Online Appendix for "Asset Pricing Implications of Short-sale Constraints in Imperfectly Competitive Markets"

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Abstract

In this online Appendix, we check the robustness of our results by deriving the equilibrium with reduced information revelation, extending the model to incorporate endogenous information acquisition, extending the model to allow initial endowment of the risky security, liquidity shocks, and risk-aversions to differ across investors, and extending the model to a dynamic setting.

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Model extensions with more heterogeneity and reduced information revelation

In the above model with asymmetric information, all of the informed (i.e., hedgers) have the same information and in equilibrium all submit orders that reveal the composite signal $\hat{S}$. We now extend our model to include multiple informed investors with different private information and their orders might not fully reveal the composite signal $\hat{S}$. We also allow initial endowment of the risky security, liquidity shocks, and risk-aversions to differ across investors.

C.1 Endogenous information acquisition

We next examine whether our results can still hold when aggregate information quality is affected by imposing short-sale constraints. To this extent, we assume that, on date 0, the informed can acquire a costly signal $\hat{s}$ as defined in (29) with precision of $\rho_\varepsilon = \frac{1}{\sigma_\varepsilon^2}$ at a cost of $c(\rho_\varepsilon) := k\rho_\varepsilon^2$, where k is a positive constant.

For a given precision ρ_ε, we solve for the equilibrium prices and quantities as previously. We then solve for the optimal precision with and without short-sale constraints. We find that the optimal precision of private information for the informed in the presence of short-sale constraints tends to be lower than that in the absence of short-sale constraints, as shown in the upper panel of Figure C.1. Intuitively, the presence of short-sale constraints may reduce the incentive of investors to acquire more precise information because short-sale constraints prevent them from fully benefiting from the private information in some states. Interestingly, Figure C.1 illustrates that more public disclosure (i.e., smaller σ_η) might actually increase the incentive of the informed to acquire more precise private information. This is because public disclosure reduces information asymmetry and thus the loss of the informed from the adverse selection problem decreases. Figure C.1 also suggests that the optimal precision increases with liquidity shock volatility. Intuitively, high liquidity shock volatility tends to increase the informed's trading volume and thus make them benefit more from more precise information.

More importantly, the lower panels of Figure C.1 show that short-sale constraints may still

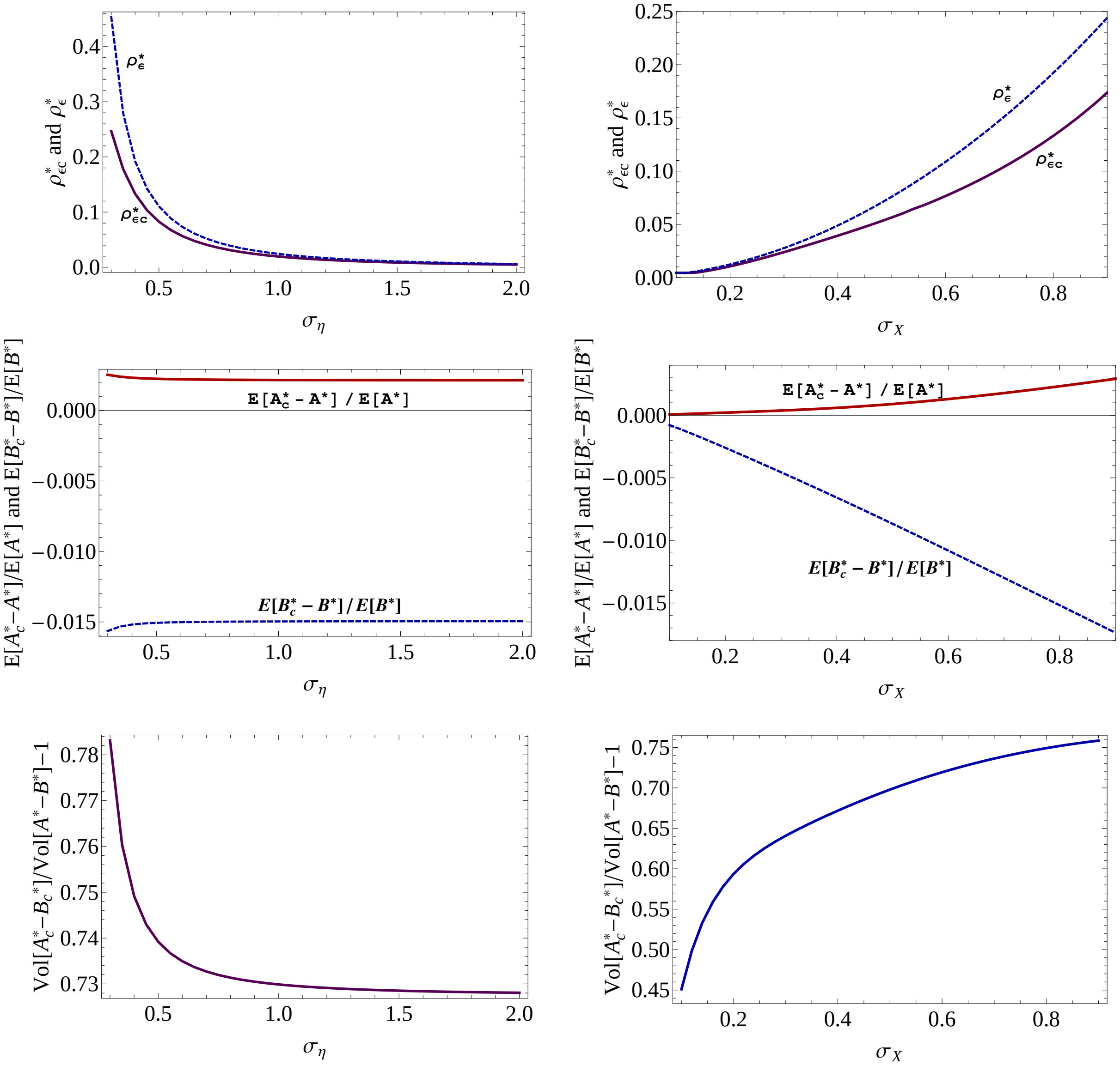

Figure C.1: The optimal precision of information with and without short-sale constraints, and the percentage changes of expected bid, expected ask, and the spread volatility. The default parameters are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.9$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.8$, $N_h = 1$, $N_m = 1$, $N_n = 10$, $\bar{\theta} = 0.01$, $\kappa_h = \kappa_n = 0$, and $k = 0.001$.

decrease the expected bid price and increase the expected ask price, and the spread volatility even with endogenous information acquisition. In addition, as in the case with exogenous information acquisition, as the liquidity shock volatility increases, the impact of short-sale constraints increases, while as the information asymmetry increases, the impact tends to decrease. For a large set of parameter values, we obtain similar patterns to those shown in Figure C.1. This demonstrates that our main results still hold even with endogenous information acquisition.

C.2 Robustness with reduced information revelation

Theorem 2 of our paper implies that when the market-maker can separate bid and ask markets, i.e., charge a positive spread, the only equilibrium is the one in which the market prices fully reveal the composite signal $\hat{S}$. Next, we demonstrate that even when the market-maker cannot charge a positive spread and short-sale constraints reduce information revelation, the average equilibrium sale (bid) price with short-sale constraints may still be lower than that without the constraints. For this purpose, we consider the model where hedgers (i.e., the informed) have zero initial endowment and non-hedgers (i.e., the uninformed) have different amount of initial endowment from that of the market-maker. As shown in Theorem 4 of our paper, the informed's demand increases with the composite signal $\hat{S}$ which combines the hedging-demand and information-motivated demand. When the composite signal $\hat{S}$ is smaller than a threshold $\underline{\mathrm{S}}$, the informed would like to short sell, but a no-short-sale constraint prevents such trading. We assume that the informed do not submit any order in this case, and thus do not reveal the value of $\hat{S}$. The uninformed and the market-maker accordingly update their beliefs, conditional on the composite signal $\hat{S} < \underline{\mathrm{S}}$, and in this sense, information revelation is reduced by the presence of short-sale constraints. We provide the analysis details of this case at the end of this subsection.

Figure C.2 shows that the equilibrium price is a constant for all $\hat{S} < \underline{\mathrm{S}}$ and there is a discontinuous movement downward at $\hat{S} = \underline{\mathrm{S}}$. In other words, when the uninformed and the market-maker only know that $\hat{S} < \underline{\mathrm{S}}$, they use the conditional average of $\hat{S}$ for the estimation of $\hat{S}$, and therefore they underestimate $\hat{S}$ for $\underline{\underline{S}} < \hat{S} < \underline{\mathrm{S}}$, which is reflected by the downward discontinuity at $\underline{\mathrm{S}}$. On the other hand, they overestimate $\hat{S}$ for $\hat{S} < \underline{\underline{S}}$, as shown in Figure C.2.

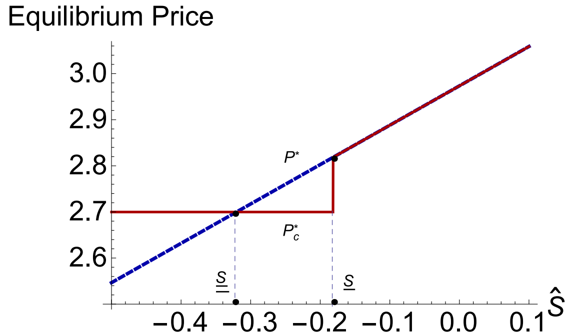

Figure C.2: The unconstrained equilibrium price P^* and the constrained equilibrium price P_c^* against $\hat{S}$. The default parameters are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.4$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.3$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta}_n = 0.1$, $\bar{\theta}_m = 0.6$, and $\kappa_h = \kappa_n = 0$.

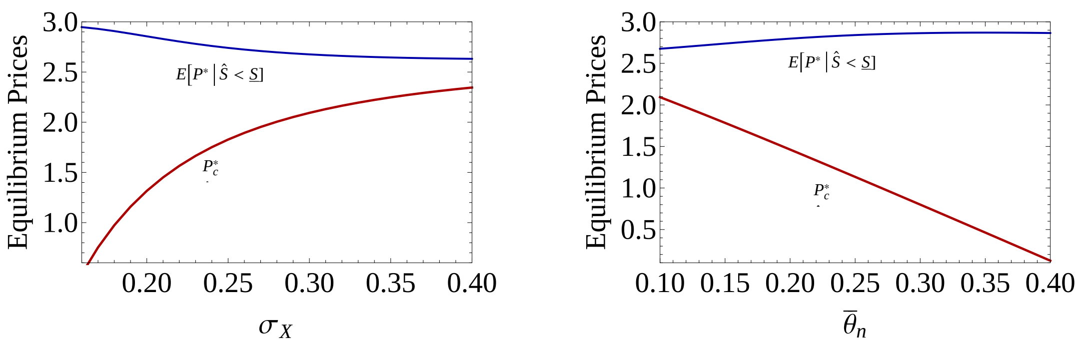

Figure C.3: The expected unconstrained equilibrium price P^* and the constrained equilibrium price P_c^* conditional on $\hat{S} < \underline{S}$ against σ_X and $\bar{\theta}_n$ when the market-maker sells in equilibrium. The default parameters are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.4$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.3$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta}_n = 0.1$, $\bar{\theta}_m = 0.6$, and $\kappa_h = \kappa_n = 0$.

Figure C.3, in which the market-maker sells in equilibrium, and Figure C.4, in which the market-maker buys in equilibrium, imply that even when short-sale constraints prevent some of the information of the informed from being revealed, the expected trading price with short-sale constraints can still be lower than that without the constraints. This result is true for a large set of parameter values. Intuitively, because the uninformed and the market-maker underestimate $\hat{S}$ for $\underline{\underline{S}} < \hat{S} < \underline{\mathrm{S}}$, but overestimate $\hat{S}$ for $\hat{S} < \underline{\underline{S}}$, this translates to a lower equilibrium price for $\underline{\underline{S}} < \hat{S} < \underline{\mathrm{S}}$, but a higher equilibrium price for $\hat{S} < \underline{\underline{S}}$ than the unconstrained equilibrium price, as

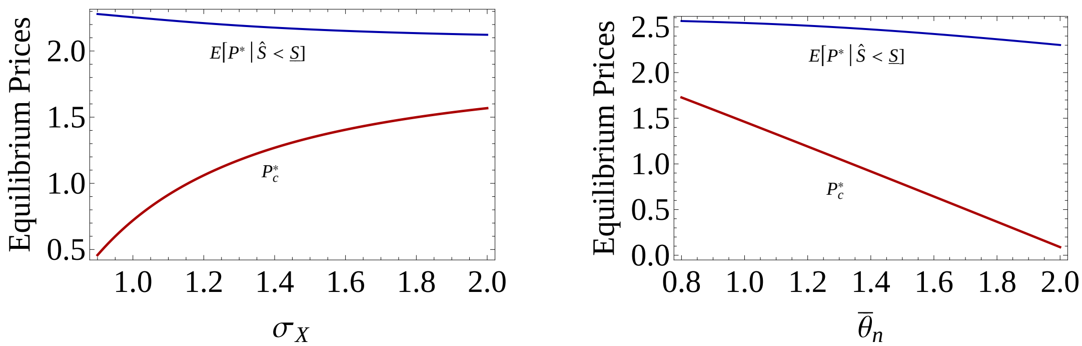

Figure C.4: The expected unconstrained equilibrium price P^* and the constrained equilibrium price P_c^* conditional on $\hat{S} < \underline{S}$ against σ_X and $\bar{\theta}_n$ when the market-maker buys in equilibrium. The default parameters are: $\delta = 1$, $\sigma_u = 0.4$, $\sigma_v = 0.4$, $\sigma_L = 0.9$, $\sigma_{VL} = 0.3$, $\bar{V} = 3$, $\sigma_X = 0.3$, $N_h = 10$, $N_m = 1$, $N_n = 100$, $\bar{\theta}_n = 2$, $\bar{\theta}_m = 0.2$, and $\kappa_h = \kappa_n = 0$.

shown in Figure C.2. Therefore, as long as the probability of $\underline{\underline{S}} < \hat{S} < \underline{\mathrm{S}}$ is significantly higher than the probability of $\hat{S} < \underline{\underline{S}}$, the equilibrium price with short-sale constraints for $\hat{S} < \underline{\mathrm{S}}$ is lower than the expected price without the constraints, conditional on $\hat{S} < \underline{\mathrm{S}}$. Because the equilibrium prices are the same with and without constraints for $\hat{S} \geq \underline{\mathrm{S}}$, the (unconditional) expected equilibrium price may also be lower with short-sale constraints.

Analysis Details for the case with reduced information revelation when hedgers are constrained

We assume that the informed (hedgers) are not endowed with any shares of the stock, the market-maker is endowed with $\bar{\theta}_m$ shares of the stock, and each uninformed trader (nonhedger) is endowed with $\bar{\theta}_n$ shares of the stock. For tractability, we study the case in which the market-maker has to post $A = B = P$. To simplify computations, we also assume that there is no public signal $\hat{S}_s$, *i.e.*, $\sigma_\eta = \infty$.

It can be demonstrated that hedgers are constrained by short-sale constraints and thus are not trading when

$$\hat{S} \leq \underline{S} := -\frac{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_h]\nu \left((N_h\nu + N_n)(\bar{\theta}_m + N_n\bar{\theta}_n) + N_n\bar{\theta}_n\right)}{(1-\rho_n)(N_h\nu(N_n+1) + N_n(N_n+2))}.$$

When the informed are constrained by short-sale constraints, in equilibrium, there are two cases: 1) if $\bar{\theta}_m > \bar{\theta}_n$, the uninformed buy from the market-maker; and 2) if $\bar{\theta}_m < \bar{\theta}_n$, then (i) the uninformed sell, and they are not constrained by short-sale constraints when $\underline{S}_n < \hat{S} < \underline{S}$, (ii) the uninformed buy when $\hat{S} < \underline{S}_n$, where

$$\underline{S}_n = \frac{\delta \text{Var}[\tilde{V}|\mathcal{I}_h] \left((N_h \nu + N_n) \bar{\theta}_m - (N_n + N_h \nu (N_h \nu + N_n + 2)) \bar{\theta}_n \right)}{(1 - \rho_n) N_h (N_h \nu + N_n + 1)}.$$

We present the details of case (1), *i.e.*, $\bar{\theta}_m > \bar{\theta}_n$, the uninformed buy from the market-maker. Case (2) can be solved similarly.

When the informed are constrained by short-sale constraints and they are not endowed with any shares of a risky asset, informed traders are not trading. The market-maker and the uninformed only know that $\hat{S} \leq \underline{S}$, and they cannot observe $\hat{S}$. The uninformed's problem becomes

$$\max_{\theta_n} \; E[-e^{-\delta\left(-\theta_n P + (\bar{\theta}_n + \theta_n)(\tilde{v} + \tilde{u})\right)} | \hat{S} \leq \underline{S}]. \tag{C-1}$$

It can be shown that (C-1) is equivalent to

$$\min_{\theta_n} \; e^{\delta \theta_n P + \frac{1}{2}\delta^2 (\bar{\theta}_n + \theta_n)^2 (\sigma_u^2 + \sigma_v^2) - \delta(\bar{\theta}_n + \theta_n)\bar{V}} \mathbf{N} \left(\frac{\underline{S} + \delta(\bar{\theta}_n + \theta_n)\rho_h \sigma_v^2}{\sqrt{\rho_h \sigma_v^2 + \omega^2 \sigma_X^2}} \right) / \mathbf{N} \left(\frac{\underline{S}}{\sqrt{\rho_h \sigma_v^2 + \omega^2 \sigma_X^2}} \right). \tag{C-2}$$

Taking the first-order condition with respect to θ_n in equation (C-2) yields

$$(P + \delta(\bar{\theta}_n + \theta_n)(\sigma_u^2 + \sigma_v^2) - \bar{V}) \mathbf{N} \left(\frac{\underline{S} + \delta(\bar{\theta}_n + \theta_n)\rho_h \sigma_v^2}{\sqrt{\rho_h \sigma_v^2 + \omega^2 \sigma_X^2}} \right) + \frac{1}{\sqrt{2\pi}} \frac{\rho_h \sigma_v^2}{\sqrt{\rho_h \sigma_v^2 + \omega^2 \sigma_X^2}} e^{-\frac{(\underline{S} + \delta(\bar{\theta}_n + \theta_n)\rho_h \sigma_v^2)^2}{2(\rho_h \sigma_v^2 + \omega^2 \sigma_X^2)}} = 0. \tag{C-3}$$

The market-maker's problem becomes

$$\max_{P} \; E[-e^{-\delta\left(\alpha P + (\bar{\theta}_m - \alpha)(\tilde{v} + \tilde{u})\right)} | \hat{S} \leq \underline{S}]. \tag{C-4}$$

It can be shown that (C-4) is equivalent to

$$\min_{P}\; e^{-\delta P\alpha+\frac{1}{2}\delta^2(\bar{\theta}_m-\alpha)^2(\sigma_u^2+\sigma_v^2)-\delta(\bar{\theta}_m-\alpha)\bar{V}}\mathbf{N}\left(\frac{\underline{S}+\delta(\bar{\theta}_m-\alpha)\rho_h\sigma_v^2}{\sqrt{\rho_h\sigma_v^2+\omega^2\sigma_X^2}}\right)/\mathbf{N}\left(\frac{\underline{S}}{\sqrt{\rho_h\sigma_v^2+\omega^2\sigma_X^2}}\right). \quad \text{(C-5)}$$

Taking the first order condition with respect to P in equation (C-5) yields

$$\left(\alpha+\frac{\partial\alpha}{\partial P}(P+\delta(\bar{\theta}_m-\alpha)(\sigma_u^2+\sigma_v^2)-\bar{V})\right)\mathbf{N}\left(\frac{\underline{S}+\delta(\bar{\theta}_m-\alpha)\rho_h\sigma_v^2}{\sqrt{\rho_h\sigma_v^2+\omega^2\sigma_X^2}}\right)$$
$$+\frac{1}{\sqrt{2\pi}}\frac{\rho_h\sigma_v^2}{\sqrt{\rho_h\sigma_v^2+\omega^2\sigma_X^2}}\frac{\partial\alpha}{\partial P}e^{-\frac{(\underline{S}+\delta(\bar{\theta}_m-\alpha)\rho_h\sigma_v^2)^2}{2(\rho_h\sigma_v^2+\omega^2\sigma_X^2)}}=0. \quad \text{(C-6)}$$

From (C-3), we can express P and $\frac{\partial\theta_n}{\partial P}$ as functions of θ_n. Substituting $\alpha = N_n\theta_n$ and $\frac{\partial\alpha}{\partial P} = N_n\frac{\partial\theta_n}{\partial P}$ into (C-6) yields a function of θ_n which can be solved numerically and then we obtain the equilibrium price. *Q.E.D.*

C.3 Robustness to a dynamic setting

One concern with our main model is that it is static. In this subsection, we demonstrate that it is likely that our main results still hold in a dynamic setting. For this purpose, we consider a two-period setting with trading dates 0, 1, and 2. Hedgers, non-hedgers, and the market-maker can trade a risk-free asset and a risky security on dates 0 and 1 to maximize their CARA utility from the terminal wealth on date 2. The date 2 payoff of each share is $\tilde{V}+\tilde{\mu}$, where $\tilde{V}$ is observable on date 1, and $\tilde{\mu}$ is observable on date 2, $\tilde{V}$ and $\tilde{\mu}$ are independent, $\tilde{V}\sim\mathbf{N}(\bar{V},\sigma_V^2)$, $\tilde{\mu}\sim\mathbf{N}(0,\sigma_\mu^2)$, $\bar{V}$ is a constant, $\sigma_V>0$, $\sigma_\mu>0$, and $\mathbf{N}(\cdot)$ denotes a normal distribution.

As previously, hedgers are subject to a liquidity shock that is modeled as a random endowment of $\hat{X}_h\sim\mathbf{N}(0,\sigma_X^2)$ units of a non-tradable risky asset on date 0. The non-traded asset has a per-unit payoff of $\tilde{L}+\tilde{e}$, where $\tilde{L}\sim\mathbf{N}(0,\sigma_L^2)$ has a covariance of $\sigma_{VL}>0$ with $\tilde{V}$ and becomes public on date 1, $\tilde{e}\sim\mathbf{N}(0,\sigma_e^2)$ has a covariance of $\sigma_{\mu e}>0$ with $\tilde{\mu}$ and becomes public on date 2, and $\tilde{L}$ and $\tilde{e}$ are independent.

Let θ_{it} denote the number of shares that an investor holds in the stock immediately after date t, $t = 0, 1$.[1] We assume that neither hedgers nor non-hedgers can short-sell, i.e.,

$$\theta_{it} \geq 0, \quad i = h, n, \ t = 0, 1. \tag{C-7}$$

Let P_t be the stock price on date t. For $i \in \{h, n\}$, investor i's problem is

$$\max_{\theta_{i0}, \theta_{i1}} E[-e^{-\delta \tilde{W}_{i2}}], \tag{C-8}$$

subject to the budget constraints

$$\tilde{W}_{i2} = \tilde{W}_{i1} - \theta_{i1} P_1 + \theta_{i1}(\tilde{V} + \tilde{\mu}) + \hat{X}_i(\tilde{L} + \tilde{e}), \tag{C-9}$$

and

$$\tilde{W}_{i1} = (\bar{\theta} - \theta_{i0}) P_0 + \theta_{i0} P_1, \tag{C-10}$$

and short-selling constraints (C-7), where $\delta > 0$ is the absolute risk-aversion parameter.

Both hedgers and non-hedgers have to trade through the market-maker, *i.e.*,

$$\theta_{mt} = N\bar{\theta} - (N_h \theta_{ht} + N_n \theta_{nt}), \quad t = 0, 1. \tag{C-11}$$

The market-maker's problem is then

$$\max_{P_0, P_1} E[-e^{-\delta \tilde{W}_{m2}}], \tag{C-12}$$

subject to the budget constraints

$$\tilde{W}_{m2} = \tilde{W}_{m1} - \theta_{m1} P_1 + \theta_{m1}(\tilde{V} + \tilde{\mu}), \tag{C-13}$$

[1] To simplify exposition, we use θ_i to denote the holdings after trading instead of the trading amount as used previously.

and

$$\tilde{W}_{m1} = (\bar{\theta} - \theta_{m0})P_0 + \theta_{m0}P_1. \tag{C-14}$$

Definition 3 *An equilibrium $(\theta^*_{ht}, \theta^*_{nt}, P^*_t, t = 0, 1)$ is such that*

1. *Given P^*_t, θ^*_{it} $(i \in \{h, n\})$ solves investor i's Problem (C-8)–(C-10) for $t = 0, 1$;*

2. *Given θ^*_{ht} and θ^*_{nt}, P^*_t solves the market-maker's Problem (C-12)–(C-14) for $t = 0, 1$ subject to the market-clearing condition (C-11).*

In general, there are 16 possible cases, depending on when the constraints bind for whom (hedgers or non-hedgers). To simplify the exposition and save space, we report the equilibrium results only for the case in which there are no non-hedgers, i.e., $N_n = 0$, because in this case, there are only four possible cases in equilibrium.[2]

Define

$$C_0 := \frac{\sigma_\mu^2 + (N_h+2)^2\sigma_V^2}{(N_h+2)\sigma_{VL}}\bar{\theta} > D_0 := \frac{(N_h+2)\sigma_V^2}{\sigma_{VL}}\bar{\theta}, \tag{C-15}$$

$$C_1 := \frac{(N_h+2)\sigma_\mu^2(\sigma_\mu^2 + (N_h+2)^2\sigma_V^2)}{(N_h+2)^2\sigma_V^2\sigma_{\mu e} + \sigma_\mu^2((N_h+2)\sigma_{VL} + \sigma_{\mu e})}\bar{\theta}, \quad D_1 := \frac{(N_h+1)\sigma_\mu^2}{\sigma_{\mu e}}\bar{\theta}. \tag{C-16}$$

The following theorem provides the equilibrium prices and equilibrium security demand in closed form.[3]

Theorem C.1 *1. If $\hat{X}_h < \min\{C_0, C_1\}$, then short-sale constraints do not bind for hedgers at either time 0 or time 1,*

(a) the equilibrium prices at time 0 and 1 are

$$P_0^* = \bar{V} - \frac{\delta(N_h+1)\left(\sigma_\mu^2\sigma_{\mu e} + (N_h+2)^2\sigma_V^2(\sigma_{VL} + \sigma_{\mu e})\right)\hat{X}_h}{(N_h+2)\left(\sigma_\mu^2 + (N_h+2)^2\sigma_V^2\right)} - \delta(\sigma_\mu^2 + \sigma_V^2)\bar{\theta}, \tag{C-17}$$

[2]The presence of non-hedgers is important for the existence of different bid and ask prices, but not for the determination of equilibrium prices when bid-ask spread is restricted to be zero, as we consider here. Indeed, we also solved the equilibrium with short-sale constraints when $N_n \neq 0$. The qualitative results are the same and available from the authors.

[3]Because the second period problem is essentially the same as the one-period problem except for different initial endowments, and the first period problem is similar to the one-period problem except for different continuation utilities, the proof is straightforward and thus omitted.

$$P_1^* = \tilde{V} + \delta \left(\frac{\sigma_\mu^2 \sigma_{VL}}{\sigma_\mu^2 + (N_h+2)^2 \sigma_V^2} - \frac{(N_h+1)\sigma_{\mu e}}{N_h+2} \right) \hat{X}_h - \delta \sigma_\mu^2 \bar{\theta}, \tag{C-18}$$

(b) and the equilibrium quantities demanded at time 0 and 1 are

$$\theta_{h0}^* = \bar{\theta} \left(1 - \frac{\hat{X}_h}{C_0} \right), \quad \theta_{m0}^* = (N_h+1)\bar{\theta} - N_h \theta_{h0}^*, \tag{C-19}$$

$$\theta_{h1}^* = \bar{\theta} \left(1 - \frac{\hat{X}_h}{C_1} \right), \quad \theta_{m1}^* = (N_h+1)\bar{\theta} - N_h \theta_{h1}^*; \tag{C-20}$$

2. If $C_1 \leq \hat{X}_h < D_0$, then short-sale constraints bind for hedgers only at time 1,

(a) the equilibrium prices at time 0 and 1 are

$$P_{0c1}^* = \bar{V} - \delta \left(\sigma_{\mu e} + \frac{N_h+1}{N_h+2} \sigma_{VL} \right) \hat{X}_h - \delta \sigma_V^2 \bar{\theta}, \quad P_{1c1}^* = \tilde{V} - \delta \sigma_{\mu e} \hat{X}_h, \tag{C-21}$$

(b) and the equilibrium quantities demanded at time 0 and 1 are

$$\theta_{h0c1}^* = \bar{\theta} \left(1 - \frac{\hat{X}_h}{D_0} \right), \quad \theta_{m0c1}^* = (N_h+1)\bar{\theta} - N_h \theta_{h0c1}^*, \quad \theta_{h1c1}^* = 0, \quad \theta_{m1c1} = (N_h+1)\bar{\theta}; \tag{C-22}$$

3. If $C_0 \leq \hat{X}_h < D_1$, then short-sale constraints bind for hedgers only at time 0,

(a) the equilibrium prices at time 0 and 1 are

$$P_{0c2}^* = \bar{V} - \delta \sigma_{VL} \hat{X}_h - \frac{N_h+1}{N_h+2} \delta \sigma_{\mu e} \hat{X}_h - \frac{N_h+1}{N_h+2} \delta \sigma_\mu^2 \bar{\theta}, \tag{C-23}$$

$$P_{1c2}^* = \tilde{V} - \frac{N_h+1}{N_h+2} \delta \sigma_{\mu e} \hat{X}_h - \frac{N_h+1}{N_h+2} \delta \sigma_\mu^2 \bar{\theta}, \tag{C-24}$$

(b) and the equilibrium quantities demanded at time 0 and 1 are

$$\theta_{h0c2}^* = 0, \quad \theta_{m0c2}^* = (N_h+1)\bar{\theta}, \tag{C-25}$$

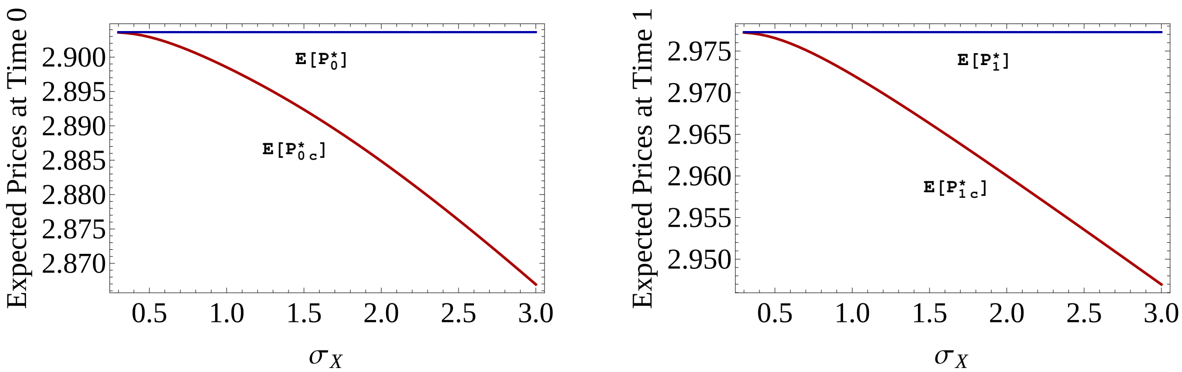

Figure C.5: The expected prices at time 0 and 1 with and without short-sale constraints against σ_X. The parameter values are: $\delta = 1$, $\sigma_V = 0.9$, $\sigma_{VL} = 0.3$, $\sigma_\mu = 0.5, \sigma_{\mu e} = 0.4$, $\bar{V} = 3$, $\tilde{V} = 3$, $N_h = 10$, $\bar{\theta} = 1/(N_h + 1)$.

$$\theta^*_{h1c2} = \frac{N_h + 1}{N_h + 2}\bar{\theta}\left(1 - \frac{\hat{X}_h}{D_1}\right), \quad \theta^*_{m1c2} = (N_h + 1)\bar{\theta} - N_h\theta^*_{h1c2}; \tag{C-26}$$

4. Otherwise, short-sale constraints bind for hedgers at both time 0 and 1,

(a) the equilibrium prices at time 0 and 1 are

$$P^*_{0c3} = \bar{V} - \delta\left(\sigma_{VL} + \sigma_{\mu e}\right)\hat{X}_h, \quad P^*_{1c3} = \tilde{V} - \delta\sigma_{\mu e}\hat{X}_h, \tag{C-27}$$

(b) and the equilibrium quantities demanded at time 0 and 1 are

$$\theta^*_{h0c2} = \theta^*_{h1c2} = 0, \quad \theta^*_{m0c2} = \theta^*_{m1c2} = (N_h + 1)\bar{\theta}. \tag{C-28}$$

Theorem C.1 implies that, as in the main model, short-sale constraints can lower the equilibrium trading price when a buyer has market power. For example, in Case 2, in which hedgers are constrained at time 1 only, one can show that the equilibrium prices at both time 0 and time 1 are lower than those without short-sale constraints respectively. To examine the average impact of short-sale constraints on equilibrium prices, we next plot in Figure C.5 the expected equilibrium prices (across $\hat{X}_h$) at time 0 (left subfigure) and at time 1 (right subfigure) with and without short-sale constraints against the liquidity shock volatility σ_X. Figure C.5 shows that the expected prices at time 0 and time 1 with short-sale constraints are lower than the prices without short-sale constraints, as in our one-period model. These results hold because the market power of the market-maker constitutes the main driving force, which is still present in a dynamic setting.

C.4 A generalized model

To simplify exposition, in the main model studied above we assume that all investors have the same risk aversion, the same initial inventory, the same date 1 resale value of the security, and only hedgers have private information and liquidity shocks. In this section, we relax these assumptions and still, the generalized model is tractable and solved in closed-form.

Let $\bar{\theta}_i$, δ_i, $\hat{X}_i$, $\tilde{V}_i$ and $\mathcal{I}_i$ denote respectively the initial inventory, the risk-aversion coefficient, the liquidity shock, the date 1 resale value of the security, and the information set for a type i investor for $i \in \{h, n, m\}$. Then by the same argument as before, a type i investor's reservation price can be written as

$$P_i^R = E[\tilde{V}_i|\mathcal{I}_i] - \delta_i \text{Cov}[\tilde{V}_i, \tilde{L}|\mathcal{I}_i]\hat{X}_i - \delta_i \text{Var}[\tilde{V}_i|\mathcal{I}_i]\bar{\theta}_i, \quad i \in \{h, n, m\}. \tag{C-29}$$

Let $\Delta_{ij} := P_i^R - P_j^R$ denote the reservation price difference between type i and type j investors for $i, j \in \{h, n, m\}$. In this generalized model, there are eight cases corresponding to eight different trading direction combinations of hedgers and non-hedgers.[4] The trading directions are determined by the ratio of the reservation price difference between the hedgers and the non-hedgers (Δ_{hn}) to the reservation price difference between the non-hedgers and the market maker (Δ_{nm}). When the magnitude of this ratio is large enough (Cases (1) and (5)), the hedgers and the non-hedgers trade in opposite directions. If it is small enough (Cases (3) and (8)), on the other hand, they trade in the same direction. In between, either the hedgers or the non-hedgers do not trade.

Define

$$b_1 = \frac{\delta_m \nu_2 N_n + 2\delta_n \nu_1}{2\delta_n \nu_1}, \quad b_2 = \frac{\delta_m \nu_2 N_h}{2\delta_h}, \tag{C-30}$$

[4] The case where both hedgers and non-hedgers do not trade is a measure zero event, which occurs only when the reservation prices of *all* investors are exactly the same.

$$b_3 = \frac{1}{2\delta_h}\left(\delta_m \nu_2 N_h + \sqrt{\frac{\delta_h(2\delta_h + \delta_m\nu_2 N_h)(\hat{N}+1)}{\delta_h N_n/(\delta_n \nu_1 N_h)+1}}\right) > b_2, \tag{C-31}$$

$$b_4 = \frac{N_h \delta_n \nu_1}{\sqrt{N_n\delta_h + N_h\delta_n\nu_1}\left(\sqrt{N_n\delta_h + N_h\delta_n\nu_1} - \sqrt{\frac{N_n\delta_h\delta_n\nu_1(\hat{N}+1)}{2\delta_n\nu_1 + N_n\delta_m\nu_2}}\right)} (> b_1), \tag{C-32}$$

where

$$\nu_1 = \frac{\mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]}{\mathrm{Var}[\tilde{V}_h|\mathcal{I}_h]},\ \nu_2 = \frac{\mathrm{Var}[\tilde{V}_m|\mathcal{I}_m]}{\mathrm{Var}[\tilde{V}_h|\mathcal{I}_h]},\ \hat{N} := \frac{\delta_m}{\delta_h}\nu_2 N_h + 1 + \frac{\delta_m \nu_2}{\delta_n \nu_1} N_n.$$

Theorem C.2 *For the generalized model, we have:*

1. *The hedgers buy and the non-hedgers sell (Case (1)) if and only if*

$$-b_1\Delta_{hn} < \Delta_{nm} < b_2\Delta_{hn}. \tag{C-33}$$

For Case (1), in the presence of short-sale constraints, we have the following two subcases.

(a) *If* $b_2\Delta_{hn} - (\kappa_n + \bar{\theta}_n)(\hat{N}+1)\delta_n\, Var[\tilde{V}_n|\mathcal{I}_n] < \Delta_{nm} < b_2\Delta_{hn}$*, then short-sale constraints do not bind.*

The equilibrium bid and ask prices are

$$A^* = P_n^R + \frac{b_2}{\hat{N}+1}\Delta_{hn} - \frac{\Delta_{nm}}{\hat{N}+1} + \frac{\Delta_{hn}}{2}, \tag{C-34}$$

$$B^* = P_n^R + \frac{b_2}{\hat{N}+1}\Delta_{hn} - \frac{\Delta_{nm}}{\hat{N}+1}, \tag{C-35}$$

and the bid-ask spread is

$$A^* - B^* = \frac{\Delta_{hn}}{2}; \tag{C-36}$$

the equilibrium security quantities demanded are

$$\theta_h^* = \frac{\Delta_{nm} + b_1\Delta_{hn}}{(\hat{N}+1)\delta_h\, Var[\tilde{V}_h|\mathcal{I}_h]}, \quad \theta_n^* = \frac{\Delta_{nm} - b_2\Delta_{hn}}{(\hat{N}+1)\delta_n\, Var[\tilde{V}_n|\mathcal{I}_n]}, \quad \theta_m^* = -\left(N_h\theta_h^* + N_n\theta_n^*\right); \tag{C-37}$$

the equilibrium quote depths are

$$\alpha^* = N_h\theta_h^*, \quad \beta^* = -N_n\theta_n^*. \tag{C-38}$$

(b) *If* $-b_1\Delta_{hn} < \Delta_{nm} \le b_2\Delta_{hn} - (\kappa_n + \bar{\theta}_n)(\hat{N}+1)\delta_n\, Var[\tilde{V}_n|\mathcal{I}_n]$, *then short-sale constraints bind for non-hedgers. The equilibrium bid and ask prices are*

$$A_c^* = P_h^R - \frac{\delta_h}{2\delta_h + \delta_m N_h \nu_2}\Delta_{hm} - \frac{\delta_h\delta_m N_n(\kappa_n + \bar{\theta}_n)\nu_2\, Var[\tilde{V}_h|\mathcal{I}_h]}{2\delta_h + \delta_m N_h\nu_2}, \tag{C-39}$$

$$B_c^* = P_n^R + \delta_n(\kappa_n + \bar{\theta}_n)\, Var[\tilde{V}_n|\mathcal{I}_n]; \tag{C-40}$$

the equilibrium security quantities demanded are

$$\theta_{hc}^* = \frac{\Delta_{hm} + \delta_m N_n(\kappa_n + \bar{\theta}_n)\nu_2\, Var[\tilde{V}_h|\mathcal{I}_h]}{(2\delta_h + \delta_m N_h\nu_2)\, Var[\tilde{V}_h|\mathcal{I}_h]}, \quad \theta_{nc}^* = -(\kappa_n + \bar{\theta}_n), \quad \theta_{mc}^* = -\left(N_h\theta_{hc}^* + N_n\theta_{nc}^*\right); \tag{C-41}$$

the equilibrium quote depths are

$$\alpha_c^* = N_h\theta_{hc}^*, \quad \beta_c^* = -N_n\theta_{nc}^*. \tag{C-42}$$

2. *The hedgers buy and the non-hedgers do not trade (Case (2)) if and only if*

$$b_2\Delta_{hn} \le \Delta_{nm} \le b_3\Delta_{hn}. \tag{C-43}$$

For Case (2), the equilibrium bid and ask prices are

$$A^* = P_h^R - \frac{\Delta_{hm}}{2 + N_h\nu_2\delta_m/\delta_h}, \quad B^* \le P_n^R; \tag{C-44}$$

the equilibrium security quantities demanded are

$$\theta_h^* = \frac{\Delta_{hm}}{(2\delta_h + N_h\nu_2\delta_m)\, Var[\tilde{V}_h|\mathcal{I}_h]}, \ \theta_n^* = 0, \ \theta_m^* = -N_h\theta_h^*; \tag{C-45}$$

the equilibrium quote depths are

$$\alpha^* = N_h \theta_h^*, \quad \beta^* = 0. \tag{C-46}$$

3. *Both the hedgers and non-hedgers buy (Case (3)) if and only if*

$$\Delta_{nm} \geq \max\{-b_4 \Delta_{hn},\ b_3 \Delta_{hn}\}. \tag{C-47}$$

For Case (3), the equilibrium prices are

$$A^* = \frac{N_h \nu_1 \delta_n P_h^R + N_n \delta_h P_n^R}{N_h \nu_1 \delta_n + N_n \delta_h} - \frac{N_h \nu_1 \delta_n \Delta_{hm} + N_n \delta_h \Delta_{nm}}{(\hat{N}+1)(N_h \nu_1 \delta_n + N_n \delta_h)}, \quad B^* \leq A^*; \tag{C-48}$$

the equilibrium security quantities demanded are

$$\theta_h^* = \frac{\Delta_{nm} + \left(1 + \frac{N_n \delta_h}{N_n \delta_h + N_h \delta_n \nu_1} + \frac{N_n \delta_m \nu_2}{\delta_n \nu_1}\right) \Delta_{hn}}{(\hat{N}+1)\delta_h \, Var[\tilde{V}_h | \mathcal{I}_h]}, \tag{C-49}$$

$$\theta_n^* = \frac{\Delta_{nm} - \left(\frac{N_h \delta_n \nu_1}{N_n \delta_h + N_h \delta_n \nu_1} + \frac{N_h \delta_m \nu_2}{\delta_h}\right) \Delta_{hn}}{(\hat{N}+1)\delta_n \, Var[\tilde{V}_n | \mathcal{I}_n]}, \tag{C-50}$$

$$\theta_m^* = -N_h \theta_h^* - N_n \theta_n^*; \tag{C-51}$$

and the equilibrium depths are

$$\alpha^* = N_h \theta_h^* + N_n \theta_n^*, \quad \beta^* = 0. \tag{C-52}$$

4. *The hedgers do not trade and non-hedgers buy (Case (4)) if and only if*

$$-b_1 \Delta_{hn} \leq \Delta_{nm} \leq -b_4 \Delta_{hn}. \tag{C-53}$$

For Case (4), the equilibrium prices are

$$A^* = P_n^R - \frac{\Delta_{nm}}{2 + N_n \nu_2 \delta_m / (\nu_1 \delta_n)}, \quad B^* \leq P_h^R; \tag{C-54}$$

the equilibrium security quantities demanded are

$$\theta_h^* = 0,\ \theta_n^* = \frac{\Delta_{nm}}{(2 + N_n \nu_2 \delta_m / (\nu_1 \delta_n)) \delta_n \, Var[\tilde{V}_n | \mathcal{I}_n]},\ \theta_m^* = -N_n \theta_n^*; \tag{C-55}$$

and the equilibrium depths are

$$\alpha^* = N_n \theta_n^*, \quad \beta^* = 0. \tag{C-56}$$

5. *The hedgers sell and the non-hedgers buy (Case (5)) if and only if*

$$b_2 \Delta_{hn} < \Delta_{nm} < -b_1 \Delta_{hn}. \tag{C-57}$$

For Case (5), in the presence of short-sale constraints, we have the following two subcases.

(a) *If* $-b_1 \Delta_{hn} - (\kappa_h + \bar{\theta}_h)(\hat{N} + 1) \delta_h \, Var[\tilde{V}_h | \mathcal{I}_h] < \Delta_{nm} < -b_1 \Delta_{hn}$*, then short-sale constraints do not bind.*

The equilibrium bid and ask prices are

$$A^* = P_n^R + \frac{\nu_2 N_h \delta_m}{2 \delta_h \left(\hat{N} + 1 \right)} \Delta_{hn} - \frac{\Delta_{nm}}{\hat{N} + 1}, \tag{C-58}$$

$$B^* = P_n^R + \frac{\nu_2 N_h \delta_m}{2 \delta_h \left(\hat{N} + 1 \right)} \Delta_{hn} - \frac{\Delta_{nm}}{\hat{N} + 1} + \frac{\Delta_{hn}}{2}, \tag{C-59}$$

and the bid-ask spread is

$$A^* - B^* = -\frac{\Delta_{hn}}{2}; \tag{C-60}$$

the equilibrium security quantities demanded are

$$\theta_h^* = \frac{\Delta_{nm} + b_1 \Delta_{hn}}{(\hat{N}+1)\delta_h \, Var[\tilde{V}_h|\mathcal{I}_h]}, \quad \theta_n^* = \frac{\Delta_{nm} - b_2 \Delta_{hn}}{(\hat{N}+1)\delta_n \, Var[\tilde{V}_n|\mathcal{I}_n]}, \quad \theta_m^* = -\left(N_h \theta_h^* + N_n \theta_n^*\right); \tag{C-61}$$

the equilibrium quote depths are

$$\alpha^* = N_n \theta_n^*, \quad \beta^* = -N_h \theta_h^*. \tag{C-62}$$

(b) *If* $b_2 \Delta_{hn} < \Delta_{nm} \leq -b_1 \Delta_{hn} - (\kappa_h + \bar{\theta}_h)(\hat{N}+1)\delta_h \, Var[\tilde{V}_h|\mathcal{I}_h]$*, then short-sale constraints bind for hedgers. The equilibrium bid and ask prices are*

$$A_c^* = P_n^R - \frac{\delta_n \nu_1}{2\delta_n \nu_1 + \delta_m N_n \nu_2} \Delta_{nm} - \frac{\delta_n \delta_m \nu_1 N_h (\kappa_h + \bar{\theta}_h) \nu_2 \, Var[\tilde{V}_h|\mathcal{I}_h]}{2\delta_n \nu_1 + \delta_m N_n \nu_2}, \tag{C-63}$$

$$B_c^* = P_h^R + \delta_h (\kappa_h + \bar{\theta}_h) \, Var[\tilde{V}_h|\mathcal{I}_h]; \tag{C-64}$$

the equilibrium security quantities demanded are

$$\theta_{hc}^* = -(\kappa_h + \bar{\theta}_h), \quad \theta_{nc}^* = \frac{\Delta_{nm} + \delta_m N_h (\kappa_h + \bar{\theta}_h) \nu_2 \, Var[\tilde{V}_h|\mathcal{I}_h]}{(2\delta_n \nu_1 + \delta_m N_n \nu_2) \, Var[\tilde{V}_h|\mathcal{I}_h]}, \quad \theta_{mc}^* = -\left(N_h \theta_{hc}^* + N_n \theta_{nc}^*\right); \tag{C-65}$$

the equilibrium quote depths are

$$\alpha_c^* = N_n \theta_{nc}^*, \quad \beta_c^* = -N_h \theta_{hc}^*. \tag{C-66}$$

6. *The hedgers sell and the non-hedgers do not trade (Case (6)) if and only if*

$$b_3 \Delta_{hn} \leq \Delta_{nm} \leq b_2 \Delta_{hn}. \tag{C-67}$$

For Case (6), in the presence of short-sale constraints, we have the following two subcases.

(a) *If* $-\Delta_{hn} - (2\delta_h + \delta_m \nu_2 N_h)(\kappa_h + \bar{\theta}_h) \, Var[\tilde{V}_h|\mathcal{I}_h] < \Delta_{nm} < b_2 \Delta_{hn}$*, then short-sale constraints* *do not bind. The equilibrium prices are*

$$B^* = P_h^R - \frac{\Delta_{hm}}{2 + N_h\nu_2\delta_m/\delta_h}, \quad A^* \geq P_n^R; \tag{C-68}$$

the equilibrium security quantities demanded are

$$\theta_h^* = \frac{\Delta_{hm}}{(2 + N_h\nu_2\delta_m/\delta_h)\delta_h\, Var[\tilde{V}_h|\mathcal{I}_h]}, \;\; \theta_n^* = 0, \;\; \theta_m^* = -N_h\theta_h^*; \tag{C-69}$$

and the equilibrium depths are

$$\alpha^* = 0, \quad \beta^* = -N_h\theta_h^*. \tag{C-70}$$

(b) If $b_3\Delta_{hn} \leq \Delta_{nm} \leq -\Delta_{hn} - (2\delta_h + \delta_m\nu_2N_h)(\kappa_h + \bar{\theta}_h)\, Var[\tilde{V}_h|\mathcal{I}_h]$*, then short-sale constraints bind for hedgers. The equilibrium prices are*

$$B_c^* = P_h^R + \delta_h(\kappa_h + \bar{\theta}_h)\, Var[\tilde{V}_h|\mathcal{I}_h], \quad A_c^* \geq P_n^R; \tag{C-71}$$

the equilibrium security quantities demanded are

$$\theta_{hc}^* = -(\kappa_h + \bar{\theta}_h), \;\; \theta_{nc}^* = 0, \;\; \theta_{mc}^* = -N_h\theta_{hc}^*; \tag{C-72}$$

and the equilibrium depths are

$$\alpha_c^* = 0, \quad \beta_c^* = -N_h\theta_{hc}^*. \tag{C-73}$$

7. *The hedgers do not trade and the non-hedgers sell (Case (7)) if and only if*

$$-b_4\Delta_{hn} \leq \Delta_{nm} \leq -b_1\Delta_{hn}. \tag{C-74}$$

For Case (7), in the presence of short-sale constraints, we have the following two subcases.

(a) If $-(2\delta_n + \delta_m N_n\nu_2/\nu_1)(\kappa_n + \bar{\theta}_n)\, Var[\tilde{V}_n|\mathcal{I}_n] < \Delta_{nm} \leq -b_1\Delta_{hn}$*, then short-sale constraints*

do not bind. The equilibrium prices are

$$B^* = P_n^R - \frac{\Delta_{nm}}{2 + N_n \nu_2 \delta_m / (\nu_1 \delta_n)}, \quad A^* \geq P_h^R. \tag{C-75}$$

the equilibrium security quantities demanded are

$$\theta_h^* = 0,\ \theta_n^* = \frac{\Delta_{nm}}{(2 + N_n \nu_2 \delta_m / (\nu_1 \delta_n))\delta_n\, Var[\tilde{V}_n | \mathcal{I}_n]},\ \theta_m^* = -N_n \theta_n^*; \tag{C-76}$$

and the equilibrium depths are

$$\alpha^* = 0, \quad \beta^* = -N_n \theta_n^*. \tag{C-77}$$

(b) If $-b_4 \Delta_{hn} \leq \Delta_{nm} \leq -(2\delta_n + \delta_m N_n \nu_2/\nu_1)(\kappa_n + \bar{\theta}_n)\, Var[\tilde{V}_n | \mathcal{I}_n]$*, then short-sale constraints bind for non-hedgers. The equilibrium prices are*

$$B^* = P_n^R + \delta_n(\kappa_n + \bar{\theta}_n)\, Var[\tilde{V}_n | \mathcal{I}_n], \quad A^* \geq P_h^R. \tag{C-78}$$

the equilibrium security quantities demanded are

$$\theta_h^* = 0,\ \theta_n^* = -(\kappa_n + \bar{\theta}_n),\ \theta_m^* = -N_n \theta_n^*; \tag{C-79}$$

and the equilibrium depths are

$$\alpha^* = 0, \quad \beta^* = -N_n \theta_n^*. \tag{C-80}$$

8. Both the hedgers and non-hedgers sell (Case (8)) if and only if

$$\Delta_{nm} \leq \min\{-b_4 \Delta_{hn},\ b_3 \Delta_{hn}\}. \tag{C-81}$$

Define $\bar{P}_h$ *(*$\bar{P}_n$*) as the critical price above which short-sale constraints bind for h (n) investors,*

$$\bar{P}_h = P_h^R + \delta_h(\kappa_h + \bar{\theta}_h)\mathrm{Var}[\tilde{V}_h|\mathcal{I}_h], \quad \bar{P}_n = P_n^R + \delta_n(\kappa_n + \bar{\theta}_n)\mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]. \tag{C-82}$$

Define

$$B^* = \varphi_m P_m^R + \varphi_n P_n^R + (1 - \varphi_m - \varphi_n)P_h^R, \tag{C-83}$$

$$B_c^h = \lambda_m P_m^R + (1 - \lambda_m)P_n^R - \lambda_h \delta_h(\kappa_h + \bar{\theta}_h)\mathrm{Var}[\tilde{V}_h|\mathcal{I}_h], \tag{C-84}$$

$$B_c^n = \gamma_m P_m^R + (1 - \gamma_m)P_h^R - \gamma_n \delta_n(\kappa_n + \bar{\theta}_n)\mathrm{Var}[\tilde{V}_n|\mathcal{I}_n], \tag{C-85}$$

where φ_m*,* φ_n*,* λ_m*,* λ_h*,* γ_m*, and* γ_n *are as defined in* (C-92)-(C-95)*.* B^* *is the equilibrium bid price in the absence of short-sale constraints, and* B_c^h *(*B_c^n*) is the equilibrium bid price given that hedgers (non-hedgers) short-sell* κ_h *(*κ_n*). In addition, let* $V(B^*)$ *denote the market-maker's expected utility in the absence of short-sale constraints and* $V_c^h(B_c^h)$ *be the market-maker's expected utility given that hedgers short-sell* κ_h*.*

Suppose $\bar{P}_h \leq \bar{P}_n$*.*[5] *For Case (8), in the presence of short-sale constraints, we have the following subcases.*

(a) If $B^* \leq \bar{P}_h$ *and* $B_c^h \leq \bar{P}_h$*, short-sale constraints do not bind for any investor, and the equilibrium price is* $B_c^* = B^*$*; the equilibrium security quantities demanded are* [6]

$$\theta_h^* = \frac{\Delta_{nm} + \left(1 + \frac{N_n\delta_h}{N_n\delta_h + N_h\delta_n\nu_1} + \frac{N_n\delta_m\nu_2}{\delta_n\nu_1}\right)\Delta_{hn}}{(\hat{N}+1)\delta_h\, Var[\tilde{V}_h|\mathcal{I}_h]},$$

$$\theta_n^* = \frac{\Delta_{nm} - \left(\frac{N_h\delta_n\nu_1}{N_n\delta_h + N_h\delta_n\nu_1} + \frac{N_h\delta_m\nu_2}{\delta_h}\right)\Delta_{hn}}{(\hat{N}+1)\delta_n\, Var[\tilde{V}_n|\mathcal{I}_n]},$$

and the equilibrium depths are

$$\alpha^* = 0, \quad \beta^* = -N_h\theta_h^* - N_n\theta_n^*.$$

[5] The case $\bar{P}_h \geq \bar{P}_n$ is symmetric (i.e., switching "h" and "n" in the notations) and thus omitted to save space.

[6] The quantities demanded for cases (b)-(g) are straightforward and thus omitted to save space.

(b) If $B^* \le \bar{P}_h$ *and* $\bar{P}_h < B_c^h \le \bar{P}_n$,[7] *then the equilibrium price is*

$$B_c^* = \begin{cases} B^* & \text{if } V(B^*) > V_c^h(B_c^h), \\ B^* \text{ or } B_c^h & \text{if } V(B^*) = V_c^h(B_c^h), \\ B_c^h \ (> B^*) & \text{if } V(B^*) < V_c^h(B_c^h), \end{cases} \tag{C-86}$$

and if the equilibrium price is equal to B_c^h*, then short-sale constraints bind for hedgers;*

(c) If $B^* > \bar{P}_h \ge B_c^h$*, then short-sale constraints bind for hedgers, and the equilibrium price is* $B_c^* = \bar{P}_h$ $(< B^*)$*;*

(d) If $B^* > B_c^h > \bar{P}_h$ *and* $B_c^h \le \bar{P}_n$*, then short-sale constraints bind for hedgers, and the equilibrium price is* $B_c^* = B_c^h$ $(< B^*)$*;*

(e) If $\bar{P}_n \ge B_c^h \ge B^* > \bar{P}_h$*, then short-sale constraints bind for hedgers, and the equilibrium price is* $B_c^* = B_c^h$ $(\ge B^*)$*;*

(f) If $B_c^h > \bar{P}_n > B^* > \bar{P}_h$*, then short-sale constraints bind for both hedgers and non-hedgers, and the equilibrium price is* $B_c^* = \bar{P}_n$ $(> B^*)$*;*

(g) If $B^* \ge \bar{P}_n$ *and* $B_c^h > \bar{P}_n$*, then short-sale constraints bind for both hedgers and non-hedgers, and the equilibrium price is* $B_c^* = \bar{P}_n$ $(\le B^*)$*.*

It can be demonstrated that as the reservation price of the market-maker varies from low to high, all seven subcases in Case (8) of Theorem C.2 can occur. It is clear from Equations (C-82) through (C-85) that $B^* - \bar{P}_h$ and $B_c^h - \bar{P}_h$ both increase with P_m^R. The intuition for Theorem C.2 is that if the reservation price of the market-maker is small enough relative to those of hedgers and non-hedgers, then the market-maker will be the only seller in equilibrium and thus short-sale constraints do not bind for any investor and the equilibrium price is the same as the case without short-sale constraints (subcase (a) in Case (8)). In the other extreme, if the reservation price of the market-maker is sufficiently large, then the market-maker is the only buyer and short-sale constraints bind for both hedgers and non-hedgers, which implies that the equilibrium price is equal to $\bar{P}_n$.

[7] It can be shown that $B^* \le \bar{P}_h$ and $B_c^h > \bar{P}_n$ cannot occur under the assumption that $\bar{P}_h \le \bar{P}_n$.

The subcases (c), (d), and (g) in Case (8) reveal that the equilibrium price can still be lower with short-sale constraints. On the other hand, Case (8) of Theorem C.2 implies that the equilibrium price with short-sale constraints can be greater than the equilibrium price without the constraints. In addition, it is also possible that the equilibrium trade price becomes higher with the constraints than without, even when the market-maker is a buyer in equilibrium. For example, in Case (f), both hedgers and non-hedgers are sellers and constrained, and thus the market-maker is the buyer. However, Theorem C.2 implies that in this case, the equilibrium price with the constraints is higher than that without. This occurs because the constraints reduce the amount that the market-maker can buy from the hedgers who are constrained at B^*, and the benefit of buying more from the non-hedgers outweighs the cost of a higher price than B^*.

Proof of Theorem C.2: The proofs of Cases (1)-(7) are similar to the proof of Theorem 1. We only sketch the main steps. First, for each case, conditional on the trading directions (or no trade), we derive the equilibrium depths, prices, and trading quantities, similar to the proof of Theorem 1. Then we verify that under the specified conditions the assumed trading directions are indeed optimal. The proof of Case (8) is straightforward. Here we only outline the proof.

Given bid price B, for $i \in \{h, n\}$, a type i investor's problem is to choose θ_i to solve

$$\max E[-e^{-\delta_i \tilde{W}_i} | \mathcal{I}_i], \tag{C-87}$$

subject to the budget constraint

$$\tilde{W}_i = -\theta_i B + (\bar{\theta}_i + \theta_i)\tilde{V}_i + \hat{X}_i \tilde{L}, \tag{C-88}$$

and the short-sale constraint

$$\theta_i + \bar{\theta}_i \geq -\kappa_i. \tag{C-89}$$

The designated market-maker's problem is to choose price B to solve

$$\max E\left[-e^{-\delta_m \tilde{W}_m} | \mathcal{I}_m\right], \tag{C-90}$$

subject to

$$\tilde{W}_m = -\sum_{i=h,n} \min\left[\frac{B - P_i^R}{\delta_i \mathrm{Var}[\tilde{V}_i|\mathcal{I}_i]}, \kappa_i + \bar{\theta}_i\right] B + \left(\bar{\theta}_m + \sum_{i=h,n} \min\left[\frac{B - P_i^R}{\delta_i \mathrm{Var}[\tilde{V}_i|\mathcal{I}_i]}, \kappa_i + \bar{\theta}_i\right]\right) \tilde{V}_m. \quad \text{(C-91)}$$

Define

$$\varphi_m = \frac{\delta_h \delta_n \nu_1}{\delta_m \delta_n \nu_1 \nu_2 N_h + 2\delta_h \delta_n \nu_1 + \delta_m \delta_h \nu_2 N_n}, \quad \text{(C-92)}$$

$$\varphi_n = N_n \left(\frac{\delta_m \nu_2}{\delta_n \nu_1} + \frac{\delta_h}{N_n \delta_h + N_h \nu_1 \delta_n}\right) \varphi_m, \quad \text{(C-93)}$$

$$\lambda_m = \frac{\delta_n \nu_1}{N_n \delta_m \nu_2 + 2\delta_n \nu_1}, \lambda_h = \frac{N_n \delta_m \nu_2 + \delta_n \nu_1}{N_n \delta_m \nu_2 + 2\delta_n \nu_1} \frac{\delta_n \nu_1 N_h}{\delta_h N_n}, \quad \text{(C-94)}$$

$$\gamma_m = \frac{\delta_h}{N_h \delta_m \nu_2 + 2\delta_h}, \gamma_n = \frac{N_h \delta_m \nu_2 + \delta_h}{N_h \delta_m \nu_2 + 2\delta_h} \frac{\delta_h N_n}{\delta_n \nu_1 N_h}. \quad \text{(C-95)}$$

First, assuming that there are no short-sale constraints, then the market-maker's problem is equivalent to

$$\max_B \quad N_h \frac{P_h^R - P}{\delta_h \mathrm{Var}[\tilde{V}_h]} B + N_n \frac{P_n^R - B}{\delta_n \mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]} B + \left(\bar{\theta}_m + N_h \frac{B - P_h^R}{\delta_h \mathrm{Var}[\tilde{V}_h|\mathcal{I}_h]} + N_n \frac{B - P_n^R}{\delta_n \mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]}\right)$$
$$\times (P_m^R + \delta_m \mathrm{Var}[\tilde{V}_m|\mathcal{I}_m]\bar{\theta}_m) - \tfrac{1}{2}\delta_m \mathrm{Var}[\tilde{V}_m|\mathcal{I}_m] \left(\bar{\theta}_m + N_h \frac{B - P_h^R}{\delta_h \mathrm{Var}[\tilde{V}_h|\mathcal{I}_h]} + N_n \frac{B - P_n^R}{\delta_n \mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]}\right)^2.$$

The first order condition then yields B^*. Conditional on hedgers always selling $\kappa_h + \bar{\theta}_h$, the market-maker's problem is equivalent to

$$\max_B \quad -N_h(\kappa_h + \bar{\theta}_h) B + N_n \frac{P_n^R - B}{\delta_n \mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]} B + \left(\bar{\theta}_m + N_h(\kappa_h + \bar{\theta}_h) + N_n \frac{B - P_n^R}{\delta_n \mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]}\right)$$
$$\times (P_m^R + \delta_m \mathrm{Var}[\tilde{V}_m|\mathcal{I}_m]\bar{\theta}_m) - \tfrac{1}{2}\delta_m \mathrm{Var}[\tilde{V}_m|\mathcal{I}_m] \left(\bar{\theta}_m + N_h(\kappa_h + \bar{\theta}_h) + N_n \frac{B - P_n^R}{\delta_n \mathrm{Var}[\tilde{V}_n|\mathcal{I}_n]}\right)^2.$$

The first order condition then yields B_c^h. Similarly, conditional on non-hedgers always selling $\kappa_n + \bar{\theta}_n$, we can derive the equilibrium price B_c^n. Then the comparison of the maximum expected utility with the constraint that $B^* \le \bar{P}_h$ and the maximum expected utility with the constraint that $\bar{P}_h \le B^* \le \bar{P}_n$, while noting that the maximum expected utility with the constraint that $B^* \ge \bar{P}_n$ is equal to the expected utility at $B^* = \bar{P}_n$, yields the equilibrium prices under different conditions. *Q.E.D.*