Supply Disruptions and Optimal Network Structures

Online Appendix

Kostas Bimpikis
Graduate School of Business, Stanford University, Stanford, CA 94305, USA.
Ozan Candogan
Booth School of Business, University of Chicago, Chicago, IL 60637, USA.
Shayan Ehsani
Management Science and Engineering, Stanford University, Stanford, CA 94305, USA.

Appendix A: Proofs

Proof of Theorem 1

The proof proceeds in three parts that (1) establish the essential uniqueness of the equilibrium under Assumption 1(i); (2) provide a characterization of equilibrium prices and procurement quantities when both Assumptions 1(i) and (ii) hold; and (3) show that if Assumption 1(ii) is violated, then there exist realizations of the disruptions for which the realized price for the output of at least one of the $K+1$ tiers is equal to zero, respectively.

The proof relies on formulating a convex optimization problem (given in (1)) and showing that its (unique) solution along with the solution to the corresponding dual problem can be used to construct a supply equilibrium. In particular, in establishing all three parts that follow, we use the problem's KKT conditions (that are necessary and sufficient for optimality).

We start by introducing some notation, stating the aforementioned optimization problem, and establishing two auxiliary lemmas. Then, we proceed to the proof of claims (1)-(3) above.

To simplify the exposition, we let $\eta_k(\omega) \triangleq \mathbb{P}(\hat{\omega}_k = \omega)$ denote the probability that state ω is realized after tiers $k+1, \cdots, K+1$ complete production. Note that $\eta(\omega_k) > 0$ for all $\omega_k \in \Omega_k$, since by definition Ω_k consists of states that are realized with nonzero probability. Also, given a state ω_k, with some abuse of notation, we let $\omega_{k,j}$ denote whether firm $j \in Tier(k+1)$ experiences a disruptive event in state ω_k (i.e., $\omega_{k,j}$ is equal to the binary variable $z_{k+1,j}$ associated with

$\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}_i\}$). In addition, we use notation $\omega_{k+1} \to \omega_k$ when state ω_k can be realized with positive probability assuming that the state describing the realizations of disruptions in tiers $k+2, \cdots, K+1$ is ω_{k+1}. Finally, we let $\mathcal{S}$ denote the set of valid states.

We consider the following optimization problem:

$$\begin{aligned}
\max \sum_{\omega_0 \in \Omega_0} \eta_0(\omega_0)\Big(\alpha x_0(\omega_0) - \frac{\beta}{2} x_0(\omega_0)^2\Big) - p_c \sum_{i \in Tier(K+1)} x_{K+1,i}(\omega_{K+1}) - \sum_{k=1}^{K+1} \sum_{i \in Tier(k)} \sum_{\omega_k \in \Omega_k} \eta_k(\omega_k) c(k) x_{k,i}(\omega_k)^2 \\
s.t. \sum_{i \in Tier(k)} x_{k,i}(\omega_k) \leq \sum_{j \in Tier(k+1) | \omega_{k+1} \to \omega_k, \omega_{k,j}=1} x_{k+1,j}(\omega_{k+1}), \qquad \text{for } 1 \leq k \leq K \text{ and } \forall \omega_k \in \Omega_k, \\
x_0(\omega_0) \leq \sum_{j \in Tier(1) | \omega_1 \to \omega_0, \omega_{0,j}=1} x_{1,j}(\omega_1), \qquad \forall \omega_0 \in \Omega_0, \\
x_0(\omega_0) \geq 0, \qquad x_{k,i}(\omega_k) \geq 0, \qquad \text{for } 1 \leq k \leq K+1, \text{ and } \forall i \in Tier(k), \forall \omega_0 \in \Omega_0, \forall \omega_k \in \Omega_k.
\end{aligned} \tag{1}$$

The following lemma summarizes a number of properties of this optimization problem that we use in our subsequent analysis.

Lemma 1. *Suppose Assumption 1(i) holds.*

(a) Optimization problem (1) admits a unique optimal solution, which we denote by $\mathbf{x}^\star$. *Moreover, this solution satisfies* $x^\star_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k)$ *for all* $k \in \{1, \ldots K+1\}$ *and* $i, j \in Tier(k)$.

(b) The nonnegativity constraints are not binding in the optimal solution for the decision variables that correspond to valid states, i.e., $x^\star_{k,i}(\omega_k) > 0$ *and* $x^\star_0(\omega_0) > 0$ *for* $\omega_k, \omega_0 \in \mathcal{S}$ *at the optimal solution.*

Proof:

(a) Note that the objective function is additively separable over the decision variables, i.e., $\{x_{k,i}(\omega_k)\}$ for $1 \leq k \leq K+1$, $i \in Tier(k)$, and $\omega_k \in \Omega_k$, and $x_0(\omega_0)$ for $\omega_0 \in \Omega_0$. Moreover, it is strictly concave in all the decision variables. Since the feasible set is convex, these observations imply that there is a unique optimal solution.[1]

We proceed by establishing that in the optimal solution

$$x^\star_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k), \tag{2}$$

for all $i, j \in Tier(k)$, $\omega_k \in \Omega_k$, and $1 \leq k \leq K+1$. Suppose that this is not the case and $x^\star_{k,i}(\omega_k) > x^\star_{k,j}(\omega_k)$ for some $i, j \in Tier(k)$ and $\omega_k \in \Omega_k$. Then, another feasible solution $\mathbf{x}'$ with the same value for the objective function can be constructed by setting $x'_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k)$ and $x'_{k,j}(\omega_k) = x^\star_{k,i}(\omega_k)$, and $\mathbf{x}'$ equal to the original solution $\mathbf{x}^\star$ for the remaining decision variables. This solution is also optimal; thus, we obtain a contradiction to its uniqueness.

[1] It can be readily seen that when the decision variables take large enough values, the objective becomes negative. Thus, it is sufficient to restrict attention to a bounded interval for the vector of decision variables $\mathbf{x}$. Thus, the existence of an optimal solution follows from the Weierstrass theorem.

(b) Next, we show that the nonnegativity constraints are not binding in the optimal solution for the decision variables that correspond to valid states. By way of contradiction, assume that this is not the case and consider the most upstream tier $k \in \{1, \cdots, K+1\}$ such that in an optimal solution $x^\star_{k,i}(\omega_k) = 0$ for some $i \in Tier(k)$, and valid state $\omega_k \in \Omega_k$. Then, by Expression (2), one of the following two cases should hold:

- $k < K+1$, i.e., $x^\star_{k,i}(\omega_k) = 0$ for all $i \in Tier(k)$ and $x^\star_{k+1,j}(\omega_{k+1}) > 0$ for any $j \in Tier(k+1)$ where $\omega_{k+1} \to \omega_k$; or
- $k = K+1$, i.e., $x^\star_{K+1,i}(\omega_{K+1}) = 0$ for all $i \in Tier(K+1)$.

Consider any valid states $\omega'_{k-1} \in \Omega_{k-1}, \ldots, \omega'_0 \in \Omega_0$ that can be reached (with nonzero probability) after ω_k is realized, i.e., $\omega_k \to \omega'_{k-1} \to \cdots \to \omega'_0$. Note that by feasibility in optimization problem (1), we have $x^\star_{\ell,i'}(\omega'_\ell) = 0 = x^\star_0(\omega'_0)$ for all $i' \in Tier(\ell)$ and $1 \le \ell < k$. Next, for each tier $\ell \le k$, fix a firm $j(\ell) \in Tier(\ell)$. Note that another feasible solution $\mathbf{x}'$ can be constructed by setting:

(i) $x'_{k,j(k)}(\omega_k) = \epsilon$,

(ii) $x'_{\ell,j(\ell)}(\omega'_\ell) = \epsilon$ for $1 \le \ell < k$ and every state ω'_ℓ such that firms $\{j(m)\}_{m=\ell+1}^{k}$ do not experience a disruption,

(iii) $x'_0(\omega'_0) = \epsilon$ for every state ω'_0 such that firms $\{j(m)\}_{m=1}^{k}$ do not experience a disruption,

while leaving the remaining decision variables unchanged.

By Assumption 1(i), we have $\alpha \prod_{\ell=1}^{K+1} q(\ell) = \alpha(K+2) > p_c$. Hence, it can be seen that this feasible solution yields a strictly higher value for the objective function for sufficiently small ϵ, thereby leading to a contradiction to $x^\star_{k,i}(\omega_k) = 0$ for some $1 < k \le K+1$, $i \in Tier(k)$, and valid state $\omega_k \in \Omega_k$ at the optimal solution of (1). Similarly, if all $x^\star_{k,i}(\omega_k) > 0$ for valid $\omega_k \in \Omega_k$, and $x^\star_0(\omega_0) = 0$ for some valid ω_0, it can be readily seen that one can construct another solution $\mathbf{x}'$ with a strictly higher value for the objective function by setting $x'_0(\omega_0) = \epsilon$ for a sufficiently small $\epsilon > 0$. Thus, we conclude that in the optimal solution of optimization problem (1), for valid states the nonnegativity constraints are not binding. □

Introducing Lagrange multipliers $\{\lambda_k(\omega_k)\}$ and $\{\lambda_0(\omega_0)\}$ for the first two sets of constraints, respectively, and $\{\mu_0(\omega_0)\}$ and $\{\mu_{k,i}(\omega_k)\}$ for the nonnegativity constraints, the necessary and sufficient (given that the problem's constraints are linear) KKT optimality conditions corresponding to optimization problem (1) can be written as follows:

$$-\eta_0(\omega_0)(\alpha - \beta x^\star_0(\omega_0)) + \lambda_0(\omega_0) - \mu_0(\omega_0) = 0, \tag{3a}$$

$$p_c + 2c(K+1)x^\star_{K+1,i}(\omega_{K+1}) - \sum_{\omega_K | \omega_{K+1} \to \omega_K, \omega_{K,i}=1} \lambda_K(\omega_K) - \mu_{K+1,i}(\omega_{K+1}) = 0, \tag{3b}$$

$$2c(k)\eta_k(\omega_k)x^\star_{k,i}(\omega_k) + \lambda_k(\omega_k) - \sum_{\omega_{k-1} | \omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \lambda_{k-1}(\omega_{k-1}) - \mu_{k,i}(\omega_k) = 0, \tag{3c}$$

$$\lambda_k(\omega_k) \geq 0 \quad \perp \quad \sum_{i \in Tier(k)} x^\star_{k,i}(\omega_k) \leq \sum_{j \in Tier(k+1)|\omega_{k+1}\to\omega_k, \omega_{k,j}=1} x^\star_{k+1,j}(\omega_{k+1}), \tag{3d}$$

$$\lambda_0(\omega_0) \geq 0 \quad \perp \quad x^\star_0(\omega_0) \leq \sum_{j \in Tier(1)|\omega_1\to\omega_0, \omega_{0,j}=1} x^\star_{1,j}(\omega_1), \tag{3e}$$

$$\mu_{k,i}(\omega_k) \geq 0 \quad \perp \quad x^\star_{k,i}(\omega_k) \geq 0, \ \mu_0(\omega_0) \geq 0 \quad \perp \quad x^\star_0(\omega_0) \geq 0. \tag{3f}$$

Our next lemma provides a characterization of the optimal dual multipliers and relates the solution of the above system to the supply equilibrium.

LEMMA 2. *Suppose Assumption 1(i) holds.*

(a) The dual multipliers $\{\lambda_k(\omega_k)\}$ and $\{\lambda_0(\omega_0)\}$ corresponding to valid states and satisfying conditions (3a)–(3f) are unique. Moreover, $\mu_0(\omega_0) = \mu_{k,i}(\omega_k) = 0$ for valid states $\omega_0 \in \Omega_0 \cap \mathcal{S}, \omega_k \in \Omega_k \cap \mathcal{S}$, where $k \in \{1, \ldots, K+1\}$ and $i \in Tier(k)$.

(b) Let $\{x^\star_{k,i}(\omega_k)\} \cup \{x^\star_0(\omega_0)\}$ and $\{\lambda_k(\omega_k)\} \cup \{\lambda_0(\omega_0)\} \cup \{\mu_k(\omega_k)\} \cup \{\mu_0(\omega_0)\}$ denote a solution of (3a)–(3f), and define $p_k(\omega_{k-1}) = \lambda_{k-1}(\omega_{k-1})/\eta_{k-1}(\omega_{k-1})$. Then $\{p_k(\omega_{k-1}), x^\star_{k,i}(\omega_k)\}$ is a supply equilibrium.

(c) Conversely, suppose that $\{p_k(\omega_{k-1}), x^\star_{k,i}(\omega_k)\}$ is a supply equilibrium. Define

$$\lambda_k(\omega_k) = p_{k+1}(\omega_k)\eta_k(\omega_k), \tag{4}$$

and $x^\star_0(\omega_0) = D(p_1(\omega_0))$. Also, let $\mu_{k,i}(\omega_k) = 0$ if $x^\star_{k,i}(\omega_k) > 0$ for $1 \leq k \leq K+1$, and, $\mu_0(\omega_0) = 0$ if $x^\star_0(\omega_0) > 0$. Otherwise, if $x^\star_{k,i}(\omega_k) = 0$, let

$$\mu_{k,i}(\omega_k) = \lambda_k(\omega_k) - \sum_{\omega_{k-1}|\omega_k\to\omega_{k-1}, \omega_{k-1,i}=1} \lambda_{k-1}(\omega_{k-1}), \tag{5}$$

when $1 \leq k \leq K$ and $\mu_{k,i}(\omega_k) = p_c - \sum_{\omega_{k-1}|\omega_k\to\omega_{k-1},\omega_{k-1,i}=1} \lambda_{k-1}(\omega_{k-1})$ when $k = K+1$. Similarly, let $\mu_0(\omega_0) = \lambda_0(\omega_0) - \eta_0(\omega_0)\alpha$, if $x^\star_0(\omega_0) = 0$.

Then, $\{x^\star_{k,i}(\omega_k)\} \cup \{x^\star_0(\omega_0)\}$ and $\{\lambda_k(\omega_k)\} \cup \{\lambda_0(\omega_0)\} \cup \{\mu_k(\omega_k)\} \cup \{\mu_0(\omega_0)\}$ is a solution of (3a)–(3f).

Proof:

(a) By Lemma 1, optimization problem (1) admits a unique optimal solution $\{x^\star_{k,i}(\omega_k)\} \cup \{x^\star_0(\omega_0)\}$. Lemma 1 also implies that $x^\star_0(\omega_0) > 0$ for $\omega_0 \in \Omega_0 \cap \mathcal{S}$ and $x^\star_{k,i}(\omega_k) > 0$ for $i \in Tier(k)$, $\omega_k \in \Omega_k \cap \mathcal{S}$. Thus, by (3f), we have $\mu_0(\omega_0) = \mu_k(\omega_k) = 0$. Thus, given $x^\star_0(\omega_0)$, condition (3a) implies that $\lambda_0(\omega_0)$ is uniquely defined for $\omega_0 \in \Omega_0 \cap \mathcal{S}$. Suppose that $\{\lambda_\ell(\omega_\ell)\}_{\ell<k, \omega_\ell \in \Omega_\ell \cap \mathcal{S}}$ are uniquely defined for some k. Since $\mu_k(\omega_k) = 0$ for $\omega_k \in \Omega_k \cap \mathcal{S}$, given $x^\star_{k,i}(\omega_k)$ and $\{\lambda_\ell(\omega_\ell)\}_{\ell<k,\omega_\ell\in\Omega_\ell\cap\mathcal{S}}$, by conditions (3b) and (3c), dual multiplier $\lambda_k(\omega_k)$ is also uniquely defined for $\omega_k \in \Omega_k \cap \mathcal{S}$. Proceeding inductively, we conclude that dual variables $\{\lambda_k(\omega_k)\}$ and $\{\lambda_0(\omega_0)\}$ corresponding to valid states and satisfying KKT conditions (3a)–(3f) are uniquely defined.

(b) First consider $x_0^\star(\omega_0)$. Note that if $x_0^\star(\omega_0) > 0$, then by (3f) and (3a) we have $\mu_0(\omega_0) = 0$ and, in turn, $p_1(\omega_0) = (\alpha - \beta x_0^\star(\omega_0))$. Hence,

$$D(p_1(\omega_0)) = x_0^\star(\omega_0) \text{ for } \omega_0 \in \mathcal{S} \text{ and } x_0^\star(\omega_0) > 0. \tag{6}$$

On the other hand, if $x_0^\star(\omega_0) = 0$, by (3a), we obtain directly that $p_1(\omega_0) \geq \alpha$. Hence,

$$D(p_1(\omega_0)) = 0 = x_0^\star(\omega_0). \tag{7}$$

Expressions (6) and (7) together with KKT condition (3e) imply condition (iii) from the definition of a supply equilibrium (Definition 1). In addition, KKT condition (3d) readily implies condition (ii) from the definition of a supply equilibrium.

Finally, we establish that the combination of KKT conditions (3f), (3b), and (3c) imply condition (i) from the definition of a supply equilibrium. To see this, recall that the expected profit of firm $i \in Tier(k)$ when the realized state is ω_k after firms in tiers $k+1, \cdots, K+1$ complete production and firm i procures $y_{k,i}$ can be written as follows

$$\begin{aligned} \bar{\pi}(i, \omega_k, y_{k,i}) &= \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \eta_{k-1}(\omega_{k-1}|\omega_k) p_k(\omega_{k-1}) y_{k,i} - p_{k+1}(\omega_k) y_{k,i} - c(k) y_{k,i}^2 \\ &= \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \frac{\eta_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)} p_k(\omega_{k-1}) y_{k,i} - p_{k+1}(\omega_k) y_{k,i} - c(k) y_{k,i}^2, \end{aligned}$$

where $\eta_{k-1}(\omega_{k-1}|\omega_k) = \frac{\eta_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)}$ is the probability state ω_{k-1} is realized, conditional on state ω_k being realized.

For $k < K+1$, by taking the derivative with respect to $y_{k,i}$ and substituting $p_{k+1}(\omega_k) = \lambda_k(\omega_k)/\eta_k(\omega_k)$ and $p_k(\omega_{k-1}) = \lambda_{k-1}(\omega_{k-1})/\eta_{k-1}(\omega_{k-1})$, we obtain

$$\left.\frac{\partial \bar{\pi}(i, \omega_k, y_{k,i})}{\partial y_{k,i}}\right|_{y_{k,i} = x_{k,i}^\star(\omega_k)} = \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \frac{\lambda_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)} - \frac{\lambda_k(\omega_k)}{\eta_k(\omega_k)} - 2c(k) x_{k,i}^\star(\omega_k) = -\frac{\mu_{k,i}(\omega_k)}{\eta_k(\omega_k)}, \tag{8}$$

where the last equality follows from (3c). Note that this equality implies that

$$x_{k,i}^\star(\omega_k) \in \arg\max_{y_{k,i} \geq 0} \bar{\pi}(i, \omega_k, y_{k,i}),$$

for all $k \in \{1, \cdots, K+1\}$. This is because if $x_{k,i}^\star(\omega_k) > 0$, then by (3f) we have $\mu_{k,i}(\omega_k) = 0$, and (8) implies that $x_{k,i}^\star(\omega_k)$ satisfies the first order optimality conditions. On the other hand, if $x_{k,i}^\star(\omega_k) = 0$, then from (8) we obtain that $\frac{\partial \bar{\pi}(i, \omega_k, y_{k,i})}{\partial y_{k,i}}\Big|_{y_{k,i} = x_{k,i}^\star(\omega_k)} \leq 0$, which readily implies that $x_{k,i}^\star(\omega_k) \in \arg\max_{y_{k,i} \geq 0} \bar{\pi}(i, \omega_k, y_{k,i})$.

Following a similar approach but using $p_{K+2}(\omega_{K+1}) = p_c$ (and the fact that $\eta(\omega_{K+1}) = 1$), the derivative of the expected profit of a firm $i \in Tier(K+1)$ is given by:

$$\left.\frac{\partial \bar{\pi}(i, \omega_{K+1}, y_{K+1,i})}{\partial y_{K+1,i}}\right|_{y_{K+1,i}=x^\star_{K+1,i}(\omega_{K+1})} = \sum_{\omega_K | \omega_{K+1} \to \omega_K, \omega_{K,i}=1} \lambda_K(\omega_K) - p_c - 2c(K+1)x^\star_{K+1,i}(\omega_{K+1})$$
$$= -\mu_{K+1,i}(\omega_{K+1}), \tag{9}$$

where the last equality follows from (3b). Once again this implies that $x^\star_{K+1,i}(\omega_{K+1}) \in \arg\max_{y_{K+1,i} \geq 0} \bar{\pi}(i, \omega_{K+1}, y_{K+1,i})$. Hence, the tuple $\{p_k(\omega_{k-1}), x^\star_{k,i}(\omega_k)\}$ also satisfies condition (i) of Definition 1; thus, it constitutes a supply equilibrium.

(c) First, note that $\mu_{k,i}(\omega_k)$ and $\mu_0(\omega_0)$ are nonnegative. This is immediate when $x^\star_{k,i}(\omega_k) > 0$ and $x^\star_0(\omega_0) > 0$. If, on the other hand, $x^\star_{k,i}(\omega_k) = 0$, condition (i) of a supply equilibrium in Definition 1 implies that

$$\mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i} - p_{k+1}(\hat{\omega}_k)\middle|\hat{\omega}_k = \omega_k\right] \leq 0.$$

Using $\lambda_k(\omega_k) = p_{k+1}(\omega_k)\eta_k(\omega_k)$ from (4), we have

$$\mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i} - p_{k+1}(\hat{\omega}_k)\middle|\hat{\omega}_k = \omega_k\right] = \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \frac{\lambda_{k-1}(\omega_{k-1})}{\eta_{k-1}(\omega_{k-1})} \frac{\eta_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)} - \frac{\lambda_k(\omega_k)}{\eta_k(\omega_k)}$$
$$= \frac{1}{\eta_k(\omega_k)} \left(\sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \lambda_{k-1}(\omega_{k-1}) - \lambda_k(\omega_k) \right) \leq 0. \tag{10}$$

Combining (10) with (5) implies that $\mu_{k,i}(\omega_k) \geq 0$ for all ω_k and $1 \leq k \leq K$. A similar argument readily applies to establish that $\mu_{K+1,i}(\omega_{K+1}) \geq 0$. Furthermore, if $x^\star_0(\omega_0) = D(p_1(\omega_0)) = 0$, then $p_1(\omega_0) \geq \alpha$. In turn, this implies

$$\mu_0(\omega_0) = \lambda_0(\omega_0) - \eta_0(\omega_0)\alpha = \eta_0(\omega_0)(p_1(\omega_0) - \alpha) \geq 0.$$

In addition to being nonnegative as we established above, by construction, variable $\mu_{k,i}(\omega_k)$ (similarly, $\mu_0(\omega_0)$) is equal to zero when $x^\star_{k,i}(\omega_k) > 0$ (similarly, $x^\star_0(\omega_0) > 0$). Thus, condition (3f) holds.

Moreover, the primal and dual variables as constructed above, satisfy conditions (3d) and (3e) given conditions (ii) and (iii) from the definition of a supply equilibrium, the nonnegativity of prices, and the fact that $x^\star_0(\omega_0) = D(p_1(\omega_0))$. In a similar fashion, note that for the given construction of variables $\mu_{k,i}(\omega_k)$ and $\mu_0(\omega_0)$, KKT conditions (3a)–(3c) are trivially satisfied when $x^\star_{k,i}(\omega_k) = 0$ (or $x^\star_0(\omega_0) = 0$). On the other hand, if $x^\star_{k,i}(\omega_k) > 0$ (or $x^\star_0(\omega_0) > 0$), condition (i) in Definition 1 implies that

$$\mathbb{E}\left[Z_{k,i}p_k(\hat{\omega}_{k-1}) - p_{k+1}(\hat{\omega}_k)\middle|\hat{\omega}_k = \omega_k\right] = 2c(k)x^\star_{k,i}(\omega_k),$$

for $1 \leq k \leq K$. Thus, (10) implies condition (3c) (a similar argument readily applies to establish that the constructed vector of dual variables satisfies also condition (3b)). Finally, when $x_0^\star(\omega_0) > 0$, condition (3a) readily follows by noting that $x_0^\star(\omega_0) = D(p_1(\omega_0))$.

Thus we conclude that $\{x_{k,i}^\star(\omega_k)\} \cup \{x_0^\star(\omega_0)\}$ together with the constructed $\{\lambda_k(\omega_k)\} \cup \{\lambda_0(\omega_0)\} \cup \{\mu_k(\omega_k)\} \cup \{\mu_0(\omega_0)\}$ is a solution of (3a)–(3f).

□

Using these auxiliary lemmas, we next complete the proof of the theorem.

Part (1): *Suppose that Assumption 1(i) holds. Then, the supply equilibrium is essentially unique.*

Let $\{p_k(\omega_{k-1}), x_{k,i}(\omega_k)\}$ be a supply equilibrium, and consider the construction in Lemma 2(c). It follows by the lemma that $\{x_{k,i}(\omega_k)\} \cup \{x_0(\omega_0)\}$ and $\{\lambda_k(\omega_k)\} \cup \{\lambda_0(\omega_0)\} \cup \{\mu_k(\omega_k)\} \cup \{\mu_0(\omega_0)\}$ constitute a solution of the KKT conditions (3a)–(3f) associated with problem (1). Lemma 1(a) and Lemma 2(a) establish that the tuple of $\{x_{k,i}(\omega_k), x_0(\omega_0)\}$ and $\{\lambda_k(\omega_k), \lambda_0(\omega_0)\}$ for valid ω_0, ω_k that satisfy (3a)–(3f) is unique. In turn, this implies that all supply equilibria share the same $\{x_{k,i}(\omega_k)\}$. Moreover, since $\lambda_k(\omega_k) = p_{k+1}(\omega_k)\eta_k(\omega_k)$ by construction (and $\eta_k(\omega_k) > 0$), it also follows that all supply equilibria share the same $\{p_{k+1}(\omega_k)\}$ for all valid states $\{\omega_k\}$. Thus, we conclude that the supply equilibrium is essentially unique.

Part (2): *Suppose that Assumptions 1(i) and (ii) hold. Then, the essentially unique equilibrium is characterized by Expressions (5), (6), and (7).*

By the expression for the procurement quantity of firm $i \in Tier(k)$ given in the statement of the theorem, i.e., (7), we have $\sum_{i \in Tier(k)} x_{k,i}(\omega_k) = s \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)}$ and

$$\sum_{i \in Tier(k+1)} z_{k+1,i} x_{k+1,i}(\omega_{k+1}) = s \frac{n(k+1,\omega_k)}{n(k+1)} \left(\prod_{\ell=k+2}^{K+1} \frac{n(\ell,\omega_{k+1})}{n(\ell)} \right) = s \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)},$$

where we make use of the fact that $\sum_{i \in Tier(k+1)} z_{k+1,i} = n(k+1,\omega_k)$, and $\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}_i\}$ and ω_{k+1} capture the same set of disruptions in tiers $\ell > k+1$, and hence $n(\ell,\omega_{k+1}) = n(\ell,\omega_k)$. These observations readily imply

$$\sum_{i \in Tier(k)} x_{k,i}(\omega_k) = \sum_{i \in Tier(k+1)} z_{k+1,i} x_{k+1,i}(\omega_{k+1}), \tag{11}$$

for $\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}\} \in \Omega_k$, and

$$D(p_1(\omega_0)) = \frac{\alpha}{\beta} - \frac{p_1(\omega_0)}{\beta} = s \prod_{\ell=1}^{K+1} \frac{n(\ell,\omega_0)}{n(\ell)} = \sum_{i \in Tier(1)} z_{1,i} x_{1,i}(\omega_1), \tag{12}$$

for $\omega_0 = \{\omega_1, \{z_{1,i}\}_i\} \in \Omega_0$. Thus, the constructed tuple satisfies conditions (ii) and (iii) defining a supply equilibrium (Definition 1). To complete the proof, it suffices to establish that the prices are nonnegative and

$$x_{k,i}(\omega_k) \in \arg\max_{y_{k,i}\geq 0} \bar{\pi}(i,\omega_k,y_{k,i}), \tag{13}$$

for all $k \in \{1,\ldots,K+1\}$, $i \in Tier(k)$, and $\omega_k \in \Omega_k$.

We first establish that (13) holds. To this end, note that $\bar{\pi}(i,\omega_k,y_{k,i})$ is concave in $y_{k,i}$, as can be seen from (2). Thus, to verify (13), it suffices to check that the first order optimality conditions are satisfied by $y_{k,i} = x_{k,i}(\omega_k)$. In other words, it suffices to show that

$$0 \geq \left.\frac{\partial \bar{\pi}(i,\omega_k,y_{k,i})}{y_{k,i}}\right|_{y_{k,i}=x_{k,i}(\omega_k)} = \mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i} - p_{k+1}(\hat{\omega}_k) - 2c(k)x_{k,i}(\hat{\omega}_k)\middle|\hat{\omega}_k = \omega_k\right], \tag{14}$$

where the inequality holds with equality for $x_{k,i}(\omega_k) > 0$. Using the expression for the prices as given in (6), we have

$$\begin{aligned}
\mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i}\middle|\hat{\omega}_k=\omega_k\right] &= \mathbb{E}\left[Z_{k,i}\left(\alpha(k) - \beta(k)s\prod_{\ell=k}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)}\right)\middle|\hat{\omega}_k=\omega_k\right] \\
&= \mathbb{E}\left[Z_{k,i}\left(\alpha(k) - \beta(k)s\frac{n(k,\hat{\omega}_{k-1})}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\middle|\hat{\omega}_k=\omega_k\right] \\
&= \alpha(k)q(k) - \beta(k)sq(k)\frac{1+q(k)(n(k)-1)}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}.
\end{aligned} \tag{15}$$

Here, the last equality follows since $n(k,\hat{\omega}_{k-1}) = \sum_{j\in Tier(k)} Z_{k,j}$ and

$$\mathbb{E}[Z_{k,i}n(k,\hat{\omega}_{k-1}) \mid \hat{\omega}_k = \omega_k] = \mathbb{E}\left[Z_{k_i} + \sum_{j\in Tier(k)|j\neq i} Z_{k,i}Z_{k,j}\right] = q(k) + q(k)^2(n(k)-1). \tag{16}$$

Similarly, we have

$$\begin{aligned}
\mathbb{E}\left[p_{k+1}(\hat{\omega}_k){+}2c(k)x_{k,i}(\hat{\omega}_k)\middle|\hat{\omega}_k=\omega_k\right] &= p_{k+1}(\omega_k) + 2c(k)x_{k,i}(\omega_k) \\
&= \alpha(k+1) - \beta(k+1)s\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)} + 2c(k)\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}.
\end{aligned} \tag{17}$$

Using these two equalities and the fact that $\alpha(k)q(k) = \alpha(k+1)$, we can rewrite (14) as follows

$$0 \geq \beta(k+1)s\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)} - \beta(k)s\frac{q(k)^2(n(k)-1+\frac{1}{q(k)})}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)} - 2c(k)\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}. \tag{18}$$

Canceling out common terms and observing that $s > 0$ under Assumption 1(i) yields

$$0 \geq \beta(k+1) - \beta(k)\frac{q(k)^2(n(k)-1+1/q(k))}{n(k)} - \frac{2c(k)}{n(k)}. \tag{19}$$

Note that this inequality always holds with equality as can be seen from the definition of $\beta(k+1)$ in (4). Thus, we conclude that (14) and, hence, (13) hold.

It remains to show that the prices are nonnegative. Note that for any ω_k, we have

$$p_{k+1}(\omega_k) \geq \alpha(k+1) - \beta(k+1)s = \alpha(k+1) - \beta(k+1)\frac{\alpha(K+2)-p_c}{\beta(K+2)} > 0, \tag{20}$$

where the inequality follows from Assumption 1(ii). Thus, the prices given in (6) are, in fact, strictly positive. Therefore, the essentially unique equilibrium is characterized by Expressions (5), (6), and (7) when Assumptions 1(i) and (ii) hold.

Part (3): *If Assumption 1(i) holds but (ii) does not, then at any supply equilibrium, there exists at least one tier $k' \in \{1,\ldots,K+1\}$, such that $p_{k'}(\bar{\omega}_{k'-1}) = 0$, where $\bar{\omega}_{k'-1} \in \Omega_{k'-1}$ is the state where no firm experiences a disruption in tiers $\{k',\ldots,K+1\}$.*

First, assume that $\frac{\alpha(k)}{\beta(k)} \geq \frac{\alpha(K+2)-p_c}{\beta(K+2)}$, for $k \in \{1,\ldots,K+1\}$, with equality for some $k = k'$. Consider the characterization provided by Expressions (5), (6), and (7). The proof of Part (2) implies (after changing the inequality in (20) to a weak inequality) that Expressions (5), (6), and (7) once again describe the essentially unique equilibrium. However, since $\alpha(k')/\beta(k') = (\alpha(K+2)-p_c)/\beta(K+2)$, it can be readily seen that in state $\bar{\omega}_{k'-1}$ where no firm in tiers $k \geq k'$ experiences a disruption, we have $p_{k'}(\omega_{k'}) = 0$; hence the claim follows.

Suppose instead that there exists some k' such that $\frac{\alpha(k')}{\beta(k')} < \frac{\alpha(K+2)-p_c}{\beta(K+2)}$ and let $\{x^\star_{k,i}(\omega_k)\}$ for $1 \leq k \leq K+1$ and $\{x^\star_0(\omega_0)\}$ denote the unique solution of optimization problem (1). We claim that there exists some valid state $\omega_\ell \in \Omega_\ell$ for $0 \leq \ell \leq K$, for which one of the first two constraints of (1) is not binding, i.e.,

$$\begin{aligned} &\sum_{i \in Tier(\ell)} x^\star_{\ell,i}(\omega_\ell) < \sum_{j \in Tier(\ell+1) | \omega_{\ell+1} \to \omega_\ell, \omega_{\ell,j}=1} x^\star_{\ell+1,j}(\omega_{\ell+1}), \qquad \text{or} \\ &x^\star_0(\omega_0) < \sum_{j \in Tier(1) | \omega_1 \to \omega_0, \omega_{0,j}=1} x^\star_{1,j}(\omega_1), \qquad \text{if } \ell = 0. \end{aligned} \tag{21}$$

By way of contradiction, suppose that the first two constraints in (1) are binding for any valid state. Recall that, as established in Lemma 1(b), the nonnegativity constraints are not binding at the optimal solution for valid states; thus, $s = \sum_{i \in Tier(K+1)} x^\star_{K+1,i}(\omega_{K+1}) > 0$. Since the first two constraints are binding in (1) for any valid state (and the optimal solution is such that $x^\star_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k)$ for all $i,j \in Tier(k)$, by Lemma 1(a)), we obtain that

$$x^\star_{k,i}(\omega_k) = \frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)} \text{ for } 1 \leq k \leq K+1 \quad \text{and} \quad x^\star_0(\omega_0) = s \prod_{\ell=1}^{K+1} \frac{n(\ell,\omega_0)}{n(\ell)}. \tag{22}$$

Let $\{\lambda_k(\omega_k), \lambda_0(\omega_0), \mu_{k,i}(\omega_k), \mu_0(\omega_0)\}$ denote the set of dual variables that satisfy (3a)–(3f). Observe that by (22) we have $x^\star_{k,i}(\omega_k), x^\star_0(\omega_0) > 0$ for valid states ω_0, ω_k. Hence, by (3f) we obtain $\mu_0(\omega_0) = \mu_k(\omega_k) = 0$ for valid states.

We claim that

$$\lambda_k(\omega_k) = \eta_k(\omega_k)\alpha(k+1) - \eta_k(\omega_k)\beta(k+1)s\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}, \tag{23}$$

for any valid state ω_k and $0 \le k \le K$. We prove that (23) holds by induction. First, consider the case when $k=0$ and let $\omega_0 \in \Omega_0$ be some valid state. Then, using the expression for $x_0^\star(\omega_0)$ in (22) (and recalling $\mu_0(\omega_0)=0$) we obtain from (3a) that

$$\lambda_0(\omega_0) = \eta_0(\omega_0)\alpha - \eta_0(\omega_0)\beta x_0^\star(\omega_0) = \eta_0(\omega_0)\alpha(1) - \eta_0(\omega_0)\beta(1)s\prod_{\ell=1}^{K+1}\frac{n(\ell,\omega_0)}{n(\ell)}.$$

Suppose that (23) holds for some $k \in \{0,1,\ldots,K-1\}$ (induction hypothesis). We will establish that it holds for $k+1$, as well. Note that for a valid state ω_{k+1}, we have

$$\begin{aligned}
\lambda_{k+1}(\omega_{k+1}) &= \sum_{\omega_k|\omega_{k+1}\to\omega_k,\omega_{k,i}=1} \lambda_k(\omega_k) - 2c(k+1)\eta_{k+1}(\omega_{k+1})x^\star_{k+1,i}(\omega_{k+1}) \\
&= \sum_{\omega_k|\omega_{k+1}\to\omega_k,\omega_{k,i}=1}\left(\eta_k(\omega_k)\alpha(k+1) - \eta_k(\omega_k)\beta(k+1)s\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}\right) - 2c(k+1)\eta_{k+1}(\omega_{k+1})x^\star_{k+1,i}(\omega_{k+1}) \\
&= \alpha(k+1)q(k+1)\eta_{k+1}(\omega_{k+1}) - s\beta(k+1)\left(\prod_{\ell=k+2}^{K+1}\frac{n(\ell,\omega_{k+1})}{n(\ell)}\right)\left(\sum_{\omega_k|\omega_{k+1}\to\omega_k,\omega_{k,i}=1}\eta_k(\omega_k)\frac{n(k+1,\omega_k)}{n(k+1)}\right) \\
&\qquad - 2c(k+1)\eta_{k+1}(\omega_{k+1})\frac{s}{n(k+1)}\prod_{\ell=k+2}^{K+1}\frac{n(\ell,\omega_{k+1})}{n(\ell)} \\
&= \alpha(k+1)q(k+1)\eta_{k+1}(\omega_{k+1}) - s\beta(k+1)\left(\prod_{\ell=k+2}^{K+1}\frac{n(\ell,\omega_{k+1})}{n(\ell)}\right)\eta_{k+1}(\omega_{k+1})\left(\frac{q(k+1)(1+q(k+1)(n(k+1)-1))}{n(k+1)}\right) \\
&\qquad - 2c(k+1)\eta_{k+1}(\omega_{k+1})\frac{s}{n(k+1)}\prod_{\ell=k+2}^{K+1}\frac{n(\ell,\omega_{k+1})}{n(\ell)} \\
&= \alpha(k+2)\eta_{k+1}(\omega_{k+1}) - s\eta_{k+1}(\omega_{k+1})\beta(k+2)\prod_{\ell=k+2}^{K+1}\frac{n(\ell,\omega_{k+1})}{n(\ell)}.
\end{aligned} \tag{24}$$

Here, the first equality follows by (3c) and the fact that $\mu_{k+1,i}(\omega_{k+1})=0$ for any valid state ω_{k+1}. The second equality follows from the induction hypothesis. The third equality follows by using (22) and the following two observations:

(i) $\sum_{\omega_k|\omega_{k+1}\to\omega_k,\omega_{k,i}=1}\eta_k(\omega_k)$ is equal to the probability that state ω_{k+1} is realized and firm $i \in Tier(k+1)$ does not experience a disruptive event, which can alternatively be expressed by $\eta_{k+1}(\omega_{k+1})q(k+1)$;

(ii) Also, $n(\ell,\omega_{k+1}) = n(\ell,\omega_k)$ for any $\ell \ge k+2$ and $\omega_{k+1}\to\omega_k$.

The fourth equality follows by observing that

$$\sum_{\omega_k|\omega_{k+1}\to\omega_k,\omega_{k,i}=1}\eta_k(\omega_k)n(k+1,\omega_k) = \mathbb{P}(Z_{k+1,i}=1)\eta_{k+1}(\omega_{k+1})\sum_{\omega_k|\omega_{k+1}\to\omega_k,\omega_{k,i}=1}\left(\eta_k(\omega_k|\omega_{k+1})\frac{n(k+1,\omega_k)}{\mathbb{P}(Z_{k+1,i}=1)}\right)$$

$$
\begin{aligned}
&= \mathbb{P}(Z_{k+1,i}=1)\eta_{k+1}(\omega_{k+1})\mathbb{E}[n(k+1,\hat{\omega}_k)|Z_{k+1,i}=1,\hat{\omega}_{k+1}=\omega_{k+1}] \\
&= \mathbb{P}(Z_{k+1,i}=1)\eta_{k+1}(\omega_{k+1})\mathbb{E}[\sum_{j\in Tier(k+1)} Z_{k+1,j}|Z_{k+1,i}=1] \\
&= \eta_{k+1}(\omega_{k+1})q(k+1)(1+q(k+1)(n(k+1)-1)). \qquad (25)
\end{aligned}
$$

The fifth equality follows from (3) and (4). Thus, the induction hypothesis holds for $k+1$ as well; i.e., Expression (23) holds for any $k\in\{1,\cdots,K\}$. Finally, by Expression (3b) we have

$$
p_c = \sum_{\omega_K|\omega_{K+1}\to\omega_K,\omega_{K,i}=1} \lambda_K(\omega_K) - 2c(K+1)x^\star_{K+1,i}(\omega_{K+1}),
$$

given that $\mu_{K+1,i}(\omega_{K+1})=0$. Using (23) to express $\lambda_K(\omega_K)$ and following a similar reasoning as in (24), yields

$$
p_c = \alpha(K+2)-\beta(K+2)s, \text{ or alternatively } s=\frac{\alpha(K+2)-p_c}{\beta(K+2)}.
$$

Let $\bar{\omega}_{k'-1}$ denote the state where no firms experience a disruption in tiers $k',\ldots,K+1$. Given that we have assumed that $\frac{\alpha(k')}{\beta(k')}<\frac{\alpha(K+2)-p_c}{\beta(K+2)}$, using (23) we obtain

$$
\frac{\lambda_{k'-1}(\bar{\omega}_{k'-1})}{\eta_{k'-1}(\bar{\omega}_{k'-1})} = \alpha(k')-\beta(k')s = \alpha(k')-\beta(k')\frac{\alpha(K+2)-p_c}{\beta(K+2)} < 0.
$$

However, given that $\lambda_{k'-1}(\omega_{k'-1})$ is nonnegative, we arrive at a contradiction to our original hypothesis that at the unique solution of (1) the first two constraints of the optimization problem are binding for any valid state. In other words, we conclude that (21) holds.

To finalize the claim, let ω_k denote a valid state where the first constraint (second if $k=0$) of (1) is not binding at an optimal solution. If there are multiple such states, let ω_k denote the one that corresponds to the largest k. Also, let $\bar{\omega}_k$ denote the state where no firm in any tier $l>k$ experiences a disruption. We claim the aforementioned constraint is also not binding for $\bar{\omega}_k$.

First, note that if $\bar{\omega}_k=\omega_k$, the claim trivially follows. Next, suppose $\bar{\omega}_k\neq\omega_k$, and the aforementioned constraint is binding for $\bar{\omega}_k$. Observe that by our choice of k and $\bar{\omega}_k$, we have

$$
s = \sum_{i\in Tier(k)} x^\star_{k,i}(\bar{\omega}_k) > \sum_{i\in Tier(k)} x^\star_{k,i}(\omega_k). \qquad (26)
$$

Let $\mathcal{T}$ denote the set of states reachable from ω_k, i.e., $\mathcal{T}=\{\omega_l|\omega_k\to\omega_{k-1}\to\cdots\to\omega_l, l<k\}\cup\{\omega_k\}$. In addition, for any $\omega_l\in\mathcal{T}$, let $m(\omega_l)$ denote the state that is identical to ω_l in terms of disruption realizations in tiers $l+1,\ldots,k$, but involves no disruptions in tiers $k+1,\ldots,K+1$. Note that $m(\omega_k)=\bar{\omega}_k$. Let $\mathcal{T}'=\{m(\omega_l)|\omega_l\in\mathcal{T}\}$. Clearly $m(\cdot)$ is a bijection between $\mathcal{T}$ and $\mathcal{T}'$, and $m(\omega_l)$ is reachable from $\bar{\omega}_k$ if and only if ω_l is reachable from ω_k.

Observe that in (1), by our choice of ω_k, the upper bound constraint on $\sum_{i\in Tier(k)} x_{k,i}(\omega_k)$ is not binding. Since (1) is a convex optimization problem, $\mathbf{x}^\star$ remains the unique optimal solution even when the constraint that sets an upper bound to $\sum_{i\in Tier(k)} x_{k,i}(\omega_k)$ is removed. We refer to the optimization problem that is obtained from (1) after removing the aforementioned constraint as (1'). We construct another solution $\mathbf{x}'$ for (1'), by setting

- $x'_{l,j}(\omega_l) = x^\star_{l,j}(\omega_l)$ for any $\omega_l \notin \mathcal{T}\cup\mathcal{T}'$,
- $x'_{l,j}(\omega_l) = \dfrac{\eta_k(\omega_k)x^\star_{l,j}(\omega_l)+\eta_k(\bar{\omega}_k)x^\star_{l,j}(m(\omega_l))}{\eta_k(\omega_k)+\eta_k(\bar{\omega}_k)}$ for $\omega_l \in \mathcal{T}$,
- $x'_{l,j}(\omega_l) = \dfrac{\eta_k(\bar{\omega}_k)x^\star_{l,j}(\omega_l)+\eta_k(\omega_k)x^\star_{l,j}(m^{-1}(\omega_l))}{\eta_k(\omega_k)+\eta_k(\bar{\omega}_k)}$ for $\omega_l \in \mathcal{T}'$,

for any tier l, $j\in Tier(l)$, and defining $x'_0(\omega_0)$ similarly. Intuitively, $\mathbf{x}'$ is obtained from $\mathbf{x}^\star$ by taking convex combinations of the decision variables between states in $\mathcal{T}$ and $\mathcal{T}'$ (and weighing decision variables corresponding to $\mathcal{T}$ by $\eta_k(\omega_k)$ and those in $\mathcal{T}'$ by $\eta_k(\bar{\omega}_k)$), while leaving the remaining decision variables intact.

The feasibility of $\mathbf{x}'$ in (1') readily follows from (26) and the fact that in (1') no constraint imposes an upper bound on $\sum_{i\in Tier(k)} x'_{k,i}(\omega_k)$. Furthermore, it can be seen that in (1) (and similarly in (1')) the objective function can be expressed as follows:

$$f+\eta_k(\omega_k)g(\{x_0\}_{\omega_0\in\mathcal{T}},\{x_{l,i}(\omega_l)\}_{i\in Tier(l),\omega_l\in\mathcal{T},l>0})+\eta_k(\bar{\omega}_k)g(\{x_0\}_{\omega_0\in\mathcal{T}'},\{x_{l,i}(\omega_l)\}_{i\in Tier(l),\omega_l\in\mathcal{T}',l>0}), \tag{27}$$

where f is a function of only $(\{x_0\}_{\omega_0\notin\mathcal{T}\cup\mathcal{T}'},\{x_{l,i}(\omega_l)\}_{i\in Tier(l),\omega_l\notin\mathcal{T}\cup\mathcal{T}',l>0})$ and f and g are concave functions. On the other hand, because of concavity and the construction of $\mathbf{x}'$, it follows that $\mathbf{x}'$ has a weakly higher objective value in (1') than $\mathbf{x}^\star$. This contradicts that fact that $\mathbf{x}^\star$ is the unique optimal solution in (1'). Therefore, we obtain a contradiction, and conclude that the first constraint (second if $k=0$) of (1) is not binding at $\bar{\omega}_k$.

Note that this implies by (3d)-(3e) that $\lambda_k(\bar{\omega}_k)=0$ (or $\lambda_0(\bar{\omega}_0)=0$). Since $\bar{\omega}_k$ is a valid state its equilibrium price is uniquely defined by $p_{k+1}(\bar{\omega}_k)=\lambda_k(\bar{\omega}_k)/\eta(\bar{\omega}_k)$ (by Lemma 2(b)). Hence, we obtain $p_{k+1}(\bar{\omega}_k)=0$. Thus, we conclude that for a state where no disruptions take place in tiers $\{k+1,\dots,K+1\}$, the equilibrium price is zero, and the claim follows. Q.E.D.

Proof of Corollary 1

Before proving the corollary, we introduce additional notation and state and prove an auxiliary lemma. Let $\mu(k)\triangleq\mathbb{E}[Z_{k,i}x_{k,i}(\hat{\omega}_k)]$ denote the expected equilibrium production output of firm i in tier k. Using the fact that $\hat{\omega}_k$ and $Z_{k,i}$ are independent (and the fact that $\mathbb{E}[Z_{k,i}]=q(k)$), $\mu(k)$ can alternatively be expressed as $\mu(k)=q(k)\mathbb{E}[x_{k,i}(\hat{\omega}_k)]$. Furthermore, let

$$\hat{\theta}(k)\triangleq\mathbb{E}\Big[\big(Z_{k,i}x_{k,i}(\hat{\omega}_k)\big)^2\Big]=q(k)\mathbb{E}\Big[x^2_{k,i}(\hat{\omega}_k)\Big],$$

for any $i\in Tier(k)$. Finally, for any tier k such that $n(k)>1$, let

$$\theta(k)\triangleq\mathbb{E}[Z_{k,i}Z_{k,j}x_{k,i}(\hat{\omega}_k)x_{k,j}(\hat{\omega}_k)]=q(k)^2\mathbb{E}[x_{k,i}(\hat{\omega}_k)x_{k,j}(\hat{\omega}_k)],$$

where $i, j \in Tier(k)$ and $i \neq j$. We next state an auxiliary lemma, which we then use in the proof of the corollary.

LEMMA 3. *Suppose Assumption 1 holds. In the equilibrium, for any tier $1 \leq k \leq K+1$ we have*

(i) $\mu(k) = \frac{s}{n(k)} \prod_{\ell=k}^{K+1} q(\ell)$;

(ii) $\theta(k) = \left(\frac{sq(k)}{n(k)}\right)^2 \prod_{\ell=k+1}^{K+1} \left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell) - 1 + \frac{1}{q(\ell)}\right)$, *for* $n(k) > 1$;

(iii) $\hat{\theta}(k) = \frac{s^2 q(k)}{n(k)^2} \prod_{\ell=k+1}^{K+1} \left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell) - 1 + \frac{1}{q(\ell)}\right)$, *and thus for* $n(k) > 1$, *we have* $\hat{\theta}(k) = \frac{\theta(k)}{q(k)}$.

Proof:

(i) By Theorem 1, we have that $x_{k,i}(\omega_k) = \frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)}$, for every tier k when the realized state is ω_k. This immediately implies that

$$\mu(k) = q(k)\mathbb{E}[x_{k,i}(\hat{\omega}_k)] = q(k)\mathbb{E}\left[\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right] = q(k)\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} q(\ell) = \frac{s}{n(k)} \prod_{\ell=k}^{K+1} q(\ell),$$

given that disruptive events are independent and $\mathbb{E}[n(\ell,\hat{\omega}_k)] = \mathbb{E}\big[\sum_{i\in Tier(\ell)} Z_{\ell,i}\big] = q(\ell)n(\ell)$, for $\ell \geq k+1$.

(ii), (iii) First, note that for tier $K+1$ we have $\theta(K+1) = \left(\frac{sq(K+1)}{n(K+1)}\right)^2$ if $n(K+1) > 1$ and $\hat{\theta}(K+1) = \frac{q(K+1)}{n(K+1)^2}s^2$, given that $s_i = \frac{s}{n(K+1)}$ for any firm $i \in Tier(K+1)$ and $\hat{\omega}_{K+1} = \omega_{K+1} = \emptyset$. For any other tier k, we can write the following recursive equation for $\theta(k)$

$$\begin{aligned}
\theta(k) &= q(k)^2\mathbb{E}[x_{k,i}(\hat{\omega}_k)x_{k,j}(\hat{\omega}_k)] = q(k)^2\mathbb{E}\left[\left(\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\left(\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\right] \\
&= q(k)^2\mathbb{E}\left[\left(\frac{s}{n(k)}\frac{n(k+1,\hat{\omega}_k)}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\left(\frac{s}{n(k)}\frac{n(k+1,\hat{\omega}_k)}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\mathbb{E}\left[\left(n(k+1,\hat{\omega}_k)\frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\left(n(k+1,\hat{\omega}_k)\frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\mathbb{E}\left[\left(\sum_{t\in Tier(k+1)} Z_{k+1,t}x_{k+1,t}(\hat{\omega}_{k+1})\right)\left(\sum_{t\in Tier(k+1)} Z_{k+1,t}x_{k+1,t}(\hat{\omega}_{k+1})\right)\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\mathbb{E}\left[\sum_{t,u\in Tier(k+1),t\neq u} Z_{k+1,t}Z_{k+1,u}x_{k+1,t}(\hat{\omega}_{k+1})x_{k+1,u}(\hat{\omega}_{k+1}) + \sum_{t\in Tier(k+1)} \left(Z_{k+1,t}x_{k+1,t}(\hat{\omega}_{k+1})\right)^2\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\Big(n(k+1)\big(n(k+1)-1\big)\theta(k+1) + n(k+1)\hat{\theta}(k+1)\Big),
\end{aligned} \tag{28}$$

where the expression in the fourth line follows from the fact that $n(k+1,\hat{\omega}_k) = \sum_{t\in Tier(k+1)} Z_{k+1,t}$ and

$$\frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)} = \frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_{k+1})}{n(\ell)} = x_{k+1,t}(\hat{\omega}_{k+1}), \tag{29}$$

for every firm $t \in Tier(k+1)$ (since $\hat{\omega}_k$ and $\hat{\omega}_{k+1}$ both encode the same disruptions in tiers above $k+1$). Following the same approach for $\hat{\theta}(k)$ we obtain that

$$\hat{\theta}(k) = \Big(\frac{q(k)}{n(k)^2}\Big)\Big(n(k+1)\big(n(k+1)-1\big)\theta(k+1) + n(k+1)\hat{\theta}(k+1)\Big). \tag{30}$$

Expressions (28) and (30) imply that $\theta(k) = q(k)\hat{\theta}(k)$. Exploiting this observation, (30) can be written as

$$\hat{\theta}(k) = \Big(\frac{q(k)}{n(k)^2}\Big)\Big(n(k+1)\big(n(k+1)-1\big)q(k+1) + n(k+1)\Big)\hat{\theta}(k+1). \tag{31}$$

Using this equality recursively with the boundary condition $\hat{\theta}(K+1) = \frac{q(K+1)}{n(K+1)^2}s^2$, part (iii) of the lemma follows. Together with the fact that $\theta(k) = q(k)\hat{\theta}(k)$, this implies part (ii); thus, completing the proof of the lemma. □

Next, using Lemma 3 we provide an expression for the (unconditional) expected profits of firms in different tiers. Consider firm i in tier $k \in \{1, \dots, K\}$. Recall that its expected profit is given as follows:

$$\pi(k) = \mathbb{E}\Bigg[p_k(\hat{\omega}_{k-1})Z_{k,i}x_{k,i}(\hat{\omega}_k) - p_{k+1}(\hat{\omega}_k)x_{k,i}(\hat{\omega}_k) - c(k)x_{k,i}^2(\hat{\omega}_k)\Bigg].$$

As before we have $n(k, \hat{\omega}_{k-1}) = \sum_{t\in Tier(k)} Z_{k,t}$ and

$$\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)} = \frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_{k})}{n(\ell)} = x_{k,t}(\hat{\omega}_k),$$

for every firm $t \in Tier(k)$ (see Expression (29)). Using these observations, we obtain

$$s\prod_{\ell=k}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)} = n(k,\hat{\omega}_{k-1})\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)} = \sum_{t\in Tier(k)} Z_{k,t}x_{k,t}(\hat{\omega}_k) = \sum_{t\in Tier(k-1)} x_{k-1,t}(\hat{\omega}_{k-1}), \tag{32}$$

where the last equality follows from Definition 1 after noting that under Assumption 1 $p_{k+1}(\omega_k) > 0$ for any state $\omega_k \in \Omega_k$. Together with Theorem 1 this equality in turn implies that

$$p_k(\hat{\omega}_{k-1}) = \alpha(k) - \beta(k)\sum_{t\in Tier(k)} Z_{k,t}x_{k,t}(\hat{\omega}_k) = \alpha(k) - \beta(k)\sum_{t\in Tier(k-1)} x_{k-1,t}(\hat{\omega}_{k-1}).$$

Using these expressions (and the corresponding expressions for $p_{k+1}(\hat{\omega}_k)$), the profits can be expressed as follows:

$$\begin{aligned}
\pi(k) &= \mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i}x_{k,i}(\hat{\omega}_k) - p_{k+1}(\hat{\omega}_k)x_{k,i}(\hat{\omega}_k) - c(k)x_{k,i}^2(\hat{\omega}_k)\right] \\
&= \mathbb{E}\left[\alpha(k)Z_{k,i}x_{k,i}(\hat{\omega}_k)\right] - \mathbb{E}\left[\beta(k)Z_{k,i}x_{k,i}(\hat{\omega}_k)\sum_{t\in Tier(k)} Z_{k,t}x_{k,t}(\hat{\omega}_k)\right] - \mathbb{E}\left[\alpha(k+1)x_{k,i}(\hat{\omega}_k)\right] \\
&\quad + \mathbb{E}\left[\beta(k+1)x_{k,i}(\hat{\omega}_k)\sum_{t\in Tier(k)} x_{k,t}(\hat{\omega}_k)\right] - \mathbb{E}\left[c(k)x_{k,i}^2(\hat{\omega}_k)\right] \\
&= \mathbb{E}\left[\alpha(k)Z_{k,i}x_{k,i}(\hat{\omega}_k)\right] - \mathbb{E}\left[\beta(k)\left(\sum_{t|t\neq i} Z_{k,i}Z_{k,t}x_{k,i}(\hat{\omega}_k)x_{k,t}(\hat{\omega}_k) + \left(Z_{k,i}x_{k,i}(\hat{\omega}_k)\right)^2\right)\right] \\
&\quad - \mathbb{E}\left[\alpha(k+1)x_{k,i}(\hat{\omega}_k)\right] + \mathbb{E}\left[\beta(k+1)x_{k,i}(\hat{\omega}_k)\sum_{t\in Tier(k)} x_{k,t}(\hat{\omega}_k)\right] - \mathbb{E}\left[c(k)x_{k,i}^2(\hat{\omega}_k)\right].
\end{aligned} \tag{33}$$

Recalling the definitions of $\mu(k), \theta(k)$, and $\hat{\theta}(k)$, we can rewrite (33) as follows:

$$\begin{aligned}
\pi(k) = \alpha(k)\mu(k) - \beta(k)\left((n(k)-1)\,\theta(k) + \hat{\theta}(k)\right) - \frac{\alpha(k+1)}{q(k)}\mu(k) \\
+ \frac{\beta(k+1)}{q(k)^2}\left[(n(k)-1)\theta(k) + q(k)\hat{\theta}(k)\right] - \frac{c(k)}{q(k)}\hat{\theta}(k).
\end{aligned} \tag{34}$$

Canceling common terms using (3) and noting that $\theta(k) = q(k)\hat{\theta}(k)$, the expression for $\pi(k)$ simplifies to

$$\begin{aligned}
\pi(k) &= \frac{\beta(k+1)}{q(k)}n(k)\hat{\theta}(k) - \beta(k)\left((n(k)-1)\,\hat{\theta}(k)q(k) + \hat{\theta}(k)\right) - \frac{c(k)}{q(k)}\hat{\theta}(k) \\
&= \hat{\theta}(k)\left(\frac{\beta(k+1)}{q(k)}n(k) - \beta(k)\Big((n(k)-1)q(k) + 1\Big) - \frac{c(k)}{q(k)}\right).
\end{aligned} \tag{35}$$

Note that the expression defining $\beta(k)$, i.e., Expression (4), yields

$$\beta(k+1)n(k) - \beta(k)(n(k)-1)q(k)^2 - \beta(k)q(k) = 2c(k).$$

Using this observation, (35) can be written as $\pi(k) = \hat{\theta}(k)\frac{c(k)}{q(k)}$. Combining this with the expression for $\hat{\theta}(k)$, i.e., $\hat{\theta}(k) = \frac{s^2 q(k)}{n(k)^2}\prod_{\ell=k+1}^{K+1}\left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell) - 1 + \frac{1}{q(\ell)}\right)$, yields

$$\pi(k) = s^2\frac{c(k)}{n(k)^2}\prod_{\ell=k+1}^{K+1} q(\ell)^2\left(\frac{n(\ell) - 1 + \frac{1}{q(\ell)}}{n(\ell)}\right), \tag{36}$$

which completes the proof of the claim for firms in any tier $k \in \{1, \ldots, K\}$.

Finally, consider tier $k = K+1$, and recall that by Theorem 1, we have

$$p_{K+2}(\hat{\omega}_{K+1}) = p_c = \alpha(K+2) - \beta(K+2)s = \alpha(K+2) - \beta(K+2)\sum_{t\in Tier(K+1)} x_{K+1,t}(\hat{\omega}_{K+1}).$$

Using this observation the profits can once again be expressed as in (33). Hence, following the same steps as before yields the profit expression in (36) for $k = K+1$. Thus, the claim follows for $k = K+1$ as well. Q.E.D.

Proof of Proposition 1

Recall that according to Theorem 1, equilibrium supply s is given by $s = \frac{\alpha(K+2)-p_c}{\beta(K+2)}$. Using (4) recursively to state $\beta(K+2)$ more explicitly, supply s can be expressed as follows:

$$s = \left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\right)\left(\sum_{r=1}^{K+1} \frac{2c(r)}{n(r)} \prod_{\ell=r+1}^{K+1} \frac{q(\ell)^2}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right) + \beta \prod_{\ell=1}^{K+1} \frac{q(\ell)^2}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right)\right)^{-1}.$$

The first part of the proposition follows immediately by observing that the cost coefficients appear only in the denominator; thus, increasing the cost coefficient corresponding to any tier decreases equilibrium supply s (provided that both before and after the update of the cost coefficient Assumption 1 is satisfied). Similarly, increasing the number of firms $n(\ell)$ in tier ℓ leads to a decrease in the term $\frac{n(\ell)-1+1/q(\ell)}{n(\ell)}$; thus, equilibrium supply is increasing in $n(\ell)$ for any tier ℓ.

To establish the last part of the proposition, we consider tier k and define the following terms to simplify exposition:

$$A_k \triangleq \sum_{r=k}^{K+1} \frac{2c(r)}{n(r)} \prod_{\ell=r+1}^{K+1} \frac{q(\ell)^2}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right) \tag{37}$$

$$B_k \triangleq \beta \prod_{\ell=1,\ell\neq k}^{K+1} \frac{q(\ell)^2}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right) + \sum_{r=1}^{k-1} \frac{2c(r)}{n(r)} \prod_{\ell=r+1,\ell\neq k}^{K+1} \frac{q(\ell)^2}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right). \tag{38}$$

Note that A_k and B_k are strictly positive and independent of $q(k)$. We can rewrite s in terms of A_k and B_k as follows:

$$s = \frac{\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c}{\left(A_k + B_k \frac{q(k)^2}{n(k)}\left(n(k)-1+\frac{1}{q(k)}\right)\right)} = \frac{n(k)\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\right)}{\left(A_k n(k) + B_k q(k)\left(1+(n(k)-1)q(k)\right)\right)}. \tag{39}$$

The partial derivative of s with respect to $q(k)$ can be expressed as follows:

$$\frac{\partial s}{\partial q(k)} = \frac{n(k)\left(A_k n(k)\alpha \prod_{\ell=1,\ell\neq k}^{K+1} q(\ell) + B_k\left(\left(1+2(n(k)-1)q(k)\right)p_c - \left(n(k)-1\right)\alpha q(k)^2 \prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)\right)\right)}{\left(A_k n(k) + B_k q(k)\left(1+(n(k)-1)q(k)\right)\right)^2}. \tag{40}$$

First, note that the denominator of (40) is always positive. Second, the numerator is concave (quadratic) in $q(k)$ and positive at $q(k)=0$. Thus, it follows that there exists some $\tilde{q} > 0$ such that for $q(k) \leq \tilde{q}$, Expression (40) is positive, whereas for $q(k) > \tilde{q}$, (40) is negative. For some modeling primitives, $\tilde{q} < 1$ and, thus, aggregate supply s at equilibrium first increases ((40) is positive) and then decreases in $q(k)$ ((40) is negative). We provide an illustration of the (potential) nonmonotonicity of s in $q(k)$ in Figure 2 of the paper. Q.E.D.

Proof of Proposition 2

According to Corollary 1, the expression for the profits of firms in tier ℓ is given as follows:

$$\pi(\ell) = s^2 \frac{c(\ell)}{n(\ell)^2} \prod_{m=\ell+1}^{K+1} \frac{q(m)^2}{n(m)} \left(n(m) - 1 + \frac{1}{q(m)} \right). \tag{41}$$

We analyze the effect of increasing $q(k)$ on $\pi(\ell)$ by considering the following cases separately: (a) $\ell < k$ and (b) $\ell \geq k$.

Case (a) $\ell < k$: Recall that s can be stated in terms of A_k and B_k defined in Expressions (37) and (38), respectively, as follows (see (39)):

$$s = \frac{\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c}{\left(A_k + B_k q(k)^2 \left(\frac{n(k)-1+1/q(k)}{n(k)} \right) \right)}.$$

Moreover, we can rewrite $\pi(\ell)$ as

$$\pi(\ell) = s^2 \frac{c(\ell)}{n(\ell)^2} \frac{q(k)^2}{n(k)} \left(n(k) - 1 + \frac{1}{q(k)} \right) \prod_{m=\ell+1, m \neq k}^{K+1} \frac{q(m)^2}{n(m)} \left(n(m) - 1 + \frac{1}{q(m)} \right),$$

or, equivalently, as

$$\pi(\ell) = \Gamma \left(q(k)^2 \left(n(k) - 1 + \frac{1}{q(k)} \right) \right) \left[\frac{\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c}{\left(A_k + B_k q(k)^2 \left(\frac{n(k)-1+1/q(k)}{n(k)} \right) \right)} \right]^2,$$

where Γ is a constant independent of $q(k)$. The partial derivative of $\pi(\ell)$ with respect to $q(k)$ can be written as follows:

$$\begin{aligned}
\frac{\partial \pi(\ell)}{\partial q(k)} = {} & \Gamma \Big(A_k n(k) + B_k q(k) \big(q(k)(n(k)-1) + 1 \big) \Big)^{-3} \\
& n(k)^2 (\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c) \Bigg[B_k q(k)(q(k)(n(k)-1)+1) \left(\alpha \prod_{\ell=1}^{K+1} q(\ell) + p_c + 2(n(k)-1) q(k) p_c \right) + \\
& A_k n(k) \left(\alpha \prod_{\ell=1}^{K+1} q(\ell) \big(3 + 4q(k)(n(k)-1) \big) - p_c \big(1 + 2q(k)(n(k)-1) \big) \right) \Bigg] > 0,
\end{aligned}$$

where the fact that $\frac{\partial \pi(\ell)}{\partial q(k)} > 0$ follows since both the numerator and denominator in the expression above are positive.

Case (b) $\ell \geq k$: In this case, Expression (41) directly implies that firms' profits in tier ℓ depend on $q(k)$ only through term s. Therefore, from Proposition 1, we conclude that $\pi(\ell)$ is, in general, nonmonotonic in $q(k)$. For completeness, we provide an example that illustrates this nonmonotonicity in Figure 2 of the paper. Q.E.D.

Proof of Proposition 3

According to Corollary 1, the expression for the profits of firms in tier ℓ is given as follows:

$$\pi(\ell) = s^2 \frac{c(\ell)}{n(\ell)^2} \prod_{m=\ell+1}^{K+1} \frac{q(m)^2}{n(m)} \left(n(m) - 1 + \frac{1}{q(m)} \right). \tag{42}$$

We analyze the effect of increasing $n(k)$ on $\pi(\ell)$ by considering the following three cases separately: (a) $k > \ell$, (b) $k = \ell$, and (c) $k < \ell$.

Case (a) $k > \ell$ *:* Collecting the terms that depend on $n(k)$ in (42), it can be seen that for $k > \ell$, $\pi(\ell)$ depends on $n(k)$ only through term $s^2 \frac{n(k)-1+1/q(k)}{n(k)}$. Thus, the sign of $\frac{\partial \pi(\ell)}{\partial n(k)}$ is the same as the sign of $\frac{\partial}{\partial n(k)} \left(s^2 \frac{n(k)-1+1/q(k)}{n(k)} \right)$ and this holds for all tiers ℓ downstream of k. Note that we can write

$$s^2 \frac{n(k) - 1 + 1/q(k)}{n(k)} = \frac{\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c \right)^2 \left(n(k)^2 - n(k) + n(k)/q(k) \right)}{\left(A_{k+1} n(k) + D_k + B_k q(k)^2 \left(n(k) - 1 + \frac{1}{q(k)} \right) \right)^2},$$

where A_k and B_k are defined as in (37) and (38); and

$$D_k \triangleq 2c(k) \prod_{m=k+1}^{K+1} \frac{q(m)^2}{n(m)} \left(n(m) - 1 + \frac{1}{q(m)} \right). \tag{43}$$

Thus,

$$\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c \right)^{-2} \frac{\partial}{\partial n(k)} \left(s^2 \frac{n(k) - 1 + 1/q(k)}{n(k)} \right) =$$
$$= \frac{(q(k) - 1) \Big[A_{k+1} n(k) - B_k q(k)(q(k) n(k) + 1 - q(k)) \Big] + D_k \Big(2n(k) q(k) - q(k) + 1 \Big)}{q(k) \left(A_{k+1} n(k) + D_k + B_k q(k)^2 \left(n(k) - 1 + \frac{1}{q(k)} \right) \right)^3},$$

which is positive for large enough $q(k) < 1$ (since it is a continuous function and takes strictly positive values for $q(k) = 1$). Conversely, one can identify conditions under which the numerator is negative. For example, it can be readily seen that this is the case if $q(k) \ll 1$ and $c(k)$ is sufficiently small.[2] In this case, the terms in the numerator involving B_k and D_k are small, and the first (negative) term dominates, thereby yielding nonmonotonicity of profits in $n(k)$.

[2] Note that it is straightforward to construct problem instances where having small $q(k)$ and $c(k)$ does not violate Assumption 1. In particular, Lemma 3 provides sufficient conditions under which the assumption holds for any network structure and choice of cost parameters. Furthermore, this lemma shows that even for $q(k) \ll 1$ for sufficiently small p_c the assumption continues to hold.

Case (b) $k=\ell$: Using A_k, B_k, D_k (as defined in (37), (38), and (43), respectively), we can rewrite s as follows:

$$s = \frac{\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c}{\left(A_{k+1} + \frac{D_k}{n(k)} + B_k q(k)^2 \left(\frac{n(k)-1+\frac{1}{q(k)}}{n(k)}\right)\right)}$$
$$= \frac{n(k)\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\right)}{\left(A_{k+1} n(k) + D_k + B_k q(k)^2 \left(n(k) - 1 + \frac{1}{q(k)}\right)\right)}.$$

Thus, the expression for the firms' profits in tier k can be written as

$$\pi(k) = \frac{\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\right)^2}{\left(A_{k+1} n(k) + D_k + B_k q(k)^2 \left(n(k) - 1 + \frac{1}{q(k)}\right)\right)^2} \frac{D_k}{2}.$$

Finally, by noting that A_{k+1}, B_k, and D_k are independent of $n(k)$, we obtain that $\frac{\partial \pi(k)}{\partial n(k)} < 0$ (since $n(k)$ appears only in the denominator of the expression).

Case (c) $k<\ell$: Ignoring the terms that do not depend on $n(k)$ in (42), it can be seen that for $k<\ell$, $\pi(\ell)$ depends on $n(k)$ only through term s. Thus, the claim follows directly by noting that the aggregate supply of raw materials s is increasing in $n(k)$ as shown in Proposition 1. Q.E.D.

Proof of Lemmas 1 and 2

First, note that $\mathbb{E}[U] = \mu(1)$, and according to Lemma 3 we have $\mathbb{E}[U] = s\prod_{\ell=1}^{K+1} q(\ell)$. In addition, note that according to the definition of the coefficient of variation, we have

$$CV(U) = \sqrt{\frac{\mathrm{Var}(U)}{\mathbb{E}[U]^2}} = \sqrt{\frac{\mathbb{E}[U^2] - \mathbb{E}[U]^2}{\mathbb{E}[U]^2}} = \sqrt{\frac{\hat{\theta}(1)\left(n(1) + n(1)(n(1)-1)q(1)\right)}{\left(s\prod_{\ell=1}^{K+1} q(\ell)\right)^2} - 1}. \tag{44}$$

Observe that Lemma 3 implies that $\hat{\theta}(1) = \frac{s^2 q(1)}{n(1)^2} \prod_{\ell=2}^{K+1} \left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell) - 1 + \frac{1}{q(\ell)}\right)$. Substituting this expression for $\hat{\theta}(1)$ in (44), it immediately follows that

$$CV(U) = \left(\prod_{\ell=1}^{K+1} \frac{1}{n(\ell)}\left(n(\ell) - 1 + \frac{1}{q(\ell)}\right) - 1\right)^{\frac{1}{2}}. \tag{45}$$

Finally, Corollary 1 together with Expression (45) yield for the profits of a retailer:

$$\pi(1) = s^2 \frac{c(1)}{n(1)} \frac{1}{n(1) - 1 + 1/q(1)} \left(CV(U)^2 + 1\right) \prod_{\ell=2}^{K+1} q(\ell)^2.$$

Q.E.D.

Proof of Lemma 3

First, note that $p_c < \alpha\prod_{\ell=1}^{K+1} q(\ell)$ in (10) implies the first part of Assumption 1. To show that (10) guarantees that the second part of Assumption 1 also holds, note that

$$\frac{\beta(k)}{\beta(K+2)} < \frac{1}{\prod_{\ell=k}^{K+1}\frac{q(\ell)^2}{n(\ell)}\Bigg(n(\ell)-1+1/q(\ell)\Bigg)} \le \frac{1}{\prod_{\ell=k}^{K+1} q(\ell)^2}, \tag{46}$$

where the first inequality follows since $c(\ell) > 0$ for $1 \le \ell \le K+1$ and the second follows since $\frac{1}{n(\ell)}\Bigg(n(\ell)-1+1/q(\ell)\Bigg) \ge 1$ for any $q(\ell)$ such that $0 < q(\ell) \le 1$.

On the other hand, when $p_c > \alpha\Bigg(\prod_{\ell=1}^{K+1} q(\ell) - \prod_{\ell=1}^{K+1} q(\ell)^2\Bigg)$, we have

$$\frac{\alpha(k)}{\alpha(K+2)-p_c} > \frac{\alpha\prod_{\ell=1}^{k-1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell)^2} = \frac{1}{\prod_{\ell=1}^{k-1} q(\ell)}\frac{1}{\prod_{\ell=k}^{K+1} q(\ell)^2} > \frac{\beta(k)}{\beta(K+2)},$$

where the last inequality follows directly from (46) given that $0 < q(\ell) \le 1$ for $1 \le \ell \le K+1$. Q.E.D.

Proof of Proposition 4

According to Corollary 1, a raw materials supplier's profits are given as

$$\pi(K+1) = s^2\frac{c(K+1)}{n(K+1)^2}.$$

Thus, since $n(K+1)$ is assumed to be fixed and given, it follows that the networks that maximize s also maximize $\pi(K+1)$. Q.E.D.

Proof of Proposition 5

The fact that $\mathcal{V}_S = \mathcal{V}_{supply}$ follows directly from Proposition 4. Note that in the absence of any disruption risk the expression for a retailer's expected profit simplifies to $\pi(1) = s^2\frac{c(1)}{n(1)^2}$ (see Corollary 1). Thus, the network that maximizes the retailers' profits is the same as the one that maximizes the supply procured in the upstream tier, which, in turn, is the same as the one that maximizes the suppliers' profits.

Finally, when $q = 1$, the expression for the supply level s simplifies as follows (see Theorem 1):

$$s = \frac{\alpha - p_c}{\beta(K+2)}.$$

Thus, when the cost coefficients for all tiers are equal, the supply level s is maximized when

$$\beta(K+2) = \sum_{r=1}^{K+1}\frac{2c}{n(r)} + \beta, \tag{47}$$

is minimized. In turn, this directly implies that, given $n(1), n(K+1)$, the optimal network is such that the number of firms in any two intermediate tiers differs by at most 1, i.e., Expression (47) is minimized when the number of firms in any two tiers $k, \ell \in \{2, \ldots, K\}$ satisfies $|n(k) - n(\ell)| \le 1$. Q.E.D.

Proof of Theorem 2

Proposition 4 readily implies that $s_{\mathcal{N}_R} \leq s_{\mathcal{N}_S}$. By way of contradiction, assume that $CV(U_{\mathcal{N}_S}) > CV(U_{\mathcal{N}_R})$. Then, since the supply s is weakly larger and CV is strictly larger in $\mathcal{N}_S$, Lemma 2 implies that retailers can obtain a strictly larger expected profit in $\mathcal{N}_S$. This contradicts our assumption about the optimality of $\mathcal{N}_R$ for retailers' profits. Thus, we must have $CV(U_{\mathcal{N}_S}) \leq CV(U_{\mathcal{N}_R})$, which concludes the proof of part (i) of the theorem.

To establish part (ii), consider $\mathcal{N}_R$ and assume by way of contradiction that there exist two consecutive tiers $k-1$ and k (with $2<k<K+1$) such that $n(k)<n(k-1)$. In what follows, we show that swapping the number of firms in tiers $k-1$ and k leads to a network that generates higher (expected) profits for the retailers. When the disruption risk is the same for all tiers, Lemma 1 implies that such a swap does not change $CV(U)$. Hence, it follows from Lemma 2 that retailers' profits are only affected through the s term in the expression for $\pi(1)$. Theorem 1 implies that when $q(\ell)=q$ and $c(\ell)=c$ for every tier ℓ, $s=(pq^{K+1}-p_c)/\beta(K+2)$, where by using Theorem 1 to express $\beta(K+2)$ more explicitly we obtain:

$$\beta(K+2)=\left(\sum_{r=1}^{K+1}\frac{2cq^{2(K-r+1)}}{n(r)}\prod_{\ell=r+1}^{K+1}\frac{n(\ell)-1+1/q}{n(\ell)}+\beta q^{2(K+1)}\prod_{\ell=1}^{K+1}\frac{n(\ell)-1+1/q}{n(\ell)}\right).$$

It is straightforward to see that when $q<1$ the aforementioned swap strictly decreases term $\sum_{r=1}^{K+1}\frac{2cq^{2(K-r+1)}}{n(r)}\prod_{\ell=r+1}^{K+1}\frac{n(\ell)-1+1/q}{n(\ell)}$. Given that the last term in the expression for s remains the same after the swap, this implies that both supply s and (through Lemma 2) the retailers' profits strictly increase. Thus, we obtain a contradiction to the optimality of $\mathcal{N}_R$ for retailers' profits. Hence, it cannot be the case that $n(k)<n(k-1)$ for two consecutive tiers.

A similar argument can be used to establish that $\mathcal{N}_S$ also takes the form of an inverted pyramid. Suppose that this is not the case. The same approach as before implies that a swap between two consecutive tiers $k-1$ and k with $n(k)<n(k-1)$ (and $2<k<K+1$) strictly increases s. However, by Proposition 4, $\mathcal{N}_S$ maximizes s, thereby yielding a contradiction. Q.E.D.

Proof of Proposition 6

Let the probability of a disruptive event be equal to q and the production cost coefficients be equal to c for all tiers. To simplify exposition, we use the shorthand notation

$$\phi\triangleq\beta(K+2)=\left(\sum_{r=1}^{K+1}\frac{2c}{n(r)}q^{2(K+1-r)}\prod_{\ell=r+1}^{K+1}\frac{n(\ell)-1+1/q}{n(\ell)}+\beta q^{2(K+1)}\prod_{\ell=1}^{K+1}\frac{n(\ell)-1+1/q}{n(\ell)}\right), \tag{48}$$

where the expression for $\beta(K+2)$ is obtained by recursively using (4). Note that by Theorem 1 we have $s=\frac{\left(\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c\right)}{\phi}$. Thus, the structure that maximizes s also minimizes ϕ. The following lemma is useful in the analysis that follows.

LEMMA 4. *Consider two network structures $\mathcal{N}$ and $\mathcal{N}'$ with the same total number of firms. Suppose that the numbers of firms in different tiers satisfy $n(k-1) \le n(k)$, $n'(k-1) \le n'(k)$ for $2 < k < K+1$ and $n(k) = n'(k)$ for $k \in \{1, K+1\}$. Let ϕ and ϕ' be the corresponding values of Expression (48) for $\mathcal{N}$ and $\mathcal{N}'$ respectively. Assume that there exist two tiers k_1, k_2 such that $K+1 > k_2 > k_1 > 1$ and*

$$n(k_1) < n'(k_1) \text{ and } n(k_2) > n'(k_2),$$
$$n(\ell) = n'(\ell) \quad \forall \ell \in \{k_1+1, \ldots, k_2-1\},$$
$$\text{and } n(\ell) \le n'(\ell) \ \forall \ell < k_1.$$

If in $\mathcal{N}$ removing one firm from k_2 and adding it to k_1 weakly increases ϕ, then in $\mathcal{N}'$ removing one firm from k_1 and adding it to k_2 weakly decreases ϕ'; thus, it increases supply s.

Proof: To simplify exposition, we define the following set of terms:

$$\gamma_1 \triangleq \left(\sum_{r=1}^{k_1-1} \frac{2c}{n(r)} q^{2(K+1-r)} \prod_{\ell=r+1}^{k_1-1} \frac{n(\ell)-1+1/q}{n(\ell)} + \beta q^{2(K+1)} \prod_{\ell=1}^{k_1-1} \frac{n(\ell)-1+1/q}{n(\ell)} \right),$$
$$\gamma_2 \triangleq \sum_{r=k_1+1}^{k_2-1} \frac{2c}{n(r)} q^{2(K+1-r)} \prod_{\ell=r+1}^{k_2-1} \frac{n(\ell)-1+1/q}{n(\ell)},$$
$$\gamma_3 \triangleq \sum_{r=k_2+1}^{K+1} \frac{2c}{n(r)} q^{2(K+1-r)} \prod_{\ell=r+1}^{K+1} \frac{n(\ell)-1+1/q}{n(\ell)},$$
$$\zeta_1 \triangleq \prod_{\ell=k_1+1}^{k_2-1} \frac{n(\ell)-1+1/q}{n(\ell)},$$
$$\zeta_2 \triangleq \prod_{\ell=k_2+1}^{K+1} \frac{n(\ell)-1+1/q}{n(\ell)}.$$

Similarly, let γ_1', γ_2', γ_3', ζ_1', and ζ_2' denote the corresponding quantities for network structure $\mathcal{N}'$. Finally, let $\hat{\phi}$ denote the value of Expression (48) corresponding to the network that results from moving one firm from tier k_2 to tier k_1 in network $\mathcal{N}$. Then, according to the assumption in the lemma, we have $\hat{\phi} \ge \phi$. Next, we write ϕ and $\hat{\phi}$ in terms of γ_1, γ_2, γ_3, ζ_1 and ζ_2. We have

$$\phi = \gamma_1 \left(\frac{\Big(n(k_1)-1+1/q\Big)\Big(n(k_2)-1+1/q\Big)}{n(k_1)n(k_2)} \right) \zeta_1\zeta_2 + 2c\left(\frac{q^{2(K+1-k_1)}}{n(k_1)}\right)\zeta_1\left(\frac{n(k_2)-1+1/q}{n(k_2)}\right)\zeta_2$$
$$+ \gamma_2\left(\frac{n(k_2)-1+1/q}{n(k_2)}\right)\zeta_2 + 2c\left(\frac{q^{2(K+1-k_2)}}{n(k_2)}\right)\zeta_2 + \gamma_3.$$

Similarly, $\hat{\phi}$ is given by

$$\hat{\phi} = \gamma_1 \left(\frac{\Big(n(k_1)+1/q\Big)\Big(n(k_2)-2+1/q\Big)}{\Big(n(k_1)+1\Big)\Big(n(k_2)-1\Big)} \right) \zeta_1\zeta_2 + 2c\left(\frac{q^{2(K+1-k_1)}}{\big(n(k_1)+1\big)} \right)\zeta_1\left(\frac{n(k_2)-2+1/q}{\big(n(k_2)-1\big)} \right)\zeta_2 \\ + \gamma_2\left(\frac{n(k_2)-2+1/q}{\big(n(k_2)-1\big)} \right)\zeta_2 + 2c\left(\frac{q^{2(K+1-k_2)}}{\big(n(k_2)-1\big)} \right)\zeta_2 + \gamma_3.$$

Note that $\phi - \hat{\phi} \le 0$ implies $(\phi - \hat{\phi})/\zeta_2 \le 0$ which using the above expressions can be expressed as follows:

$$\gamma_1\zeta_1\left(\frac{\Big(n(k_1)-1+1/q\Big)\Big(n(k_2)-1+1/q\Big)}{n(k_1)n(k_2)} - \frac{\Big(n(k_1)+1/q\Big)\Big(n(k_2)-2+1/q\Big)}{\Big(n(k_1)+1\Big)\Big(n(k_2)-1\Big)} \right) \\ + \left(\frac{2cq^{2(K+1-k_1)}\zeta_1\big(n(k_2)-1+1/q\big) + n(k_1)\Big(\gamma_2\big(n(k_2)-1+1/q\big) + 2cq^{2(K+1-k_2)}\Big)}{n(k_1)n(k_2)} \right) \\ - \left(\frac{2cq^{2(K+1-k_1)}\zeta_1\big(n(k_2)-2+1/q\big) + \big(n(k_1)+1\big)\Big(\gamma_2\big(n(k_2)-2+1/q\big) + 2cq^{2(K+1-k_2)}\Big)}{\big(n(k_1)+1\big)\big(n(k_2)-1\big)} \right) \le 0.$$

Multiplying this expression by $q^2\Big(n(k_1)n(k_2)\big(n(k_1)+1\big)\big(n(k_2)-1\big)\Big)$ and rearranging terms, we conclude that the above inequality holds if and only if the following holds:

$$\gamma_1\zeta_1(1-q)\Big(n(k_2)-n(k_1)-1\Big)\Big(q\big(n(k_2)+n(k_1)-1\big)+1\Big) - \gamma_2(1-q)qn(k_1)\big(n(k_1)+1\big) \\ + \zeta_1 2cqq^{2(K+1-k_1)}\Big(\Big((n(k_2)-1)^2q + n(k_2)-1\Big) - n(k_1)(1-q)\Big) \\ - 2cq^2q^{2(K+1-k_2)}n(k_1)\big(n(k_1)+1\big) \le 0. \tag{49}$$

Consider network structure $\mathcal{N}'$. Let $\hat{\phi}'$ denote the value of Expression (48) corresponding to the network that results from moving one firm from tier k_1 to tier k_2 in network $\mathcal{N}'$. By way of contradiction, assume that by moving one firm from tier k_1 to k_2, we have $\phi' < \hat{\phi}'$ or equivalently

$$0 < \hat{\phi}' - \phi'.$$

Once again writing the difference in terms of γ_1', γ_2', γ_3', ζ_1', ζ_2', and rearranging terms we obtain that this inequality holds if and only if

$$\gamma_1'\zeta_1'(1-q)\Big(n'(k_2)-n'(k_1)+1\Big)\Big(q\big(n'(k_2)+n'(k_1)-1\big)+1\Big) - \gamma_2'(1-q)q\big(n'(k_1)-1\big)n'(k_1) \\ + \zeta_1' 2cqq^{2(K+1-k_1)}\Big(\Big(\big(n'(k_2)\big)^2q + n'(k_2)\Big) - (n'(k_1)-1)(1-q)\Big) \\ - 2cq^2q^{2(K+1-k_2)}\big(n'(k_1)-1\big)n'(k_1) > 0. \tag{50}$$

However, the inequality above cannot hold in light of inequality (49). In particular, note that since $n(\ell) = n'(\ell)$ for every $\ell \in \{k_1+1, \ldots, k_2-1\}$, we have $\gamma_2 = \gamma_2'$, and $\zeta_1 = \zeta_1'$. Also since $n(\ell) \le n'(\ell)$, for every tier $\ell < k_1$, we have $\gamma_1 \ge \gamma_1'$. Finally, since $n'(k_2)+1 \le n(k_2)$ and $n'(k_1)-1 \ge n(k_1)$, the left-hand side of the inequality (50) is less than or equal to the left-hand side of inequality (49), which is a contradiction. So, it must be the case that $\phi' \geq \hat{\phi}'$ and, thus, the proof of the lemma is complete. □

Using the lemma above we complete the proof of the proposition. Let k be the first tier such that $n_S(k) \neq n_R(k)$ (thus, for every $\ell < k$, we have $n_S(\ell) = n_R(\ell)$). By way of contradiction, assume that $n_S(k) < n_R(k)$. Since the total number of firms in the two networks is the same, there should be a tier $k_2 > k$ such that $n_S(k_2) > n_R(k_2)$. Consider the most downstream such tier k_2. Also consider the largest $k_1 < k_2$ such that $n_S(k_1) < n_R(k_1)$. Note that for every tier between k_1 and k_2 the number of firms in the two networks is equal, and also for every $k' < k_1$ we have $n_S(k') \leq n_R(k')$. Moreover, by Proposition 4, $\mathcal{N}_S$ is a network that maximizes the supply level s, and moving one firm from tier k_2 and adding it to tier k_1 should not strictly decrease ϕ (and strictly increase the supply s). Note that by Theorem 2, for any $\ell > \ell'$ such that $\ell, \ell' \in \{2, \ldots, K\}$, we have $n_S(\ell) \geq n_S(\ell')$ and $n_R(\ell) \geq n_R(\ell')$. Thus, we can use Lemma 4 and conclude that in network $\mathcal{N}_R$ moving one firm from tier k_1 to tier k_2 does not decrease the supply level s.

Note that since $k_1 < k_2$, we have $n_R(k_1) \leq n_R(k_2)$. We claim that in $\mathcal{N}_R$ if we move a firm from tier k_1 to tier k_2, the following term strictly increases:

$$\prod_{\ell=1}^{K+1} \left(\frac{n_R(\ell) - 1 + 1/q}{n_R(\ell)} \right).$$

Note that by Lemma 1 this implies that the aforementioned change in number of firms in different tiers strictly increases CV of the output of the chain. In order to show that the term above increases, it is enough to show that

$$\left(\frac{n_R(k_1) - 1 + 1/q}{n_R(k_1)} \right) \left(\frac{n_R(k_2) - 1 + 1/q}{n_R(k_2)} \right) < \left(\frac{n_R(k_1) - 2 + 1/q}{n_R(k_1) - 1} \right) \left(\frac{n_R(k_2) + 1/q}{n_R(k_2) + 1} \right).$$

The inequality above holds since for $q < 1$ we have

$$\left(\frac{n_R(k_1) - 2 + 1/q}{n_R(k_1) - 1} \right) \left(\frac{n_R(k_2) + 1/q}{n_R(k_2) + 1} \right) - \left(\frac{n_R(k_1) - 1 + 1/q}{n_R(k_1)} \right) \left(\frac{n_R(k_2) - 1 + 1/q}{n_R(k_2)} \right)$$
$$= \frac{\big(n_R(k_2) - n_R(k_1) + 1\big)\big(1 - q\big)\Big(\big(n_R(k_2) + n_R(k_1) - 1\big)q + 1\Big)}{\big(n_R(k_1) - 1\big) n_R(k_1) n_R(k_2) \big(n_R(k_2) + 1\big) q^2} > 0.$$

Summarizing, in network $\mathcal{N}_R$ moving one firm from tier k_1 to tier k_2 weakly increases s and strictly increases the CV of the output. By Lemma 2, we conclude that this increases the retailers' profits. This contradicts our original assumption that $\mathcal{N}_R$ is a network that maximizes the profits of the retailer. Thus, we obtain a contradiction, and conclude that $n_S(k) \geq n_R(k)$. Hence, the claim follows. Q.E.D.

Proof of Corollary 2

Observe that the claim trivially holds if $\mathcal{N}_S = \mathcal{N}_R$. Suppose this is not the case. Then, the fact that $\mathcal{N}_S$ and $\mathcal{N}_R$ have the same number of firms, together with Proposition 6 implies the claim. Q.E.D.

Proof of Theorem 3

Note that Expression (1) implies that in the end consumer market at most α/β units of final goods are sold. Thus, the total monetary transfer from the end consumers to the rest of the supply chain is bounded by α^2/β. In turn, this implies that at most $\bar{N} = \alpha^2/\beta\kappa$ firms find it optimal to enter the supply chain at equilibrium.

In addition, recall from Proposition 3 that for a given network there exists $q' < 1$ such that if $q \geq q'$, then adding a firm to a tier of the supply chain increases the profits of firms in all other tiers. Since at most $\bar{N}$ firms participate in the production process, there are finitely many networks to consider. Hence, there exists some $\hat{q} < 1$ such that if $q \geq \hat{q}$, then adding a firm to a tier of the supply chain increases the profits of firms in all other tiers for networks with at most $\bar{N}$ firms. Note that by construction all networks in $\mathcal{W}_\kappa$ have at most $\bar{N}$ firms, and satisfy the aforementioned property. In the remainder of the proof we focus on $q \geq \hat{q}$, and exploit this observation.

In particular, start with a network in $\mathcal{W}_\kappa$ (recall that $\mathcal{W}_\kappa$ is nonempty), and assume that a firm can enter some tier i and earn more than κ (if there is no such firm, then the network we start with is an equilibrium network). Assume that this firm joins tier i; then Proposition 3 implies that no firm in any other tier finds it optimal to leave the chain. Furthermore, the profits in tier i are decreasing yet they have to be at least as high as the cost of entry κ (as otherwise the new firm would not enter). Thus, we obtain another network structure in $\mathcal{W}_\kappa$. Proceeding sequentially (and keeping in mind that for any network in $\mathcal{W}_\kappa$, Proposition 3 applies by our choice of $\hat{q}$), after a finite number of firms enter the chain, we reach an outcome where additional entry would not be profitable (as the maximum total profits for the firms are bounded, and the entry cost $\kappa > 0$).[3] Note that this is an equilibrium outcome (hence, an equilibrium exists) since at each step we obtain a new network in $\mathcal{W}_\kappa$, and reach an outcome where no further entry is profitable.

[3] The number of firms that enter each tier cannot grow to infinity. To see this note that according to Corollary 1, we have for the firms' profits in tier $1 \leq k \leq K+1$,

$$\pi(k) = s^2 \frac{c(k)}{n(k)^2} \prod_{\ell=k+1}^{K+1} \frac{q(\ell)^2}{n(\ell)} \left(n(\ell) - 1 + \frac{1}{q(\ell)} \right).$$

In addition, the supply s is bounded from above for any allocation of firms into the different tiers. Thus, as the number of firms in tier k increases, $\pi(k)$ approaches zero and thus at some point firms' expected profits become smaller than the entry cost κ.

Finally, we show that there exists an equilibrium network that is maximal. The claim trivially follows if the equilibrium is unique. Suppose that there are two arbitrary equilibrium structures $\mathcal{N}$ and $\mathcal{N}'$, and define structure $\mathcal{N}_{\max}$ as follows: for every tier $1 \le \ell \le K+1$, let $n_{\max}(\ell) = \max\{n(\ell), n'(\ell)\}$. We claim that in the structure $\mathcal{N}_{\max}$ we have $\pi_{\max}(\ell) \ge \kappa$ for all $1 \le \ell \le K+1$. To see this, consider an arbitrary tier $1 \le \ell \le K+1$, and without loss of generality assume that $n(\ell) \ge n'(\ell)$. Then since $n_{\max}(\ell) = n(\ell)$ and for every other tier $k \ne \ell$ we have $n_{\max}(k) \ge n(k)$, it follows from Proposition 3 that $\pi_{\max}(\ell) \ge \pi(\ell) \ge \kappa$. Since ℓ is arbitrary it follows that $\mathcal{N}_{\max}$ belongs to $\mathcal{W}_\kappa$. Note that if no other firm finds it profitable to enter $\mathcal{N}_{\max}$, then this network is an equilibrium. Otherwise, after a finite number of (sequential) entries to $\mathcal{N}_{\max}$ as explained before, we reach an equilibrium structure where each tier has weakly more number of firms than the corresponding tier in $\mathcal{N}$ and $\mathcal{N}'$. The above argument implies that given any equilibrium structures $\mathcal{N}$ and $\mathcal{N}'$, we can obtain another equilibrium structure that has (weakly) more firms than both networks in all tiers. Since firms pay an entry cost $\kappa > 0$, it follows that the number of firms in an equilibrium network is bounded, and there are finitely many equilibria. Thus, we conclude that the maximal equilibrium structure exists. Q.E.D.

Proof of Proposition 7

Consider an equilibrium chain $\mathcal{N}_\mathcal{E}$. Let $\hat{\pi}_\mathcal{E}(k)$ and $\hat{s}_\mathcal{E}$ denote the profits of firms in tier k and the level of supply procured by tier $K+1$, respectively, in the network that results after an additional firm enters tier k of $\mathcal{N}_\mathcal{E}$. Using Corollary 1, we obtain

$$\begin{aligned}\kappa > \hat{\pi}_\mathcal{E}(k) &= \hat{s}_\mathcal{E}^2 \frac{c(k)}{(n_\mathcal{E}(k)+1)^2} \prod_{\ell=k+1}^{K+1} q^2\Big(\frac{n_\mathcal{E}(\ell)-1+1/q}{n_\mathcal{E}(\ell)}\Big) \\ &> s_\mathcal{E}^2 \frac{c(k)}{(n_\mathcal{E}(k)+1)^2} \prod_{\ell=k+1}^{K+1} q^2\Big(\frac{n_\mathcal{E}(\ell)-1+1/q}{n_\mathcal{E}(\ell)}\Big),\end{aligned} \tag{51}$$

where the second inequality follows from the fact that $\hat{s}_\mathcal{E} > s_\mathcal{E}$ (the level of supply procured by the firms in tier $K+1$ increases when a firm enters the chain as can be directly seen from Proposition 1). On the other hand, since $\mathcal{N}_\mathcal{E}$ is an equilibrium network structure, we have that the profits $\pi_\mathcal{E}(k+1)$ of firms in tier $k+1$ (at the corresponding supply equilibrium) are such that $\pi_\mathcal{E}(k+1) \ge \kappa$. This further implies that

$$\pi_\mathcal{E}(k+1) = s_\mathcal{E}^2 \frac{c(k+1)}{n_\mathcal{E}(k+1)^2} \prod_{\ell=k+2}^{K+1} q^2\Big(\frac{n_\mathcal{E}(\ell)-1+1/q}{n_\mathcal{E}(\ell)}\Big) \ge \kappa. \tag{52}$$

Expressions (51) and (52) yield the following inequality:

$$\frac{c(k+1)}{n_\mathcal{E}(k+1)^2} \prod_{\ell=k+2}^{K+1} q^2\Big(\frac{n_\mathcal{E}(\ell)-1+1/q}{n_\mathcal{E}(\ell)}\Big) > \frac{c(k)}{(n_\mathcal{E}(k)+1)^2} \prod_{\ell=k+1}^{K+1} q^2\Big(\frac{n_\mathcal{E}(\ell)-1+1/q}{n_\mathcal{E}(\ell)}\Big),$$

which, in turn, yields

$$\frac{(n_{\mathcal{E}}(k)+1)^2}{n_{\mathcal{E}}(k+1)^2} > \frac{c(k)}{c(k+1)} q^2 \Big(\frac{n_{\mathcal{E}}(k+1)-1+1/q}{n_{\mathcal{E}}(k+1)}\Big). \tag{53}$$

Finally, inequality (53) implies that

$$n_{\mathcal{E}}(k) > q n_{\mathcal{E}}(k+1) \sqrt{\frac{c(k)}{c(k+1)}} \sqrt{\frac{n_{\mathcal{E}}(k+1)-1+1/q}{n_{\mathcal{E}}(k+1)}} - 1. \tag{54}$$

Following a similar approach and noting that since $\mathcal{N}_{\mathcal{E}}$ is an equilibrium chain, no firm has an incentive to enter tier $k+1$ (i.e., using an inequality similar to (51) for tier $k+1$, and an inequality similar to (52) for tier k), we can establish that

$$n_{\mathcal{E}}(k) < q(n_{\mathcal{E}}(k+1)+1) \sqrt{\frac{c(k)}{c(k+1)}} \sqrt{\frac{n_{\mathcal{E}}(k+1)-1+1/q}{n_{\mathcal{E}}(k+1)}}, \tag{55}$$

which completes the first part of the proposition.

For the second part, we assume that $c(k) = c$ for all $1 \le k \le K+1$. Then, note that (54) directly implies that

$$\Big\lfloor q \cdot n_{\mathcal{E}}(k+1) \Big\rfloor \le n_{\mathcal{E}}(k),$$

since $\frac{n_{\mathcal{E}}(k+1)-1+1/q}{n_{\mathcal{E}}(k+1)} \ge 1$. Also note that if

$$q\Big(n_{\mathcal{E}}(k+1)+1\Big) \sqrt{\frac{n_{\mathcal{E}}(k+1)-1+\frac{1}{q}}{n_{\mathcal{E}}(k+1)}} - q n_{\mathcal{E}}(k+1) \le 1, \tag{56}$$

then by noting that $n_{\mathcal{E}}(k)$ is an integer, (55) immediately implies

$$\Big\lceil q \cdot n_{\mathcal{E}}(k+1) \Big\rceil \ge n_{\mathcal{E}}(k).$$

Straightforward algebra implies that inequality (56) holds for any $q \le 1$. Summarizing, we have established that

$$\Big\lfloor q \cdot n_{\mathcal{E}}(k+1) \Big\rfloor \le n_{\mathcal{E}}(k) \le \Big\lceil q \cdot n_{\mathcal{E}}(k+1) \Big\rceil. \tag{57}$$

Q.E.D.

Appendix B: Discussion and Additional Results

Appendix B discusses three results that were omitted from the main body of the paper. In particular, Appendix B.1 establishes a connection between balanced networks and the coefficient of variation of the output that reaches the downstream consumer market. Appendix B.2 considers the extension of our benchmark model to the case where disruptive events in a tier are correlated. Finally, in Appendix B.3 we illustrate potential prescriptive implications of our results.

B.1. Balanced Networks and Coefficient of Variation

In this section, we establish that more balanced structures induce smaller CV. Throughout the section we consider networks with a fixed number N of firms. Before we state our results formally, we define a preorder on network structures with the same number of firms that aims to capture how “balanced” they are.

Definition 1. Suppose networks $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same number of firms. Network $\mathcal{N}_1$ is more balanced than network $\mathcal{N}_2$ if for any two tiers k_1 and k_2 we have

$$|n_1(k_2) - n_1(k_1)| \leq |n_2(k_2) - n_2(k_1)|,$$

where $n_1(k)$ and $n_2(k)$ denote the number of firms in tier k in networks $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. The result that follows relates this preorder to the coefficient of variation of the corresponding outputs.

Lemma 5. *Let $q(k) = q < 1$ for all tiers $1 \leq k \leq K+1$ and assume that network $\mathcal{N}_1$ is more balanced than network $\mathcal{N}_2$. Then, the coefficient of variation of the supply chain's output corresponding to $\mathcal{N}_1$ is (weakly) smaller than that corresponding to $\mathcal{N}_2$, i.e., $CV(U_{\mathcal{N}_1}) \leq CV(U_{\mathcal{N}_2})$.*

Proof: The claim trivially holds if $\mathcal{N}_1 = \mathcal{N}_2$. Suppose this is not the case. Consider the following sets S_1 and S_2:

$$\begin{aligned} S_1 &\triangleq \{1 \leq k \leq K+1 | n_2(k) < n_1(k)\}, \\ S_2 &\triangleq \{1 \leq k \leq K+1 | n_2(k) > n_1(k)\}. \end{aligned} \tag{58}$$

Since $\mathcal{N}_1 \neq \mathcal{N}_2$ and these networks have the same number N of firms, it follows that $S_1, S_2 \neq \emptyset$.

First, note that for any two tiers $k_1 \in S_1$ and $k_2 \in S_2$ we should have $n_2(k_1) < n_2(k_2)$, since otherwise we would have $|n_2(k_1) - n_2(k_2)| < |n_1(k_1) - n_1(k_2)|$, which contradicts the assumption that network $\mathcal{N}_1$ is more balanced than $\mathcal{N}_2$. To see this, assume, by way of contradiction, that there exist tiers $k_1 \in S_1$ and $k_2 \in S_2$ such that $n_2(k_1) \geq n_2(k_2)$. Since we have $n_1(k_1) > n_2(k_1)$ and $n_1(k_2) < n_2(k_2)$ we obtain

$$n_1(k_2) < n_2(k_2) \leq n_2(k_1) < n_1(k_1),$$

which in turn implies that $|n_2(k_1) - n_2(k_2)| < |n_1(k_1) - n_1(k_2)|$, contradicting the assumption that $\mathcal{N}_1$ is more balanced.

Fix some $k_1 \in S_1$ and $k_2 \in S_2$. We claim:

$$\left(\frac{n_2(k_1)-1+\frac{1}{q}}{n_2(k_1)}\right)\left(\frac{n_2(k_2)-1+\frac{1}{q}}{n_2(k_2)}\right) \geq \left(\frac{n_2(k_1)+\frac{1}{q}}{n_2(k_1)+1}\right)\left(\frac{n_2(k_2)-2+\frac{1}{q}}{n_2(k_2)-1}\right). \tag{59}$$

To see this, note that we have

$$\begin{aligned}\left(\frac{n_2(k_1)-1+\frac{1}{q}}{n_2(k_1)}\right)\left(\frac{n_2(k_2)-1+\frac{1}{q}}{n_2(k_2)}\right) - \left(\frac{n_2(k_1)+\frac{1}{q}}{n_2(k_1)+1}\right)\left(\frac{n_2(k_2)-2+\frac{1}{q}}{n_2(k_2)-1}\right) \\ = \frac{\big(n_2(k_2)-n_2(k_1)-1\big)\big(1-q\big)\Big(\big(n_2(k_1)+n_2(k_2)-1\big)q+1\Big)}{n_2(k_1)\big(n_2(k_1)+1\big)\big(n_2(k_2)-1\big)n_2(k_2)q^2} \geq 0,\end{aligned} \tag{60}$$

where the inequality holds since $n_2(k_2) > n_2(k_1)$.

Finally, Expression (45) together with the inequality above imply that in $\mathcal{N}_2$ after moving a firm from k_2 to k_1, the CV of the resulting network is at most as high as the CV of the original network. Finally, noting that network $\mathcal{N}_1$ can be obtained from $\mathcal{N}_2$ after moving a certain number of firms belonging to tiers in S_2 to tiers in S_1 one after the other, we conclude that $CV(U_{\mathcal{N}_2}) \geq CV(U_{\mathcal{N}_1})$ (note that after each such move, the following still holds $|n_1(k_1)-n_1(k_2)| \leq |n_2(k_1)-n_2(k_2)|$ for $k_1 \in S_1$ and $k_2 \in S_2$.) □

This result establishes that when a network becomes less balanced, e.g., due to having too few firms in one tier, while having many more in another, then the coefficient of variation of the output increases. The increase in the output variability is intuitive, since if the network has a tier with a few firms, in case of disruption in this tier, the output is significantly reduced.

Our next result complements the previous one by showing that the CV minimizing networks are the most balanced ones, in the sense that the number of firms in different tiers is (almost) the same.[4]

LEMMA 6. *Let $q(k) = q < 1$ for all tiers $1 \leq k \leq K+1$, and denote by $\mathcal{N}$ the network with N firms that minimizes CV. The network $\mathcal{N}$ is such that for any two tiers $1 \leq k_1, k_2 \leq K+1$, we have $|n(k_1) - n(k_2)| \leq 1$.*

Proof: By way of contradiction, assume that for two tiers k_1 and k_2, $|n(k_1)-n(k_2)| \geq 2$, and without loss of generality assume $n(k_2) > n(k_1)$. Then moving one firm from tier k_2 to tier k_1 decreases CV. Note that to establish this claim it is enough to show that

$$\left(\frac{n_2(k_1)-1+\frac{1}{q}}{n_2(k_1)}\right)\left(\frac{n_2(k_2)-1+\frac{1}{q}}{n_2(k_2)}\right) > \left(\frac{n_2(k_1)+\frac{1}{q}}{n_2(k_1)+1}\right)\left(\frac{n_2(k_2)-2+\frac{1}{q}}{n_2(k_2)-1}\right), \tag{61}$$

which follows from equality (60) by noting that $n(k_2) > n(k_1)+1$. □

[4] Note that for a given number N of firms, it may not be possible to have the same number of firms in all tiers. To account for this possibility, our result allows for the number of firms in different tiers to differ by one.

B.2. Correlated Disruptions

Assume that for any two firms i, j in $Tier(k)$ we have $Cov(Z_{k,i}, Z_{k,j}) = \delta(k)$, where we recall that $Z_{k,i}$ is a Bernoulli random variable that captures whether a disruptive event has occurred in firm i. Suppose that independence of disruptions across tiers is still preserved (i.e., $Cov(Z_{k,i}, Z_{\ell,j}) = 0$ for $k \neq \ell$). Theorem 1 below generalizes Theorem 1 to this case where disruptive events in a tier may be correlated. Before stating the theorem, we introduce some notation (analogous to (3) and (4)) and state an assumption analogous to Assumption 1.

In particular, we let

$$\hat{\alpha}(k) = \alpha \prod_{\ell=1}^{k-1} q(\ell),$$

$$\hat{\beta}(k) = \begin{cases} \beta & \text{if } k = 1, \\ \hat{\beta}(k-1)\dfrac{\delta(k-1)+q(k-1)^2}{n(k-1)}\left(n(k-1) - 1 + \dfrac{q(k-1)}{\delta(k-1)+q(k-1)^2}\right) + \dfrac{2c(k-1)}{n(k-1)} & \text{if } 1 < k \leq K+2. \end{cases}$$

In addition, we state the following assumption.

ASSUMPTION 1. *The supply chain network is such that:*

(i) $\hat{\alpha}(K+2) > p_c$,

(ii) $\dfrac{\hat{\alpha}(k)}{\hat{\beta}(k)} > \dfrac{\hat{\alpha}(K+2) - p_c}{\hat{\beta}(K+2)}$, *for* $k \in \{1, \dots, K+1\}$.

Then, we obtain the following theorem, which is analogous to Theorem 1.

THEOREM 1. *Suppose that Assumption 1(i) holds. Then, the supply equilibrium is essentially unique. In addition, if Assumption 1(ii) holds, the (essentially unique) equilibrium can be characterized as follows:*

(i) The aggregate supply s of raw materials is given by

$$s = \frac{\hat{\alpha}(K+2) - p_c}{\hat{\beta}(K+2)}. \tag{62}$$

(ii) The price for the output of tier k when the state is ω_{k-1} is given by

$$p_k(\omega_{k-1}) = \hat{\alpha}(k) - \hat{\beta}(k) s \prod_{\ell=k}^{K+1} \frac{n(\ell, \omega_{k-1})}{n(\ell)} > 0, \tag{63}$$

for all $k \in \{1, \dots, K+1\}$ and $\omega_{k-1} \in \Omega_{k-1}$. Here, we let $n(\ell, \omega_{k-1})$ denote the number of firms in tier $\ell \geq k$ that did not experience a disruption at state ω_{k-1}.

(iii) The procurement quantity of firm i in tier k when the state is ω_k is given by

$$x_{k,i}(\omega_k) = \frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell, \omega_k)}{n(\ell)}, \tag{64}$$

for all $k \in \{1, \dots, K+1\}$ and $\omega_k \in \Omega_k$.

Finally, if Assumption 1(ii) does not hold, then at any supply equilibrium, there exists at least one tier $k' \in \{1,\dots,K+1\}$, such that $p_{k'}(\bar{\omega}_{k'-1}) = 0$, where $\bar{\omega}_{k'-1} \in \Omega_{k'-1}$ is the state where no firm experiences a disruption in tiers $\{k',\dots,K+1\}$.

It can be readily seen that the proofs of the essential uniqueness of the equilibrium (under Assumption 1(i)) and its characterization when Assumption 1(ii) does not hold follow using essentially the same arguments as in the proofs of the corresponding claims in Theorem 1 (with the only difference that we need to take into account the correlation between disruptions in the same tier when writing Expression (25)). Similarly, in order to establish that the prices and procurement quantities given in Expressions (62), (63), and (64) constitute an equilibrium, as in the proof of Theorem 1, we use the first order optimality conditions for the firms' expected profit maximization problems. In the setting of Theorem 1, when evaluating firms' expected profits, we need to take into account the correlation in disruptions affecting firms in the same tier. In particular, this implies that instead of Expression (16), we have

$$\mathbb{E}[Z_{k,i} n(k,\hat{\omega}_{k-1}) \mid \hat{\omega}_k = \omega_k] = \mathbb{E}\left[Z_{k_i} + \sum_{j \in Tier(k) | j \neq i} Z_{k,i} Z_{k,j}\right] = q(k) + \left(q(k)^2 + \delta(k)\right)(n(k)-1).$$

Since the proof generally follows the same steps as the proof of Theorem 1, it is omitted for brevity.

B.3. Prescriptive Implications

The closed-form characterization of profits given in Corollary 1 is useful for understanding how implementing a strategic initiative may impact firms' profits, and consequently inform managerial decision making. In this section, we illustrate this observation by investigating how investing in improving the monitoring/production reliability of a given tier (or other interventions that effectively lead to a higher $q(k)$) may affect the profits of a firm, and how a manager should prioritize investing in such interventions among different tiers of the chain.

In particular, we consider a retailer investing in decreasing the disruption risk associated with one of the intermediate stages of production. Assuming that such efforts incur the same cost irrespective of the tier they are targeted to, we provide some understanding of the potential return investing in a tier would yield as a function of the primitives of the environment.

Formally, we say that it is optimal for a downstream retailer to invest in decreasing the disruption risk of tier k if $\frac{d \log \pi(1)}{dq(k)}$ is maximized for k. Intuitively, this is the tier where a marginal improvement in disruption probability has the largest impact on the retailers' profits. The proposition below provides a characterization of such an optimal tier for the retailers as a function of the modeling primitives. To simplify the exposition and obtain a crisp characterization, the proposition focuses on the setting where the variable costs associated with production are positive but negligible for all intermediate stages of productions, i.e., $c(k) \to 0$ for $k > 1$, whereas $c(1) > 0$ (note that this implies that roughly the entire surplus generated by the supply network goes to the downstream retailers).[5]

Proposition 1. *Consider a setting where $c(k) \to 0$ for all $1 < k \leq K+1$. Then, it is optimal for the downstream retailer to invest in decreasing the disruption risk of tier k that maximizes the following expression:*

$$\frac{1}{q(k)}\left(\frac{2\alpha\prod_{\ell=1}^{K+1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c} - 2 + \frac{1/q(k)}{n(k) - 1 + \frac{1}{q(k)}}\right). \tag{65}$$

Proof: The profits for a downstream retailer are given by the following expression from Corollary 1:

$$\pi(1) = s^2 \frac{c(1)}{n(1)^2} \prod_{\ell=2}^{K+1} \frac{q(\ell)^2}{n(\ell)} \Big(n(\ell) - 1 + 1/q(\ell)\Big). \tag{66}$$

Also, the aggregate supply s at equilibrium is given by

$$s = \frac{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c}{\beta(K+2)}.$$

We are interested in finding the tier k for which the derivative $\frac{d\pi(1)}{dq(k)}$ is maximized. Note that this is equivalent to finding k for which $\frac{d\log\pi(1)}{dq(k)}$ is maximized. We can write the logarithm of Expression (66) as

$$\begin{aligned}\log\pi(1) =& 2\log\left(\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c\right) - 2\log\left(\beta\prod_{\ell=1}^{K+1} q(\ell)^2 \frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)} + \sum_{r=1}^{K+1}\frac{2c(r)}{n(r)}\prod_{\ell=r+1}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)}\right)\\ &+\log\left(\frac{c(1)}{n(1)^2}\prod_{\ell=2,\ell\neq k}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)}\right) + \log\left(q(k)^2\frac{n(k)-1+\frac{1}{q(k)}}{n(k)}\right).\end{aligned}$$

Thus, we can write $\frac{d\log\pi(1)}{dq(k)}$ as follows:

$$\begin{aligned}\frac{\partial\log\pi(1)}{\partial q(k)} =& \frac{2\alpha\prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c} + \frac{2q(k)(n(k)-1)+1}{q(k)^2\big(n(k)-1+\frac{1}{q(k)}\big)}\\ &-2\frac{\beta\frac{2q(k)\left(n(k)-1\right)+1}{n(k)}\prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)} + \sum_{r=1}^{k-1}\frac{2c(r)}{n(r)}\frac{2q(k)\left(n(k)-1\right)+1}{n(k)}\prod_{\ell=r+1,\ell\neq k}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)}}{\beta\prod_{\ell=1}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)} + \sum_{r=1}^{K+1}\frac{2c(r)}{n(r)}\prod_{\ell=r+1}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)}}.\end{aligned}$$

Then, by using $c(k) \to 0$ for $k > 1$, we can rewrite the above as follows:

$$\frac{\partial\log\pi(1)}{\partial q(k)} = \frac{2\alpha\prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c} - \frac{2q(k)(n(k)-1)+1}{q(k)^2\big(n(k)-1+\frac{1}{q(k)}\big)},$$

[5] Similar insights hold in the general setting as well but obtaining analytical expressions is much more challenging.

which, in turn, implies that it is optimal for a downstream retailer to invest in decreasing the disruption risk of tier k for which the following expression is maximized

$$\frac{1}{q(k)}\left(\frac{2\alpha\prod_{\ell=1}^{K+1}q(\ell)}{\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c}-\frac{2(n(k)-1)+1/q(k)}{n(k)-1+\frac{1}{q(k)}}\right).$$

Finally, we can rewrite the above as

$$\frac{1}{q(k)}\left(\frac{2\alpha\prod_{\ell=1}^{K+1}q(\ell)}{\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c}-2+\frac{1/q(k)}{n(k)-1+\frac{1}{q(k)}}\right),$$

which completes the proof of the claim. □

As one would intuitively expect, if the number of firms in each tier is the same, Expression (65) implies that it is optimal to invest in the tier for which the disruption risk is the highest. On the other hand, this conclusion no longer holds if the number of firms in different tiers is different. In particular, it may no longer be optimal to invest in the tier that is most prone to disruptions, i.e., the tier with the lowest $q(k)$, or the retailers' direct suppliers, i.e., firms in tier 2. In general, retailers may find it optimal to concentrate their efforts on a different tier depending on the environment they operate in. We illustrate this with an example where the chain consists of $K+1=3$ tiers with $n(3)=5$ firms, $n(2)\in\{3,5,7\}$ firms, and $n(1)=1$ retailer. For each value of $n(2)$ we compare (65) for tiers $k=2$ and $k=3$, and obtain $(q(2),q(3))$ pairs for which it is optimal to invest in tier $k=2$ or $k=3$. The results are illustrated in Figure 1 below.

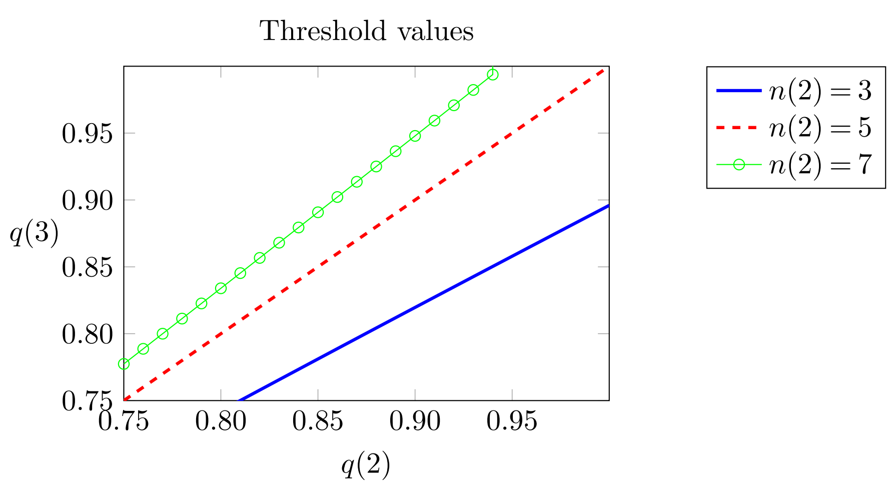

Figure 1 **Each of the curves in the figure corresponds to a different value for $n(2)$, i.e., $n(2)\in\{3,5,7\}$, when $n(3)=5$. For $(q(2),q(3))$ pairs that lie above the corresponding curve, it is optimal to invest in tier 2, whereas otherwise it is optimal to invest in tier 3. Here, the probability of successful production in tier 1 is $q(1)=1$, $p_c=0.5$, and $\alpha=2$.**

Despite its simplicity, this example clearly illustrates the importance of results such as Theorem 1 and Corollary 1 in prescribing managerial decision-making in multi-tier supply chains: although intuitively it may seem best for a firm to focus on its direct suppliers (as their operations have an immediate impact on the firm's profits), this often leads to suboptimal returns. Thus, developing an understanding of the firm's supply chain structure and taking into account how relationships between its direct and indirect suppliers affect its profits may be necessary for its strategic planning.