Online Appendix

O.A Relationship Other Measures of Codependence: Additional Results

O.A.1 Proofs

Proof of Proposition A1. Consider Taylor expansion series of $e^{s\log Z}$ around $s=0$:

$$k(s) \;=\; \log\left(E\left(e^{s\log Z}\right)\right)=\sum_{j=1}^{\infty}\frac{1}{j!}\kappa_{jt}s^{j}$$

where κ_{1t} is the mean of $\log Z$, κ_{2t} is the variance of $\log Z$, κ_{3t} is the skewness of $\log Z$, and κ_{4t} is the excess kurtosis of $\log z$. Thus, using the definition $Z=X/Y$, we show that

$$\begin{aligned}
L[X/Y] \;=\;& \frac{1}{2!}\left(Cov\left((x-Ex),(x-Ex)\right)-Cov\left((x-Ex),(y-Ey)\right)\right)\\
&-Cov\left((y-Ey),(x-Ex)-(y-Ey)\right)+\frac{1}{3!}\frac{Cov\left((x-Ex)^{2},x-Ex\right)}{\left(Var\left(x-y\right)\right)^{\frac{3}{2}}}\\
&+\frac{1}{3!}\frac{Cov\left((y-Ey)^{2},x-Ex\right)}{\left(Var\left(x-y\right)\right)^{\frac{3}{2}}}-\frac{2}{3!}\frac{Cov\left((x-Ex)(y-Ey),x-Ex\right)}{\left(Var\left(x-y\right)\right)^{\frac{3}{2}}}\\
&-\frac{1}{3!}\frac{Cov\left((x-Ex)^{2},y-Ey\right)}{\left(Var\left(x-y\right)\right)^{\frac{3}{2}}}-\frac{1}{3!}\frac{Cov\left((y-Ey)^{2},y-Ey\right)}{\left(Var\left(x-y\right)\right)^{\frac{3}{2}}}\\
&+\frac{2}{3!}\frac{Cov\left((x-Ex)(y-Ey),y-Ey\right)}{\left(Var\left(x-y\right)\right)^{\frac{3}{2}}}+\frac{1}{4!}\frac{Cov\left(((x-Ex))^{3},(x-Ex)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}\\
&-\frac{1}{4!}\frac{Cov\left((y-Ey)^{3},(x-Ex)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}-\frac{3}{4!}\frac{Cov\left((x-Ex)^{2}(y-Ey),(x-Ex)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}\\
&+\frac{3}{4!}\frac{Cov\left((x-Ex)(y-Ey)^{2},(x-Ex)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}+\frac{1}{4!}\frac{Cov\left(((y-Ey))^{3},(y-Ey)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}\\
&+\frac{3}{4!}\frac{Cov\left((x-Ex)^{2}(y-Ey),(y-Ey)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}-\frac{3}{4!}\frac{Cov\left((x-Ex)(y-Ey)^{2},(y-Ey)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}\\
&-\frac{1}{4!}\frac{Cov\left((x-Ex)^{3},(y-Ey)\right)}{\left(Var\left(x-y\right)\right)^{\frac{4}{2}}}-\frac{3}{4!}+\ldots
\end{aligned}\tag{O.A1}$$

Define:

$$
\begin{aligned}
\lambda^0_{x,y} &= \left[1-\frac{1}{2}\frac{Var(x)+Var(y)}{L[X]+L[Y]}\right], \quad \lambda^1_{x,y}=\frac{\sqrt{Var\left(x\right)}\sqrt{Var\left(y\right)}}{L\left[X\right]+L\left[Y\right]}, \\
\omega_{x^i,y^j} &= \frac{\left(\sqrt{Var\left(x\right)}\right)^i\left(\sqrt{Var\left(y\right)}\right)^j}{\left(\sqrt{Var\left(x-y\right)}\right)^{(i+j)}\left(L\left[X\right]+L\left[Y\right]\right)}, \quad \rho_{x,y}=\frac{Cov\left(x,y\right)}{\sqrt{Var\left(x\right)}\sqrt{Var\left(y\right)}},
\end{aligned}
$$

which, combined with (O.A1) allows us to obtain:

$$
\begin{aligned}
\varrho_{X,Y} &= 1-\frac{L\left[X/Y\right]}{L\left[X\right]+L\left[Y\right]} \\
&\approx \lambda^0_{x,y}+\lambda^1_{x,y}\cdot\rho_{x,y}-\frac{1}{3!}\left[\omega_{x^3}\cdot S\left(x,x,x\right)-\omega_{y^3}\cdot S\left(y,y,y\right)+3\cdot\omega_{x,y^2}\cdot S\left(y,y,x\right)\right. \\
&\quad \left.-3\cdot\omega_{x,y^2}\cdot S\left(x,x,y\right)\right]-\frac{1}{4!}\left[\omega_{x^4}\cdot K\left(x,x,x,x\right)+\omega_{y^4}\cdot K\left(y,y,y,y\right)\right. \\
&\quad \left.-4\cdot\omega_{x^3,y}\cdot K\left(x,x,x,y\right)-4\omega_{x,y^3}\cdot K\left(y,y,y,x\right)+6\cdot\omega_{x^2,y^2}\cdot K\left(x,x,y,y\right)-3\right],
\end{aligned}
$$

□

O.A.2 Comparison of Entropy Correlation Measures

In this section we compare our entropy-based correlation with an alternative based on the measure proposed by Backus et al. (2016) (henceforth BBC). BBC define the coentropy of two random variables X and Y as $L[XY]-L[X]-L[Y]$. In order to make all correlation measures comparable with each other, we divide BBC's coentropy measure by the sum of the two random variables' entropies, $L[X]+L[Y]$. This is the same rescaling that we apply to the entropy-based correlation matrix that we propose in this paper. For completeness, we also report the Pearson correlation for all the cases that we analyze.

We analyze the behavior of the various correlation measures under four alternative forms of codependence. To highlight the differences of the distributions, we report their contours in figure O.A1. The first case (top-left corner) is the one in which the logarithms of the random variables are jointly normally distributed. The other three cases correspond to situations in which there is tail codependence (and thus a departure from log-normality). Specifically, the top-right panel represents a case of symmetric tail codependence; the bottom-left panel represents a case of left-tail codependence; and the bottom-right panel presents a case of right tail codependence. The distributions of the random variables that display tail codependence were generated as a mixture of normal random variables, following the same methodology described in the examples in section 2 of the main text.

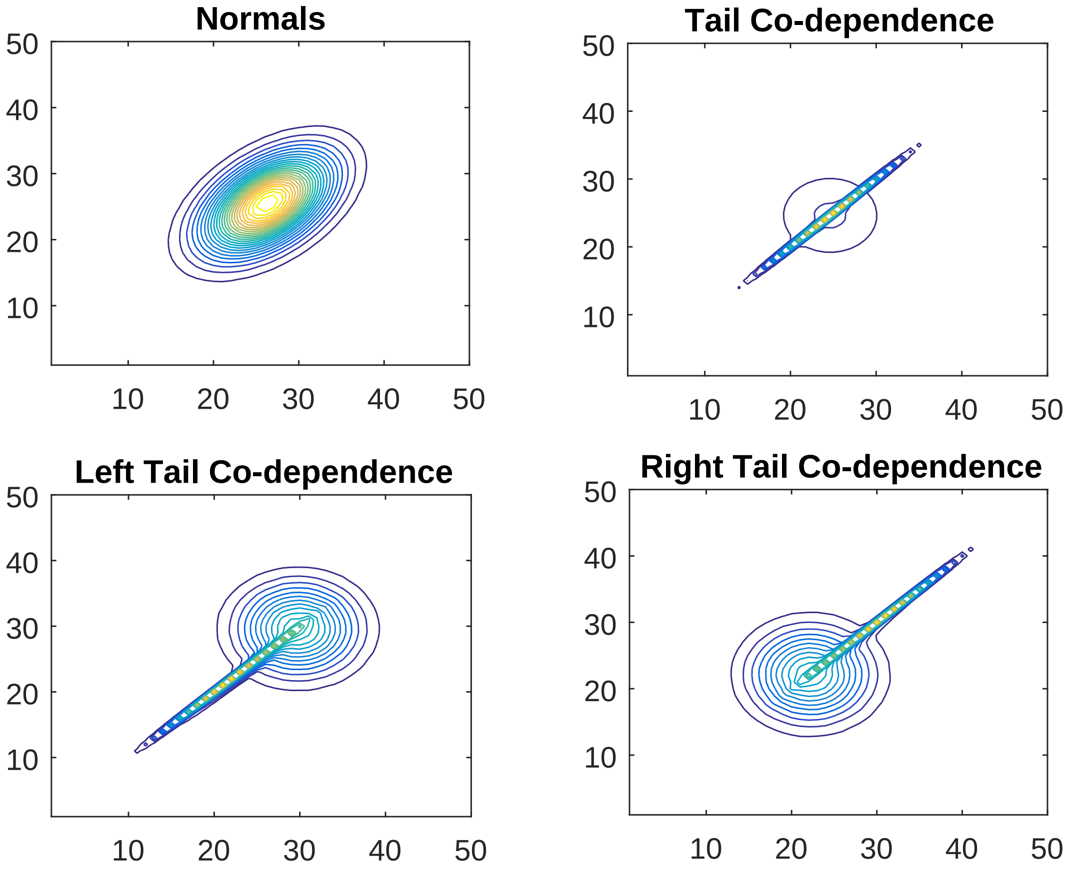

FIG. O.A1. Each panel shows the contours of the joint probability density functions of two random variables. The plot in the panel in the top-left corner ("Normals") refers to the case in which the logarithm of the two random variables are distributed according to a joint normal. The plot in the top-right corner ("Tail Co-dependence") refers to the case in which the two random variables are distributed according to a mixture of two bivariate standard normals: for one of the distributions the Pearson correlation is equal to zero, while for the other it is equal to one. The plot in the bottom-left corner ("Left Tail Co-dependence") is identical to the one in the top-right corner, except that the perfectly correlated bivariate normals have mean of -2. Similarly, the plot in the bottom-right corner ("Right Tail Co-dependence") is identical to the one in the top-right corner, but with the means of the perfectly correlated normals in the mixture set to $+2$. In all the reported cases the Pearson correlation of the logarithm of the two random variables is equal to 0.5.

The results of our comparison of correlations for the various distributions are reported in table O.A1. Case 1 confirms that all the entropy-based correlation measures are exactly identical to the Pearson correlation of the two variables in log-units. This is consistent with our findings in section 2 of the main text. For the symmetric tail dependence case (second panel of table O.A1), our measure of entropy correlation between X and Y (third column) is generally more conservative compared to the BBC entropy-based correlation matrix. In absolute value, our coentropy measure is very similar to BBC's correlation between X and $1/Y$. In general, it seems to be the case that all correlation measures have some desirable properties, including the monotonicity in the extent of codependence.

The situation changes when we consider the cases of one-sided tail dependence. For

Table O.A1: Comparison of Correlations

Type of correlation	**Pearson**		**Chabi-Yo and Colacito**			**Backus et al.**	
Variables	(X,Y)	(X,Y)	(X,Y)	(Y,X)	(X,1/Y)	(X,Y)	(X,1/Y)
Units	logs	levels	levels	levels	levels	levels	levels
Case 1: Normals							
	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.25	0.17	0.25	0.25	-0.25	0.25	-0.25
	0.50	0.38	0.50	0.50	-0.50	0.50	-0.50
	0.75	0.65	0.75	0.75	-0.75	0.75	-0.75
	1.00	1.00	1.00	1.00	-1.00	1.00	-1.00
Case 2: Tail codependence							
	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.25	0.24	0.17	0.17	-0.35	0.35	-0.17
	0.50	0.49	0.38	0.38	-0.61	0.61	-0.38
	0.75	0.74	0.64	0.64	-0.82	0.82	-0.64
	1.00	1.00	1.00	1.00	-1.00	1.00	-1.00
Case 3: Left-Tail Codependence							
	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.25	0.03	0.16	0.16	0.07	0.05	-0.26
	0.50	0.08	0.38	0.38	0.03	0.11	-0.46
	0.75	0.19	0.67	0.67	-0.27	0.26	-0.67
	1.00	1.00	1.00	1.00	-1.00	1.00	-1.00
Case 4: Right-Tail Codependence							
	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.25	0.84	0.33	0.33	-1.39	1.15	-0.26
	0.50	0.95	0.52	0.52	-1.24	0.97	-0.46
	0.75	0.99	0.67	0.67	-0.77	0.77	-0.67
	1.00	1.00	1.00	1.00	-1.00	1.00	-1.00

Notes. This table reports the correlations computed for the four different types of codependence depicted in figure O.A1. The first two columns refer to the Pearson correlation of the two random variables (in logarithmic and level units); the next three columns refer to the entropy-based correlation presented in this paper; and the last two columns refer to entropy correlation based on the coentropy measure proposed by Backus et al. (2016). All correlations are computed from simulations of 1,000,000 observations.

the case of left-tail codependence, both the correlation measure proposed in this paper (third column) and BBC's correlation (sixth column) are increasing with the Pearson correlation of the random variables. However, we note a marked degree of nonlinearity for the BBC correlation: this correlation increases slowly for Pearson correlation values up to 0.75 and then converges quickly to 1, as the variables become perfectly correlated. The measure of entropy correlation proposed in this paper, in contrast, displays a smooth rate of increase, while still providing a conservative assessment of the degree of codependence of the two random variables.

For the case of right-tail codependence, we point out that BBC's entropy correlation displays a nonmonotonic pattern with respect to Pearson correlation: it increases sharply for a small degree of codependence, reaching correlation values in excess of 1; it decreases for intermediate values of correlation; and ultimately it increases again, as the extent of the two random variables' codependence approaches 1. The entropy correlation measure proposed in this paper does not display this pattern, thus making it a more desirable measure to quantitatively assessing the bounds on the codependence of international SDFs, which is the main objective of our analysis.

Additionally, notice that in all our simulations, the coentropy measure that we propose is exactly identical regardless of whether we compute the correlation between X and Y or the correlation between Y and X.

O.B Additional Empirical Results

In this section we provide a robustness exercise for the empirical results reported in section 4 of the main text. Specifically, table B1 documents that excluding Australia and New Zealand, i.e., two countries that are commonly featured in the long side of carry-trade strategies, does not alter our main findings. As noted in the main text, Australia and New Zealand do not provide monthly inflation data, which we therefore interpolated from quarterly observations. Evidently, the way in which we construct monthly inflation series for these two countries is not a main driver of our findings.

In table B2 we further restrict our cross section by focusing on the subset of G-10 countries that belong to the G-7, namely Canada, Germany, Japan, the UK, and the US. The results are virtually identical to those reported in main text.

We report the results for the bilateral case of the US and the UK in table B3. These two countries have a long history of bilateral trade and feature highly integrated financial markets, making them a recurrent object of analysis in the international finance literature (see Colacito and Croce (2011), among others). The results in table B3 document that our main conclusions also apply to this specific country pair.

Lastly, table B4 documents that the same upward-sloping pattern of the codependence of the transitory components is also present when the Pearson correlation measure is used. This includes the ability to reject the null hypothesis that the slope of the term structure of the correlation of the transitory components is less than or equal to zero. This is an interesting result for at least two reasons. First, it speaks to the robustness of our empirical findings. Second, it can be interpreted as supporting the idea that departures from normality play a relatively smaller role when it comes

TABLE B1: Coentropy Indices (Excluding Australia and New Zealand)

	Horizon				
	1	**3**	**12**	**48**	**Slope**
Panel A: Co-entropy of the Total SDF					
ϱ_{M,M^*}	0.96 [0.91, 0.98]	0.94 [0.81, 0.97]	0.93 [0.82, 0.96]	0.93 [0.84, 0.97]	−0.02 [0.76]
Panel B: Co-entropy Decomposition					
ϱ_{M^{*P},M^P}	0.93 [0.8, 0.97]	0.91 [0.7, 0.97]	0.90 [0.69, 0.96]	0.91 [0.73, 0.96]	−0.02 [0.65]
$\varrho_{M^T,M^{*T}}$	0.31 [0.22, 0.41]	0.44 [0.29, 0.58]	0.58 [0.44, 0.71]	0.66 [0.38, 0.85]	0.35 [0.05]
$\varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	0.10 [0.02, 0.19]	0.05 [−0.12, 0.22]	0.00 [−0.22, 0.21]	−0.02 [−0.35, 0.32]	−0.13 [0.72]
Panel C: Hypothesis Testing					
$H_0 : \varrho_{M,M^*} \leq \varrho_{M^P,M^{*P}}$	[0.23]	[0.35]	[0.31]	[0.32]	
$H_0 : \varrho_{M,M^*} \leq \varrho_{M^T,M^{*T}}$	[0.00]	[0.00]	[0.01]	[0.03]	
$H_0 : \varrho_{M,M^*} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.00]	[0.00]	[0.00]	[0.00]	
$H_0 : \varrho_{M^P,M^{*P}} \leq \varrho_{M^T,M^{*T}}$	[0.00]	[0.01]	[0.03]	[0.07]	
$H_0 : \varrho_{M^P,M^{*P}} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.00]	[0.00]	[0.00]	[0.00]	
$H_0 : \varrho_{M^T,M^{*T}} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.00]	[0.00]	[0.00]	[0.01]	

Notes. Panel A reports the estimated median coentropy indices of the total SDFs at horizons of 1, 3, 12, and 48 months. Panel B reports the estimated decomposition of the median coentropy indices of the total SDFs reported in equation (5) at horizons of 1, 3, 12, and 48 months. The first row in the panel refers to the coentropy of the permanent components; the second row refers to the coentropy of the transitory components; the third row refers to the coentropy of the ratio of the permanent and transitory components. For panels A and B, the numbers in square brackets denote the 95 percent confidence intervals obtained by bootstrapping 10,000 independent samples. For panels A and B, the column "Slope" reports the difference between the coentropy index at the 48-month horizon and the corresponding coentropy index at the 1-month horizon, and the number in square brackets is the p-value for the null hypothesis that the slope is less than or equal to zero. Panel C shows the p-values for the null hypotheses reported in the first column at horizons of 1, 3, 12, and 48 months.

to the dynamics of the transitory components. This is consistent with the theoretical model that we propose, in which the transitory components are entirely driven by the normally distributed predictive components of consumption growth.

TABLE B2: Coentropy Indices (G-10 Countries Belonging to the G-7 Group)

	Horizon				
	1	**3**	**12**	**48**	**Slope**
Panel A: Coentropy of the Total SDFs					
ϱ_{M,M^*}	0.96 $[0.91, 0.98]$	0.94 $[0.82, 0.98]$	0.93 $[0.82, 0.97]$	0.93 $[0.81, 0.99]$	-0.03 $[0.77]$
Panel B: Coentropy Decomposition					
ϱ_{M^{*P},M^P}	0.93 $[0.78, 0.98]$	0.92 $[0.72, 1]$	0.89 $[0.65, 0.97]$	0.90 $[0.67, 0.97]$	-0.04 $[0.68]$
$\varrho_{M^T,M^{*T}}$	0.34 $[0.23, 0.45]$	0.50 $[0.33, 0.64]$	0.69 $[0.57, 0.8]$	0.78 $[0.58, 0.93]$	0.44 $[0.01]$
$\varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	0.13 $[0.01, 0.24]$	0.08 $[-0.16, 0.29]$	0.05 $[-0.24, 0.32]$	0.04 $[-0.42, 0.46]$	-0.09 $[0.62]$
Panel C: Hypothesis Testing					
$H_0 : \varrho_{M,M^*} \leq \varrho_{M^P,M^{*P}}$	[0.24]	[0.35]	[0.30]	[0.33]	
$H_0 : \varrho_{M,M^*} \leq \varrho_{M^T,M^{*T}}$	[0.00]	[0.02]	[0.04]	[0.11]	
$H_0 : \varrho_{M,M^*} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.00]	[0.00]	[0.00]	[0.00]	
$H_0 : \varrho_{M^P,M^{*P}} \leq \varrho_{M^T,M^{*T}}$	[0.00]	[0.04]	[0.11]	[0.21]	
$H_0 : \varrho_{M^P,M^{*P}} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.00]	[0.00]	[0.00]	[0.01]	
$H_0 : \varrho_{M^T,M^{*T}} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.01]	[0.01]	[0.00]	[0.01]	

Notes. Panel A reports the estimated median coentropy indices of the total SDFs at horizons of 1, 3, 12, and 48 months. Panel B reports the estimated decomposition of the median coentropy indices of the total SDFs reported in equation (5) at horizons of 1, 3, 12, and 48 months. The first row in the panel refers to the coentropy of the permanent components; the second row refers to the coentropy of the transitory components; the third row refers to the coentropy of the ratio of the permanent and transitory components. For panels A and B, the numbers in square brackets denote the 95 percent confidence intervals obtained by bootstrapping 10,000 independent samples. For panels A and B, the column "Slope" reports the difference between the coentropy index at the 48-month horizon and the corresponding coentropy index at the 1-month horizon, and the number in square brackets is the p-value for the null hypothesis that the slope is less than or equal to zero. Panel C shows the p-values for the null hypotheses reported in the first column at horizons of 1, 3, 12, and 48 months.

TABLE B3: Coentropy Indices (US vs. UK)

	Horizon				
	1	**3**	**12**	**48**	**Slope**
Panel A: Coentropy of the Total SDFs					
ϱ_{M,M^*}	0.96 $[0.93, 0.98]$	0.95 $[0.87, 1]$	0.93 $[0.87, 0.97]$	0.95 $[0.87, 0.99]$	-0.02 $[0.69]$
Panel B: Coentropy Decomposition					
ϱ_{M^{*P},M^P}	0.94 $[0.86, 0.97]$	0.93 $[0.77, 1]$	0.90 $[0.73, 0.96]$	0.91 $[0.76, 1]$	-0.04 $[0.69]$
$\varrho_{M^T,M^{*T}}$	0.28 $[0.05, 0.49]$	0.51 $[0.12, 0.79]$	0.76 $[0.58, 0.9]$	0.86 $[0.72, 0.99]$	0.58 $[0]$
$\varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	0.17 $[-0.09, 0.39]$	0.17 $[-0.33, 0.58]$	0.16 $[-0.36, 0.59]$	0.24 $[-1, 0.71]$	0.07 $[0.46]$
Panel C: Hypothesis Testing					
$H_0 : \varrho_{M,M^*} \leq \varrho_{M^P,M^{*P}}$	[0.23]	[0.36]	[0.26]	[0.30]	
$H_0 : \varrho_{M,M^*} \leq \varrho_{M^T,M^{*T}}$	[0.00]	[0.02]	[0.03]	[0.17]	
$H_0 : \varrho_{M,M^*} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.00]	[0.01]	[0.00]	[0.00]	
$H_0 : \varrho_{M^P,M^{*P}} \leq \varrho_{M^T,M^{*T}}$	[0.01]	[0.04]	[0.13]	[0.35]	
$H_0 : \varrho_{M^P,M^{*P}} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.01]	[0.02]	[0.02]	[0.05]	
$H_0 : \varrho_{M^T,M^{*T}} \leq \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$	[0.29]	[0.18]	[0.01]	[0.03]	

Notes. Panel A reports the estimated median coentropy indices of the total SDFs at horizons of 1, 3, 12, and 48 months. Panel B reports the estimated decomposition of the median coentropy indices of the total SDFs reported in equation (5) at horizons of 1, 3, 12, and 48 months. The first row in the panel refers to the coentropy of the permanent components; the second row refers to the coentropy of the transitory components; the third row refers to the coentropy of the ratio of the permanent and transitory components. For panels A and B, the numbers in square brackets denote the 95 percent confidence intervals obtained by bootstrapping 10,000 independent samples. For panels A and B, the column "Slope" reports the difference between the coentropy index at the 48-month horizon and the corresponding coentropy index at the 1-month horizon, and the number in square brackets is the p-value for the null hypothesis that the slope is less than or equal to zero. Panel C shows the p-values for the null hypotheses reported in the first column at horizons of 1, 3, 12, and 48 months.

TABLE B4: Pearson Correlations of the Transitory Components

	Horizon				
	1	**3**	**12**	**48**	**Slope**
$\varrho_{M^T, M^{*T}}$	0.31 [0.20, 0.41]	0.42 [0.26, 0.57]	0.58 [0.43, 0.74]	0.68 [0.41, 0.92]	0.37 [0.03]

Notes. This table reports the estimated Pearson correlations of the transitory components of the SDFs at horizons of 1, 3, 12, and 48 months. The numbers in square brackets denote the 95 percent confidence intervals obtained by bootstrapping 10,000 independent samples. The column "Slope" reports the difference between the correlation at the 48-month horizon and the corresponding coentropy index at the 1-month horizon, and the number in square brackets is the p-value for the null hypothesis that the slope is less than or equal to zero.