Online Appendix

Constrained Assortment Optimization under the Markov Chain based Choice Model

Appendix C: A Mixed-Integer Programming Formulation

We show that the following mixed-integer program (MIP) is an exact reformulation of Cardinality-Assort.

$$
\begin{aligned}
\max \ & \sum_{i=1}^{n} \alpha_i p_i \\
\text{s.t. } & \alpha_i + \beta_i - \sum_{j=1}^{n} \rho_{ji}\beta_j = \lambda_i, && \forall i = 1,\dots,n \\
& y_i \geq \alpha_i, && \forall i = 1,\dots,n \\
& \sum_{i=1}^{n} y_i \leq k \\
& \alpha_i \geq 0, \beta_i \geq 0, y_i \in \{0,1\}, && \forall i = 1,\dots,n.
\end{aligned}
\tag{A.3}
$$

LEMMA 11. *The mixed-integer program* (A.3) *is an exact reformulation of Cardinality-Assort.*

Proof. Consider the following LP:

$$
\begin{aligned}
\max \ & \sum_{i=1}^{n} \alpha_i p_i \\
\text{s.t. } & \alpha_i + \beta_i - \sum_{j=1}^{n} \rho_{ji}\beta_j = \lambda_i, && \forall i = 1,\dots,n \\
& \alpha_i \geq 0, \beta_i \geq 0, && \forall i = 1,\dots,n.
\end{aligned}
\tag{A.4}
$$

Let (α, β) be an extreme point solution to the above LP, and let $S = \{i : \alpha_i > 0\}$. Feldman and Topaloglu (2017b) show that α_i is the choice probability $\pi(i, S)$ when the assortment S is offered under the Markov chain choice model. Hence, the objective value $\sum_{i=1}^{n} \alpha_i p_i$ equals to $R(S)$. By adding the indicator variables y_i, we are restricting ourselves to the subset of feasible solutions where at most k of the α_i-s are allowed to be strictly positive. Note that the extreme points of this polytope, corresponding to the projection of the feasible space of the MIP down to the (α, β) coordinates, are exactly the set of assortments S with cardinality at most k. Hence, (A.3) is a mixed-integer formulation of the cardinality constrained assortment optimization problem. □

Appendix D: Dynamic Setting with a Cardinality Constraint

In this section, we show that augmenting the dynamic optimization problem with a cardinality constraint changes the fundamental nature of this problem. In particular, technical ideas such as those of Feldman and Topaloglu (2017b) in the context of unconstrained dynamic assortment optimization do not apply anymore. We then explain how, building on the work of Gallego et al. (2016), our algorithm can be leveraged to derive a constant factor approximation for the dynamic assortment optimization problem with an additional cardinality constraint. The more general case of a capacity constraint can be handled in a similar way.

D.1. Model and notation

We consider the network revenue management problem with parallel flights under the Markov chain model. In this setting, we have a network of L resources, where each resource ℓ has a limited inventory level (or capacity) of C_ℓ. The total selling horizon consists of T discrete time periods and at each time period t in the selling horizon, we need to decide which subset of items S_t should be offered to customers. Customers arrive into the system one by one and choose among the offered items according to the Markov chain choice model. When an item i is sold, we generate a revenue of r_i and consume $a_{i,\ell}$ units of resource ℓ. We assume that $a_{i,\ell}$ is non zero only for one resource, i.e. we only consider the setting where the network consists of parallel flights. This is similar to Gallego et al. (2016). Finally, we operate under an additional cardinality constraint, stating that each subset S_t may consist of at most k items. Note that the latter constraint restricts the offer set size in each time period, which is different than the capacity C_ℓ, that restricts the number of times each resource ℓ may be purchased over the entire selling horizon. The goal is to compute a policy to dynamically decide which subsets of items to make available over the selling horizon so as to maximize the total expected revenue.

D.2. Structural properties for the single resource revenue management problem

In what follows, consider the special case of a single resource (i.e., $L = 1$) without a cardinality constraint. In this case, the optimal sequence of assortments prescribes an assortment $S^t_{OPT}(x)$ for every time period t and remaining capacity x of the single resource available. Interestingly, Feldman and Topaloglu (2017b) show that there is an optimal sequence of assortments such that:

1. The optimal assortments are nested over time, i.e., $S^{t-1}_{OPT}(x) \subseteq S^t_{OPT}(x)$ for any capacity x.
2. The optimal assortments are nested with the remaining capacity, i.e., $S^t_{OPT}(x-1) \subseteq S^t_{OPT}(x)$ for any time period t.

In turn, this characterization yields an efficient procedure to solve the single resource revenue management problem. Their result relies on the key observation that, if we decrease the price of each item by the same positive amount, the optimal solution to the static unconstrained assortment optimization problem becomes a smaller subset. We restate this observation for completeness.

LEMMA 12 **(Lemma 3 of Feldman and Topaloglu (2017b))**. *For $\eta \geq 0$, let S^η be an optimal solution to the static unconstrained assortment optimization problem when we decrease the price of each item by the same amount η. Then, there exists an optimal solution S^* to the original unconstrained assortment optimization problem (i.e., when $\eta = 0$) such that $S^\eta \subseteq S^*$.*

We show that this key observation no longer holds when we place a cardinality constraint on the size of the offer set. This highlights the fact that adding a cardinality constraint changes the fundamental nature of this optimization problem, and that key structural properties used in previous work no longer hold.

Counterexample. Consider the next instance, with a single time period, in which one item can be offered out of $\{a, b\}$, with the following attributes:

- Item a: $\lambda_a = 1/3$, $r_a(\eta) = 6 - \eta$, and $\rho_{a,0} = 1$.
- Item b: $\lambda_b = 1/12$, $r_b(\eta) = 12 - \eta$, and $\rho_{b,0} = 1$.

Let $R^{\eta}(S)$ denote the expected revenue of the assortment S with respect to the item prices $r_{\cdot}(\eta)$. Then, for $\eta = 0$, we have $R^0(\{a\}) = 2$ and $R^0(\{b\}) = 1$, meaning that the optimal (original) single-item assortment is $\{a\}$. On the other hand, when $\eta = 6$, we obtain $R^6(\{a\}) = 0$ and $R^6(\{b\}) = 0.5$, so the optimal single-item assortment is $\{b\}$. This example demonstrates that $S^6 \not\subseteq S^*$; i.e., due to having a capacity constraint, decreasing all revenues by the same amount might lead to an optimal assortment which is not a subset of the original optimal assortment. In fact, we can actually show that Lemma 12 does not hold under a cardinality constraint even with respect to the MNL model.

D.3. Constant factor approximation for network revenue management with parallel flights

In this section, we show how to leverage our algorithmic approach together with the methodology of Gallego et al. (2016) to obtain a constant factor approximation for the network revenue management problem with an additional cardinality constraint. For this purpose, we follow the technical ideas of Gallego et al. (2016) and adapt their arguments to account for the additional cardinality constraint.

D.3.1. Choice-based deterministic linear program. The following linear program can be considered as a proxy to the network revenue management problem (Feldman and Topaloglu 2017b, Gallego et al. 2016). In fact, Gallego et al. (2016) show that the value of this choice-based deterministic linear program (CDLP) provides an upper bound on the optimal revenue of the network revenue management problem.

$$\begin{aligned} V^{CDLP} = \max \ & \sum_{S \in \mathcal{S}} x(S) \cdot R(S) \\ \text{s.t.} \ & \sum_{S \in \mathcal{S}} x(S) \cdot \sum_{i \in S} a_{i,\ell} \cdot \pi(i,S) \leq C_{\ell}, \quad \forall \ell = 1, \ldots, L \\ & \sum_{S \in \mathcal{S}} x(S) = 1 \\ & x(S) \geq 0, \qquad \forall S \in \mathcal{S}. \end{aligned}$$

Here, recall that we have L resources and that $a_{i,\ell}$ corresponds to the amount of resource ℓ consumed by item i. We use $\mathcal{S}$ to denote the set of feasible assortments, i.e., $\mathcal{S} = \{S : |S| \leq k\}$.

Clearly, the CDLP is a linear program with exponentially many variables. To overcome this difficulty, we adapt the column generation approach of Gallego et al. (2016). As formally explained in Algorithm 6, in each step d, we maintain a set $\mathcal{A}^d \subseteq \mathcal{S}$ of active assortments (each corresponding to a *column* in our LP) which is expanded in an iterative way. We start with a limited number of columns and solve a reduced linear program that only involves these columns. Given the optimal dual values of the reduced LP in the current iteration, we then calculate the reduced costs of potential assortments that have not yet been considered. If there is an assortment with strictly positive reduced cost, we incorporate the corresponding column into the active set and iterate. If no potential assortment has a positive reduced cost, the current solution is optimal. The key observation is that checking if there exists an assortment whose reduced cost is strictly positive can be expressed as a static assortment optimization problem. The latter is unconstrained in the setting considered by Gallego et al. (2016), but becomes constrained in our case; this is precisely where we employ Algorithm 3.

We show that, given access to a $1/2$-approximation for the static constrained assortment optimization problem, the column generation procedure guarantees an $1/2$-approximation to the CDLP.

Lemma 13. *Algorithm 6 returns a* $1/2$*-approximate solution to CDLP.*

Algorithm 6 Column generation algorithm for CDLP

1: Initially, let $\mathcal{A}^1$ be any collection of feasible assortments ($\mathcal{A}^1 \subset \mathcal{S}$). Set $d \leftarrow 1$.

2: Solve the following reduced linear program

$$\begin{aligned} V^d = \max \; & \sum_{S\in\mathcal{A}^d} x(S)\cdot R(S) \\ & \sum_{S\in\mathcal{A}^d} x(S) \sum_{i\in S} a_{i,\ell}\cdot\pi(i,S) \le C_\ell, \quad \forall \ell = 1,\ldots,L \\ & \sum_{S\in\mathcal{A}^d} x(S) = 1 \\ & x(S) \ge 0, \qquad \forall S\in\mathcal{A}^d. \end{aligned}$$

3: For each $\ell = 1,\ldots,L$, let $\nu(\ell)$ be the dual variable associated with resource ℓ. Let σ be the dual variable associated with the equality constraint.

4: Let $\hat{S}$ be an approximate solution computed using Algorithm 3 to the following cardinality constrained assortment optimization problem:

$$\max_{S\in\mathcal{S}} \sum_{i\in S}\left(r_i - \sum_{\ell} a_{i,\ell}\cdot\nu(\ell)\right)\cdot\pi(i,S).$$

5: If $\sum_{i\in\hat{S}}(r_i - \sum_\ell a_{i,\ell}\cdot\nu(\ell))\cdot\pi(i,\hat{S}) \le \sigma$, STOP.

6: Otherwise, set $\mathcal{A}^{d+1} \leftarrow \mathcal{A}^d \cup \hat{S}$ and return to step 2.

Proof. Consider the dual formulation of CDLP:

$$\begin{aligned} \min \sum_\ell & C_\ell\cdot y(\ell) + z \\ & \sum_{i\in S}\left(r_i - \sum_\ell a_{i,\ell}\cdot y(\ell)\right)\cdot\pi(i,S) - z \le 0, \quad \forall S\in\mathcal{S} \\ & y(\ell)\ge 0, \qquad \forall \ell. \end{aligned}$$

At termination of the algorithm, let $\nu(1),\ldots,\nu(L),\sigma$ be the corresponding dual variables. Since the assortment $\hat{S}$ computed in step 4 provides a 1/2-approximation for cardinality constrained instance being solved, when $\sum_{i\in\hat{S}}(r_i - \sum_\ell a_{i,\ell}\cdot\nu(\ell))\cdot\pi(i,\hat{S}) \le \sigma$ we necessarily have $\max_{S\in\mathcal{S}}\sum_{i\in S}(r_i - \sum_\ell a_{i,\ell}\cdot\nu(\ell))\cdot\pi(i,S) \le 2\sigma$. Consequently, the dual variables $\nu(1),\ldots,\nu(L),\sigma$ are feasible for the next relaxed LP:

$$\begin{aligned} \min \sum_\ell & C_\ell\cdot y(\ell) + z \\ & \sum_{i\in S}\left(r_i - \sum_\ell a_{i,\ell}\cdot y(\ell)\right)\cdot\pi(i,S) - 2\cdot z \le 0, \quad \forall S\in\mathcal{S} \\ & y(\ell)\ge 0, \qquad \forall \ell. \end{aligned}$$

whose dual is

$$\begin{aligned} V^R = \max \; & \sum_{S \in \mathcal{S}} x(S) \cdot R(S) \\ & \sum_{S \in \mathcal{S}} x(S) \sum_{i \in S} a_{i,\ell} \cdot \pi(i,S) \le C_\ell, \quad \forall \ell = 1,\dots,L \\ & 2 \cdot \sum_{S \in \mathcal{S}} x(S) = 1 \\ & x(S) \ge 0, \qquad \forall S \in \mathcal{S}. \end{aligned}$$

Finally, we introduce the following related LP:

$$\begin{aligned} V^{R'} = \max \; & \sum_{S \in \mathcal{S}} x(S) \cdot R(S) \\ & 2 \cdot \sum_{S \in \mathcal{S}} x(S) \sum_{i \in S} a_{i,\ell} \cdot \pi(i,S) \le C_\ell, \quad \forall \ell = 1,\dots,L \\ & 2 \cdot \sum_{S \in \mathcal{S}} x(S) = 1 \\ & x(S) \ge 0, \qquad \forall S \in \mathcal{S}. \end{aligned}$$

The important observation is that

$$\frac{1}{2} V^{CDLP} = V^{R'} \le V^R \le \sum_\ell C_\ell \nu(\ell) + \sigma.$$

The first equality holds since the difference between $V^{R'}$ and V^{CDLP} is just a rescaling of all constraints. The first inequality is obtained by noting that the constraint associated with each resource ℓ is tighter in $V^{R'}$ than in V^R. Finally, the last inequality is implies by weak duality. Therefore, when the algorithm terminates, the value obtained is at least $V^{CDLP}/2$. □

D.3.2. Constant factor approximation. As previously mentioned, Gallego et al. (2016) propose a 1/2-approximation for the network revenue management problem without a cardinality constraint. Their "primal routing" algorithm exploits the CDLP solution as a guide to build a dynamic policy. It is not difficult to verify that their analysis goes through when we replace the exact CDLP solution by an approximate one, as given by Algorithm 6. The only difference is that their column generation algorithm returns a near-optimal solution to CDLP whereas our adapted column generation returns a 1/2-approximation to the CDLP. Consequently, we lose an additional factor of 1/2 compared to them. This still yields a constant factor approximation for the network revenue management with an additional cardinality constraint.

Theorem 7. *Under the Markov chain model, the network revenue management with an additional cardinality constraint can be approximated within factor $1/4-\epsilon$, for any fixed $\epsilon > 0$.*