Online Appendix

Pricing Ancillary Service Subscriptions

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All technical proofs for the paper titled "Pricing Ancillary Service Subscriptions" are provided in this Online Appendix.

Proof of Proposition 1. Under the i.i.d. Gumbel distribution for ϵ_k's, we have a closed-form expression for π_k^S and π_k^T as follows (see, e.g., Anderson et al. 1992):

$$\begin{aligned}\pi_k^S &= \lambda_k \mu E\big[\log\big(u_{k0} + \exp(u_{mk} - p + W_k \cdot u_{ak})\big)\big] - S \\ &= \lambda_k \mu \cdot \big(w_k \log(u_{k0} + \exp(u_{mk} - p + u_{ak})) + (1 - w_k)\log(u_{k0} + \exp(u_{mk} - p))\big) - S, \\ \pi_k^T &= \lambda_k \cdot \mu E\big[\log\big(u_{k0} + \exp(u_{mk} - p + W_k \cdot (u_{ak} - T))\big)\big] \\ &= \lambda_k \mu \cdot \big(w_k \log(u_{k0} + \exp(u_{mk} - p + u_{ak} - T)) + (1 - w_k)\log(u_{k0} + \exp(u_{mk} - p)\big].\end{aligned}$$

By the utility equivalence between with and without service subscription, a type-k customer will subscribe to the ancillary service if and only if $\pi_k^S \geq \pi_k^T$ or equivalently the subscription price S is less than $S_k^o(p;w)$, where $S_k^o(p;w)$ is defined as follows:

$$S_k^o(p;w) := w_k \cdot \lambda_k \mu \log\left(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\right) = w_k \cdot S_k^o(p).$$

Note that $S_k^o(p)$ can be viewed as the willingness to pay for service subscription when customers always consume the main product and ancillary service together. □

Proof of Lemma 1. Let function $f_1(\alpha)$ be the following

$$f_1(\alpha) := \log\left(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p - (1-\alpha)T)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\right) + \frac{(1-\alpha)T/\mu \cdot \exp((u_k - p - (1-\alpha)T)/\mu)}{\exp(u_{k0}) + \exp((u_k - p - (1-\alpha)T)/\mu)}.$$

Consider its first-order derivative as follows:

$$\frac{\partial f_1(\alpha)}{\partial \alpha} = (1-\alpha)(T/\mu)^2 \cdot \frac{\exp(u_{k0}/\mu) \cdot \exp((u_k - p - (1-\alpha)T)/\mu)}{(\exp(u_{k0}/\mu) + \exp((u_k - p - (1-\alpha)T)/\mu))^2} \geq 0.$$

Then, $f_1(0) \leq f_1(1)$, i.e.,

$$\frac{T/\mu \cdot \exp((u_k - p - T)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)} \leq \log\left(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\right).$$

Similarly, let function $f_2(\alpha)$ be the following

$$f_2(\alpha) := \log\left(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - (1-\alpha)T)/\mu)}\right) + \frac{\alpha T/\mu \cdot \exp((u_k - p - (1-\alpha)T)/\mu)}{\exp(u_{k0}) + \exp((u_k - p - (1-\alpha)T)/\mu)}.$$

Consider its first-order derivative as follows:

$$\frac{\partial f_2(\alpha)}{\partial \alpha} = \alpha (T/\mu)^2 \cdot \frac{\exp(u_{k0}/\mu)\cdot \exp((u_k - p - (1-\alpha)T)/\mu)}{(\exp(u_{k0}/\mu) + \exp((u_k - p - (1-\alpha)T)/\mu))^2} \geq 0.$$

Then, $f_2(0) \leq f_2(1)$, i.e.,

$$\log\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big) \leq \frac{T/\mu \cdot \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}.$$

It completes the proof. □

Proof of Proposition 2. By Lemma 1, Part (a) holds immediately. For part (b), consider the derivative of $S_k^o(p)$ w.r.t. T,

$$\frac{\partial S_k^o(p)}{\partial T} = \frac{\lambda_k \cdot \exp((u_k - p - T)/\mu)}{\exp((u_{k0} - T)/\mu) + \exp((u_k - p - T)/\mu)} \geq 0,$$

and $\partial S_k^o(p)/\partial T \leq \lambda_k$. Consider the derivative of $S_k^o(p)$ w.r.t. u_{k0} as follows:

$$\frac{\partial}{\partial u_{k0}}\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big) = -\frac{(\exp((u_k - p)/\mu) - \exp((u_k - p - T)/\mu))\cdot \exp(u_{k0}/\mu)}{\mu(\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu))^2} \leq 0,$$

then $S_k^o(p)$ is decreasing in the utility of outside option u_{k0}. Similarly,

$$\frac{\partial}{\partial p}\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big) = -\frac{\exp(u_{k0}/\mu)\cdot \big(\exp((u_k - p)/\mu) - \exp((u_k - p - T)/\mu)\big)}{\mu(\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu))^2} \leq 0,$$

For the monotonicity of $S_k^o(p)$ w.r.t. μ, consider the derivative as follows:

$$\begin{aligned}\Delta \ :=& \frac{\partial}{\partial \mu}\mu \log\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big) = \log\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big)\\ &- \frac{u_{k0}\cdot \exp(u_{k0}/\mu) + (u_k - p)\cdot \exp((u_k - p)/\mu)}{\mu\big(\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)\big)} + \frac{u_{k0}\cdot \exp(u_{k0}/\mu) + (u_k - p)\cdot \exp((u_k - p - T)/\mu)}{\mu\big(\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)\big)}.\end{aligned}$$

The second and third terms together can be rewritten as follows:

$$-\frac{T/\mu \cdot \exp((u_k - p - T)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)} + \frac{(u_{k0} - (u - p))\cdot \exp(u_{k0}/\mu)\cdot \big(\exp((u_k - p)/\mu) - \exp((u_k - p - T)/\mu)\big)}{\mu\big(\exp((u_{k0} - T)/\mu) + \exp((u_k - p - T)/\mu)\big)\cdot \big(\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)\big)}.$$

By Lemma 1, it holds that

$$\log\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big) \geq \frac{T/\mu \cdot \exp((u_k - p - T)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}.$$

If $u_k - p \leq u_{k0}$, then

$$\Delta \geq \frac{(u_{k0} - (u_k - p))\cdot (1 - \exp(-T/\mu))\cdot \exp(u_{k0}/\mu)\cdot \exp((u_k - p)/\mu)}{\mu\big(\exp((u_{k0} - T)/\mu) + \exp((u_k - p - T)/\mu)\big)\cdot \big(\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)\big)} \geq 0.$$

Thus, $S_k^o(p)$ is increasing in μ if $u_k - p \leq u_{k0}$.

By Lemma 1,

$$\log\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big) = \log\Big(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\Big)$$

$$\leq \frac{T/\mu \cdot \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}.$$

If $u_k - p - T \geq u_{k0}$, then

$$\begin{aligned}\Delta \leq\ & \frac{T/\mu \cdot \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)} - \frac{T/\mu \cdot \exp((u_k - p - T)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)} \\ & + \frac{(u_{k0} - (u - p)) \cdot \exp(u_{k0}/\mu) \cdot \big(\exp((u_k - p)/\mu) - \exp((u_k - p - T)/\mu)\big)}{\mu\big(\exp((u_{k0} - T)/\mu) + \exp((u_k - p - T)/\mu)\big) \cdot \big(\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)\big)} \\ =\ & \frac{(u_{k0} - (u_k - p - T)) \cdot \exp(u_{k0}/\mu) \cdot \big(\exp((u_k - p)/\mu) - \exp((u_k - p - T)/\mu)\big)}{\mu\big(\exp((u_{k0} - T)/\mu) + \exp((u_k - p - T)/\mu)\big) \cdot \big(\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)\big)} \leq 0.\end{aligned}$$

Thus, $S_k^o(p)$ is decreasing in μ if $u_k - p - T \geq u_{k0}$.

We consider the limits of $S_k^o(p)$.

$$\begin{aligned}\lim_{\mu\to\infty} S_k^o(p) &= \lambda \cdot \frac{\lim_{\mu\to\infty} \frac{\partial}{\partial\mu} \cdot \log\left(\frac{\exp(u_{k0}/\mu)+\exp((u_k-p)/\mu)}{\exp(u_{k0}/\mu)+\exp((u_k-p-T)/\mu)}\right)}{\lim_{\mu\to\infty} -1/\mu^2} \\ &= \lambda \cdot \lim_{\mu\to\infty} \frac{\exp(u_{k0}/\mu) \cdot u_{k0} + \exp((u_k - p)/\mu) \cdot (u_k - p)}{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)} \\ &\quad - \frac{\exp(u_{k0}/\mu) \cdot u_{k0} + \exp((u_k - p - T)/\mu) \cdot (u_k - p - T)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)} \\ &= \lambda \cdot \left(\frac{u_{k0} + u_k - p}{2} - \frac{u_{k0} + u_k - p - T}{2}\right) = \lambda T/2.\end{aligned}$$

If $u_k - p \leq u_{k0}$, then

$$\lim_{\mu\to 0} S^o = \lambda \lim_{\mu\to 0} \mu \cdot \log\left(\frac{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\right) = \lambda \lim_{\mu\to 0} \mu \cdot \log\left(\frac{1 + \exp((u_k - p - u_{k0})/\mu)}{1 + \exp((u_k - p - u_{k0} - T)/\mu)}\right) = 0.$$

If $u_k - p > u_{k0}$, similarly, we have

$$\begin{aligned}\lim_{\mu\to 0} S^o &= \lambda \cdot \lim_{\mu\to\infty} \frac{\exp(u_{k0}/\mu) \cdot u_{k0} + \exp((u_k - p)/\mu) \cdot (u_k - p)}{\exp(u_{k0}/\mu) + \exp((u_k - p)/\mu)} - \frac{\exp(u_{k0}/\mu) \cdot u_{k0} + \exp((u_k - p - T)/\mu) \cdot (u_k - p - T)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)} \\ &= \lambda \cdot \left(u_k - p - \lim_{\mu\to 0} \frac{\exp(u_{k0}/\mu) \cdot u_{k0} + \exp((u_k - p - T)/\mu) \cdot (u_k - p - T)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)}\right).\end{aligned}$$

If $u_k - p - T \geq u_{k0}$, then

$$\lim_{\mu\to 0} \frac{\exp(u_{k0}/\mu) \cdot u_{k0} + \exp((u_k - p - T)/\mu) \cdot (u_k - p - T)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)} = u_k - p - T;$$

if $u_k - p - T < u_{k0}$, then

$$\lim_{\mu\to 0} \frac{\exp(u_{k0}/\mu) \cdot u_{k0} + \exp((u_k - p - T)/\mu) \cdot (u_k - p - T)}{\exp(u_{k0}/\mu) + \exp((u_k - p - T)/\mu)} = u_{k0}.$$

Therefore, if $u_k - p - T \geq u_{k0}$, we have $\lim_{\mu\to 0} S_k^o(p) = \lambda_k T$; if $u_k - p - T < u_{k0}$, we have $\lim_{\mu\to 0} S_k^o(p) = \lambda(u_k - p - u_{k0})$. □

Proof of Theorem 1. We will show that for any pair (i,j), the equation on the willingness to pay for type-i and type-j customers, i.e., $S_i^o(p) = S_j^o(p)$, has at most one finite solution w.r.t. p. Consider the derivative for $S_i^o(p)$ w.r.t. p as follows:

$$\frac{\partial S_i^o(p)}{\partial p} = -\gamma_i \cdot \frac{\exp(u_i - p) - \exp(u_i - p - T)}{(1 + \exp(u_i - p)) \cdot (1 + \exp(u_i - p - T))}.$$

Then, after some algebra, the equation $\partial S_i^o(p)/\partial p = \partial S_j^o(p)/\partial p$ can be rewritten as follows:

$$\begin{aligned}\big(\gamma_i \exp(u_j) - \gamma_j \exp(u_i)\big) \cdot \exp(u_i + u_j - T) \cdot \exp(-2p) + (\gamma_i - \gamma_j) \cdot \exp(u_i + u_j) \\ \cdot (1 + \exp(-T)) \exp(-p) + \big(\gamma_i \exp(u_i) - \gamma_j \exp(u_j)\big) = 0.\end{aligned}$$

The left-hand side (LHS) is a quadratic function in terms of $\exp(-p)$, so the above equation has at most two distinct solutions w.r.t. p, denoted by $p_1 \le p_2$. Therefore, $S_i^o(p) = S_j^o(p)$ has at most *three* distinct solutions w.r.t. p. Next, we will show it has at most *one* finite solution.

Without loss of generality, suppose $\gamma_i \exp(u_j) - \gamma_j \exp(u_i) \ge 0$. Then, $S_i^o(p) - S_j^o(p)$ is increasing in p for $p \le p_1$; then is decreasing in p for $p_1 \le p \le p_2$; and thereafter is increasing in p for $p \ge p_2$. Consider $S_i^o(p) = S_j^o(p)$,

$$\gamma_i \cdot \log\left(\frac{1 + \exp(u_i - p)}{1 + \exp(u_i - p - T)}\right) = \gamma_j \cdot \log\left(\frac{1 + \exp(u_j - p)}{1 + \exp(u_j - p - T)}\right).$$

Note that $p = \infty$ is a solution to the above equation. Therefore, there exist at most *two* finite solutions to the equation $S_i^o(p) = S_j^o(p)$. Moreover, at $p = -\infty$, $S_i^o(-\infty) = \gamma_i T$ and $S_j^o(p) = \gamma_j T$. We consider two cases: (a) $\gamma_i \ge \gamma_j$; (b) $\gamma_i < \gamma_j$.

Case (a): $\gamma_i \ge \gamma_j$. Then, $S_i^o(p) - S_j^o(p) > 0$ for $p \le p_1$ because $S_i^o(p) - S_j^o(p)$ is increasing in p for $p \le p_1$. Therefore, there exists at most one finite solution to $S_i^o(p) = S_j^o(p)$ and the solution is between p_1 and p_2.

Case (b): $\gamma_i < \gamma_j$. By $\gamma_i \exp(u_j) - \gamma_j \exp(u_i) \ge 0$, we have $\exp(u_i - u_j) \le \gamma_i/\gamma_j < 1$. Then, $u_i < u_j$. By part (b) of Proposition 2, $S_i^o(p)$ and $S_j^o(p)$ are increasing respectively in u_i and u_j, so we have $S_i^o(p) < S_j^o(p)$ for any p. In other words, there is no finite solution to $S_i^o(p) = S_j^o(p)$ in this case.

Thus, we have shown that for any pair (i,j), the equation $S_i^o(p) = S_j^o(p)$ has at most *one* finite solution, or their willingness to pay crosses at most once as the main product price p changes. □

Proof of Proposition 3. For a given main product price p and service subscription fee S, a type-k customer subscribes to the service if and only if her willingness to pay is greater than the subscription fee, i.e., $S_k^o(p) \ge S$. Thus, the total expected revenue for the firm serving heterogeneous customers can be expressed as follows:

$$R(p, S; T) = \sum_{k=1}^{K} \alpha_k \cdot R_k(p, S; T),$$

where $R_k(p,S;T)$ is the revenue for each type-k customer. In particular, if $S \le S_k^o(p)$, the type-k customers will subscribe to the ancillary service and

$$R_k(p,S;T) = R_k^S(p,S;T) = \lambda_k p \cdot \frac{\exp(u-p)}{1+\exp(u-p)} + S;$$

If $S > S_k^o(p;T)$, the type-k customers will not subscribe to the ancillary service and

$$R_k(p,S;T) = R_k^T(p;T) = \lambda_k (p+T) \cdot \frac{\exp(u-p-T)}{1+\exp(u-p-T)}.$$

By Lemma 1, for any given main product price p, we have

$$\log\Big(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\Big) \ge \frac{T\cdot\exp(u-p-T)}{1+\exp(u-p-T)}.$$

Therefore, the firm earns more revenue by providing service subscription. In particular, we show $R_k^S(p,S_k^o(p);T) \ge R_k^T(p;T)$ in detail as follows:

$$\begin{aligned}
R_k^S(p,S_k^o(p);T) - R_k^T(p;T) &= \frac{\lambda_k p\cdot\exp(u-p)}{1+\exp(u-p)} + \lambda_k\cdot\log\Big(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\Big) - \frac{\lambda_k(p+T)\cdot\exp(u-p-T)}{1+\exp(u-p-T)}\\
&\ge \frac{\lambda_k p\cdot\exp(u-p)}{1+\exp(u-p)} + \frac{\lambda_k T\cdot\exp(u-p-T)}{1+\exp(u-p-T)} - \frac{\lambda_k(p+T)\cdot\exp(u-p-T)}{1+\exp(u-p-T)}\\
&= \lambda_k p\cdot(q^S(p) - q^T(p)) \ge 0.
\end{aligned}$$

The last inequality holds because $q^S(p) \ge q^T(p)$. Moreover,

$$q^S(p) - q^T(p) = \frac{\exp(u-p)}{1+\exp(u-p)} - \frac{\exp(u-p-T)}{1+\exp(u-p-T)} \ge \min\big(q^T(1-q^T), q^S(1-q^S)\big)\cdot T.$$

Then, we have

$$R_k^S(p,S^o(p);T) - R_k^T(p;T) \ge \lambda_k p\cdot\min\big(q^T(p)\cdot(1-q^T(p)), q^S(p)\cdot(1-q^S(p))\big)\cdot T,$$

which provides a lower bound for the revenue improvement of the main product with subscription of ancillary service, compared to that without subscription.

(a) For the monotonicity of $R_k^S(p,S_k^o(p);T) - R_k^T(p;T)$, consider its derivative w.r.t. T as follows:

$$\frac{\partial[R_k^S(p,S_k^o(p);T) - R_k^T(p;T)]}{\partial T} = \lambda_k(p+T)\cdot\frac{\exp(u-p-T)}{\big(1+\exp(u-p-T)\big)^2} > 0.$$

Thus, $R_k^S(p,S_k^o(p);T) - R_k^T(p;T)$ is strictly increasing in T.

(b) Similarly, we have the following

$$\frac{\partial[R_k^S(p,S_k^o(p);T) - R_k^T(p;T)]}{\partial p} = -\frac{\lambda_k p\cdot\exp(u-p)}{\big(1+\exp(u-p)\big)^2} + \frac{\lambda_k(p+T)\cdot\exp(u-p-T)}{\big(1+\exp(u-p-T)\big)^2}.$$

The first-order condition $\partial\big(R_k^S(p,S_k^o(p);T) - R_k^T(p;T)\big)/\partial p = 0$ can be rewritten as follows:

$$\frac{p}{p+T}\cdot\Big(\frac{1+\exp(u-p-T)}{1+\exp(u-p)}\Big)^2 = \exp(-T).$$

Because $p/(p+T) = 1 - T/(p+T)$ is increasing in p and $(1+\exp(u-p-T))/(1+\exp(u-p)) = 1-(\exp(u)-\exp(u-T))/(\exp(p)+\exp(u))$ is also increasing in p, then the above equation has at most one solution w.r.t. p, denoted by p^o. Then, for any $p \le p^o$, we have $\partial\big(R_k^S(p, S_k^o(p); T) - R_k^T(p;T)\big)/\partial p \ge 0$; for any $p \ge p^o$, we have $\partial\big(R_k^S(p, S_k^o(p); T) - R_k^T(p;T)\big)/\partial p \le 0$. Therefore, $R_k^S(p, S_k^o(p); T) - R_k^T(p;T)$ is strictly increasing in the main product price p if $p \le p^o$; and is strictly decreasing in p for $p \ge p^o$. □

Proof of Theorem 2. Consider the derivative of $R(p, S_k^o(p); T)$ defined in equation (10) w.r.t. T as follows:

$$\frac{\partial R(p, S_k^o(p); T)}{\partial T} = \sum_{i=1}^{K} \alpha_i \lambda_i \big(q^T(p)\big)^2 - \sum_{i=k}^{K} \alpha_i \big(\lambda_i q^T(p) - \lambda_k\big) \cdot q^T(p),$$

where $q^T(p)$ is defined in equation (2), i.e., $q^T(p) = \exp(u-p-T)/(1+\exp(u-p-T))$. Notice that λ_k is increasing in k, i.e., $\lambda_1 < \lambda_2 < \cdots < \lambda_K$. Therefore, for any given main product price p and the pay-per-use service cost T, the derivative $\partial R(p, S_k^o(p); T)/\partial T$ is increasing in k, i.e.,

$$\frac{\partial R(p, S_1^o(p); T)}{\partial T} < \frac{\partial R(p, S_2^o(p); T)}{\partial T} < \cdots < \frac{\partial R(p, S_K^o(p); T)}{\partial T}.$$

Thus, for any pair (k, k'), lines $R(p, S_k^o(p); T)$ and $R(p, S_{k'}^o(p); T)$ cross at most once or there exists at most one solution to equation $R(p, S_k^o(p); T) = R(p, S_{k'}^o(p); T)$ w.r.t. T, denoted by $x_{k,k'}$.

Next, we show that among all the intersections $x_{k,k'}$'s, there exist at most $K-1$ points, denoted by $\infty = T_{\tau(0)} > T_{\tau(1)} > \cdots > T_{\tau(K^o-1)} > T_{\tau(K^o)} = 0$, $K_o \le K$ such that for any $T_{\tau(k)} \le T < T_{\tau(k-1)}$, we have $R(p, S_{\tau(k)}^o(p); T) \ge R(p, S_{k'}^o(p); T)$ for any $k' \ne \tau(k)$.

By the definition of $S_k^o(p;T)$ (we indicate the dependence of T explicitly to show the threshold results) in equation (4), we have $S_k^o(p;\infty) = \lim_{T\to\infty} S_k^o(p;T) = \lambda_k \log(1+\exp(u-p))$. Then, all type-$i$ customers will subscribe to the service, $i \ge k$; all other customers will neither subscribe to service nor purchase main product with pay-per-use option, so the total revenue can be rewritten as follows:

$$R(p, S_k^o(p;\infty); \infty) = \sum_{i=k}^{K} \alpha_i \Big(S_k^o(p;\infty) + \lambda_i p \cdot \frac{\exp(u-p)}{1+\exp(u-p)}\Big).$$

Let $\tau(1)$ be the index of the highest $R(p, S_k^o(p;\infty); \infty)$, defined as follows:

$$\tau(1) = \arg\max \Big\{ R(p, S_i^o(p;\infty); \infty) : \; i = 1, 2, \ldots, K \Big\}.$$

Suppose that the largest point among $x_{\tau(1),k}$ for any $k \ne \tau(1)$ is $x_{\tau(1),\tau(2)}$, which is the solution to $R(p, S_{\tau(1)}^o(p); T) = R(p, S_{\tau(2)}^o(p); T)$. Let $T_{\tau(1)} = x_{\tau(1),\tau(2)}$. Note that $R(p, S_{\tau(1)}^o(p); T) \ge R(p, S_k^o(p); T)$ for $k \ne \tau(1)$ and any $T \ge T_{\tau(1)}$; $R(p, S_{\tau(2)}^o(p); T) \ge R(p, S_{\tau(1)}^o(p); T)$ for any $T < T_{\tau(1)}$. Moreover, $\partial R(p, S_{\tau(2)}^o(p); T)/\partial T < \partial R(p, S_{\tau(1)}^o(p); T)/\partial T$ for any given p and T and therefore $\tau(1) > \tau(2)$ because $\partial R(p, S_k^o(p); T)/\partial T$ is increasing in k for any given p and T. Therefore, for any $T \ge T_{\tau(1)}$, it is optimal to charge $S^o_{\tau(1)}(p;T)$ for the service subscription and all type-k customers will subscribe to the service, $k \geq \tau(1)$.

Now, remove $\tau(1)$ from the consideration set. Denote the largest point among $x_{\tau(2),k}$ for any $k \neq \tau(1), \tau(2)$ by $x_{\tau(2),\tau(3)}$. Let $T_{\tau(2)} = x_{\tau(2),\tau(3)}$. By similar arguments, we have $R(p, S^o_{\tau(2)}(p);T) \geq R(p, S^o_k(p);T)$ for $k \neq \tau(2)$ and any $T_{\tau(1)} > T \geq T_{\tau(2)}$; $R(p, S^o_{\tau(3)}(p);T) \geq R(p, S^o_{\tau(2)}(p);T) \geq R(p, S^o_{\tau(1)}(p);T)$ for any $T < T_{\tau(2)}$. Moreover, $\tau(2) > \tau(3)$. Therefore, for any $T_{\tau(1)} > T \geq T_{\tau(2)}$, it is optimal to charge $S^o_{\tau(2)}(p;T)$ for the service subscription and all type-k customers will subscribe to the service, $k \geq \tau(2)$.

Repeat the above procedure and stop until only one is left, denoted by $R(p, S^o_{\tau(K^o)}(p);T)$. Let $T_{\tau(K^o)} = 0$. Thus, we have show that for any $T_{\tau(k)} \leq T < T_{\tau(k-1)}$, it is optimal to charge $S^o_{\tau(k)}(p;T)$ for the service subscription and all type-i customers will subscribe to it, $i \geq \tau(k)$.

We remark that K^o may be strictly smaller than K, i.e., it may not be optimal to charge the willingness to pay of a certain type of customers as T changes. It is the case if that type of customers do not contribute a significant amount of market share. □

Proof of Theorem 3. (a) Solving equation $S^o_k(p) = S$ yields p_k, i.e.,

$$\lambda_k \cdot \log\Big(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\Big) = S \quad \Longrightarrow \quad p_k = u - \log\Big(\frac{\exp(S/\lambda_k) - 1}{1 - \exp(S/\lambda_k - T)}\Big).$$

If $\lambda_k \leq S/T$, the type-k customer will not subscribe to the service, we set $p_k = 0$. Notice that λ_k is assumed to be increasing in k, so p_k is also increasing in k, i.e., $p_1 \leq p_2 \leq \cdots \leq p_K$. For any product price $p_k < p \leq p_{k+1}$, we have $S^o_k(p) < S$ and $S^o_{k+1}(p) \geq S$. Therefore, all type-j customers, $j \leq k$, will *not* subscribe to the service, whereas all type-j customers, $j \geq k+1$, will subscribe to the service.

(b) We compare the values of $R(p,S;T)$ at the left and right sides of point p_k:

$$\lim_{p \nearrow p_k} R(p,S;T) = \sum_{i=1}^{k-1} \alpha_i \lambda_i (p+T) \cdot \frac{\exp(u-p-T)}{1+\exp(u-p-T)} + \sum_{i=k}^{K} \alpha_i \Big(S^o_k(p) + \lambda_i p \cdot \frac{\exp(u-p)}{1+\exp(u-p)}\Big),$$
$$\lim_{p \searrow p_k} R(p,S;T) = \sum_{i=1}^{k} \alpha_i \lambda_i (p+T) \cdot \frac{\exp(u-p-T)}{1+\exp(u-p-T)} + \sum_{i=k+1}^{K} \alpha_i \Big(S^o_k(p) + \lambda_i p \cdot \frac{\exp(u-p)}{1+\exp(u-p)}\Big).$$

Note that $S^o_k(p) = \lambda_k \cdot \log\Big(\frac{1+\exp(u-p_k)}{1+\exp(u-p_k-T)}\Big)$, and therefore, the comparison between $\lim_{p \nearrow p_k} R(p,S;T)$ and $\lim_{p \searrow p_k} R(p,S;T)$ becomes the comparison between $R_k(p_k, S^o_k(p);T)$ and $R^T_k(p;T)$, which are defined in equations (6) and (7). By Proposition 3, we have $R_k(p_k, S^o_k(p);T) > R^T_k(p;T)$, and therefore $\lim_{p \nearrow p_k} R(p, S^o_k(p);T) > \lim_{p \searrow p_k} R(p, S^o_k(p);T)$.

(c) Note that the unique solution to $p + T = 1 + \exp(u-p-T)$ w.r.t. p is p^*_T. Suppose $p_{j^o} \leq p^*_T < p_{j^o+1} \leq p_{k^o} \leq p^*_T + T < p_{k^o+1}$. For any product price $p_k < p \leq p_{k+1}$, the total revenue can be expressed as follows:

$$R(p,S;T) = \sum_{i=1}^{k} \alpha_i \lambda_i (p+T) \cdot \frac{\exp(u-p-T)}{1+\exp(u-p-T)} + \sum_{i=k+1}^{K} \alpha_i \Big(S + \lambda_i p \cdot \frac{\exp(u-p)}{1+\exp(u-p)}\Big).$$

Consider its derivative w.r.t. p as follows:

$$\frac{\partial R(p,S;T)}{\partial p} = q^T\Big(\sum_{i=1}^{k}\alpha_i\lambda_i\big(1-\frac{p+T}{1+\exp(u-p-T)}\big)+\sum_{i=k+1}^{K}\alpha_i\lambda_i\big(1-\frac{p}{1+\exp(u-p)}\big) \cdot\big(1+\frac{\exp(T)-1}{1+\exp(u-p)}\big)\Big).$$

For any $p \le p_T^*$, we have $\frac{p}{1+\exp(u-p)} < \frac{p+T}{1+\exp(u-p-T)} \le 1$, and therefor $\partial R(p,S;T)/\partial p > 0$. Note that $\lim_{p\nearrow p_k} R(p,S;T) < \lim_{p\searrow p_k} R(p,S;T)$. Thus, given service subscription fee S, $R(p,S;T)$ is increasing in p for $p \le p_T^*$.

Similarly, for any $p \ge p_T^*+T$, we have $\frac{p+T}{1+\exp(u-p-T)} > \frac{p}{1+\exp(u-p)} \le 1$, and therefor $\partial R(p,S;T)/\partial p < 0$, so $R(p,S;T)$ is decreasing in p for $\max(p_k, p_T^*+T) \le p < p_{k+1}$. □

Proof of Proposition 4. Customers subscribe to the service if and only if $S^o(p) \ge S$, i.e.,

$$\lambda_k\cdot\log\Big(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\Big) \ge S \implies p_k \le u-\log\Big(\frac{\exp(S/\lambda_k)-1}{1-\exp(S/\lambda_k-T)}\Big).$$

The revenue comparison yields

$$S+\lambda_k p\cdot\frac{\exp(u-p)}{1+\exp(u-p)} \ge \max_p \lambda_k(p+T)\cdot\frac{\exp(u-p-T)}{1+\exp(u-p-T)} := R_T^*.$$

Note that $R_k^S(p,S_k^o(p);T) = \lambda_k\cdot\log\Big(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\Big)+\lambda_k p\cdot\frac{\exp(u-p)}{1+\exp(u-p)}$. We consider its derivative as follows:

$$\frac{\partial R_k^S(p,S_k^o(p);T)}{\partial p} = \lambda_k q^S(p)\cdot\big(q^T(p)/q^S(p)-p(1-q^S(p))\big).$$

The ratio $q^T(p)/q^S(p)$ is decreasing in p because

$$\frac{\partial q^T(p)/q^S(p)}{\partial p} = \exp(-T)\cdot\frac{\partial}{\partial p}\frac{1+\exp(u-p)}{1+\exp(u-p-T)} = -\frac{(1-\exp(-T))\cdot\exp(u-p-T)}{\big(1+\exp(u-p-T)\big)^2} \le 0.$$

Notice that $p(1-q^S(p))$ is increasing in p. Thus, $q^T(p)/q^S(p)-p(1-q^S(p))$ is decreasing in p from positive to negative as p increases from zero to infinity, and therefore $R^S(p,S^o(p);T)$ is strictly unimodal in p. Moreover, $\max_p R_k^S(p,S^o(p);T) \ge R_T^*$. Then, equation $R_k^S(p,S^o(p);T) = R_T^*$ has two distinct solutions, denoted by $\underline{p} \le \overline{p}$. Let $\underline{S} = S_k^o(\overline{p})$ and $\overline{S} = S_k^o(\underline{p})$. For any $\underline{S} \le S \le \overline{S}$, we have

$$\max_{p:S^o(p)\ge S} S+\lambda_k p\cdot\frac{\exp(u-p)}{1+\exp(u-p)} \le \max_{p:S^o(p)\ge S} S^o(p)+\lambda_k p\cdot\frac{\exp(u-p)}{1+\exp(u-p)} \le \max_p\; R_k^S(p,S^o(p);T).$$

Let $S_k^o(p') = S$. Then, we have $\underline{p} \le p' \le \overline{p}$ and

$$\max_{p:S_k^o(p)\ge S}\; S+\lambda_k p\cdot\frac{\exp(u-p)}{1+\exp(u-p)} \ge R_k^S(p',S^o(p');T) > R_T^*.$$

The last inequality holds because $R^S(p,S_k^o(p);T)$ is strictly unimodal in p, and $R_k^S(\underline{p},S_k^o(\underline{p});T) = R_k^S(\overline{p},S_k^o(\overline{p});T) = R_T^*$.

For any $S < \underline{S}$ (resp., $S > \overline{S}$), we have the following

$$\max_{p:S_k^o(p)\geq S} S + \lambda_k p \cdot \frac{\exp(u-p)}{1+\exp(u-p)} \leq \max_{p:S_k^o(p)\geq S} S_k^o(p) + \lambda_k p \cdot \frac{\exp(u-p)}{1+\exp(u-p)} = \max_{p:S_k^o(p)\geq S} R_k^S(p, S_k^o(p); T) < R_T^*.$$

The last inequality holds because $R_k^S(p, S_k^o(p); T)$ is strictly unimodal in p, $R_k^S(\underline{p}, S_k^o(\underline{p}); T) = R_k^S(\overline{p}, S_k^o(\overline{p}); T) = R_T^*$, and $p > \overline{p}$ (resp., $p < \underline{p}$). □

Proof of Theorem 4. By equation (10), if the service subscription charges $S_k^o(p)$, all type-i customers will not subscribe to ancillary service and use the pay-per-use option, $i < k$, and their revenue can be expressed as follows:

$$\sum_{i=1}^{k-1} \alpha_i \lambda_i (p+T) \cdot \frac{\exp(u-p-T)}{1+\exp(u-p-T)}.$$

Considering the first-order condition for the above equation yields

$$1 - (p+T)/(1+\exp(u-p-T)) = 0.$$

Thus, the total revenue is strictly unimodal in p and the optimal product price is equal to the unique solution to the first-order condition, denoted by p_T^*. All other type-i customers will subscribe to the ancillary service, $i \geq k$, and their revenue can be expressed as follows:

$$\sum_{i=k}^{K} \alpha_i \Big(S_k^o(p; T) + \lambda_i p \cdot \frac{\exp(u-p)}{1+\exp(u-p)} \Big).$$

Considering the first-order condition for the above equation yields

$$1 - \frac{p}{1+\exp(u-p)} - \frac{\lambda_k \sum_{i=k}^K \alpha_i}{\sum_{i=k}^K \lambda_i \alpha_i} \cdot \frac{1-\exp(-T)}{1+\exp(u-p-T)} = 0.$$

The LHS is decreasing in p, so the objective function is strictly unimodal and has the unique maximizer, denoted by p_k^S.

Let $\overline{\lambda}_k$ be the weighted average demand rate, i.e., $\overline{\lambda}_k = \sum_{i=k}^K \lambda_i \alpha_i / \sum_{i=k}^K \alpha_i$. Notice that λ_k is increasing in k, so it is straightforward to show that $\overline{\lambda}_k$ is increasing in k. Moreover, if $\lambda_k/\overline{\lambda}_k \geq \lambda_{k+1}/\overline{\lambda}_{k+1}$ (resp., $\lambda_k/\overline{\lambda}_k \leq \lambda_{k+1}/\overline{\lambda}_{k+1}$), then we have $p_k^S \geq p_{k+1}^S$ (resp., $p_k^S \leq p_{k+1}^S$). □

Proof of Theorem 5. To show that charging S_L^o earns more revenue than charging S_H^o for service subscription, we need to show the following inequality

$$\big(\alpha_L \lambda_L \cdot q_L^S(p) + \alpha_H \lambda_H \cdot q_H^S(p)\big) \cdot p + S_L^o(p) \geq \alpha_L \lambda_L (p+T) \cdot q_L^T(p) + \alpha_H \big(\lambda_H p \cdot q_H^S(p) + S_H^o(p)\big) \quad \text{(EC.1)}$$

The above inequality holds if $\alpha_L > \underline{\alpha}_L$ or $\lambda_L > \underline{\lambda}_L$, where $\underline{\alpha}_L$ and $\underline{\lambda}_L$ are defined as follows:

$$\underline{\alpha}_L = \frac{S_H^o(p) - S_L^o(p)}{\alpha_L \big(q_L^S(p) \cdot p - q_L^T(p) \cdot (p+T)\big) + S_H^o(p)} \text{ and } \underline{\lambda}_L = \frac{(1-\alpha_L) S_H^o(p) - S_L^o(p)}{\alpha_L \big(q_L^S(p) \cdot p - q_L^T(p) \cdot (p+T)\big)}.$$

Similarly, the equation $R(p, S_H^o(p)) = R(p, S_L^o(p))$ can be rewritten as follows:

$$\frac{\alpha_L \lambda_L \cdot \left(q_L^S(p) \cdot p - q_L^T(p) \cdot (p+T)\right) + S_L^o(p)}{(1-\alpha_L) \cdot \lambda_H} = \log\left(\frac{1+\exp(u_H - p)}{1+\exp(u_H - p - T)}\right).$$

By Proposition 2, the RHS is strictly increasing in u_H, so denote the unique solution w.r.t. u_H by $\overline{u}_H$. Then, for any $u_H < \overline{u}_H$, we have $R(p, S_H^o(p)) < R(p, S_L^o(p))$.

The inequality (EC.1) can be rewritten as follows $\alpha_L \cdot \left(\lambda_L p \cdot q_S(p) + S_H^o(p) - \lambda_L(p+T) \cdot q_T(p)\right) \geq S_H^o(p) - S_L^o(p)$. Define $\underline{\alpha}_L$ as follows:

$$\underline{\alpha}_L = \frac{S_H^o(p) - S_L^o(p)}{\lambda_L p \cdot q_S(p) + S_H^o(p) - \lambda_L(p+T) \cdot q_T(p)}.$$

Noting that $\lambda_L p \cdot q_S(p) + S_H^o(p) - \lambda_L(p+T) \cdot q_T(p) > \lambda_L p \cdot q_S(p) + S_L^o(p) - \lambda_L(p+T) \cdot q_T(p) > 0$, then, $\underline{\alpha}_L > 0$ and if and only if $\alpha_L \geq \underline{\alpha}_L$, the desired inequality (EC.1) holds.

The inequality (EC.1) can also be rewritten as follows:

$$\lambda_L\left(\alpha_L\left(p \cdot q_S(p) - (p+T) \cdot q_T(p)\right) + \log\left(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\right)\right) \geq \alpha_H \lambda_H \log\left(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\right).$$

Let $\underline{\lambda}_L$ be defined as follows:

$$\underline{\lambda}_L = \frac{\alpha_H \lambda_H \log\left(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\right)}{\alpha_L\left(p \cdot q_S(p) - (p+T) \cdot q_T(p)\right) + \log\left(\frac{1+\exp(u-p)}{1+\exp(u-p-T)}\right)}.$$

By similar arguments, we know $\underline{\lambda}_L > 0$. Therefore, if and only if $\lambda_L \geq \underline{\lambda}_L$, the desired inequality (EC.1) holds. □

Proof of Corollary 2. Consider the first-order derivative of $R^S(p, S^o(p))$ as follows:

$$\frac{\partial R^S(p, S^o(p))}{\partial p} = \lambda q^S(p) \cdot \left(q^T(p)/q^S(p) - p(1 - q^S(p))\right).$$

The ratio $q^T(p)/q^S(p)$ is decreasing in p because

$$\frac{\partial q^T(p)/q^S(p)}{\partial p} = \exp(-T) \cdot \frac{\partial}{\partial p} \frac{1+\exp(u-p)}{1+\exp(u-p-T)} = -\frac{(1-\exp(-T)) \cdot \exp(u-p-T)}{\left(1+\exp(u-p-T)\right)^2} \leq 0.$$

Notice that $p(1-q^S(p))$ is increasing in p. Therefore, $q^T(p)/q^S(p) - p(1-q^S(p))$ is decreasing in p from positive to negative as p increases from zero to infinity, and $R^S(p, S^o(p))$ is strictly unimodal in p. Solving the following first-order condition yields the unique optimal price p_S^* for the main product in the presence of service subscription, i.e., $q^T(p)/q^S(p) - p(1-q^S(p)) = 0$.

Without service subscription, we maximize the total revenue $R^T(p)$ w.r.t. p. Given the pay-per-use price T of the ancillary service, the optimal price of the main product p_T^* can be found by solving the corresponding first-order condition

$$1 - (p+T)(1-q_T(p))\Big|_{p=p_T^*} = 1 - \frac{(p_T^* + T)}{1+\exp((u - p_T^* - T)/\mu)} = 0.$$

Note that $q^T(p)|_{p=p_T^*} = q^S(p)|_{p=p_T^*+T}$ and $q^T \le q^S$ for any price p and $T \ge 0$. Therefore,

$$q^T(p)/q^S(p) - p(1-q^S(p))\Big|_{p=p_T^*+T} \le 0.$$

Thus, the optimal price of the main product in the presence of service subscription is lower than $p_T^* + T$, i.e., $p_S^* \le p_T^* + T$. For the other side of the inequalities, consider the following

$$\begin{aligned}
q^T(p)/q^S(p) - p(1-q^S(p))\Big|_{p=p_T^*} &= \frac{\exp(-T)+\exp(u-p_T^*-T)}{1+\exp(u-p_T^*-T)} - \frac{p_T^*}{1+\exp(u-p_T^*)} \\
= 1 - \frac{1-\exp(-T)}{1+\exp(u-p_T^*-T)} - \frac{p_T^*}{1+\exp(u-p_T^*)} &= \frac{p_T^*}{1+\exp(u-p_T^*-T)} - \frac{p_T^*}{1+\exp(u-p_T^*)} \\
+ \frac{T-(1-\exp(-T))}{1+\exp(u-p_T^*-T)} \ge 0.
\end{aligned}$$

The third equality holds because $1 - (p_T^*+T)/\big(1+\exp(u-p_T^*-T)\big) = 0$; the inequality holds because $x \ge 1-\exp(-x)$ for any $x \ge 0$. Recall that $q^T(p)/q^S(p) - p(1-q^S(p))$ is decreasing in p, then therefore we have $p_S^* \ge p_T^*$.

Consider the cross derivative of $R^S(p, S^o(p))$ w.r.t. p and T as follows:

$$\frac{\partial^2 R^S(p,S^o(p))}{\partial p \partial T} = \lambda \partial q^T(p)/\partial T = -\lambda/\mu \cdot q^T(p)(1-q^T(p)) \le 0.$$

Then, $R^S(p, S^o(p))$ has decreasing differences in (p,T), and therefore p_S^* is decreasing in T (see, e.g., Theorem 2.3 in Vives 2001). □

Proof of Corollary 3. Suppose $u_H = u_L = u$, then $q_k^S(p)$ and $q_k^T(p)$ are invariant in k, so we omit the subscript k. By Proposition 2, $S_H^o(p) \le \lambda_H T \cdot q_S(p)$ and $S_L^o(p) \ge \lambda_L T \cdot q_T(p)$. The inequality (EC.1) holds in a stronger version as follows:

$$\alpha_L\lambda_L(pq^S(p) - (p+T)q^T(p)) + \alpha_L S_L^o(p) \ge \alpha_L\lambda_L p(q^S(p) - q^T(p)) \ge \alpha_H(\lambda_H - \lambda_L)T \cdot q^S(p),$$

which is equivalent to

$$p\big(1 - q^T(p)/q^S(p)\big) \ge \frac{\alpha_H(\lambda_H-\lambda_L)T}{\alpha_L\lambda_L}.$$

By the proof of Corollary 2, $\big(1-q^T(p)/q^S(p)\big)$ is increasing in p. Note that $p(1-q^T(p)/q^S(p)) \to \infty$ as $p \to \infty$. Let $\underline{p}$ be the unique solution to

$$p\big(1 - q^T(p)/q^S(p)\big) = \frac{\alpha_H(\lambda_H-\lambda_L)T}{\alpha_L\lambda_L}.$$

Then, if $p \ge \underline{p}$, the desired inequality (EC.1) holds.

(b) Consider the first-order derivative of $R(p, S_L^o(p))$ as follows:

$$\frac{\partial R(p,S_L^o(p))}{\partial p} = (\alpha_L\lambda_L + \alpha_H\lambda_H)q^S(p) \cdot \left(\frac{\alpha_H(\lambda_H-\lambda_L)}{\alpha_L\lambda_L+\alpha_H\lambda_H} + q^T(p)/q^S(p) - p(1-q^S(p))\right).$$

We can show that $q^T(p)/q^S(p) - p(1-q^S(p))$ is decreasing in p from positive to negative as p increases from zero to infinity, and $R^S(p, S^o(p))$ is strictly unimodal in p. Solving the following first-order condition yields the unique optimal price p_L^* for the main product in the presence of service subscription, i.e.,

$$\frac{\alpha_H(\lambda_H - \lambda_L)}{\alpha_L\lambda_L + \alpha_H\lambda_H} + q^T(p)/q^S(p) - p(1-q^S(p)) = 0.$$

For the case of charging $S_H^o(p)$ for the service subscription, $R(p, S_H^o(p)) = \alpha_L\lambda_L(p+T)\cdot q^T(p) + \alpha_H\big(\lambda_H p \cdot q^S(p) + S_H^o(p)\big)$. Moreover, $(p+T)\cdot q^T(p)$ is unimodal in p and it achieves its maximum at p_T^*; $\lambda_H p\cdot q^S(p) + S_H^o(p)$ is also unimodal in p and it achieves its maximum at p_S^* by solving the corresponding first-order condition

$$q^T(p)/q^S(p) - p(1-q^S(p)) = 0.$$

Moreover, we have $p_T^* < p_S^*$. Because $R(p, S_H^o(p))$ is the sum of two unimodal functions, which may not be unimodal, it must achieve its maximum at a point between p_T^* and p_S^*, i.e., $p_T^* < p_H^* < p_S^*$.

Furthermore, by comparing the above two first-order conditions and noting $\lambda_H \geq \lambda_L$, we have $p_L^* \geq p_S^*$. Therefore, $p_H^* < p_L^*$. □

Proof of Theorem 6. We first show that there exists a unique equilibrium, under which each firm i offers subscription at price $S_i^*(p_i, p_j)$ and all customers will subscribe to the ancillary service with both firms.

Suppose that each firm i offers subscription at price S_i. A customer faces four options when deciding whether to subscribe to the ancillary service with each firm: (1) subscribe neither; (2) subscribe to firm i only; (3) subscribe to firm j only; (4) subscribe to both firms i and j. We will investigate all the cases one by one.

Case (1): customers subscribe to neither ancillary service. The revenue of firm i is

$$R_i = \lambda(p_i + T)\cdot \frac{\exp(u_i - p_i - T)}{1 + \exp(u_i - p_i - T) + \exp(u_j - p_j - T)}.$$

By Proposition 3, firm i's revenue will be higher if offering subscription of ancillary service at price

$$S_i = \lambda \log\Big(\frac{1 + \exp(u_i - p_i) + \exp(u_j - p_j - T)}{1 + \exp(u_i - p_i - T) + \exp(u_j - p_j - T)}\Big).$$

Therefore, case (1) is not an equilibrium.

Cases (2) and (3): customers subscribe to ancillary service with either of the two firms only, e.g., firm i. By similar arguments, firm j's revenue will be higher if it also offers subscription at an appropriate price as follows:

$$S_j = \lambda \log\Big(\frac{1 + \exp(u_i - p_i) + \exp(u_j - p_j)}{1 + \exp(u_i - p_i) + \exp(u_j - p_j - T)}\Big).$$

Therefore, neither cases (2) nor (3) is a Nash equilibrium.

Case (4): customers subscribe to ancillary service with both providers. Apparently, $\big(S_i^*(p_i,p_j), S_j^*(p_i,p_j)\big)$ is an equilibrium by now familiar arguments. Then, we argue that any pair of subscription fees (S_i, S_j) other than $\big(S_i^*(p_i,p_j), S_j^*(p_i,p_j)\big)$ is not an equilibrium. Note that customers subscribe to ancillary service with both firms, so by similar argument to equation (2) the following inequalities hold:

$$S_i \leq S_i^*(p_i,p_j) \text{ and } S_j \leq S_j^*(p_i,p_j).$$

If one of the above inequalities holds strictly, e.g., $S_i < S_i^*(p_i,p_j)$, firm i can earn strictly more revenue by increasing the subscription fee to $S_i^*(p_i,p_j)$ because all customers still subscribe to its service and the main product price is fixed. Therefore, $\big(S_i^*(p_i,p_j), S_j^*(p_i,p_j)\big)$ is the unique equilibrium.

At the equilibrium, the customer's surplus is equal to the expected maximum utility subtracted by the upfront subscription fees, i.e., $\lambda \log\big(1+\exp(u_i-p_i)+\exp(u_j-p_j)\big) - S_i^*(p_i,p_j) - S_j^*(p_i,p_j)$. To show that customers are better-off at the equilibrium, we compare customer surplus without and with subscription as follows:

$$\begin{aligned}
&\lambda \log\big(1+\exp(u_i-p_i)+\exp(u_j-p_j)\big) - S_i^*(p_i,p_j) - S_j^*(p_i,p_j) - \lambda \log\big(1+\exp(u_i-p_i-T) \\
&\quad +\exp(u_j-p_j-T)\big) \\
&= \lambda \log\left(\frac{\big(1+\exp(u_i-p_i-T)+\exp(u_j-p_j)\big)\cdot\big(1+\exp(u_i-p_i)+\exp(u_j-p_j-T)\big)}{\big(1+\exp(u_i-p_i)+\exp(u_j-p_j)\big)\cdot\big(1+\exp(u_i-p_i-T)+\exp(u_j-p_j-T)\big)}\right) > 0.
\end{aligned}$$

The inequality holds because

$$\begin{aligned}
&\big(1+\exp(u_i-p_i-T)+\exp(u_j-p_j)\big)\cdot\big(1+\exp(u_i-p_i)+\exp(u_j-p_j-T)\big) \\
&\quad - \big(1+\exp(u_i-p_i)+\exp(u_j-p_j)\big)\cdot\big(1+\exp(u_i-p_i-T)+\exp(u_j-p_j-T)\big) \\
&= \exp(u_i-p_i-T)\cdot\exp(u_j-p_j-T)\cdot(\exp(T)-1)^2 > 0.
\end{aligned}$$

The inequality holds strictly for any $T > 0$.

We will show that the total revenue of firm i at equilibrium is higher than that without service subscription. By Lemma 1,

$$\log\Big(\frac{1+\exp(u_i-p_i)+\exp(u_j-p_j)}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j)}\Big) \geq \frac{T\cdot\exp(u_i-p_i-T)}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j)}.$$

For the desired inequality

$$\begin{aligned}
&\frac{p_i\cdot\exp(u_i-p_i)}{1+\exp(u_i-p_i)+\exp(u_j-p_j)} + \log\Big(\frac{1+\exp(u_i-p_i)+\exp(u_j-p_j)}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j)}\Big) \\
&\geq \frac{(p_i+T)\cdot\exp(u_i-p_i-T)}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j-T)},
\end{aligned}$$

it is sufficient to show

$$\frac{p_i}{\exp(-T)+\exp(u_i-p_i-T)+\exp(u_j-p_j-T)}+\frac{T}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j)}$$
$$\geq \frac{p_i+T}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j-T)},$$

Or equivalently

$$p_i\cdot\big(1+\exp(u_i-p_i-T)+\exp(u_j-p_j)\big)\geq T\cdot\exp(u_i-p_i)\cdot\big(\exp(-T)+\exp(u_i-p_i-T)$$
$$+\exp(u_j-p_j-T)\big).$$

The above inequality holds if $T\leq p_i/\exp(u_i-p_i)$. Thus, the competition in service subscription results in a "Win-Win-Win" situation for both firms and customers. □

Proof of Theorem 7. For $S_i>S_i^o(p_i,p_j)$, customers do not subscribe to the ancillary service. In this case, it is a price competition under the MNL model, i.e., the choice probabilities $q_i^T(p_i,p_j;S_i)$ and $q_j^T(p_i,p_j;S_i)$ are

$$q_i^T(p_i,p_j;S_i):=\frac{\exp(u_i-p_i-T)}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j-T)} \text{ and } q_j^T(p_i,p_j;S_i):=\frac{\exp(u_j-p_j-T)}{1+\exp(u_i-p_i-T)+\exp(u_j-p_j-T)}.$$

The revenue functions for firms i and j are $R_i^T(p_i,p_j;S_i)=\lambda(p_i+T)\cdot q_i^T(p_i,p_j;S_i)$ and $R_j^T(p_i,p_j;S_i)=\lambda(p_j+T)\cdot q_j^T(p_i,p_j;S_i)$. The Nash equilibrium exists and is unique, denoted by $(p_i^\sharp,p_j^\sharp)$. The detailed analysis and closed-form expression can be found in the proof of Theorem 8.

For $S_i\leq S_i^o(p_i,p_j)$, customers subscribe to the ancillary service with firm i. Then, a customer will choose the main product of firm i with probability $q_i^S(p_i,p_j;S_i)$,

$$q_i^S(p_i,p_j;S_i):=\frac{\exp(u_i-p_i)}{1+\exp(u_i-p_i)+\exp(u_j-p_j-T)} \text{ and } q_j^S(p_i,p_j;S_i):=\frac{\exp(u_j-p_j-T)}{1+\exp(u_i-p_i)+\exp(u_j-p_j-T)}.$$

The revenue functions are $R_i^S(p_i,p_j;S_i)=\lambda p_i\cdot q_i^S(p_i,p_j;S_i)+S_i$ and $R_j^S(p_i,p_j;S_i)=\lambda(p_j+T)\cdot q_j^S(p_i,p_j;S_i)$. Comparing the two cases, p_i in Case 2 plays exactly the same role of (p_i+T) in Case 1, so the unique Nash equilibrium is $(p_i^\sharp+T,p_j^\sharp)$.

(a) If customers do not subscribe to the ancillary service, $(p_i^\sharp,p_j^\sharp)$ is the unique Nash equilibrium. While firm j has no incentive to deviate, firm i may lower price such that customers will subscribe to its ancillary service. If so, he must set his price p_i such that the willingness to pay for the service subscription is greater than or equal to S_i, i.e., $S_i^o(p_i,p_j^\sharp)\geq S_i$.

Similar to the proof of Corollary 2, $S_i^o(p_i,p_j^\sharp)+R_i^S(p_i,p_j^\sharp)$ is unimodal in p_i and is increasing in p_i for $p_i\leq p_i^\sharp$. Denote the unique solution to the following equation w.r.t. $p_i\leq p_i^\sharp$ by $\overline{p}_i$:

$$S_i^o(p_i,p_j^\sharp)+R_i^S(p_i,p_j^\sharp)=R_i^T(p_i^\sharp,p_j^\sharp).$$

Let $\overline{S}_i=S_i^o(\overline{p}_i,p_j^\sharp)$. Therefore, if $S_i>\overline{S}_i$, firm i has no incentive to deviate either and then $(p_i^\sharp,p_j^\sharp)$ is indeed the unique equilibrium.

(b) If S_i is low enough, customers subscribe to the ancillary service with firm i. Then, the unique Nash equilibrium in the price competition is $(p_i^\sharp + T, p_j^\sharp)$, because S_i is the sunk cost and p_i plays exactly the same role of $p_i + T$ in the case without subscription. By Proposition 2, $S_i^o(p_i^\sharp + T, p_j)$ is increasing in p_j. While firm i has no incentive to deviate, firm j may lower its price such that customers will not subscribe to the ancillary service with firm i. If so, its price p_j must satisfy $S_i^o(p_i^\sharp + T, p_j) < S_i$. Denote the unique solution to the following equation w.r.t. $p_j \le p_j^\sharp$ by $\underline{p}_j$:

$$R_j^T(p_i^\sharp + T, p_j) = R_j^S(p_i^\sharp + T, p_j^\sharp).$$

Let $\underline{S}_i = S_i^o(p_i^\sharp + T, \underline{p}_j)$. Therefore, if $S_i < \underline{S}_i$, firm j has no incentive to deviate either, so $(p_i^\sharp + T, p_j^\sharp)$ is the unique Nash equilibrium.

Apparently, we have $\overline{S}_i > \underline{S}_i$ by Proposition 2. Each firm captures exactly the same amount of market share, so firm j earns the same amount of revenue. Firm i collects more revenue by exactly the amount of S_i, because customers subscribe to its ancillary service. Whereas, customer surplus decreases also by the exact same amount.

(c) For $\underline{S}_i \le S_i \le \overline{S}_i$, there are two candidates for the pure-strategy Nash equilibrium, i.e., $(p_i^\sharp, p_j^\sharp)$ and $(p_i^\sharp + T, p_j^\sharp)$. However, by similar arguments, we find that either firm i or j has incentive to deviate from $(p_i^\sharp, p_j^\sharp)$ or $(p_i^\sharp + T, p_j^\sharp)$, e.g., firm i may lower its price such that customers will subscribe to its ancillary service if customers have not yet; firm j may also lower its price such that customers will unsubscribe to firm i if they have already. Thus, there does not exist a pure-strategy Nash equilibrium. □

Proof of Theorem 8. As shown in Theorem 6, for any pair of prices (p_i, p_j), each firm will price its service subscription such that all customers will subscribe to it. Therefore, the two-dimensional problem reduces to a single-dimension price competition. The payoff function for firm i is

$$R_i^S(p_i, p_j) = \frac{\lambda p_i \cdot \exp(u_i - p_i)}{1 + \exp(u_i - p_i) + \exp(u_j - p_j - T)} + \lambda \log\Big(\frac{1 + \exp(u_i - p_i) + \exp(u_j - p_j - T)}{1 + \exp(u_i - p_i - T) + \exp(u_j - p_j - T)}\Big). \quad \text{(EC.2)}$$

By similar arguments to Corollary 2, given any price p_j, function $R_i^S(p_i, p_j)$ is strictly unimodal w.r.t. p_i, which is equivalent to the quasi-concavity in a single-dimension space. Therefore, there exists a Nash equilibrium (see, e.g., Vives 2001). Next, we will establish its uniqueness and find the closed-form expression. Without loss of generality, let $\lambda = 1$ for notational simplicity in this proof.

Define the corresponding attractiveness as follows: $a_i^T = \exp(u_i - p_i - T)$, $a_i^S = \exp(u_i - p_i)$ and $a_j^T = \exp(u_j - p_j - T)$. We change decision variables to the attractiveness. First, we consider the case without a service subscription option. The revenue functions for firms i and j are

$$R_i^T(a_i^T, a_j^T) = \frac{\big(u_i - \log(a_i^T)\big) \cdot a_i^T}{1 + a_i^T + a_j^T} \text{ and } R_j^T(a_i^T, a_j^T) = \frac{\big(u_j - \log(a_j^T)\big) \cdot a_j^T}{1 + a_i^T + a_j^T}.$$

The first-order condition of $R_i^T(a_i^T, a_j^T)$ w.r.t. a_i^T is

$$\frac{\partial R_i^T(a_i^T, a_j^T)}{\partial a_i^T} = -\frac{1}{1 + a_i^T + a_j^T} + \frac{(1 + a_j^T)(u_i - \log(a_i^T))}{(1 + a_i^T + a_j^T)^2} = 0.$$

Using some algebra yields

$$(a-a_i^T)\cdot(u_i-\log(a_i^T))=a, \tag{EC.3}$$

where $a=1+a_i^T+a_j^T$ represents the total attractiveness of the outside option, products i and j. For any given a in a feasible region, the left-hand side (LHS) of equation (EC.3) is decreasing in a_i^T, so the corresponding a_i^T is unique, denoted by $a_i^T=A_i^T(a)$. To study the monotonicity of $A_i^T(a)$ with respect to a, taking the full differential for equation (EC.3) and rearranging it results in

$$\frac{\partial A_i^T(a)}{\partial a}=\frac{(u_i-\log(a_i^T))-1}{(u_i-\log(a_i^T))+(a-a_i^T)/a_i^T}>0.$$

The inequality holds because $u_i-\log(a_i^T)>1$ for any a in the implicit function (EC.3). We will next show that $A_i^T(a)/a$ is decreasing in a. Equation (EC.3) can be rewritten as follows

$$(1-A_i^T(a)/a)\cdot(u_i-\log(A_i^T(a)))=1.$$

Note that $A_i^T(a)$ is increasing in a, so $A_i^T(a)/a$ must be decreasing in a.

Similarly, we can define $A_j^T(a)$ for the one-to-one mapping between a and a_j^T. Moreover, $A_j^T(a)/a$ is also decreasing in a. Consider the following equation w.r.t. a:

$$\frac{1}{a}+\frac{A_i^T(a)}{a}+\frac{A_j^T(a)}{a}=1. \tag{EC.4}$$

The LHS is strictly decreasing in a, so there exists a unique solution to the above equation, denoted by $a^\sharp$. Therefore, there exists a unique Nash equilibrium $(a_i^\sharp,a_j^\sharp)$ in the attractiveness competition without service subscription, where $a_i^\sharp=A_i^T(a^\sharp)$ and $a_j^\sharp=A_j^T(a^\sharp)$. Immediately, the unique Nash equilibrium in price competition is $(p_i^\sharp,p_j^\sharp)$, where $p_i^\sharp=u_i-T-\log(a_i^\sharp)$ and $p_j^\sharp=u_j-T-\log(a_j^\sharp)$.

If only firm i offers service subscription, its revenue can be rewritten as follows

$$R_i^S(a_i^S,a_j^S)=\frac{(u_i-\log(a_i^S))\cdot a_i^S}{1+a_i^S+a_j^T}+\log\Big(\frac{1+a_i^S+a_j^T}{1+a_i^S/\exp(T)+a_j^T}\Big),$$

while firm j's revenue function does not change. By the first-order condition, we obtain

$$(a-a_i^S)\cdot(u_i-\log(a_i^S))\cdot\big(a-(1-1/\exp(T))a_i^S\big)=a^2/\exp(T), \tag{EC.5}$$

where again $a=1+a_i^S+a_j^T$. Apparently, for given a and T there also exists a unique corresponding a_i^S, denoted by $A_i^S(a;T)$. For given T, taking the full differential for equation (EC.5) and using some algebra yields

$$\begin{aligned}\frac{\partial A_i^S(a;T)}{\partial a}=&\Big((u_i-\log(a_i^S))\cdot\big(a-(1-1/\exp(T))a_i^S\big)+(a-a_i^S)\cdot(u_i-\log(a_i^S))-2a/\exp(T)\Big)\\ &\Big/\Big((u_i-\log(a_i^S))\cdot\big(a-(1-1/\exp(T))a_i^S\big)+(a-a_i^S)/a_i^S\cdot\big(a-(1-1/\exp(T))a_i^S\big)\\ &+(a-a_i^S)\cdot(u_i-\log(a_i^S))\cdot(1-1/\exp(T))\Big).\end{aligned}$$

By equation (EC.5), and $a - a_i^S < a$, $a - (1 - 1/\exp(T))a_i^S < a$, we have $(u_i - \log(a_i^S)) \cdot \big(a - (1 - 1/\exp(T))a_i^S\big) > a/\exp(T)$ and $(a - a_i^S) \cdot (u_i - \log(a_i^S)) > a/\exp(T)$. Then, for any given T, $\partial A_i^S(a;T)/\partial a > 0$ and therefore $A_i^S(a;T)$ is strictly increasing in a.

Equation (EC.5) can also be rewritten as follows

$$\big(1 - A_i^S(a;T)/a\big) \cdot \big(u_i - \log(A_i^S(a;T))\big) \cdot \big(\exp(T) - (\exp(T) - 1) \cdot A_i^S(a;T)/a\big) = 1.$$

Note that $A_i^S(a;T)$ is increasing in a, so $\big(u_i - \log(A_i^S(a;T))\big)$ is decreasing in a. Therefore, by the above equation, $A_i^S(a;T)/a$ must be decreasing in a. The unique Nash equilibrium is the solution to the following equation:

$$\frac{1}{a} + \frac{A_i^S(a;T)}{a} + \frac{A_j^T(a)}{a} = 1. \tag{EC.6}$$

The LHS of the above equation is strictly decreasing in a, so there exists a unique solution, denoted by $a^\flat$. Therefore, for any given T, there exists a unique Nash equilibrium $(a_i^\flat, a_j^\flat)$, where $a_i^\flat = A_i^S(a^\flat;T)$ and $a_j^\flat = A_j^T(a^\flat)$.

Next, we study the comparative statics of the equilibrium w.r.t. T. Equation (EC.5) can also be reexpressed as follow:

$$(a - a_i^S) \cdot (u_i - \log(a_i^S)) \cdot \big(\exp(T) - (\exp(T) - 1) \cdot a_i^S/a\big) = a. \tag{EC.7}$$

Note that $\exp(T) - (\exp(T) - 1) \cdot a_i^S/a = (1 - a_i^S/a)\exp(T) + a_i^S/a \geq 1$ for any $T \geq 0$, and it is increasing in T for any given a. Therefore, for each given a, the corresponding $A_i^S(a;T)$ by equation (EC.7) is increasing in T. The unique solution to equation (EC.6), denoted by $a^\flat(T)$, is increasing in T. Therefore, the equilibrium $a_j^\flat = A_j^T(a^\flat)$ is increasing in T and $a_i^\flat = A_i^S(a^\flat;T)$ is also increasing in T. Immediately, the equilibrium prices, expressed by $p_i^\flat = u_i - \log(a_i^\flat)$ and $p_j^\flat = u_j - T - \log(a_j^\flat)$, are decreasing in T.

In addition, $A_i^S(a;T) \geq A_i^T(a)$ for any a. By comparing equations (EC.4) and (EC.6), we have $a^\flat \geq a^\sharp$ and therefore $(a_i^\flat, a_i^\flat) \succeq (a_i^\sharp, a_i^\sharp)$. Furthermore, for the corresponding prices, $p_i^\flat < p_i^\sharp + T$ and $p_j^\flat < p_j^\sharp$ for any given pay-per-use fee T.

Moreover, we have

$$\frac{A_i^S(a^\flat)}{a^\flat} > \frac{A_i^T(a^\sharp)}{a^\sharp}, \quad \frac{A_j^T(a^\flat)}{a^\flat} < \frac{A_j^T(a^\sharp)}{a^\sharp} \text{ and } \frac{A_i^S(a^\flat)}{a^\flat} + \frac{A_j^T(a^\flat)}{a^\flat} > \frac{A_i^T(a^\sharp)}{a^\sharp} + \frac{A_j^T(a^\sharp)}{a^\sharp}.$$

That is, firm i will capture more customers, firm j will lose some market share, and the total market share served by both firms will increase. Recall that firm j's attractiveness is higher, i.e., $a_j^\flat > a_j^\sharp$, so the corresponding price is lower and therefore firm j's revenue at the equilibrium is lower, compared to that without service subscription for firm i. □

Proof of Theorem 9. Continue with the proof of Theorem 8. Similarly, the revenue function of firm j with service subscription can be rewritten as follows:

$$R_j^S(a_i^S, a_j^S) = \frac{(u_j - \log(a_j^S)) \cdot a_j^S}{1 + a_i^S + a_j^S} + \log\Big(\frac{1 + a_i^S + a_j^S}{1 + a_i^S + a_j^S / \exp(T)}\Big).$$

By the first-order condition, we obtain the following equation:

$$(a - a_j^S) \cdot (u_j - \log(a_j^S)) \cdot \big(a - (1 - 1/\exp(T))a_j^S\big) = a^2 / \exp(T), \tag{EC.8}$$

where again $a = 1 + a_i^S + a_j^S$. Similarly, we can define the corresponding $A_j^S(a;T)$ for each a and T, and $A_j^S(a;T)$ is increasing in a and T, respectively. Moreover, $A_j^S(a;T)/a$ is decreasing in a. The unique Nash equilibrium is the solution to the following equation:

$$\frac{1}{a} + \frac{A_i^S(a;T)}{a} + \frac{A_j^S(a;T)}{a} = 1. \tag{EC.9}$$

The LHS of the above equation is decreasing in a, so for each given T, there exists a unique solution, denoted by $a^\natural(T)$. Therefore, there exists a unique Nash equilibrium $(a_i^\natural, a_j^\natural)$, where $a_i^\natural = A_i^S(a^\natural;T)$ and $a_j^\natural = A_j^S(a^\natural;T)$. The unique Nash equilibrium in price competition is $(p_i^\natural, p_j^\natural)$, where $p_i^\natural = u_i - \log(a_i^\natural)$ and $p_j^\natural = u_j - \log(a^\natural)$. By similar arguments, $a^\natural(T)$ is increasing in T, and therefore $a_i^\natural$ and $a_j^\natural$ are also increasing in T, respectively. Therefore, $p_i^\natural$ and $p_j^\natural$ are decreasing in T.

Next, we compare the equilibria with and without service subscription. Equation (EC.8) can also be rewritten as follow:

$$(a - a_i^S) \cdot (u_i - \log(a_i^S)) \cdot \big(\exp(T) - (\exp(T) - 1) \cdot a_i^S / a\big) = a. \tag{EC.10}$$

Note that $\exp(T) - (\exp(T) - 1) \cdot a_i^S/a = (1 - a_i^S/a)\exp(T) + a_i^S/a \geq 1$ for any $T \geq 0$, where the inequality holds because $(1 - a_i^S/a)\exp(T)$ is increasing in T. Comparing equations (EC.3) to (EC.10), for each a, the corresponding $a_i^S \geq a_i^T$, i.e., $A_i^S(a;T) \geq A_i^T(a)$ and $A_j^S(a;T) \geq A_j^T(a)$ for each a. Thus, comparing equations (EC.4) and (EC.9) yields $a^\natural \geq a^\sharp$. Therefore, $(a_i^\natural, a_i^\natural) \succeq (a_i^\sharp, a_i^\sharp)$. Immediately, we have $(p_i^\natural, p_j^\natural) \preceq (p_i^\sharp, p_j^\sharp) + T$. $\square$