Appendix. Appendix

Proofs of Auxiliary Results.

A. Proof of Proposition 1.

First suppose that the nominal mean $\boldsymbol{\mu}$ increases by $\boldsymbol{\Delta\mu} \geq \mathbf{0}$. Then there exists $\boldsymbol{\Delta}\mathbf{s}^* \geq \mathbf{0}$ and $\boldsymbol{\Delta}\mathbf{x}^* \geq \mathbf{0}$ that solves the nominal problem with deterministic demand $\boldsymbol{\Delta\mu}$, with non-negative profit $P \geq 0$. Thus the worst-case profit with $\boldsymbol{\mu} + \boldsymbol{\Delta\mu}$ increases at least by P. On the other hand, one can show that the set $\mathcal{U}$ increases in any of $\lambda_1, \ldots, \lambda_l$, followed by that the objective value decreases. □

B. Proof of Proposition 2.

For fixed $\{\mathbf{w}_t, \mathbf{W}_{\tau,t}\}$ and $\mathbf{D}_{[1:T]}$, the inner maximization in problem (6) with respect to $\mathbf{X}_{[1:T]}$ is a linear program in which $\{\mathbf{w}_t, \mathbf{W}_{\tau,t}\}$ are on the right-hand side. It follows that for any $\mathbf{D}_{[1:T]}$, the inner maximization problem (6) is concave in $\{\mathbf{w}_t, \mathbf{W}_{\tau,t}\}$. Hence the objective function in (6) is concave as well, because it is a pointwise infimum of concave functions. Moreover, the problem is always feasible with assigning zero vectors and matrices to $\{\mathbf{w}_t, \mathbf{W}_{\tau,t}\}$. Finally, applying strong duality to constraint (7) shows that a feasible set of $\{\mathbf{w}_t, \mathbf{W}_{\tau,t}\}$ is a polyhedron and hence, convex. □

C. Proof of Proposition 3.

$\Phi(s_0, d_0) := \max_{\pi} V\Big(\pi(w_t, W_{\tau,t}); s_0, d_0\Big)$, where $V\Big(\pi(w_t, W_{\tau,t}); s_0, d_0\Big)$ is defined as

$$\begin{aligned} V\Big(\pi(w_t, W_{\tau,t}); s_0, d_0\Big) &:= \min_{\mathbf{d}\in\mathcal{U}^T} \max_{\mathbf{x},\mathbf{s}} P\Big(\pi(w_t, W_{\tau,t}), \mathbf{d}, \mathbf{x}; s_0, d_0\Big) \\ &= \min_{\mathbf{d}\in\mathcal{U}^T} \max_{\mathbf{x},\mathbf{s}} \Bigg[-c_S\left(\sum_{t=1}^T s_t\right) - c_H \sum_{t=1}^T \left(s_0 + \sum_{\tau=1}^t (s_\tau - x_\tau)\right) \\ &\qquad - c_P \sum_{t=1}^T \left(d_0 + \sum_{\tau=1}^t (d_\tau - x_\tau)\right) + r\left(\sum_{t=1}^T x_t\right)\Bigg], \end{aligned}$$

where the inner maximization problem has a feasible set $\mathcal{X}(\pi, \mathbf{d}, s_0, d_0)$.

We first claim that the following equation holds for any affine policy $\pi(w_t, W_{\tau,t})$ such that $s_0 \leq w_1$,

$$V\Big(\overline{\pi}(\overline{w}_t, \overline{W}_{\tau,t}); s_0, d_0\Big) = V\Big(\pi(w_t, W_{\tau,t}); 0, 0\Big) + c_S s_0 + (r - c_S) d_0, \tag{24}$$

where $\overline{\pi}(\overline{w}_t, \overline{W}_{\tau,t})$ is defined as (11). For any demand realization $\mathbf{d} \in \mathcal{U}^T$, let $\widetilde{s}_t = \widetilde{s}_t(\pi, \mathbf{d})$ and $\widetilde{x}_t = \widetilde{x}_t(\pi, \mathbf{d})$ be an aggregated order quantity and the corresponding optimal processing activity at time t with zero initial input and demand, i.e., it solves the inner maximization problem of $V(\pi; 0, 0)$ with $\mathbf{d}$. Let $\widetilde{\overline{s}}_t = \widetilde{\overline{s}}_t(\overline{\pi}, \mathbf{d})$ be an order quantity by the policy $\overline{\pi}$ for $V(\overline{\pi}; s_0, d_0)$, and define $\widetilde{\overline{x}}_t = \widetilde{\overline{x}}_t(\mathbf{d}) = \widetilde{x}_t(\mathbf{d}) + d_0$. Then by (11), $\widetilde{\overline{s}}_t = \widetilde{s}_t - s_0 + d_0$ for every $t = 1, \ldots, T$, and thus

$$\begin{aligned} &P\Big(\pi(w_t, W_{\tau,t}), \mathbf{d}, \mathbf{x}; 0, 0\Big) + c_S s_0 + (r - c_S) d_0 \\ &= \Big(-c_S \widetilde{s}_T - c_H \sum_{t=1}^T (\widetilde{s}_t - \widetilde{x}_t) - c_P \sum_{t=1}^T (\widetilde{d}_t - \widetilde{x}_t) + r\widetilde{x}_T\Big) + (r - c_S) d_0 + c_S s_0 \\ &= -c_S(\widetilde{\overline{s}}_T - s_0 + d_0) - c_H \sum_{t=1}^T (\widetilde{\overline{s}}_t + d_0) - c_P \sum_{t=1}^T (d_0 + \widetilde{d}_t) + (c_H + c_P) \sum_{t=1}^T (\widetilde{x}_t + d_0) + r(\widetilde{x}_T + d_0) \\ &= -c_S \widetilde{\overline{s}}_T - c_H \sum_{t=1}^T (s_0 + \widetilde{\overline{s}}_t - \widetilde{\overline{x}}_t) - c_P \sum_{t=1}^T (d_0 + \widetilde{d}_t - \widetilde{\overline{x}}_t) + r\widetilde{\overline{x}}_T \\ &= P\Big(\overline{\pi}(\overline{w}_t, \overline{W}_{\tau,t}), \mathbf{d}, \overline{\mathbf{x}}; s_0, d_0\Big). \end{aligned}$$

Since $\overline{\mathbf{x}}$ is a feasible solution of the inner maximization problem in $V(\overline{\pi};s_0,d_0)$, the LHS of (24) is greater than or equal to the RHS. Similar argument can be made to show the contrary and this concludes the proof of (24).

Now suppose $\overline{\pi}^*(\overline{w}_t^*,\overline{W}_{\tau,t}^*)$ is not optimal, and let $\overline{\varphi}^*(\overline{v}_t^*,\overline{V}_{\tau,t}^*)$ be an optimal solution of $\Phi(s_0,d_0)$. Now one can easily check that by (24),

$$\begin{aligned}\Phi(s_0,d_0) = V(\overline{\varphi}^*;s_0,d_0) &= V(\varphi^*;0,0) + c_S s_0 + (r-c_S)d_0 \\ &\le V(\pi^*;0,0) + c_S s_0 + (r-c_S)d_0 = V(\overline{\pi}^*;s_0,d_0) < \Phi(s_0,d_0),\end{aligned}$$

which makes a contradiction. Finally, the proof of Eq. (24) directly shows that both $\Phi(0,0)$ and $\Phi(s_0,d_0)$ shares a common worst-case scenario among $\mathcal{U}^T$. □

D. Proof of Proposition 4.

Let $\overline{\pi}_{\text{PA}} = (\overline{\pi}_1,\dots,\overline{\pi}_N)$, where each $\overline{\pi}_j$ solves the j^{th} subproblem. Then the worst-case scenario $(\mathbf{d}_1^*,\dots,\mathbf{d}_N^*)$ solves an optimization problem

$$\min_{\mathbf{d}_1,\dots,\mathbf{d}_N}\ \max_{\mathbf{x}_1,\dots,\mathbf{x}_N}\ \left[\sum_{j=1}^N P_j(\overline{\pi}_j,\mathbf{d}_j,\mathbf{x}_j;u^s_{t_{j-1}},u^d_{t_{j-1}})\right]. \tag{25}$$

Since $\overline{\pi}_{\text{PA}}$ is a well-defined periodic-affine policy, (25) is rewritten as

$$\begin{aligned}&\min_{\mathbf{d}_1,\dots,\mathbf{d}_N}\ \max_{\mathbf{x}_1,\dots,\mathbf{x}_N}\ \left[\sum_{j=1}^N P_j(\overline{\pi}_j,\mathbf{d}_j,\mathbf{x}_j;u^s_{t_{j-1}},u^d_{t_{j-1}})\right]\\ &=\min_{\mathbf{d}_1,\dots,\mathbf{d}_N}\ \max_{\mathbf{x}_1,\dots,\mathbf{x}_N}\ \left[\sum_{j=1}^N \widetilde{P}_j(\overline{\pi}_j,\mathbf{d}_j,\mathbf{x}_j;u^s_{t_{j-1}},u^d_{t_{j-1}})\right]\\ &=\min_{\mathbf{d}_1,\dots,\mathbf{d}_N}\ \max_{\mathbf{x}_1,\dots,\mathbf{x}_N}\ \left[\sum_{j=1}^N P_j^{\text{PA}}(\overline{\pi}_j,\mathbf{d}_j,\mathbf{x}_j)\right]\\ &=\sum_{j=1}^N\left[\min_{\mathbf{d}_j\in\mathcal{U}^j}\max_{\mathbf{x}_j}\ P_j^{\text{PA}}(\overline{\pi}_j,\mathbf{d}_j,\mathbf{x}_j)\right]\\ &=\sum_{j=1}^N\max_{\mathbf{x}_j}\ P_j^{\text{PA}}(\overline{\pi}_j,\mathbf{d}_j^*,\mathbf{x}_j),\end{aligned}$$

where $\widetilde{P}_j$ and P_j^{PA} are defined in (18) and (19). As a result, the overall objective function is separable for each subproblem, and hence, the worst-case scenario consists of those of the subperiods. □

E. Proof of Theorem 1.

Since every affine policy is feasible to the DP problem (12), proving $V_{\text{Aff}}^* \le V_{\text{DP}}^*$ directly follows. Thus it suffices to show $V_{\text{PA}}^* = V_{\text{DP}}^*$, and let $\overline{\pi}_{\text{PA}} = (\overline{\pi}_1,\dots,\overline{\pi}_N)$ be an output of the periodic-affine algorithm. That is, $\overline{\pi}_j$ is an affine-IBS policy associated with $\pi_j = \pi_j(w_t^{(j)},W_{\tau,t}^{(j)})$, where π_j solves (20) for each $j=1,\dots,N$. Define $V_j(u^s,u^d)$ as a worst-case optimal profit from the j^{th} subperiod to the last period, where u^s and u^d are current on-hand input and backlogged demand. Our framework justifies using the (robust) optimality equation and a (worst-case) value function approach in robust dynamic programming scheme; we refer Iyengar (2005) to readers for technical details. We will show for every $j=1,\dots,N$,

(a) $V_j(u^s,u^d)$ is concave in (u^s,u^d), and

(b) $V_j(u^s,u^d)=V_j(0,0)+c_S u^s+(r-c_S)u^d$ for every $0\le u^s\le w_1^{(j)}$, $u^d\ge 0$.

by mathematical induction.

Now consider $j=N$ and suppose $u^s\le w_1^{(N)}$. Then $V_N(u^s,u^d)$ can be written as

$$\begin{aligned}V_N(u^s,u^d):=\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^N)}\ \min_{\mathbf{d}_N}\ \max_{\mathbf{x}_N,\mathbf{s}_N}\ & P_N(\pi,\mathbf{d}_N,\mathbf{x}_N;u^s,u^d)\\ \text{s.t.}\quad & (\mathbf{x}_N,\mathbf{s}_N)\in\mathcal{X}(\pi,\mathbf{d}_N,u^s,u^d),\end{aligned}$$

where $\pi=\pi(w_t,W_{\tau,t})$ and the constraints in $\mathcal{X}(\pi,\mathbf{d}_N,u^s,u^d)$ can be rearranged so that the right hand sides are linear in $(u^s,u^d,w_t,W_{\tau,t})$. Since P_N is concave in $(\mathbf{x}_N,w_t,W_{\tau,t},u^s,u^d)$ and $\mathcal{X}(\pi,\mathbf{d}_N,u^s,u^d)$ defines a polyhedron for any π, u^s, and u^d, the objective function within the min operator is concave in $(u^s,u^d,w_t,W_{\tau,t})$ by concavity preservation under maximization. Since a pointwise infimum of concave functions are concave and applying concavity preservation under maximization again to the outermost max operator, we finally have that $V_N(u^s,u^d)$ is concave in (u^s,u^d). On the other hand, (b) follows directly from Proposition 3 for $j=N$.

Now suppose that both (a) and (b) hold for any $1<j\le N$, and let $u^s\le w_1^{(j-1)}$ and $u^d\ge 0$. Then from the optimality equation, we have

$$V_{j-1}(u^s,u^d)=\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})}\ \min_{\mathbf{d}_{j-1}}\ \max_{\mathbf{x}_{j-1}}\left[P_{j-1}\Big(\pi,\mathbf{d}_{j-1},\mathbf{x}_{j-1};u^s,u^d\Big)+V_j(u^s_j,u^d_j)\right]. \tag{26}$$

Since $V_j(u^s_j,u^d_j)$ is concave in (u^s_j,u^d_j) and (u^s_j,u^d_j) can be expressed as affine functions of $(w_t,W_{\tau,t},u^s,u^d)$, applying the above argument shows that $V_{j-1}(u^s,u^d)$ is also concave in (u^s,u^d). In addition, we have

$$\begin{aligned}V_{j-1}(u^s,u^d)&=\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})}\ \min_{\mathbf{d}_{j-1}}\ \max_{\mathbf{x}_{j-1}}\left[P_{j-1}\Big(\pi,\mathbf{d}_{j-1},\mathbf{x}_{j-1};u^s,u^d\Big)+V_j(u^s_j,u^d_j)\right]\\ &\le\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})}\ \min_{\mathbf{d}_{j-1}}\ \max_{\mathbf{x}_{j-1}}\left[P_{j-1}\Big(\pi,\mathbf{d}_{j-1},\mathbf{x}_{j-1};u^s,u^d\Big)+V_j(0,0)+c_S u^s_j+(r-c_S)u^d_j\right]\\ &=\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})}\ \min_{\mathbf{d}_{j-1}}\ \max_{\mathbf{x}_{j-1}}\left[P_{j-1}\Big(\pi,\mathbf{d}_{j-1},\mathbf{x}_{j-1};u^s,u^d\Big)+c_S u^s_j+(r-c_S)u^d_j\right]+V_j(0,0)\\ &=\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})}\ \min_{\mathbf{d}_{j-1}}\ \max_{\mathbf{x}_{j-1}}\left[\widetilde{P}_{j-1}\Big(\pi,\mathbf{d}_{j-1},\mathbf{x}_{j-1};u^s,u^d\Big)+c_S u^s+(r-c_S)u^d\right]+V_j(0,0)\\ &=c_S u^s+(r-c_S)u^d+\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})}\ \min_{\mathbf{d}_{j-1}}\ \max_{\mathbf{x}_{j-1}}\left[\widetilde{P}_{j-1}\Big(\pi,\mathbf{d}_{j-1},\mathbf{x}_{j-1};0,0\Big)\right]+V_j(0,0)\\ &\le V_{j-1}(u^s,u^d).\end{aligned}$$

The first inequality comes from that both (a) and (b) hold for V_j, and the third equality is from the definition of $\widetilde{P}_{j-1}$. Finally, the last inequality is a worst-case profit from j^{th} subperiod with a policy $\overline{\pi}_{\text{PA}}$, by Assumption 1. Since $\overline{\pi}_{\text{PA}}$ is a feasible policy to the DP, the last inequality follows. This shows that whenever $u^s\le w_1^{(j-1)}$, then the value function $V_{j-1}(u^s,u^d)$ is achieved with a policy $(\overline{\pi}_{j-1},\dots,\overline{\pi}_N)$. Finally we have

$$\begin{aligned}V_{j-1}(u^s,u^d)&=c_S u^s+(r-c_S)u^d+\max_{\pi\in\Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})}\ \min_{\mathbf{d}_{j-1}}\ \max_{\mathbf{x}_{j-1}}\left[\widetilde{P}_{j-1}\Big(\pi,\mathbf{d}_{j-1},\mathbf{x}_{j-1};0,0\Big)\right]+V_j(0,0)\\ &=V_{j-1}(0,0)+c_S u^s+(r-c_S)u^d,\end{aligned}$$

and thus (b) holds for $j-1$. By Assumption 1, $\overline{\pi}_{\text{PA}}=(\overline{\pi}_1,\dots,\overline{\pi}_N)$ satisfies $u^s_{t_j}\le w_1^{(j+1)}$ for every $j=1,\dots,N-1$ and every realization of demands, and hence, $\overline{\pi}_{\text{PA}}$ is Bellman-optimal to the DP (12) and this concludes with $V^*_{\text{PA}}=V^*_{\text{DP}}$. □

F. Proof of Proposition 5.

It suffices to show that an optimal periodic-affine policy is indeed an affine policy. Using the same notations in Theorem 1 and without loss of generality, we may assume that $N=2$ and let $\overline{\pi}^*_{\mathrm{PA}} = (\overline{\pi}_1, \overline{\pi}_2)$, where $\pi_j = (w_t^{(j)}, W_{\tau,t}^{(j)})$ for $j=1,2$. By definition of periodic-affine policies, we only need to check that if an order quantity at time t_1+1 is affine in $\mathcal{U} = \mathcal{U}^1 \times \mathcal{U}^2$. Recall that $\overline{\pi}^*_{\mathrm{PA}}$ determines order quantity at t_1+1 as

$$\begin{aligned} w_1^{(2)} - u^s_{t_1} + u^d_{t_1} &= w_1^{(2)} - \max\left(\sum_{t=1}^{t_1}(s_t - d_t),\ 0\right) + \max\left(\sum_{t=1}^{t_1}(d_t - s_t),\ 0\right) \\ &= w_1^{(2)} + \left(\sum_{t=1}^{t_1}(d_t - s_t)\right). \end{aligned}$$

It is affine in $\mathbf{d}_1$, since s_t is affine in $\mathbf{d}_1$, and this concludes the proof. □

G. Proof of Theorem 2.

We use the value function $V_j(u^s, u^d)$ defined in Theorem 1. From concavity $V_j(u^s, u^d)$ and using (b), we have

$$V_j(u^s, u^d) \le V_j(0,0) + c_S u^s + (r - c_S) u^d$$

for every $u^s \ge 0$ and $u^d \ge 0$. We will show that

$$V_j(u^s, u^d) \le c_S u^s + (r - c_S) u^d + \sum_{k=j}^{N} \widetilde{f}^*_k \qquad \forall j = 1, \ldots, N \tag{27}$$

by induction, and plugging $j=1$ and $u^s = u^d = 0$ into (27) concludes the proof.

From $V_N(0,0) = \widetilde{f}^*_N$, we have that (27) holds for $j = N$. Now suppose $1 < j \le N$. Then from the optimality equation we have

$$\begin{aligned} V_{j-1}(u^s, u^d) &= \max_{\pi \in \Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})} \ \min_{\mathbf{d}_{j-1}} \ \max_{\mathbf{x}_{j-1}} \left[P_{j-1}\Big(\pi, \mathbf{d}_{j-1}, \mathbf{x}_{j-1}; u^s, u^d\Big) + V_j(u^s_j, u^d_j) \right] \\ &\le \max_{\pi \in \Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})} \ \min_{\mathbf{d}_{j-1}} \ \max_{\mathbf{x}_{j-1}} \left[P_{j-1}\Big(\pi, \mathbf{d}_{j-1}, \mathbf{x}_{j-1}; u^s, u^d\Big) + c_S u^s_j + (r - c_S) u^d_j + \sum_{k=j}^{N} \widetilde{f}^*_k \right] \\ &\le \ c_S u^s + (r - c_S) u^d + \max_{\pi \in \Pi_{\mathrm{aff}}(\mathcal{U}^{j-1})} \ \min_{\mathbf{d}_{j-1}} \ \max_{\mathbf{x}_{j-1}} \left[\widetilde{P}_{j-1}\Big(\pi, \mathbf{d}_{j-1}, \mathbf{x}_{j-1}; 0, 0\Big) \right] + \sum_{k=j}^{N} \widetilde{f}^*_k \\ &= \ c_S u^s + (r - c_S) u^d + \sum_{k=j-1}^{N} \widetilde{f}^*_k, \end{aligned}$$

where the first inequality holds from the induction hypothesis and the third equality is from definition of $\widetilde{P}_{j-1}$. One can show that the maximization problem in the third equality is concave in u^s and u^d, as similar in the proof of Theorem 1 and this concludes the proof. □

H. Proof of Theorem 3.

All the proofs of Theorem 1 and 2 can be extended into multi-station networks, by using a basis matrix $\mathbf{R}_B$ to replace $c_S u^s$ and $(r - c_S) u^d$ terms in the proof with $\mathbf{c}_S^\top \mathbf{u}^s$ and $(\mathbf{R}_B^\top \mathbf{r} - \mathbf{R}_B^\top \mathbf{R}_S^\top \mathbf{c}_S)^\top \mathbf{u}^d$, respectively. This expressions are still linear in $\mathbf{u}^s$ and $\mathbf{u}^d$, hence all the arguments in the proof are valid. □

I. Proof of Theorem 4.

It suffices to show for single-station cases, since it is straightforward to extend the result to general multi-station networks, as in Theorem 3. Note that the optimality equation for the infinite horizon problem is written as

$$V_\infty(u^s,u^d) = \max_{\pi\in\Pi_{\text{aff}}(\mathcal{U}^{T_0})} \min_{D_{[1:T_0]}} \max_{X_{[1:T_0]}} \left[P\Big(\pi, D_{[1:T_0]}, X_{[1:T_0]}; u^s, u^d\Big) + \beta^{T_0} V_\infty(\overline{u}^s, \overline{u}^d_j)\right], \tag{28}$$

where $\overline{u}^s$ and $\overline{u}^d$ denotes on-hand input and backorders after T_0 periods (one stage).

We impose mild conditions so that the optimality equation (28) defines a contraction mapping and there exists V_∞ which is the unique fixed point. (See Iyengar (2005) for details.) Hence the value iteration algorithm is well-defined, and let $V_n(u^s,u^d)$ be a value function after n iterations. Recalling that $\overline{u}^s$ and $\overline{u}^d$ are expressed as linear functions of u^s and u^d, one can show by applying concavity preservation under maximization as similar in Theorem 1 that if $V_n(u^s,u^d)$ is concave, then so $V_{n+1}(u^s,u^d)$ is. Since we can start with any bounded continuous function for the value iteration algorithm, we conclude that $V_\infty(u^s,u^d)$ is concave in (u^s,u^d).

In this setting, there exists a stationary optimal policy $\pi_\infty = (\pi,\pi,\ldots)$ where $\pi = \pi(u^s,u^d)$ is defined for each subperiod of length T_0. By Proposition 3, we can see that $V_\infty(u^s,u^d) = V_\infty(0,0) + c_S u^s + (r - c_S)u^d$ for $u^s \le w_1$ and with concavity of V_∞, we have

$$V_\infty(u^s,u^d) \le V_\infty(0,0) + c_S u^s + (r-c_S)u^d$$

for every $u^s \ge 0$ and $u^d \ge 0$.

Let $V_\infty(\pi_\infty)$ be a worst-case objective value under policy π_∞, and V_∞^* be an optimal value of the DP problem. Since π_∞ is feasible to the DP by Assumption 1, we have $V_\infty(\pi_\infty) \le V_\infty^*$. On the other hand,

$$\begin{aligned}
V_\infty^* = &\max_{\pi\in\Pi_{\text{aff}}(\mathcal{U}^{T_0})} \min_{D_{[1:T_0]}} \max_{X_{[1:T_0]}} \left[P\Big(\pi, D_{[1:T_0]}, X_{[1:T_0]}; 0,0\Big) + \beta^{T_0} V_\infty(\overline{u}^s,\overline{u}^d)\right] \\
\le &\max_{\pi\in\Pi_{\text{aff}}(\mathcal{U}^{T_0})} \min_{D_{[1:T_0]}} \max_{X_{[1:T_0]}} \left[P\Big(\pi, D_{[1:T_0]}, X_{[1:T_0]}; u^s,u^d\Big) + \beta^{T_0}\left(c_S\overline{u}^s + (r-c_S)\overline{u}^d + V_\infty(0,0)\right)\right] \\
= &\max_{\pi\in\Pi_{\text{aff}}(\mathcal{U}^{T_0})} \min_{D_{[1:T_0]}} \max_{X_{[1:T_0]}} P_\infty^{\text{PA}}\Big(\pi, D_{[1:T_0]}, X_{[1:T_0]}\Big) + \beta^{T_0} V_\infty(0,0) \\
= &V_\infty(\pi_\infty),
\end{aligned}$$

by Assumption 1 (this step is similar to Theorem 1), and this implies that an optimal value to the DP is achieved by π_∞, where the stationary policy π is obtained by solving the optimization problem (22). □