Appendix

This file is the electronic companion of the paper "Managing Appointment Booking under Customer Choices" by Nan Liu, Peter M. van de Ven, and Bo Zhang.

A. Proof of the Results in Section 3

A.1. Preliminarily results

We first state and prove an auxiliary lemma on the structural results of the value function for the non-sequential offering model. This lemma will be used in proving other results in the paper.

LEMMA 3. *Let Ω be a preference matrix, $\mathbf{m} \geq 0$, $j = 1, \ldots, J$ and $n \in \{1, \ldots, N\}$, then the value function $V_n(\mathbf{m})$ satisfies*
(i) $0 \leq V_{n+1}(\mathbf{m}) - V_n(\mathbf{m}) \leq 1; \quad \forall n = 0, 1, 2, \ldots;$
(ii) $0 \leq V_n(\mathbf{m} + \mathbf{e}_j) - V_n(\mathbf{m}) \leq 1; \quad \forall n = 0, 1, 2, \ldots;$
(iii) if $\lambda_0 > 0$, *then* $V_n(\mathbf{m} + \mathbf{e}_j) - V_n(\mathbf{m}) < 1; \quad \forall n = 1, 2, \ldots.$

These monotonicity results are quite intuitive. Properties (i) and (ii) state that the optimal expected reward is increasing in the number of customers and the number of slots left and the changes in the optimal expected reward are bounded by the changes in the number of customers to go and the number of slots available. Property (iii) suggests that if there is a strictly positive probability that no customers would come in each period, then the increase of the optimal expected reward is strictly smaller than that of the available slots.

Proof. We use induction to prove this lemma. We first prove the first two properties. For $n = 0$, these two properties hold trivially. Suppose that they also hold up to $n = t$. Consider $n = t + 1$. Let $\mathbf{g}_t^*(\mathbf{m})$ represent the optimal decision rule in period t when the system state is $\mathbf{m}$. Let $V_s^{\mathbf{f}}(\mathbf{m})$ be the expected number of slots filled given that the decision rule $\mathbf{f}$ is taken at stage s and from stage $s-1$ onwards the optimal decision rule is used. Let $p_k(\mathbf{m}, \mathbf{f})$ be the probability that a type k slot is booked at state $\mathbf{m}$ if action $\mathbf{f}$ is taken. It follows that

$$\begin{aligned}
V_{t+1}(\mathbf{m}) \geq V_{t+1}^{\mathbf{g}_t^*(\mathbf{m})}(\mathbf{m}) =& \sum_{k=0}^{J} p_k(\mathbf{m}, \mathbf{g}_t^*(\mathbf{m}))[\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} - \mathbf{e}_k)] \\
\geq& \sum_{k=0}^{J} p_k(\mathbf{m}, \mathbf{g}_t^*(\mathbf{m}))[\mathbb{1}_{\{k>0\}} + V_{t-1}(\mathbf{m} - \mathbf{e}_k)] \\
=& V_t(\mathbf{m}),
\end{aligned}$$

where the first inequality is due to the definition of $V_{t+1}(\mathbf{m})$ and the second inequality follows from the induction hypothesis. Following a similar argument and fixing $j \in \{1, 2, \ldots, J\}$, we have

$$\begin{aligned}
V_{t+1}(\mathbf{m} + e_j) \geq& V_{t+1}^{\mathbf{g}_{t+1}^*(\mathbf{m})}(\mathbf{m} + e_j) \\
=& \sum_{k=0}^{J} p_k(\mathbf{m} + \mathbf{e}_j, \mathbf{g}_{t+1}^*(\mathbf{m}))[\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} + \mathbf{e}_j - \mathbf{e}_k)] \\
=& \sum_{k=0}^{J} p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m}))[\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} + \mathbf{e}_j - \mathbf{e}_k)] \\
\geq& \sum_{k=0}^{J} p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m}))[\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} - \mathbf{e}_k)] \\
=& V_{t+1}(\mathbf{m}),
\end{aligned}$$

where the second equality results from the decision rules and the state transition probability (2).

To show the RHS of the inequality in (i) for $n = t+1$, note that

$$\begin{aligned}
&V_{t+1}(\mathbf{m}) - V_t(\mathbf{m}) \\
=&\sum_{k=0}^{J} p_k(\mathbf{m}, \mathbf{g}^*_{t+1}(\mathbf{m}))[\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m}-\mathbf{e}_k)] - V_t(\mathbf{m}) \\
=&\sum_{k=1}^{J} p_k(\mathbf{m}, \mathbf{g}^*_{t+1}(\mathbf{m})) + \sum_{k=0}^{J} p_k(\mathbf{m}, \mathbf{g}^*_{t+1}(\mathbf{m}))[V_t(\mathbf{m}-\mathbf{e}_k) - V_t(\mathbf{m})] \\
\le&\sum_{k=1}^{J} p_k(\mathbf{m}, \mathbf{g}^*_{t+1}(\mathbf{m})) \\
\le&1,
\end{aligned}$$

where the first inequality follows from that $V_t(\mathbf{m}-\mathbf{e}_k) \le V_t(\mathbf{m})$, which has been shown above.

To show the RHS of the inequality in (ii) for $n = t+1$, we define a decision rule $\mathbf{h}$ in period $t+1$ such that $\mathbf{h} = \mathbf{g}^*_{t+1}(\mathbf{m}+\mathbf{e}_j)$ except $h_j = 0$. It follows that

$$V_{t+1}(\mathbf{m}+\mathbf{e}_j) - V_{t+1}(\mathbf{m}) \le V_{t+1}(\mathbf{m}+\mathbf{e}_j) - V^{\mathbf{h}}_{t+1}(\mathbf{m}), \tag{17}$$

because $\mathbf{h}$ may not be the optimal given system state $\mathbf{m}$ at period $t+1$. For $u = 1, 2, \ldots, J$, let

$$q_u = p_u(\mathbf{m}+\mathbf{e}_j, \mathbf{g}^*_{t+1}(\mathbf{m}+\mathbf{e}_j))$$

and

$$q'_u = p_u(\mathbf{m}, \mathbf{h}).$$

It is easy to check that $q_u \le q'_u, \forall u \ne 0, j$ and $q'_j = 0$. Now, let $\Omega_{i\cdot} = (\Omega_{i1}, \Omega_{i2}, \ldots, \Omega_{iJ})$ and use $\langle \cdot, \cdot \rangle$ to represent the inner product. We have that

$$\sum_{u=1}^{J} q_u = \sum_{i=1}^{I} \lambda_i \mathbb{1}_{\{\langle \Omega_{i\cdot}, \mathbf{g}_{t+1}(\mathbf{m}+\mathbf{e}_j)\rangle > 0\}} \ge \sum_{i=1}^{I} \lambda_i \mathbb{1}_{\{\{\langle \Omega_{i\cdot}, \mathbf{h}\rangle > 0\}} = \sum_{u=1}^{J} q'_u,$$

because $\mathbf{h} = \mathbf{g}^*_{t+1}(\mathbf{m}+e_j)$ except $h_j = 0$. Therefore, $q_0 = 1 - \sum_{u=1}^{J} q_u \le 1 - \sum_{u=1}^{J} q'_u = q'_0$. Define $\delta_u = q'_u - q_u$ for $u \ne j$. It is clear that $\delta_u \ge 0, \forall u \ne j$, and we note the following relationship.

$$q_j = 1 - \sum_{u \ne j} q_u = 1 - \sum_{u \ne j} (q'_u - \delta_u) = \sum_{u \ne j} \delta_u.$$

Now, we can continue the inequality (17) as follows.

$$\begin{aligned}
&V_{t+1}(\mathbf{m}+\mathbf{e}_j) - V^{\mathbf{h}}_{t+1}(\mathbf{m}) \\
=&\sum_{u=0}^{J} q_u[\mathbb{1}_{\{u>0\}} + V_t(\mathbf{m}+\mathbf{e}_j-\mathbf{e}_u)] - \sum_{u=0}^{J} q'_u[\mathbb{1}_{\{u>0\}} + V_t(\mathbf{m}-\mathbf{e}_u)] \\
=&(1-q_0) - (1-q'_0) + \sum_{u=0}^{J} q_u V_t(\mathbf{m}+\mathbf{e}_j-\mathbf{e}_u) - \sum_{u=0}^{J} q'_u V_t(\mathbf{m}-\mathbf{e}_u)] \\
=&\delta_0 + \sum_{u \ne j} q_u (V_t[\mathbf{m}+\mathbf{e}_j-\mathbf{e}_u) - V_t(\mathbf{m}-\mathbf{e}_u)] + q_j V_t(\mathbf{m}) - \sum_{u \ne j} \delta_u V_t(\mathbf{m}-\mathbf{e}_u)
\end{aligned}$$

$$
\begin{aligned}
&=\delta_0+\sum_{u\neq j}q_u(V_t[\mathbf{m}+\mathbf{e}_j-\mathbf{e}_u)-V_t(\mathbf{m}-\mathbf{e}_u)]+\sum_{u\neq j}\delta_u V_t(\mathbf{m})-\sum_{u\neq j}\delta_u V_t(\mathbf{m}-\mathbf{e}_u)\\
&=\delta_0+\sum_{u\neq j}q_u(V_t[\mathbf{m}+\mathbf{e}_j-\mathbf{e}_u)-V_t(\mathbf{m}-\mathbf{e}_u)]+\sum_{u\neq 0,j}\delta_u[V_t(\mathbf{m})-V_t(\mathbf{m}-\mathbf{e}_u)]\\
&\leq\delta_0+\sum_{u\neq j}q_u+\sum_{u\neq 0,j}\delta_u\\
&=1,
\end{aligned}
$$

where the last inequality comes from the induction hypothesis for property (ii).

As for property (iii), first note that it trivially holds for $n=1$. We can then follow similar induction steps as those used to prove the RHS of the inequality in property (ii) to complete the proof. □

A.2. Proof of Proposition 1

Proof. We focus on the W model instance here, as the N Model instance is a special case of this. For $n\geq 1$ and any system state $(x,y)\geq(1,1)$, the optimality equation for the W model instance reads.

$$
V_n(x,y)=\max\left\{\begin{array}{l}1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{n-1}(x-1,y)+(\frac{1}{2}\lambda_2+\lambda_3)V_{n-1}(x,y-1)+\lambda_0V_{n-1}(x,y),\\ (1-\lambda_3-\lambda_0)+(\lambda_3+\lambda_0)V_{n-1}(x,y)+(\lambda_1+\lambda_2)V_{n-1}(x-1,y),\\ (1-\lambda_1-\lambda_0)+(\lambda_1+\lambda_0)V_{n-1}(x,y)+(\lambda_2+\lambda_3)V_{n-1}(x,y-1)\end{array}\right\}, \quad (18)
$$

where the three terms in the max operator correspond to the action of offering slot types $\{1,2\}$, $\{1\}$ and $\{2\}$, respectively. For the boundary conditions, it is easy to see that $V_0(x,y)=0$ regardless of x and y. When one type of the slots are depleted, it is optimal to offer the other type of the slots. To calculate $V_n(x,0)$, note that type 1 slots are accepted only by type 1 and type 2 customers and the number of type 1 and type 2 customers in the last n customers yet to come has a binomial distribution with parameters n and $\lambda_1+\lambda_2$. Denote this random variable by $X_1\sim\mathrm{Bin}(n,\lambda_1+\lambda_2)$. It follows that

$$
V_n(x,0)=\mathbf{E}(\min\{x,X_1\})=\sum_{k=0}^{n}\min(x,k)\binom{n}{k}(\lambda_1+\lambda_2)^k(1-\lambda_1-\lambda_2)^{n-k}. \quad (19)
$$

Similarly, with $X_2\sim\mathrm{Bin}(n,\lambda_2+\lambda_3)$

$$
V_n(0,y)=\mathbf{E}(\min\{y,X_2\})=\sum_{k=0}^{n}\min(y,k)\binom{n}{k}(\lambda_2+\lambda_3)^k(1-\lambda_2-\lambda_3)^{n-k}. \quad (20)
$$

For ease of presentation, we define $\Delta_n^{ij}(x,y)$ to be the difference of the ith and jth terms in the max operator (18) above, $i,j\in\{1,2,3\}$. In particular, we have

$$
\Delta_n^{12}(x,y)=\lambda_3-\frac{1}{2}\lambda_2V_{n-1}(x-1,y)+(\frac{1}{2}\lambda_2+\lambda_3)V_{n-1}(x,y-1)-\lambda_3V_{n-1}(x,y), \quad (21)
$$

and

$$
\Delta_n^{13}(x,y)=\lambda_1-\frac{1}{2}\lambda_2V_{n-1}(x,y-1)+(\frac{1}{2}\lambda_2+\lambda_1)V_{n-1}(x-1,y)-\lambda_1V_{n-1}(x,y). \quad (22)
$$

It suffices to show that $\Delta_n^{12}(x,y),\Delta_n^{13}(x,y)\geq 0$ for any $x,y\geq 1$ (the case when x or y equals 0 is trivial as it meets the boundary conditions discussed above; see (19) and (20)). We use induction below to prove this. When $n=1$, it is a trivial proof as it is optimal to offer all available slots with one period left. Suppose that (21) and (22) hold up to $n=k$ and for any $x,y\geq 1$. Now, consider $n=k+1$ and $x,y\geq 1$. We have four cases to check: (1) $x=y=1$; (2) $y=1$ and $x\geq 2$; (3) $x=1$ and $y\geq 2$; and (4) $x,y\geq 2$. We start with case (1) and evaluate the term $\Delta^{13}_{k+1}(x,1)$ below.

$$\begin{aligned}
\Delta^{13}_{k+1}(1,1) =&\lambda_1+(\lambda_1+\frac{1}{2}\lambda_2)V_k(0,1)-\frac{1}{2}\lambda_2V_k(1,0)-\lambda_1V_k(1,1)\\
=&\lambda_1+(\lambda_1+\frac{1}{2}\lambda_2)[1-(\lambda_1+\lambda_0)^k]-\frac{1}{2}\lambda_2[1-(\lambda_3+\lambda_0)^k]\\
&-\lambda_1[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(0,1)+(\lambda_3+\frac{1}{2}\lambda_2)V_{k-1}(1,0)+\lambda_0V_{k-1}(1,1)]\\
=&\lambda_0\Delta^{13}_k(1,1)+(\lambda_1+\frac{1}{2}\lambda_2)[1-(\lambda_1+\lambda_0)^k]-\frac{1}{2}\lambda_2[1-(\lambda_3+\lambda_0)^k]\\
&-\lambda_1[(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(0,1)+(\lambda_3+\frac{1}{2}\lambda_2)V_{k-1}(1,0)]\\
&-\lambda_0(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(0,1)+\frac{1}{2}\lambda_0\lambda_2V_{k-1}(1,0)\\
=&\lambda_0\Delta^{13}_k(1,1)+(\lambda_1+\frac{1}{2}\lambda_2)[1-(\lambda_1+\lambda_0)]-\frac{1}{2}\lambda_2[1-(\lambda_3+\lambda_0)^k]\\
&+[\frac{1}{2}\lambda_0\lambda_2-\lambda_1(\lambda_3+\frac{1}{2}\lambda_2)][1-(\lambda_3+\lambda_0)^{k-1}]\\
=&\lambda_0\Delta^{13}_k(1,1)+[\frac{1}{2}\lambda_2(\lambda_3+\lambda_0)-\frac{1}{2}\lambda_0\lambda_2+\lambda_1(\lambda_3+\frac{1}{2}\lambda_2)](\lambda_3+\lambda_0)^{k-1}\\
=&\lambda_0\Delta^{13}_k(1,1)+[\frac{1}{2}\lambda_2(\lambda_1+\lambda_3)+\lambda_1\lambda_3](\lambda_3+\lambda_0)^{k-1}\geq 0
\end{aligned}$$

where the second equality follow from (19), (20) and the induction hypothesis. Observing the symmetry, we can show $\Delta^{12}_{k+1}(1,1)\geq 0$.

We now study case (2). We can evaluate the term $\Delta^{13}_{k+1}(x,1)$ as below.

$$\begin{aligned}
\Delta^{13}_{k+1}(x,1) =&\lambda_1+\lambda_1V_k(x-1,1)-\lambda_1V_k(x,1)+\frac{1}{2}\lambda_2V_k(x-1,1)-\frac{1}{2}\lambda_2V_k(x,0)\\
=&\lambda_1+\lambda_1[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-2,1)+(\frac{1}{2}\lambda_2+\lambda_3)V_{k-1}(x-1,0)+\lambda_0V_{k-1}(x-1,1)]\\
&-\lambda_1[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-1,1)+(\frac{1}{2}\lambda_2+\lambda_3)V_{k-1}(x,0)+\lambda_0V_{k-1}(x,1)]\\
&+\frac{1}{2}\lambda_2[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-2,1)+(\frac{1}{2}\lambda_2+\lambda_3)V_{k-1}(x-1,0)+\lambda_0V_{k-1}(x-1,1)]\\
&-\frac{1}{2}\lambda_2[1-\lambda_3-\lambda_0+(\lambda_1+\lambda_2)V_{k-1}(x-1,0)+\lambda_3V_{k-1}(x,0)+\lambda_0V_{k-1}(x,0)],
\end{aligned}$$

where the second equality follows from the induction hypothesis. Note that $\lambda_1=\lambda_1(\lambda_0+\lambda_1+\lambda_2+\lambda_3)$. We can continue the equality chain above as follows.

$$\begin{aligned}
&\Delta^{13}_{k+1}(x,1)\\
=&(\lambda_1+\frac{1}{2}\lambda_2)[\lambda_1+\lambda_1V_{k-1}(x-2,1)-\lambda_1V_{k-1}(x-1,1)+\frac{1}{2}\lambda_2V_{k-1}(x-2,1)-\frac{1}{2}\lambda_2V_{k-1}(x-1,0)]\\
&+\lambda_0[\lambda_1+\lambda_1V_{k-1}(x-1,1)-\lambda_1V_{k-1}(x,1)+\frac{1}{2}\lambda_2V_{k-1}(x-1,1)-\frac{1}{2}\lambda_2V_{k-1}(x,0)]\\
&+(\frac{1}{2}\lambda_2+\lambda_3)[\lambda_1+\lambda_1V_{k-1}(x-1,0)-\lambda_1V_{k-1}(x,0)+\frac{1}{2}\lambda_2V_{k-1}(x-1,0)]\\
&+\frac{1}{2}\lambda_2\lambda_3-\frac{1}{2}\lambda_2\frac{1}{2}\lambda_2V_{k-1}(x-1,0)-\frac{1}{2}\lambda_2\lambda_3V_{k-1}(x,0)\\
=&(\lambda_1+\frac{1}{2}\lambda_2)\Delta^{13}_k(x-1,1)+\lambda_0\Delta^{13}_k(x,1)+(\frac{1}{2}\lambda_2\lambda_1+\lambda_3\lambda_1+\frac{1}{2}\lambda_2\lambda_3)[1+V_{k-1}(x-1,0)-V_{k-1}(x,0)]\geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis (22) and Lemma 3. Following a similar proof, we can show that $\Delta^{12}_{k+1}(x,1),\Delta^{12}_{k+1}(1,y),\Delta^{13}_{k+1}(1,y)\geq 0$ for $x,y\geq 2$.

Finally, we consider case (4) and evaluate the term $\Delta^{13}_{k+1}(x,y)$ below.

$$\begin{aligned}
&\Delta^{13}_{k+1}(x,y)\\
=&\lambda_1+\lambda_1 V_k(x-1,y)-\lambda_1 V_k(x,y)+\frac{1}{2}\lambda_2 V_k(x-1,y)-\frac{1}{2}\lambda_2 V_k(x,y-1)\\
=&\lambda_1+\lambda_1[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-2,y)+(\frac{1}{2}\lambda_2+\lambda_3)V_{k-1}(x-1,y-1)+\lambda_0 V_{k-1}(x-1,y)]\\
&-\lambda_1[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-1,y)+(\frac{1}{2}\lambda_2+\lambda_3)V_{k-1}(x,y-1)+\lambda_0 V_{k-1}(x,y)]\\
&+\frac{1}{2}\lambda_2[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-2,y)+(\frac{1}{2}\lambda_2+\lambda_3)V_{k-1}(x-1,y-1)+\lambda_0 V_{k-1}(x-1,y)]\\
&-\frac{1}{2}\lambda 2[1-\lambda_0+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-1,y-1)+(\frac{1}{2}\lambda_2+\lambda_3)V_{k-1}(x,y-2)+\lambda_0 V_{k-1}(x,y-1)],
\end{aligned}$$

where the second equality follows from the induction hypothesis. Recall that $\sum_{i=0}^{3}\lambda_i=1$ and thus $\lambda_1=\lambda_1(\lambda_0+\lambda_1+\lambda_2+\lambda_3)$. We can continue the equality chain above as follows.

$$\begin{aligned}
\Delta^{13}_{k+1}(x,y)=&(\lambda_1+\frac{1}{2}\lambda_2)[\lambda_1+\lambda_1 V_{k-1}(x-2,y)-\lambda_1 V_{k-1}(x-1,y)\\
&+\frac{1}{2}\lambda_2 V_{k-1}(x-2,y)-\frac{1}{2}\lambda_2 V_{k-1}(x-1,y-1)]\\
&+(\frac{1}{2}\lambda_2+\lambda_3)[\lambda_1+\lambda_1 V_{k-1}(x-1,y-1)-\lambda_1 V_{k-1}(x,y-1)\\
&+\frac{1}{2}\lambda_2 V_{k-1}(x-1,y-1)-\frac{1}{2}\lambda_2 V_{k-1}(x,y-2)]\\
&+\lambda_0[\lambda_1-\frac{1}{2}\lambda_2 V_{k-1}(x,y-1)+(\lambda_1+\frac{1}{2}\lambda_2)V_{k-1}(x-1,y)-\lambda_1 V_{k-1}(x,y)]\\
=&(\lambda_1+\frac{1}{2}\lambda_2)\Delta^{13}_{k}(x-1,y)+(\frac{1}{2}\lambda_2+\lambda_3)\Delta^{12}_{k}(x,y-1)+\lambda_0\Delta^{13}_{k}(x,y)\geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis. Using similar arguments, we can show that $\Delta^{12}_{k+1}(x,y)\geq 0$ for $x,y\geq 2$. Combining the four cases above, we prove the desired result. □

A.3. Proof of Proposition 2

Before we prove Proposition 2, we first present an auxiliary result.

LEMMA 4. *Consider the "M" network and let $n\in\mathbb{N}$. Then*

$$V_n(0,m_2,m_3-1)\geq V_n(0,m_2-1,m_3),\quad m_2\geq 1,\ m_3\geq 1, \tag{23}$$

$$V_n(m_1-1,m_2,0)\geq V_n(m_1,m_2-1,0),\quad m_1\geq 1,\ m_2\geq 1. \tag{24}$$

Proof. We will prove (23) by induction; this immediately implies (24) due to symmetry.

First, we can see by inspection that

$$V_1(0,m_2,m_3-1)=1-\lambda_0\geq V_1(0,m_2-1,m_3).$$

Now, let $t\in\mathbb{N}$ and assume that (23) holds for all $n\leq t$. In order to show that (23) holds for $n=t+1$ as well, first observe that for $m_1=0$, the M model reduces to the N model, and by Proposition 1 we know that it is optimal to offer all slots:

$$\begin{aligned}
V_n(0,m_2,m_3)=&(1-\lambda_0)+(\lambda_1+\frac{1}{2}\lambda_2)V_{n-1}(0,m_2-1,m_3)+\frac{1}{2}\lambda_2 V_{n-1}(0,m_2,m_3-1)\\
&+\lambda_0 V_{n-1}(0,m_2,m_3),\quad m_2\geq 1,\ m_3\geq 1,\ n\in\mathbb{N},
\end{aligned}\tag{25}$$

and

$$V_n(0,m_2,0) = (1-\lambda_0) + (1-\lambda_0)V_{n-1}(0,m_2-1,0) + \lambda_0 V_{n-1}(0,m_2,0), \quad m_2 \geq 1,\ n \in \mathbb{N}. \tag{26}$$

We first prove that (23) holds for $m_2 \geq 2$ and $m_3 \geq 2$, and treat the boundary cases separately. Using (25) we can write

$$\begin{aligned}
&V_{t+1}(0,m_2,m_3-1)\\
&= (1-\lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0,m_2-1,m_3-1) + \frac{1}{2}\lambda_2 V_t(0,m_2,m_3-2) + \lambda_0 V_t(0,m_2,m_3-1)\\
&\geq (1-\lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0,m_2-2,m_3) + \frac{1}{2}\lambda_2 V_t(0,m_2-1,m_3-1) + \lambda_0 V_t(0,m_2-1,m_3)\\
&= V_{t+1}(0,m_2-1,m_3).
\end{aligned}$$

Here we use the induction hypothesis (23) (with $n=t$) for the inequality, and use (25) for the second equality.

For the case $m_2 \geq 2$ and $m_3 = 1$ we use (26) to obtain

$$\begin{aligned}
&V_{t+1}(0,m_2,0)\\
=&(1-\lambda_0) + (1-\lambda_0)V_t(0,m_2-1,0) + \lambda_0 V_t(0,m_2,0)\\
\geq&(1-\lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0,m_2-2,1) + \frac{1}{2}\lambda_2 V_t(0,m_2-1,0) + \lambda_0 V_t(0,m_2-1,1)\\
=&V_{t+1}(0,m-1,1),
\end{aligned}$$

where the inequality follows from the induction hypothesis (23), and the final equality from our knowledge on the optimal control for $n = t+1$, see (25).

For the case $m_2 = 1$ and $m_3 \geq 2$ we write, using (25),

$$\begin{aligned}
&V_{t+1}(0,1,m_3-1)\\
&= (1-\lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0,0,m_3-1) + \frac{1}{2}\lambda_2 V_t(0,1,m_3-2) + \lambda_0 V_t(0,1,m_3-1)\\
&= (1-\lambda_0-\lambda_1) + \frac{1}{2}\lambda_2 V_t(0,0,m_3-1) + \frac{1}{2}\lambda_2 V_t(0,1,m_3-2) + \lambda_1\big(1 + V_t(0,0,m_3-1)\big)\\
&\quad + \lambda_0 V_t(0,1,m_3-1)\\
&\geq (1-\lambda_0-\lambda_1) + \lambda_2 V_t(0,0,m_3-1) + (\lambda_0+\lambda_1)V_t(0,0,m_3)\\
&=V_{t+1}(0,0,m_3).
\end{aligned}$$

For the second inequality, we use the induction hypothesis (23) and apply Lemma 3(ii) to show that $1 + V_t(0,0,m_3-1) \geq V_t(0,0,m_3)$.

The case $m_2 = m_3 = 1$ we can do directly, by observing that

$$V_{t+1}(0,1,0) = 1-(1-\lambda_0)^{t+1} \geq 1-(1-\lambda_0-\lambda_2)^{t+1} = V_{t+1}(0,0,1),$$

completing the proof. □

With Lemma 4, we can now prove Proposition 2.

Proof of Proposition 2. From the boundary conditions, it is easy to see that $V_0(\mathbf{m}) = 0$ regardless of $\mathbf{m}$. When $m_2 = 0$ the problem degenerates into two separate problems with a single customer type and single slot type where the straightforward optimal decision is to offer all slots to customers. When either $m_1 = 0$ or $m_3 = 0$, the problem reduces to an "N" model and it is optimal to offer all available slots (see Proposition 1). Thus, what remains to be shown is that when none of the slots are depleted, it is optimal to offer type-1 and type-3 slots, but block type-2 slots.

Throughout this proof we assume that $\mathbf{m} \geq (1,1,1)$, unless stated otherwise. In this case, the Bellman equation can be written as

$$V_n(\mathbf{m}) = \max \left\{ \begin{array}{l} 1-\lambda_0 + \frac{1}{2}\lambda_1 V_{n-1}(\mathbf{m}-\mathbf{e}_1) + \frac{1}{2}(\lambda_1+\lambda_2)V_{n-1}(\mathbf{m}-\mathbf{e}_2) + \frac{1}{2}\lambda_2 V_{n-1}(\mathbf{m}-\mathbf{e}_3) \\ +\lambda_0 V_{n-1}(\mathbf{m}), \\ 1-\lambda_0 + \frac{1}{2}\lambda_1 V_{n-1}(\mathbf{m}-\mathbf{e}_1) + (\frac{1}{2}\lambda_1+\lambda_2)V_{n-1}(\mathbf{m}-\mathbf{e}_2) + \lambda_0 V_{n-1}(\mathbf{m}), \\ 1-\lambda_0 + \lambda_1 V_{n-1}(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_{n-1}(\mathbf{m}-\mathbf{e}_3) + \lambda_0 V_{n-1}(\mathbf{m}), \\ 1-\lambda_0 + (\lambda_1+\frac{1}{2}\lambda_2)V_{n-1}(\mathbf{m}-\mathbf{e}_2) + \frac{1}{2}\lambda_2 V_{n-1}(\mathbf{m}-\mathbf{e}_3) + \lambda_0 V_{n-1}(\mathbf{m}), \\ \lambda_1 + \lambda_1 V_{n-1}(\mathbf{m}-\mathbf{e}_1) + (\lambda_0+\lambda_2)V_{n-1}(\mathbf{m}), \\ 1-\lambda_0 + (\lambda_1+\lambda_2)V_{n-1}(\mathbf{m}-\mathbf{e}_2) + \lambda_0 V_{n-1}(\mathbf{m}), \\ \lambda_2 + \lambda_2 V_{n-1}(\mathbf{m}-\mathbf{e}_3) + (\lambda_0+\lambda_1)V_{n-1}(\mathbf{m}) \end{array} \right\}, \tag{27}$$

where the seven terms in the max operator correspond to the action of offering slot types $\{1,2,3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1\}$, $\{2\}$ and $\{3\}$, respectively.

For ease of notation we define $\Delta_n^{ij}(\mathbf{m})$ to be the difference of the ith and jth terms in the max operator (27) above, $i,j \in \{1,2,\ldots,7\}$. To prove the desired result it suffices to show for any $n \in$ that $\Delta_n^{3,j} \geq 0$, $j \neq 3$.

First, by writing out the definition,

$$\Delta_n^{35}(\mathbf{m}) = \lambda_2[1 + (V_{n-1}(\mathbf{m}-\mathbf{e}_3) - V_{n-1}(\mathbf{m}))] \geq 0, \tag{28}$$

$$\Delta_n^{37}(\mathbf{m}) = \lambda_1[1 + (V_{n-1}(\mathbf{m}-\mathbf{e}_1) - V_{n-1}(\mathbf{m}))] \geq 0. \tag{29}$$

The equalities follow from the fact that $V_{n-1}(\mathbf{m}) - V_{n-1}(\mathbf{m}-\mathbf{e}_3) \leq 1$ (for (28)) and $V_{n-1}(\mathbf{m}) - V_{n-1}(\mathbf{m}-\mathbf{e}_1) \leq 1$ (for (29)), see Lemma 3.(i).

The other four inequalities can be written as

$$\Delta_{n+1}^{31} \geq 0 \Leftrightarrow \lambda_1 V_n(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_n(\mathbf{m}-\mathbf{e}_3) \geq (\lambda_1+\lambda_2)V_n(\mathbf{m}-\mathbf{e}_2), \tag{30}$$

$$\Delta_{n+1}^{32} \geq 0 \Leftrightarrow \frac{1}{2}\lambda_1 V_n(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_n(\mathbf{m}-\mathbf{e}_3) \geq \big(\frac{1}{2}\lambda_1+\lambda_2\big)V_n(\mathbf{m}-\mathbf{e}_2), \tag{31}$$

$$\Delta_{n+1}^{34} \geq 0 \Leftrightarrow \lambda_1 V_n(\mathbf{m}-\mathbf{e}_1) + \frac{1}{2}\lambda_2 V_n(\mathbf{m}-\mathbf{e}_3) \geq (\lambda_1+\frac{1}{2}\lambda_2)V_n(\mathbf{m}-\mathbf{e}_2), \tag{32}$$

$$\Delta_{n+1}^{36} \geq 0 \Leftrightarrow \lambda_1 V_n(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_n(\mathbf{m}-\mathbf{e}_3) \geq (\lambda_1+\lambda_2)V_n(\mathbf{m}-\mathbf{e}_2). \tag{33}$$

Note that (30) and (33) are equivalent, as are (31) and (32), due to symmetry. Thus, we limit ourselves to showing that (30) and (31) hold, which we will do by induction.

Let $n = 1$, then it is readily seen that for (30),

$$\lambda_1 V_1(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_1(\mathbf{m}-\mathbf{e}_3) = (\lambda_1+\lambda_2)(1-\lambda_0) = (\lambda_1+\lambda_2)V_1(\mathbf{m}-\mathbf{e}_2),$$

and for (31),

$$\frac{1}{2}\lambda_1 V_1(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_1(\mathbf{m}-\mathbf{e}_3) = (\frac{1}{2}\lambda_1+\lambda_2)(1-\lambda_0) = (\frac{1}{2}\lambda_1+\lambda_2)V_1(\mathbf{m}-\mathbf{e}_2),$$

so both hold.

Next we let $t \in \mathbb{N}$ and assume that (30)-(33) hold for all $n \le t-1$, i.e.,

$$\lambda_1 V_n(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_n(\mathbf{m}-\mathbf{e}_3) \ge (\lambda_1+\lambda_2) V_n(\mathbf{m}-\mathbf{e}_2), \quad n \le t-1, \tag{34}$$

$$\frac{1}{2}\lambda_1 V_n(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_n(\mathbf{m}-\mathbf{e}_3) \ge (\frac{1}{2}\lambda_1+\lambda_2) V_n(\mathbf{m}-\mathbf{e}_2), \quad n \le t-1. \tag{35}$$

In this case we know that g_n in (4) provides an optimal policy for all $n \le t$. We shall now demonstrate that (34) and (35) hold for $n = t$ as well, which implies that g_n is also optimal for $n = t+1$. Since we know an optimal control policy for $n \le t$, we also know the transition probabilities given that we use optimal control.

$$\begin{aligned}
p_0(\mathbf{m}) &= \lambda_0 + \lambda_1 \mathbb{1}_{\{m_1=m_2=0\}} + \lambda_2 \mathbb{1}_{\{m_2=m_3=0\}}, \\
p_1(\mathbf{m}) &= \lambda_1 \mathbb{1}_{\{m_1 \ge 1\}}, \\
p_2(\mathbf{m}) &= \lambda_1 \mathbb{1}_{\{m_1=0,\ m_2\ge 1\}} + \lambda_2 \mathbb{1}_{\{m_2 \ge 1,\ m_3=0\}}, \\
p_3(\mathbf{m}) &= \lambda_2 \mathbb{1}_{\{m_3 \ge 1\}}.
\end{aligned}$$

Using the above transition probabilities we can compute

$$V_t(\mathbf{m}-\mathbf{e}_1) = 1-\lambda_0 + \lambda_1 V_{t-1}(\mathbf{m}-2\mathbf{e}_1) + \lambda_2 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_1), \tag{36}$$

$$V_t(\mathbf{m}-\mathbf{e}_3) = 1-\lambda_0 + \lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m}-2\mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_3). \tag{37}$$

Moreover, we know from the induction hypothesis (34) that

$$\lambda_1 V_{t-1}(\mathbf{m}-2\mathbf{e}_1) + \lambda_2 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_3) \ge (\lambda_1+\lambda_2) V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2), \tag{38}$$

$$\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m}-2\mathbf{e}_3) \ge (\lambda_1+\lambda_2) V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3). \tag{39}$$

Using (36)-(39), we can write

$$\begin{aligned}
&\lambda_1 V_t(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_t(\mathbf{m}-\mathbf{e}_3) \\
&\ge (\lambda_1+\lambda_2)(1-\lambda_0) + \lambda_1(\lambda_1+\lambda_2) V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + \lambda_2(\lambda_1+\lambda_2) V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3) \\
&+ \lambda_0\big(\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_{t-1}(\mathbf{m}-\mathbf{e}_3)\big) \\
&\ge (\lambda_1+\lambda_2)(1-\lambda_0) + \lambda_1(\lambda_1+\lambda_2) V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + \lambda_2(\lambda_1+\lambda_2) V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3) \\
&+ \lambda_0(\lambda_1+\lambda_2) V_{t-1}(\mathbf{m}-\mathbf{e}_2) \\
&= (\lambda_1+\lambda_2) V_t(\mathbf{m}-\mathbf{e}_2),
\end{aligned}$$

where the second inequality follows from the induction hypothesis (34). This proves the desired inequality.

Similarly, to verify (31) we use (36) and (37) and apply the induction hypothesis (35) to obtain, after some rearranging,

$$\begin{aligned}
&\frac{1}{2}\lambda_1 V_t(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_t(\mathbf{m}-\mathbf{e}_3) \\
\ge&(\frac{1}{2}\lambda_1+\lambda_2)(1-\lambda_0) + (\frac{1}{2}\lambda_1+\lambda_2)\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + (\frac{1}{2}\lambda_1+\lambda_2)\lambda_2 V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3) \\
&+ (\frac{1}{2}\lambda_1+\lambda_2)\lambda_0(\mathbf{m}-\mathbf{e}_2) \qquad (40) \\
&= (\frac{1}{2}\lambda_1+\lambda_2) V_t(\mathbf{m}-\mathbf{e}_2). \qquad (41)
\end{aligned}$$

Next, we verify the induction hypotheses for the various boundary cases. First, it is readily verified, using our knowledge of the optimal control for $n=t$, that for $m_1=1$

$$\begin{aligned} V_t(\mathbf{m}-\mathbf{e}_1) &= (1-\lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_{t-1}(0,m_2-1,m_3) + \frac{1}{2}\lambda_2 V_{t-1}(0,m_2,m_3-1) + \lambda_0 V_{t-1}(0,m_2,m_3) \\ &\geq 1-\lambda_0 + (\lambda_1+\lambda_2)V_{t-1}(0,m_2-1,m_3) + \lambda_0 V_{t-1}(0,m_2,m_3), \end{aligned} \tag{42}$$

where the inequality follows from Lemma 4. Analogously, we derive

$$V_t(\mathbf{m}-\mathbf{e}_3) \geq 1-\lambda_0 + (\lambda_1+\lambda_2)V_{t-1}(m_1,m_2-1,0) + \lambda_0 V_{t-1}(m_1,m_2,0), \quad m_3=1. \tag{43}$$

First we treat the case $m_1=1$ and $m_3\geq 2$. Combining (37) and (42) yields

$$\begin{aligned} &\lambda_1 V_t(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_t(\mathbf{m}-\mathbf{e}_3) \\ \geq &\lambda_1[(1-\lambda_0) + (\lambda_1+\lambda_2)V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + \lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_1)] \\ &+ \lambda_2[(1-\lambda_0) + \lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m}-2\mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_3)] \\ \geq &(\lambda_1+\lambda_2)(1-\lambda_0) + (\lambda_1+\lambda_2)\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + (\lambda_1+\lambda_2)\lambda_2 V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3) \\ &+ (\lambda_1+\lambda_2)\lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_2) \\ &= (\lambda_1+\lambda_2)V_t(\mathbf{m}-\mathbf{e}_2), \end{aligned}$$

with the second inequality due to the induction hypothesis (34).

In order to show (35) we can again use (37) and (42), and do some rearranging to show that

$$\begin{aligned} &\frac{1}{2}\lambda_1 V_t(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_t(\mathbf{m}-\mathbf{e}_3) \\ \geq &\frac{1}{2}\lambda_1[(1-\lambda_0) + (\lambda_1+\lambda_2)V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + \lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_1)] \\ &+ \lambda_2[(1-\lambda_0) + \lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m}-2\mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_3)] \\ \geq &(\frac{1}{2}\lambda_1+\lambda_2)(1-\lambda_0) + (\lambda_1+\lambda_2)\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + \lambda_2[\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_3) \\ &+ (\frac{1}{2}\lambda_1+\lambda_2)V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3)] + (\frac{1}{2}\lambda_1+\lambda_2)\lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_2) \\ \geq &(\frac{1}{2}\lambda_1+\lambda_2)(1-\lambda_0) + (\lambda_1+\lambda_2)\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) + \lambda_2[\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) \\ &+ (\frac{1}{2}\lambda_1+\lambda_2)V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3)] + (\frac{1}{2}\lambda_1+\lambda_2)\lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_2) \\ = &(\frac{1}{2}\lambda_1+\lambda_2)V_{t-1}(\mathbf{m}-\mathbf{e}_2), \end{aligned} \tag{44}$$

where the second and third equalities follows from the induction hypothesis (35) and Lemma 4, respectively. This shows that the (35) holds for $m_1=1$, $m_3\geq 2$.

The proof for the case $m_1\geq 2$, $m_3=1$ follows from symmetry. Finally, we verify the case $m_1=m_3=1$. We first bound, using (42) and (43),

$$\begin{aligned} &\lambda_1 V_t(\mathbf{m}-\mathbf{e}_1) + \lambda_2 V_t(\mathbf{m}-\mathbf{e}_3) \geq (\lambda_1+\lambda_2)(1-\lambda_0) + (\lambda_1+\lambda_2)\lambda_1 V_{t-1}(\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2) \\ &+ (\lambda_1+\lambda_2)\lambda_2 V_{t-1}(\mathbf{m}-\mathbf{e}_2-\mathbf{e}_3) + (\lambda_1+\lambda_2)\lambda_0 V_{t-1}(\mathbf{m}-\mathbf{e}_2) \\ &= (\lambda_1+\lambda_2)V_t(\mathbf{m}-\mathbf{e}_2). \end{aligned}$$

Using these same inequalities we can show

$$\begin{aligned}
&\frac{1}{2}\lambda_1 V_t(\mathbf{m}-\mathbf{e}_1)+\lambda_2 V_t(\mathbf{m}-\mathbf{e}_3)\\
&\geq \frac{1}{2}\lambda_1[(1-\lambda_0)+(\lambda_1+\lambda_2)V_{t-1}(0,m_2-1,1)+\lambda_0 V_{t-1}(0,m_2,1)]\\
&+\lambda_2[(1-\lambda_0)+(\lambda_2+\frac{1}{2}\lambda_1)V_{t-1}(1,m_2-1,0)+\frac{1}{2}\lambda_1 V_{t-1}(0,m_2,0)+\lambda_0 V_{t-1}(1,m_2,0)]\\
&\geq (\frac{1}{2}\lambda_1+\lambda_2)(1-\lambda_0)+(\frac{1}{2}\lambda_1+\lambda_2)\lambda_2 V_{t-1}(1,m_2-1,0)+(\frac{1}{2}\lambda_1+\lambda_2)\lambda_1 V_{t-1}(0,m_2-1,1)\\
&+\lambda_0[\frac{1}{2}\lambda_1 V_{t-1}(0,m_2,1)+\lambda_2 V_{t-1}(1,m_2,0)]\\
&\geq (\frac{1}{2}\lambda_1+\lambda_2)(1-\lambda_0)+(\frac{1}{2}\lambda_1+\lambda_2)\lambda_2 V_{t-1}(1,m_2-1,0)+(\frac{1}{2}\lambda_1+\lambda_2)\lambda_1 V_{t-1}(0,m_2-1,1)\\
&+\lambda_0(\frac{1}{2}\lambda_1+\lambda_2)V_{t-1}(1,m_2-1,1)\\
&=\Big(\frac{1}{2}\lambda_1+\lambda_2\Big)V_t(\mathbf{m}-\mathbf{e}_2),
\end{aligned}$$

with the second inequality using Lemma 4.(i). and the third inequality due to the induction hypothesis (35). This completes the proof. □

A.4. Proof of Corollary 1

We prove by contradiction. Suppose (5) does not hold and thus

$$V_n(\mathbf{m}-e_2)>V_n(\mathbf{m}-e_1) \text{ and } V_n(\mathbf{m}-e_2)>V_n(\mathbf{m}-e_3). \tag{45}$$

In period $n+1$ and at state $\mathbf{m}$, action $\mathbf{d}_1:=(1,0,1)$ yields the value-to-go of

$$p_1(\mathbf{m},\mathbf{d}_1)V_n(\mathbf{m}-e_1)+p_3(\mathbf{m},\mathbf{d}_1)V_n(\mathbf{m}-e_3)+\lambda_0 V_n(\mathbf{m}),$$

which is strictly less than the value-to-go under action $\mathbf{d}_2=(0,1,0)$ given by

$$p_2(\mathbf{m},\mathbf{d}_2)V_n(\mathbf{m}-e_2)+\lambda_0 V_n(\mathbf{m}),$$

by using (45) and $p_1(\mathbf{m},\mathbf{d}_1)+p_3(\mathbf{m},\mathbf{d}_1)=p_2(\mathbf{m},\mathbf{d}_2)=1-\lambda_0$. This contradicts the result in Proposition 2 on the optimality of $\mathbf{d}_1$. □

A.5. Proof of Theorem 1

This proof entails a few key steps. First, we show that the optimal amount of the customers scheduled in the fluid model is an upper bound to that in the corresponding stochastic model (see Proposition 5 below). Then, we construct a lower bound for the objective value of the stochastic model under any static randomized policy (see Lemma 6 below). Finally, we show that under the static randomized policy π^{p^*} this lower bound, after normalization (i.e., divided by the scaling factor K), converges to the optimal objective value of the fluid model, which is a constant upper bound for the stochastic model. To economize our notation in the proof below, we let $\mathcal{I}=\{1,2,\ldots,I\}$ be the set of customer types and $\mathcal{J}=\{1,2,\ldots,J\}$ be the set of slot types.

Lemma 5.

$$Z_n(\mathbf{m})\geq V_n(\mathbf{m}),\ \forall n=1,2,\ldots,N,\ \mathbf{m}\in\mathbb{Z}_+^J.$$

Proof. We first show that Problem (P1) has an equivalent dynamic programming (DP) formulation. This DP formulation will facilitate our proof that the fluid model provides an upper bound for the stochastic model. To differentiate from the stochastic model, we let $\tilde{V}_n(\mathbf{m})$ be the maximum amount of fluid that can be served given n periods to go and the capacity vector $\mathbf{m}$. Consider the following DP formulation.

$$\tilde{V}_n(\mathbf{m}) = \max\{\sum_{j\in\mathcal{J}} y_j(n) + \tilde{V}_{n-1}(\mathbf{m}-\mathbf{y}(n))\}, \tag{46}$$

$$\text{subject to:}\quad y_j(n) = \sum_{i\in\mathcal{I}} y_{i,j}(n),\ j\in\mathcal{J}, \tag{47}$$

$$\mathbf{m}-\mathbf{y}(n)\geq 0, \tag{48}$$

$$\tilde{V}_0(\mathbf{x}) = 0,\ \forall \mathbf{x}\geq 0, \tag{49}$$

$$\text{and (6), (7), (8), (9), (10) defined for } n \text{ only.} \tag{50}$$

Recall that $Z_n(\mathbf{m})$ is the optimal objective value to Problem (P1) with $M_j(n) = m_j$ and n periods left to go. We claim that

$$Z_n(\mathbf{m}) = \tilde{V}_n(\mathbf{m}),\ \forall n = 1,2,\ldots,N,\ \mathbf{m}\in\mathbb{Z}_+^J. \tag{51}$$

We use induction to prove this claim. It is easy to check the cases for $n=1$. Now suppose that $Z_n(\mathbf{m}) = \tilde{V}_n(\mathbf{m})$ holds for $n = 2,3,\ldots,N-1$, and consider that $n = N$. Consider an optimal solution f^* under the LP formulation. Following the decision at period N specified by f^* in both the LP and DP formulations. We see that the amount of fluid served in period N is the same under both formulations, and that the capacity left for period $N-1$ is also the same for both formulations. Following the induction hypothesis, we know that the total amount of fluid served from $N-1$ periods onward is the same under both formulations. Now, the optimal action for period N under the LP formulation is clearly feasible for the DP formulation, but not necessarily optimal. Thus we have $Z_N(\mathbf{m}) \leq \tilde{V}_N(\mathbf{m})$.

Taking the optimal action in period N under the DP formulation, and apply it to both the DP and LP formulations. Following a similar argument above, we can show that $Z_N(\mathbf{m}) \geq \tilde{V}_N(\mathbf{m})$. It thus follows that $Z_N(\mathbf{m}) = \tilde{V}_N(\mathbf{m})$, as desired.

Now, to prove Lemma 5, it suffices to show that

$$\tilde{V}_n(\mathbf{m}) \geq V_n(\mathbf{m}),\ \forall n = 1,2,\ldots,N,\ \mathbf{m}\in\mathbb{Z}_+^J.$$

We use induction to show this result. We first check the case when $n=1$. Suppose that action k corresponds to the optimal action taken in the stochastic model at $n=1$. In the fluid model, we use the same action. That is, we set $z_k(n) = 1$ and set $z_s(n) = 0$ for $s\neq k$. The feasibility of the optimal action in the stochastic model implies that for each type of the slots opened, there is at least 1 unit of capacity. Thus, in the fluid model, we can set $\tau_{k,j}(n) = \mathbf{w}_j^k$ for all $j\in\mathcal{J}$ as the draining speed for each type of slots is bounded by 1 implied by constraint (9). Then, one can algebraically check that the expected number of customers served in the stochastic model is the same as the amount of the fluid served in the fluid model.

Now suppose $\tilde{V}_n(\mathbf{m}) \geq V_n(\mathbf{m})$ holds up to $n = 2,3,\ldots,N-1$ and consider $n = N$. Again, we apply the optimal action in the stochastic model at period N, say $\mathbf{d}$, to the fluid model at period N. Here we let

$p_k(\mathbf{m},\mathbf{d})$ be the probability that a type k slot is booked at state $\mathbf{m}$ if action $\mathbf{d}$ is taken. Using (2), one can check that

$$\sum_{j\in\mathcal{J}} p_j(\mathbf{m},\mathbf{d}) = \sum_{j\in\mathcal{J}} y_j(n) \tag{52}$$

and

$$\sum_{j=0}^{J} p_j(\mathbf{m},\mathbf{d})(\mathbf{m}-\mathbf{e}_j) = \mathbf{m}-\mathbf{y}(N). \tag{53}$$

The first equation (52) above suggests that the amount of customers served in both models are the same. The second equation (53) implies that the system state at period $N-1$ in the fluid model is a convex combination of the possible states that a stochastic model may reach, in which the weights are the associated state transition probabilities. To simplify notation, we let $p_j = p_j(\mathbf{m},\mathbf{d}),\ j=0,1,\ldots,J$. We claim that, for the fluid model,

$$\tilde{V}_{N-1}(\mathbf{m}-\mathbf{y}(n)) \ge \sum_{j=0}^{J} p_j \tilde{V}_{N-1}(\mathbf{m}-\mathbf{e}_j), \tag{54}$$

which will be proved at the end. By the induction hypothesis, we have that

$$\tilde{V}_{N-1}(\mathbf{m}-\mathbf{e}_j) \ge V_{N-1}(\mathbf{m}-\mathbf{e}_j),\ \forall j=0,1,\ldots,J. \tag{55}$$

It follows that

$$\tilde{V}_N(\mathbf{m}) \ge \sum_{j\in\mathcal{J}} y_j(N) + \tilde{V}_{N-1}(\mathbf{m}-\mathbf{y}(N)) \tag{56}$$

$$= \sum_{j\in\mathcal{J}} p_j + \tilde{V}_{N-1}(\mathbf{m}-\mathbf{y}(N)) \tag{57}$$

$$\ge \sum_{j=0}^{J} p_j[\mathbb{1}_{\{j>0\}} + \tilde{V}_{N-1}(\mathbf{m}-\mathbf{e}_j)] \tag{58}$$

$$\ge \sum_{j=0}^{J} p_j[\mathbb{1}_{\{j>0\}} + V_{N-1}(\mathbf{m}-\mathbf{e}_j)] \tag{59}$$

$$= V_N(\mathbf{m}). \tag{60}$$

Inequality (56) holds as the optimal action $\mathbf{d}$ for the stochastic model may not be optimal for the fluid model; (57) holds because of (52); inequalities (58) and (59) follow from (54) and (55), respectively; equality (60) holds by definition.

Finally, we prove our claim (54) for $\mathbf{y}(N)$ that satisfies (53). To do this, we turn to the LP formulation (P1) for the fluid model. So $\tilde{V}_{N-1}(\cdot)$ in (54) is equal to the optimal objective value of the corresponding LP formulation by claim (51). To simplify the notation, we can imagine that this fluid model can be written into the following standard form of LP:

$$\begin{aligned} \tilde{V}_{N-1}(\mathbf{h}) = \max\ & \mathbf{c}\mathbf{x}, \\ \text{subject to:}\quad & A\mathbf{x} = \mathbf{h} \\ & B\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge 0, \end{aligned}$$

in which $\mathbf{x}$ is the vector of decision variables, $\mathbf{h}$ is the vector for slots capacity, $\mathbf{b}$ is the vector for other right-hand-side coefficients, and A, B and C are properly constructed matrices representing the coefficients for $\mathbf{x}$ in the constraint sets. Denote the optimal decision to this LP formulation when $\mathbf{h}=\mathbf{m}-\mathbf{e}_j$ as $\mathbf{x}_j$, $j=0,1,\ldots,J$. It is easy to check that a solution $\sum_{j=0}^{J} p_j\mathbf{x}_j$ is feasible (but not necessarily optimal) to the LP when $\mathbf{h}$ is replaced by $\mathbf{m}-\mathbf{y}(N)$ and other coefficients are fixed, due to (53) and that $\sum_{j=0}^{J} p_j\mathbf{x}_j$ is a convex combination of $\mathbf{x}_j$'s. Thus we have

$$\tilde{V}_{N-1}(\mathbf{m}-\mathbf{y}(N)) \geq \mathbf{c}\sum_{j=0}^{J} p_j\mathbf{x}_j = \sum_{j=0}^{J} p_j\mathbf{c}\mathbf{x}_j = \sum_{j=0}^{J} p_j\tilde{V}_{N-1}(\mathbf{m}-\mathbf{e}_j),$$

proving the claim (54) and completing the whole proof. □

Before presenting Lemma 6, we introduce a few ancillary notations first. Recall that a static randomized policy π^p offers $\mathbf{w}^k$ with probability p_k. Define $\mathcal{I}_j=\{i:\Omega_{ij}=1\}$ be the set of customer types who accept type j slots. Recall that $\mathcal{K}_j=\{s:w_j^s=1, s\in\mathcal{K}\}$ be the index set of actions that offer type j slots. Let

$$\Upsilon_j=\sum_{k\in\mathcal{K}_j} p_k\Big(\sum_{i\in\mathcal{I}_j}\frac{\lambda_i}{\sum_{l,l\in\mathcal{J}}\Omega_{il}w_l^k}\Big)$$

be the probability that a type j slot will be taken under policy π^p when $\mathbf{m}>0$. To simplify notations below, let $\mathcal{I}_j(\mathbf{m})=\mathcal{I}_j$ if $m_j>0$ and $\mathcal{I}_j(\mathbf{m})=\emptyset$ if $m_j=0$. Define

$$\Upsilon_j(\mathbf{m})=\sum_{k\in\mathcal{K}_j} p_k\Big(\sum_{i\in\mathcal{I}_j(\mathbf{m})}\frac{\lambda_i}{\sum_{l:m_l>0,l\in\mathcal{J}}\Omega_{il}w_l^k}\Big) \tag{61}$$

be the probability that a type j slot will be taken under policy π^p when some of m_j's are zeros. Note that Υ_j is a constant while $\Upsilon_j(\mathbf{m})$ depends on $\mathbf{m}$. Also, $\Upsilon_j(\mathbf{m})\geq\Upsilon_j$ for j such that $m_j>0$.

LEMMA 6. *For any static randomized policy π^p, $V_n^{\pi^p}(\mathbf{m})\geq\sum_{j\in\mathcal{J}}\mathbb{E}[Bin(n,\Upsilon_j)\wedge m_j]$, $\forall\mathbf{m}\geq 0$ and $n=1,2,\ldots,N$.*

Proof. We prove this result by induction. Consider the case when $n=1$. The above inequality holds as equality if $\mathbf{m}>0$. If there are some $m_j=0$, then

$$\begin{aligned} V_1^{\pi^p}(\mathbf{m}) &= \sum_{j:m_j>0}\mathbb{E}[Bin(1,\Upsilon_j(\mathbf{m}))] \\ &= \sum_{j:m_j>0}\mathbb{E}[Bin(1,\Upsilon_j(\mathbf{m}))\wedge m_j] \\ &\geq \sum_{j:m_j>0}\mathbb{E}[Bin(1,\Upsilon_j)\wedge m_j] \\ &= \sum_j \mathbb{E}[Bin(n,\Upsilon_j)\wedge m_j], \end{aligned}$$

where the first inequality above follows from (61).

Now, assume that the desired inequality holds up to $n-1$ and consider the case of n. If $\mathbf{m}>0$, then

$$\begin{aligned} V_n^{\pi^p}(\mathbf{m}) &= \sum_j \Upsilon_j(1+V_{n-1}^{\pi^p}(\mathbf{m}-e_j)) + (1-\sum_j\Upsilon_j)V_{n-1}^{\pi^p}(\mathbf{m}) \\ &\geq \sum_j \Upsilon_j[1+\sum_{s\neq j}E(Bin(n-1,\Upsilon_s)\wedge m_s)+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))] \end{aligned}$$

$$
\begin{aligned}
&+(1-\sum_j \Upsilon_j)\sum_t E(Bin(n-1,\Upsilon_t)\wedge m_t)\\
&=\sum_j \Upsilon_j[1+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]+\sum_t E(Bin(n-1,\Upsilon_t)\wedge m_t)\\
&+\sum_j \Upsilon_j[\sum_{s\neq j} E(Bin(n-1,\Upsilon_s)\wedge m_s)-\sum_t E(Bin(n-1,\Upsilon_t)\wedge m_t)]\\
&=\sum_j \Upsilon_j[1+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]\\
&+\sum_j E(Bin(n-1,\Upsilon_j)\wedge m_j)-\sum_j \Upsilon_j E(Bin(n-1,\Upsilon_j)\wedge m_j)\\
&=\sum_j \Upsilon_j[1+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]+\sum_j(1-\Upsilon_j)E(Bin(n-1,\Upsilon_j)\wedge m_j)\\
&=\sum_j E(Bin(n,\Upsilon_j)\wedge m_j).
\end{aligned}
$$

If there are some $m_j=0$, then

$$
\begin{aligned}
V_n^{\pi^p}(\mathbf{m}) &= \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})(1+V_{n-1}^{\pi^p}(\mathbf{m}-e_j))+(1-\sum_j \Upsilon_j(\mathbf{m}))V_{n-1}^{\pi^p}(\mathbf{m})\\
&\geq \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})[1+\sum_{s\neq j}E(Bin(n-1,\Upsilon_s)\wedge m_s)+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]\\
&\quad +(1-\sum_{j:m_j>0}\Upsilon_j(\mathbf{m}))\sum_t E(Bin(n-1,\Upsilon_t)\wedge m_t)\\
&= \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})[1+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]+\sum_t E(Bin(n-1,\Upsilon_t)\wedge m_t)\\
&+ \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})[\sum_{s\neq j}E(Bin(n-1,\Upsilon_s)\wedge m_s)-\sum_t E(Bin(n-1,\Upsilon_t)\wedge m_t)]\\
&= \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})[1+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]+\sum_{t:m_t>0} E(Bin(n-1,\Upsilon_t)\wedge m_t)\\
&+ \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})[\sum_{s\neq j}E(Bin(n-1,\Upsilon_s)\wedge m_s)-\sum_{t:m_t>0} E(Bin(n-1,\Upsilon_t)\wedge m_t)]\\
&= \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})[1+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]\\
&+ \sum_{j:m_j>0}E(Bin(n-1,\Upsilon_j)\wedge m_j)-\sum_{j:m_j>0}\Upsilon_j(\mathbf{m})E(Bin(n-1,\Upsilon_j)\wedge m_j)\\
&= \sum_{j:m_j>0}\Upsilon_j(\mathbf{m})[1+E(Bin(n-1,\Upsilon_j)\wedge(m_j-1))]+\sum_{j:m_j>0}(1-\Upsilon_j(\mathbf{m}))E(Bin(n-1,\Upsilon_j)\wedge m_j)\\
&= \sum_{j:m_j>0}E\{[Bin(1,\Upsilon_j(\mathbf{m}))+Bin(n-1,\Upsilon_j)]\wedge m_j\}\\
&\geq \sum_{j:m_j>0}E[Bin(Bin(n,\Upsilon_j)\wedge m_j]\\
&= \sum_j E(Bin(n,\Upsilon_j)\wedge m_j),
\end{aligned}
$$

where the last inequality results from (61). This completes the proof. □

Before presenting the proof of Theorem 1, we need two more ancillary results. The first result states that for the Kth problem, its objective value of the fluid model is K times that of the base model with $K=1$. The second is a convergence result, and we let $\xrightarrow{D}$ denote convergence in distribution.

LEMMA 7. $Z_{NK}(\mathbf{m}K) = KZ_N(\mathbf{m}),\ \forall \mathbf{m} \geq 0,\ K = 1, 2, 3, \ldots,.$

Proof. It suffices to show that

$$K^{-1}Z_{NK}(\mathbf{m}K) \leq Z_N(\mathbf{m}),\ \forall \mathbf{m} \geq 0,\ K = 1, 2, 3, \ldots, \tag{62}$$

and

$$Z_{NK}(\mathbf{m}K) \geq KZ_N(\mathbf{m}),\ \forall \mathbf{m} \geq 0,\ K = 1, 2, 3, \ldots. \tag{63}$$

To show (62), we let $z_k^*(i, K),\ \forall i = 1, 2, \ldots, NK, \forall k = 1, 2, \ldots, 2^J$ be the optimal solution for the Kth fluid model. Define

$$z_k(n, 1) = \frac{\sum_{i=(n-1)K+1}^{nK} z_k^*(i, K)}{K},\ \forall n = 1, 2, \ldots, N,\ k = 1, 2, \ldots, 2^J.$$

It suffices to show that $z_k(n, 1)$ is a feasible solution for the base fluid model with $K = 1$, and gives an objective value of $K^{-1}Z_{NK}(\mathbf{m}K)$. It is easy to check that $z_k(n, 1)$ satisfies (6), because that $0 \leq z_k^*(i, K) \leq 1$ by definition. To check that $z_k(n, 1)$ satisfies (7), we have that

$$\begin{aligned}
\sum_k z_k(n, 1) &= \frac{1}{K} \sum_k \sum_{i=(n-1)K+1}^{nK} z_k^*(i, K) \\
&= \frac{1}{K} \sum_{i=(n-1)K+1}^{nK} \underbrace{\sum_k z_k^*(i, K)}_{=1 \text{ by definition of } z_k^*(i,K)} \\
&= \frac{1}{K} \cdot K = 1.
\end{aligned}$$

Constraints (8)-(10) hold as they are simply definitions of $\tau_{k,j}(n)$ and $y_{i,j}(n)$.

Now, let $M_j(n, K)$ be the capacity left for slot type j with n periods to go in the Kth fluid model under its respective solution under consideration. To show that $z_k(n, 1)$ gives an objective value of $K^{-1}Z_{NK}(\mathbf{m}K)$, it suffices to show

$$M_j(n, 1) = \frac{1}{K} M_j(nK, K),\ \forall n = 1, 2, \ldots, N. \tag{64}$$

We prove (64) by induction. First consider $n = N$. By definition, we have

$$M_j(N, 1) = m_j = \frac{1}{K}(m_j K) = \frac{1}{K} M_j(NK, K).$$

Assume that (64) holds for $N-1$, $N-2$, ..., n. Consider the case of $n-1$.

$$\begin{aligned}
M_j(n-1, 1) &= M_j(n, 1) - \sum_i \sum_{k \in \mathcal{K}_j} z_k(n, 1) \mathbf{w}_j^k \frac{\lambda_i}{\sum_{l \in \mathcal{J}} \min\{\Omega_{i,l}, \mathbf{w}_l^k\}} \\
&= M_j(n, 1) - \sum_i \sum_{k \in \mathcal{K}_j} \frac{\sum_{s=(n-1)K+1}^{nK} z_k^*(s, K)}{K} \mathbf{w}_j^k \frac{\lambda_i}{\sum_{l \in \mathcal{J}} \min\{\Omega_{i,l}, \mathbf{w}_l^k\}} \\
&= \frac{1}{K}(\underbrace{KM_j(n, 1)}_{= M_j(nK,K) \text{ by induction}} - \underbrace{\sum_{s=(n-1)K+1}^{nK} \sum_i \sum_{k \in \mathcal{K}_j} z_k^*(s, K) \mathbf{w}_j^k \frac{\lambda_i}{\sum_{l \in \mathcal{J}} \min\{\Omega_{i,l}, \mathbf{w}_l^k\}}}_{\text{fluid taking type } j \text{ slots from periods } nK \text{ to } (n-1)K+1 \text{ in the } K\text{th fluid model}}) \\
&= \frac{1}{K} M_j(nK - K, K) \\
&= \frac{1}{K} M_j((n-1)K, K),
\end{aligned}$$

which proves (64). And thus (62) holds.

Next, we prove (63). Let $z_k^*(i,1)$ be the optimal solution to the base fluid model with $K=1$. For $i=1,2,\ldots,N$, define

$$z_k(n,K)=z_k^*(i,1), \text{ if } (i-1)K+1\le n\le iK.$$

It suffices to show that $z_k(n,K)$ is a feasible solution to the Kth fluid model and gives rise to an objective value of $KZ_N(\mathbf{m})$. It is easy to check that $z_k(n,K)$ satisfies (6), (8), (9) and (10). To check (7), note that

$$\sum_k z_k(n,K)=\sum_k z_k^*(i,1)=1, \text{ for } i=1,2,\ldots,N \text{ and } (i-1)K+1\le n\le iK.$$

To show that $z_k(n,K)$ gives rise to an objective value of $KZ_N(\mathbf{m})$, it suffices to show that

$$M_j(nK,K)=KM_j(n,1),\ \forall n=1,2,\ldots,N. \tag{65}$$

We prove (65) by induction. For $n=N$, (65) holds by definition. Assume that (65) holds for $N-1$, $N-2$, ..., n. Consider the case of $n-1$.

$$\begin{aligned}
M_j((n-1)K,K) &= M_j(nK,K)-\sum_{s=(n-1)K+1}^{nK}\sum_i\sum_{k\in\mathcal{K}_j} z_k(s,K)\mathbf{w}_j^k\frac{\lambda_i}{\sum_{l\in\mathcal{J}}\min\{\Omega_{i,l},\mathbf{w}_l^k\}}\\
&= \underbrace{M_j(nK,K)}_{=KM_j(n,1)\text{ by induction}}-\sum_{s=(n-1)K+1}^{nK}\sum_i\sum_{k\in\mathcal{K}_j} z_k^*(n,1)\mathbf{w}_j^k\frac{\lambda_i}{\sum_{l\in\mathcal{J}}\min\{\Omega_{i,l},\mathbf{w}_l^k\}}\\
&= KM_j(n,1)-K\underbrace{\sum_i\sum_{k\in\mathcal{K}_j} z_k^*(n,1)\mathbf{w}_j^k\frac{\lambda_i}{\sum_{l\in\mathcal{J}}\min\{\Omega_{i,l},\mathbf{w}_l^k\}}}_{\text{fluid taking type } j \text{ slots in period } n \text{ for model with } K=1}\\
&= KM_j(n-1,1),
\end{aligned}$$

which proves (65). Thus (63) holds. This completes the proof. □

LEMMA 8. (Billingsley 1968, p. 34) *Suppose that X and $\{X_k\}$ are $\mathbb{R}^n$-valued random variables such that $X_k\xrightarrow{D}X$, and suppose that the functions $h_k:\mathbb{R}^n\to\mathbb{R}$ converge uniformly on compact sets to a continuous function $h:\mathbb{R}^n\to\mathbb{R}$. Then $h_k(X_k)\xrightarrow{D}h(X)$.*

We are now in a position to prove Theorem 1, the main result in this section.

Proof of Theorem 1. Consider the stochastic scheduling policy π^{p^*} defined above. To simplify notations below, we define

$$\Upsilon_j^*=\sum_{k\in\mathcal{K}_j}p_k^*\Big(\sum_{i\in\mathcal{I}_j}\frac{\lambda_i}{\sum_{l,l\in\mathcal{J}}\Omega_{il}\mathbf{w}_l^k}\Big) \tag{66}$$

be the probability that a type j slot will be taken under policy π^{p^*} when $\mathbf{m}>0$. We have

$$K^{-1}\sum_j\mathbb{E}[Bin(NK,\Upsilon_j^*)\wedge m_jK]\le K^{-1}V_{NK}^{\pi^{p^*}}(\mathbf{m}K)\le K^{-1}Z_{NK}(\mathbf{m}K)=Z_N(\mathbf{m}), \tag{67}$$

where the first inequality follows from Lemma 6, the second inequality follows from Lemma 5, and the last equality follows from Lemma 7. The LHS of (67) can be rewritten as

$$K^{-1}\sum_j\mathbb{E}[Bin(NK,\Upsilon_j^*)\wedge m_jK]=\sum_j\mathbb{E}[K^{-1}Bin(NK,\Upsilon_j^*)\wedge m_j] \tag{68}$$

The strong law of large numbers implies that

$$K^{-1}Bin(NK,\Upsilon_j^*) \xrightarrow{D} N\Upsilon_j^* \quad \text{as } K \to \infty.$$

Applying Lemma 8, we conclude that

$$\sum_j [K^{-1}Bin(NK,\Upsilon_j^*) \wedge m_j] \xrightarrow{D} \sum_j [N\Upsilon_j^* \wedge m_j] \quad \text{as } K \to \infty.$$

Because that the random variable $\sum_j [K^{-1}Bin(NK,\Upsilon_j^*) \wedge m_j]$ is uniformly bounded by $\sum_j m_j$, we have that

$$\lim_{K\to\infty} \mathbb{E} \sum_j [K^{-1}Bin(NK,\Upsilon_j^*) \wedge m_j] = \sum_j (N\Upsilon_j^* \wedge m_j) = Z_N(\mathbf{m}), \tag{69}$$

where the last equality follows from the definition of p^* and Υ_j^*. Specifically, Υ_j^* is defined based on the fluid model, via the use of p_k^* which is the proportion of time in which slot type k is offered in the fluid model; see equations (14) and (66). Note that in the fluid model, we have constraints ensuring that a slot type can only be offered if it is still available. Thus Υ_j^* matches exactly the proportion of time in which slot type j is being drawn. This quantity times N is exactly the amount of type j slots being taken in the fluid model. In fact, $(N\Upsilon_j^* \wedge m_j) = N\Upsilon_j^*$ and $\sum_j N\Upsilon_j^* = Z_N(\mathbf{m})$. Combining (67), (68) and (69) gives the desired result and completes the whole proof. □

A.6. Proof of Theorem 2

Proof. We prove this by induction. Let Ω be any preference matrix. For $n=1$, π_0 is optimal and thus $V_1(\mathbf{m}) = V_{1,\pi_0}(\mathbf{m}) \le 2V_{1,\pi_0}(\mathbf{m})$. Suppose the desired result holds for any $n \le k-1$ and state $\mathbf{m}$.

Now we consider two systems, one under an optimal policy and the other under π_0, both starting from state $\mathbf{m}$ in period $n=k$ and operating independently from each other. We denote by $L_k^*(\mathbf{m})$ the slot type filled in period k in the first system (i.e., using an optimal policy), and by $L_k^{\pi_0}(\mathbf{m})$ the slot type filled in period k in the second system (i.e., using π_0). These two random variables are independent and we shall next condition on them. Specifically, let $V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m}))$ denote the value attained in the first system conditioning on these two random variables. types. We have that

$$\begin{aligned}
V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})) &= E[\mathbb{1}_{\{L_k^*(\mathbf{m})>0\}}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})] + V_{k-1}(\mathbf{m}-\mathbf{e}_{L_k^*(\mathbf{m})}) \\
&= \mathbb{1}_{\{L_k^*(\mathbf{m})>0\}} + V_{k-1}(\mathbf{m}-\mathbf{e}_{L_k^*(\mathbf{m})}) \\
&\le \mathbb{1}_{\{L_k^*(\mathbf{m})>0\}} + V_{k-1}(\mathbf{m}) \qquad (70)\\
&\le \mathbb{1}_{\{L_k^*(\mathbf{m})>0\}} + \mathbb{1}_{\{l>0\}} + V_{k-1}(\mathbf{m}-\mathbf{e}_l),\ \forall l \in \bar{S}(\mathbf{m}) \cup \{0\}, \qquad (71)
\end{aligned}$$

where inequality (70) follows from the left inequality of Lemma 3 (ii) and inequality (71) holds due to the right inequality of Lemma 3 (ii). We now let $l = L_k^{\pi_0}(\mathbf{m})$ in (71) and in turn have

$$V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})) \le \mathbb{1}_{\{L_k^*(\mathbf{m})>0\}} + \mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m})>0\}} + V_{k-1}(\mathbf{m}-\mathbf{e}_{L_k^{\pi_0}(\mathbf{m})}).$$

Further applying the induction hypothesis to the above inequality, we obtain

$$V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})) \le \mathbb{1}_{\{L_k^*(\mathbf{m})>0\}} + \mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m})>0\}} + 2V_{k-1,\pi_0}(\mathbf{m}-\mathbf{e}_{L_k^{\pi_0}(\mathbf{m})}).$$

Finally, taking expectations on both sides of the above inequality leads to

$$V_k(\mathbf{m}) \le E[\mathbb{1}_{\{L_k^*(\mathbf{m})>0\}}] + E[\mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m})>0\}}] + 2E[V_{k-1,\pi_0}(\mathbf{m} - \mathbf{e}_{L_k^{\pi_0}(\mathbf{m})})].$$

Now note that $E[\mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m})>0\}}] \ge E[\mathbb{1}_{\{L_k^*(\mathbf{m})>0\}}]$ by the definition of the greedy policy, and hence we arrive at

$$V_k(\mathbf{m}) \le 2E[\mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m})>0\}}] + 2E[V_{k-1,\pi_0}(\mathbf{m} - \mathbf{e}_{L_k^{\pi_0}(\mathbf{m})})] = 2V_{k,\pi_0}(\mathbf{m}).$$

□

B. Proof of the Results in Section 4

B.1. Proof of Lemma 1

Proof. The proof follows that of Lemma 3 with some minor modifications. In particular, to prove (17), we define a decision rule $\mathbf{h}$ in period $t+1$ which acts the same as $\mathbf{g}_{t+1}^*(\mathbf{m}+\mathbf{e}_j)$ regarding all slot types but type j. For type j, $\mathbf{h}$ does not offer it in any subsets it offers. That is, $\mathbf{h} = \mathbf{g}_{t+1}^*(\mathbf{m}+\mathbf{e}_j)$ except that we enforce $h_{kj} = 0, \ ,\forall k$. All other parts of the proof readily follow. □

B.2. Proof of Lemma 2

Proof. For convenience, we introduce some notation here. We say the decision rule in period n given the system occupies state $\mathbf{m} \in S$ can be described by a matrix-valued function: $\mathbf{g}_n : S \to \mathbf{d}$ in which $\mathbf{d} = \{d_{kj}\}$ is a J by J matrix, $d_{kj} \in \{0,1\}$. If $d_{kj} = 1$, type j slots are offered in the kth subset. Since these subsets offered are mutually exclusive, $\sum_{k=1}^K d_{kj} \le 1, \ \forall j$. As before, depleted slot types cannot be offered: $d_{kj} \le m_j$.

Let $\hat{\mathbf{d}}$ denote an optimal decision rule. Without loss of generality, we assume that $m_j > 0, \ \forall j \in \mathcal{J}$. Otherwise we would consider a network where the preference matrix has been modified by removing empty slots. Let $\hat{\mathcal{J}} = \{j : \ \sum_{k=1}^{K-1} d_{kj} = 1, \ j \in \mathcal{J}\}$ be the set of slot types offered by $\hat{\mathbf{d}}$ collectively in all subsets it offers. Assume that $\mathcal{J} \setminus \hat{\mathcal{J}} \neq \emptyset$. Consider another decision rule $\tilde{\mathbf{d}}$ which follows exactly the same sequential offering rule as $\hat{\mathbf{d}}$, and then offers all slots types in $\mathcal{J} \setminus \hat{\mathcal{J}}$ as the Kth offer set. So $\tilde{\mathbf{d}}$ eventually offers all slot types. To prove the desired result, it suffices to show that $\tilde{\mathbf{d}}$ is no worse than $\hat{\mathbf{d}}$, and thus must be optimal as well.

First consider a policy that uses $\hat{\mathbf{d}}$ in the first slot, and then follows the optimal scheduling rule. Let $V_n^{\hat{d}}(\mathbf{m})$ denote the expected objective value following such a policy. Recall that $V_n(\mathbf{m})$ is the optimal expected objective value, and that $p_j(\mathbf{m}, \mathbf{d})$ denotes the probability that a type j slot will be booked in state $\mathbf{m}$ if decision rule $\mathbf{d}$ is used. It follows that

$$V_n^{\hat{d}}(\mathbf{m}) = \sum_{j\in\hat{\mathcal{J}}} p_j(\mathbf{m}, \hat{\mathbf{d}}) + \sum_{j\in\hat{\mathcal{J}}} p_j(\mathbf{m}, \hat{\mathbf{d}}) V_{n-1}(\mathbf{m} - \mathbf{e}_j) + [1 - \sum_{j\in\hat{\mathcal{J}}} p_j(\mathbf{m}, \hat{\mathbf{d}})] V_{n-1}(\mathbf{m}). \tag{72}$$

Then, consider a a policy that uses $\tilde{\mathbf{d}}$ first, and then follow the optimal scheduling rule. The expected objective valuing of this policy is

$$V_n^{\tilde{d}}(\mathbf{m}) = \sum_{j\in\mathcal{J}} p_j(\mathbf{m}, \tilde{\mathbf{d}}) + \sum_{j\in\mathcal{J}} p_j(\mathbf{m}, \tilde{\mathbf{d}}) V_{n-1}(\mathbf{m} - \mathbf{e}_j) + [1 - \sum_{j\in\mathcal{J}} p_j(\mathbf{m}, \tilde{\mathbf{d}})] V_{n-1}(\mathbf{m}) \tag{73}$$

It is easy to check that $p_j(\mathbf{m}, \hat{\mathbf{d}}) = p_j(\mathbf{m}, \tilde{\mathbf{d}})$ for $j \in \hat{\mathcal{J}}$, as $\hat{\mathbf{d}}$ acts the same as $\tilde{\mathbf{d}}$ in the first $K-1$ offer sets that cover slots types in $\hat{\mathcal{J}}$. Subtracting (72) from (73) and simplifying, we arrive at

$$V_n^{\tilde{d}}(\mathbf{m}) - V_n^{\hat{d}}(\mathbf{m}) = \sum_{j\in\mathcal{J}\setminus\hat{\mathcal{J}}} p_j(\mathbf{m}, \tilde{\mathbf{d}})(1 + V_{n-1}(\mathbf{m} - \mathbf{e}_j) - V_{n-1}(\mathbf{m})) \ge 0,$$

where the last inequality directly follows from Lemma 1, proving the desired result. □

B.3. Proof of Theorem 3

Proof. Lemma 2 suggests that there exists an optimal decision rule $\mathbf{S}^* = S_1^*\text{-}\ldots\text{-}S_K^*$ such that $\cup_{i=1}^K S_i^* = J$. Suppose that $S_1^*\text{-}\ldots\text{-}S_K^*$ does not take the form as desired, we will show below that the objective value obtained by partitioning $\mathbf{S}^*$ into singletons $\{j_1\}\text{-}\ldots\text{-}\{j_J\}$ is no worse than that of $S_1^*\text{-}\ldots\text{-}S_K^*$.

If there exists some k that $|S_k^*| > 1$, let us consider an alternative decision rule

$$\hat{\mathbf{S}}^* = S_1^* - \cdots - S_{k-1}^* - \{t_1\} - S_k^* \setminus \{t_1\} - \cdots - S_K^*, \tag{74}$$

such that

$$V_{n-1}(\mathbf{m} - \mathbf{e}_{t_1}) \geq V_{n-1}(\mathbf{m} - \mathbf{e}_t), \ \forall t \in S_k^* \setminus \{t_1\}. \tag{75}$$

This new decision rule follows the same offering sequence as the original rule, except that it splits the offer set S_k^* into two sub-offer sets S_{k-1}^* and $\{t_1\}$.

Now, we will show that $\hat{\mathbf{S}}^*$ does no worse than $\mathbf{S}^*$. To do that, let $V_n^1(\mathbf{m})$ be the expected number of slots filled at the end of the booking horizon by following decision rule $\hat{\mathbf{S}}^*$ at period n and then following the optimal decision afterwards. Let $\Delta^1 = V_n(\mathbf{m}) - V_n^{\hat{\mathbf{S}}^*}(\mathbf{m})$. Let $I^* = \{i : \Omega_{it_1} = 1, \sum_{j \in S_k^* \setminus t_1} \Omega_{ij} \geq 1, \sum_{j \in \cup_{l=1}^{k-1} S_l^*} \Omega_{ij} = 0\}$ be the set of customer types that accept type t_1 slots and also at least one slot type in the set of $S_k^* \setminus t_1$, but do not accept any slot type that has been offered so far in sets S_1^* through S_{k-1}^*. Let $J^*(i) = \{j : j \in S_k^*, \Omega_{ij} = 1\}$ be the subset of slots type in S_k^* that are acceptable by customer type i, $i \in I^*$. Clearly, $t_1 \in J^*(i)$. One can find that

$$\Delta^1 = \sum_{i \in I^*} \frac{\lambda_i}{|J^*(i)|} \sum_{j \in J^*(i)} V_{n-1}(\mathbf{m} - \mathbf{e}_j) - \sum_{i \in I^*} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{t_1}) \leq 0,$$

where the last equality follows from (75), proving that $\hat{\mathbf{S}}^*$ does no worse than $\mathbf{S}^*$.

Following the procedure above to keep splitting offer sets that contain more than one slot types, we can obtain an optimal action of form $\{j_1'\}\text{-}\ldots\text{-}\{j_J'\}$ so that each sequential offer set contains exactly one slot type. Suppose that $\{j_1'\}\text{-}\ldots\text{-}\{j_J'\}$ does not follow the order desired. That is, there exists $1 \leq u \leq J+1$ such that $V_{n-1}(\mathbf{m} - \mathbf{e}_{j_u'}) < V_{n-1}(\mathbf{m} - \mathbf{e}_{j_{u+1}'})$. Consider another decision rule with only j_u' and j_{u+1}' switched and others remained the same order.

$$\{j_1'\} - \cdots - \{j_{u+1}'\} - \{j_u'\} - \cdots - \{j_J'\}. \tag{76}$$

It suffices to show the claim that (76) either provides the same objective value as $\{j_1'\}\text{-}\ldots\text{-}\{j_J'\}$, or strictly higher, which contradicts with the optimality of $\{j_1'\}\text{-}\ldots\text{-}\{j_J'\}$, and thus for all $1 \leq u \leq J+1$, $V_{n-1}(\mathbf{m} - \mathbf{e}_{j_u'}) \geq V_{n-1}(\mathbf{m} - \mathbf{e}_{j_{u+1}'})$ as desired.

To show the claim above, let $I' = \{i : \Omega_{ij_u'} = 1, \Omega_{ij_{u+1}'} = 1, \sum_{v=1}^{u-1} \Omega_{ij_v} = 0\}$ be the set of customer types that accept both types j_u' and j_{u+1}' slots, but do not accept any slot type that has been offered so far. Let $V_n^2(\mathbf{m})$ be the expected number of slots filled at the end of the booking horizon by following decision rule (76) at period n and then following the optimal decision afterwards. We consider

$$\Delta^2 = V_n(\mathbf{m}) - V_n^2(\mathbf{m}) = \sum_{i \in I'} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j_u'}) - \sum_{i \in I^*} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j_{u+1}'}).$$

If $\sum_{i \in I'} \lambda_i = 0$, then $\Delta^2 = 0$ and thus (76) is optimal. However, if $\sum_{i \in I'} \lambda_i > 0$, then $\Delta^2 < 0$ leading to the contradiction desired. This proves our claim and completes the proof. □

B.4. Proof of Theorem 4

Proof. For notational convenience, here we consider the case when $\lambda_0 = 0$ and all customer types $i \in \mathcal{I}$ can be covered by at least one slot type left in $\mathbf{m}$. Proofs of other cases follow a similar procedure.

It is trivial that $V_n^s(\mathbf{m}) = V_n^f(\mathbf{m})$, for $n = 0, 1$ and for all $\mathbf{m} \geq 0$. Assume the desired equality holds up to $n = t-1$, and consider $n = t$. Let $V_n^f(\mathbf{m}|i)$ be the optimal value function with system state $\mathbf{m}$, the current arrival being customer type $i \in \mathcal{I}$ and n periods to go. Then $\forall i \in \mathcal{I}$,

$$V_n^f(\mathbf{m}|i) = \max_{\mathbf{d}} \{\sum_{j=1}^{J} p_{ij}(\mathbf{m}, \mathbf{d})[1 + V_{n-1}^f(\mathbf{m} - \mathbf{e}_j)],$$

where $\mathbf{d}$ is the offered set and $p_{ij}(\mathbf{m}, \mathbf{d})$ is the probability that slot type j will be taken if $\mathbf{d}$ is offered and the arrival is type i customer. It is not difficult to see that the optimal offer set will be the slot type $j^*(i)$ (which is a function of i) such that

$$j^*(i) = \underset{j \in \{k: \Omega_{ik}=1, k \in \mathcal{J}\}}{\arg\max} V_{n-1}^f(\mathbf{m} - \mathbf{e}_j). \tag{77}$$

That is, for any arriving customer type, the optimal action is to offer the slot type that is acceptable by this customer type and that leads to the largest value-to-go. It follows that

$$V_n^f(\mathbf{m}) = \sum_{i \in \mathcal{I}} \lambda_i V_n^f(\mathbf{m}|i) = 1 + \sum_{i \in \mathcal{I}} \lambda_i V_{n-1}^f(\mathbf{m} - \mathbf{e}_{j^*(i)}) = 1 + \sum_{i \in \mathcal{I}} \lambda_i V_{n-1}^s(\mathbf{m} - \mathbf{e}_{j^*(i)}) = V_n^s(\mathbf{m}).$$

To see the last equality, note that the optimal action stipulated by Theorem 3 ensures that (i) for any arriving customer type, it will find an acceptable slot type and (ii) this accepted slot type leads to the largest value-to-go among all slot types accepted by this customer type. This is exactly enforced by (77). □

B.5. Proof of Theorem 5

Proof. For notational convenience, here we consider the case when $\lambda_0 = 0$. The proof for the case when $\lambda_0 > 0$ follows similar steps. In light of Theorem 3, it suffices to show that for any j_1, j_2 such that $I(j_1) \subset I(j_2)$,

$$V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1}) \geq V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2}), \forall\ n = 1, 2, \ldots, N. \tag{78}$$

It is easy to see that (78) holds for $n = 1$. Assume it holds up to $n > 1$ and consider $n + 1$. Let us consider a few cases below.

Case 1: $\mathbf{m}_{j_1} \geq 2, \mathbf{m}_{j_2} \geq 1$. Let g^* be an optimal action for the state $\mathbf{m} - \mathbf{e}_{j_2}$. Let $B_{ij}^g(\mathbf{m})$ denote the probability that a type i customers will choose type j slots in state $\mathbf{m}$ when action g is taken. If $j = 0$, then no slots are chosen. Note that g^* is always feasible for state $\mathbf{m} - \mathbf{e}_{j_1}$, and that $B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_1}) = B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2})$. Thus,

$$V_n(\mathbf{m} - \mathbf{e}_{j_1}) \geq V_n^{g^*}(\mathbf{m} - \mathbf{e}_{j_1}) = \sum_{i=1}^{I} \lambda_i \sum_{j=0}^{J} B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_1})[\mathbb{1}_{j>0} + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_j)]$$
$$\geq \sum_{i=1}^{I} \lambda_i \sum_{j=0}^{J} B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2})[\mathbb{1}_{j>0} + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_j)] = V_n^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) = V_n(\mathbf{m} - \mathbf{e}_{j_2}),$$

where the second inequality follows from the induction hypothesis.

Case 2: $\mathbf{m}_{j_1} = 1, \mathbf{m}_{j_2} \geq 2$. Again, let g^* be an optimal action for the state $\mathbf{m} - \mathbf{e}_{j_2}$. Following induction hypothesis, we choose g^* so that a slot type with a smaller set of covered customer types will be offered before any other slot type with a larger covered set of customer types. Thus, g^* offers j_1 before offering j_2. Let $\tilde{g}$ be an action that follows exactly as g^* except that $\tilde{g}$ does not offer type j_1 slots. It is clear that $\tilde{g}$ is feasible for state $\mathbf{m}-\mathbf{e}_{j_1}$. There are two subcases.

Case 2a: None of the slot types if any offered between j_1 and j_2 by g^* are acceptable by customer type i_1, $\forall i_1 \in \underline{I}(j_1)$ where $\underline{I}(j_1)$ represent the set of customer types who would choose slot type j_1 when it is offered by g^* at state $\mathbf{m}-\mathbf{e}_{j_2}$. The following inequalities hold.

$$B_{ij}^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1}) = B_{ij}^{g^*}(\mathbf{m}-\mathbf{e}_{j_2}),\ \forall i \notin \underline{I}(j_1),\ \forall j;$$
$$B_{ij_1}^{g^*}(\mathbf{m}-\mathbf{e}_{j_2}) = B_{ij_2}^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1}) = 1,\ \forall i \in \underline{I}(j_1).$$

It follows that

$$\begin{aligned}
&V_n(\mathbf{m}-\mathbf{e}_{j_1}) \geq V_n^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1}) = \sum_{i=1}^{I}\lambda_i \sum_{j=1}^{J} B_{ij}^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1})[1+V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_j)]\\
=&\sum_{i\notin \underline{I}(j_1)}\lambda_i \sum_{j=1}^{J} B_{ij}^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1})[1+V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_j)] + \sum_{i\in \underline{I}(j_1)}\lambda_i B_{ij_2}^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1})[1+V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_{j_2})]\\
\geq&\sum_{i\notin \underline{I}(j_1)}\lambda_i \sum_{j=1}^{J} B_{ij}^{g^*}(\mathbf{m}-\mathbf{e}_{j_2})[1+V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}-\mathbf{e}_j)] + \sum_{i\in \underline{I}(j_1)}\lambda_i B_{ij_1}^{g^*}(\mathbf{m}-\mathbf{e}_{j_2})[1+V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}-\mathbf{e}_{j_1})]\\
=&V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}),
\end{aligned}$$

where the last inequality follows from the induction hypothesis.

Case 2b: Let $\underline{I}(j_k)$ be the set of the customer types who will actually choose slot type j_k under g^* when it is offered at state $\mathbf{m}-\mathbf{e}_{j_2}$. One slot type, say j_3, offered between j_1 and j_2 by g^* is acceptable by some customer type $i_1 \in \underline{I}(j_1)$. Consider j_3 to be the only one of such slots (extension to multiple of such slots uses a similar but more tedious proof). Because slot types j_1, j_2 and j_3 can cover some same customer types, we know that either $I(j_3)\subset I(j_k)$ or $I(j_3)\supset I(j_k)$, $k=1,2$, by the preassumption of the theorem. Because we choose g^* based on the size of covered set of customer types, we have that $I(j_1)\subset I(j_2)\subset I(j_3)$. Also note that $\cap_{k=1}^{3}\underline{I}(j_k)=\emptyset$ because any customer type can choose at most one slot type. Let $j(i)$ be the slot type chosen by customer type i, $i\notin \underline{I}(j_1)\cup\underline{I}(j_2)\cup\underline{I}(j_3)$. Note that $j(i)$ is the same under g^* or $\tilde{g}$, $\forall i\notin \underline{I}(j_1)\cup\underline{I}(j_2)\cup\underline{I}(j_3)$. It follows that

$$\begin{aligned}
&V_n(\mathbf{m}-\mathbf{e}_{j_1}) - V_n(\mathbf{m}-\mathbf{e}_{j_2}) \geq V_n^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1}) - V_{n-1}^{g^*}(\mathbf{m}-\mathbf{e}_{j_2})\\
=&\sum_{i\notin \underline{I}(j_1)\cup\underline{I}(j_2)\cup\underline{I}(j_3)}\lambda_i[V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_{j(i)}) - V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}-\mathbf{e}_{j(i)})]\\
&+\sum_{i\in \underline{I}(j_1)\cup\underline{I}(j_3)}\lambda_i V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_{j_3}) + \sum_{i\in \underline{I}(j_2)}\lambda_i V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_{j_2})\\
&-\{\sum_{i\in \underline{I}(j_1)}\lambda_i V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}-\mathbf{e}_{j_1}) + \sum_{i\in \underline{I}(j_3)}\lambda_i V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}-\mathbf{e}_{j_3}) + \sum_{i\in \underline{I}(j_2)}\lambda_i V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}-\mathbf{e}_{j_2})\} \geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis.

Case 3: $\mathbf{m}_{j_1}=1, \mathbf{m}_{j_2}=1$. Let g^* be an optimal action for the state $\mathbf{m}-\mathbf{e}_{j_2}$ (g^* does not offer slot type j_2 because none is available). Let $\tilde{g}$ be an action that follows exactly as g^* except that $\tilde{g}$ does not offer type j_1 slots but offers type j_2 at the end. It is clear that $\tilde{g}$ is feasible for state $\mathbf{m}-\mathbf{e}_{j_1}$. Let $\underline{I}(j_2)$ be the customer types who choose j_2 under $\tilde{g}$; these customers do not book any appointments under g^*. Let $j(i)$ be the slot type actually chosen by customer type i, $i \notin \underline{I}(j_2)$. Note that $j(i)$ is the same under g^* or $\tilde{g}$. It follows that

$$\begin{aligned}
&V_n(\mathbf{m}-\mathbf{e}_{j_1}) - V_n(\mathbf{m}-\mathbf{e}_{j_2}) \geq V_n^{\tilde{g}}(\mathbf{m}-\mathbf{e}_{j_1}) - V_n^{g^*}(\mathbf{m}-\mathbf{e}_{j_2})\\
=&\sum_{i\notin \underline{I}(j_2)} \lambda_i[V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_{j(i)}) + V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2}-\mathbf{e}_{j(i)})] + \sum_{i\in \underline{I}(j_2)} \lambda_i[1+V_{n-1}(\mathbf{m}-\mathbf{e}_{j_1}-\mathbf{e}_{j_2}) - V_{n-1}(\mathbf{m}-\mathbf{e}_{j_2})]\\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis and Lemma 1. This completes the whole proof. □

B.6. Proof of Proposition 3

Proof. It suffices to show the following monotonic results for the "W" model with sequential offers: $V_n(\mathbf{m}-\mathbf{e}_1) - V_n(\mathbf{m}-\mathbf{e}_2)$ increases as m_1 increases and that $V_n(\mathbf{m}-\mathbf{e}_2) - V_n(\mathbf{m}-\mathbf{e}_1)$ increases as m_2 increases, $\forall n \geq 1,\ \forall \mathbf{m} \geq (1,1)$.

That is,

$$V_n(\mathbf{m}) - V_n(\mathbf{m}+\mathbf{e}_1-\mathbf{e}_2) \geq V_n(\mathbf{m}-\mathbf{e}_1) - V_n(\mathbf{m}-\mathbf{e}_2),\ \forall n\geq 1,\ \forall \mathbf{m}\geq(1,1), \tag{79}$$

and

$$V_n(\mathbf{m}) - V_n(\mathbf{m}+\mathbf{e}_2-\mathbf{e}_1) \geq V_n(\mathbf{m}-\mathbf{e}_2) - V_n(\mathbf{m}-\mathbf{e}_1),\ \forall n\geq 1,\ \forall \mathbf{m}\geq(1,1). \tag{80}$$

To facilitate the proof of (79) and (80), we introduce a few notations. Let $\Delta_n^A(\mathbf{m}) = V_n(\mathbf{m}-\mathbf{e}_1) - V_n(\mathbf{m}-\mathbf{e}_2)$ and $\Delta_n^B(\mathbf{m}) = V_n(\mathbf{m}-\mathbf{e}_2) - V_n(\mathbf{m}-\mathbf{e}_1)$. Note that (79) and (80) are symmetric, and thus we limit ourselves to just prove (79).

Consider the case for $n=1$. At $\mathbf{m}=(1,1)$, $\Delta_1^A(\mathbf{m}) = V_1(0,1) - V_1(1,0) = (\lambda_2+\lambda_3) - (\lambda_1+\lambda_2) = \lambda_3 - \lambda_1$. For $\mathbf{m}=(m_1,1)$ and $m_1\geq 2$, we have $\Delta_1^A(\mathbf{m}) = V_1(m_1-1,1) - V_1(m_1,0) = (1-\lambda_0) - (\lambda_1+\lambda_2) = \lambda_3$. Thus, (79) holds for $n=1$, $\mathbf{m}=(m_1,1)$ and $m_1\geq 1$. Now, for $n=1$ and $\mathbf{m}=(1,m_2)$ and $m_2\geq 2$, we have $\Delta_1^A(\mathbf{m}) = V_1(0,m_2) - V_1(1,m_2-1) = (\lambda_2+\lambda_3) - (1-\lambda_0) = -\lambda_1$. Consider $n=1$, $\mathbf{m}=(m_1,m_2)$, and $m_1,m_2\geq 2$. In this case, we have that $\Delta_1^A(\mathbf{m}) = V_1(m_1-1,m_2) - V_1(m_1,m_2-1) = (1-\lambda_0) - (1-\lambda_0) = 0$. Thus, (79) holds for $n=1$, $\mathbf{m}=(m_1,m_2)$ and $m_1\geq 1, m_2\geq 2$. This completes the proof of (79) for $n=1$.

Assume that (79) holds up to $n=k$ for $\mathbf{m}\geq(1,1)$. We will use induction below to show that this is also true for $n=k+1$. We start by witting the Bellman's equation below.

$$V_{k+1}(\mathbf{m}) = \max\left\{\begin{array}{l} 1-\lambda_0 + (\lambda_1+\frac{1}{2}\lambda_2)V_k(\mathbf{m}-\mathbf{e}_1) + (\frac{1}{2}\lambda_2+\lambda_3)V_k(\mathbf{m}-\mathbf{e}_2) + \lambda_0 V_k(\mathbf{m}),\\ 1-\lambda_0 + (\lambda_1+\lambda_2)V_k(\mathbf{m}-\mathbf{e}_1) + \lambda_3 V_k(\mathbf{m}-\mathbf{e}_2) + \lambda_0 V_k(\mathbf{m}),\\ 1-\lambda_0 + \lambda_1 V_k(\mathbf{m}-\mathbf{e}_1) + (\lambda_2+\lambda_3)V_k(\mathbf{m}-\mathbf{e}_2) + \lambda_0 V_k(\mathbf{m}). \end{array}\right\}, \tag{81}$$

where the three terms in the max operator correspond to actions $\{1,2\}$, $\{1\}$-$\{2\}$ and $\{2\}$-$\{1\}$, respectively. Action $\{S_1\}$-$\{S_2\}$ offers subset S_1 followed by subset S_2. For ease of notation, we define $\Delta_{k+1}^{ij}(\mathbf{m})$ to be the difference of the ith and jth terms in the max operator (81) above, $i,j\in\{1,2,3\}$. It follows that

$$\Delta_{k+1}^{21}(\mathbf{m}) = \frac{1}{2}\lambda_2[V_k(\mathbf{m}-\mathbf{e}_1) - V_k(\mathbf{m}-\mathbf{e}_2)] = \frac{1}{2}\lambda_2\Delta_k^A(\mathbf{m}),$$

and

$$\Delta_{k+1}^{31}(\mathbf{m}) = \frac{1}{2}\lambda_2[V_k(\mathbf{m}-\mathbf{e}_2) - V_k(\mathbf{m}-\mathbf{e}_1)] = \frac{1}{2}\lambda_2\Delta_k^B(\mathbf{m}).$$

Because $\Delta_{k+1}^{21}(\mathbf{m}) + \Delta_{k+1}^{31}(\mathbf{m}) = 0$, one of these two terms must be non-negative suggesting one of the corresponding actions is optimal. In particular, if $\Delta_{k+1}^{21}(\mathbf{m}) \geq 0$, or equivalently, $\Delta_k^A(\mathbf{m}) \geq 0$, the the optimal action is {1}-{2}; otherwise, it would be {2}-{1}.

To prove the desired result, we need to consider the following cases. Case (1): $\mathbf{m} = (1,1)$; case (2): $\mathbf{m} = (m_1, 1),\ m_1 \geq 2$; case (3): $\mathbf{m} = (1, m_2),\ m_2 \geq 2$; and case (4), $\mathbf{m} \geq (2,2)$.

For Case (1) with $\mathbf{m} = (1,1)$, we have

$$\Delta_{k+1}^A(1,1) = V_{k+1}(0,1) - V_{k+1}(1,0) = [(1-\lambda_1-\lambda_0) + (\lambda_1+\lambda_0)V_k(0,1)] - [(1-\lambda_3-\lambda_0) + (\lambda_3+\lambda_0)V_k(1,0)].$$

We consider two subcases. Case (1a): if at state $(1,1)$ the optimal action is {1}-{2}, then

$$\begin{aligned}
&\Delta_{k+1}^A(2,1) = V_{k+1}(1,1) - V_{k+1}(2,0) \\
=&[(1-\lambda_0) + (\lambda_1+\lambda_2)V_k(0,1) + \lambda_3 V_k(1,0) + \lambda_0 V_k(1,1)] \\
&- [(1-\lambda_3-\lambda_0) + (\lambda_1+\lambda_2)V_k(1,0) + (\lambda_3+\lambda_0)V_k(2,0)].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Delta_{k+1}^A(2,1) - \Delta_{k+1}^A(1,1) \\
=&\lambda_1[1 - V_k(1,0)] + \lambda_2[V_k(0,1) - V_k(1,0)] + \lambda_3[2V_k(1,0) - V_k(2,0)] + \lambda_0[\Delta_k^A(2,1) - \Delta_k^A(1,1)] \\
=&\lambda_1[1 - V_k(1,0)] + \lambda_2\Delta_k^A(1,1) + \lambda_3[2V_k(1,0) - V_k(2,0)] + \lambda_0[\Delta_k^A(2,1) - \Delta_k^A(1,1)].
\end{aligned}$$

It is trivial that $1 - V_k(1,0) \geq 0$. We also know that $\Delta_k^A(1,1) \geq 0$ in this case because the optimal action is {1}-{2}; and that $\Delta_k^A(2,1) - \Delta_k^A(1,1) \geq 0$ by the induction hypothesis. Finally, we claim that

$$2V_k(1,0) - V_k(2,0) \geq 0,\ \forall k \geq 1, \tag{82}$$

which will be shown at the end of this proof. Thus, $\Delta_{k+1}^A(2,1) - \Delta_{k+1}^A(1,1) \geq 0$ if the optimal action is {1}-{2} at state $(1,1)$.

Case (1b): if the optimal action at state $(m_1, 1)$ is {2}-{1}, then

$$\begin{aligned}
&\Delta_{k+1}^A(2,1) = V_{k+1}(1,1) - V_{k+1}(2,0) \\
=&[(1-\lambda_0) + \lambda_1 V_k(0,1) + (\lambda_2+\lambda_3)V_k(1,0) + \lambda_0 V_k(1,1)] \\
&- [(1-\lambda_3-\lambda_0) + (\lambda_1+\lambda_2)V_k(1,0) + (\lambda_3+\lambda_0)V_k(2,0)].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Delta_{k+1}^A(2,1) - \Delta_{k+1}^A(1,1) \\
=&\lambda_1[1 - V_k(1,0)] + \lambda_3[2V_k(1,0) - V_k(2,0)] + \lambda_0[\Delta_k^A(2,1) - \Delta_k^A(1,1)] \geq 0.
\end{aligned}$$

In summary, cases (1a) and (1b) collectively show that $\Delta_{k+1}^A(2,1) - \Delta_{k+1}^A(1,1) \geq 0$.

For Case (2) with $\mathbf{m} = (m_1, 1),\ m_1 \geq 2$, we evaluate $V_{k+1}(m_1-1,1) - V_{k+1}(m_1,0)$ in the following two subcases. Case(2a): if the optimal action at state $(m_1, 1)$ is $\{1\}$-$\{2\}$, then

$$\begin{aligned}
&V_{k+1}(m_1-1,1) - V_{k+1}(m_1,0)\\
=&[(1-\lambda_0) + (\lambda_1+\lambda_2)V_k(m_1-2,1) + \lambda_3 V_k(m_1-1,0) + \lambda_0 V_k(m_1-1,1)]\\
&-[(1-\lambda_3-\lambda_0) + (\lambda_1+\lambda_2)V_k(m_1-1,0) + (\lambda_3+\lambda_0)V_k(m_1,0)]\\
=&\lambda_3 + (\lambda_1+\lambda_2)\Delta_k^A(m_1-1,1) + \lambda_0\Delta_k^A(m_1,1) + \lambda_3[(1-(\lambda_1+\lambda_2)^{m_1-1}) - (1-(\lambda_1+\lambda_2)^{m_1})]\\
=&\lambda_3 + (\lambda_1+\lambda_2)\Delta_k^A(m_1-1,1) + \lambda_0\Delta_k^A(m_1,1) - \lambda_3(\lambda_3+\lambda_0)(\lambda_1+\lambda_2)^{m_1-1},
\end{aligned}$$

which increases as m_1 increases by the induction hypothesis.

Case (2b): if the optimal action at state $(m_1, 1)$ is $\{2\}$-$\{1\}$, then for $\mathbf{m} = (m_1, 1),\ m_1 \geq 2$,

$$\begin{aligned}
&V_{k+1}(m_1-1,1) - V_{k+1}(m_1,0)\\
=&[(1-\lambda_0) + \lambda_1 V_k(m_1-2,1) + (\lambda_2+\lambda_3)V_k(m_1-1,0) + \lambda_0 V_k(m_1-1,1)]\\
&-[(1-\lambda_3-\lambda_0) + (\lambda_1+\lambda_2)V_k(m_1-1,0) + (\lambda_3+\lambda_0)V_k(m_1,0)]\\
=&\lambda_3 + \lambda_1\Delta_k^A(m_1-1,1) + \lambda_0\Delta_k^A(m_1,1) + \lambda_3[(1-(\lambda_1+\lambda_2)^{m_1-1}) - (1-(\lambda_1+\lambda_2)^{m_1})]\\
=&\lambda_3 + \lambda_1\Delta_k^A(m_1-1,1) + \lambda_0\Delta_k^A(m_1,1) - \lambda_3(\lambda_3+\lambda_0)(\lambda_1+\lambda_2)^{m_1-1},
\end{aligned}$$

which also increases as m_1 increases by the induction hypothesis. Thus, cases (1a) though (1d) shows that $\Delta_n^A(\mathbf{m})$ increases in m_1 for $n \geq 1$ and $\mathbf{m} = (m_1, 1), m_1 \geq 1$.

Case (3): $\mathbf{m} = (1, m_2),\ m_2 \geq 2$. We want to show that

$$\Delta_{k+1}^A(2,m_2) - \Delta_{k+1}^A(1,m_2) = [V_{k+1}(1,m_2) - V_{k+1}(2,m_2-1)] - [V_{k+1}(0,m_2) - V_{k+1}(1,m_2-1)] \geq 0.$$

Again, we separate into a few subcases. If the optimal action at state $(1, m_2-1)$ is $\{1\}$-$\{2\}$, then $\Delta_k^A(1, m_2 - 1) \geq 0$. It follows that $\Delta_k^A(2, m_2-1) \geq 0$ by the induction hypothesis, and the optimal action at state $(2, m_2-1)$ is also $\{1\}$-$\{2\}$. But the optimal actions at state $(1, m_2)$ can still be either $\{1\}$-$\{2\}$ or $\{2\}$-$\{1\}$. Following this logic, we need to consider four subcases. Case (3a): the optimal actions at state $(1, m_2-1)$, $(2, m_2-1)$ and $(1, m_2)$ are all $\{1\}$-$\{2\}$. Case (3b): the optimal actions at state $(1, m_2-1)$, $(2, m_2-1)$ and $(1, m_2)$ are $\{1\}$-$\{2\}$, $\{1\}$-$\{2\}$ and $\{2\}$-$\{1\}$, respectively. Case (3c): the optimal actions at state $(1, m_2-1)$, $(2, m_2-1)$ and $(1, m_2)$ are all $\{2\}$-$\{1\}$. Case (3d): the optimal actions at state $(1, m_2-1)$, $(2, m_2-1)$ and $(1, m_2)$ are $\{2\}$-$\{1\}$, $\{1\}$-$\{2\}$ and $\{2\}$-$\{1\}$, respectively.

For case (3a), we have

$$\begin{aligned}
&\Delta_{k+1}^A(2,m_2) - \Delta_{k+1}^A(1,m_2)\\
=&\lambda_1[1 - V_k(0,m_2-1)] + (\lambda_1+\lambda_2)\Delta_k^A(1,m_2)\\
&+\lambda_3[\Delta_k^A(2,m_2-1) - \Delta_k^A(1,m_2-1)] + \lambda_0[\Delta_k^A(2,m_2) - \Delta_k^A(1,m_2)] \geq 0,
\end{aligned}$$

where the inequality follows from the fact that the first term is trivially nonnegative, the second term is positive as the optimal actions at state $(1, m_2)$ is $\{1\}$-$\{2\}$ and the last two terms are nonnegative by the induction hypothesis.

For case (3b), we have

$$\begin{aligned}&\Delta^A_{k+1}(2,m_2)-\Delta^A_{k+1}(1,m_2)\\=&\lambda_1[-V_k(1,m_2-1)+1+V_k(0,m_2-1)]\\&+\lambda_3[\Delta^A_k(2,m_2-1)-\Delta^A_k(1,m_2-1)]+\lambda_0[\Delta^A_k(2,m_2)-\Delta^A_k(1,m_2)]\geq 0,\end{aligned}$$

where the first term is nonnegative following Lemma 1 and the other two terms are nonnegative following the induction hypothesis.

For case (3c), we have

$$\begin{aligned}&\Delta^A_{k+1}(2,m_2)-\Delta^A_{k+1}(1,m_2)\\=&\lambda_1[-V_k(1,m_2-1)+1+V_k(0,m_2-1)]\\&+(\lambda_2+\lambda_3)[\Delta^A_k(2,m_2-1)-\Delta^A_k(1,m_2-1)]+\lambda_0[\Delta^A_k(2,m_2)-\Delta^A_k(1,m_2)]\geq 0,\end{aligned}$$

following a similar argument of case (3b).

For case (3d), we have

$$\begin{aligned}&\Delta^A_{k+1}(2,m_2)-\Delta^A_{k+1}(1,m_2)\\=&\lambda_1[-V_k(1,m_2-1)+1+V_k(0,m_2-1)]\\&-\lambda_2\Delta^A_k(1,m_2-1)+\lambda_3[\Delta^A_k(2,m_2-1)-\Delta^A_k(1,m_2-1)]+\lambda_0[\Delta^A_k(2,m_2)-\Delta^A_k(1,m_2)]\geq 0,\end{aligned}$$

following a similar logic of case (3b) and the fact that $\Delta^A_k(1,m_2-1)\leq 0$ (because the optimal action at state $(1,m_2-1)$ is {2}-{1}). This completes the proof of case (3).

For case (4) $\mathbf{m}\geq(2,2)$, we evaluate $V_n(\mathbf{m}-\mathbf{e}_1)-V_n(\mathbf{m}-\mathbf{e}_2)$ and need to consider four subcases. Case (4a): if the optimal actions at states $(\mathbf{m}-\mathbf{e}_1)$ and $(\mathbf{m}-\mathbf{e}_2)$ are both {1}-{2}. Then

$$\begin{aligned}\Delta^A_{k+1}(\mathbf{m})&=V_{k+1}(\mathbf{m}-\mathbf{e}_1)-V_{k+1}(\mathbf{m}-\mathbf{e}_2)\\&=(\lambda_1+\lambda_2)\Delta^A_k(\mathbf{m}-\mathbf{e}_1)+\lambda_3\Delta^A_k(\mathbf{m}-\mathbf{e}_2)+\lambda_0\Delta^A_k(\mathbf{m}),\end{aligned}$$

which increases in m_1 by the induction hypothesis. Case (4b): if the optimal actions at states $(\mathbf{m}-\mathbf{e}_1)$ and $(\mathbf{m}-\mathbf{e}_2)$ are both {2}-{1}. Then,

$$\begin{aligned}\Delta^A_{k+1}(\mathbf{m})&=V_{k+1}(\mathbf{m}-\mathbf{e}_1)-V_{k+1}(\mathbf{m}-\mathbf{e}_2)\\&=\lambda_1\Delta^A_k(\mathbf{m}-\mathbf{e}_1)+(\lambda_2+\lambda_3)\Delta^A_k(\mathbf{m}-\mathbf{e}_2)+\lambda_0\Delta^A_k(\mathbf{m}),\end{aligned}$$

which increases in m_1 by the induction hypothesis. Case (4c): if the optimal actions at states $(\mathbf{m}-\mathbf{e}_1)$ and $(\mathbf{m}-\mathbf{e}_2)$ are {1}-{2} and {2}-{1}, respectively. Then,

$$\begin{aligned}&\Delta^A_{k+1}(\mathbf{m})=V_{k+1}(\mathbf{m}-\mathbf{e}_1)-V_{k+1}(\mathbf{m}-\mathbf{e}_2)\\=&\lambda_1\Delta^A_k(\mathbf{m}-\mathbf{e}_1)+\lambda_2[V_k(\mathbf{m}-2\mathbf{e}_1)-V_k(\mathbf{m}-2\mathbf{e}_2)]+\lambda_3\Delta^A_k(\mathbf{m}-\mathbf{e}_2)+\lambda_0\Delta^A_k(\mathbf{m})\\=&\lambda_1\Delta^A_k(\mathbf{m}-\mathbf{e}_1)+\lambda_2[\Delta^A_k(\mathbf{m}-\mathbf{e}_1)+\Delta^A_k(\mathbf{m}-\mathbf{e}_2)]+\lambda_3\Delta^A_k(\mathbf{m}-\mathbf{e}_2)+\lambda_0\Delta^A_k(\mathbf{m}),\end{aligned}$$

which increases in m_1 by the induction hypothesis. Case (4d): if the optimal actions at states $(\mathbf{m}-\mathbf{e}_1)$ and $(\mathbf{m}-\mathbf{e}_2)$ are {2}-{1} and {1}-{2}, respectively. Then,

$$\Delta_{k+1}^A(\mathbf{m}) = V_{k+1}(\mathbf{m}-\mathbf{e}_1) - V_{k+1}(\mathbf{m}-\mathbf{e}_2) = \lambda_1\Delta_k^A(\mathbf{m}-\mathbf{e}_1) + \lambda_3\Delta_k^A(\mathbf{m}-\mathbf{e}_2) + \lambda_0\Delta_k^A(\mathbf{m}),$$

which increases in m_1 by the induction hypothesis.

Finally, we show our claim (82), which can be easily done by induction. When $k=1$, $2V_k(1,0) - V_k(2,0) = 2(\lambda_1+\lambda_2) - [1-(1-\lambda_1-\lambda_2)^2] = (\lambda_1+\lambda_2)^2 \geq 0$. Assume this holds up to $k=u$. Consider $k=u+1$. We have that

$$\begin{aligned}
&2V_{u+1}(1,0) - V_{u+1}(2,0)\\
=&2[(\lambda_1+\lambda_2)+(\lambda_3+\lambda_0)V_u(1,0)] - [(\lambda_1+\lambda_2)+(\lambda_1+\lambda_2)V_u(1,0)+(\lambda_3+\lambda_0)V_u(2,0)]\\
=&(\lambda_1+\lambda_2)[1-V_u(1,0)] + (\lambda_3+\lambda_0)[2V_u(1,0)-V_u(2,0)] \geq 0,
\end{aligned}$$

proving our claim (82) and completing the whole proof. □

C. Scheduling in a Rolling-Horizon Setting

Our scheduling policy is based on a model that looks at how appointment slots are depleted in a single day, and implicitly assumes that customer demand to a single day is independent from other days. In practice, customer demand for different days may be correlated because customers who do not find an acceptable slot in one day may opt for another day. To incorporate this effect, we develop a model to evaluate the potential benefits of using our scheduling policies in a multi-day rolling-horizon setting. In Section C.1 we discuss how to extend our single-day model to a rolling-horizon setting, and argue why it is mathematically challenging, if not intractable. In Section C.2 we present numerical results for the multi-day model, and demonstrate that the insights obtained from our analysis for the single-day setting remain valid in the multi-day case.

C.1. Rolling Horizon Model Outline

There is a variety of ways to extend the single-day model to a rolling-horizon setting. Here we discuss only one such possibility, although we believe our takeaways hold more generally. Where possible we aim to follow and extend the notation introduced in Section 3. For ease of presentation, we assume that each day some deterministic number N customers arrive looking to book an appointment, and we limit ourselves to the non-sequential offering case; a similar model can be constructed when customer arrivals in different periods are mutually independent and identically distributed and when sequential offering is used. In fact, we consider the case with random arrivals and sequential offering numerically in Section C.2.

Suppose that the scheduling window is T days. In other words, customers are allowed to make appointments up to T days in advance, i.e., customers who arrive on day t can book appointments on day $t+1$, $t+2$, ..., $t+T$. For each day, available slots are divided into J types as before, and we denote by $\mathbf{b}=(b_1,\ldots,b_J)$ the initial capacity of slots for each day. Each customer has $D \leq T$ acceptable days, which are selected uniformly at random from the scheduling window. (For example, if $T=10$ and $D=3$ then one customer may accept day 3, 5 and 6 from her arrival day.) Upon arrival of a new customer, she will ask the provider for potential slots in each of these D days, one day at a time in a random order. For each day the provider will reveal some offer set. If this set contains at least one acceptable slot the customer books the appointment, ending the scheduling process for this customer. Otherwise the provider will present the offer set for the next day in the permutation, until either the customer finds an acceptable slot, or all D days have been covered. If none of the offer sets for the D days contain acceptable slots, the customers leaves without selecting a slot.

Let $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_D)$ denote the days and the order requested by (and thus offered to) an arbitrary arriving customer, drawn uniformly at random from all possible permutations of size D in $\{1,2,\ldots,T\}$ (each with probability $(T-D)!/T!$). We denote by $\mathbf{S}^{\sigma(t)}=(S_{t+\sigma_1},\ldots,S_{t+\sigma_D})$ the corresponding offer set on day t when a customer arrives with requested day order $\boldsymbol{\sigma}$, and $\mathbf{S}^{\sigma(t)}$ is our decision variable. The system state is captured by a vector $\mathbf{m}=(\mathbf{m}^{(1)},\ldots,\mathbf{m}^{(T)})$, representing the number of remaining slots for each day of the booking horizon. Here, $\mathbf{m}^{(s)}=(m_1^s,\ldots,m_J^s)$ where $s=1,\ldots,T$, and m_j^s represents the number of slots remaining for type j on day s in the booking horizon. The offer set for the sth day can only contain slots that are available, i.e., $S_{t+s}\subseteq \bar{S}(\mathbf{m}^{(s)})$, where $\bar{S}(\mathbf{m}^{(s)})=\{j=1,\ldots,J\ :\ m_j^s>0\}$. In vector notation we write $\mathbf{S}^{\sigma(t)}\subseteq \bar{\mathbf{S}}(\mathbf{m}^{\sigma})=(\bar{S}(\mathbf{m}^{(\sigma_1)}),\ldots,\bar{S}(\mathbf{m}^{(\sigma_D)}))$.

We consider I customer types, characterized by which slot types they accept. We assume that among acceptable days customer do not have strong preferences on which day to get service, although we could for instance easily incorporate customer preferences on days in our model. As before, we represent customer types by a matrix Ω, where $\Omega_{ij}=1$ if a type-i customer accepts type-j slots. Type-i customers arrive with probability λ_i. Let us denote by $q^{\boldsymbol{\sigma}}_{ijl}(\mathbf{S}^{\sigma(t)})$ the probability that a type-i customer with permutation $\boldsymbol{\sigma}$ selects a type-j slot on day $t+\sigma_l$, so

$$q^{\boldsymbol{\sigma}}_{ijl}(\mathbf{S}^{\sigma(t)})=\begin{cases}\frac{\Omega_{ij}}{\sum_{k\in S_{t+\sigma_l}}\Omega_{i}k} & \text{if } \sum_{k\in S_{t+\sigma_l}}\Omega_{ik}>0 \text{ and } \sum_{l'=1}^{l-1}\sum_{k\in S_{t+\sigma'_l}}\Omega_{ik}=0\\ 0 & \text{otherwise.}\end{cases}$$

The probability that an arbitrary customer with permutation $\boldsymbol{\sigma}$ selects slot j on day $t+\sigma_l$, given offer set $\mathbf{S}^{\sigma(t)}$ can then be obtained as

$$q^{\boldsymbol{\sigma}}_{jl}(\mathbf{S}^{\sigma(t)})=\sum_{i=1}^{I}\lambda_i q^{\boldsymbol{\sigma}}_{ijl}(\mathbf{S}^{\sigma(t)}).$$

Suppose that we follow a booking policy f, which specifies $\mathbf{S}^{\sigma(t)}$ when a customer arrives and reveals her permutation $\boldsymbol{\sigma}$. Let $V^f_{n,t}(\mathbf{m})$ denote the expected total number of slots filled given that there are n customers yet to come on day t and the current system state is $\mathbf{m}$, under policy f. Then for $n\geq 2$,

$$V^f_{n,t}(\mathbf{m})=\frac{(T-D)!}{T!}\sum_{\boldsymbol{\sigma}}\left[\sum_{l=1}^{D}\sum_{j=1}^{J}q^{\boldsymbol{\sigma}}_{jl}(\mathbf{S}^{\sigma(t)})(1-V^f_{n-1,t}(\mathbf{m})+V^f_{n-1,t}(\mathbf{m}-\mathbf{e}_{\sigma_l,j}))\right]+V^f_{n-1,t}(\mathbf{m}), \tag{83}$$

where $\mathbf{e}_{\sigma_l,j}$ denotes the all-zero vector with a 1 at (σ_l,j). When $n=1$ we have to shift the capacity vector by one day in the next step, so

$$V^f_{1,t}(\mathbf{m})=\frac{(T-D)!}{T!}\sum_{\boldsymbol{\sigma}}\left[\sum_{l=1}^{D}\sum_{j=1}^{J}q^{\boldsymbol{\sigma}}_{j,l}(\mathbf{S}^{\sigma(t)})\big(1-V^f_{N,t-1}(L(\mathbf{m}))+V^f_{N,t-1}(L(\mathbf{m}-\mathbf{e}_{\sigma_l,j}))\big)\right]+V^f_{N,t-1}(L(\mathbf{m})), \tag{84}$$

where $L(\mathbf{m})=(\mathbf{m}^{(2)},\dots,\mathbf{m}^{(D)},\mathbf{b})$, the operator that adds a new day to the capacity vector, and removes the current day that is just past.

One may consider a finite horizon problem that finds f to maximize $V^f_{N,\tau}(\mathbf{m})$ for a given, finite planning horizon of τ days. Or, one may consider an infinite horizon problem with the long-average reward criteria as follows. Let $\psi_f(\mathbf{m})$ be the long-run expected number of slots booked each day under policy f given the initial state $\mathbf{m}$, i.e.,

$$\psi_f(\mathbf{m})=\lim_{t\to\infty}\frac{V^f_{N,t}(\mathbf{m})}{t}.$$

A booking policy f^* is said to be optimal if

$$\psi_{f^*}(\mathbf{m})=\sup_f\psi_f(\mathbf{m}),\ \forall\mathbf{m}.$$

For either case, the problem is very challenging, if not intractable, due to its aperiodic nature. It is readily seen that the recursive equations in (83)-(84) are much more complex than those for the single-day case presented in (3), given the additional parameter t and the summation over D and $\boldsymbol{\sigma}$. In the multi-day setting, the offering decision is a function of a $J\times T$-dimensional vector that keeps track of the remaining capacity of all days on the planning horizon, and the space of potential policies has size 2^{JT}.

For instance, recall that for the single-day case we derived the optimal offering policy for the M model instance in Proposition 2, by showing that it is optimal to offer only slots 1 and 3, unless one has been depleted, in which case all remaining slots should be offered. However, in the multi-day setting we may wish to offer different slots depending on how close it is to the present day. This makes it highly unlikely to obtain any analytical results for this multi-day setting. Moreover, even numerically solving for f^* through policy iteration is unlikely to be possible, given the size of the state space in this model. Instead, we now use simulation to test certain heuristics derived in the single-day setting.

C.2. Numerical Results

In these experiments we assume that the number of daily arrivals is either deterministic N or a Poisson random variable with mean N. Daily capacity of the service provider is N slots, which may be allocated over different types. We consider an M model for within-day preferences (because offering all is not optimal in the single-day setting under this model). That is, each customer will either be type 1 or 2, and there are three slot types in each day.

The system evolves as described in Section C.1. The scheduling window is T days. Upon each customer's arrival, she has D acceptable days, and these D acceptable days are randomly generated within the scheduling window. The customer will then ask the provider for potential slots in each of these D days (one day at a time in a random order). The provider offers slots following one of the three scheduling policies discussed in the paper: offering-all, non-sequential optimal (blocking type 2 if available), and sequential optimal in the M model instance. If the customer finds an acceptable slot in a day, the customer will take it and the scheduling is done for this customer; if the customer cannot find acceptable slots in all D acceptable days, she will leave without booking an appointment.

In our experiments, we let $(T, N) \in \{(15, 30), (30, 50)\}$ to consider systems at different scales. We vary the arrival probabilities and the initial capacity vectors in each day (similar to Table 12). We also vary $D = 1, 2, 3, 4$ to study the impact of customer flexibility in their choices of days (a larger D implies that customers are more flexible in their choices). We run simulations for 1200 days, and use the first 200 days as warm-up periods. Based on the results of the last 1000 days, we calculate the percentage improvement, if any, in the slot fill count for non-sequential optimal and sequential optimal against offering-all for each combination of parameters. For each D and the arrival probability vector, we report the max, mean and median percentage improvement among the initial capacity vectors we consider.

Tables C1 and C2 show the comparison results with deterministic daily arrivals for systems with $(T, N) = (15, 30)$ and $(T, N) = (30, 50)$, respectively. We observe that the optimal non-sequential and sequential scheduling policies obtained in our single-day model still bring sizable benefits to the multi-day scheduling setting we consider. Consistent with earlier findings, sequential offering brings much higher efficiency gains compared to non-sequential offering. The maximum improvement in fill count by sequential offering compared to offering-all can be as high as 12%. We also observe that when customers become more flexible in their day choices (i.e., when D increases), the benefits due to "smart" scheduling decrease. This can be explained by that when customers are more flexible, their preferences are immaterial and thus taking customers' preferences into account when making scheduling decisions becomes less valuable.

Results when daily arrivals are Poisson random variables are similar; see Tables C3 and C4.

Table C1 Policy Comparison in a Multi-day Scheduling Setting (with deterministic arrivals, $T = 15$, $N = 30$).

	D	# of Scenarios	$(\lambda_1, \lambda_2) = (1/2, 1/2)$			$(\lambda_1, \lambda_2) = (1/3, 2/3)$			$(\lambda_1, \lambda_2) = (1/4, 3/4)$		
			Max	Average	Median	Max	Average	Median	Max	Average	Median
Non-sequential Optimal vs. Offering-all	1	91	3.9%	2.1%	2.0%	4.3%	2.4%	2.2%	3.7%	2.4%	2.5%
	2	91	3.7%	1.9%	1.7%	3.3%	2.0%	2.2%	2.8%	2.0%	2.1%
	3	91	3.3%	1.6%	1.7%	2.7%	1.7%	1.9%	2.5%	1.6%	1.7%
	4	91	2.7%	1.3%	1.3%	2.3%	1.4%	1.5%	2.0%	1.4%	1.4%
Sequential Optimal vs. Offering-all	1	91	9.5%	4.0%	3.3%	11.0%	5.1%	4.0%	9.8%	5.5%	5.6%
	2	91	9.8%	3.8%	2.7%	9.5%	4.5%	4.6%	7.6%	4.9%	5.2%
	3	91	8.9%	3.4%	2.4%	7.7%	3.8%	4.4%	6.1%	4.0%	4.2%
	4	91	7.6%	2.8%	1.8%	6.6%	3.1%	3.5%	5.2%	3.3%	3.4%

Table C2 Policy Comparison in a Multi-day Scheduling Setting (with deterministic arrivals, $T = 30$, $N = 50$).

	D	# of Scenarios	$(\lambda_1, \lambda_2) = (1/2, 1/2)$			$(\lambda_1, \lambda_2) = (1/3, 2/3)$			$(\lambda_1, \lambda_2) = (1/4, 3/4)$		
			Max	Average	Median	Max	Average	Median	Max	Average	Median
Non-sequential Optimal vs. Offering-all	1	231	4.7%	2.3%	2.1%	4.7%	2.7%	2.6%	4.0%	2.7%	3.0%
	2	231	4.1%	2.1%	2.3%	3.8%	2.3%	2.6%	3.2%	2.3%	2.4%
	3	231	3.4%	1.7%	1.8%	3.1%	1.9%	2.1%	2.5%	1.9%	1.9%
	4	231	3.0%	1.4%	1.5%	2.6%	1.5%	1.7%	2.3%	1.5%	1.5%
Sequential Optimal vs. Offering-all	1	231	11.2%	4.2%	3.3%	12.3%	5.6%	4.9%	10.6%	6.3%	6.6%
	2	231	11.2%	4.1%	3.0%	9.9%	5.0%	5.5%	8.2%	5.4%	5.7%
	3	231	9.7%	3.4%	2.4%	8.2%	4.1%	4.7%	6.7%	4.4%	4.4%
	4	231	8.2%	2.9%	1.9%	6.9%	3.4%	3.8%	5.7%	3.6%	3.7%

Table C3 Policy Comparison in a Multi-day Scheduling Setting (with Poisson Arrivals, $T = 15$, $N = 30$).

	D	# of Scenarios	$(\lambda_1, \lambda_2) = (1/2, 1/2)$			$(\lambda_1, \lambda_2) = (1/3, 2/3)$			$(\lambda_1, \lambda_2) = (1/4, 3/4)$		
			Max	Average	Median	Max	Average	Median	Max	Average	Median
Non-sequential Optimal vs. Offering-all	1	91	4.1%	2.1%	2.0%	4.2%	2.4%	2.1%	3.7%	2.4%	2.5%
	2	91	3.7%	1.9%	1.8%	3.4%	2.0%	2.2%	2.9%	2.0%	2.1%
	3	91	3.0%	1.5%	1.5%	1.6%	1.7%	2.3%	2.3%	1.6%	1.6%
	4	91	2.7%	1.2%	1.2%	2.3%	1.3%	1.6%	2.1%	1.4%	1.4%
Sequential Optimal vs. Offering-all	1	91	9.6%	4.0%	3.4%	11.2%	5.1%	4.2%	10.0%	5.6%	5.8%
	2	91	9.8%	3.7%	2.8%	9.0%	4.5%	4.4%	7.6%	4.8%	5.1%
	3	91	8.6%	3.2%	2.2%	7.5%	3.7%	4.2%	6.1%	4.0%	4.1%
	4	91	7.5%	2.6%	1.7%	6.4%	3.1%	3.5%	5.2%	3.3%	3.3%

Table C4 Policy Comparison in a Multi-day Scheduling Setting (with Poisson Arrivals, $T = 30$, $N = 50$).

	D	# of Scenarios	$(\lambda_1, \lambda_2) = (1/2, 1/2)$			$(\lambda_1, \lambda_2) = (1/3, 2/3)$			$(\lambda_1, \lambda_2) = (1/4, 3/4)$		
			Max	Average	Median	Max	Average	Median	Max	Average	Median
Non-sequential Optimal vs. Offering-all	1	231	4.5%	2.3%	2.1%	5.1%	2.7%	2.6%	4.0%	2.7%	2.9%
	2	231	4.0%	2.0%	2.0%	3.7%	2.2%	2.5%	3.1%	2.2%	2.3%
	3	231	3.4%	1.7%	1.7%	3.0%	1.8%	2.1%	2.6%	1.8%	1.8%
	4	231	2.9%	1.4%	1.4%	2.6%	1.5%	1.7%	2.3%	1.5%	1.5%
Sequential Optimal vs. Offering-all	1	231	11.1%	4.1%	3.2%	12.3%	5.6%	4.7%	10.5%	6.2%	6.5%
	2	231	10.9%	3.9%	2.8%	9.8%	4.9%	5.3%	7.9%	5.3%	5.5%
	3	231	9.5%	3.3%	2.3%	8.0%	4.0%	4.6%	6.6%	4.3%	4.4%
	4	231	8.3%	2.7%	1.7%	6.8%	3.3%	3.8%	5.6%	3.6%	3.6%