derivative with respect to e and r, we obtain:

$$\frac{\partial\mu}{\partial e} = -\frac{2\beta^2\theta^2\xi}{(r-2\beta\theta+e\theta\xi(\beta-1))^2} < 0, \quad \text{and} \quad \frac{\partial\mu}{\partial r} = -\frac{2\beta^2\theta}{(\beta-1)(r-2\beta\theta+e\theta\xi(\beta-1))^2} < 0.$$

$\underline{\text{Case 1B:}}$ From (8), the piracy rate in this case is given by:

$$\mu = \frac{2(\beta\theta-r)(r-\beta\theta(1-e\xi))}{2r\beta\theta(2-e\xi)-\beta^2\theta^2(2-e\xi)-2r^2}.$$

Taking partial derivative with respect to e and r, we get:

$$\frac{\partial\mu}{\partial e} = -\frac{2\beta^2\theta^2\xi(\beta\theta-r)^2}{\left(2(\beta\theta-r)^2-e\beta\theta\xi(\beta\theta-2r)\right)^2} < 0, \quad \text{and} \quad \frac{\partial\mu}{\partial r} = -\frac{2e\beta^2\theta^2\xi(\beta\theta(2-e\xi)-2r)}{\left(2(\beta\theta-r)^2-e\beta\theta\xi(\beta\theta-2r)\right)^2}.$$

Now, to show that $\frac{\partial\mu}{\partial r}$ is also negative, we need to prove that $\beta\theta(2-e\xi) > 2r$, which follows directly by multiplying the last inequality below by $\beta\theta > 0$:

$$e\xi < 1-\frac{r}{\beta\theta} \quad \Leftrightarrow \quad 2-e\xi > 2-\left(1-\frac{r}{\beta\theta}\right) = 1+\frac{r}{\beta\theta} > \frac{2r}{\beta\theta}.$$

ii) We now show that the manufacturer's profit, as given in (9), is monotonically increasing in both e and r.

<u>$\beta < 1$, Case 1A:</u> In this case, $\pi^* = \frac{(r+\theta(1-\beta)(1+e\beta\xi))^2}{4\theta(1-\beta)} - \frac{c\theta^2}{2}$. We simply differentiate π^* with respect to e and r to get:

$$\frac{\partial \pi^*}{\partial e} = \frac{\beta\xi(r+\theta(1-\beta)(1+e\beta\xi))}{2} > 0 \quad \text{and} \quad \frac{\partial \pi^*}{\partial r} = \frac{1+e\beta\xi}{2} + \frac{r}{2\theta(1-\beta)} > 0.$$

<u>$\beta \geq 1$, Case 1A:</u> Now, $\pi^* = \frac{(r+e\theta\xi(\beta-1))^2}{4\beta\theta(\beta-1)} - \frac{c\theta^2}{2}$. Differentiate π^* with respect to e and r to get:

$$\frac{\partial \pi^*}{\partial e} = \frac{\xi(r+e\theta\xi(\beta-1))}{2\beta} > 0 \quad \text{and} \quad \frac{\partial \pi^*}{\partial r} = \frac{e\xi}{2\beta} + \frac{r}{2\beta\theta(\beta-1)} > 0.$$

<u>Case 1B:</u> $\pi^* = \frac{e\xi\beta\theta^2}{4(\beta\theta-r)} - \frac{c\theta^2}{2}$. As before, we differentiate π^* with respect to e and r:

$$\frac{\partial \pi^*}{\partial e} = \frac{\beta\theta^2\xi}{4(\beta\theta-r)} > 0 \quad \text{and} \quad \frac{\partial \pi^*}{\partial r} = \frac{e\beta\theta^2\xi}{4(\beta\theta-r)^2} > 0.$$

iii) We now show that consumer surplus, as in (10), is monotonically decreasing in both e and r.

<u>$\beta < 1$, Case 1A:</u> In this case:

$$\frac{\partial (CS)}{\partial e} = -\frac{\xi(1-\beta)\left(2(\beta\theta-r)+e\beta^2\theta\xi\right)}{4} < 0 \quad \text{and} \quad \frac{\partial (CS)}{\partial r} = \frac{r(4-3\beta)}{4\beta\theta(1-\beta)} + \frac{e\xi(1-\beta)}{2} - \frac{3}{4}.$$

The first inequality follows from $\frac{r}{\beta\theta} < 1$. To show that $\frac{\partial (CS)}{\partial r}$ is negative, we note that it is a linear increasing function of e and r, and its maximum is obtained at the extreme point $(\bar{r}, \bar{e})$, where $\frac{\bar{r}}{\beta\theta} = \frac{1-\beta}{2-\beta}$ and $\bar{e} = h_2(r)$. However, even this maximum value is negative because:

$$\left.\frac{\partial (CS)}{\partial r}\right|_{\substack{e=\bar{e}\\ r=\bar{r}}} = -\frac{1}{2(2-\beta)} < 0.$$

Therefore, $\frac{\partial (CS)}{\partial r}$ must be negative over the entire range.

<u>$\beta \geq 1$, Case 1A:</u> Here:

$$\frac{\partial (CS)}{\partial e} = -\frac{\xi(\beta-1)(2(\beta\theta-r)+e\theta\xi)}{4\beta} < 0 \quad \text{and} \quad \frac{\partial (CS)}{\partial r} = -1 + \frac{e\xi(\beta-1)}{2\beta} + \frac{r(4\beta-3)}{4\beta\theta(\beta-1)}.$$

The first one is clearly negative as $\frac{r}{\beta\theta} < 1$. To prove that the second one is also negative, we note that it is a linearly increasing function of e and r, and, once again, its maximum is obtained at the extreme point $(\bar{r}, \bar{e})$, where $\frac{\bar{r}}{\beta\theta} = \frac{2(\beta-1)}{2\beta-1}$ and $\bar{e} = k_2(r)$. It turns out that this maximum value is negative because:

$$\left.\frac{\partial (CS)}{\partial r}\right|_{\substack{e=\bar{e}\\ r=\bar{r}}} = -\frac{1}{2(2\beta-1)} < 0.$$

Therefore, $\frac{\partial (CS)}{\partial r}$ must also be negative over the entire range.

<u>Case 1B:</u> In this case:

$$\frac{\partial (CS)}{\partial e} = \frac{\xi\left(8r\beta\theta+\beta\theta^2(1-4\beta)-4r^2\right)}{8(\beta\theta-r)} \quad \text{and} \quad \frac{\partial (CS)}{\partial r} = -1 + \frac{r}{\beta\theta} + e\xi\left(\frac{1}{2} + \frac{\beta\theta^2}{8(\beta\theta-r)^2}\right).$$

To show that the first one is negative, we must prove that $X(r) = 8r\beta\theta + \beta\theta^2(1-4\beta) - 4r^2 < 0$. Now $X(r)$ is clearly a concave function of r and its maximum, $r = \beta\theta$, is not a valid interior solution, as

$r < \beta\theta$. Therefore, the maximum of $X(r)$ occurs at the upper bound of r. First, for $\theta_{1\mathrm{B}}$ to be a valid solution, r must always satisfy $\frac{r}{\beta\theta} < \frac{1}{3}$. In addition, if $\beta < 1$, r must also satisfy $\frac{r}{\beta\theta} < \frac{2\beta-1}{2\beta}$. We consider two cases. When $\beta < 1$, we set $\bar{r} = \frac{\beta\theta(2\beta-1)}{2\beta}$ and find that $X(\bar{r}) < -(1-\beta)\theta^2 < 0$. On the other hand, when $\beta \geq 1$, we set $\bar{r} = \frac{\beta\theta}{3}$ and find that $X(\bar{r}) < -\frac{\beta\theta^2(16\beta-9)}{9} < 0$.

To show that the second expression is also negative, we note that it is a linearly increasing function of both r and e, and must have its maximum at the largest possible values of r and e. Now, since $e\xi < 1 - \frac{r}{\beta\theta}$, the upper bound for e is simply $\bar{e} < \frac{\beta\theta - r}{\beta\theta\xi}$. Once again, we consider two cases. When $\beta \leq \frac{3}{4}$, we set $\bar{r} = \frac{\beta\theta(2\beta-1)}{2\beta}$ and get:

$$\left.\frac{\partial(CS)}{\partial r}\right|_{\substack{e=\bar{e}\\ r=\bar{r}}} = -\frac{1-\beta}{4\beta} < 0.$$

Now, when $\beta > \frac{3}{4}$, we set $\bar{r} = \frac{\beta\theta}{3}$ and get:

$$\left.\frac{\partial(CS)}{\partial r}\right|_{\substack{e=\bar{e}\\ r=\bar{r}}} = \frac{9-16\beta}{48\beta} < 0.$$

iv) We now show that the legal social welfare, SW_{L}, is monotonically increasing in both e and r.

<u>$\beta < 1$, Case 1A:</u> By differentiating the legal social welfare in this case, it can be easily shown that:

$$\frac{\partial(SW_{\mathrm{L}})}{\partial e} = \frac{\beta\xi(r + \theta(1-\beta+e\beta\xi(3-2\beta)))}{4} > 0 \quad \text{and} \quad \frac{\partial(SW_{\mathrm{L}})}{\partial r} = \frac{e\beta\xi}{4} + \frac{\theta(1-\beta)-r}{4\theta(1-\beta)^2}.$$

The second derivative would also be positive if $(1-\beta)\theta - r > 0$, which is indeed true because:

$$\frac{r}{\beta\theta} < \frac{1-\beta}{2-\beta} \Rightarrow (1-\beta)\theta - r > \frac{r(2-\beta)}{\beta} - r = \frac{2r(1-\beta)}{\beta} > 0.$$

<u>$\beta \geq 1$, Case 1A:</u> Here:

$$\frac{\partial(SW_{\mathrm{L}})}{\partial e} = \frac{\xi\left(r + \beta\theta(2-e\xi)\right)}{4\beta} \quad \text{and} \quad \frac{\partial(SW_{\mathrm{L}})}{\partial r} = \frac{1}{8}\left(\frac{2e\xi}{\beta} + \frac{2r(3\beta-2)}{\beta\theta(\beta-1)^2}\right).$$

Since $e\xi < 1$ and $\beta > 1$, the above derivatives are both positive.

<u>Case 1B:</u> In this case, we have:

$$\frac{\partial(SW_{\mathrm{L}})}{\partial e} = \frac{3\beta\theta^2\xi}{8(\beta\theta - r)} > 0 \quad \text{and} \quad \frac{\partial(SW_{\mathrm{L}})}{\partial r} = \frac{3e\beta\theta^2\xi}{8(\beta\theta-r)^2} > 0,$$

which completes the proof. ■

Proof of Theorem 2

<u>Case 1A:</u> From (FOC1A), we get:

$$\pi' = \frac{(1-\beta)(1+e\beta\xi)^2}{4} - \frac{r^2}{4\theta^2(1-\beta)} - c\theta.$$

Clearly,

$$\frac{\partial\pi'}{\partial e} = \frac{\beta\xi(1-\beta)(1+e\beta\xi)}{2} > 0, \quad \text{and} \quad \frac{\partial\pi'}{\partial r} = -\frac{r}{2(1-\beta)\theta^2} < 0.$$

Finally, the second order condition must be satisfied in optimality, that is, $\left.\frac{\partial \pi'}{\partial \theta}\right|_{\theta=\theta^*} < 0$. It now follows from the Implicit Function Theorem that:

$$\frac{d\theta^*}{de} = -\frac{\left.\frac{\partial \pi'}{\partial e}\right|_{\theta=\theta^*}}{\left.\frac{\partial \pi'}{\partial \theta}\right|_{\theta=\theta^*}} > 0 \quad \text{and} \quad \frac{d\theta^*}{dr} = -\frac{\left.\frac{\partial \pi'}{\partial r}\right|_{\theta=\theta^*}}{\left.\frac{\partial \pi'}{\partial \theta}\right|_{\theta=\theta^*}} < 0.$$

<u>Case 1B:</u> In this case, (FOC1B) gives:

$$\pi' = \frac{e\beta\theta\xi(\beta\theta - 2r)}{4(\beta\theta - r)^2} - c\theta,$$

differentiating which, we get:

$$\frac{\partial \pi'}{\partial e} = \frac{\beta\theta\xi(\beta\theta - 2r)}{4(\beta\theta - r)^2} > 0 \quad \text{and} \quad \frac{\partial \pi'}{\partial r} = -\frac{er\beta\theta\xi}{2(\beta\theta - r)^3} < 0.$$

Once again, the second order condition must be satisfied in optimality, implying $\left.\frac{\partial \pi'}{\partial \theta}\right|_{\theta=\theta^*} < 0$. The rest follows from the Implicit Function Theorem, in a manner similar to the proof for 1A above. ■

Proof of Theorem 3

To prove this theorem, we will show that:

i) If $\beta < 1$, social welfare is increasing in e over the entire primary piracy region. However, social welfare is not monotonic in r and is increasing either only at a low r or for a low β and a high e. In particular, in Case 1A, social welfare is increasing in r if and only if $r < \sigma_{1\text{A}}(\theta^*)$, where $\sigma_{1\text{A}}(\cdot)$ is as defined in Lemma C1. Similarly, in Case 1B, social welfare is increasing in r if $\beta < \frac{3}{4}$ and $e > \frac{8r(2\beta\theta^*-r)(\beta\theta^*-r)^3}{\beta^2\theta^{*2}\xi(20r^3+4r^2\theta^*-44r^2\beta\theta^*-11r\beta\theta^{*2}+28r\beta^2\theta^{*2}+3\beta^2\theta^{*3}-4\beta^3\theta^{*3})}$; otherwise, it is decreasing.

ii) If $\beta \geq 1$, social welfare is increasing in e except when e is large. Specifically, it is increasing if $e < \frac{4(\beta\theta^*-r)^2(\beta\theta^*-2r)\left(\beta^2\theta^{*2}+r^2\right)}{\beta^2\theta^{*2}\xi(4r^3+4r^2\theta^*(4\beta-1)-r\beta\theta^{*2}(28\beta-13)+4\beta^2\theta^{*3}(2\beta-1))}$; it is decreasing otherwise. However, social welfare is decreasing in r over the entire primary piracy region.

We now proceed to prove each case separately.

i) We start with $\beta < 1$. When $\beta < 1$, Cases 1A and 1B can both occur, and we need to consider them separately.

<u>$\beta < 1$, Case 1A:</u> We first consider the long-run impact of e. Using the chain rule, we get:

$$\left.\frac{d(SW)}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial(SW)}{\partial e}\right|_{\theta=\theta^*} + \left.\frac{\partial(SW)}{\partial \theta}\right|_{\theta=\theta^*} \frac{d\theta^*}{de}.$$

The expression for $\frac{\partial(SW)}{\partial e}$ is already available from the proof of Theorem 1. Also, we use the Implicit Function Theorem and the intermediate steps from the proof of Theorem 2 to find:

$$\frac{d\theta^*}{de} = -\frac{\left.\frac{\partial \pi'}{\partial e}\right|_{\theta=\theta^*}}{\left.\frac{\partial \pi'}{\partial \theta}\right|_{\theta=\theta^*}} = \frac{2\beta\theta^{*3}\xi(1-\beta)^2(1+e\beta\xi)}{(1-\beta)^2\theta^{*2}(1+e\beta\xi)^2 - 3r^2}.$$

Finally, we can also find that:

$$\frac{\partial(SW)}{\partial\theta} = \frac{r^2(4-\beta)}{8\beta\theta^2(1-\beta)} + \frac{1+3\beta}{8} - \frac{e\beta\xi(1-\beta)}{2} - \frac{e^2\beta^2\xi^2(1-\beta)}{8}.$$

Combining all of these, we get:

$$\left.\frac{d(SW)}{de}\right|_{\theta=\theta^*} = \frac{\xi(1-\beta)X}{4Y}, \text{ where}$$

$$X = 6r^3 + 4r^2\beta\theta^*\left(\frac{4-\beta}{4\beta}+e\xi(1-\beta)\right) - 2r{\theta^*}^2(1-\beta)^2(1+e\beta\xi)^2 + 3\beta^2{\theta^*}^3(1-\beta)(1+e\beta\xi)\left(\frac{1+3\beta}{3\beta}-e\xi(1-\beta)\right),$$

and $Y = {\theta^*}^2(1-\beta)^2(1+e\beta\xi)^2 - 3r^2$.

Now, we show that $Y > 0$. First, (FOC1A) implies that $Y = 4c(1-\beta){\theta^*}^3 - 2r^2$. Then, from the second order condition, we find that $\frac{1}{4}\left(\frac{2r^2}{(1-\beta){\theta^*}^3} - 4c\right) < 0$ must hold, immediately implying that $Y > 0$. Therefore, to complete the proof, we only need to show that X is positive. We do so now.

Since $r < \frac{\beta\theta^*}{2}$, we can write:

$$\begin{aligned} X &> 6r^3 + 4r^2\beta\theta^*\left(\frac{4-\beta}{4\beta}+e\xi(1-\beta)\right) - \beta{\theta^*}^3(1-\beta)^2(1+e\beta\xi)^2 + 3\beta^2{\theta^*}^3(1-\beta)(1+e\beta\xi)\left(\frac{1+3\beta}{3\beta}-e\xi(1-\beta)\right) \\ &= 6r^3 + 4r^2\beta\theta^*\left(\frac{4-\beta}{4\beta}+e\xi(1-\beta)\right) + 4\beta^2{\theta^*}^3(1-\beta)(1+e\beta\xi)(1-e\xi(1-\beta)) > 0, \end{aligned}$$

because $(1-e\xi(1-\beta)) > 0$.

We now consider the long-run impact of r. As before, we employ the chain rule:

$$\left.\frac{d(SW)}{dr}\right|_{\theta=\theta^*} = \left.\frac{\partial(SW)}{\partial r}\right|_{\theta=\theta^*} + \left.\frac{\partial(SW)}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{dr}.$$

From the proof of Theorem 1, we can find the expression for $\frac{\partial(SW)}{\partial r}$. Also, the Implicit Function Theorem and the intermediate steps in the proof of Theorem 2 give us:

$$\frac{d\theta^*}{dr} = -\frac{2r\theta^*}{Y}, \text{ where, as before, } Y = {\theta^*}^2(1-\beta)^2(1+e\beta\xi)^2 - 3r^2 > 0.$$

Since we already know $\frac{\partial(SW)}{\partial\theta}$ from above, we combine all of these to obtain:

$$\left.\frac{d(SW)}{dr}\right|_{\theta=\theta^*} = \frac{X(r;\theta^*)}{4\beta\theta^* Y}, \text{ where}$$

$$\begin{aligned} X(r;\theta) = 8r^3 &- 3r^2\beta\theta(1-2e\xi(1-\beta)) - 2r\theta^2\left(2-3\beta(1-\beta)+e\beta\xi(4-5\beta)(1-\beta) + 2e^2\beta^2\xi^2(1-\beta)^2\right) \\ &+ \beta\theta^3(1-2e\xi(1-\beta))(1-\beta)^2(1+e\beta\xi)^2. \end{aligned}$$

Since $Y > 0$, the sign of the derivative above depends on the sign of $X(r;\theta^*)$ alone. The proof then follows directly from Lemma C1.

$\underline{\beta < 1, \text{Case 1B:}}$ In this case, we have:

$$\frac{\partial(SW)}{\partial e} = \frac{\xi\left(4r^2 + \beta\theta^2(3-4\beta)\right)}{8(\beta\theta - r)}, \qquad \text{[from the proof of Theorem 1]}$$

$$\frac{\partial(SW)}{\partial \theta} = \frac{1}{8}\left(4\beta + \frac{4r^2}{\beta\theta^2} + e\xi\left(1 - 4\beta - \frac{r^2}{(\beta\theta - r)^2}\right)\right), \text{ and}$$

$$\frac{d\theta^*}{de} = -\frac{\left.\frac{\partial\pi'}{\partial e}\right|_{\theta=\theta^*}}{\left.\frac{\partial\pi'}{\partial\theta}\right|_{\theta=\theta^*}} = \frac{\theta^*}{e} + \frac{2r^2}{e\beta(\beta\theta^* - 3r)}. \qquad \text{[from the Implicit Function Theorem]}$$

Combining all these with the help of the chain rule, we get:

$$\left.\frac{d(SW)}{de}\right|_{\theta=\theta^*} = \frac{X}{8e\beta^2{\theta^*}^2(\beta\theta^* - r)(\beta\theta^* - 3r)}, \text{ where}$$

$$X = -e\beta^2{\theta^*}^2\xi\left(4r^3 + 4r^2\theta^*(4\beta-1) - r\beta{\theta^*}^2(28\beta-13) + 4\beta^2{\theta^*}^3(2\beta-1)\right) + 4(\beta\theta^* - r)^2(\beta\theta^* - 2r)\left(\beta^2{\theta^*}^2 + r^2\right).$$

Since its denominator is clearly positive, $\frac{d(SW)}{de}$ is positive at $\theta = \theta^*$ if and only if $X > 0$. Now, it is clear that X is a linear function of e and attains its minimum at one of the extreme values of e. Because $0 \le e \le \frac{\beta\theta - r}{\beta\theta\xi}$ in Case 1B, we now find:

$$X|_{e=0} = 4(\beta\theta^* - r)^2(\beta\theta^* - 2r)\left(\beta^2{\theta^*}^2 + r^2\right) > 0, \text{ and}$$

$$X|_{e=\frac{\beta\theta^* - r}{\beta\theta^*\xi}} = (\beta\theta^* - r)\left(8r^4 + \frac{16r\beta\theta^*(\beta\theta^* + 3r)(\beta\theta^* - 3r)}{9} + \frac{11r\beta^3{\theta^*}^3}{9} + \frac{\beta{\theta^*}^2(1-\beta)\left(\beta\theta^*(\beta\theta^* + 31(\beta\theta^* - 3r)) + 4(\beta\theta^* - 3r)^2\right)}{9}\right) > 0.$$

X is, therefore, always positive in the region of interest, which means $\frac{d(SW)}{de}$ is positive at $\theta = \theta^*$.

We now consider the long-run impact of r on social welfare. We know:

$$\frac{\partial(SW)}{\partial r} = -\frac{r}{\beta\theta} - \frac{e\xi}{2} + \frac{3e\beta\theta^2\xi}{8(\beta\theta - r)^2}, \qquad \text{[from the proof of Theorem 1]}$$

$$\frac{\partial(SW)}{\partial \theta} = \frac{1}{8}\left(4\beta + \frac{4r^2}{\beta\theta^2} + e\xi\left(1 - 4\beta - \frac{r^2}{(\beta\theta - r)^2}\right)\right), \text{ and}$$

$$\frac{d\theta^*}{dr} = -\frac{\left.\frac{\partial\pi'}{\partial r}\right|_{\theta=\theta^*}}{\left.\frac{\partial\pi'}{\partial\theta}\right|_{\theta=\theta^*}} = -\frac{2r}{\beta(\beta\theta^* - 3r)}. \qquad \text{[from the Implicit Function Theorem]}$$

Once again, combining these using the chain rule, we get:

$$\left.\frac{d(SW)}{dr}\right|_{\theta=\theta^*} = \frac{X}{8\beta^2{\theta^*}^2(\beta\theta^* - r)^2(\beta\theta^* - 3r)}, \text{ where}$$

$$X = e\beta^2{\theta^*}^2\xi Y - Z,$$

$$Y = 20r^3 + 4r^2\theta^* - 44r^2\beta\theta^* - 11r\beta{\theta^*}^2 + 28r\beta^2{\theta^*}^2 + 3\beta^2{\theta^*}^3 - 4\beta^3{\theta^*}^3, \text{ and}$$

$$Z = 8r(2\beta\theta^* - r)(\beta\theta^* - r)^3 > 0.$$

Now, for $\frac{d(SW)}{dr}$ to be positive at $\theta = \theta^*$, both its numerator and denominator must have the same sign. Since the denominator is clearly positive, this implies that X must be greater than zero to make the derivative positive. To complete the proof then, we will show that: (a) if $\beta \geq \frac{3}{4}$, then $X < 0$, implying that the derivative can never be positive, and (b) if $\beta < \frac{3}{4}$, then $Y > 0$, implying $X > 0$ iff $e > \frac{Z}{\beta^2\theta^2\xi Y}$.

(a) We first consider the situation where $\frac{3}{4} \leq \beta < 1$. We note that X is a linear function of e, and the maximum value of X must be attained at one of the extreme values of e. Of course, $X|_{e=0} = -Z < 0$, so we consider the other extreme, $e = \frac{1}{\xi}$. After some algebra, we find:

$$X|_{e=\frac{1}{\xi}} = -r^3\left(8r^2-40r\beta\theta^*+13\beta^2{\theta^*}^2\right)-\beta^2{\theta^*}^2\left(39r^3-4r^2\theta^*(1+3\beta)-r\beta{\theta^*}^2(12\beta-11)+\beta^2{\theta^*}^3(4\beta-3)\right).$$

Since both roots of $\left(8r^2 - 40r\beta\theta^* + 13\beta^2{\theta^*}^2\right) = 0$ are larger than $\frac{\beta\theta^*}{3}$, $\left(8r^2 - 40r\beta{\theta^*}^2 + 13\beta^2{\theta^*}^2\right) > 0$ in the region of interest. On the other hand, $\left(39r^3-4r^2\theta^*(1+3\beta)-r\beta{\theta^*}^2(12\beta-11)+\beta^2{\theta^*}^3(4\beta-3)\right) = 0$ is a cubic equation in r, whose discriminant, $\beta^2{\theta^*}^6(\beta(9\beta(24\beta(302\beta-1445)+38461)-113468)+1168)$, is negative at all $\beta \in \left[\frac{3}{4}, 1\right)$. So, there is only one real root. Furthermore, since the cubic polynomial approaches $-\infty$ as $r \to -\infty$, and becomes equal to the non-negative number $\beta^2{\theta^*}^3(4\beta - 3)$ at $r = 0$, its only real root cannot be positive. Therefore, the polynomial is non-negative at all $r \geq 0$. Taken together, we have just shown that $X|_{e=\frac{1}{\xi}} < 0$ whenever $\beta \geq \frac{3}{4}$, implying that social welfare is decreasing in r for $\beta \geq \frac{3}{4}$.

(b) Now, we move to the case of $\beta < \frac{3}{4}$. Here, we first solve $\frac{\partial Y}{\partial r} = 0$, and observe that the unique minimum and maximum of Y happen at:

$$r_{\min} = \frac{\theta^*}{30}\left(2(11\beta-1)+\sqrt{\beta(64\beta+77)+4}\right), \text{ and } r_{\max} = \frac{\theta^*}{30}\left(2(11\beta-1)-\sqrt{\beta(64\beta+77)+4}\right).$$

Since $\beta > \frac{1}{2}$ in Case 1B, it is immediate that $r_{\min}$ is larger than $\frac{\beta\theta^*}{3}$ and that $Y|_{r=\frac{\beta\theta^*}{3}} > 0$. These, taken together with the fact that $Y|_{r=0} > 0$ for $\beta < \frac{3}{4}$, imply that Y must be positive in $\left[0, \frac{\beta\theta^*}{3}\right)$. Viewed differently, if Y actually became negative in this region, the minimum would have occurred prior to the function becoming positive again at $r = \frac{\beta\theta^*}{3}$. Thus, $Y > 0$ for all $\beta < \frac{3}{4}$ in Case 1B, implying that social welfare is increasing in r if and only if $e > \frac{Z}{\beta^2{\theta^*}^2\xi Y}$.

ii) Now, we consider $\beta \geq 1$. Recall that, in the long run, Case 1A is not possible here. So, we need to consider only Case 1B.

$\beta \geq 1$, Case 1B: The expression for social welfare in this case is basically the same as the previous case (i.e., $\beta < 1$, Case 1B).

Therefore, we have:

$$\left.\frac{d(SW)}{de}\right|_{\theta=\theta^*} = \frac{-e\beta^2{\theta^*}^2\xi Y + Z}{8e\beta^2{\theta^*}^2(\beta\theta^* - r)(\beta\theta^* - 3r)}, \text{ where}$$

$Y = \left(4r^3 + 4r^2\theta^*(4\beta-1) - r\beta{\theta^*}^2(28\beta-13) + 4\beta^2{\theta^*}^3(2\beta-1)\right)$ and $Z = 4(\beta\theta^*-r)^2(\beta\theta^*-2r)\left(\beta^2{\theta^*}^2+r^2\right)$. Clearly, $Z > 0$. Furthermore, since the denominator is positive, $\frac{d(SW)}{de}$ is positive at $\theta = \theta^*$ iff $(-e\beta^2{\theta^*}^2\xi Y + Z) > 0$. Assuming $Y > 0$, this condition is equivalent to:

$$e < \frac{Z}{\beta^2{\theta^*}^2\xi Y} = \frac{4(\beta\theta^* - r)^2(\beta\theta^* - 2r)\left(\beta^2{\theta^*}^2 + r^2\right)}{\beta^2{\theta^*}^2\xi\left(4r^3 + 4r^2\theta^*(4\beta - 1) - r\beta{\theta^*}^2(28\beta - 13) + 4\beta^2{\theta^*}^3(2\beta - 1)\right)},$$

and the desired result would follow directly. Therefore, to complete the proof, we only need to show that $Y > 0$. Since $\frac{\partial^2 Y}{\partial r^2} = 8(3r + \theta^*(4\beta - 1)) > 0$, the minimum of Y can happen either at an interior point or at one of the extreme points of $r \in \left[0, \frac{\beta\theta^*}{3}\right)$. Solving the first order condition, we get two roots, of which we only consider the positive one: $\frac{r}{\beta\theta^*} = \frac{2-8\beta+\sqrt{4+\beta(148\beta-71)}}{6\beta}$, which is larger than the maximum possible value of $\frac{1}{3}$. Clearly, the interior minimum is beyond the admissible range of r. Therefore, the minimum must occur at one of the extreme points. We find that $Y > 0$ at both these points:

$$Y|_{r=0} = 4\beta^2{\theta^*}^3(2\beta - 1) > 0, \quad \text{and} \quad Y|_{r=\frac{\beta\theta^*}{3}} = \frac{\beta^2{\theta^*}^3(16\beta-3)}{27} > 0.$$

Since this implies that $Y > 0$ over the entire valid range, the result, as stated in the theorem, has been proved.

We now consider the long-run impact of r on social welfare. We know from Theorem 1 that $\frac{\partial(SW)}{\partial r} < 0$, and from Theorem 2 that $\frac{d\theta^*}{dr} < 0$. Furthermore:

$$\frac{\partial(SW)}{\partial\theta} = \frac{1}{8}\left(4\beta + \frac{4r^2}{\beta\theta^2} + e\xi\left(1 - 4\beta - \frac{r^2}{(\beta\theta - r)^2}\right)\right),$$

which is a linear function of $e \in \left[0, \frac{\beta\theta - r}{\beta\theta\xi}\right]$ and is clearly positive at $e = 0$. At the other extreme, $e = \frac{\beta\theta - r}{\beta\theta\xi}$, we get:

$$\left.\frac{\partial(SW)}{\partial\theta}\right|_{e=\frac{\beta\theta-r}{\beta\theta\xi}} = \frac{4r(\beta\theta + r)(\beta\theta - r) + \beta\theta^2(\beta\theta - 2r)}{8\beta\theta^2(\beta\theta - r)} > 0.$$

Therefore, $\frac{\partial(SW)}{\partial\theta}$ is always positive in this case. Using the chain rule now, we get:

$$\left.\frac{d(SW)}{dr}\right|_{\theta=\theta^*} = \underbrace{\left.\frac{\partial(SW)}{\partial r}\right|_{\theta=\theta^*}}_{<0} + \underbrace{\left.\frac{\partial(SW)}{\partial\theta}\right|_{\theta=\theta^*}}_{>0} \underbrace{\frac{d\theta^*}{dr}}_{<0} < 0.$$

Proof of Proposition 4

To prove this proposition, we will show that:

i) The piracy rate is monotonically decreasing in r, but it is not monotonic in e. In Case 1A, the piracy rate is increasing in e if $e < \frac{3r-\theta^*(1-\beta)}{\beta\theta^*\xi(1-\beta)}$, and decreasing otherwise. In Case 1B, however, the piracy rate is increasing in e if and only if $e < \frac{4r^3-\beta\theta^*(\beta\theta^*-3r)^2}{r\beta\theta^*\xi(\beta\theta^*-2r)}$.

ii) The manufacturer's profit is monotonically increasing in both e and r.

iii) The consumer surplus is monotonically decreasing in r, but not in e. It is increasing in e for moderate or low values of e. Specifically, in Case 1A, it is increasing if and only if $e < \gamma_{1\text{A}}(\theta^*)$, where $\gamma_{1\text{A}}(\cdot)$ is as defined in Lemma C2. In Case 1B, the consumer surplus is increasing in e if and only if $e < \frac{4(\beta\theta^*-r)^3(\beta\theta^*-2r)(\beta\theta^*+r)}{\beta^2{\theta^*}^2\xi(8\beta^3{\theta^*}^3-2\beta^2{\theta^*}^3-36r\beta^2{\theta^*}^2+7r\beta{\theta^*}^2+48r^2\beta\theta^*-4r^2\theta^*-20r^3)}$.

iv) The legal social welfare is monotonically increasing in e, but not in r unless r is small. More specifically, it is increasing in r if and only if $r < \tau_{1\text{A}}(\theta^*)$ in Case 1A or $r < \frac{\beta\theta^*(11-\sqrt{73})}{8}$ in Case 1B, where $\tau_{1\text{A}}(\cdot)$ is as defined in Lemma C2.

We now prove each result one by one:

i) We again start with the piracy rate and show that it is monotonically decreasing in r.

Case 1A: We know from Proposition 3 that $\left.\frac{\partial\mu}{\partial r}\right|_{\theta=\theta^*} < 0$. Also, from Theorem 2, we know that

$\frac{d\theta^*}{dr} < 0$. Finally,

$$\left.\frac{\partial\mu}{\partial\theta}\right|_{\theta=\theta^*} = \frac{r\beta}{2(1-\beta)(r-\beta\theta^*(1-e\xi(1-\beta)))^2} > 0.$$

Combining everything using the chain rule, we get:

$$\left.\frac{d\mu}{dr}\right|_{\theta=\theta^*} = \underbrace{\left.\frac{\partial\mu}{\partial r}\right|_{\theta=\theta^*}}_{<0} + \underbrace{\left.\frac{\partial\mu}{\partial\theta}\right|_{\theta=\theta^*}}_{>0}\underbrace{\frac{d\theta^*}{dr}}_{<0} < 0.$$

Case 1B: This is quite similar to Case 1A in the long run. In this case, we get:

$$\left.\frac{\partial\mu}{\partial\theta}\right|_{\theta=\theta^*} = \frac{2er\beta^2\theta\xi(\beta\theta^*(1-e\xi)+(\beta\theta^*-2r))}{\left(2(\beta\theta^*-r)^2+e\beta\theta^*\xi(\beta\theta^*-2r)\right)^2} > 0.$$

As before, using the chain rule, we can easily show that μ is monotonic in r in the long run.

We now consider the long-run impact of e on piracy rate. Again, we use the chain rule:

$$\left.\frac{d\mu}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial\mu}{\partial e}\right|_{\theta=\theta^*} + \left.\frac{\partial\mu}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{de}.$$

Case 1A: In this case:

$$\frac{\partial\mu}{\partial e} = -\frac{\beta^2\theta^2\xi}{2(r-\beta\theta(1-e\xi(1-\beta)))^2}, \quad \frac{d\theta^*}{de} = \frac{2\beta{\theta^*}^3\xi(1-\beta)^2(1+e\beta\xi)}{(1-\beta)^2{\theta^*}^2(1+e\beta\xi)^2-3r^2},$$

and $\frac{\partial\mu}{\partial\theta}$ is as given above. Combining, we get:

$$\left.\frac{d\mu}{de}\right|_{\theta=\theta^*} = \frac{\beta^2{\theta^*}^2\xi(r+\theta^*(1-\beta)(1+e\beta\xi))(3r-\theta^*(1-\beta)(1+e\beta\xi))}{2(\beta\theta^*(1-e\xi(1-\beta))-r)^2\left({\theta^*}^2(1-\beta)^2(1+e\beta\xi)^2-3r^2\right)}.$$

Since, as before, ${\theta^*}^2(1-\beta)^2(1+e\beta\xi)^2-3r^2>0$, the denominator is clearly positive, implying that the above derivative will be positive iff $3r-\theta^*(1-\beta)(1+e\beta\xi)>0$ or $e<\frac{3r-\theta^*(1-\beta)}{\beta\theta^*\xi(1-\beta)}$.

Case 1B: In this case:

$$\frac{\partial\mu}{\partial e} = -\frac{2\beta^2\theta^2\xi(\beta\theta-r)^2}{\left(2(\beta\theta-r)^2-e\beta\theta\xi(\beta\theta-2r)\right)^2}, \quad \frac{d\theta^*}{de} = \frac{\theta^*}{e}+\frac{2r^2}{e\beta(\beta\theta^*-3r)},$$

and $\frac{\partial\mu}{\partial\theta}$ is as given above. Combining, we get:

$$\left.\frac{d\mu}{de}\right|_{\theta=\theta^*} = \frac{2\beta\theta^*\xi(\beta\theta^*-r)\left(4r^3-\beta\theta^*(\beta\theta^*-3r)^2-er\beta\theta^*\xi(\beta\theta^*-2r)\right)}{(\beta\theta^*-3r)\left(2(\beta\theta^*-r)^2-e\beta\theta^*\xi(\beta\theta^*-2r)\right)^2}.$$

Since the denominator is clearly positive, the above derivative will be positive iff $4r^3-\beta\theta^*(\beta\theta^*-3r)^2-er\beta\theta^*\xi(\beta\theta^*-2r)>0$ or $e<\frac{4r^3-\beta\theta^*(\beta\theta^*-3r)^2}{r\beta\theta^*\xi(\beta\theta^*-2r)}$.

ii) Next, we show that the manufacturer's profit is monotonically increasing in both e and r. This is straightforward. From the Envelope Theorem, we get:

$$\left.\frac{d\pi^*}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial\pi^*}{\partial e}\right|_{\theta=\theta^*} > 0 \quad \text{and} \quad \left.\frac{d\pi^*}{dr}\right|_{\theta=\theta^*} = \left.\frac{\partial\pi^*}{\partial r}\right|_{\theta=\theta^*} > 0,$$

because the last part of each inequality follows from Proposition 3.

iii) We now show that the consumer surplus is monotonically decreasing in r. We first use the chain rule:

$$\left.\frac{d(CS)}{dr}\right|_{\theta=\theta^*} = \left.\frac{\partial(CS)}{\partial r}\right|_{\theta=\theta^*} + \left.\frac{\partial(CS)}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{dr}.$$

We know from Proposition 3 that $\left.\frac{\partial(CS)}{\partial r}\right|_{\theta=\theta^*} < 0$. Furthermore, from Theorem 2, we know that $\frac{d\theta^*}{dr} < 0$. Therefore, to complete the proof, we need to show that $\left.\frac{\partial(CS)}{\partial\theta}\right|_{\theta=\theta^*} > 0$. We now do so for each case.

<u>Case 1A:</u> In this case, we already know that $e < \frac{1}{\xi}$ and $\frac{r}{\beta\theta} < \frac{1-\beta}{2-\beta}$. Furthermore, since $\beta < 1$, we must have $\beta(1-\beta) < \frac{1}{4}$. Therefore, we get:

$$\begin{aligned}
\left.\frac{\partial(CS)}{\partial\theta}\right|_{\theta=\theta^*} &= \frac{1}{8}\left(1+3\beta-4e\beta\xi(1-\beta)-e^2\beta^2\xi^2(1-\beta)-\frac{r^2(4-3\beta)}{\beta{\theta^*}^2(1-\beta)}\right)\\
&> \frac{1}{8}\left(1+3\beta-e\xi-\frac{e^2\beta\xi^2}{4}-\frac{r^2(4-3\beta)}{\beta\theta^2(1-\beta)}\right)\\
&> \frac{1}{8}\left(1+3\beta-1-\frac{\beta}{4}-\frac{\beta(1-\beta)^2(4-3\beta)}{(2-\beta)^2(1-\beta)}\right) > \frac{1}{8}\left(\frac{3\beta}{4}+\frac{\beta\left(4-\beta-\beta^2\right)}{(2-\beta)^2}\right) > 0.
\end{aligned}$$

<u>Case 1B:</u> In this case, $e < \frac{\beta\theta-r}{\beta\theta\xi}$ and $\frac{r}{\beta\theta} < \frac{1}{3}$. Then:

$$\begin{aligned}
\left.\frac{\partial(CS)}{\partial\theta}\right|_{\theta=\theta^*} &= \frac{1}{8}\left(4\beta-\frac{4r^2}{\beta{\theta^*}^2}-e\xi\left(4\beta+\frac{r^2}{(\beta\theta^*-r)^2}-1\right)\right)\\
&> \frac{1}{8}\left(4\beta-\frac{4r^2}{\beta{\theta^*}^2}-\frac{\beta\theta^*-r}{\beta\theta^*}\left(4\beta+\frac{r^2}{(\beta\theta^*-r)^2}-1\right)\right) = \frac{1}{8}\left(\frac{4r}{\theta^*}\left(1-\frac{r}{\beta\theta^*}\right)+\frac{\beta\theta^*-2r}{\beta\theta^*-r}\right) > 0.
\end{aligned}$$

We now consider the long-run impact of e. Again, we make use of the chain rule:

$$\left.\frac{d(CS)}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial(CS)}{\partial e}\right|_{\theta=\theta^*} + \left.\frac{\partial(CS)}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{de}.$$

<u>Case 1A:</u> In this case:

$$\frac{\partial(CS)}{\partial e} = \frac{\xi(1-\beta)(2r-\beta\theta(2+e\beta\xi))}{4}, \quad \frac{d\theta^*}{de} = \frac{2\beta\theta^3\xi(1-\beta)^2(1+e\beta\xi)}{\theta^2(1-\beta)^2(1+e\beta\xi)^2-3r^2},$$

and $\frac{\partial(CS)}{\partial\theta}$ is as given above. Combining, we get:

$$\left.\frac{d(CS)}{de}\right|_{\theta=\theta^*} = \frac{\xi(1-\beta)X(e;\theta^*)}{4Y}, \text{ where}$$

$$\begin{aligned}
X(e;\theta) = &-6r^3-r^2\theta(4-9\beta+2e\beta\xi(2-3\beta))+2r\theta^2(1-\beta)^2(1+e\beta\xi)^2\\
&-\beta\theta^3(1-\beta)(1+e\beta\xi)\left(1-5\beta+7e\beta\xi(1-\beta)+2e^2\beta^2\xi^2(1-\beta)\right)
\end{aligned}$$

and, as before, $Y = \theta^2(1-\beta)^2(1+e\beta\xi)^2-3r^2 > 0$. Therefore, the above derivative would be positive iff $X(e;\theta^*) > 0$ which, according to Lemma C2, is true if $e < \gamma_{1\mathrm{A}}(\theta)$.

Case 1B: Here,

$$\frac{\partial(CS)}{\partial e} = \frac{\xi\left(8r\beta\theta - \beta\theta^2(4\beta-1) - 4r^2\right)}{8(\beta\theta - r)}, \qquad \frac{d\theta^*}{de} = \frac{\theta^*}{e} + \frac{2r^2}{e\beta(\beta\theta^* - 3r)},$$

and $\frac{\partial(CS)}{\partial\theta}$ is as given above. Combining, we get:

$$\left.\frac{d(CS)}{de}\right|_{\theta=\theta^*} = \frac{-e\beta^2{\theta^*}^2\xi X + Y}{8e\beta^2{\theta^*}^2(\beta\theta^* - 3r)(\beta\theta^* - r)}, \text{ where}$$

$X = 8\beta^3{\theta^*}^3 – 2\beta^2{\theta^*}^3 – 36r\beta^2{\theta^*}^2 + 7r\beta{\theta^*}^2 + 48r^2\beta\theta^* – 4r^2\theta^* – 20r^3$, and $Y = 4(\beta\theta^* – r)^3(\beta\theta^* – 2r)(\beta\theta^* + r) > 0$.

Now, assuming $X > 0$, it is clear that the above derivative would be positive if and only if $e < \frac{Y}{\beta^2{\theta^*}^2\xi X}$. Therefore, the proof can be completed by simply showing that $X > 0$; we do so now.

We first solve $\frac{\partial X}{\partial r} = 0$ and find two roots. From the second order condition, we find that the first one is a minimum and the second, a maximum:

$$r_{\min} = \frac{\theta^*}{30}\left(2(12\beta - 1) - \sqrt{4 + 9\beta(1 + 4\beta)}\right), \text{ and } r_{\max} = \frac{\theta^*}{30}\left(2(12\beta - 1) + \sqrt{4 + 9\beta(1 + 4\beta)}\right).$$

Since $\beta > \frac{1}{2}$ in Case 1B, it is immediate that $r_{\min}$ and $r_{\max}$ are both larger than $\frac{\beta\theta^*}{3}$. These, taken together with the facts that $X|_{r=0} = 2\beta^2\theta^3(4\beta - 1) > 0$ and $X|_{r=\frac{\beta\theta^*}{3}} = \frac{\beta^2\theta^3}{27}(16\beta - 3) > 0$, imply that $X > 0$ in $\left[0, \frac{\beta\theta^*}{3}\right)$.

iv) Finally, we show that the legal social welfare is monotonically increasing in e. Again, we make use of the chain rule:

$$\left.\frac{d(SW_{\mathrm{L}})}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial(SW_{\mathrm{L}})}{\partial e}\right|_{\theta=\theta^*} + \left.\frac{\partial(SW_{\mathrm{L}})}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{de}.$$

We know from Proposition 3 that $\left.\frac{\partial(SW_{\mathrm{L}})}{\partial e}\right|_{\theta=\theta^*} > 0$. Furthermore, from Theorem 2, we know that $\frac{d\theta^*}{de} > 0$. Therefore, to complete the proof, we only need to show that $\left.\frac{\partial(SW_{\mathrm{L}})}{\partial\theta}\right|_{\theta=\theta^*} > 0$.

Case 1A: In this case, $\beta < 1$. Thus, we get:

$$\left.\frac{\partial(SW_{\mathrm{L}})}{\partial\theta}\right|_{\theta=\theta^*} = \frac{3}{8} + \frac{r^2}{8{\theta^*}^2(1-\beta)^2} + \frac{e\beta\xi}{8}\left(2(1-\beta) + e\beta\xi(3 - 2\beta)\right) > 0.$$

Case 1B: In this case, $\frac{r}{\beta\theta} < \frac{1}{3}$. Therefore:

$$\left.\frac{\partial(SW_{\mathrm{L}})}{\partial\theta}\right|_{\theta=\theta^*} = \frac{3e\beta\theta^*\xi(\beta\theta^* - 2r)}{8(\beta\theta^* - r)^2} > 0.$$

We now consider the long-run impact of r. Once again, we make use of the chain rule:

$$\left.\frac{d(SW_{\mathrm{L}})}{dr}\right|_{\theta=\theta^*} = \left.\frac{\partial(SW_{\mathrm{L}})}{\partial r}\right|_{\theta=\theta^*} + \left.\frac{\partial(SW_{\mathrm{L}})}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{dr}.$$

Case 1A: In this case:

$$\frac{\partial(SW_{\mathrm{L}})}{\partial r} = \frac{e\beta\xi}{4} + \frac{\theta(1-\beta) - r}{4\theta(1-\beta)^2}, \qquad \frac{d\theta^*}{dr} = -\frac{2r\theta}{\theta^2(1-\beta)^2\left(1 + e\beta\xi\right)^2 - 3r^2},$$

and $\frac{\partial(SW_{\mathrm{L}})}{\partial\theta}$ is as above. Combining, we get:

$$\left.\frac{d(SW_{\mathrm{L}})}{dr}\right|_{\theta=\theta^*} = \frac{X(r;\theta^*)}{4\theta^* Y(1-\beta)^2}, \text{ where}$$

$$\begin{aligned} X(r;\theta) = {} & 2r^3\beta - 3r^2\theta(1-\beta)(1+e\beta\xi(1-\beta)) - 2r\theta^2(1-\beta)^2\left(1+\beta+e\beta^2\xi(1+e\xi)\right) \\ & + \theta^3(1-\beta)^3(1+e\beta\xi)^2(1+e\beta\xi(1-\beta)), \end{aligned}$$

and, as before, $Y = {\theta^*}^2(1-\beta)^2(1+e\beta\xi)^2 - 3r^2 > 0$. Therefore, the derivative in question would be positive if and only if $X(r;\theta^*) > 0$, which, by Lemma C2, is true if $r < \tau_{1\mathrm{A}}(\theta^*)$.

<u>Case 1B:</u> In this case:

$$\frac{\partial(SW_{\mathrm{L}})}{\partial r} = \frac{3e\beta\theta^2\xi}{8(\beta\theta-r)^2}, \quad \frac{\partial(SW_{\mathrm{L}})}{\partial\theta} = \frac{3e\beta\theta\xi(\beta\theta-2r)}{8(\beta\theta-r)^2}, \text{ and } \quad \frac{d\theta^*}{dr} = -\frac{2r}{\beta(\beta\theta^*-3r)}.$$

Combining, we get:

$$\left.\frac{d(SW_{\mathrm{L}})}{dr}\right|_{\theta=\theta^*} = \frac{e\theta^*\xi\left(4r^2 - 11r\beta\theta^* + 3\beta^2{\theta^*}^2\right)}{8(\beta\theta^*-r)^2(\beta\theta^*-3r)},$$

which would be positive if and only if $4r^2 - 11r\beta\theta^* + 3\beta^2{\theta^*}^2 > 0$, or $r < \frac{\beta\theta^*(11-\sqrt{73})}{8} \approx 0.307\beta\theta^*$. ■

Proof of Lemma A1

We will prove this by contradiction. In Case 1B, the manufacturer's profit is given by:

$$\pi = p\lambda\left(1-\frac{p}{\theta}\right) - \frac{c\theta^2}{2},$$

where $\lambda = \frac{e\theta\xi}{\theta-\frac{r}{\beta}}$. When $\frac{p}{\theta} = \delta$, we can substitute $p = \delta\theta$ in the above expression and differentiate w.r.t. θ to obtain the following first order condition:

$$\frac{d\pi}{d\theta} = \frac{\theta^2\left(e\beta^2\delta\xi(1-\delta)\right) - \theta(2er\beta\delta\xi(1-\delta))}{(\beta\theta-r)^2} - c\theta = 0.$$

Since $\theta > 0$ and $\frac{r}{\beta\theta} < 1$, we can multiply both sides by $\frac{(\beta\theta-r)^2}{\theta}$ and solve the resulting quadratic equation to obtain the following two roots:

$$\theta^* = \left\{\frac{2cr-e\beta\delta\xi(\delta-1)+\sqrt{e\beta\delta\xi(\delta-1)}\sqrt{4cr+e\beta\delta\xi(\delta-1)}}{2c\beta}, \frac{2cr-e\beta\delta\xi(\delta-1)-\sqrt{e\beta\delta\xi(\delta-1)}\sqrt{4cr+e\beta\delta\xi(\delta-1)}}{2c\beta}\right\}.$$

However, for either root to be real, δ must be greater than 1, which contradicts the fact that $\delta < \frac{1}{2}$. Thus, $\frac{p}{\theta} = \delta$ cannot be valid in Case 1B. ■

Proof of Proposition A1

In Case 1A, the manufacturer's profit is given by:

$$\pi = p\left(\lambda\left(1-\frac{p}{\theta}\right) + (1-\lambda)\left(1-\frac{p-r}{(1-\beta)\theta}\right)\right) - \frac{c\theta^2}{2}, \quad \text{where} \quad \lambda = \begin{cases} \frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}} & \text{if } \beta < 1, \\ \frac{e\theta\xi}{\theta-\frac{r-p}{\beta-1}} & \text{otherwise.} \end{cases}$$

When $\frac{p}{\theta} = \delta$, we can substitute $p = \delta\theta$ in the above expressions, differentiate w.r.t. θ, and solve the appropriate first order condition to obtain the desired result. It is easy to verify that the second order condition holds as well. ■

Proof of Corollary A1

It is easy to see that θ^* is independent of r. Also, differentiating θ^* w.r.t. e, we get:

$$\frac{d\theta^*}{de} = \begin{cases} \frac{\beta\delta\xi}{c} & \text{if } \beta < 1, \\ \frac{\delta\xi}{c} & \text{otherwise.} \end{cases}$$

Since both terms are positive, the result follows. ■

Proof of Theorem A1

According to Lemma A1, Case 1B is no longer possible. Hence, we simply limit our attention to Case 1A.

i) We start by showing the manufacturer's profit monotonically increases in both e and r. The manufacturer's profit is given by:

$$\pi = p\left(\lambda\left(1-\frac{p}{\theta}\right) + (1-\lambda)\left(1-\frac{p-r}{(1-\beta)\theta}\right)\right) - \frac{c\theta^2}{2}, \quad \text{where} \quad \lambda = \begin{cases} \frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}} & \text{if } \beta < 1, \\ \frac{e\theta\xi}{\theta-\frac{r-p}{\beta-1}} & \text{otherwise.} \end{cases}$$

$\underline{\beta < 1}$: From the Envelope Theorem, we get:

$$\left.\frac{d\pi}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial\pi}{\partial e}\right|_{\theta=\theta^*} = \beta\delta\xi\theta^* > 0 \quad \text{and} \quad \left.\frac{d\pi}{dr}\right|_{\theta=\theta^*} = \left.\frac{\partial\pi}{\partial r}\right|_{\theta=\theta^*} = \frac{\delta}{1-\beta} > 0.$$

$\underline{\beta \geq 1}$: Again, from the Envelope Theorem, we get:

$$\left.\frac{d\pi}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial\pi}{\partial e}\right|_{\theta=\theta^*} = \delta\xi\theta^* > 0 \quad \text{and} \quad \left.\frac{d\pi}{dr}\right|_{\theta=\theta^*} = \left.\frac{\partial\pi}{\partial r}\right|_{\theta=\theta^*} = \frac{\delta}{\beta-1} > 0.$$

ii) We now show how the consumer surplus changes with e and r.

$\underline{\beta < 1}$: The consumer surplus (CS) is given by:

$$CS = \frac{r(r+\beta\theta(e\xi(1-\beta)-2\delta)) + \beta\theta^2\left(\delta^2-(1-\beta)(\delta(2+e\beta\xi)-1)\right)}{2\beta\theta(1-\beta)}.$$

We first investigate the effect of e on CS. By differentiating CS with respect to e and θ, we get:

$$\frac{\partial(CS)}{\partial e} = -\frac{1}{2}\xi(\beta\theta\delta - r) \quad \text{and} \quad \frac{\partial(CS)}{\partial\theta} = \frac{\beta\theta^2\left(\delta^2-(1-\beta)(\delta(2+e\beta\xi)-1)\right) - r^2}{2\beta\theta^2(1-\beta)}.$$

Using the chain rule, we get:

$$\begin{aligned} \left.\frac{d(CS)}{de}\right|_{\theta=\theta^*} &= \left.\frac{\partial(CS)}{\partial e}\right|_{\theta=\theta^*} + \left.\frac{\partial(CS)}{\partial\theta}\right|_{\theta=\theta^*} \frac{d\theta^*}{de} \\ &= \frac{\xi\left(cr\theta^{*2}(1-\beta)+\delta X\right)}{2\left(c\theta^{*2}(1-\beta)\right)}, \end{aligned}$$

where $X = \beta\theta^{*2}\left(2\delta^2+(1-\delta(3+2e\beta\xi))(1-\beta)\right) - r^2$. From the condition $\lambda = \frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}} < 1$, we get $r < \beta\theta(\delta - e\xi(1-\beta))$. Hence, $X > \beta\theta^{*2}\left(2\delta^2+(1-(3+2e\beta\xi)\delta)(1-\beta)\right) - (\beta\theta^{*}(\delta-e\xi(1-\beta)))^2 = \beta\theta^{*2}Y$, where $Y = (1-3\delta)(1-\beta)+2\delta^2-\beta\delta^2-e^2\beta\xi^2(1-\beta)^2$. Since $r>0$, $\beta\theta(\delta-e\xi(1-\beta))>0$, which shows that $e^2\beta\xi^2(1-\beta)^2 < \beta\delta^2$. Thus, $Y > (1-3\delta)(1-\beta)+2\delta^2-\beta\delta^2-\beta\delta^2 = (1-\beta)(1-\delta)(1-2\delta) > 0$. Finally, since $Y>0$, $X>0$, and thus, $\frac{d(CS)}{de}|_{\theta=\theta^*} > 0$.

Now, we show that *CS* decreases with r. By differentiating *CS* with respect to r, we get:

$$\frac{\partial(CS)}{\partial r} = \frac{r-\beta\theta\delta}{\beta\theta(1-\beta)} + \frac{e\xi}{2}.$$

From the condition $\lambda = \frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}} < 1$, we get $\frac{e\xi}{2} < -\frac{r-\beta\theta\delta}{2\beta\theta(1-\beta)}$, and from the condition $\frac{r}{\beta\theta} < \frac{p}{\theta} = \delta$, $r < \beta\theta\delta$. Thus, $\frac{\partial(CS)}{\partial r} = \frac{r-\beta\theta\delta}{\beta\theta(1-\beta)} + \frac{e\xi}{2} < \frac{r-\beta\theta\delta}{\beta\theta(1-\beta)} - \frac{r-\beta\theta\delta}{2\beta\theta(1-\beta)} = \frac{r-\beta\theta\delta}{2\beta\theta(1-\beta)} < 0$. Since $\frac{d\theta^*}{dr} = 0$, $\left.\frac{d(CS)}{dr}\right|_{\theta=\theta^*} = \left.\frac{\partial(CS)}{\partial r}\right|_{\theta=\theta^*} < 0$.

$\underline{\beta \geq 1}$: In this case, the consumer surplus is given by:

$$CS = \frac{r^2 + r\theta(e\xi(\beta-1) - 2(\beta+\delta-1)) + \theta^2\left(\beta\left(\beta+\delta^2-1\right) - e\xi(\beta-1)(\beta+\delta-1)\right)}{2\theta(\beta-1)}.$$

Hence, at $\theta = \theta^*$, $\frac{d(CS)}{dr} = \frac{\partial(CS)}{\partial r} = \frac{r-(\beta+\delta-1)\theta}{(\beta-1)\theta} + \frac{e\xi}{2}$. Since $p = \delta\theta$ and $\lambda = \frac{e\theta\xi}{\theta - \frac{r-p}{\beta-1}} < 1$, we can easily show that $\frac{r-(\beta+\delta-1)\theta}{(\beta-1)\theta} < -e\xi$, which implies that $\frac{d(CS)}{dr} < -e\xi + \frac{e\xi}{2} = -\frac{e\xi}{2} < 0$, at $\theta = \theta^*$.

On the other hand, *CS* is not a monotonic function of e. First, we write using the chain rule:

$$\begin{aligned}\left.\frac{d(CS)}{de}\right|_{\theta=\theta^*} &= \left.\frac{\partial(CS)}{\partial e}\right|_{\theta=\theta^*} + \left.\frac{\partial(CS)}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{de} \\ &= \frac{c\xi(\beta-1)\left(\frac{r\delta(e\xi(\beta-1)-\beta\delta)^2}{c(\beta-1)} + \frac{\delta^2\left(\beta\left((\beta-1)(1+\delta)+2\delta^2\right)-2e\xi(\beta-1)(\beta+\delta-1)\right)(e\xi(\beta-1)-\beta\delta)^2}{c^2(\beta-1)^2} - r^2\right)}{2\delta(e\xi(\beta-1)-\beta\delta)^2}.\end{aligned}$$

Next, we solve $\frac{d(CS)}{de} = 0$, a quadratic equation in r, to obtain the only valid root:

$$\hat{r} = \frac{\delta(e\xi(\beta-1)-\beta\delta)\left(e\xi(\beta-1)-\beta\delta-\sqrt{\beta(\beta(2+\delta)^2-4(1-\delta)(1+2\delta))-2e\xi(\beta-1)(\beta(4+\delta)-4(1-\delta))+e^2\xi^2(\beta-1)^2}\right)}{2c(\beta-1)}.$$

Now, it can be shown that, within the valid region for Case 1A, $\frac{d(CS)}{de}$ crosses zero only once, and it is a decreasing function of r at that point. Therefore, $\frac{d(CS)}{de}$ is positive if and only if $r < \hat{r}$. In other words, *CS* is increasing in e as long as $r < \hat{r}$ and is decreasing otherwise.

iii) Finally, we show how social welfare changes with e and r.

$\underline{\beta < 1}$: The social welfare (SW) is given by:

$$SW = \frac{r(\beta\theta(2\delta-e\xi(1-\beta))-r)+\beta\theta^2\left((1-\beta)(1-c\theta)+e\beta\xi\delta(1-\beta)-\delta^2\right)}{2\beta\theta(1-\beta)}.$$

We first investigate the effect of e on SW. Using the chain rule, we get:

$$\left.\frac{d(SW)}{de}\right|_{\theta=\theta^*} = \left.\frac{\partial(SW)}{\partial e}\right|_{\theta=\theta^*} + \left.\frac{\partial(SW)}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{de}.$$

By differentiating SW with respect to e and θ, we get:

$$\frac{\partial(SW)}{\partial e}=\frac{\xi}{2}(\beta\theta\delta-r)\quad\text{and}\quad\frac{\partial(SW)}{\partial\theta}=\frac{r^2+\beta\theta^2((1-\beta)(1-2c\theta)+\delta(e\beta\xi(1-\beta)-\delta))}{2\beta\theta^2(1-\beta)}.$$

Since $\frac{r}{\beta\theta}<\frac{p}{\theta}=\delta$, $r<\beta\theta\delta$. Thus, $\frac{\partial(SW)}{\partial e}>0$.
Now, $\left.\frac{\partial(SW)}{\partial\theta}\right|_{\theta=\theta^*}=\frac{X}{2\beta\theta^{*2}(1-\beta)}$, where $X=r^2+\beta\theta^{*2}(1-\beta)(1-c\theta^*-\delta)$. Substituting θ^* in X with the expression from Proposition A1, we get:

$$X=\frac{(rc(1-\beta))^2+Y\beta\delta^2(1-\beta)(1+e\beta\xi)-\delta)^2}{c^2(1-\beta)^2},$$

where $Y=(1–\beta)(1–\delta)^2+\beta\delta^2–e\beta\xi\delta(1–\beta)$. From the condition $\lambda=\frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}}<1$, we get $r<\beta\theta(\delta–e\xi(1–\beta))$, and since $r>0$, $\delta-e\xi(1-\beta)>0$. Multiplying both sides by $\beta\delta$, we get $\beta\delta^2-e\beta\xi\delta(1-\beta)>0$, implying that $Y>0$. Finally, since $Y>0$, X must be positive, implying $\left.\frac{\partial(SW)}{\partial\theta}\right|_{\theta=\theta^*}>0$. Since, $\frac{d\theta^*}{de}>0$ as well, each term in the above chain rule is positive, so $\left.\frac{d(SW)}{de}\right|_{\theta=\theta^*}>0$.
Now, we show that SW increases with r. By differentiating SW with respect to r, we get:

$$\frac{\partial(SW)}{\partial r}=\frac{\beta\delta\theta-r}{\beta\theta(1-\beta)}-\frac{e\xi}{2}.$$

From the condition $\lambda=\frac{e\theta\xi(1-\beta)}{p-\frac{r}{\beta}}<1$, we get $\frac{\beta\theta\delta-r}{\beta\theta(1-\beta)}>e\xi$. Thus, $\frac{\partial(SW)}{\partial r}=\frac{\beta\delta\theta–r}{\beta\theta(1–\beta)}-\frac{e\xi}{2}>e\xi-\frac{e\xi}{2}=\frac{e\xi}{2}>0$.
Since $\frac{d\theta^*}{dr}=0$, $\left.\frac{d(SW)}{dr}\right|_{\theta=\theta^*}=\left.\frac{\partial(SW)}{\partial r}\right|_{\theta=\theta^*}>0$.
$\underline{\beta\geq 1}$: In this case, social welfare is given by:

$$SW=\frac{r\theta(2\delta-e\xi(\beta-1))-r^2+\theta^2\left(\beta\left(\beta-1-\delta^2\right)-e\xi(\beta-1)(\beta-1-\delta)\right)}{2\theta(\beta-1)}.$$

Therefore, at $\theta=\theta^*$, $\frac{d(SW)}{dr}=\frac{\partial(SW)}{\partial r}=-\frac{r-\delta\theta}{\theta(\beta-1)}-\frac{e\xi}{2}$, which is clearly negative, because $r>\beta\delta\theta>\delta\theta$.
To show that SW is not a monotonic function of e, we write using the chain rule:

$$\begin{aligned}\left.\frac{d(SW)}{de}\right|_{\theta=\theta^*}&=\left.\frac{\partial(SW)}{\partial e}\right|_{\theta=\theta^*}+\left.\frac{\partial(SW)}{\partial\theta}\right|_{\theta=\theta^*}\frac{d\theta^*}{de}\\&=\frac{c\xi(\beta-1)\left(r^2-\frac{r\delta(e\xi(\beta-1)-\beta\delta)^2}{c(\beta-1)}-\frac{\delta^2\left(\beta\left(2\delta^2-(\beta-1)(1+\delta)\right)+2e\xi(\beta-1)(\beta-\delta-1)\right)(e\xi(\beta-1)-\beta\delta)^2}{c^2(\beta-1)^2}\right)}{2\delta(e\xi(\beta-1)-\beta\delta)^2}.\end{aligned}$$

Solving $\frac{d(SW)}{de}=0$, a quadratic equation in r, we obtain two roots of which only one is valid:

$$\hat{r}=\frac{\delta(e\xi(\beta-1)-\beta\delta)\left(e\xi(\beta-1)-\beta\delta-\sqrt{\beta(\delta^2(8+\beta)-4(\beta-1)(1+\delta))-2e\xi(\beta-1)(4(1-\beta+\delta)+\beta\delta)+e^2\xi^2(\beta-1)^2}\right)}{2c(\beta-1)}.$$

Since $\frac{d(SW)}{de}$ can be shown to be a decreasing function of r within the valid region of Case 1A, $\frac{d(SW)}{de}$ is positive if and only if $r<\hat{r}$. Therefore, SW is increasing in e if $r<\hat{r}$ and decreasing otherwise. ■

Proof of Theorem A2

It turns out that, in the primary piracy region (1A and 1B), e and ξ always appear together in the product form, that is, as $e\xi$. In other words, every expression in Sections 3 and 4 can be rewritten by replacing $e\xi$

with $E = e\xi$. In Section 4, we defined several performance metrics—social welfare (including and excluding pirate consumption), piracy rate, quality, profit, and consumer surplus—on which we performed comparative statics to get our results. Let $M(E)$ denote any of these metrics, both short- and long-run, as a function of $E = e\xi$. In Section 4, when we did comparative statics with respect to e, we did so for a fixed ξ and examined the sign of:

$$\frac{dM(E)}{de} = \frac{dM(E)}{dE} \times \frac{d(e\xi)}{de} = \xi M'(E). \tag{C1}$$

Since, $\xi > 0$ by definition, our comparative statics results in Section 4 simply depended on the sign of $M'(E)$. For example, when we proved in Section 4 that the optimal quality level is increasing in e, we automatically proved that θ^* is increasing in E too, that is, $\theta^{*\prime}(E) = \frac{d\theta^*(E)}{dE} > 0$.

In the extension in Section 5.2, we replaced the fixed ξ by a function $\xi(e)$. Therefore, (C1) should now be rewritten as:

$$\frac{dM(E)}{de} = \frac{dM(E)}{dE} \times \frac{d(e\xi(e))}{de} = (\xi + e\xi')M'(E). \tag{C2}$$

Therefore, a necessary and sufficient condition for all our earlier results to carry over is that the right-hand sides of (C1) and (C2) must have exactly the same sign, positive or negative. A quick comparison between (C1) and (C2) tells us that they would have exactly the same sign if and only if $\xi + e\xi' > 0$. Now, e is positive, and so is $\xi = \mathbf{e}^{\frac{1}{\omega T(1+eB)}} - 1$; also:

$$\xi' = \frac{d\xi}{de} = -\frac{B(\xi + 1)}{\omega T(1 + eB)^2}.$$

As a result, when $B < 0$, $\xi + e\xi' > 0$ is trivially satisfied.

On the other hand, when $B > 0$, $\xi + e\xi' > 0$ holds if and only if $\left(1 - \mathbf{e}^{-\frac{1}{y(1+x)}} - \frac{x}{(1+x)^2 y}\right) > 0$, where $x = eB$ and $y = \omega T$. Therefore, if $y = f(x)$ is the solution to

$$1 - \mathbf{e}^{-\frac{1}{y(1+x)}} - \frac{x}{(1+x)^2 y} = 0, \tag{C3}$$

then $f(x)$ would partition the (x, y) space into two regions, one where $\xi + e\xi'$ is positive and the other where it is not. Also the left-hand of (C3) approaches $-\infty$ when y approaches zero, but remain positive for large values of y. Clearly, $\xi + e\xi' > 0$ is equivalent to $y > f(x)$, that is, to $\omega T > f(eB)$. ■

Proof of Corollary A2

We start with the observation that $F'_a(z) = \frac{\mathbf{e}^{-z}}{a} > 0$ and $F''_a(z) = -\frac{\mathbf{e}^{-z}}{a} < 0$, making $F_a(z)$ in (A2) a concave and strictly increasing function of z. Therefore, $F_a(z)$ cuts the 45°-line exactly twice, once from below and a second time from above. The crossing from below happens at $z = 0$, which is of no interest, since $z = 0$ makes $y = \frac{1}{z(1+x)}$ infinitely large. Therefore, $z = F_a(z)$ has a unique non-zero solution.

To further characterize this solution, we write using (A2):

$$1 - az = \mathbf{e}^{-z} \leq 1 - z + \frac{z^2}{2},$$

the last inequality coming from a Taylor series approximation of $\mathbf{e}^{-z}$. Since $z > 0$, the above simply reduces to: $z \geq 2(1 - a) = \frac{2}{1+x}$, or equivalently to $y \leq 0.5$. In other words, the function $y = f(x)$ is bounded above by the asymptote $y = 0.5$. Therefore, $\omega T \geq 0.5$ is a sufficient condition for $\omega T > f(eB)$ to hold, and the rest follows directly from Theorem A2. ■

Proof of Lemma C1

To prove this result, let us first recognize that $X(r;\theta)$ is a cubic polynomial in r satisfying:

$$X(-\infty;\theta) = -\infty, \quad X(+\infty;\theta) = +\infty, \quad X(0;\theta) = (1-2e\xi(1-\beta))(1-\beta)^2\beta(1+e\beta\xi)^2\theta^3, \text{ and}$$
$$X\left(\frac{\beta\theta}{2};\theta\right) = -\frac{\beta\theta^3}{4}\left(4(1-\beta)+7\beta^2+2e\xi(1-\beta)(4-\beta(4+5\beta))+4e^2\beta\xi^2(1-\beta)^2(4-3\beta)+8e^3\beta^2\xi^3(1-\beta)^3\right).$$

Now, recall that $2e\xi(1-\beta) < 1$, which immediately implies that $X(0;\theta) > 0$. Moving on to $X\left(\frac{\beta\theta}{2};\theta\right)$, note that $\left(4e^2\beta\xi^2(1-\beta)^2(4-3\beta)+8e^3\beta^2\xi^3(1-\beta)^3\right) > 0$ holds trivially. Further, $\big(4(1-\beta)+7\beta^2+2e\xi(1-\beta)(4-\beta(4+5\beta))\big)$ is linear in e, its value at $e=0$ is positive trivially, and its value at $e=\frac{1}{\xi(2-\beta)}$ is $\frac{16(1+\beta)}{2-\beta}-\beta(22+3\beta)$, which can be easily shown to be positive for all $\beta\in[0,1]$. Therefore, $\big(4(1-\beta)+7\beta^2+2e\xi(1-\beta)(4-\beta(4+5\beta))\big) > 0$ always, implying that $X\left(\frac{\beta\theta}{2};\theta\right) < 0$.

The above signs, taken together, imply that the polynomial has three real roots, one negative and two positive, and also that exactly one of the two positive roots is bigger than $\frac{\beta\theta}{2}$ while the other one is smaller. Note that $r\in\left[0,\frac{\beta\theta(1-\beta)}{2-\beta}\right]$ implies that $\beta\theta > 2r$ (as $\frac{1-\beta}{2-\beta}<\frac{1}{2}$ for $0<\beta<1$). Therefore, we are at most interested in $r\in\left[0,\frac{\beta\theta}{2}\right]$, which, in turn, means that the only root of interest is the smaller of the two positive roots. Let us denote it by $\sigma_{1\mathrm{A}}(\theta)$. Then, $X(r;\theta)$ is positive if $0\le r<\sigma_{1\mathrm{A}}(\theta)$. ■

Proof of Lemma C2

i) It is easy to see that $X_1(e;\theta)$, given by:

$$\begin{aligned}X_1(e;\theta) = &-6r^3 - r^2\theta(4-9\beta+2e\beta\xi(2-3\beta)) + 2r\theta^2(1-\beta)^2(1+e\beta\xi)^2\\ &-\beta\theta^3(1-\beta)(1+e\beta\xi)\left(1-5\beta+7e\beta\xi(1-\beta)+2e^2\beta^2\xi^2(1-\beta)\right),\end{aligned}$$

is a cubic expression in e. Its discriminant can be written as $4\beta^{10}\theta^{12}\xi^6(1-\beta)^2Y\left(\frac{r}{\beta\theta},\beta\right)$, where the function $Y(\cdot,\cdot)$ is given by:

$$\begin{aligned}Y(z,\beta) = {}&4(1-\beta)^3(41-9\beta) - 48z(1-\beta)^4 + z^2(1-\beta)^2(16-\beta(488-\beta(1177-27\beta(22-3\beta))))\\ &+12z^3(1-\beta)^3\beta(8-3\beta(37-9\beta)) - 4z^4(1-\beta)\beta(8-\beta(349-\beta(977-3\beta(283-63\beta))))\\ &-48z^5(2-\beta)(1-\beta)^2\beta^2(3-14\beta) + 16z^6(2-\beta)^2\beta^2(1-12(1-\beta)\beta).\end{aligned}$$

Since $r\in\left[0,\frac{\beta\theta(1-\beta)}{2-\beta}\right]$, $\beta\theta>2r$ as well (as $\frac{1-\beta}{2-\beta}<\frac{1}{2}$ for $0<\beta<1$). Therefore, we are only interested in values of z below $\frac{1}{2}$. We will first show that $Y>0$ for $0<\beta<1$ and $0<z<\frac{1}{2}$. To do so, we differentiate Y with respect to β multiple times to find that $\frac{\partial^6 Y}{\partial\beta^6}>0$. Therefore, $\frac{\partial^5 Y}{\partial\beta^5}$ is an increasing function of β and is maximized at $\beta=1$. Since even this maximum value is negative, we conclude that $\frac{\partial^5 Y}{\partial\beta^5}<0$, implying that $\frac{\partial^4 Y}{\partial\beta^4}$ is a decreasing function of β. It is, therefore, minimized at $\beta=1$, and this minimum value is found to be positive. Therefore, $\frac{\partial^4 Y}{\partial\beta^4}>0$, and $\frac{\partial^3 Y}{\partial\beta^3}$ is an increasing function of β. Its maximum value at $\beta=1$ turns out to be negative, indicating that $\frac{\partial^3 Y}{\partial\beta^3}<0$ and $\frac{\partial^2 Y}{\partial\beta^2}$ is a decreasing function of β. Continuing this alternating pattern, the minimum of $\frac{\partial^2 Y}{\partial\beta^2}$ occurring at $\beta=1$ is positive, making $\frac{\partial^2 Y}{\partial\beta^2}>0$ and $\frac{\partial Y}{\partial\beta}$ an increasing function of β. Therefore, $\frac{\partial Y}{\partial\beta}$ is maximized at $\beta=1$, and this maximum value happens to be negative. This implies that the original function, Y, is decreasing in

β. Since its minimum value, occurring at $\beta = 1$, is positive, Y must be positive over the entire valid range.

This, of course, means that the discriminant itself is positive and $X_1 = 0$ must have three real roots. We now investigate the nature of these three roots. Solving $\frac{\partial X_1}{\partial e} = 0$, and checking the second order condition, we get the two critical points of X_1:

$$e_{\min} = \frac{-U - Q}{6\beta^2\theta\xi(1-\beta)}, \text{ and } e_{\max} = \frac{-U + Q}{6\beta^2\theta\xi(1-\beta)},$$

where $Q = \sqrt{4r^2(1-2\beta(4-5\beta)) - 12r\beta\theta(1-\beta)^2 + 3\beta^2\theta^2(1-\beta)(11-3\beta)} > 0$ and $U = (1-\beta)(9\beta\theta - 2r) > 0$. Clearly, $e_{\min}$ is negative, making the first (smallest) of the three real roots negative and of little interest. It turns out that the second root is negative as well. To see this, we consider two separate cases. First, if $0 < \beta \leq \frac{1}{5}$, it can be shown that $e_{\max}$ is negative as well, implying that the second root is negative. This is because $U > Q$:

$$U^2 - Q^2 = 12\beta\left(r^2(2-3\beta) + 2r\beta\theta(1-\beta) + \theta(1-\beta)(\beta\theta-2r) + \frac{9\beta\theta^2}{5}(1-\beta) + 6\beta\theta^2(1-\beta)\left(\frac{1}{5}-\beta\right)\right) > 0.$$

If, on the other hand, $\frac{1}{5} \leq \beta < 1$, then we find:

$$\begin{aligned}
X_1(0;\theta) &= -6r^3 - r^2\theta(4-9\beta) + 2r\theta^2(1-\beta)^2 + \beta\theta^3(5\beta-1)(1-\beta)\\
&> -3r^2\beta\theta - r^2\theta(4-9\beta) + 2r\theta^2(1-\beta)^2 + \beta\theta^3(5\beta-1)(1-\beta) \quad [\text{since } \beta\theta > 2r]\\
&= -4r^2\theta(1-\beta) + 2r^2\beta\theta + 2r\theta^2(1-\beta)^2 + \beta\theta^3(5\beta-1)(1-\beta)\\
&> -2r\beta\theta^2(1-\beta) + 2r^2\beta\theta + 2r\theta^2(1-\beta)^2 + \beta\theta^3(5\beta-1)(1-\beta) \quad [\text{since } \beta\theta > 2r]\\
&= 2r\theta^2(1-\beta)(1-2\beta) + 2r^2\beta\theta + \beta\theta^3(5\beta-1)(1-\beta)\\
&> 2r\theta^2(1-\beta)(1-2\beta) + 2r^2\beta\theta + 2r\theta^2(5\beta-1)(1-\beta) \quad [\text{since } \beta\theta > 2r]\\
&= 2r\theta^2(1-\beta)(3\beta) + 2r^2\beta\theta > 0.
\end{aligned}$$

Therefore, $X_1(0;\theta) > 0$, implying, once again, that the second root is negative. Hence, only the largest root, denoted $e = \gamma_{1\text{A}}(\theta)$, is of interest to us, and it is obvious that X_1 can be positive only to the left of this root, as X_1 approaches $-\infty$ when e is very large.

ii) Note that:

$$\begin{aligned}
X_2(r;\theta) = {} & 2r^3\beta - 3r^2\theta(1-\beta)(1+e\beta\xi(1-\beta)) - 2r\theta^2(1-\beta)^2\left(1+\beta+e\beta^2\xi(1+e\xi)\right)\\
& + \theta^3(1-\beta)^3(1+e\beta\xi)^2(1+e\beta\xi(1-\beta))
\end{aligned}$$

is a cubic polynomial in r. Further, its discriminant is:

$$\begin{aligned}
& 16\theta^6(1-\beta)^6\left(\left(9+22\beta+21\beta^2+12\beta^3+4\beta^4\right) + e\xi\left(45\beta+36\beta^2+12\beta^3+6\beta^4+12\beta^5\right) + \right.\\
& \quad e^2\xi^2\left(108\beta^2-15\beta^3+15\beta^4-12\beta^5+21\beta^6\right) + e^3\xi^3\left(144\beta^3-153\beta^4+69\beta^5-12\beta^6+22\beta^7\right) +\\
& \quad \left. e^4\xi^4\left(108\beta^4-159\beta^5+102\beta^6-15\beta^7+9\beta^8\right) + e^5\xi^5\left(45\beta^5-81\beta^6+57\beta^7-9\beta^8\right) + e^6\xi^6\left(9\beta^6-14\beta^7+9\beta^8\right)\right).
\end{aligned}$$

When the discriminant is expressed this way as a polynomial of e, all the coefficients of that polynomial turn out to be trivially positive if $\beta \in (0,1)$. Since $e > 0$, the discriminant must be positive, implying that $X_2(r;\theta)$ has three real roots. And, since (i) $X_2(-\infty;\theta) \to -\infty$, (ii) $X_2(0;\theta) > 0$, and (iii) the coefficient of r^2 is negative, $X_2(r;\theta)$ must have exactly one negative and two positive roots. Between these two positive roots occurs the minimum of the function, at $r_{\min} = \frac{\theta(1-\beta)(3(1+e\beta\xi(1-\beta))+Q)}{6\beta}$, where $Q = \sqrt{9+3\beta(4(1+\beta)+e\xi(6-\beta(6-4\beta-e\xi(3-\beta(2-3\beta)))))} > 0$. Now, we check whether $r_{\min}$ is larger than $\frac{\beta\theta(1-\beta)}{2-\beta}$, the maximum permissible value of r in Case 1A. Indeed, it would be larger iff $\frac{3(1+e\beta\xi(1-\beta))+Q}{6\beta} > \frac{\beta}{2-\beta}$ or $(2-\beta)Q > U$, where $U = 6\beta^2 - 3(1+e\beta\xi(1-\beta))(2-\beta)$. If $U \leq 0$, $(2-\beta)Q > U$ holds trivially. On the other hand, if $U > 0$, this is equivalent to:

$$(2-\beta)^2Q^2 > U^2 \quad \Leftrightarrow \quad 4(1-\beta)+\beta(10-2\beta(3+\beta))+e\beta^2\xi(2-\beta)(5-4\beta+e\xi(2-\beta)) > 0.$$

Now, the last inequality obviously holds since $0 < \beta < 1$, immediately implying that $r_{\min}$ and the largest root of $X_2(r;\theta) = 0$ are both larger than $\frac{\beta\theta(1-\beta)}{2-\beta}$. The largest root, therefore, is of no interest to us. So, the only root of interest is the smaller of the two positive roots, which we denote by $\tau_{1\mathrm{A}}(\theta)$. Thus, $X_2(r;\theta)$ is positive if $r < \tau_{1\mathrm{A}}(\theta)$, and it is negative otherwise. ■