Proofs of Propositions

A High $\pi_i\alpha_i$

1 Status Quo

Lemma A.1 When $\pi_i\alpha_i > 1$, under the status quo, i.e., $\delta = 1$, the complete threshold characterization of the consumer market equilibrium is as follows:

(I) $(0 < v_a < 1)$, where $v_a = \frac{p + c_a}{1 - \pi_a\alpha_a}$:

(A) $p + c_a + \pi_a\alpha_a < 1$

(B) $c_p \geq c_a + \pi_a\alpha_a$

(C) $c_p \geq \frac{c_a + p\pi_a\alpha_a}{1 - \pi_a\alpha_a}$

(II) $(0 < v_a < v_p < 1)$, where $v_a = \frac{p + c_a}{1 - \pi_a\alpha_a}$ and $v_p = \frac{c_p - c_a}{\pi_a\alpha_a}$:

(A) $c_p < c_a + \pi_a\alpha_a$

(B) $c_p > \frac{c_a + p\pi_a\alpha_a}{1 - \pi_a\alpha_a}$

(III) $(0 < v_p < 1)$, where $v_p = p + c_p$:

(A) $c_p + p < 1$

(B) $c_p \leq \frac{c_a + p\pi_a\alpha_a}{1 - \pi_a\alpha_a}$

Proof of Lemma A.1: This is a sub-case in the proof of Lemma A.2, by setting $\delta = 1$. □

Proof of Lemma 2: We prove that if $\pi_i\alpha_i > 1$ and $\delta = 1$ (i.e., when patching rights are not priced), then if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$, we have that

$$p^* = \frac{1 - \pi_a\alpha_a - c_a}{2},$$

(A.1)
and \( \sigma^* \) is characterized by \( 0 < v_a < v_p < 1 \) such that the lower tier of users prefers automated patching. On the other hand, if \( c_a \geq 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a} \), then

\[
p^* = \frac{1 - c_p}{2},
\]

(A.2)

and \( \sigma^* \) is characterized by \( 0 < v_a < v_p < 1 \) such that there is no user of automated patching in equilibrium.

Suppose \( 0 < v_a < 1 \) is induced. Then the profit function is \( \Pi_I(p) = p(1 - v_a) \). Using Lemma A.1, we have that \( v_a = \frac{p + c_a}{1 - \pi_a \alpha_a} \). The optimal price is found to be \( p_1^* = \frac{1}{2} (1 - c_a - \pi_a \alpha_a) \) with the corresponding profit \( \Pi_I^* = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)} \).

Similarly, suppose instead that \( 0 < v_a < v_p < 1 \) is induced. Then the profit function is \( \Pi_{II}(p) = p(1 - v_a) \). Using Lemma A.1, we have that \( v_a = \frac{p + c_a}{1 - \pi_a \alpha_a} \). Again, the optimal price is found to be \( p_2^* = \frac{1}{2} (1 - c_a - \pi_a \alpha_a) \) with the corresponding profit \( \Pi_{II}^* = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)} \).

Lastly, suppose that \( 0 < v_p < 1 \) is induced. Then the profit function is \( \Pi_{III}(p) = p(1 - v_p) \). Using Lemma A.1, we have that \( v_p = c_p + p \). Now, the optimal price is found to be \( p_3^* = \frac{1 - c_p}{2} \) with the corresponding profit \( \Pi_{III}^* = \frac{(1 - c_p)^2}{4} \).

We next find conditions under which the maximizing price for each case indeed induces that market structure. For \( 0 < v_a < 1 \), we need the set of conditions for Case (I) in Lemma A.1 to hold for \( p_1^* \). To satisfy the first condition, we need \( p + c_a + \pi a \alpha_a < 1 \) for \( p = p_1^* = \frac{1}{2} (1 - c_a - \pi a \alpha_a) \). This simplifies to \( c_a + \pi a \alpha_a < 1 \), which is a preliminary model assumption. Then we need \( c_p \geq \frac{c_a + c_p \pi a \alpha_a}{1 - \pi a \alpha_a} \) to hold for \( p = p_1^* \) as well, which simplifies to \( c_a \leq \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \). We also needed the condition \( c_a \leq c_p - \pi a \alpha_a \) for this case to hold. Since \( \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} > c_p - \pi a \alpha_a \) follows from \( 0 < c_p < 1 \) and \( 0 < \pi a \alpha_a < 1 \), the condition under which \( p_1^* \) would induce \( 0 < v_a < 1 \) is \( c_a \leq c_p - \pi a \alpha_a \).

Similarly, for Case (II), the condition under which \( p_2^* \) would induce \( 0 < v_a < v_p < 1 \) is \( c_p - \pi a \alpha_a < \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \). And lastly, for Case (III), the condition under which \( p_3^* \) would induce \( 0 < v_p < 1 \) is \( c_a \leq c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a \). Note that \( c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a < \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \), so that when \( c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a < c_a < \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \), we’ll need to compare \( \Pi_{II}^* \) and \( \Pi_{III}^* \).

Next, we find the conditions under which the maximal profits of each case dominate the other cases. In particular, when \( c_a < c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a \), then \( \Pi_I^* = \Pi_{II}^* > \Pi_{III}^* \). Since \( c_p - \pi a \alpha_a < c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a \) from \( c_p < 1 \), this implies that \( 0 < v_a < 1 \) will be the resulting consumer market structure for \( c_a \leq c_p - \pi a \alpha_a \). Also, \( 0 < v_a < v_p < 1 \) will be the resulting consumer market structure for \( c_p - \pi a \alpha_a < c_a < c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a \).

Also, for \( c_a > \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \), we have that \( \Pi_{II}^* > \Pi_{III}^* \), so that \( 0 < v_p < 1 \) will be the resulting market structure when \( c_a > \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \).

In between \( c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a < c_a < \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \), we find the conditions under which \( \Pi_{II}^* \) dominates \( \Pi_{III}^* \). Comparing the profits, we find that \( \Pi_{II}^* \geq \Pi_{III}^* \) when \( c_a \leq 1 - \pi a \alpha_a - (1 - c_p)\sqrt{1 - \pi a \alpha_a} \). Then we note that \( c_p - \frac{1}{2} (1 + c_p) \pi a \alpha_a < 1 - \pi a \alpha_a - (1 - c_p)\sqrt{1 - \pi a \alpha_a} < \frac{(2c_p - \pi a \alpha_a)(1 - \pi a \alpha_a)}{2(1 - \pi a \alpha_a)} \) always holds, so that the resulting market structure when \( c_p - \pi a \alpha_a < c_a < 1 - \pi a \alpha_a - (1 - c_p)\sqrt{1 - \pi a \alpha_a} \) is \( 0 < v_a < v_p < 1 \), and the resulting market structure when \( c_a \geq 1 - \pi a \alpha_a - (1 - c_p)\sqrt{1 - \pi a \alpha_a} \) is \( 0 < v_p < 1 \). □
2 Pricing Patching Rights

Lemma A.2 When $\pi_i \alpha_i > 1$, under PPR, the complete threshold characterization of the consumer market equilibrium is as follows:

(I) $(0 < v_a < 1)$, where $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$:
   
   (A) $\delta p + c_a + \pi_a \alpha_a < 1$
   (B) $c_p + (1 - \delta) p \geq c_a + \pi_a \alpha_a$
   (C) $c_p \geq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$

(II) $(0 < v_a < v_p < 1)$, where $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$ and $v_p = \frac{(1 - \delta) p + c_p - c_a}{\pi_a \alpha_a}$:
   
   (A) $c_p + (1 - \delta) p < c_a + \pi_a \alpha_a$
   (B) $c_p \geq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$

(III) $(0 < v_p < 1)$, where $v_p = p + c_p$:

   (A) $c_p + p < 1$
   (B) $c_p \leq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$

Proof of Lemma A.2: First, we establish the general threshold-type equilibrium structure. The proof of this is a sub-case of the argument in Lemma A.4, with the size of the unpatched user population $u = 0$ since $\pi_i \alpha_i > 1$. This establishes the threshold-type consumer market equilibrium structure.

Next, we characterize in more detail each outcome that can arise in equilibrium, as well as the corresponding parameter regions. For Case (I), in which all consumers who purchase choose the automated patching option, i.e., $0 < v_a < 1$, based on the threshold-type equilibrium structure, we have $u = 0$. For this market structure to be an equilibrium, we need $v_a > 0$, $v_a < 1$, the consumer $v = 1$ weakly preferring $(B, AP)$ over $(B, P)$, and the consumer $v = v_a$ weakly preferring $(NB, NP)$ over $(B, P)$.

The condition $v_a > 0$ is satisfied by our assumption that $\pi_a \alpha_a < 1$, since $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. Then for $v_a < 1$, we need $\delta p + c_a + \pi_a \alpha_a < 1$. For $v = 1$ to weakly prefer $(B, AP)$ over $(B, P)$, it needs to be the case that $v - \delta p - c_a - \pi_a \alpha_a v \geq v - p - c_p$ for $v = 1$. This simplifies to $c_p + (1 - \delta) p \geq c_a + \pi_a \alpha_a$. For $v = v_a$ to weakly prefer $(NB, NP)$ over $(B, P)$, it needs to be the case that $0 \geq v - p - c_p$ for $v = v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. This simplifies to $c_p \geq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$.

Next, for case (II), in which the top tier purchases $(B, P)$ but the lower tier of consumers purchase $(B, AP)$, i.e., $0 < v_a < v_p < 1$, we have $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$ and $v_p = \frac{(1 - \delta) p + c_p - c_a}{\pi_a \alpha_a}$. Following the same steps as before, we find the corresponding conditions for which case (II) arises. For this case to arise, we need $v_a > 0$, $v_p > v_a$, and $v_p < 1$. Again, $v_a > 0$ is satisfied under $\pi_a \alpha_a < 1$, one of the preliminary assumptions of the model. To have $v_p > v_a$, we need $\frac{(1 - \delta) p + c_p - c_a}{\pi_a \alpha_a} > \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. This simplifies to $c_p > \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$. Lastly, to have $v_p < 1$, we need $\frac{(1 - \delta) p + c_p - c_a}{\pi_a \alpha_a} < 1$. This simplifies to $c_p + (1 - \delta) p < c_a + \pi_a \alpha_a$. 

A.3
Lastly, for case (III), in which consumers who purchase are all standard patching, choosing $(B, P)$, $v_p = p + c_p$. For this case to be an equilibrium, we need $v_p > 0$, $v_p < 1$, $v = v_p$ preferring $(NB, NP)$ over $(B, AP)$, and $v = v_p$ preferring $(B, P)$ over $(B, AP)$. The condition $v_p > 0$ is satisfied. For $v_p < 1$, we need the condition $c_p + p < 1$. For $v = v_p$ to prefer $(NB, NP)$ over $(B, AP)$, we need $0 \geq v - \delta p - c_a - \pi_a v$ for $v = v_p = c_p + p$. This becomes $c_p + (1 - \delta) p \leq c_a + (c_p + p) \pi_a$. Lastly, for $v = v_p$ to prefer $(B, P)$ over $(B, AP)$, we need $v - p - c_p \geq v - \delta p - c_a - \pi_a v$ for $v = v_p = c_p + p$. This also simplifies to $c_p \leq \frac{c_a + p(\pi_a \alpha_a(1 - \delta))}{1 - \pi_a \alpha_a}$). This concludes the proof of the consumer market equilibrium for the PPR case when $\pi_i \alpha_i > 1$. \hfill \Box

**Proof of Lemma 3:** We prove that if $\pi_i \alpha_i > 1$ and patching rights are priced by the vendor, then if $c_p - \pi_a \alpha_a < c_a \leq c_p(1 - \pi_a \alpha_a)$, we have

$$p^* = \frac{1 - c_p}{2}, \quad (A.3)$$

$$\delta^* = \frac{1 - c_a - \pi_a \alpha_a}{1 - c_p}, \quad (A.4)$$

and $\sigma^*$ is characterized by $0 < v_a < v_p < 1$ such that the lower tier of users prefers automated patching. On the other hand, if $c_a > c_p(1 - \pi_a \alpha_a)$, then

$$p^* = \frac{1 - c_p}{2}, \quad (A.5)$$

and $\sigma^*$ is characterized by $0 < v_a < v_p < 1$ such that there is no user of automated patching in equilibrium.

Suppose $0 < v_a < 1$ is induced. Then the profit function is $\Pi_I(p, \delta) = \delta p(1 - v_a)$. Using Lemma A.2, we have that $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. Similar to the status quo case, the optimal price and discount satisfies $\delta^*_1 p^*_1 = \frac{1 - c_a - \pi_a \alpha_a}{2}$ with the corresponding profit $\Pi^*_I = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)}$. Notice in this case that there is not a unique maximizer, and in fact, the optimal $(p^*, \delta^*)$ traces out an isoprofit curve.

Next, suppose that $0 < v_a < v_p < 1$ is induced. Then the profit function is $\Pi_H(p, \delta) = p(1 - v_p) + \delta p(v_p - v_a)$. Using Lemma A.2, we have that $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$ and $v_p = \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}$. From the first-order condition for $p$, we have $p^*_2(\delta) = \frac{(1 - \pi_a \alpha_a)(1 - c_a)}{(1 - \delta)^2 - \pi_a \alpha_a}$ with the corresponding profit $\Pi^*_H = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)}$. Notice in this case that there is not a unique maximizer, and in fact, the optimal $(p^*, \delta^*)$ traces out an isoprofit curve.

Lastly, suppose that $0 < v_p < 1$ is induced. Then the profit function is $\Pi_{III}(p, \delta) = p(1 - v_p)$. Using Lemma A.2, we have that $v_p = c_p + p$. As in the case when patching rights aren’t priced, the optimal price is found to be $p^*_3 = \frac{1 - c_p}{2}$ with the corresponding profit $\Pi^*_{III} = \frac{(1 - c_p)^2}{4}$.

We next find conditions under which the maximizing price for each case indeed induces that market structure. For $0 < v_a < 1$, we need the set of conditions for Case (I) in Lemma A.2 to hold for $p^*_1$. For $\delta^* + c_a + \pi_a \alpha_a < 1$ to hold for the $p^*_1$ and $\delta^*_1$, we need $c_a + \pi_a \alpha_a < 1$, which is one of the preliminary assumptions of the model to not rule out automated patching for every consumer. Secondly, for $c_p + (1 - \delta)p \geq c_a + \pi_a \alpha_a$, we need $1 - c_a - \pi_a \alpha_a \geq (1 + c_a - \pi_a \alpha_a - 2c_p)$. If $1 + c_a - 2c_p + \pi_a \alpha_a \leq 0$, then any $\delta$ satisfies this condition. Otherwise, we need $\delta \leq \frac{1 - c_a - \pi_a \alpha_a - 2c_p}{1 + c_a + \pi_a \alpha_a - 2c_p}$. Note that in this case, $\frac{1 - c_a - \pi_a \alpha_a}{1 + c_a + \pi_a \alpha_a - 2c_p} > 0$ so that such a $\delta$ (the corresponding $p^*_1(\delta)$) can be found. Last, we need $c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a$ to hold for the profit-maximizing $p^*_1$ and $\delta^*_1$. This

A.4
simplifies to $\delta(c_a + (1-2c_p)(1-\pi_a\alpha_a)) \leq (1-\pi_a\alpha_a)(1-c_a-\pi_a\alpha_a)$. Then if $c_a + (1-2c_p)(1-\pi_a\alpha_a) \leq 0$, any $\delta$ satisfies this condition. Otherwise, we’ll need $\delta \leq \frac{(1-\pi_a\alpha_a)(1-c_a-\pi_a\alpha_a)}{c_a+(1-2c_p)(1-\pi_a\alpha_a)}$. Note in this case that 

$$
\frac{(1-\pi_a\alpha_a)(1-c_a-\pi_a\alpha_a)}{c_a+(1-2c_p)(1-\pi_a\alpha_a)} > 0
$$

so that such a $\delta$ (and the corresponding $p^*_p(\delta)$) can be found. In summary, $0 < v_a < 1$ can always be induced in equilibrium by some $p$ and $\delta$, given a set of parameters $c_a, \pi_a\alpha_a$, and $c_p$ that satisfy the preliminary model assumptions.

Similarly, for Case (II), the condition under which $p^*_p$ would induce $0 < v_a < v_p < 1$ is $c_p - \pi_a\alpha_a < c_a \leq c_p(1-\pi_a\alpha_a)$. And, lastly, for Case (III), we need $\delta \geq \frac{-2c_a + (1+c_p)(1-\pi_a\alpha_a)}{1-c_p}$ for $c_p \leq c_a + (1-\delta)p + (c_p + p)\pi_a\alpha_a$ to hold for $p = p^*_p$. This means that $0 < v_p < 1$ can always be induced in equilibrium using $p^*_p$ by setting a high enough $\delta$, given a set of parameters $c_a, \pi_a\alpha_a$, and $c_p$ that satisfy the preliminary model assumptions.

Next, we find the conditions under which the maximal profits of each case dominate each other. First, note that $\Pi^*_I \geq \Pi^*_III$ iff $c_a \leq 1-\pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$.

Next, note that $\Pi^*_II - \Pi^*_I = \frac{c_a + \pi_a\alpha_a - c_p}{4\pi_a\alpha_a}$, so that if $0 < v_a < v_p < 1$ can be induced, then it will dominate $0 < v_a < 1$. Also, $\Pi^*_II - \Pi^*_III = \frac{(c_a - c_p)(1-\pi_a\alpha_a)}{4\pi_a\alpha_a(1-\pi_a\alpha_a)}$, so that if $0 < v_a < v_p < 1$ can be induced, then it will dominate $0 < v_p < 1$ as well. Therefore, when $c_p - \pi_a\alpha_a < c_a \leq c_p(1-\pi_a\alpha_a)$, then $0 < v_a < v_p < 1$ will be the equilibrium market structure.

Furthermore, consider the boundaries of this region. When $c_a = c_p - \pi_a\alpha_a$, then the profit of the adjacent region is $\Pi^*_I = \frac{(1-c_p)^2}{4(1-\pi_a\alpha_a)}$ while $\Pi^*_II = \frac{(1-c_p)^2}{4(1-\pi_a\alpha_a)}$ as well. Similarly, at the other end, when $c_a = c_p(1-\pi_a\alpha_a)$, then $\Pi^*_II = \frac{1}{4}(1-c_p)^2 = \Pi^*_I$. This means that $0 < v_a < 1$ will be the equilibrium market structure for $c_a \leq c_p - \pi_a\alpha_a$ and $0 < v_p < 1$ will be the equilibrium market structure for $c_a \geq c_p(1-\pi_a\alpha_a)$. Note that if $c_a \geq c_p(1-\pi_a\alpha_a)$, then $\frac{-2c_a + (1+c_p)(1-\pi_a\alpha_a)}{1-c_p} \leq 1$ so that $\delta^*_3 = 1$ can be chosen to induce $0 < v_p < 1$ in equilibrium. □

**Proof of Proposition 1:** We show that for $\pi_i\alpha_i > 1$, if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, the increase in profitability under PPR is given by

$$
\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1-\pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1-c_a-\pi_a\alpha_a)^2}. \quad (A.6)
$$

First, note that $1-\pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a} < c_p(1-\pi_a\alpha_a)$, since $0 < c_p < 1$ and $0 < \pi_a\alpha_a < 1$. Hence, when $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, in both the status quo case and when patching rights are priced, the equilibrium consumer market structure is $0 < v_a < v_p < 1$. Then from the proof of Lemma 3 above, the profit under PPR is $\Pi_P = \frac{1}{4}(1-2c_p + \frac{(c_a-c_p)^2}{\pi_a\alpha_a} + \frac{c_a^2}{1-\pi_a\alpha_a})$ and from the proof of Lemma 2, the status quo case has $\Pi_{SQ} = \frac{(1-c_a-\pi_a\alpha_a)^2}{4(1-\pi_a\alpha_a)}$. Simplifying, we have

$$
\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1-\pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1-c_a-\pi_a\alpha_a)^2}. \quad \square
$$
B Low $\pi_i \alpha_i$

1 Status Quo

Lemma A.3 Under the status quo, i.e., $\delta = 1$, the complete threshold characterization of the consumer market equilibrium is as follows:

(I) $0 < v_a < 1$, where $v_a = \frac{p + c_a}{1 - \pi_a \alpha_a}$:

(A) $p + c_a + \pi_a \alpha_a < 1$
(B) $c_p \geq c_a + \pi_a \alpha_a$
(C) $(c_a + p)\pi_i \alpha_i \geq c_a + p(\pi_a \alpha_a)$
(D) $c_p \geq c_a + (c_p + p)\pi_a \alpha_a$
(E) $\pi_i \alpha_i > \pi_a \alpha_a$ and $c_a + p(\pi_a \alpha_a) \leq (c_a + p)\pi_i \alpha_i$

(II) $0 < v_b < 1$, where $v_b = \frac{1}{2} + \frac{-1 + \pi_a \alpha_i + \sqrt{(1 - \pi_a \alpha_a - \pi_i \alpha_i)^2 + 4p\pi_a \alpha_s}}{2p\pi_a \alpha_s}$

(A) $1 - 2c_p + \pi_i \alpha_i + \pi_s \alpha_s \leq \sqrt{4p\pi_a \alpha_s + (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2}$
(B) $\pi_i \alpha_i < 1 - p$
(C) $(1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) + (-1 + 2c_a + 2p + \pi_a \alpha_a)\pi_s \alpha_s \geq (1 - \pi_a \alpha_a)\sqrt{4p\pi_s \alpha_s + (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2}$
(D) Either $1 + \pi_i \alpha_i + \pi_s \alpha_s - 2\pi_a \alpha_a > 0$ and $p < \frac{(1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)}{\pi_s \alpha_s}$ and $2(\pi_a \alpha_a + c_a) + \sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s} \geq \pi_i \alpha_i + \pi_s \alpha_s + 1$, or

\[
-2\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + 1 < 0 \text{ or } p > \frac{(1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)}{\pi_s \alpha_s}
\]

and $\pi_a \alpha_a (\pi_i \alpha_i + \pi_s \alpha_s) + 2\pi_s \alpha_s (c_a + p) + 1 \geq \pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + (1 - \pi_a \alpha_a)\sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s}$, or

\[
p = \frac{(1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)}{\pi_s \alpha_s}
\]

(III) $0 < v_b < v_a < 1$, where $v_b$ is the most positive root of the cubic $f_1(x) = (1 - \pi_a \alpha_a)\pi_s \alpha_s x^3 + ((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_s \alpha_s - p\pi_s \alpha_s)x^2 + p(-1 + \pi_a \alpha_a) + p(-1 + \pi_i \alpha_i)x + p^2$ and

\[
v_a = \frac{c_a v_b}{v_b (1 - \pi_a \alpha_a) - p}:
\]

(A) $(\pi_a \alpha_a + c_a - 1)(\pi_a \alpha_a - \pi_i \alpha_i + c_a) > \pi_s \alpha_s (\pi_a \alpha_a + c_a + p - 1)$
(B) $\pi_i \alpha_i (c_a + p) < c_a + p(\pi_a \alpha_a)$
(C) $\pi_a \alpha_a \leq c_p - c_a$

(IV) $0 < v_b < v_a < v_p < 1$, where $v_b$ is the most positive root of $f_1(x)$ and $v_a = \frac{c_a v_b}{v_b (1 - \pi_a \alpha_a) - p}$ and $v_p = \frac{c_p - c_a}{\pi_a \alpha_a}$:

(A) $c_p < c_a + \pi_a \alpha_a$
(B) $c_p (1 - \pi_a \alpha_a) > c_a$
(C) $c_p (\pi_a \alpha_a)^2 (-c_a + c_p (1 - \pi_a \alpha_a)) + \pi_a \alpha_s (-c_a + c_p)^2 (c_a - c_p (1 - \pi_a \alpha_a) + \pi_a \alpha_a p) - (c_a - c_p)(c_a + c_p (1 - \pi_a \alpha_a)) \pi_a \alpha_a \pi_i \alpha_i < 0$
(D) $c_a + p(\pi_a \alpha_a) > (c_a + p)\pi_i \alpha_i$

A.6
Proof of Lemma A.3: This is proven as a sub-case in the proof of Lemma A.4. □

Proof of Lemma 4: Technically, we prove the existence of \( \hat{\alpha}_1 \) such that if \( \pi_s \alpha_i < \min \left[ \frac{c_p}{1 + c_p \pi_s \alpha_a}, \frac{c_p - \pi_a \alpha_a}{1 - \pi_a \alpha_a} \right] \), then for \( \alpha_s > \hat{\alpha}_1 \), \( p^* \) is set so that

1. if \( c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1 - c_p) \sqrt{1 - \pi_a \alpha_a} \), then \( \sigma^*(v) \) is characterized by \( 0 < v_b < v_a = v_p < 1 \) under optimal pricing, and

2. if \( c_a > 1 - \pi_a \alpha_a - (1 - c_p) \sqrt{1 - \pi_a \alpha_a} \), then \( \sigma^*(v) \) is characterized by \( 0 < v_b < v_p < 1 \) under optimal pricing.

The sketch of the proof is as follows. From Lemma A.3, a unique consumer market equilibrium arises, given a price \( p \). Within each region of the parameter space defined by Lemma A.3, the thresholds \( v_a, v_b, \) and \( v_p \) are smooth functions of the parameters, including \( p \). In the cases where the thresholds are given in closed-form, this is clear. In the cases where these thresholds are implicitly defined as the root of some cubic equation, then the smoothness of the thresholds in the parameters follows from the Implicit Function Theorem. Specifically, for each of those cases, the threshold defined was the most positive root \( v_b^* \) of a cubic function of \( v_b \), \( f(v_b, p) = 0 \). Moreover,
the cubic $f(v_b, p)$ has two local extrema in $v_b$ and is negative to the left of $v_b^*$ and positive to the right of it ($f(v_b^* - \epsilon, p) < 0$ and $f(v_b^* + \epsilon, p) > 0$ for arbitrarily small $\epsilon > 0$). Therefore, $\frac{\partial f}{\partial v_b}(v_b, p) \neq 0$ so that the Implicit Function Theorem applies. The thresholds being smooth in $p$ implies that the profit function for each case of the parameter space defined by Lemma A.3 is smooth in $p$. We find the profit-maximizing price within the compact closure of each case, so that the price that induces the largest profit among the cases will be the equilibrium price set by the vendor.

Having given the sketch of the proof, we now proceed with the proof. The conditions of this lemma precludes candidate market structures from arising in equilibrium. Specifically, $c_p - \pi_a \alpha_a < c_a$ rules out Cases (I) and (III) of Lemma A.3, and $\pi_i \alpha_i < \min \left[ \frac{c_p \pi_a \alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p} \right]$ rules out Cases (VI) and (VII). We consider the remaining possible consumer equilibria that can be induced when the vendor sets prices optimally. Suppose $0 < v_b < 1$ is induced. By part (II) of Lemma A.3, we obtain $v_b = \frac{1}{2} + \frac{-1 + (\alpha + \frac{1}{\pi})}{\pi} \frac{1 + \frac{1}{\pi} - \frac{1}{\pi_\alpha}}{\pi_\alpha}$. The profit function in this case is $\Pi(p) = p(1 - v_b(p))$. Let $C_{II}$ be the compact closure of the region of the parameter space defining $0 < v_b < 1$, given in part (II) of Lemma A.3. By the Weierstrass extreme value theorem, there exists a $p$ in $C_{II}$ that maximizes $\Pi(p)$. This $p$ may be on the boundary, and we show that the vendor’s profit function is continuous across region boundaries later. Otherwise, if this $p$ is interior, the unconstrained maximizer satisfies the first-order condition $\Pi'(p) = 0$.

Using the first-order condition and letting

$$Q_1 \triangleq \sqrt{\left(-\pi_i \alpha_i + \pi_a \alpha_a + 1\right)^2 \left(\pi_a \alpha_a (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_a \alpha_a)^2\right)},$$

the roots of $\Pi'(p) = 0$ are $-\frac{(\pi_i \alpha_i)^2 + 2\pi_i \alpha_i (1 - 2\pi_a \alpha_a) + \pi_a \alpha_a (4 - \pi_a \alpha_a) - 1}{9\pi_a \alpha_a} \pm \frac{Q_1}{9\pi_a \alpha_a}.$

However, $-\frac{(\pi_i \alpha_i)^2 + 2\pi_i \alpha_i (1 - 2\pi_a \alpha_a) + \pi_a \alpha_a (4 - \pi_a \alpha_a) - 1 - Q_1}{9\pi_a \alpha_a} < 0$ for $\pi_a \alpha_a > 1 - \pi_i \alpha_i$, so for $\pi_a \alpha_a > 1 - \pi_i \alpha_i$, the unconstrained maximizer is given by

$$p_{II} = -\frac{(\pi_i \alpha_i)^2 + 2\pi_i \alpha_i (1 - 2\pi_a \alpha_a) + \pi_a \alpha_a (4 - \pi_a \alpha_a) - 1 + Q_1}{9\pi_a \alpha_a}. \quad (A.7)$$

The second-order condition is satisfied if $Q_1 + 2 \left(\pi_a \alpha_a (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_a \alpha_a)^2\right) > 0$, which holds when $\pi_a \alpha_a > 1 - \pi_i \alpha_i$. Substituting (A.7) into the profit function, we obtain

$$\Pi_{II} = \frac{1}{54 (\pi_a \alpha_a)^2} \left( (\pi_i \alpha_i)^2 - Q_1 + 2\pi_i \alpha_i (2\pi_a \alpha_a - 1) + \pi_a \alpha_a (\pi_a \alpha_a - 4) + 1 \right) \left( \sqrt{5 (\pi_i \alpha_i)^2 + 4Q_1 + 2\pi_i \alpha_i (\pi_a \alpha_a - 5) + \pi_a \alpha_a (5\pi_a \alpha_a - 2) + 5 + 3\pi_i \alpha_i - 3\pi_a \alpha_a - 3} \right). \quad (A.8)$$

On the other hand, suppose $0 < v_b < v_a < v_p < 1$ is induced. By part (IV) of Lemma A.3, we obtain that $v_b$ is the most positive root of the cubic

$$f_1(x) \triangleq (1 - \pi_a \alpha_a)\pi_a \alpha_a x^3 + \left((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_a \alpha_a - p\pi_a \alpha_a\right)x^2 + \left(p(-1 + \pi_a \alpha_a) + p(-1 + \pi_i \alpha_i)\right)x + p^2. \quad (A.9)$$
The profit function is $\Pi_{IV}(p) = p(1 - v_b(p))$. Let $C_{IV}$ be the compact closure of the region of the parameter space defining $0 < v_b < v_a < v_p < 1$, given in part (IV) of Lemma A.3. Again, by the Weierstrass extreme value theorem, there exists a $p$ in $C_{IV}$ that maximizes $\Pi(p)$. This $p$ may be on the boundary, and we show that the vendor’s profit function is continuous across region boundaries later. Otherwise, if this $p$ is interior, the unconstrained maximizer satisfies the first-order condition $\Pi'(p) = 0$. The first-order condition is given by

$$
\Pi'_{IV}(p) = (1 - v_b(p)) -pv'_b(p) = 0. \tag{A.10}
$$

By equating (A.9) to 0 and implicitly differentiating, we have that

$$
v'_b(p) = \frac{v_b(p)(\pi_a \alpha_i + \pi_i \alpha_i - \pi_s \alpha_s v_b(p) - 2) + 2p}{v_b(2(\pi_a \alpha_a - 1)(\pi_i \alpha_i - 1) + 2\pi_s \alpha_s(c_a + p) + 3\pi_s \alpha_s(\pi_a \alpha_a - 1)v_b(p) - p(\pi_a \alpha_a + \pi_i \alpha_i - 2)}.
$$

Substituting this into (A.10) and re-writing (A.9), we have that $v_b(p^*)$ and $p^*$ simultaneously need to solve

$$
(1 - \pi_a \alpha_a)\pi_s \alpha_s v_b^3 + ((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_s \alpha_s - p\pi_s \alpha_s)v_b^2 + (p(-1 + \pi_a \alpha_a) + p(-1 + \pi_i \alpha_i))v_b + p^2 = 0, \text{ and } \tag{A.12}
$$

$$
1-v_b - \frac{p(2p + v_b(\pi_a \alpha_a + \pi_i \alpha_i - \pi_s \alpha_s v_b - 2))}{v_b(2(\pi_a \alpha_a - 1)(\pi_i \alpha_i - 1) + 2\pi_s \alpha_s(c_a + p) + 3\pi_s \alpha_s v_b(\pi_a \alpha_a - 1)) - p(\pi_a \alpha_a + \pi_i \alpha_i - 2)} = 0. \tag{A.13}
$$

Letting

$$
Q_2 \triangleq \sqrt{(\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s(v_b - 2)v_b - 2)^2 + 8(v_b - 1)v_b(\pi_a \alpha_a(2c_a + 3v_b(\pi_a \alpha_a - 1)) - 2(\pi_a \alpha_a - 1)(\pi_i \alpha_i - 1))}
$$

and solving (A.13) for $p$, we have that $p$ is either $\frac{1}{4} \left(-\pi_a \alpha_a - \pi_i \alpha_i - \pi_s \alpha_s v_b^2 + 2\pi_s \alpha_s v_b - 2 - Q_2\right)$ or $\frac{1}{4} \left(-\pi_a \alpha_a - \pi_i \alpha_i - \pi_s \alpha_s v_b^2 + 2\pi_s \alpha_s v_b + 2 + Q_2\right)$. We can rule out the larger root when $\pi_s \alpha_s > 2 + \pi_a \alpha_a + \pi_i \alpha_i$ since when $\pi_s \alpha_s > 2 + \pi_a \alpha_a + \pi_i \alpha_i$, then $Q_2 > -2 + 4v_b + \pi_a \alpha_a + \pi_i \alpha_i - (2 - v_b)v_b\pi_s \alpha_s$. This is equivalent to the larger root for $p^*$ being greater than $v_b$, which can’t happen in equilibrium. Therefore,

$$
p(v_b) = \frac{1}{4} \left(-\pi_a \alpha_a - \pi_i \alpha_i - \pi_s \alpha_s v_b^2 + 2\pi_s \alpha_s v_b + 2 - Q_2\right). \tag{A.14}
$$

Substituting this into (A.12), we have that $v_b(p^*)$ solves

$$
((\pi_a \alpha_a + \pi_i \alpha_i - 2)^2 + 4v_b^2(\pi_a \alpha_a(4\pi_i \alpha_i + 5\pi_s \alpha_s - 4) + 2\pi_i \alpha_i(\pi_s \alpha_s - 2) + \pi_s \alpha_s(\pi_s \alpha_s - 4c_a - 7) + 4) - 4\pi_s \alpha_s v_b(\pi_a \alpha_a + \pi_i \alpha_i - 2c_a - 2) + 3(\pi_s \alpha_s)^2v_b^4 - 2\pi_s \alpha_s v_b^3(11\pi_a \alpha_a + \pi_i \alpha_i + 4\pi_s \alpha_s - 12) - 16v_b - 2v_b \left((\pi_a \alpha_a)^2 + 2\pi_a \alpha_a(3\pi_i \alpha_i - 4) + \pi_i \alpha_i(\pi_i \alpha_i - 8)\right) + (\pi_s \alpha_s v_b(3v_b - 2) - (2v_b - 1)(\pi_a \alpha_a + \pi_i \alpha_i - 2))Q_2 = 0.
$$

A generalization of the Implicit Function Theorem gives that $v_b$ is not only a smooth function of the parameters, but it’s also an analytic function of the parameters so that it can be represented.
locally as a Taylor series of its parameters (Brillinger 1966). More specifically, since \( f_1(x) \neq 0 \) at the root for which \( v_b \) is defined, there exists an \( \alpha_1 > 0 \) such that for \( \alpha_s > \alpha_1 \), \( v_b = \sum_{k=0}^{\infty} \frac{a_k}{(\pi_s \alpha_s)^k} \) for some coefficients \( a_k \).

Substituting this into (A.15), we have \( \pi_s \alpha_s \left( a_0^2(2a_0-1)\pi_s \alpha_s - 2a_0 + c_a + 1 \right) + \sum_{k=0}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} = 0 \). Then for \( a_0 = 0 \) or \( a_0 = \frac{1+\pi_a-\pi_s \alpha_s}{2(1-\pi_s \alpha_s)} \), the only solutions for \( a_0 \) that make the first term \( 0 \). However, if \( a_0 = 0 \), then \( \pi_s \alpha_s \left( a_0^2(2a_0-1)\pi_s \alpha_s - 2a_0 + c_a + 1 \right) + \sum_{k=0}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} \) becomes \( c_a^2 + \sum_{k=1}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} > 0 \) for large enough \( \pi_s \alpha_s \) so that there exists no coefficients \( A_k \) such that \( c_a^2 + \sum_{k=1}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} = 0 \) for \( \alpha_s > \alpha_1 \), a contradiction.

Thus \( a_0 = \frac{1+\pi_a-\pi_s \alpha_s}{2(1-\pi_s \alpha_s)} \). Substituting \( v_b = \frac{1+\pi_a-\pi_s \alpha_s}{2(1-\pi_s \alpha_s)} + \sum_{k=1}^{\infty} a_k \xi^k \) into (A.15), we have that \( \frac{a_1(-\pi_s \alpha_s + c_a + 1)^3 + 2c_a(\pi_s \alpha_s - 1)((\pi_s \alpha_s - 1)(\pi_s \alpha_s - \pi_i \alpha_i) + c_a^2 + c_a(2\pi_s \alpha_s + \pi_i \alpha_i - 3))}{2(\pi_s \alpha_s - 1)(3\pi_s \alpha_s - c_a)} + \sum_{k=1}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} = 0 \). Solving for \( a_1 \) to make this first term zero, we have \( a_1 = \frac{2c_a(1-\pi_a \alpha_s)((1-\pi_a \alpha_s)(\pi_s \alpha_s + \pi_i \alpha_i) + c_a^2 + c_a(2\pi_s \alpha_s + \pi_i \alpha_i - 3))}{(-\pi_s \alpha_s + c_a + 1)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} \frac{a_k}{(\pi_s \alpha_s)^k} \).

Using \( v_b = \frac{1+\pi_a-\pi_s \alpha_s}{2(1-\pi_s \alpha_s)} + \frac{2c_a(1-\pi_a \alpha_s)((1-\pi_a \alpha_s)(\pi_s \alpha_s + \pi_i \alpha_i) + c_a^2 + c_a(2\pi_s \alpha_s + \pi_i \alpha_i - 3))}{(-\pi_s \alpha_s + c_a + 1)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} \frac{a_k}{(\pi_s \alpha_s)^k} \), we can solve for \( a_2, a_3, \) and so on recursively by repeatedly substituting this expression for \( v_b \) into (A.15) and solving for the coefficients to make the expression zero. Doing this, we find that the threshold \( v_b \) is

\[
v_b = \frac{1+\pi_a-\pi_s \alpha_s}{2(1-\pi_s \alpha_s)} + \frac{2c_a(1-\pi_s \alpha_s)((1-\pi_s \alpha_s)(\pi_s \alpha_s + \pi_i \alpha_i) + c_a^2 + c_a(2\pi_s \alpha_s + \pi_i \alpha_i - 3))}{(-\pi_s \alpha_s + c_a + 1)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} \frac{a_k}{(\pi_s \alpha_s)^k},
\]

(A.16)

and substituting this into (A.14), we have that the optimal price set by the vendor is

\[
p_{IV}^* = \frac{1}{2}(1-\pi_s \alpha_a - c_a) + \frac{2c_a^2(\pi_s \alpha_a - 1)((\pi_s \alpha_a - 1)(2\pi_s \alpha_a - \pi_i \alpha_i - 1) + c_a(2\pi_s \alpha_a + \pi_i \alpha_i - 3))}{(-\pi_s \alpha_a + c_a + 1)^2 \pi_s \alpha_s} + \sum_{k=2}^{\infty} b_k \left( \frac{1}{\pi_s \alpha_s} \right)^k.
\]

(A.17)

The corresponding profit is given as

\[
\Pi_{IV}^* = \frac{(\pi_s \alpha_a + c_a - 1)^2}{4(1-\pi_s \alpha_a)} + c_a(\pi_s \alpha_a + c_a - 1)((\pi_s \alpha_a - 1)(\pi_s \alpha_a - \pi_i \alpha_i) + c_a(\pi_s \alpha_a + \pi_i \alpha_i - 2))}{(-\pi_s \alpha_a + c_a + 1)^2 \pi_s \alpha_s} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k.
\]

(A.18)

As a matter of notation, we will use \( a_k, b_k, \) and \( c_k \) to denote coefficients in the Taylor expansions without referring to specific expressions throughout the appendix. These will be used across different cases, and they don’t refer to the same quantities or expressions across cases.

Lastly, suppose \( 0 < v_b < v_p < 1 \) is induced. The profit function in this case is \( \Pi_V(p) = p(1-v_b(p)) \), where \( v_b \) is the most positive root of \( f_2(x) \triangleq \pi_s \alpha_s x^3 + (1-\pi_i \alpha_i - (c_p + p)\pi_s \alpha_s) x^2 - p(2-\pi_i \alpha_i) x + p^2 \) by part (V) of Lemma A.3. Omitting the algebra (similar to the previous case), there exists an \( \alpha_2 > 0 \) such that for \( \alpha_s > \alpha_2 \), the unconstrained maximizer is given as

\[
p_V^* = \frac{1-c_p}{2} - \frac{2c_p^2(1-3c_p + \pi_i \alpha_i(1+c_p))}{(1+c_p)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} b_k \left( \frac{1}{\pi_s \alpha_s} \right)^k,
\]

(A.19)
and the profit induced is given as

$$\Pi_V = \frac{1}{4}(1-c_p)^2 + \frac{(1-c_p)cp}{(1+c_p)^2\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s\alpha_s} \right)^k. \quad \text{(A.20)}$$

To compare profits, we note that there exists $\alpha_3 > 0$ such that for $\alpha_s > \alpha_3$, (A.8) can be expressed as the Taylor series

$$\Pi_{II} = \frac{(1-\pi_i\alpha_i)^2}{4\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s\alpha_s} \right)^k. \quad \text{(A.21)}$$

Then comparing (A.21) with either (A.18) or (A.20), we find that (A.8) is dominated by the other two profits when $\alpha_s$ exceeds an implicit bound (say, $\alpha_s > \alpha_1$, for some $\alpha_1 > 0$).

Next, using (A.17) with Lemma A.3, we find the conditions under which the interior optimal price of $0 < v_b < v_a < v_p < 1$ would indeed induce this market structure. First, we still need the conditions $c_p < c_a + \pi_s\alpha_s$ and $c_p(1-\pi_s\alpha_s) > c_a$. Next, for $c_p(\pi_s\alpha_s)^2(-c_a + cp(1-\pi_s\alpha_s)) + \pi_s\alpha_s(-c_a + cp)^2(c_a - cp(1-\pi_s\alpha_s) + \pi_s\alpha_s) < 0$ to hold for $p = p_{IV}^*$, we need $-\frac{1}{2}(cp - c_a)^2(-2c_a + 2cp + (-1 + c_a - 2cp)\pi_s\alpha_s + (\pi_s\alpha_s)^2)\pi_s\alpha_s + \sum B_k \left( \frac{1}{\pi_s\alpha_s} \right)^k < 0$ for some coefficients $B_k$. There exists $\alpha_4 > 0$ such that if $\alpha_s > \alpha_4$, then $c_a < \frac{(2cp - \pi_s\alpha_s)(1-\pi_s\alpha_s)}{2-\pi_s\alpha_s}$ is sufficient for this condition to hold. Note that $\frac{(2cp - \pi_s\alpha_s)(1-\pi_s\alpha_s)}{2-\pi_s\alpha_s} < cp(1-\pi_s\alpha_s)$, so $c_a < \frac{(2cp - \pi_s\alpha_s)(1-\pi_s\alpha_s)}{2-\pi_s\alpha_s}$ is the tighter bound on $c_a$. Lastly, for $c_a + p(\pi_s\alpha_s) > (c_a + p)\pi_s\alpha_s$ to hold for $p = p_{IV}^*$, we need $c_a - \frac{1}{2}\pi_s\alpha_s(-1 + c_a + \pi_s\alpha_s) - \frac{1}{2}\pi_s\alpha_s(1 + c_a - \pi_s\alpha_s) + \sum B_k \left( \frac{1}{\pi_s\alpha_s} \right)^k > 0$ for some coefficients $B_k > 0$. It suffices to have $c_a > \frac{(1-\pi_s\alpha_s)(\pi_s\alpha_s - \pi_s\alpha_i)}{2\pi_s\alpha_s + (\pi_s\alpha_s)^2}$. Since $\pi_s\alpha_i < \pi_s\alpha_a$ follows from one of the assumptions of this lemma ($\pi_i\alpha_i < \frac{cp\pi_s\alpha_s}{c_p - c_a}$), this condition is automatically satisfied since $c_a > 0$.

In summary, the optimal price $0 < v_b < v_a < v_p < 1$ indeed induces this market structure when $c_p - \pi_s\alpha_a < c_a < \frac{(2cp - \pi_s\alpha_s)(1-\pi_s\alpha_a)}{2-\pi_s\alpha_a}$ for $\alpha > \alpha_4$.

Similarly, using (A.19) with Lemma A.3, there exists $\alpha_5 > 0$ such that when $\alpha_s > \alpha_5$, the optimal price of $0 < v_b < v_p < 1$ indeed induces the correct market structure when $c_a > cp - \frac{1}{2}(1 + cp)\pi_s\alpha_a$ and $\pi_i\alpha_i < \frac{c_p}{1+cp}$.

Let $\hat{\alpha}_1$ be the max of $\hat{\alpha}_1$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, and $\alpha_5$. Then for $\alpha_s > \hat{\alpha}_1$, if $c_a > cp - \pi_s\alpha_a$, we have that (A.8) is dominated. Moreover, since $c_p - \frac{1}{2}(1 + cp)\pi_s\alpha_a < \frac{(2cp - \pi_s\alpha_s)(1-\pi_s\alpha_a)}{2-\pi_s\alpha_a}$, there will be a region in which the interior maximizers of both $0 < v_b < v_a < v_p < 1$ and $0 < v_b < v_p < 1$ induce their corresponding cases.

Comparing (A.18) with (A.20), we see that (A.18) is greater when $2cp(1-\pi_s\alpha_a) < \frac{c_p^2}{c_a} + (1 - \pi_s\alpha_a)(cp(2-c_p) - \pi_s\alpha_a)$, which can be written as $c_a < 1 - \pi_s\alpha_a - (1-c_p)\sqrt{1 - \pi_s\alpha_a}$. Note that for any $c_p \in [0, 1]$, we have that $c_p - \frac{1}{2}(1 + cp)\pi_s\alpha_a < 1 - \pi_s\alpha_a - (1-c_p)\sqrt{1 - \pi_s\alpha_a} < \frac{(2cp - \pi_s\alpha_s)(1-\pi_s\alpha_a)}{2-\pi_s\alpha_a}$ from $\pi_s\alpha_a \in (0, 1)$. Therefore, for $\alpha_s > \hat{\alpha}_1$, if $cp - \pi_s\alpha_a < c_a \leq 1 - \pi_s\alpha_a - (1-c_p)\sqrt{1 - \pi_s\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing, and if $c_a > 1 - \pi_s\alpha_a - (1-c_p)\sqrt{1 - \pi_s\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing. □
2 Pricing Patching Rights

Lemma A.4 Under PPR, the complete threshold characterization of the consumer market equilibrium is as follows:

(I) \(0 < v_a < 1\), where \(v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\):
   
   \(A\) \(\delta p + c_a + \pi_a \alpha_a < 1\)
   
   \(B\) \((1 - \delta)p + c_p \geq c_a + \pi_a \alpha_a\)
   
   \(C\) \((c_a + \delta p)\pi_i \alpha_i \geq c_a + p(1 + \delta + \pi_a \alpha_a)\)
   
   \(D\) \(c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a\)
   
   \(E\) Either \(\pi_i \alpha_i < \pi_a \alpha_a\) and \(\pi_a \alpha_a - \pi_i \alpha_i + c_a \leq (1 - \delta)p\), or
   
   \(\pi_i \alpha_i > \pi_a \alpha_a\) and \(c_a + p(1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i\), or
   
   \(\pi_i \alpha_i = \pi_a \alpha_a\) and \((1 - \delta)p - c_a \geq 0\)

(II) \(0 < v_a < v_b < 1\), where \(v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\) and \(v_b = \frac{-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s (c_a + (\delta - 1)p)}}{2\pi_s \alpha_s}\):

   \(A\) \(\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s > 0\)
   
   \(B\) \((\pi_a \alpha_a c_a + \pi_s \alpha_s x c_p) < -\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s (c_a + (\delta - 1)p)}\)
   
   \(C\) \(1 - \delta)p < \pi_a \alpha_a - \pi_i \alpha_i + c_a\)

(III) \(0 < v_a < v_b < v_p < 1\), where \(v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\), \(v_b\) is the most positive root of the cubic

\(f_3(x) = \pi_s \alpha_s x^2 (c_a - c_p + (\delta - 1)p + \pi_a \alpha_a x) + (c_a + (\delta - 1)p + \pi_a \alpha_a x)(c_a + (\delta - 1)p + x(\pi_a \alpha_a - \pi_i \alpha_i))\),

and \(v_p = \frac{c_{a,1} v_b}{c_{a,2} + c_{a,3} v_b}\):

   \(A\) \((1 - \delta)p - c_a \geq 0\)

   \(B\) \(\pi_a \alpha_a > c_p\)

   \(C\) \(\pi_s \alpha_s (\pi_a \alpha_a - c_a + c_p - \delta p + p) < (c_p - \pi_a \alpha_a) (c_p - \pi_i \alpha_i)\)

   \(D\) \(\pi_i \alpha_i (c_a + \delta p) + \pi_a \alpha_a c_p > \pi_i \alpha_i (c_p + p)\)

   \(E\) Either \(c_a + p(\pi_a \alpha_a + \delta) \leq p\) and \(\pi_s \alpha_s (c_a + \delta p)^2 (c_a + \pi_a \alpha_a (c_p + p) - c_p + (\delta - 1)p) + (\pi_a \alpha_a - 1) (c_a + p(\pi_a \alpha_a + \delta - 1)) (\pi_i \alpha_i (c_a + \delta p) - c_a) - p(\pi_a \alpha_a + \delta - 1) > 0\)

(IV) \(0 < v_b < 1\), where \(v_b = \frac{1}{2} + \frac{-1 + \pi_i \alpha_i + \sqrt{(1 - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4\pi_s \alpha_s \pi_i \alpha_i}}{2\pi_s \alpha_s}\):

   \(A\) \(1 - 2c_p + \pi_i \alpha_i + \pi_s \alpha_s \leq \sqrt{4p\pi_s \alpha_s + (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2}\)

   \(B\) \(\pi_i \alpha_i < 1 - p\)

   \(C\) \((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) + (-1 + 2c_a + 2\delta p + \pi_a \alpha_a)\pi_s \alpha_s \geq (1 - \pi_a \alpha_a) (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2\)
(D) Either \( 1 + \pi_i \alpha_i + \pi_s \alpha_s - 2 \pi_a \alpha_a > 0 \) and \( p < \frac{1-\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s}{\pi_s \alpha_s} \) and \( 2(\pi_a \alpha_a + c_a - (1 - \delta)p) + \sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s} \geq \pi_i \alpha_i + \pi_s \alpha_s + 1 \), or
\[
\left( -2\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + 1 < 0 \right) \text{ or } p > \frac{1-\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s}{\pi_s \alpha_s}
\] and \( \pi_a \alpha_a(\pi_i \alpha_i + \pi_s \alpha_s) + 2\pi_s \alpha_s (c_a + \delta p) + 1 \geq \pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + (1 - \pi_a \alpha_a)\sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s} \), or
\[
p = \frac{1-\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s}{\pi_s \alpha_s} \text{ and } (1 - \delta)p - c_a \leq 0
\]

(V) \((0 < v_b < v_a < 1)\), where \(v_b\) is the most positive root of the cubic \(f_4(x) \triangleq (1 - \pi_a \alpha_a)\pi_s \alpha_s x^3 + ((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_s \alpha_s - \delta p \pi_s \alpha_s)^2 + (p(-1 + \pi_a \alpha_a) + p(-1 + \pi_i \alpha_i))x + p^2\) and \(v_a = \frac{(c_a - (1 - \delta)p)v_b}{v_b(1 - \pi_a \alpha_a) - p}\);

(A) \((1 - \delta)p - c_a < 0\)

(B) \((\pi_a \alpha_a + c_a - 1 - (1 - \delta)p)(\pi_a \alpha_a - \pi_i \alpha_i + c_a - (1 - \delta)p) > \pi_s \alpha_s (\pi_a \alpha_a + c_a + \delta p - 1)\)

(C) \(\pi_a \alpha_i (c_a + \delta p) < c_a + p(-1 + \delta + \pi_a \alpha_a)\)

(D) \(\pi_a \alpha_a \leq (1 - \delta)p + c_p - c_a\)

(VI) \((0 < v_b < v_a < v_p < 1)\), where \(v_b\) is the most positive root of \(f_4(x)\) and \(v_a = \frac{(c_a - (1 - \delta)p)v_b}{v_b(1 - \pi_a \alpha_a) - p}\) and \(v_p = \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}\);

(A) \((1 - \delta)p - c_a < 0\)

(B) \(c_p + (1 - \delta)p < c_a + \pi_a \alpha_a\)

(C) \(c_p(1 - \pi_a \alpha_a) + (1 - \delta)p > c_a\)

(D) \(c_p(\pi_a \alpha_a)^2(-c_a + (1 - \delta)p + c_p(-1 + \pi_a \alpha_a)) + \pi_s \alpha_s(-c_a + (1 - \delta)p + c_p^2(c_a - c_p(1 - \pi_a \alpha_a) + (\pi_a \alpha_a - 1 + \delta)p) - (c_a - c_p - (1 - \delta)p)(c_a - (1 - \delta)p - c_p(1 - \pi_a \alpha_a))\pi_a \alpha_a \pi_i \alpha_i < 0\)

(E) \(c_a + p(-1 + \delta + \pi_a \alpha_a) > (c_a + \delta p)\pi_i \alpha_i\)

(VII) \((0 < v_b < v_p < 1)\), where \(v_b\) is the most positive root of \(f_5(x) \triangleq \pi_s \alpha_s x^3 + (1 - \pi_i \alpha_i - (c_p + \delta p)\pi_s \alpha_s x^2 - p(2 - \pi_i \alpha_i)x + p^2\) and \(v_p = \frac{c_p v_b}{v_b - p}\);

(A) \((-1 + c_p + p)\pi_s \alpha_s < (1 - c_p)(-c_p + \pi_i \alpha_i)\)

(B) \(\pi_i \alpha_i < \frac{c_p}{c_p + p}\)

(C) \((1 - \pi_a \alpha_a)(c_a + (\pi_a \alpha_a - (1 - \delta)p))(c_a + p(\pi_a \alpha_a - (1 - \delta)p) - (c_a + \delta p)\pi_i \alpha_i) + (c_a + \delta p)^2(c_a - c_p - (1 - \delta)p + (c_p + \pi_a \alpha_a)\pi_s \alpha_s \geq 0\)

(D) \(c_a + p(\delta + \pi_a \alpha_a) > p\)

(E) Either \(c_a - (1 - \delta)p + c_p(-1 + \pi_a \alpha_a) \geq 0\), or \(c_a + c_p(-1 + \pi_a \alpha_a) < 0 \) and \(\pi_a \alpha_a(c_a - (1 - \delta)p + c_p(-1 + \pi_a \alpha_a))(c_p \pi_a \alpha_a + (c_a - c_p - (1 - \delta)p)\pi_i \alpha_i) \leq (-c_a + c_p + (1 - \delta)p)^2(c_a - c_p - (1 - \delta)p + (c_p + \pi_a \alpha_a)\pi_s \alpha_s\)

(VIII) \((0 < v_a < v_p < 1)\), where \(v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\) and \(v_p = \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}\);

(A) \(c_p + (1 - \delta)p < c_a + \pi_a \alpha_a\)
Given the size of unpatched user population \( u \), first we establish the general threshold-type equilibrium structure. With valuation \( v > v^* \), \( v \in V \), consumer purchases one of the alternatives, if and only if

\[ (0 < v_p < 1), \text{ where } v_p = p + c_p. \]

\( (A) \) \( c_p + p < 1 \)
\( (B) \) \( c_a + (c_p + p)\pi_a \alpha_a \geq c_p + (1 - \delta)p \)
\( (C) \) \( (c_p + p)\pi_i \alpha_i \geq c_p \)

**Proof of Lemma A.4:** First, we establish the general threshold-type equilibrium structure. Given the size of unpatched user population \( u \), the net payoff of the consumer with type \( v \) for strategy profile \( \sigma \) is written as

\[ U(v, \sigma) \equiv \begin{cases} 
  v - p - c_p & \text{if } \sigma(v) = (B, P); \\
  v - p - \pi_s \alpha_s u v - \pi_i \alpha_i v & \text{if } \sigma(v) = (B, NP); \\
  v - \delta p - c_a - \pi_a \alpha_a v & \text{if } \sigma(v) = (B, AP); \\
  0 & \text{if } \sigma(v) = (NB, NP). 
\end{cases} \quad (A.22) \]

Note \( \sigma(v) = (B, P) \) if and only if

\[ v - p - c_p \geq v - p - \pi_s \alpha_s u v - \pi_i \alpha_i v \iff v \geq \frac{c_p}{\pi_s \alpha_s u + \pi_i \alpha_i}, \text{ and} \]
\[ v - p - c_p \geq v - \delta p - c_a - \pi_a \alpha_a v \iff v \geq \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}, \text{ and} \]
\[ v - p - c_p \geq 0 \iff v \geq c_p + p, \]

which can be summarized as

\[ v \geq \max \left( \frac{c_p}{\pi_s \alpha_s u + \pi_i \alpha_i}, \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}, c_p + p \right). \quad (A.23) \]

By (A.23), if a consumer with valuation \( v_0 \) buys and patches the software, then every consumer with valuation \( v > v_0 \) will also do so. Hence, there exists a threshold \( v_p \in (0, 1] \) such that for all \( v \in V \), \( \sigma^*(v) = (B, P) \) if and only if \( v \geq v_p \). Similarly, \( \sigma(v) \in \{(B, P), (B, NP), (B, AP)\} \), i.e., the consumer purchases one of the alternatives, if and only if

\[ v - p - c_p \geq 0 \iff v \geq c_p + p, \text{ or} \]
\[ v - p - \pi_s \alpha_s u v - \pi_i \alpha_i v \geq 0 \iff v \geq \frac{p}{1 - \pi_s \alpha_s u - \pi_i \alpha_i}, \text{ or} \]
\[ v - \delta p - c_a - \pi_a \alpha_a v \geq 0 \iff v \geq \frac{\delta p + c_a}{1 - \pi_a \alpha_a}. \]
which can be summarized as

\[ v \geq \min \left( c_p + p, \frac{p}{1 - \pi_a \alpha_s u - \pi_f \alpha_i}, \frac{\delta p + c_a}{1 - \pi_a \alpha_a} \right). \] (A.24)

Let \( 0 < v_1 \leq 1 \) and \( \sigma^*(v) \in \{(B, P), (B, NP), (B, AP)\} \), then by (A.24), for all \( v > v_1 \), \( \sigma^*(v) \in \{(B, P), (B, NP), (B, AP)\} \), and hence there exists a \( v \in (0, 1) \) such that a consumer with valuation \( v \in V \) will purchase if and only if \( v \geq v \).

By (A.23) and (A.24), \( v \leq v_p \) holds. Moreover, if \( v < v_p \), consumers with types in \([v, v_p]\) choose either \((B, NP)\) or \((B, AP)\). A purchasing consumer with valuation \( v \) will prefer \((B, NP)\) over \((B, AP)\) if and only if

\[ v - p - \pi_a \alpha_s uv - \pi_f \alpha_i v > v - \delta p - c_a - \pi_a \alpha_a v \iff v(\pi_a \alpha_a - \pi_a \alpha_s u - \pi_f \alpha_i) > (1 - \delta)p - c_a. \] (A.25)

This inequality can be either \( v > \frac{(1-\delta)p-c_a}{\pi_a \alpha_a - \pi_a \alpha_s u - \pi_f \alpha_i} \) or \( v < \frac{(1-\delta)p-c_a}{\pi_a \alpha_a - \pi_a \alpha_s u - \pi_f \alpha_i} \), depending on the sign of \( \pi_a \alpha_a - \pi_a \alpha_s u - \pi_f \alpha_i \). Consequently, there can be two cases for \((B, NP)\) and \((B, AP)\) in equilibrium: first, there exists \( v_u \in [v, v_p] \) such that \( \sigma(v) = (B, NP) \) for all \( v \in [v_u, v_p] \), and \( \sigma(v) = (B, AP) \) for all \( v \in [v_d, v_u] \) where \( v_d = v \). In the second case, there exists \( v_d \in [v, v_p] \) such that \( \sigma(v) = (B, AP) \) for all \( v \in [v_d, v_p] \), and \( \sigma(v) = (B, NP) \) for all \( v \in [v_u, v_d] \), where \( v_u = v \). If \( \pi_a \alpha_a - \pi_a \alpha_s u - \pi_f \alpha_i = 0 \), then depending on the sign of \( (1 - \delta)p - c_a \), all consumers unilaterally prefer either \((B, NP)\) or \((B, AP)\); e.g., if \( (1 - \delta)p > c_a \), all consumers prefer \((B, AP)\), and if \( (1 - \delta)p < c_a \), then all consumers prefer \((B, NP)\). Finally, if \( (1 - \delta)p = c_a \), then all consumers are indifferent between \((B, NP)\) and \((B, AP)\), in which case only the size of the consumer population \( u \) matters in equilibrium, i.e., \( \pi_a \alpha_a - \pi_a \alpha_s u - \pi_f \alpha_i = 0 \) in equilibrium. Technically, there are multiple equilibria in this case; however, utility of each consumer and the vendor’s profit are the same in all equilibria. So, without loss of generality, we focus on the threshold-type equilibrium in this case. In summary, we have established the threshold-type consumer market equilibrium structure.

Next, we characterize in more detail each outcome that can arise in equilibrium, as well as the corresponding parameter regions. For Case (I), in which all consumers who purchase choose the automated patching option, i.e., \( 0 < v_a < 1 \), based on the threshold-type equilibrium structure, we have \( u = 0 \). We prove the following claim related to the corresponding parameter region in which Case (I) arises.

**Claim 1** The equilibrium that corresponds to case (I) arises if and only if the following conditions are satisfied:

\[ \delta p + c_a + \pi_a \alpha_a < 1 \text{ and } (1 - \delta)p + c_p \geq c_a + \pi_a \alpha_a \text{ and } \]

\[ (c_a + \delta p)\pi_i \alpha_i \geq c_a + p(-1 + \delta + \pi_a \alpha_a) \text{ and } c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a \text{ and } \]

\[ \left\{ \begin{array}{l}
\pi_i \alpha_i < \pi_a \alpha_a \text{ and } \pi_a \alpha_a - \pi_i \alpha_i + c_a \leq (1 - \delta)p, \text{ or } \\
\pi_i \alpha_i > \pi_a \alpha_a \text{ and } c_a + p(-1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i, \text{ or } \\
\pi_i \alpha_i = \pi_a \alpha_a \text{ and } (1 - \delta)p - c_a \geq 0 
\end{array} \right\}. \] (A.26)

In this case, the threshold for the consumer indifferent between purchasing the automated
patching option and not purchasing at all, \( v_a \), satisfies

\[
v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}. \quad (A.27)
\]

For this to be an equilibrium, it is necessary and sufficient to have \( 0 < v_a < 1 \), type \( v = 1 \) prefers \((B, AP)\) over \((B, P)\), and all \( v \) prefer \((B, AP)\) over \((B, NP)\). Note that type \( v = 1 \) preferring \((B, AP)\) over \((B, P)\) implies that \( v < 1 \) does the same, by (A.23). Also, the type \( v = v_a \) preferring \((NB, NP)\) over both \((B, NP)\) and \((B, P)\) implies that all \( v < v_a \) do the same, by (A.24).

We always have \( v_a > 0 \) from our model assumptions, namely \( \pi_a \alpha_a < 1 \). To have \( v_a < 1 \), a necessary and sufficient condition is \( \delta p + c_a + \pi_a \alpha_a < 1 \).

For \( v = 1 \) to weakly prefer \((B, AP)\) over \((B, P)\), a necessary and sufficient condition is \( 1 - \delta p - c_a - \pi_a \alpha_a \geq 1 - p - c_p \), which reduces to \( (1 - \delta)p \geq \pi_a \alpha_a + c_a - c_p \).

The condition for all \( v \) to prefer \((B, AP)\) over \((B, NP)\) depends on the magnitude of \( \pi_i \alpha_i \). If \( \pi_i \alpha_i < \pi_a \alpha_a \), then if \( v = 1 \) prefers \((B, AP)\) over \((B, NP)\), then all \( v < 1 \) do too. Therefore, a necessary and sufficient condition is \( v - \delta p - c_a - \pi_a \alpha_a v \geq v - p - (\pi_s \alpha_s u(\sigma) + \pi_i \alpha_i) v \) for \( v = 1 \). With \( u(\sigma) = 0 \), this becomes \( c_a + p\delta + \pi_a \alpha_a \leq p + \pi_i \alpha_i \). So if \( \pi_i \alpha_i < \pi_a \alpha_a \), then we need the condition \( c_a + p\delta + \pi_a \alpha_a \leq p + \pi_i \alpha_i \).

On the other hand, if \( \pi_i \alpha_i > \pi_a \alpha_a \), then \( v = v_a \) preferring \((B, AP)\) over \((B, NP)\) implies that all \( v > v_a \) do too. Hence, a necessary and sufficient condition is for \( v - \delta p - c_a - \pi_a \alpha_a v \geq \delta p \pi_i \alpha_i v \), then if \( u(\sigma) = 0 \), this becomes \( c_a + p(-1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i \).

In the case of \( \pi_i \alpha_i = \pi_a \alpha_a \), we need \( (1 - \delta)p - c_a \geq 0 \) for everyone to weakly prefer \((B, AP)\) over \((B, NP)\).

We also need \( v = v_a \) to weakly prefer \((NB, NP)\) over both \((B, NP)\) and \((B, P)\), so that all \( v < v_a \) do too. We need \( 0 \geq v - p - \pi_a \alpha_a v \) and \( 0 \geq v - p - c_p \) for \( v = v_a \). These simplify to \( (c_a + \delta p)\pi_i \alpha_i \geq c_a + p(-1 + \delta + \pi_a \alpha_a) \) and \( c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a \). Altogether, Case (I) arises if and only if the condition in (A.26) occurs. \( \square \)

Next, for Case (II), in which there are no consumers choosing \((B, P)\) but the upper tier of consumers is unpatched while the bottom tier chooses automated patching, i.e., \( 0 < v_a < v_b < 1 \), we have \( u = 1 - v_b \). Following the same steps as before, we prove the following claim related to the corresponding conditions for which Case (II) arises.

**Claim 2** The equilibrium that corresponds to Case (II) arises if and only if the following conditions are satisfied:

\[
(1 - \delta)p - c_a > 0 \quad \text{and} \quad (1 - \delta)p < \pi_a \alpha_a - \pi_i \alpha_i + c_a \quad \text{and} \quad \pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s > 0 \quad \text{and} \quad (2\pi_s \alpha_s)(c_a + \delta p) \quad \text{over both (A.28)}
\]

\[
\frac{2\pi_s \alpha_s}{1 - \pi_a \alpha_a} < -\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s(c_a + (\delta - 1)p)} \quad \text{and} \quad \pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s \leq \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s(c_a + (\delta - 1)p) + 2c_p}. \quad (A.28)
\]

In this case, the threshold for the consumer indifferent between purchasing the automated patching option and not purchasing at all, \( v_a \), again satisfies

\[
v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}. \quad (A.29)
\]
The threshold for the consumer indifferent between being unpatched an purchasing the automated patching option, \( v_b \), satisfies

\[
v_b = \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i}.
\]  

(A.30)

Using \( u = 1 - v_b \), we find that \( v_b \) solves a quadratic equation. To find which root is the solution, we note that \( v_b \) must satisfy \( \pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i > 0 \) since higher types choose \((B, NP)\) over \((B, AP)\) by (A.25). This implies \((1 - \delta)p - c_a > 0\) in order for \( v_b > 0 \). Using this, we find that the root of the quadratic which specifies \( v_b \) is given by

\[
v_b = \frac{\pi_a \alpha_a + \pi_i \alpha_i - \pi_a \alpha_a + \sqrt{(\pi_a \alpha_a - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4\pi_s \alpha_s ((1 - \delta)p - c_a)}}{2\pi_s \alpha_s}.
\]  

(A.31)

For this to be an equilibrium, we need \( 0 < v_a < v_b < 1 \) and type \( v = 1 \) prefers \((B, NP)\) over \((B, P)\). Note that type \( v = 1 \) preferring \((B, NP)\) over \((B, P)\) implies that \( v < 1 \) does the same, by (A.23). This also implies that \( v_b \) prefers \((B, AP)\) over \((B, P)\), so that \( v < v_b \) also prefer \((B, AP)\) over \((B, P)\), again by (A.23). Moreover, type \( v = v_a \) preferring \((NB, NP)\) over both \((B, NP)\) and \((B, P)\) implies that all \( v < v_a \) do the same, by (A.24).

Again, we always have \( v_a > 0 \) from our model assumptions, namely \( \pi_a \alpha_a < 1 \). For \( v_b < 1 \), an equivalent condition is \((1 - \delta)p - c_a + \pi_a \alpha_a - \pi_i \alpha_i\) and \( \pi_i \alpha_i < \pi_s \alpha_s + \pi_a \alpha_a \).

For \( v_a < v_b \), it is equivalent for \( \sqrt{(\pi_a \alpha_a - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4\pi_s \alpha_s ((1 - \delta)p - c_a)} > (2\pi_s \alpha_s) \left( \frac{c_a + \delta p}{1 - \pi_a \alpha_a} \right) + \pi_a \alpha_a - \pi_s \alpha_s - \pi_i \alpha_i \).

For type \( v = 1 \) to weakly prefer \((B, NP)\) over \((B, P)\), we equivalently have the condition \( \pi_a \alpha_a + \pi_s \alpha_s + \pi_i \alpha_i - 2c_p \leq \sqrt{(\pi_a \alpha_a - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4\pi_s \alpha_s ((1 - \delta)p - c_a)} \). These conditions can all be found in (A.28).

Next, for case (III), in which all segments are represented and the middle tier is unpatched, i.e., \( 0 < v_a < v_b < v_p < 1 \), we have \( u = v_p - v_b \). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (III) arises.

**Claim 3** The equilibrium that corresponds to case (III) arises if and only if the following conditions are satisfied:

\[
(1 - \delta)p - c_a \geq 0 \text{ and } \pi_a \alpha_a > c_p \text{ and } \pi_s \alpha_s (-\pi_a \alpha_a - c_a + c_p - \delta p + p) < (c_p - \pi_a \alpha_a)(c_p - \pi_i \alpha_i) \text{ and } \pi_i \alpha_i (c_a + \delta p) + \pi_a \alpha_a c_p > \pi_i \alpha_i (c_p + p) \text{ and } \\
\left\{ \begin{array}{l} (c_a + p(\pi_a \alpha_a + \delta) \leq p) \text{ or } (c_a + p(\pi_a \alpha_a + \delta) > p \text{ and } \\
\pi_s \alpha_s (c_a + \delta p)^2 (c_a + \pi_a \alpha_a (c_p + p) - c_p + (\delta - 1)p) + \right. \\
\left. (\pi_a \alpha_a - 1)(c_a + p(\pi_a \alpha_a + \delta - 1))(\pi_i \alpha_i (c_a + \delta p) - c_a - p(\pi_a \alpha_a + \delta - 1)) > 0 \right\}. \]  

(A.32)

In this case, the threshold for the consumer indifferent between purchasing the automated
patching option and not purchasing at all, \( v_a \), again satisfies

\[
v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}.
\]

(A.33)

To solve for the thresholds \( v_b \) and \( v_p \), using \( u = v_p - v_b \), note that they solve

\[
v_b = \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s (v_p - v_b)} - \pi_t \alpha_i,
\]

and

\[
v_p = \frac{cp}{\pi_s \alpha_s (v_p - v_b) + \pi_t \alpha_i}.
\]

(A.34)

(A.35)

From (A.34), we have \( \pi_s \alpha_s (v_p - v_b) + \pi_t \alpha_i = \pi_a \alpha_a - \frac{(1 - \delta)p - c_a}{v_p} \), while from (A.35), we have \( \pi_s \alpha_s (v_p - v_b) + \pi_t \alpha_i = \frac{cp}{v_p} \). Equating these two expressions and solving for \( v_p \) in terms of \( v_b \), we have

\[
v_p = \frac{cp v_b}{c_a - (1 - \delta)p + v_b \pi_a \alpha_a}.
\]

(A.36)

Plugging this expression for \( v_p \) into (A.35) and noting that \( c_a - (1 - \delta)p + v_b \pi_a \alpha_a > 0 \) in order for \( v_p > 0 \), we find that \( v_b \) must be a zero of the cubic equation:

\[
f_1(x) \triangleq (c_a - (1 - \delta)p + x \pi_a \alpha_a)(c_a - (1 - \delta)p + x(\pi_a \alpha_a - \pi_t \alpha_i)) + x^2 \pi_s \alpha_s (c_a - cp - (1 - \delta)p + x \pi_a \alpha_a).
\]

(A.37)

To find which root of the cubic \( v_b \) must be, first note that \( \pi_a \alpha_a - \pi_s \alpha_s u - \pi_t \alpha_i > 0 \) for consumers of higher valuation to prefer \( (B, NP) \) over \( (B, AP) \) by (A.25). From that, we have that \( (1 - \delta)p - c_a > 0 \) in order for \( v_b > 0 \). To pin down the root of the cubic, note that the cubic’s highest order term is \( \pi_s \alpha_s \pi_a \alpha_a x^3 \), so \( \lim_{x \to -\infty} f_1(x) = -\infty \) and \( \lim_{x \to \infty} f_1(x) = \infty \). We find

\[
f_1(0) = ((1 - \delta)p - c_a)^2 > 0 \text{ and } f_1(1) = \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a} = -\frac{c_a \pi_s \alpha_s ((1 - \delta)p - c_a)^2}{(\pi_a \alpha_a)^2} < 0, \text{ while } 0 < \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a}.
\]

We note that from (A.34), we have that \( v_b > \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a} \), so it follows that \( v_b \) is the largest (i.e., most positive) root of the cubic. Then using (A.36), we solve for \( v_p \).

For this to be an equilibrium, a necessary and sufficient condition is \( 0 < v_a < v_b < v_p < 1 \). This tells us that all \( v \in [v_p, 1] \) have the same preferences and will purchase \( (B, P) \), all \( v \in [v_b, v_p) \) have the same preferences and will purchase \( (B, NP) \), and all \( v \in [v_a, v_b) \) have the same preferences and will purchase \( (B, AP) \). Finally, all \( v < v_a \) have the same preferences and will not purchase in equilibrium.

For \( v_p < 1 \), using (A.36), a necessary and sufficient condition for this to hold is \( (1 - \delta)p - c_a < v_b(\pi_a \alpha_a - cp) \). Since \( (1 - \delta)p - c_a > 0 \), again, from \( v_b > 0 \), we need \( \pi_a \alpha_a > cp \). To have \( v_b > \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a - cp} \), a necessary and sufficient condition is that \( f_1 \left( \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a - cp} \right) < 0 \) so that the third root of \( f_1(x) \) is greater than \( \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a - cp} \). Omitting the algebra, this simplifies to \( \pi_s \alpha_s (-\pi_a \alpha_a - c_a + cp - \delta p + p) < \frac{c_p}{(c_p - \pi_a \alpha_a)}(cp - \pi_t \alpha_i) \).

For \( v_b < v_p \), using (A.36), it is equivalent to have \( v_b < \frac{(1 - \delta)p - c_a + cp}{\pi_a \alpha_a} \). A necessary and sufficient condition for this is that \( f_1 \left( \frac{(1 - \delta)p - c_a + cp}{\pi_a \alpha_a} \right) > 0 \) so that the third root of \( f_1(x) \) is smaller than \( \frac{(1 - \delta)p - c_a + cp}{\pi_a \alpha_a} \). This condition becomes \( \pi_t \alpha_i (c_a + \delta p) + \pi_a \alpha_a cp > \pi_t \alpha_i (cp + p) \).

For \( v_a < v_b \), using (A.27), an equivalent condition is \( v_b > \frac{\delta p + c_a}{1 - \pi_a \alpha_a} \). Since \( v_b > \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a} \) (by the
construction of $v_b$ above as the largest root of the cubic), it follows that if $\frac{(1-\delta)p-c_a}{\pi_s\alpha_s} \geq \frac{\delta p + c_a}{1 - \pi_s\alpha_s}$, then we don’t need any extra conditions. The condition $\frac{(1-\delta)p-c_a}{\pi_s\alpha_s} \geq \frac{\delta p + c_a}{1 - \pi_s\alpha_s}$ simplifies to $(1-\delta)\pi_s\alpha_a)p \geq c_a$. Otherwise, if $(1-\delta - \pi_s\alpha_a)p < c_a$, then we need $f_1(\frac{\delta p + c_a}{1 - \pi_s\alpha_s}) < 0$ for $v_a < v_b$. This condition is $\pi_s\alpha_s(c_a + \delta p)^2(c_a + \pi_s\alpha_a(c_p + p) - c_p + (\delta - 1)p) + (\pi_s\alpha_a - 1)(c_a + p(\pi_s\alpha_a + \delta - 1))(\pi_s\alpha_i(c_a + \delta p) - c_a - p(\pi_s\alpha_a + \delta - 1)) > 0$, which is given in (A.32). □

Next, for case (IV), in which all consumers who purchase are unpatched, i.e., $0 < v_b < 1$, we have $u = 1 - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter conditions for which case (IV) arises.

**Claim 4** The equilibrium that corresponds to case (IV) arises if and only if the following conditions are satisfied:

$1 - 2c_p + \pi_i\alpha_i + \pi_s\alpha_s \leq \sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2}$ and $\pi_i\alpha_i < 1 - p$ and

$$(1-\pi_s\alpha_a)(1-\pi_i\alpha_i)(1-2c_a + 2\delta p + \pi_s\alpha_a)\pi_s\alpha_s \geq (1 - \pi_s\alpha_a)\sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2}$$ and

$$\left\{ \begin{array}{l}
\left(1 + \pi_i\alpha_i + \pi_s\alpha_s - 2\pi_o\alpha_o > 0 \text{ and } p < \frac{(1 - \pi_s\alpha_a)(-\pi_s\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)}{\pi_s\alpha_s} \right. \\
2(\pi_s\alpha_a + c_a - (1 - \delta)p) + \sqrt{(\pi_i\alpha_i + \pi_s\alpha_s - 1)^2 + 4p\pi_s\alpha_s} \geq \pi_i\alpha_i + \pi_s\alpha_s + 1 \right. \\
\left. \left( -2\pi_o\alpha_o + \pi_i\alpha_i + \pi_s\alpha_s + 1 < 0 \text{ or } p > \frac{(1 - \pi_s\alpha_a)(-\pi_s\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)}{\pi_s\alpha_s} \right) \right. \\
\pi_s\alpha_s(\pi_i\alpha_i + \pi_s\alpha_s) + 2\pi_s\alpha_s(c_a + \delta p) + 1 \geq \pi_s\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s + (1 - \pi_s\alpha_a)\sqrt{(\pi_i\alpha_i + \pi_s\alpha_s - 1)^2 + 4p\pi_s\alpha_s} \right. \}
$$

To solve for the threshold $v_b$, using $u = 1 - v_b$, we solve

$$v_b = \frac{p}{1 - \pi_s\alpha_s(1 - v_b) - \pi_i\alpha_i}. \quad (A.39)$$

For this to be an equilibrium, we have that $1 - \pi_s\alpha_s u - \pi_i\alpha_i > 0$, otherwise all consumers would prefer (NB, NP) over (B, NP), which can’t happen in equilibrium. Using $1 - \pi_s\alpha_s(1 - v_b) - \pi_i\alpha_i > 0$, we find the right root of the quadratic for $v_b$ to be

$$v_b = \frac{1}{2} + \frac{-1 + \pi_i\alpha_i + \sqrt{(1 - \pi_s\alpha_s - \pi_i\alpha_i)^2 + 4p\pi_s\alpha_s}}{2\pi_s\alpha_s}. \quad (A.40)$$

For this to be an equilibrium, the necessary and sufficient conditions are that $0 < v_b < 1$, type $v = 1$ weakly prefers (B, NP) to both (B, AP) over (B, P), and $v = v_b$ weakly prefers (NB, NP) over (B, AP).

For $0 < v_b < 1$, it is equivalent to have $\pi_i\alpha_i < 1 - p$.

For $v = 1$ to prefer (B, NP) over (B, P), we need $1 - 2c_p + \pi_i\alpha_i + \pi_s\alpha_s \leq \sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2}$.

For $v = v_b$ to weakly prefer (NB, NP) over (B, AP), we need $0 \geq v_b - \delta p - c_a - \pi_s\alpha_a v_b$. This simplifies to $(1 - \pi_s\alpha_a)(1 - \pi_i\alpha_i) + (1 - 2c_a + 2\delta p + \pi_s\alpha_a)\pi_s\alpha_s \geq \sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2}$.

A.19
For everyone to prefer \((B, NP)\) over \((B, AP)\), the condition needed depends on whether \(u(\sigma) > \frac{\pi_a \alpha_a - \pi_i \alpha_i}{\pi_s \alpha_s}\), as seen in (A.25). If \(u(\sigma) > \frac{\pi_a \alpha_a - \pi_i \alpha_i}{\pi_s \alpha_s}\), then lower valuation consumers would prefer \((B, NP)\) over \((B, AP)\) so that a sufficient condition for everyone to prefer \((B, NP)\) over \((B, AP)\) is that \(v = 1\) weakly prefers \((B, NP)\) over \((B, AP)\). On the other hand, if \(u(\sigma) < \frac{\pi_a \alpha_a - \pi_i \alpha_i}{\pi_s \alpha_s}\), then higher valuation consumers prefer \((B, NP)\) over \((B, AP)\) so that the condition would be \(v = v_b\) weakly prefers \((B, NP)\) over \((B, AP)\).

The condition \(u(\sigma) > \frac{\pi_a \alpha_a - \pi_i \alpha_i}{\pi_s \alpha_s}\) is equivalent to \(1 + \pi_i \alpha_i + \pi_s \alpha_s - 2\pi_a \alpha_a > 0\) and \(p < (1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_a \alpha_a)\).

The condition that \(v = 1\) weakly prefers \((B, NP)\) over \((B, AP)\) is \(v - p - (1 - v_b)(\pi_s \alpha_s v + \pi_i \alpha_i v) \geq v - \delta p - c_a - \pi_a \alpha_a v\) for \(v = v_b\). This simplifies to \(2(\pi_a \alpha_a + c_a - (1 - \delta)p) + \sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s} \geq \pi_i \alpha_i + \pi_s \alpha_s + 1\).

The condition that \(v = v_b\) weakly prefers \((B, NP)\) over \((B, AP)\) is \(\pi_a \alpha_a(\pi_i \alpha_i + \pi_s \alpha_s) + 2\pi_s \alpha_s(c_a + \delta p) + 1 \geq \pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + (1 - \pi_a \alpha_a)\sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s}\).

Lastly, if \(u(\sigma) = \frac{\pi_a \alpha_a - \pi_i \alpha_i}{\pi_s \alpha_s}\), then everyone will prefer \((B, NP)\) over \((B, AP)\) if \((1 - \delta)p - c_a \leq 0\). The conditions of these subcases are given in (A.38).

Next, for case \((V)\), in which the lower tier of purchasing consumers is unpatched while the upper tier does automated patching, i.e., \(0 < v_b < v_a < 1\), we have \(u = v_a - v_b\). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case \((V)\) arises.

**Claim 5** The equilibrium that corresponds to case \((V)\) arises if and only if the following conditions are satisfied:

\[
(1 - \delta)p - c_a < 0 \quad \text{and} \quad (\pi_a \alpha_a + c_a - 1 - (1 - \delta)p)(\pi_a \alpha_a - \pi_i \alpha_i + c_a - (1 - \delta)p) > \pi_s \alpha_s(\pi_a \alpha_a + c_a + \delta p - 1) \quad \text{and} \quad \pi_i \alpha_i(c_a + \delta p) < c_a + p(1 + \delta + \pi_a \alpha_a) \quad \text{and} \quad \pi_a \alpha_a \leq (1 - \delta)p + c_p - c_a. \quad (A.41)
\]

To solve for the thresholds \(v_b\) and \(v_a\), using \(u = v_a - v_b\), note that they solve

\[
v_b = \frac{p}{1 - \pi_s \alpha_s(v_a - v_b) - \pi_i \alpha_i}, \quad \text{and} \quad (A.42)
\]

\[
v_a = \frac{1 - (1 - \delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s(v_a - v_b) - \pi_i \alpha_i}. \quad (A.43)
\]

From (A.42), we have \(\pi_s \alpha_s(v_a - v_b) + \pi_i \alpha_i = 1 - \frac{2}{v_a}\), while from (A.43), we have \(\pi_s \alpha_s(v_a - v_b) + \pi_i \alpha_i = \pi_a \alpha_a - \frac{(1 - \delta)p - c_a}{v_a}\). Equating these two expressions and solving for \(v_a\) in terms of \(v_b\), we have

\[
v_a = \frac{v_b(-c_a + (1 - \delta)p)}{p - v_b(1 - \pi_a \alpha_a)}. \quad (A.44)
\]

Plugging this expression for \(v_a\) into (A.42), we find that \(v_b\) must be a zero of the cubic equation:

\[
f_2(x) \triangleq (1 - \pi_a \alpha_a)\pi_s \alpha_s x^3 + ((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_s \alpha_s - \delta p \pi_s \alpha_s)x^2 - p(2 - \pi_a \alpha_a - \pi_i \alpha_i)x + p^2.
\]

(A.45)
To find which root of the cubic \( v_b \) must be, first note that \( \pi_a \alpha_a - \pi_a \alpha_i u - \pi_i \alpha_i < 0 \) for consumers of higher valuation to prefer \((B, NP)\) over \((B, AP)\) by (A.25). From that, we have that \( c_a - (1-\delta)p > 0 \) in order for \( v_a > 0 \). To pin down the root of the cubic, note that the cubic’s highest order term is \( \pi_a \alpha_a (1-\pi_a \alpha_a) x^3 \), so \( \lim_{x \to -\infty} f_2(x) = -\infty \) and \( \lim_{x \to +\infty} f_2(x) = \infty \). We find \( f_2(0) = p^2 > 0 \) and \( f_2 \left( \frac{p}{1-\pi_a \alpha_a} \right) = -p^2 \pi_a \alpha_a (c_a - (1-\delta)p) \frac{1}{(1-\pi_a \alpha_a)} < 0 \). Since \( \lim_{x \to -\infty} f_2(x) = \infty \), there exists a root larger than \( \frac{p}{1-\pi_a \alpha_a} \). Note that from (A.42), we have \( v_b > \frac{p}{1-\pi_a \alpha_a} \). Therefore, \( v_b \) is the largest root of the cubic, lying past \( \frac{p}{1-\pi_a \alpha_a} \). Then using (A.44), we solve for \( v_a \).

For this to be an equilibrium, the necessary and sufficient conditions are \( 0 < v_b < v_a < 1 \) and type \( v = 1 \) prefers \((B, AP)\) over \((B, P)\). Type \( v = 1 \) preferring \((B, AP)\) over \((B, P)\) ensures \( v < 1 \) does so too, by (A.23). Moreover, since type \( v = v_a \) is indifferent between \((B, AP)\) and \((B, NP)\), and since \((B, AP)\) is preferred over \((B, P)\), by transitivity, it follows that type \( v_a \) prefers \((B, NP)\) over \((B, P)\). It follows that \( v < v_a \) prefers \((B, NP)\) over \((B, P)\) as well by (A.23).

For \( v_a < 1 \), using (A.44), an equivalent condition for this to hold is \( p + v_b (-1 + c_a - p (1-\delta) + \pi_a \alpha_a) < 0 \). If \( c_a - p (1-\delta) \geq 1 - \pi_a \alpha_a \), then this case can’t happen. Otherwise, if \( c_a - p (1-\delta) < 1 - \pi_a \alpha_a \), then this condition becomes \( v_b > \frac{p}{1-c_a+p(1-\delta)-\pi_a \alpha_a} \). This is equivalent to \( f_2 \left( \frac{c_a+p(1-\delta)}{1-\pi_a \alpha_a} \right) > 0 \), which simplifies to \( \pi_a \alpha_a + c_a - 1 - (1-\delta)p (\pi_a \alpha_a - \pi_i \alpha_i + c_a - (1-\delta)p) > \pi_s \alpha_s (\pi_a \alpha_a + c_a + \delta p - 1) \).

For \( v_b > v_b \), using (A.44), it is equivalent to require \( v_b < \frac{c_a+\delta p}{1-\pi_a \alpha_a} \). For this to happen, we need the condition \( f_2 \left( \frac{c_a+p(1-\delta)}{1-\pi_a \alpha_a} \right) > 0 \), which simplifies to \( \pi_a \alpha_a (\pi_i \alpha_i + c_a + \delta p) < c_a + p (-1 + \delta + \pi_a \alpha_a) \).

For \( v_b > 0 \), this holds by construction of \( v_b \) as the largest root of \( f_2(x) \) (which was shown to be larger than \( \left. \frac{p}{1-\pi_a \alpha_a} \right) \), so no additional conditions are needed.

Finally, for type \( v = 1 \) to prefer \((B, AP)\) over \((B, P)\), a necessary and sufficient condition is \( \pi_a \alpha_a \leq (1-\delta)p + c_p - c_a \). The conditions above are summarized in (A.41). \( \Box \)

Next, for case (VI), in which all segments are represented and the middle tier does automated patching, i.e., \( 0 < v_b < v_a < v_p < 1 \), we have \( u = v_a - v_b \). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VI) arises.

**Claim 6** The equilibrium that corresponds to case (VI) arises if and only if the following conditions are satisfied:

\[
(1-\delta)p - c_a < 0 \quad \text{and} \quad c_p + (1-\delta)p < c_a + \pi_a \alpha_a \quad \text{and} \quad c_p (1-\pi_a \alpha_a) + (1-\delta)p > c_a \quad \text{and} \quad c_p (\pi_a \alpha_a)^2 (-c_a + (1-\delta)p + c_p (1-\pi_a \alpha_a) + \pi_s \alpha_s (c_a - c_p (1-\pi_a \alpha_a) + (\pi_a \alpha_a - 1+\delta)p) - (c_a - c_p - (1-\delta)p) (c_a - (1-\delta)p - c_p (1-\pi_a \alpha_a) \pi_a \alpha_a \pi_i \alpha_i - 0 \quad \text{and} \quad c_a + p (-1 + \delta + \pi_a \alpha_a) > (c_a + \delta p) \pi_i \alpha_i. \tag{A.46}
\]

To solve for the thresholds \( v_b \) and \( v_a \), using \( u = v_a - v_b \), note that they solve

\[
v_b = \frac{p}{1-\pi_a \alpha_a (v_a - v_b) - \pi_i \alpha_i}, \tag{A.47}
\]

\[
v_a = \frac{(1-\delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s (v_a - v_b) - \pi_i \alpha_i}. \tag{A.48}
\]

These are the same as (A.42) and (A.43). Using the exact same argument, it follows that \( v_b \)
is the largest root of the cubic $f_2(x)$, lying past $\frac{c_a - (1-\delta)p}{1 - \pi_a \alpha_a}$. Note that the largest root $v_b$ is indeed larger $\frac{p}{1 - \pi_a \alpha_a}$ in this case as well since $v_b = \frac{-c_a + p(1-\delta) + v_b(1 - \pi_a \alpha_a)}{c_a + p(1-\delta) + v_b(1 - \pi_a \alpha_a)}$ and $(1 - \delta)p - c_a < 0$. Then using (A.44), we solve for $v_a$.

In this case, however, we also have a standard patching population, with the standard patching threshold given by $v_p = \frac{c_p - (c_a - (1-\delta)p)}{\pi_a \alpha_a}$.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_b < v_a < v_p < 1$. This tells us that all $v \in [v_p, 1]$ have the same preferences and will purchase $(B, P)$, all $v \in [v_a, v_p)$ have the same preferences and will purchase $(B, AP)$, and all $v \in [v_b, v_a)$ have the same preferences and will purchase $(B, NP)$. Finally, all $v < v_b$ have the same preferences and will not purchase in equilibrium.

For $v_p < 1$, the necessary and sufficient condition is $c_p + (1-\delta)p < c_a + \pi_a \alpha_a$.

For $v_a < v_p$, using (A.44) to write $v_a$ in terms of $v_b$, it is equivalent to write $v_b((1-\delta)p - c_a + c_p(1 - \pi_a \alpha_a)) > p((1-\delta)p - c_a + c_p)$. If $c_a + c_p(1 - \pi_a \alpha_a) + p(1 - \delta) > 0$, then we can rewrite this as $v_b > \frac{p((1-\delta)p - c_a + c_p)}{(1-\delta)p - c_a + c_p(1 - \pi_a \alpha_a)}$. This is equivalent to $f_2((1-\delta)p - c_a + c_p(1 - \pi_a \alpha_a)) < 0$, which simplifies to $c_p(\pi_a \alpha_a)^2(-c_a + (1 - \delta)p + c_p(1 - \pi_a \alpha_a) + \pi_a \alpha_s(-c_a + (1 - \delta)p + c_p)^2(c_a - c_p(1 - \pi_a \alpha_a) + (\pi_a \alpha_a - 1 + \delta)p) - (c_a - c_p - (1 - \delta)p)(c_a - (1 - \delta)p - c_p(1 - \pi_a \alpha_a))\pi_a \alpha_a \pi_a \alpha_i < 0$.

On the other hand, if $-c_a + c_p(1 - \pi_a \alpha_a) + p(1 - \delta) < 0$, then we need $v_b < \frac{p((1-\delta)p - c_a + c_p)}{(1-\delta)p - c_a + c_p(1 - \pi_a \alpha_a)}$. If $p((1-\delta)p - c_a + c_p) \geq 0$, then this can’t happen since the denominator is negative and $v_b > 0$. If $p((1-\delta)p - c_a + c_p) < 0$, then $\frac{p((1-\delta)p - c_a + c_p)}{(1-\delta)p - c_a + c_p(1 - \pi_a \alpha_a)} < p$. However, $v_b > \frac{p}{1 - \pi_a \alpha_a}$, so this can’t happen either. Therefore, $-c_a + c_p(1 - \pi_a \alpha_a) + p(1 - \delta) < 0$ rules out this case.

Lastly, if $-c_a + c_p(1 - \pi_a \alpha_a) + p(1 - \delta) = 0$, then this $v_b((1 - \delta)p - c_a + c_p(1 - \pi_a \alpha_a)) > p((1 - \delta)p - c_a + c_p)$ becomes $0 > ((1 - \delta)p - c_a + c_p)$. This simplifies to $0 < c_p \pi_a \alpha_a$, which can’t happen.

Therefore, the conditions for $v_a < v_p$ are $c_p(1 - \pi_a \alpha_a) + (1 - \delta)p > c_a$ and $c_p(\pi_a \alpha_a)^2(-c_a + (1 - \delta)p + c_p(1 - \pi_a \alpha_a) + \pi_a \alpha_s(-c_a + (1 - \delta)p + c_p)^2(c_a - c_p(1 - \pi_a \alpha_a) + (\pi_a \alpha_a - 1 + \delta)p) - (c_a - c_p - (1 - \delta)p)(c_a - (1 - \delta)p - c_p(1 - \pi_a \alpha_a))\pi_a \alpha_a \pi_a \alpha_i < 0$.

For $v_b < v_a$, again using (A.44) to write $v_a$ in terms of $v_b$, this simplifies to $v_b < \frac{c_a + p\delta}{1 - \pi_a \alpha_a}$. This can equivalently be expressed as $f_2\left(\frac{c_a + p\delta}{1 - \pi_a \alpha_a}\right) > 0$, which simplifies to $c_a + p(-1 + \delta + \pi_a \alpha_a) > (c_a + \delta p)\pi_a \alpha_i$.

For $0 < v_b$, no condition is needed since $v_b$ is defined to be the largest root of the cubic, which was shown to be larger than $\frac{p}{1 - \pi_a \alpha_a}$.

A summary of the above necessary and sufficient conditions is given in (A.46). □

Next, for case (VII), in which there are no automated patching users while the lower tier is unpatched and the upper tier is patch, i.e., $0 < v_b < v_p < 1$, we have $u = v_p - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VII) arises.

**Claim 7** The equilibrium that corresponds to case (VII) arises if and only if the following condi-
tions are satisfied:

\[(−1 + \alpha c + p)\pi_s^s\alpha_s < (1 − \alpha c)(−\alpha c + \pi_i^i\alpha_i) \text{ and } \pi_i^i\alpha_i < \frac{\alpha c}{\alpha c + p} \text{ and}
\]

\[\left(1 − \pi_a^a\alpha_a\right)(c_a + (\pi_a^a\alpha_a − (1 − \delta))p)(c_a + p(\pi_a^a\alpha_a − (1 − \delta)) − (c_a + \delta p)\pi_i^i\alpha_i) + \right.

\[\left(c_a + \delta p\right)^2(c_a − p − (1 − \delta)p + (c_p + p)\alpha a\pi a\pi s\alpha s ≥ 0 \text{ and } c_a + p(\delta + \pi_a^a\alpha_a) > p \text{ and}
\]

\[
\left\{(c_a − (1 − \delta)p + c_p(−1 + \pi_a^a\alpha_a) ≥ 0 \right\}
\]

\[\left(c_a + c_p(−1 + \pi_a^a\alpha_a) < 0 \text{ and } \pi_a^a\alpha_a(c_a − (1 − \delta)p + c_p(−1 + \pi_a^a\alpha_a))(c_p\pi a\alpha a + (c_a − c_p − (1 − \delta)p)\pi_i^i\alpha_i) ≤
\]

\[(-c_a + c_p + (1 − \delta)p)^2(c_a − c_p − (1 − \delta)p + (c_p + p)\pi a\alpha a)\pi s\alpha s \right\} \). (A.49)

To solve for the thresholds \(v_b\) and \(v_p\), using \(u = v_p − v_b\), note that they solve

\[v_b = \frac{p}{1 − \pi_s\alpha_s(v_p − v_b) − \pi_i^i\alpha_i}, \text{ and} \tag{A.50}
\]

\[v_p = \frac{c_p}{\pi_s\alpha_s(v_p − v_b) + \pi_i^i\alpha_i}. \tag{A.51}
\]

From (A.50), we have \(\pi_s\alpha_s(v_p − v_b) + \pi_i^i\alpha_i = 1 − \frac{p}{v_b}\), while from (A.51), we have \(\pi_s\alpha_s(v_p − v_b) + \pi_i^i\alpha_i = \frac{c_p}{v_p}\). Equating these two expressions and solving for \(v_p\) in terms of \(v_b\), we have

\[v_p = \frac{c_p v_b}{v_b − p}. \tag{A.52}
\]

Plugging this expression for \(v_p\) into (A.50), we find that \(v_b\) must be a zero of the cubic equation:

\[f_3(x) \triangleq \pi_s\alpha_s x^3 + (1 − \pi_i^i\alpha_i − \pi_s\alpha_s(c_p + p))x^2 − (2 − \pi_i^i\alpha_i)px + p^2. \tag{A.53}
\]

To find which root of the cubic \(v_b\) must be, note that the cubic’s highest order term is \(\pi_s\alpha_s x^3\), so \(\lim_{x \to −\infty} f_3(x) = −\infty\) and \(\lim_{x \to \infty} f_3(x) = \infty\). We find \(f_3(0) = p^2 > 0\), and \(f_3(p) = −\pi_s\alpha_s p^2 < 0\). Since \(v_b − p > 0\) in equilibrium, we have that \(v_b\) is the largest root of the cubic, lying past \(p\). Then using (A.52), we solve for \(v_p\).

For this to be an equilibrium, the necessary and sufficient conditions are \(0 < v_b < v_p < 1\), type \(v = v_p\) prefers \((B, P)\) over \((B, AP)\), and type \(v = v_b\) prefers \((NB, NP)\) to \((B, AP)\). Type \(v = v_p\) preferring \((B, P)\) over \((B, AP)\) ensures \(v > v_p\) also prefer \((B, P)\) over \((B, AP)\), by (A.23). Moreover, type \(v = v_b\) preferring \((NB, NP)\) over \((B, AP)\) ensures \(v < v_b\) do so too, by (A.24).

For \(v_p < 1\), a necessary and sufficient condition for this to hold is \(v_b > \frac{p}{1−c_p}\). This is equivalent to \(f_3\left(\frac{p}{1−c_p}\right) < 0\). This simplifies to \((−1 + c_p + p)\pi_s\alpha_s < (1 − c_p)(−\alpha c + \pi_i^i\alpha_i)\).

For \(v_p > v_b\), a necessary and sufficient condition is \(v_b < c_p+p\). This is equivalent to \(f_3(c_p+p) > 0\), or \(\pi_i^i\alpha_i < \frac{c_p}{c_p+p}\).

We don’t need any conditions for \(v_b > 0\), since by construction, \(v_b > p > 0\).

For type \(v = v_p\) to prefer \((B, P)\) over \((B, AP)\), a necessary and sufficient condition is \(\frac{c_p v_b}{v_b − p} ≥ \frac{c_p−(c_a−(1−\delta)p)}{\pi_a^a\alpha_a}\). This simplifies to \(v_b(c_a − (1 − \delta)p − c_p(1 − \pi_a^a\alpha_a)) ≥ p(c_a − (1 − \delta)p − c_p)\). This can be
broken down into three cases, depending on the sign of \( c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a) \) (also considering the case when the factor is zero). When \( c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a) = 0 \), the left side is 0 while the right side is negative, so this inequality holds. If \( c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a) > 0 \), then the inequality becomes \( v_b \geq \frac{c_a - (1 - \delta)p - cp}{c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a)} \). But \( \frac{c_a - (1 - \delta)p - cp}{c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a)} < 1 \), and since \( v_b > p \) by construction, this inequality holds without further conditions. On the other hand, if \( c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a) < 0 \), then we need \( v_b \leq \frac{c_a - (1 - \delta)p - cp}{c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a)} \). So we need \( f_3 \left( \frac{p(c_a - (1 - \delta)p - cp)}{c_a - (1 - \delta)p - cp(1 - \pi_a \alpha_a)} \right) \geq 0 \). Omitting the algebra, this simplifies to \( \pi_a \alpha_a (c_a - (1 - \delta)p + cp(-1 + \pi_a \alpha_a))(cp\pi_a \alpha_a + (c_a - cp - (1 - \delta)p)\pi_i \alpha_i) \leq (-c_a + cp + (1 - \delta)p)^2(c_a - cp - (1 - \delta)p + (cp + p)\pi_a \alpha_a)\pi_s \alpha_s \).

For \( v = v_b \) to prefer \((NB, NP)\) to \((B, AP)\), a necessary and sufficient condition is \( v_b \leq \frac{\delta p + c_a}{1 - \pi_a \alpha_a} \), which becomes \( f_3 \left( \frac{\delta p + c_a}{1 - \pi_a \alpha_a} \right) \geq 0 \). This simplifies to \( c_a > (1 - \delta - \pi_a \alpha_a)p \) and \( (1 - \pi_a \alpha_a)(c_a + (\pi_a \alpha_a - (1 - \delta))p(c_a + p(\pi_a \alpha_a - (1 - \delta)) - (c_a + \delta p)\pi_i \alpha_i) + (c_a + \delta p)^2(c_a - cp - (1 - \delta)p + (cp + p)\pi_a \alpha_a)\pi_s \alpha_s \geq 0 \).

The conditions are summarized in (A.49).

Next, for case (VIII), in which there are no unpatched users while the lower tier chooses automated patching and the upper tier chooses standard patching, i.e., \( 0 < v_a < v_p < 1 \), we have \( u = 0 \). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VIII) arises.

**Claim 8** The equilibrium that corresponds to case (VIII) arises if and only if the following conditions are satisfied:

\[
c_p + (1 - \delta)p < c_a + \pi_a \alpha_a \quad \text{and} \quad c_p + (1 - \delta)p > c_a + (c_p + p)\pi_a \alpha_a \quad \text{and} \quad (c_p - c_a + (1 - \delta)p)\pi_i \alpha_i \geq c_p \pi_a \alpha_a \quad \text{and} \\
\left\{ \\
\left( \pi_i \alpha_i < \pi_a \alpha_a \quad \text{and} \quad c_p \pi_a \alpha_a + \pi_i \alpha_i (c_a - cp - (1 - \delta)p) \leq 0 \right) \right. \quad \text{or} \\
\left( \pi_i \alpha_i > \pi_a \alpha_a \quad \text{and} \quad c_p + p(1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i \right) \quad \text{or} \\
\left. \pi_i \alpha_i = \pi_a \alpha_a \quad \text{and} \quad (1 - \delta)p - c_a \geq 0 \right\}. \quad (A.54)
\]

In this case, the threshold for the consumer indifferent between purchasing the automated patching option and not purchasing at all, \( v_a \), satisfies

\[
v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}, \quad (A.55)
\]

and the consumer indifferent between choosing automated patching and standard patching is given by

\[
v_p = \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}, \quad (A.56)
\]

For this to be an equilibrium, it is necessary and sufficient to have \( 0 < v_a < v_p < 1 \), no one prefers \((B, NP)\) over \((B, AP)\), and no one prefers \((B, NP)\) over \((B, P)\).

For \( v_p < 1 \), this is equivalently written as \( c_p + (1 - \delta)p < c_a + \pi_a \alpha_a \).

To have \( v_a < v_p \), a necessary and sufficient condition is \( c_p + (1 - \delta)p > c_a + (c_p + p)\pi_a \alpha_a \).

We always have \( v_a > 0 \) from our model assumptions, namely \( \pi_a \alpha_a < 1 \).

To ensure that no one prefers \((B, NP)\) over \((B, P)\), it suffices to make \( v = v_p \) weakly prefer
(B, P) over (B, NP) so that everyone of higher valuation would also have the same preference (by (A.23)). This condition then becomes \((c_p - c_a + (1 - \delta)p)\pi_i\alpha_i \geq c_p\pi_a\alpha_a\).

To ensure that no one prefers (B, NP) over (B, AP), we need \((\pi_a\alpha_a - (\pi_a\alpha_s u(\sigma) + \pi_i\alpha_i))v \leq (1 - \delta)p - c_a\). In this case, \(u(\sigma) = 0\) so that there are three cases, depending on the sign of \(\pi_a\alpha_a - \pi_i\alpha_i\).

If \(\pi_a\alpha_a > \pi_i\alpha_i\), then higher valuation consumers prefer (B, NP) over (B, AP), so a necessary and sufficient condition is for \(v = v_p\) to weakly prefer (B, AP) over (B, NP). This becomes
\[
c_p\pi_a\alpha_a + \pi_i\alpha_i(c_a - c_p - (1 - \delta)p) \leq 0.
\]

On the other hand, if \(\pi_a\alpha_a < \pi_i\alpha_i\), then lower valuation consumers prefer (B, NP) over (B, AP). In this case, a necessary and sufficient for no one to prefer (B, NP) over (B, AP) is for \(v = v_a\) to weakly prefer (B, AP) over (B, NP). This simplifies to \(c_a + p(-1 + \delta + \pi_a\alpha_a) \leq (c_a + \delta p)\pi_i\alpha_i\).

Lastly, if \(\pi_a\alpha_a = \pi_i\alpha_i\), then we need \((1 - \delta)p - c_a \geq 0\) for everyone to prefer (B, AP) over (B, NP). Altogether, Case (VIII) arises if and only if the condition in (A.54) occurs. □

Lastly, for case (IX), in which all users choose standard patching, i.e., \(0 < v_p < 1\), we have \(u = 0\). Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (IX) arises.

**Claim 9** The equilibrium that corresponds to case (IX) arises if and only if the following conditions are satisfied:
\[
c_p + p < 1 \text{ and } c_a + (c_p + p)\pi_a\alpha_a \geq c_p + (1 - \delta)p \text{ and } (c_p + p)\pi_i\alpha_i \geq c_p.
\]
(A.57)

In this case, the threshold for the consumer indifferent between choosing the standard patching option and not purchasing at all, \(v_p\), satisfies
\[
v_p = c_p + p.
\]
(A.58)

For this to be an equilibrium, it is necessary and sufficient to have \(0 < v_p < 1\), \(v = v_p\) prefers (NB, NP) over both (B, NP) and (B, AP), and \(v = v_p\) prefers (B, P) over both (B, NP) and (B, AP).

For \(v_p < 1\), this is equivalently written as \(c_p + p < 1\).

For \(v = v_p\) to weakly prefer (NB, NP) over (B, NP), we need \(0 \geq v_p - p - \pi_i\alpha_i v_p\), which simplifies to \((c_p + p)\pi_i\alpha_i \geq c_p\).

For \(v = v_p\) to weakly prefer (NB, NP) over (B, AP), we need \(c_a + (c_p + p)\pi_a\alpha_a \geq c_p + (1 - \delta)p\).

For \(v = v_p\) to weakly prefer (B, P) over (B, NP) and (B, AP), the conditions will be the same as above since \(v = v_p\) is indifferent between (NB, NP) and (B, P).

Altogether, Case (IX) arises if and only if the condition in (A.57) occurs.

This completes the proof of the general consumer market equilibrium for the proprietary case. □

**Proof of Lemma 5:** Technically, we prove that there exists an \(\bar{\alpha}_2\) such that if \(\pi_i\alpha_i < \min \left[\frac{c_p\pi_a\alpha_a}{1 + c_p - c_a}, \frac{c_p}{1 + c_p}\right]\), then for \(\alpha_s > \bar{\alpha}_2\), \(p^*\) and \(\delta^*\) are set so that

1. if \(c_a < \min \left[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)\right]\), then \(\sigma^*(v)\) is characterized by \(0 < v_a < v_b < v_p < 1\) under optimal pricing,
2. if \(|\pi_a \alpha_a - c_p| < c_a < c_p(1 - \pi_a \alpha_a)|\), then \(\sigma^*(v)\) is characterized by \(0 < v_b < v_a < v_p < 1\) under optimal pricing, and if

3. if \(c_a > c_p(1 - \pi_a \alpha_a)|\), then \(\sigma^*(v)\) is characterized by \(0 < v_b < v_p < 1\) under optimal pricing.

The sketch of the proof is similar to that of Lemma 4. Using Lemma A.4 instead of Lemma A.3, we find the profit-maximizing price within the compact closure of each case, so that the price that induces the largest profit among the cases will be the equilibrium price set by the vendor.

The conditions of this lemma preclude certain candidate market structures from arising in equilibrium. Specifically, using Lemma A.4, \(\pi_i \alpha_i < \min \left[\pi_a \alpha_a, \frac{c_p}{1 + c_p}\right]\) rules out Cases (VIII) and (IX). We consider the remaining consumer structures that can arise when the vendor sets prices optimally.

Suppose \(0 < v_a < 1\) is induced. By part (I) of Lemma A.4, we obtain \(v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\). The profit function in this case is \(\Pi_I(p, \delta) = \delta p(1 - v_a(p, \delta)\). Let \(C_l\) be the compact closure of the region of the parameter space defining \(0 < v_a < 1\), given in part (I) of Lemma A.4. By the Weierstrass extreme value theorem, there exists \(p\) and \(\delta\) in \(C_l\) that maximizes \(\Pi(p, \delta)\). This \(p\) and \(\delta\) combination may be on the boundary, and we show that the vendor’s profit function is continuous across region boundaries later. Otherwise, if this \(p\) and \(\delta\) are interior, the unconstrained maximizer satisfies the first-order conditions.

Differentiating the profit function with respect to \(p\), we have that \(p_I^*(\delta) = \frac{1 - c_a - \pi_a \alpha_a}{\pi_a \alpha_a}\). The second-order condition gives \(\frac{\partial^2}{\partial p^2}\Pi(p, \delta) = -\frac{2\delta^2}{1 - \pi_a \alpha_a}\). We see that

\[
\Pi_I = \Pi(p^*(\delta), \delta) = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)}
\]

for any \(\delta\), so this is the maximal profit of this case.

On the other hand, suppose \(0 < v_a < v_b < 1\) is induced. By part (II) of Lemma A.4, we obtain that \(v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\) and \(v_b = \frac{-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s + \sqrt{(-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s)^2 - 4\pi s \alpha s (c_a + (\delta - 1)p}}{2\pi s \alpha s}\). The profit function in this case is

\[
\Pi_{II}(p, \delta) = p(1 - v_b(p, \delta)\) + \delta p(v_b(p, \delta) - v_a(p, \delta)).
\]

The first-order condition in \(p\) yields

\[
\frac{(\delta - 1)p}{\sqrt{(-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s)^2 - 4\pi s \alpha s (c_a + (\delta - 1)p)}} + \frac{\delta}{2\pi s \alpha s}\left(\frac{-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s + \sqrt{(-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s)^2 - 4\pi s \alpha s (c_a + (\delta - 1)p}} - \frac{c_a + \delta p}{1 - \pi a \alpha a}\right) + \delta p\left(\frac{-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s + \sqrt{(-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s)^2 - 4\pi s \alpha s (c_a + (\delta - 1)p}} - \frac{c_a + \delta p}{1 - \pi a \alpha a}\right) - \frac{\delta - 1}{\sqrt{(-\pi a \alpha a + \pi a \alpha i + \pi s \alpha s)^2 - 4\pi s \alpha s (c_a + (\delta - 1)p)}} = 0.
\]
Letting \( X = \sqrt{(-\pi_s \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s(c_a + (\delta - 1)p)} \), we can rewrite this as
\[
X^2 + \left(-\pi_s \alpha_a + \pi_i \alpha_i + \frac{\pi_s \alpha_s(\pi_a \alpha_a + 2c_a + 4\delta p - 1)}{(\delta - 1)(\pi_a \alpha_a - 1)}\right) X + 2\pi_s \alpha_s(1 - \delta)p = 0. \tag{A.61}
\]

Similarly, the first-order condition in \( \delta \) can be written as
\[
X^2 + \left(-\pi_s \alpha_a + \pi_i \alpha_i + \frac{\pi_s \alpha_s(\pi_a \alpha_a + 2c_a + 4\delta p - 1)}{\pi_a \alpha_a - 1}\right) X + 2\pi_s \alpha_s(1 - \delta)p = 0. \tag{A.62}
\]

Using the first-order conditions together, we have that \( X(1 - c_a - 2p\delta - \pi_a \alpha_a) = 0 \). If \( X = 0 \), then using the definition of \( X \), we have that \( (1 - \delta)p = \frac{4\pi_s \alpha_s - (\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2}{4\pi_s \alpha_s} \). However, if \( \pi_s \alpha_s > \pi_a \alpha_a - \pi_i \alpha_i + 2\sqrt{(c_a + 1)(\pi_a \alpha_a - \pi_i \alpha_i + c_a + 1)} + 2c_a + 2 \), then \( \delta p > 1 + p \). This can’t happen in equilibrium since \( \delta p < 1 \) for consumers to be willing to pay for the automated patching option. Therefore, it cannot be the case that \( X = 0 \).

Then from \( X(1 - c_a - 2p\delta - \pi_a \alpha_a) = 0 \), we have that \( \delta^*(p) = \frac{1 - c_a - \pi_s \alpha_a}{2p} \). Plugging this back into the profit function and maximizing over \( p \) again, we have two roots for \( p \):
\[
p = -\left(\frac{1}{18\pi_s \alpha_s}\right) \left(\pi_s \alpha_s(\pi_a \alpha_a + 8\pi_i \alpha_i) + 2(\pi_a \alpha_a - \pi_i \alpha_i)^2 + 2(\pi_s \alpha_s)^2 - 3(c_a + 3)\pi_s \alpha_s \mp 2\sqrt{(\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s)^2 ((\pi_a \alpha_a - \pi_i \alpha_i)^2 + (\pi_s \alpha_s)^2 + \pi_s \alpha_s(\pi_a \alpha_a + \pi_i \alpha_i - 3c_a))}\right) \tag{A.63}
\]

However, when
\[
\pi_s \alpha_s > \frac{1}{8(1 - \pi_i \alpha_i)} \left((\pi_a \alpha_a)^2 - 2\pi_a \alpha_a + 8(\pi_i \alpha_i)^2 + c_a^2 + 2c_a \pi_a \alpha_a - 8c_a \pi_i \alpha_i + 6c_a + 9 - 16\pi_i \alpha_i - (\pi_a \alpha_a - 4\pi_i \alpha_i + c_a + 3)\sqrt{(\pi_a \alpha_a - 1)(\pi_a \alpha_a + 8\pi_i \alpha_i - 9) + c_a^2 + 2c_a(\pi_a \alpha_a - 4\pi_i \alpha_i + 3)}\right) \tag{A.64}
\]

then the smaller root will be negative while the larger root is positive. Therefore, the equilibrium price of this case is
\[
p_{II}^* = -\left(\frac{1}{18\pi_s \alpha_s}\right) \left(\pi_s \alpha_s(\pi_a \alpha_a + 8\pi_i \alpha_i) + 2(\pi_a \alpha_a - \pi_i \alpha_i)^2 + 2(\pi_s \alpha_s)^2 - 3(c_a + 3)\pi_s \alpha_s - 2\sqrt{(\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s)^2 ((\pi_a \alpha_a - \pi_i \alpha_i)^2 + (\pi_s \alpha_s)^2 + \pi_s \alpha_s(\pi_a \alpha_a + \pi_i \alpha_i - 3c_a))}\right). \tag{A.65}
\]

The equilibrium discount of this case is given as
\[
\delta_{II}^* = (9\pi_s \alpha_s(\pi_a \alpha_a + c_a - 1)) \left(\pi_s \alpha_s(\pi_a \alpha_a + 8\pi_i \alpha_i) + 2(\pi_a \alpha_a - \pi_i \alpha_i)^2 + 2(\pi_s \alpha_s)^2 - 3\pi_s \alpha_s(c_a + 3) - 2\sqrt{(\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s)^2 ((\pi_a \alpha_a - \pi_i \alpha_i)^2 + (\pi_s \alpha_s)^2 + \pi_s \alpha_s(\pi_a \alpha_a + \pi_i \alpha_i - 3c_a))}\right)^{-1}. \tag{A.66}
\]

The equilibrium profit is given as \( \Pi_{II}^* = \Pi_{II}(p_{II}^*, \delta_{II}^*) \). As we had done in Lemma 4, we can
characterize the profit of this case using Taylor series. In particular, there exists an $\alpha_6$ such for that $\alpha_s > \alpha_6$, the maximal profit of this case is

$$
\Pi_{II} = \frac{(1 - c_a - \pi_s a_{\alpha_s})^2}{4(1 - \pi_s a_{\alpha_s})} + \frac{(c_a + \pi_s a_{\alpha_s} - \pi_s i_{\alpha_i})^2}{4\pi_s a_{\alpha_s}} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s a_{\alpha_s}} \right)^k
$$

(A.67)

for some coefficients $c_k$. As done before in Lemma 4, we will use $a_k, b_k, c_k,$ and $d_k$ to denote coefficients in the Taylor expansions without referring to specific expressions. These will be used across different cases, and they don’t refer to the same quantities or expressions across cases.

On the other hand, suppose $0 < v_a < v_b < v_p < 1$ is induced. By part (III) of Lemma A.4, we obtain that $v_a = \frac{\delta p + c_x}{1 - \pi_s a_{\alpha_s}}, v_b$ is the most positive root of the cubic $f_3(x) \triangleq \pi_s a_{\alpha_s} x^2(c_a - c_p + (\delta - 1)p + \pi_s a_{\alpha_s}) + (c_a + (\delta - 1)p + \pi_s a_{\alpha_s})(c_a + (\delta - 1)p + x(\pi_s a_{\alpha_s} - \pi_s i_{\alpha_i})), \) and $v_p = \frac{c_p v_b}{c_a - (1 - \delta)p + \pi_s a_{\alpha_s} v_b}$. The profit function in this case is

$$
\Pi_{III}(p, \delta) = p(1 - v_b(p, \delta)) + \delta p(v_b(p, \delta) - v_a(p, \delta)).
$$

(A.68)

As we had done in Lemma 4, we employ asymptotic analysis to characterize the equilibrium prices and profit of this case. In particular, since $v_b$ is the most positive root of the cubic equation $f_3(x) \triangleq \pi_s a_{\alpha_s} x^2(c_a - c_p + (\delta - 1)p + \pi_s a_{\alpha_s}) + (c_a + (\delta - 1)p + \pi_s a_{\alpha_s})(c_a + (\delta - 1)p + x(\pi_s a_{\alpha_s} - \pi_s i_{\alpha_i}))$ (and since $f'(x) \neq 0$ at the value of $x$ that defines $v_b$), it follows that $v_b$ is an analytic function of the parameters. Letting $v_b = A_0 + \sum_{k=1}^{\infty} d_k \left( \frac{1}{\pi_s a_{\alpha_s}} \right)^k$, the cubic equation defining $v_b$ becomes

$$
A_0^2(c_a - c_p - p(1 - \delta) + A_0 \pi_s a_{\alpha_s} \pi_s a_{\alpha_s} + \sum_{k=0}^{\infty} e_k \left( \frac{1}{\pi_s a_{\alpha_s}} \right)^k) = 0
$$

for some coefficients $e_k$. For this equation to hold, we must have $A_0 = 0$ or $A_0 = \frac{c_p - c_a + (1 - \delta)p}{\pi_s a_{\alpha_s}}$. The double root $A_0 = 0$ corresponds to the two solutions of the cubic converging to zero, while $A_0 = \frac{c_p - c_a + (1 - \delta)p}{\pi_s a_{\alpha_s}} > 0$ corresponds to the largest root of cubic.

Then substituting $v_b = \frac{c_p - c_a + (1 - \delta)p}{\pi_s a_{\alpha_s}} + A_1 \pi_s a_{\alpha_s} + \sum_{k=2}^{\infty} d_k \left( \frac{1}{\pi_s a_{\alpha_s}} \right)^k$ into $f_3(x)$ (the cubic equation defining $v_b$), we have

$$
c_p(c_a + (\delta - 1)p)(\pi_s a_{\alpha_s}) + A_1(\pi_s a_{\alpha_s} - \pi_s i_{\alpha_i}) + \sum_{k=1}^{\infty} e_k \left( \frac{1}{\pi_s a_{\alpha_s}} \right)^k = 0
$$

for some coefficients $e_k$. Solving for $A_1$ gives $A_1 = \frac{-c_p(c_a + (\delta - 1)p) + c_p(\pi_s a_{\alpha_s} - \pi_s i_{\alpha_i})}{(c_a - c_p + (\delta - 1)p)^2}$. Successively iterating in this way, we can solve for the coefficients in the Taylor series for $v_b$, giving

$$
v_b(p, \delta) = \frac{-c_a + c_p - \delta p + p}{\pi_s a_{\alpha_s}} - \frac{(c_p(\pi_s a_{\alpha_s}) + (\delta - 1)p) + c_p(\pi_s a_{\alpha_s} - \pi_s i_{\alpha_i}))}{\pi_s a_{\alpha_s}(c_a - c_p + (\delta - 1)p)^2} + \frac{(c_p(\pi_s a_{\alpha_s}) + (\delta - 1)p)(\pi_s a_{\alpha_s} + (\delta - 1)p) + c_p(\pi_s a_{\alpha_s} - \pi_s i_{\alpha_i}))}{(\pi_s a_{\alpha_s})^2(c_a - c_p + (\delta - 1)p)^3} + \sum_{k=3}^{\infty} d_k \left( \frac{1}{\pi_s a_{\alpha_s}} \right)^k
$$

(A.69)

Substituting (A.69) into the profit function (A.68), differentiating with respect to $p$ for the first-order condition, and then substituting in $p = \sum_{k=0}^{\infty} a_k \left( \frac{1}{\pi_s a_{\alpha_s}} \right)^k$ to iteratively solve for the
coefficients $a_k$ as done above for $v_b(p, \delta)$, we get

$$p^*(\delta) = -\frac{c_a(\pi_a\alpha_a + \delta - 1) + (\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_p(\delta - 1))}{2(2\delta(\pi_a\alpha_a - 1) - \pi_a\alpha_a + \delta^2 + 1)} + \left(\pi_s\alpha_s\left(-\pi_a\alpha_a(\delta - 1)(\pi_a\alpha_a - 1) + c_a(\delta(3\pi_a\alpha_a - 2) - \pi_a\alpha_a + \delta^2 + 1) - \right.ight.$$

$$c_p(\pi_a\alpha_a\delta^2 + 2\pi_a\alpha_a\delta - \pi_a\alpha_a + \delta^2 - 2\delta + 1)\left.)\right)^3 - 1 \left((2\pi_a\alpha_a c_p(\delta - 1)(\pi_a\alpha_a - 1)(2\delta(\pi_a\alpha_a - 1) - \pi_a\alpha_a + \delta^2 + 1)

$$

$$(\pi_i\alpha_i c_a^2 (3\pi_i\alpha_i - 2) - \pi_i\alpha_i + \delta^2 + 1) + c_a(c_p((\pi_i\alpha_i)^2 (5\delta - 3) + \pi_i\alpha_i(\delta^2(3 - \pi_i\alpha_i) - \delta(5\pi_i\alpha_i + 6) + 2\pi_i\alpha_i + 3) - 2\pi_i\alpha_i(\delta - 1)^2) - \pi_i\alpha_i\pi_i\alpha_i(\delta - 1)(\pi_i\alpha_i - 1)) + c_p(\pi_i\alpha_i(\delta - 1)\pi_i\alpha_i - 1)(\pi_i\alpha_i + \pi_i\alpha_i) + c_p((\pi_i\alpha_i)^2 (\delta^2 - 6\delta + 3) + \pi_i\alpha_i(\delta^2(\pi_i\alpha_i - 3) + 2\delta(\pi_i\alpha_i + 3) - \pi_i\alpha_i - 3) +

$$

$$\pi_i\alpha_i(\delta - 1)^2)) \right) + \sum_{k=2}^{\infty} a_k \left(\frac{1}{\pi_s\alpha_s}\right)^k.$$ (A.70)

Substituting (A.70) into the profit function (A.68), differentiating with respect to $\delta$ for the first-order condition, and then substituting in $\delta = \sum_{k=0}^{\infty} b_k \left(\frac{1}{\pi_s\alpha_s}\right)^k$ to iteratively solve for the coefficients $b_k$, we get

$$\delta_{III}^* = \frac{1 - \pi_a\alpha_a - c_a}{1 - c_p} - \left(4c_p\pi_a\alpha_a(\pi_a\alpha_a + c_a - 1)(\pi_i\alpha_i c_a^2 + c_a(-\pi_a\alpha_a\pi_i\alpha_i + 3\pi_a\alpha_a c_p - 2\pi_i\alpha_i c_p) + c_p(\pi_a\alpha_a(\pi_a\alpha_a + \pi_i\alpha_i) + c_p(\pi_i\alpha_i - 3\pi_a\alpha_a))) \right)\left(\pi_s\alpha_s(c_p - 1)^2(\pi_a\alpha_a - c_a + c_p)^3\right)^{-1} + \sum_{k=2}^{\infty} b_k \left(\frac{1}{\pi_s\alpha_s}\right)^k$$ (A.71)

for some coefficients $b_k$.

Substituting this into (A.70), we have that

$$p_{III}^* = \frac{1 - c_p}{2} + \frac{2\pi_a\alpha_a c_p(\pi_i\alpha_i c_a^2 + c_a(-\pi_a\alpha_a\pi_i\alpha_i + 3\pi_a\alpha_a c_p - 2c_p\pi_i\alpha_i) + c_p(\pi_a\alpha_a(\pi_a\alpha_a + \pi_i\alpha_i) + c_p(\pi_i\alpha_i - 3\pi_a\alpha_a)))}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a - c_p)^3} + \sum_{k=2}^{\infty} a_k \left(\frac{1}{\pi_s\alpha_s}\right)^k$$ (A.72)

The second-order conditions are satisfied, and the profit at this maximizer is given as

$$\Pi_{III}^* = \frac{1}{4}\left(\frac{c_a^2}{1 - \pi_a\alpha_a} + \frac{(c_a - c_p)^2}{\pi_a\alpha_a} - 2c_p + 1\right) + \frac{c_p(\pi_a\alpha_a + c_a - c_p)(\pi_i\alpha_i(c_a - \pi_a\alpha_a) + c_p(2\pi_a\alpha_a - \pi_i\alpha_i))}{\pi_s\alpha_s(\pi_a\alpha_a - c_a + c_p)^2} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k$$ (A.73)

Next, suppose $0 < v_b < 1$ is induced. By part (IV) of Lemma A.4, we obtain that $v_b = \frac{1}{2}$ +
\[-1 + \pi_i \alpha_i + \sqrt{(1 - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4 \pi_s \alpha_s} / 2 \pi_s \alpha_s \]. The profit function in this case is

\[\Pi_{IV}(p, \delta) = p(1 - v_b(p, \delta)).\] (A.74)

For brevity of exposition, we will just quickly give the optimal prices and profits after writing the profit function for the remaining cases. The derivation is the same as these previous cases.

The optimal price is given as

\[p_{IV}^* = \frac{1}{9 \pi_s \alpha_s} \left( - (\pi_i \alpha_i)^2 + 2 \pi_i \alpha_i (1 - 2 \pi_s \alpha_s) + \pi_s \alpha_s (4 - \pi_s \alpha_s) - 1 + \sqrt{(-\pi_i \alpha_i + \pi_s \alpha_s + 1)^2 (\pi_s \alpha_s (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_s \alpha_s)^2) + 2 \pi_i \alpha_i (2 \pi_s \alpha_s - 1 + \pi_s \alpha_s (\pi_s \alpha_s - 4) + 1) (3 \pi_i \alpha_i - 3 \pi_s \alpha_s - 3 + (5 (\pi_i \alpha_i)^2 + 4 \sqrt{(-\pi_i \alpha_i + \pi_s \alpha_s + 1)^2 (\pi_s \alpha_s (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_s \alpha_s)^2) + 2 \pi_i \alpha_i (\pi_s \alpha_s - 5) + \pi_s \alpha_s (5 \pi_s \alpha_s - 5)^2 + 5)^{1/2})} \right)\] (A.75)

The discount \(\delta\) can be any \(\delta\) high enough to satisfy the conditions of part (IV) of Lemma A.4 are met under optimal pricing. The profit induced by the optimal price is given as

\[\Pi_{IV}^* = \frac{1}{54(\pi_s \alpha_s)^2} \left( \left( (\pi_i \alpha_i)^2 - \sqrt{(-\pi_i \alpha_i + \pi_s \alpha_s + 1)^2 (\pi_s \alpha_s (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_s \alpha_s)^2) + 2 \pi_i \alpha_i (2 \pi_s \alpha_s - 1 + \pi_s \alpha_s (\pi_s \alpha_s - 4) + 1) (3 \pi_i \alpha_i - 3 \pi_s \alpha_s - 3 + (5 (\pi_i \alpha_i)^2 + 4 \sqrt{(-\pi_i \alpha_i + \pi_s \alpha_s + 1)^2 (\pi_s \alpha_s (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_s \alpha_s)^2) + 2 \pi_i \alpha_i (\pi_s \alpha_s - 5) + \pi_s \alpha_s (5 \pi_s \alpha_s - 5)^2 + 5)^{1/2})} \right) \right) \] (A.76)

To compare this with the asymptotic profit expressions for the other cases, it will be helpful to also represent the above profit as a Taylor series. There exists \(\alpha_7 > 0\) such that for \(\alpha_s > \alpha_7\), the profit above can be written as

\[\Pi_{IV}^* = \frac{1}{4 \pi_s \alpha_s} - \frac{1}{8 (\pi_s \alpha_s)^2} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k\] (A.77)

Next, suppose \(0 < v_b < v_a < 1\) is induced. By part (V) of Lemma A.4, we obtain that \(v_b\) is the most positive root of the cubic \(f_4(x) \triangleq (1 - \pi_a \alpha_a) \pi_s \alpha_s x^3 + ((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_s \alpha_s - \delta p \pi_s \alpha_s) x^2 + (p(-1 + \pi_a \alpha_a) + p(-1 + \pi_i \alpha_i)) x + p^2\) and \(v_a = (v_a - (1 - \delta)p v_b / v_b (1 - \pi_a \alpha_a) - p)\). The profit function in this case is

\[\Pi_V(p, \delta) = \delta p(1 - v_a(p, \delta)) + p(v_a(p, \delta) - v_b(p, \delta)).\] (A.78)

The optimal price, if interior, is given as

\[p_V^* = \frac{(1 - \pi_i \alpha_i)(-\pi_a \alpha_a + c_a + 1)}{4 (1 - \pi_a \alpha_a)} - \frac{((\pi_i \alpha_i - 1)^2 ((\pi_a \alpha_a - 1)^2 + c_a (3 \pi_a \alpha_a - 2 \pi_i \alpha_i - 1)))}{16 (c_a \pi_s \alpha_s (\pi_a \alpha_a - 1))} + \sum_{k=2}^{\infty} a_k \left( \frac{1}{\pi_s \alpha_s} \right)^k.\] (A.79)
The optimal discount, if interior, is given as
\[
\delta^*_V = \frac{2(1 - \pi_a \alpha_a)(1 - \pi_a \alpha_a - c_a)}{(1 - \pi_i \alpha_i)(1 - \pi_a \alpha_a + c_a)} + \frac{(1 - \pi_a \alpha_a)(\pi_a \alpha_a + c_a - 1)((\pi_a \alpha_a - 1)^2 + c_a(3\pi_a \alpha_a - 2\pi_i \alpha_i - 1))}{2c_a \delta_a \alpha_a(-\pi_a \alpha_a + c_a + 1)^2} + \sum_{k=2}^{\infty} b_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \tag{A.80}
\]

The profit induced by the interior maximizer in this case is given by
\[
\Pi^*_V = \frac{(\pi_a \alpha_a + c_a - 1)^2}{4(1 - \pi_a \alpha_a)} + \frac{(c_a(1 - \pi_i \alpha_i)^2)}{4(\pi_s \alpha_s(1 - \pi_a \alpha_a))} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \tag{A.81}
\]

Next, suppose $0 < v_b < v_a < v_p < 1$ is induced. By part (VI) of Lemma A.4, we obtain that $v_b$ is the most positive root of $f_4(x)$, $v_a = \frac{(c_a - (1-\delta)p)v_b}{v_b(1-\pi_a \alpha_a) - p}$ and $v_p = \frac{(1-\delta)p + c_p - c_a}{\pi_a \alpha_a}$. The profit function in this case is
\[
\Pi_{VI}(p, \delta) = \delta p(v_p(p, \delta) - v_a(p, \delta)) + p((1 - v_p(p, \delta)) + (v_a(p, \delta) - v_b(p, \delta))). \tag{A.82}
\]

The optimal price, if interior, is given as
\[
p^*_V = \frac{1 - c_p + c_a(c_a - c_p \pi_a \alpha_a + c_p)(c_a(1 - \pi_i \alpha_i) + (1 - \pi_a \alpha_a)(-\pi_i \alpha_i + 2c_p - 1))}{\pi_s \alpha_s(1 - c_a - \pi_a \alpha_a)^3} + \sum_{k=2}^{\infty} a_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \tag{A.83}
\]

The optimal discount, if interior, is given as
\[
\delta^*_VI = \frac{1 - c_a - \pi_a \alpha_a}{1 - c_p} \frac{2c_a(c_a^2 + (\pi_a \alpha_a - 1)(\pi_a \alpha_a + c_p^2 - 2c_p \pi_a \alpha_a))}{2c_a(\pi_a \alpha_a + c_a + 1)^3} + \sum_{k=2}^{\infty} b_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \tag{A.84}
\]

The profit induced by the interior maximizer in this case is given by
\[
\Pi_{VI}^* = \frac{1}{4} \left( \frac{c_a^2}{1 - \pi_a \alpha_a} + \frac{(c_a - c_p)^2}{\pi_a \alpha_a} - 2c_p + 1 \right) + \frac{c_a(1 - c_p)(c_a(1 - \pi_i \alpha_i) + (1 - \pi_a \alpha_a)(c_p - \pi_i \alpha_i))}{\pi_s \alpha_s(-\pi_a \alpha_a + c_a + 1)^2} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \tag{A.85}
\]

Lastly, suppose $0 < v_b < v_p < 1$ is induced. By part (VII) of Lemma A.4, we obtain that $v_b$ is the most positive root of $f_5(x) = \pi_s \alpha_s x^3 + (1 - \pi_i \alpha_i - (c_p + p)\pi_s \alpha_s)x^2 - p(2 - \pi_i \alpha_i)x + p^2$ and $v_p = \frac{c_p v_b}{v_b - p}$. The profit function in this case is
\[
\Pi_{VII}(p, \delta) = p(1 - v_b(p, \delta)). \tag{A.86}
\]
The optimal price, if interior, is given as
\[ p_{\text{II}}^* = \frac{1 - c_p}{2} - \frac{(2c_p^2(\pi_i\alpha_i(c_p + 1) - 3c_p + 1))}{(c_p + 1)^3\pi_s\alpha_s} + \sum_{k=2}^{\infty} a_k \left( \frac{1}{\pi_s\alpha_s} \right)^k. \] (A.87)

The discount \( \delta \) can be any \( \delta \) high enough to satisfy the conditions of part (VII) of Lemma A.4 are met under optimal pricing. The profit induced by the interior maximizer in this case is given by
\[ \Pi_{\text{II}}^* = \frac{1}{4}(1 - c_p)^2 - \frac{c_p(1 - c_p)(\pi_i\alpha_i(c_p + 1) - 2c_p)}{(c_p + 1)^2\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s\alpha_s} \right)^k. \] (A.88)

To find the conditions under which the interior maximizer of each case indeed induces the correct market structure, we follow the same steps as in Lemma 4 by finding the conditions under which the interior maximizer satisfies the conditions of the cases in Lemma A.4. For brevity, we omit the algebra.

To have \( 0 < v_b < v_a < v_p < 1 \) be induced by the maximizing prices given by (A.83) and (A.84), the conditions are \( |c_p - \pi_i\alpha_a| < c_a < c_p(1 - \pi_a\alpha_a) \) and \( \pi_i\alpha_i < \frac{(c_a + c_p(1 - \pi_a\alpha_a))}{1 + c_a - \pi_a\alpha_a} \). Note that \( \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a} \) from \( 0 < c_p < 1 \), \( 0 < c_a < 1 \), and \( 0 < \pi_a\alpha_a < 1 \), so that the condition of the lemma \( \pi_i\alpha_i < \frac{c_p\pi_a\alpha_a}{1 + c_a - \pi_a\alpha_a} \) implies \( \pi_i\alpha_i < \frac{(c_a + c_p(1 - \pi_a\alpha_a))}{1 + c_a - \pi_a\alpha_a} \).

To have \( 0 < v_a < v_b < v_p < 1 \) be induced by the maximizing prices given by (A.72) and (A.71), the conditions are \( c_a < \min \{c_p(1 - \pi_a\alpha_a), \pi_a\alpha_a - c_p\} \) and \( \pi_i\alpha_i < \frac{2c_p\pi_a\alpha_a}{c_p - c_a + \pi_a\alpha_a} \). Note that \( \frac{2c_p\pi_a\alpha_a}{c_p - c_a + \pi_a\alpha_a} > \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a} \) from \( 0 < \pi_a\alpha_a < 1 \) and \( 0 < c_a < c_p(1 - \pi_a\alpha_a) \), so that the condition of the lemma \( \pi_i\alpha_i < \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a} \) implies \( \pi_i\alpha_i < \frac{2c_p\pi_a\alpha_a}{c_p - c_a + \pi_a\alpha_a} \).

To have \( 0 < v_b < v_p < 1 \) be induced by the maximizing price given by (A.87), the conditions are \( \pi_i\alpha_i < \frac{2c_p}{1 + c_p} \) and \( \delta \geq \frac{-2c_a + (1 + c_p)(1 - \pi_a\alpha_a)}{1 - c_p} \). Note that the condition \( \pi_i\alpha_i < \frac{2c_p}{1 + c_p} \) holds since one of the conditions of this lemma is \( \pi_i\alpha_i < \frac{c_p}{1 + c_p} \). Then given any parameters in the parameter space satisfying the lemma, this case \( 0 < v_b < v_p < 1 \) can always be induced with any \( \delta \) large enough to satisfy these conditions.

Now we compare the maximizing profits of each case to establish the lemma. By comparing (A.73) and (A.85) with (A.59), (A.67), (A.81), (A.77), and (A.88), it follows that there exists \( \alpha_8 > 0 \) such that if \( \alpha_s > \alpha_8 \), if either \( 0 < v_b < v_a < v_p < 1 \) or \( 0 < v_a < v_b < v_p < 1 \) can be induced by their maximizing prices, then they will because they dominate the profits of the other cases. Furthermore, since (A.73) can only be achieved when \( c_a < \min \{c_p(1 - \pi_a\alpha_a), \pi_a\alpha_a - c_p\} \) and (A.85) can only be achieved when \( |c_p - \pi_a\alpha_a| < c_a < c_p(1 - \pi_a\alpha_a) \) (which doesn’t overlap with the region over which (A.73) can be achieved), it follows that \( p^* \) and \( \delta^* \) are set so that

1. if \( c_a < \min \{\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)\} \), then \( \sigma^*(v) \) is characterized by \( 0 < v_a < v_b < v_p < 1 \) under optimal pricing,
2. if \( |\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a) \), then \( \sigma^*(v) \) is characterized by \( 0 < v_b < v_a < v_p < 1 \) under optimal pricing.

When \( c_a = c_p(1 - \pi_a\alpha_a) \), then the maximal profit when inducing \( 0 < v_b < v_p < 1 \) equals the maximal profits when inducing either \( 0 < v_a < v_b < v_p < 1 \) or \( 0 < v_b < v_a < v_p < 1 \). Furthermore, for \( c_a \geq c_p(1 - \pi_a\alpha_a) \), by comparing (A.88) with (A.59), (A.67), (A.81), and (A.77), it follows that
there exists $\alpha_0 > 0$ such that if $\alpha_s > \alpha_0$, then the profit of $0 < v_b < v_p < 1$ will dominate the other cases. Also, $\delta = 1$ can be set to induce this case since $\delta = 1$ satisfies $\delta \geq \frac{-2\alpha + (1+c)p((1-\pi_a\alpha_a))}{1-c_p}$ when $c_a \geq c_p(1-\pi_a\alpha_a)$. Altogether, when $\alpha_s > \hat{\alpha}_2 = \max[\alpha_0, \alpha_9]$ and if $\pi_i \alpha_i < \min \left\{ \frac{c_p \pi_a \alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p} \right\}$, then $p^*$ and $\delta^*$ are set so that

1. if $c_a < \min \left[ \pi_a \alpha_a - c_p, c_p(1-\pi_a \alpha_a) \right]$, then $\sigma^*(v)$ is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing,

2. if $\pi_a \alpha_a - c_p < c_a < c_p(1-\pi_a \alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing, and if

3. if $c_a > c_p(1-\pi_a \alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing.

\[ \square \]

**Proof of Proposition 2:** We focus on the region in which all segments are represented under optimal pricing in the base case. Specifically, for $\alpha_s > \hat{\alpha}_1$, by Lemma 4, we have that $p^*$ is set so that if $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1-c_p)\sqrt{1-\pi_a \alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing. By Lemma 5, for $\alpha_s > \hat{\alpha}_2$, when patching rights are priced under the same parameter region, there are two cases: either $0 < v_a < v_b < v_p < 1$ is induced or $0 < v_b < v_a < v_p < 1$ is induced. Specifically, $p^*$ and $\delta^*$ are set so that

(i) if $c_a < \min \left[ \pi_a \alpha_a - c_p, c_p(1-\pi_a \alpha_a) \right]$, then $\sigma^*(v)$ is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing, and

(ii) if $\pi_a \alpha_a - c_p < c_a < c_p(1-\pi_a \alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing.

In either case, since $c_p(1-\pi_a \alpha_a) > 1-\pi_a \alpha_a - (1-c_p)\sqrt{1-\pi_a \alpha_a}$ using the assumptions that $0 < c_p < 1$ and $0 < \pi_a \alpha_a < 1$, we have that $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1-c_p)\sqrt{1-\pi_a \alpha_a}$ is a subset of the union of the regions $c_a < \min \left[ \pi_a \alpha_a - c_p, c_p(1-\pi_a \alpha_a) \right]$ and $\pi_a \alpha_a - c_p < c_a < c_p(1-\pi_a \alpha_a)$. Moreover, the intersection of $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1-c_p)\sqrt{1-\pi_a \alpha_a}$ with either $c_a < \min \left[ \pi_a \alpha_a - c_p, c_p(1-\pi_a \alpha_a) \right]$ or $\pi_a \alpha_a - c_p < c_a < c_p(1-\pi_a \alpha_a)$ is non-empty.

In the first case, the induced profit under optimal pricing in the status quo case when patching rights aren’t priced is given by (A.18), and induced profit when patching rights are priced is given by (A.73). The fractional increase in profit is given by

\[
\frac{\Pi_p - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1 - \pi_a \alpha_a)(c_a - c_p + \pi_a \alpha_a)^2}{\pi_a \alpha_a (1 - c_a - \pi_a \alpha_a)^2} + \left( \pi_a \alpha_a - c_a + c_p \right)^2 M - 4\pi_a \alpha_a (1 - \pi_a \alpha_a)(1 - \pi_a \alpha_a - c_a) (-\pi_a \alpha_a + c_a + c_p) (\pi_a \alpha_a ((c_a + 2)c_p + c_a) + c_a(1 - c_p)(c_p - c_a) - 2c_p(\pi_a \alpha_a)^2) (c_a - c_p(1 - \pi_a \alpha_a)) + 4\pi_i \alpha_i (\pi_a \alpha_a - 1)(-\pi_a \alpha_a + c_a + 1)(-\pi_a \alpha_a + c_a - c_p)]^2 (c_a + c_p + 1) - \pi_a \alpha_a (c_a - 1)(c_a + c_p) + c_a(c_a - c_p)^2 (c_a + c_p(\pi_a \alpha_a - 1)) \right) \left( \pi_a \alpha_a \pi_s \alpha_s (\pi_a \alpha_a + c_a + 1)^2 (\pi_a \alpha_a + c_a - 1)^3 (\pi_a \alpha_a - c_a + c_p)^2 \right)^{-1} + K_h,
\]

(A.89)
where

\[ M = 4c_a(\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_a - c_p)\left((\pi_a\alpha_a + c_a)(\pi_a\alpha_a(2 - \pi_a\alpha_a) + c_a(\pi_a\alpha_a - 2) + 2c_p(\pi_a\alpha_a - 1)^2 - \pi_a\alpha_a)\right) \]  

(A.90)

and \( K_h \) is a term of order \( O\left(\frac{1}{(\pi_a\alpha_a)^2}\right) \).

Moreover, the reduction in the size of the unpatched population when patching rights are priced is given by

\[
\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR) = \left(\frac{c_a^2(\pi_a\alpha_a - 2) + c_a(\pi_a\alpha_a + c_p(2 - 3\pi_a\alpha_a)) + \pi_a\alpha_a(\pi_a\alpha_a - 1)(c_p - \pi_a\alpha_a)}{\pi_a\alpha_a(-\pi_a\alpha_a + c_a + 1)(\pi_a\alpha_a - c_a + c_p)} \sum_{k=2}^{\infty} d_k \left(\frac{1}{\pi_a\alpha_a}\right)^k \right), \]  

(A.91)

which is strictly positive when \( c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)] \).

Similarly, in the second case, the induced profit under optimal pricing in the status quo case when patching rights aren’t priced is given by (A.18), and induced profit when patching rights are priced is given by (A.85). The fractional increase in profit is given by

\[
\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1 - \pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1 - c_a - \pi_a\alpha_a)^2} + \left(M + (\pi_a\alpha_a + c_a + 1)(4\pi_a\alpha_a(\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_a - c_p)
(c_a + c_p(\pi_a\alpha_a - 1)))\right) \left(\pi_a\alpha_a\pi_a\alpha_a(-\pi_a\alpha_a + c_a + 1)^2(\pi_a\alpha_a + c_a - 1)\right)^{-1} K_k \), \]  

(A.92)

where \( K_k \) is a term of order \( O\left(\frac{1}{(\pi_a\alpha_a)^2}\right) \).

Moreover, the reduction in the size of the unpatched population when patching rights are priced is given by

\[
\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR) = \left(\frac{1 - \pi_a\alpha_a}{\pi_a\alpha_a(-\pi_a\alpha_a + c_a + 1)} \sum_{k=2}^{\infty} d_k \left(\frac{1}{\pi_a\alpha_a}\right)^k \right), \]  

(A.93)

which is strictly positive when \( |\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a) \).

In both cases, pricing patching rights increases the vendor’s profit and reduces the equilibrium size of the unpatched population as compared to when patching rights aren’t priced. □

**Proof of Corollary 1:** Differentiating (A.89) and (A.92) with respect to \( \alpha_i \), we have that in either case, the profit difference between pricing patching rights and the status quo case decreases in \( \alpha_i \).

In the first case when \( 0 < v_a < v_b < v_p < 1 \) is induced under pricing patching rights, the derivative of the profit difference with respect to \( \alpha_i \) is given as

\[
\frac{d}{d\alpha_i} (\Pi_P - \Pi_{SQ}) = \pi_i \left(-c_a^3 + c_a^2(2c_p + 1) + c_a(\pi_a\alpha_a(\pi_a\alpha_a - 1) - c_p^2 + c_p\pi_a\alpha_a) + c_p(\pi_a\alpha_a - 1)(c_p - \pi_a\alpha_a)\right) + 
\pi_a\alpha_a(-\pi_a\alpha_a + c_a + 1)(-\pi_a\alpha_a + \pi_a\alpha_a) \sum_{k=2}^{\infty} d_k \left(\frac{1}{\pi_a\alpha_a}\right)^k , \]  

(A.94)
which is negative when \(c_a < \min \{\pi_a \alpha_a - c_p, c_p(1 - \alpha_a)\}\).

In the second case when \(0 < v_b < v_a < v_p < 1\) is induced under pricing patching rights, the derivative of the profit difference with respect to \(\alpha_i\) is given as

\[
\frac{d}{d\alpha_i}(\Pi_P - \Pi_{SQ}) = -\frac{c_a \pi_i(\pi_a \alpha_a + c_a - c_p)}{\pi_a \alpha_a (-\pi_a \alpha_a + c_a + 1)} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_a \alpha_a} \right)^k,
\]

which is negative when \(|\pi_a \alpha_a - c_p| < c_a < c_p(1 - \alpha_a)\). □

**Proof of Corollary 2:** Following the proof of Proposition 2, the reduction in the size of the unpatched population under the conditions of the corollary is given either by (A.91) or (A.93), depending on the parameters (with the conditions also given in the proof of Proposition 2).

In the first case, \(\frac{d}{\partial c_p} [\hat{u}(\sigma|SQ) - \hat{u}(\sigma|PPR)] = -\frac{2\pi_a \alpha_a (c_a - \pi_a \alpha_a)}{(c_a - c_p - \pi_a \alpha_a)^2} + \sum_{k=2}^{\infty} d_k \left( \frac{1}{\pi_a \alpha_a} \right)^k\). Using \(c_a < \pi_a \alpha_a - c_p\) (one of the conditions for this case, from Lemma 5), there exists \(\alpha_s\) such that \(\alpha_s > \hat{\alpha}_s\) implies that \(\frac{d}{\partial c_p} [\hat{u}(\sigma|SQ) - \hat{u}(\sigma|PPR)] < 0\).

Also,

\[
\frac{d}{d(\pi_a \alpha_a)} \left[ \hat{u}(\sigma|SQ) - \hat{u}(\sigma|PPR) \right] = \frac{1}{\pi_a \alpha_a} \left( 1 + (2(c_p - c_a)(c_a^2 - (1 + 2c_a(1 + c_a))c_p) + 4(c_a - c_p) \right)
\]

\[
(c_a^2 - (1+c_a)c_p)\pi_a \alpha_a - 2(c_a^2 - c_a c_p + c_p^2)(\pi_a \alpha_a)^2((1+c_a - \pi_a \alpha_a)^2(-c_a + c_p + \pi_a \alpha_a)^2)^{-1}) + \sum_{k=2}^{\infty} d_k \left( \frac{1}{\pi_a \alpha_a} \right)^k.
\]

There exists \(\hat{\alpha}_s\) such that \(\alpha_s > \hat{\alpha}_s\) implies that this is strictly positive, since \(c_a < \pi_a \alpha_a - c_p\) in this parameter region and \(\pi_a \alpha_a + c_a < 1\) from our initial model assumptions.

In the second case, \(\frac{d}{\partial c_p} [\hat{u}(\sigma|SQ) - \hat{u}(\sigma|PPR)] = -\frac{(1-\pi_a \alpha_a)}{1+c_a - \pi_a \alpha_a} + \sum_{k=2}^{\infty} d_k \left( \frac{1}{\pi_a \alpha_a} \right)^k\). There exists \(\alpha_s\) such that \(\alpha_s > \hat{\alpha}_s\) implies that \(\frac{d}{\partial c_p} [\hat{u}(\sigma|SQ) - \hat{u}(\sigma|PPR)]\) is strictly negative.

Also, \(\frac{d}{d(\pi_a \alpha_a)} \left[ \hat{u}(\sigma|SQ) - \hat{u}(\sigma|PPR) \right] = -\frac{-c_a^2 + c_a(1+c_p - 2\pi_a \alpha_a) + (1 - \pi_a \alpha_a)^2}{(1+c_a - \pi_a \alpha_a)^2} + \sum_{k=2}^{\infty} d_k \left( \frac{1}{\pi_a \alpha_a} \right)^k\).

Note that \(-c_a^2 + c_a(1+c_p - 2\pi_a \alpha_a) + (1 - \pi_a \alpha_a)^2 > 0\), using \(c_a < c_p(1 - \pi_a \alpha_a)\) and \(c_a > \pi_a \alpha_a - c_p\) (from Lemma 5). Therefore, for sufficiently large \(\pi_a \alpha_a\), \(\frac{d}{d(\pi_a \alpha_a)} [\hat{u}(\sigma|SQ) - \hat{u}(\sigma|PPR)] > 0\). □

**Proof of Proposition 3:** Under the conditions of Proposition 3, when patching rights are priced, there are two cases in equilibrium by Lemma 5: either \(0 < v_a < v_b < v_p < 1\) is induced or \(0 < v_b < v_a < v_p < 1\) is induced. In either case, we show that the optimal discount \(\delta^* < 1\). It follows that it suffices to show that the price of the automated patching option can be higher when patching rights are priced than compared to the status quo case for low \(c_a\).

In the first region when \(c_a < \min \{\pi_a \alpha_a - c_p, c_p(1 - \pi_a \alpha_a)\}\), then \(\sigma^*(v)\) is characterized by \(0 < v_a < v_b < v_p < 1\) under optimal pricing. The discount when pricing patching rights is given in (A.71). Under the conditions \(c_a < \min \{\pi_a \alpha_a - c_p, c_p(1 - \pi_a \alpha_a)\}\), there exists \(\alpha_3 > 0\) such that \(\alpha_s > \hat{\alpha}_3\) implies the expression for \(\delta^*\) given in (A.71) is bounded above by 1. To prove the lemma for this case, it suffices to show that the price of the automated patching option can be greater than the common status quo price across both options.
The price of the automated patching option when patching rights aren’t priced is given in (A.17). Using (A.72) and (A.71), when patching rights are priced, the automated patching option has price

\[ \delta^* p^* = \frac{1}{2} (1 - c_\alpha - \pi_a \alpha_a) + \sum_{k=2}^{\infty} a_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \] (A.97)

Comparing (A.97) with (A.17), the price of the automated patching option when patching rights are priced is greater than the common price in the base case when \( c_\alpha < \frac{\pi_a \alpha_a (2 \pi_a \alpha - 1)}{2 \pi_a \alpha + \pi_a \alpha - 3} - \pi_a \alpha_a. \) The intersection of this with the parameter region of this case is non-empty when \( \pi_a \alpha_a < \frac{1 + \pi_i \alpha_i}{2}. \)

The argument for the second case (in which \( 0 < v_b < v_a < v_p < 1 \) is induced in equilibrium) is similar and is omitted for brevity. We find that \( \delta^* p^* \) is greater than the price of the status quo case if \( c_\alpha < \frac{(1 - \pi_a \alpha_a) (\pi_a \alpha - 1)}{4 \pi_a \alpha + \pi_a \alpha + 5}. \) The intersection of this with the parameter region of this case, namely with the condition \( c_\alpha > \pi_a \alpha_a - c_p, \) is non-empty also when \( \pi_a \alpha_a < \frac{1 + \pi_i \alpha_i}{2}. \)

**Proof of Proposition 4:** Using Lemma 4, the status quo pricing induces \( 0 < v_b < v_a < v_p < 1 \) market structure. Using Lemma 5, the vendor induces either \( 0 < v_a < v_b < v_p < 1 \) or \( 0 < v_b < v_a < v_p < 1 \) under PPR. Using the definition of social welfare in (35), the social welfare under status quo pricing is given by

\[ W_{SQ} = \int_{v_a(p^*)}^{1} v dv - \left( \int_{v_a(p^*)}^{v_p(p^*)} c_a + \pi_a \alpha_a v dv + \int_{v_a(p^*)}^{v_b(p^*)} ((v_a(p^*) - v_b(p^*)) \pi_s \alpha_s + \pi_i \alpha_i) v dv + c_p (1 - v_b(p^*)) \right), \] (A.98)

where \( p^* \) is the equilibrium price in the status quo case, given in (A.17). Using its asymptotic expansion, this can be written as

\[ W_{SQ} = \frac{1}{8} (\pi_a \alpha_a - \frac{3 c_a^2}{\pi_a \alpha_a - 1} + \frac{4 (c_a - c_p)^2}{\pi_a \alpha_a} + 2 c_a - 8 c_p + 3) + \sum_{k=1}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \] (A.99)

When pricing patching rights, one of the two cases will arise. In the first case, the social welfare is given as

\[ W_P = \int_{v_b(p^*, \delta^*)}^{1} v dv - \left( \int_{v_b(p^*, \delta^*)}^{v_p(p^*, \delta^*)} c_a + \pi_a \alpha_a v dv + \right. \]

\[ \left. \int_{v_b(p^*, \delta^*)}^{v_p(p^*, \delta^*)} ((v_a(p^*, \delta^*) - v_b(p^*, \delta^*)) \pi_s \alpha_s + \pi_i \alpha_i) v dv + c_p (1 - v_b(p^*, \delta^*)) \right), \] (A.100)

where \( \delta^* \) and \( p^* \) are given in (A.71) and (A.72) respectively. In the second case, the social welfare is given as

\[ W_P = \int_{v_b(p^*, \delta^*)}^{1} v dv - \left( \int_{v_b(p^*, \delta^*)}^{v_p(p^*, \delta^*)} c_a + \pi_a \alpha_a v dv + \right. \]

\[ \left. \int_{v_b(p^*, \delta^*)}^{v_a(p^*, \delta^*)} ((v_a(p^*, \delta^*) - v_b(p^*, \delta^*)) \pi_s \alpha_s + \pi_i \alpha_i) v dv + c_p (1 - v_b(p^*, \delta^*)) \right), \] (A.101)

where \( \delta^* \) and \( p^* \) are given in (A.84) and (A.83) respectively.

A.36
In both cases, the asymptotic expression for the equilibrium welfare is given as
\[
W_P = \frac{3}{8} \left( \frac{c_a^2}{1 - \pi_a \alpha_a} + \frac{(c_a - c_p)^2}{\pi_a \alpha_a} - 2c_p + 1 \right) + \sum_{k=1}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. 
\] (A.102)

Comparing (A.99) and (A.102) reveals that pricing patching rights in this region of the parameter space hurts welfare.

We further characterize which losses drive this result. We define the total attack-related losses under status quo pricing with
\[
SL_{SQ} \triangleq \int_V 1_{\{\sigma^*(v) = (B,NP)|SQ\}} (\pi_s \alpha_s u(\sigma^*|SQ) + \pi_i \alpha_i) \, dv, 
\] (A.103)
the total costs associated with automated patching under status quo pricing with
\[
AL_{SQ} \triangleq \int_V 1_{\{\sigma^*(v) = (B,AP)|SQ\}} c_a + \pi_a \alpha_a \, dv, 
\] (A.104)
and the total costs associated with standard patching under status quo pricing with
\[
PL_{SQ} \triangleq \int_V 1_{\{\sigma^*(v) = (B,P)|SQ\}} c_p \, dv. 
\] (A.105)

Specifically, the loss measures when \(0 < v_a < v_b < v_p < 1\) is induced in equilibrium under status quo pricing are given as follows.

\[
SL_{SQ} = - \left( (\pi_a \alpha_a (\pi_a \alpha_a - 1) + c_a (\pi_a \alpha_a - 2)) ((\pi_a \alpha_a - 1)(\pi_a \alpha_a - \pi_i \alpha_i) + c_a (\pi_a \alpha_a + \pi_i \alpha_i - 2)) \right) \left( 2(\pi_a \alpha_a (\pi_a \alpha_a - 1)(-\pi_a \alpha_a + c_a + 1) \right)^{-1} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k, 
\] (A.106)

\[
AL_{SQ} = \left( \pi_a \alpha_a (\pi_a \alpha_a - 1) + c_a (\pi_a \alpha_a - 2)) ((\pi_a \alpha_a - 1)^3 (\pi_a \alpha_a - \pi_i \alpha_i) + c_a^3 (3\pi_a \alpha_a + \pi_i \alpha_i - 4) + c_a^2 (\pi_a \alpha_a - 1) (3\pi_a \alpha_a - \pi_i \alpha_i - 2) + c_a (\pi_a \alpha_a - 1) \frac{2}{2(\pi_a \alpha_a (\pi_a \alpha_a - 1)(-\pi_a \alpha_a + c_a + 1)) \left( (\pi_a \alpha_a)^2 + c_a (\pi_a \alpha_a (c_a - 2c_p - 1) - 2c_a + 2c_p) (c_a (\pi_a \alpha_a - 2) + (\pi_a \alpha_a - 1)(\pi_a \alpha_a + 2c_p)) \right) \left( 8 (\pi_a \alpha_a (\pi_a \alpha_a - 1)^2) \right)^{-1} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k, 
\] (A.107)

and
\[
PL_{SQ} = \frac{c_p (\pi_a \alpha_a + c_a - c_p) \, dv + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. 
\] (A.108)
Similarly, we define the total attack-related losses under PPR with

$$SL_P \triangleq \int_{\mathcal{V}} \mathbb{I}_{\{\sigma^*(v)=(B,NP)|PPR\}}(\pi_s \alpha_s u(\sigma^*|PPR) + \pi_i \alpha_i) \, dv,$$

the total costs associated with automated patching under PPR with

$$AL_P \triangleq \int_{\mathcal{V}} \mathbb{I}_{\{\sigma^*(v)=(B,AP)|PPR\}} c_a + \pi_a \alpha_a \, dv,$$

and the total costs associated with standard patching under PPR with

$$PL_P \triangleq \int_{\mathcal{V}} \mathbb{I}_{\{\sigma^*(v)=(B,P)|PPR\}} c_p \, dv.$$

In the first case, in which $0 < v_a < v_b < v_p < 1$ is induced under PPR, the loss measures in equilibrium are given as follows.

$$SL_P = \frac{c_p(\pi_i \alpha_i (\pi_a \alpha_a - c_a + c_p) - 2\pi_a \alpha_a c_p)}{\pi_s \alpha_s (-\pi_a \alpha_a + c_a - c_p)} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k,$$  \hspace{1cm} (A.109)

$$AL_P = \left( c_p(2c_p \pi_a \alpha_a ((\pi_a \alpha_a)^2 + c_a^2 - \pi_a \alpha_a (2c_a + c_p) - 3c_a c_p + 2c_p^2) + \pi_i \alpha_i (\pi_a \alpha_a + c_a - c_p) \right) \left( \pi_s \alpha_s (-\pi_a \alpha_a + c_a - c_p) \right)$$

$$+ \frac{c_p(\pi_a \alpha_a + c_a - c_p)}{2\pi_a \alpha_a} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k,$$  \hspace{1cm} (A.110)

and

$$PL_P = \left( c_p \pi_a \alpha_a (\pi_a \alpha_a + c_a + c_p) (\pi_i \alpha_i (\pi_a \alpha_a + c_a + c_p) + c_p (\pi_a \alpha_a - c_a + c_p)) \right) \left( \pi_s \alpha_s (\pi_a \alpha_a - c_a + c_p) \right)^{-1}$$

$$+ \frac{(c_a + c_p(\pi_a \alpha_a - 1))(c_a(2\pi_a \alpha_a - 3) + (\pi_a \alpha_a - 1)(2\pi_a \alpha_a + c_p))}{8\pi_a \alpha_a (\pi_a \alpha_a - 1)^2} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k.$$  \hspace{1cm} (A.111)

Comparing these measures across status quo pricing and pricing patching rights, we have that $AL_P > AL_{SQ}$, $PL_{SQ} > PL_P$, and if $\frac{4c_p^2 \pi_a \alpha_a}{-c_a + c_p + \pi_a \alpha_a} - \frac{(c_a(2-\pi_a \alpha_a) + \pi_a \alpha_a(1-\pi_a \alpha_a))^2}{(1+c_a-\pi_a \alpha_a)(1-\pi_a \alpha_a)} > 0$, then $SL_P > SL_{SQ}$. Otherwise, $SL_P \leq SL_{SQ}$.

In the second case, in which $0 < v_b < v_a < v_p < 1$ is induced under PPR, the loss measures in equilibrium are given as follows.

$$SL_P = -\frac{(c_a - \pi_a \alpha_a c_p + c_p)(-\pi_i \alpha_i (-\pi_a \alpha_a + c_a + 1) + c_a - \pi_a \alpha_a c_p + c_p)}{2(\pi_s \alpha_s (\pi_a \alpha_a - 1) (-\pi_a \alpha_a + c_a + 1))} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k,$$  \hspace{1cm} (A.112)
Proof of Proposition 5: When \( 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a} < c_a < c_p(1 - \pi_a \alpha_a) \), then by Lemmas 4 and 5, status quo pricing induces \( 0 < v_b < v_a < v_p < 1 \) while pricing patching rights induces either \( 0 < v_a < v_b < v_p < 1 \) or \( 0 < v_b < v_a < v_p < 1 \).

So the welfare expression under PPR is the same as in the proof of Proposition 4. Now however, the welfare under status quo pricing is given as

\[
W_{SQ} = \int_{v_b(p^*)}^{v_p} vdv - \left( \int_{v_b(p^*)}^{v_p} \left( (v_p - v_b(p^*))\pi_s \alpha_s + \pi_i \alpha_i \right)vdv + c_p(1 - v_p(p^*)) \right),
\]

where \( p^* \) is the equilibrium price in the status quo case, given in (A.19). Using its asymptotic expansion, this can be written as

\[
W_{SQ} = \frac{3}{8}(1 - c_p)^2 + \sum_{k=1}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k.
\]

(A.119)

Comparing (A.119) to (A.102) reveals that, for sufficiently high \( \pi_s \alpha_s \), pricing patching rights in this region of the parameter space improves welfare. Defining the specific losses for when \( 0 < v_b < v_p < 1 \) arises under status quo pricing, we derive the following loss measures.

\[
SL_{SQ} = -\frac{c_p(\pi_i \alpha_i(c_p + 1) - 2c_p)}{(c_p + 1)\pi_s \alpha_s} + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k,
\]

\[
AL_{SQ} = 0,
\]

(A.120)
and

\[ PL_{SQ} = \frac{c_p (\pi_i \alpha_i (c_p + 1) (c_p^2 + 1) + 2c_p (-2c_p^2 + c_p - 1))}{(c_p + 1)^3 \pi_s \alpha_s} - \frac{1}{2} (c_p - 1)c_p + \sum_{k=2}^{\infty} c_k \left( \frac{1}{\pi_s \alpha_s} \right)^k. \]  

(A.122)

Comparing these with their respective measures given in the proof of Proposition 4, we find

\[ SL_P < SL_{SQ}, \quad PL_P < PL_{SQ}, \quad \text{and} \quad AL_P > AL_{SQ}. \]

\[ \square \]

References