Electronic Companion

EC.1. Computing the Exact KG Quantity

An algorithm based on enumeration of elements in $\mathcal{X}$ is proposed in Frazier et al. (2009) to compute the KG quantity (17), when it can be expressed as a one-dimensional random variable.

Given S_n and x, sort the set $\{p_y^n, q_y^n(x)\}_{x\in\mathcal{X}}$ with respect to the values $q_y^n(x)$ in ascending order, and label the sorted set by $\{p_k^n, q_k^n\}_{k=1}^K$, i.e., $q_1^n \le q_2^n \le \cdots \le q_K^n$. Let $\{c_k^n\}_{k=1}^{\bar{K}}$ be the non-decreasing sequence of breakpoints where the lines $p_k^n + q_k^n t$ intersect, for some $\bar{K} \le K$. Frazier et al. (2009) show that

$$\mathbb{E}\left[\max_{y\in\mathcal{X}}\left(p_y^n + q_y^n(x)T_{2a_n}\right) \mid S_n\right] - \max_{y\in\mathcal{X}} p_y^n = \sum_{k=1}^{\bar{K}}(q_{k+1}^n - q_k^n)\mathbb{E}\left[\max\{0, T_{2a_n} - |c_k^n|\}\right]. \quad \text{(EC.1)}$$

The expectation $\mathbb{E}\left[\max\{0, T_{2a_n} - |c_i^n|\}\right]$ can be expressed in terms of the pdf, $f_{T_{2a_n}}(\cdot)$, and cdf, $F_{T_{2a_n}}(\cdot)$, of the standard Student's t-distribution with $2a_n$ degrees of freedom, see Section 5.3 of (Ryzhov and Powell 2012),

$$\mathbb{E}\left[\max\{0, T_{2a_n} - |c_k^n|\}\right] = \left(\frac{2a_n + (c_k^n)^2}{2a_n - 1} f_{T_{2a_n}}(|c_k^n|) - |c_k^n|(1 - F_{T_{2a_n}}(|c_k^n|))\right). \quad \text{(EC.2)}$$

Applying equations (EC.1) and (EC.2) to (17), we arrive at the exact KG quantity

$$v_n^{\mathrm{KG}}(x) = \left(1 - e^{-\lambda^{\mathscr{M}}(x)}\right)\left(\sum_{k=1}^{\bar{K}}(q_{k+1}^n - q_k^n)\left(\frac{2a_n + (c_k^n)^2}{2a_n - 1} f_{T_{2a_n}}(|c_k^n|) - |c_k^n|(1 - F_{T_{2a_n}}(|c_k^n|))\right)\right). \quad \text{(EC.3)}$$

This expression indicates that when there is at least one $q_k^n < q_{k+1}^n$, we have $v_n^{\mathrm{KG}}(x) > 0$.

EC.2. Second-Order Cone Optimization for Linear Exposure Rates

This section presents a second-order cone optimization representation for problem (24), under a linearity assumption for the exposure rate. Below, $\mathrm{vec}(\cdot)$ is the vectorization operator, and $(\mu;\theta)$ denotes the column vector by stacking the vector μ on top of θ.

Proposition EC.1. *Let the exposure rate be linear in input features: $\lambda^{\mathscr{M}}(x) = \bar{\lambda}^\top x$ for some $\bar{\lambda} \in \mathbb{R}^m$, such that $\bar{\lambda}^\top x \ge 0$ for all $x \in \mathcal{X}$. Denote the finite set of possible values of $\lambda^{\mathscr{M}}(x)$ by $\{\lambda_1, \cdots, \lambda_L\}$. Then an optimal solution of problem (24) can be computed by solving L mixed-integer second-order cone optimization problems of the following type*

$$\mathcal{V}(\lambda_\ell) \stackrel{\text{def}}{=} (1 - e^{-\lambda_\ell}) \max \sum_{j=1}^{J} w_j vec\left((\mu_Z;\theta_n)\bar{\lambda}^\top\right)^\top vec(u^j) + \tau - \nu \quad \text{(EC.4)}$$

$$s.t. \quad \|P_n^{1/2}\hat{x}\|_2 \le \sum_{j=1}^{J} w_j t_j vec\left(vec(\Sigma_n)\bar{\lambda}^\top\right)^\top vec(z^j),$$

$$
\begin{aligned}
&u^j_{i,i'} \le y^j_i, \quad u^j_{i,i'} \le y^j_{i'}, \quad u^j_{i,i'} \ge y^j_i + y^j_{i'} - 1, \quad j=1,\dots,J,\ i,i'=1,\dots,m,\\
&z^j_{(i+r(i'-1)),i''} \le (y^j_B)_i, \quad z^j_{(i+r(i'-1)),i''} \le y^j_{i''}, \qquad i,i'=1,\dots,r,\ i''=1,\dots,m,\\
&z^j_{(i+r(i'-1)),i''} \le (x_B)_{i'}, \qquad i,i'=1,\dots,r,\ i''=1,\dots,m,\\
&z^j_{(i+r(i'-1)),i''} \ge (y^j_B)_i + (x_B)_{i'} + (y^j_B)_{i''} - 2, \quad i,i'=1,\dots,r,\ i''=1,\dots,m,\\
&\tau\cdot \boldsymbol{1}_m - M(\boldsymbol{1}_m - x) \le \hat{x} \le Mx, \quad \tau \le M, \quad \bar{\lambda}^\top x = \lambda_\ell,\\
&(\hat{x},\tau)\in\mathcal{X}^+,\ x, y^1,\dots,y^J \in \mathcal{X}, \quad u^j \in \{0,1\}^{m\times m},\ z^j \in \{0,1\}^{r^2\times m}.
\end{aligned}
$$

Here, ν is given in (22) *and can be computed by the following integer linear optimization problem,*

$$
\begin{aligned}
\nu = \max_{x\in\mathcal{X},\ u\in\{0,1\}^{m\times m}} \quad & vec\left((\mu_Z;\theta_n)\bar{\lambda}^\top\right)^\top vec(u), \\
s.t. \quad & u_{i,i'} \le x_i, \quad u_{i,i'} \le x_{i'}, \quad u_{i,i'} \ge x_i + x_{i'} - 1, \quad i,i'=1,\dots,m.
\end{aligned}
\tag{EC.5}
$$

Proof. When the set of possible $\lambda^{\mathscr{M}}(x)$'s is finite, i.e., $\{\lambda^{\mathscr{M}}(x)\}_{x\in\mathcal{X}} = \{\lambda_1,\cdots,\lambda_L\}$, Problem (24) can be solved by selecting the largest $\mathcal{V}(\lambda_\ell)$ for $\ell = 1,\dots,L$, where one can solve in parallel

$$
\begin{aligned}
\mathcal{V}(\lambda_\ell) \overset{\text{def}}{=} (1-e^{-\lambda_\ell}) \max_{\substack{(\hat{x},\tau)\in\mathcal{X}^+,\ \tau\le M,\ x\in\mathcal{X}\\ y^1\in\mathcal{X},\cdots,y^J\in\mathcal{X}}} \quad & \left\{\sum_{j=1}^J w_j \lambda^{\mathscr{M}}(y^j)(\mu_Z^\top y^j_Z + \theta_n^\top y^j_B) + \tau - \nu\right\} \\
\text{subject to} \quad & \lambda^{\mathscr{M}}(x) = \lambda_\ell,\\
& \tau\cdot\mathbf{1}_m - M(\mathbf{1}_m - x) \le \hat{x} \le Mx,\\
& \|P_n^{1/2}\hat{x}\|_2 \le \sum_{j=1}^J w_j t_j \lambda^{\mathscr{M}}(y^j) x_B^\top \Sigma_n y^j_B.
\end{aligned}
\tag{EC.6}
$$

Under the linearity assumption for intensity $\lambda^{\mathscr{M}}(x) = \bar{\lambda}^\top x$, for all y^j, we have

$$
\begin{aligned}
\lambda^{\mathscr{M}}(y^j)(\mu_Z;\theta_n)^\top y^j &= (y^j)^\top \bar{\lambda}(\mu_Z;\theta_n)^\top y^j \\
&= \operatorname{tr}\left(\bar{\lambda}(\mu_Z;\theta_n)^\top \cdot y^j (y^j)^\top\right)\\
&= \operatorname{vec}((\mu_Z;\theta_n)\bar{\lambda}^\top)^\top \operatorname{vec}(u^j),
\end{aligned}
\tag{EC.7}
$$

where u^j satisfies $u^j = y^j\,(y^j)^\top$. Similarly, by defining the decision variable $z^j = \operatorname{vec}\left(y^j_B x_B^\top\right)(y^j)^\top$,

$$
\lambda^{\mathscr{M}}(y^j) x_B^\top \Sigma_n y^j_B = \bar{\lambda}^\top y^j \operatorname{tr}\left(x_B^\top \Sigma_n y^j_B\right) = \operatorname{tr}\left((y^j)^\top \bar{\lambda} \operatorname{tr}\left(\Sigma_n y^j_B x_B^\top\right)\right) = \operatorname{vec}\left(\operatorname{vec}(\Sigma_n)\bar{\lambda}^\top\right)^\top \operatorname{vec}(z^j). \tag{EC.8}
$$

The constraints $u^j = y^j\,(y^j)^\top$ and $z^j = \operatorname{vec}\left(y^j_B x_B^\top\right)(y^j)^\top$ can be expressed by linear inequalities. Applying (EC.7) and (EC.8) in the maximization problem (EC.6) along with the linearized inequality constraints for u^j and z^j yields (EC.4). Equation (EC.5) comes from $\nu = \max_{x\in\mathcal{X}} \operatorname{vec}\left((\mu_Z;\theta_n)\bar{\lambda}^\top\right)^\top \operatorname{vec}\left(xx^\top\right)$. □

EC.3. Marketing Campaign Features and Constraints

First, to ensure that the set of constraints on the allowable inputs contains at least one equality constraint (see the assumption after equation (1)), we include a feature $x_0 \in \{0,1\}$ with the associated constraint $x_0 = 1$. For age, we introduce $x_1, \cdots, x_6$, where, $x_1 = 1$ if age is between 18 and 24, $x_2 = 1$ if age is between $25 - 34$, etc. The age-related features should satisfy the constraint $\sum_{i=1}^{6} x_i = 1$. Similarly, we define income-related binary features $x_7, \ldots, x_{12}$, features related to the insurance needs x_{13}, x_{14}, x_{15}, decision variables x_{16} and x_{17} for Insurance Product, x_{18}, x_{19}, x_{20} for Customer Mindset, and $x_{21}, \ldots, x_{24}$ for Time with Insurer. In summary, the set of customer segment features satisfying the requirements is

$$\mathcal{X}_{\mathrm{C}} = \left\{ x \in \{0,1\}^{38} \text{ s.t. } \sum_{i=1}^{6} x_i = 1, \sum_{i=7}^{12} x_i = 1, \sum_{i=13}^{15} x_i = 1, \sum_{i=16}^{17} x_i = 1, \sum_{i=18}^{20} x_i = 1, \sum_{i=21}^{24} x_i = 1 \right\}.$$

For Ad channels, we define variables x_{25} (agent), x_{26} (digital), x_{27} (contact center), x_{28} (agent and contact center), x_{29} (digital and contact center), x_{30} (digital and agent), x_{31} (agent and digital and contact center). For pricing innovation features, we introduce x_{34} to specify that the advertisement emphasizes on first accident forgiveness, and x_{35}, if the advertisement emphasizes on in-vehicle telematics programs. The associated constraints define the set,

$$\begin{aligned}\mathcal{X}_{\mathrm{A}} = \Big\{ x \in \{0,1\}^{38} \text{ s.t. } \quad & x_{28} = x_{25} \times x_{27},\ x_{29} = x_{26} \times x_{27},\ x_{30} = x_{25} \times x_{26},\ x_{31} = x_{25} \times x_{26} \times x_{27}, \\ & \textstyle\sum_{i=25}^{27} x_i \geq 1,\ \sum_{i=32}^{33} x_i = 1,\ \sum_{i=34}^{35} x_i = 1,\ \sum_{i=36}^{37} x_i = 1 \Big\}.\end{aligned}$$

The nonlinear constraints such as $x_{28} = x_{25} \times x_{27}$ in the definition of $\mathcal{X}_2$ above can be reformulated by linear constraints, namely for this example, $x_{28} \leq x_{25}$, $x_{28} \leq x_{27}$, $x_{25} + x_{27} - x_{28} \leq 1$.

The cross-type constraints in our study are to ensure that for price-focused customers the accident forgiveness pricing innovation property is emphasized, for customers who are with an insurer for more than three years the accident forgiveness is not mentioned, the easy service is emphasized for convenience-focused customers, while for customers older than 45 years options on online services and digital marketing channel are not highlighted, and finally the telematics programs are not emphasized for customers earning more than $100K$. These requirements imply the following set

$$\begin{aligned}\mathcal{X}_{\mathrm{I}} = \{ x \in \{0,1\}^{38} \text{ s.t. } \quad & x_{20} - x_{34} \leq 0,\ x_\ell + x_{34} \leq 1 \quad \ell = 23, 24,\ x_{19} - x_{33} \leq 0, \\ & x_i + x_{32} \leq 1,\ x_i + x_{26} \leq 1 \quad i = 4, 5, 6, \quad x_j + x_{35} \leq 1 \quad j = 11, 12 \}.\end{aligned}$$

Hence, the set of feasible marketing campaigns is given by $\mathcal{X} = \mathcal{X}_{\mathrm{C}} \cap \mathcal{X}_{\mathrm{A}} \cap \mathcal{X}_{\mathrm{I}}$ includes $K = |\mathcal{X}| = 34560$ alternatives.

EC.4. Proofs

This section presents the proofs of the results of the paper.

EC.4.1. Proof of Proposition 1

We start by establishing a few intermediate results.

LEMMA EC.1. *Suppose Assumption [I]:* $\Sigma_{n+1}^{-1} = \Sigma_n^{-1} + \alpha x_{B,n} x_{B,n}{}^\top$ *for some* $\alpha > 0$. *Then*

$$|\Sigma_{n+1}|^{-\frac{1}{2}} = |\Sigma_n|^{-\frac{1}{2}} \sqrt{1 + \alpha x_{B,n}{}^\top \Sigma_n x_{B,n}}. \tag{EC.9}$$

Proof. It follows from assumption [I] and the fact that for any invertible square matrix X, we have $|X + AB| = |X||I + BX^{-1}A|$ and $|A^{-1}| = |A|^{-1}$, that

$$|\Sigma_{n+1}|^{-1} = |\Sigma_{n+1}^{-1}| = |\Sigma_n^{-1} + \alpha x_{B,n} x_{B,n}{}^\top| = |\Sigma_n^{-1}||1 + \alpha x_{B,n}{}^\top \Sigma_n x_{B,n}|.$$

Positive semidefiniteness of Σ_Z, Σ_B, and Σ_n implies that $1 + \alpha x_{B,n}{}^\top \Sigma_n x_{B,n} > 0$. This completes the proof of equation (EC.9). □

LEMMA EC.2. *Let* Σ_{n+1} *be defined as in* (9)*. Then [I] in Lemma EC.1 holds for* $\alpha = (1 + x_{Z,n}^\top \Sigma_Z x_{Z,n} + x_{B,n}^\top \Sigma_B x_{B,n})^{-1}$.

Proof. It follows from the matrix inversion lemma, $(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$, that

$$\begin{aligned}
\Sigma_{n+1}^{-1} &= \left(\Sigma_n - \frac{\Sigma_n x_{B,n} x_{B,n}{}^\top \Sigma_n}{1 + x_{B,n}{}^\top \Sigma_n x_{B,n} + x_{Z,n}^\top \Sigma_Z x_{Z,n} + x_{B,n}{}^\top \Sigma_B x_{B,n}}\right)^{-1} \\
&= \Sigma_n^{-1} + \Sigma_n^{-1}\Sigma_n x_{B,n} \left(1 + x_{Z,n}^\top \Sigma_Z x_{Z,n} + x_{B,n}{}^\top \Sigma_B x_{B,n}\right)^{-1} x_{B,n}{}^\top \Sigma_n \Sigma_n^{-1} \\
&= \Sigma_n^{-1} + \alpha x_{B,n} x_{B,n}{}^\top.
\end{aligned} \tag{EC.10}$$

This completes the proof. □

LEMMA EC.3. *Let* $L = \theta_{n+1} - \theta_n$*. Suppose [I] holds and*

[II] there exists $\gamma > 0$ *such that* $L = \frac{\gamma \ell}{\alpha^{-1} + x_{B,n}{}^\top \Sigma_n x_{B,n}} \Sigma_n x_{B,n}$.

Then,

$$\begin{aligned}
(\mu_B - \theta_{n+1})^\top \Sigma_{n+1}^{-1} (\mu_B - \theta_{n+1}) = (\mu_B - \theta_n)^\top \Sigma_n^{-1} (\mu_B - \theta_n) + \alpha \left(\ell\gamma + (\theta_n - \mu_B)^\top x_{B,n}\right)^2 \\
- \frac{\gamma^2 \ell^2}{x_{B,n}{}^\top \Sigma_n x_{B,n} + \alpha^{-1}}.
\end{aligned} \tag{EC.11}$$

Proof. Using $L = \theta_{n+1} - \theta_n$, we have

$$\begin{aligned}
&(\mu_B - \theta_{n+1})^\top \Sigma_{n+1}^{-1} (\mu_B - \theta_{n+1}) \\
&\qquad = (\mu_B - \theta_n)^\top \Sigma_{n+1}^{-1} (\mu_B - \theta_n) - 2(\mu_B - \theta_n)^\top \Sigma_{n+1}^{-1} L + L^\top \Sigma_{n+1}^{-1} L.
\end{aligned} \tag{EC.12}$$

Next, we apply Assumption [I] for each term in (EC.12). First,

$$(\mu_B-\theta_n)^\top\Sigma_{n+1}^{-1}(\mu_B-\theta_n)=(\mu_B-\theta_n)^\top\Sigma_n^{-1}(\mu_B-\theta_n)+\alpha(\mu_B-\theta_n)^\top x_{B,n}x_{B,n}{}^\top(\mu_B-\theta_n). \quad \text{(EC.13)}$$

From Assumptions [I] and [II], we have

$$\begin{aligned} L^\top\Sigma_{n+1}^{-1}L &= \left(\frac{\gamma\ell}{x_{B,n}{}^\top\Sigma_n x_{B,n}+\alpha^{-1}}\right)^2 x_{B,n}{}^\top\Sigma_n\left(\Sigma_n^{-1}+\alpha x_{B,n}x_{B,n}{}^\top\right)\Sigma_n x_{B,n} \quad \text{(EC.14)}\\ &= \left(\frac{\gamma\ell}{x_{B,n}{}^\top\Sigma_n x_{B,n}+\alpha^{-1}}\right)^2 \alpha\left(\alpha^{-1}+x_{B,n}{}^\top\Sigma_n x_{B,n}\right)x_{B,n}{}^\top\Sigma_n x_{B,n}\\ &= \frac{\alpha\gamma^2\ell^2 x_{B,n}{}^\top\Sigma_n x_{B,n}}{x_{B,n}{}^\top\Sigma_n x_{B,n}+\alpha^{-1}} = \alpha\gamma^2\ell^2-\frac{\gamma^2\ell^2}{x_{B,n}{}^\top\Sigma_n x_{B,n}+\alpha^{-1}}. \end{aligned}$$

Next, we have

$$\begin{aligned} (\mu_B-\theta_n)^\top\Sigma_{n+1}^{-1}L &= (\mu_B-\theta_n)^\top\left(\Sigma_n^{-1}+\alpha x_{B,n}x_{B,n}{}^\top\right)\frac{\gamma\ell}{(\alpha^{-1}+x_{B,n}{}^\top\Sigma_n x_{B,n})}\Sigma_n x_{B,n}\\ &= \frac{\gamma\ell}{\alpha^{-1}+x_{B,n}{}^\top\Sigma_n x_{B,n}}(\mu_B-\theta_n)^\top\left(\Sigma_n^{-1}+\alpha x_{B,n}x_{B,n}{}^\top\right)\Sigma_n x_{B,n}\\ &= \frac{\gamma\ell}{(\alpha^{-1}+x_{B,n}{}^\top\Sigma_n x_{B,n})}(\mu_B-\theta_n)^\top x_{B,n}\left(1+\alpha x_{B,n}{}^\top\Sigma_n x_{B,n}\right)\\ &= \alpha\gamma\ell(\mu_B-\theta_n)^\top x_{B,n}. \quad \text{(EC.15)} \end{aligned}$$

By using equations (EC.13), (EC.14), and (EC.15) in (EC.12), we have

$$\begin{aligned} &(\mu_B-\theta_{n+1})^\top\Sigma_{n+1}^{-1}(\mu_B-\theta_{n+1}) = (\mu_B-\theta_n)^\top\Sigma_n^{-1}(\mu_B-\theta_n)\\ &\quad+\alpha\left[(\mu_B-\theta_n)^\top x_{B,n}x_{B,n}{}^\top(\mu_B-\theta_n)-2\ell\gamma(\mu_B-\theta_n)^\top x_{B,n}+\gamma^2\ell^2\right]-\frac{\gamma^2\ell^2}{x_{B,n}{}^\top\Sigma_n x_{B,n}+\alpha^{-1}}\\ &= (\mu_B-\theta_n)^\top\Sigma_n^{-1}(\mu_B-\theta_n)+\alpha\left(\ell\gamma-(\mu_B-\theta_n)^\top x_{B,n}\right)^2-\frac{\gamma^2\ell^2}{x_{B,n}{}^\top\Sigma_n x_{B,n}+\alpha^{-1}}, \end{aligned}$$

which leads to equation (EC.11). □

Next we apply Lemmas EC.1, EC.2, and EC.3 to prove Proposition 1.

Proof of Proposition 1. Since (μ_B,ρ) follows a multivariate normal-gamma distribution with parameters $(\theta_n,\Sigma_n,a_n,b_n)$, the joint density is given by

$$\begin{aligned} \Pr(\mu_B,\rho|S_n) &= \Pr(\mu_B|\rho,\theta_n,\Sigma_n)\ \Pr(\rho|a_n,b_n)\\ &= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}}|\Sigma_n|^{-1/2}e^{-\frac{\rho}{2}(\mu_B-\theta_n)^\top\Sigma_n^{-1}(\mu_B-\theta_n)}\times\frac{b_n^{a_n}}{\Gamma(a_n)}\rho^{a_n-1}e^{-b_n\rho}, \quad \text{(EC.16)} \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function.

Let η_{n+1} and κ_{n+1} be the observations corresponding to the measurement $x_n=(x_{Z,n},x_{B,n})$. It follows from $\zeta\sim\mathcal{N}(\mu_Z,\frac{1}{\rho}\Sigma_Z)$ and $\beta\sim\mathcal{N}(\mu_B,\frac{1}{\rho}\Sigma_B)$, and the assumption that ζ, β, and ϵ are independent given ρ, that for every given $(x_{Z,n},x_{B,n})$,

$$\left\{\zeta^\top x_{Z,n}+\beta^\top x_{B,n}+\epsilon\mid\mu_B,\rho\right\}\ \sim\mathcal{N}\left(\mu_Z^\top x_{Z,n}+\mu_B^\top x_{B,n}\ ,\ \frac{1}{\rho}\left(1+x_{Z,n}^\top\Sigma_Z x_{Z,n}+x_{B,n}^\top\Sigma_B x_{B,n}\right)\right). \quad \text{(EC.17)}$$

Define $\hat{\rho} \stackrel{\text{def}}{=} \frac{1}{1+x_{Z,n}^\top \Sigma_Z x_{Z,n} + x_{B,n}^\top \Sigma_B x_{B,n}} \rho$ and $\hat{\sigma}_n \stackrel{\text{def}}{=} 1 + x_{Z,n}^\top \Sigma_Z x_{Z,n} + x_{B,n}^\top \Sigma_B x_{B,n}$. Thus, $\hat{\rho} = \rho \hat{\sigma}_n^{-1}$. Therefore, under model (2), we have

$$\left\{\eta_{n+1} = \kappa_{n+1}\left(\zeta^\top x_{Z,n} + \beta^\top x_{B,n} + \epsilon\right) | \mu_B, \rho, x_n, \kappa_{n+1}\right\} \sim \mathcal{N}\left(\kappa_{n+1}\left(\mu_Z^\top x_{Z,n} + \mu_B^\top x_{B,n}\right), \frac{\kappa_{n+1}^2}{\hat{\rho}}\right). \quad \text{(EC.18)}$$

Consequently, given $\kappa_{n+1} > 0$,

$$\Pr(\eta_{n+1}|\mu_B, \rho, x_n, \kappa_{n+1}) = \frac{\sqrt{\hat{\rho}}}{\sqrt{2\pi}\kappa_{n+1}} \exp\left(-\frac{\hat{\rho}}{2\kappa_{n+1}^2}\left(\eta_{n+1} - \kappa_{n+1}(\mu_Z^\top x_{Z,n} + \mu_B^\top x_{B,n})\right)^2\right). \quad \text{(EC.19)}$$

Define $\ell \stackrel{\text{def}}{=} \frac{\eta_{n+1}}{\kappa_{n+1}} - (\mu_Z^\top x_{Z,n} + \theta_n^\top x_{B,n})$. Bayes' rule implies that

$$\begin{aligned}
\Pr\left(\mu_B, \rho | S_n, x_n, \kappa_{n+1}, \eta_{n+1}\right) &\propto \Pr\left(\mu_B, \rho | S_n, x_n, \kappa_{n+1}\right) \times \Pr\left(\eta_{n+1} | \mu_B, \rho, S_n, x_n, \kappa_{n+1}\right) \\
&= \Pr\left(\mu_B, \rho | S_n\right) \times \Pr\left(\eta_{n+1} | \mu_B, \rho, x_n, \kappa_{n+1}\right) && \text{(EC.20)} \\
&= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_n|^{-1/2} e^{-\frac{\rho}{2}(\mu_B - \theta_n)^\top \Sigma_n^{-1}(\mu_B - \theta_n)} \times \frac{b_n^{a_n}}{\Gamma(a_n)} \rho^{a_n - 1} e^{-b_n \rho} && \text{(EC.21)} \\
&\times \frac{\sqrt{\hat{\rho}}}{\sqrt{2\pi}\kappa_{n+1}} e^{-\frac{\hat{\rho}}{2}\left(\ell + (\theta_n - \mu_B)^\top x_{B,n}\right)^2},
\end{aligned}$$

The equality (EC.20) comes from the fact that the realized frequency κ_{n+1} does not impact the distribution of μ_B and ρ. Thus, $\Pr\left(\mu_B, \rho | S_n, x_n, \kappa_{n+1}\right) = \Pr\left(\mu_B, \rho | S_n\right)$. In addition, given μ_B and ρ, the description of η_{n+1} implies that $p\left(\eta_{n+1} | \mu_B, \rho, S_n, x_n, \kappa_{n+1}\right) = p\left(\eta_{n+1} | \mu_B, \rho, x_n, \kappa_{n+1}\right)$. Equation (EC.21) comes from (EC.16) and (EC.19).

Using the expression of θ_{n+1} in equation (8), the assumptions in Lemma EC.3 hold for $\gamma = 1$ and $\alpha = \hat{\sigma}_n^{-1}$. Therefore, it follows from Lemma EC.3 that

$$(\mu_B - \theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B - \theta_{n+1}) = (\mu_B - \theta_n)^\top \Sigma_n^{-1}(\mu_B - \theta_n) + \hat{\sigma}_n^{-1}\left(\ell + (\theta_n - \mu_B)^\top x_{B,n}\right)^2 - 2(b_{n+1} - b_n).$$

Here, we have introduced the notation b_{n+1} as defined in equation (11) to get $2(b_{n+1} - b_n) = \frac{\gamma^2 \ell^2}{x_{B,n}{}^\top \Sigma_n x_{B,n} + \alpha^{-1}}$. Therefore,

$$\begin{aligned}
&-\frac{\rho}{2}(\mu_B - \theta_n)^\top \Sigma_n^{-1}(\mu_B - \theta_n) - b_n \rho - \frac{\hat{\rho}}{2}\left(\ell - (\mu_B - \theta_n)^\top x_{B,n}\right)^2 \\
&= -\frac{\rho}{2}(\mu_B - \theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B - \theta_{n+1}) - b_{n+1}\rho. && \text{(EC.22)}
\end{aligned}$$

Using Lemma EC.1 for $\alpha = \hat{\sigma}_n^{-1}$ and equation (EC.22), the right-hand side of (EC.21) equals

$$\begin{aligned}
&\left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_n|^{-\frac{1}{2}} e^{-\frac{\rho}{2}(\mu_B - \theta_n)^\top \Sigma_n^{-1}(\mu_B - \theta_n)} \times \frac{b_n^{a_n}}{\Gamma(a_n)} \rho^{a_n - 1} e^{-b_n \rho} \frac{\sqrt{\hat{\rho}}}{\sqrt{2\pi}\kappa_{n+1}} e^{-\frac{\hat{\rho}}{2}\left(\ell + (\theta_n - \mu_B)^\top x_{B,n}\right)^2} \\
&= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} \frac{|\Sigma_{n+1}|^{-\frac{1}{2}}}{\sqrt{1 + \hat{\sigma}_n^{-1} x_{B,n}{}^\top \Sigma_n x_{B,n}}} e^{-\frac{\rho}{2}(\mu_B - \theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B - \theta_{n+1})} \times \frac{\sqrt{\hat{\sigma}_n^{-1}} b_n^{a_n}}{\sqrt{2}\kappa_{n+1} B(a_n, \frac{1}{2}) \Gamma(a_{n+1})} \rho^{a_{n+1} - 1} e^{-b_{n+1}\rho} \\
&= c_0 \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_{n+1}|^{-\frac{1}{2}} e^{-\frac{\rho}{2}(\mu_B - \theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B - \theta_{n+1})} \times \frac{b_{n+1}^{a_{n+1}}}{\Gamma(a_{n+1})} \rho^{a_{n+1} - 1} e^{-b_{n+1}\rho}. && \text{(EC.23)}
\end{aligned}$$

where we have introduced the notation $a_{n+1} = a_n + \frac{1}{2}$, see equation (10), and used the fact that $B(a_n, \frac{1}{2}) = \frac{\Gamma(a_n)\Gamma(\frac{1}{2})}{\Gamma(a_n+\frac{1}{2})} = \frac{\sqrt{\pi}\Gamma(a_n)}{\Gamma(a_{n+1})}$, where $B(a_n, \frac{1}{2})$ is the beta function. In equation (EC.23), the factor c_0 is given by

$$c_0 \stackrel{\text{def}}{=} \frac{\hat{\sigma}_n^{-1/2} b_n^{a_n}}{\sqrt{1+\hat{\sigma}_n^{-1} x_{B,n}{}^\top \Sigma_n x_{B,n}}\, b_{n+1}^{a_{n+1}} \sqrt{2}\kappa_{n+1} B(a_n, \frac{1}{2})}. \tag{EC.24}$$

The factor c_0 in equation (EC.24) is constant with respect to ρ and μ_B. Thus, we arrive at

$$p\left(\mu_B, \rho | S_n, x_n, \kappa_{n+1}, \eta_{n+1}\right) \propto \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_{n+1}|^{-1/2} e^{-\frac{\rho}{2}(\mu_B-\theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B-\theta_{n+1})} \frac{b_{n+1}^{a_{n+1}}}{\Gamma(a_{n+1})} \rho^{a_{n+1}-1} e^{-b_{n+1}\rho},$$

which is the normal-gamma density with parameters $S_{n+1} = (\theta_{n+1}, \Sigma_{n+1}, a_{n+1}, b_{n+1})$. □

EC.4.2. Proof of Proposition 2

Proof. It follows from $\lambda^{\mathscr{M}}\left(x^{\bar{k}}\right) = \lambda_{\bar{k}} \sim Gamma\left(c_n\left(x^{\bar{k}}\right), d_n\left(x^{\bar{k}}\right)\right)$ that $\Pr\left(\lambda_{\bar{k}} \mid c_n(x^{\bar{k}}), d_n(x^{\bar{k}})\right) \propto \lambda_{\bar{k}}^{c_n\left(x^{\bar{k}}\right)-1} e^{-d_n\left(x^{\bar{k}}\right)\lambda_{\bar{k}}}$. Let κ_{n+1} be drawn from $Poisson\left(\lambda^{\mathscr{M}}\left(x^{\bar{k}}\right)\right)$. The conditional distribution of $\lambda^{\mathscr{M}}\left(x^{\bar{k}}\right) = \lambda_{\bar{k}}$ given S_{n+1} is

$$\begin{aligned}\Pr\left(\lambda_{\bar{k}} \mid c_n(x^{\bar{k}}), d_n(x^{\bar{k}}), \kappa_{n+1}\right) &\propto \Pr\left(\kappa_{n+1} | \lambda_{\bar{k}}\right) \times \Pr\left(\lambda_{\bar{k}} \mid c_n(x^{\bar{k}}), d_n(x^{\bar{k}})\right) \\ &\propto \lambda_{\bar{k}}^{\kappa_{n+1}} e^{-\lambda_{\bar{k}}} \times \lambda_{\bar{k}}^{c_n(x^{\bar{k}})-1} e^{-d_n(x^{\bar{k}})\lambda_{\bar{k}}} = \lambda_{\bar{k}}^{c_n(x^{\bar{k}})+\kappa_{n+1}-1} e^{-\left(d_n(x^{\bar{k}})+1\right)\lambda_{\bar{k}}}.\end{aligned}$$

Hence, the posterior distribution remains a Gamma distribution with the shape parameter $c_n(x^{\bar{k}}) + \kappa_{n+1}$ and the rate parameter $d_n(x^{\bar{k}}) + 1$. □

EC.4.3. Proof of Proposition 3

Proof. From the description of η_{n+1} in (2), we have

$$\eta_{n+1} \mid S_n, \rho, x_n, \mu_B, \kappa_{n+1} \sim \mathcal{N}\left(\kappa_{n+1}(\mu_Z^\top x_{Z,n} + \mu_B^\top x_{B,n}), \frac{\kappa_{n+1}^2}{\rho}\hat{\sigma}_n\right). \tag{EC.25}$$

Recall that $\hat{\sigma}_n = 1 + x_{Z,n}^\top \Sigma_Z x_{Z,n} + x_{B,n}^\top \Sigma_B x_{B,n}$. Thus from (EC.25) and $\mu_B | S_n, \rho \sim \mathcal{N}(\theta_n, \frac{1}{\rho}\Sigma_n)$, the random variable $\eta_{n+1} \mid S_n, \rho, x_n, \kappa_{n+1}$ follows a multiplicative model whose mean is distributed according to another Gaussian distribution. This implies that the Gaussian distribution

$$J \stackrel{\text{def}}{=} \eta_{n+1} - \kappa_{n+1}(\mu_Z^\top x_{Z,n} + \mu_B^\top x_{B,n}) \mid S_n, \rho, x_n, \mu_B, \kappa_{n+1} \sim \mathcal{N}\left(0, \frac{\kappa_{n+1}^2}{\rho}\hat{\sigma}_n\right)$$

is independent of $\kappa_{n+1}(\mu_Z^\top x_{Z,n} + \mu_B^\top x_{B,n})$. Therefore,

$$\begin{aligned}\{\eta_{n+1} \mid S_n, \rho, x_n, \kappa_{n+1}\} &= J + \kappa_{n+1}(\mu_Z^\top x_{Z,n} + \mu_B^\top x_{B,n}) \\ &\sim \mathcal{N}\left(0 + \kappa_{n+1}(\mu_Z^\top x_{Z,n} + \theta_n^\top x_{B,n}), \frac{\kappa_{n+1}^2}{\rho}\hat{\sigma}_n + \frac{\kappa_{n+1}^2}{\rho} x_{B,n}{}^\top \Sigma_n x_{B,n}\right).\end{aligned}$$

Given $k_{n+1} = k > 0$, we have

$$\begin{aligned}\Pr\left(\eta_{n+1} = \eta \mid S_n, x_n, \kappa_{n+1} = k\right) &= \int_0^\infty \Pr\left(\eta_{n+1} \mid \rho = r, S_n, x_n, \kappa_{n+1} = k\right) \Pr(\rho = r) dr \\ &= \int_0^\infty \frac{\sqrt{r}}{k\sqrt{2\pi\left(\hat{\sigma}_n + x_{B,n}^\top \Sigma_n x_{B,n}\right)}} e^{-\frac{r\left(\eta - k(\mu_Z^\top x_{Z,n} + \theta_n^\top x_{B,n})\right)^2}{2k^2(\hat{\sigma}_n + x_{B,n}{}^\top \Sigma_n x_{B,n})}} \times \frac{b_n^{a_n}}{\Gamma(a_n)} r^{a_n-1} e^{-b_n r} dr.\end{aligned}$$

Define

$$\hat{\eta}_{n+1} \stackrel{\text{def}}{=} \sqrt{\frac{a_n}{\kappa^2 b_n(\hat{\sigma}_n + {x_{B,n}}^\top \Sigma_n x_{B,n})}} \left(\eta_{n+1} - \kappa(\mu_Z^\top x_{Z,n} + \theta_n^\top x_{B,n})\right). \tag{EC.26}$$

Therefore, given (S_n, x_n, κ_{n+1}), $\hat{\eta}_{n+1} = \hat{\eta}$ if and only if $\eta_{n+1} = g(\hat{\eta})$, where

$$g(\hat{\eta}) \stackrel{\text{def}}{=} k(\mu_Z^\top x_{Z,n} + \theta_n^\top x_{B,n}) + k\sqrt{\frac{b_n\left(\hat{\sigma}_n + {x_{B,n}}^\top \Sigma_n x_{B,n}\right)}{a_n}}\hat{\eta}.$$

Thus, $\Pr\left(\hat{\eta}_{n+1} = \hat{\eta} | S_n, x_n, \kappa_{n+1}\right) = \Pr\left(\eta_{n+1} = g(\hat{\eta}) | S_n, x_n, \kappa_{n+1}\right) \times g'(\hat{\eta})$. Denote $\hat{\sigma} \stackrel{\text{def}}{=} \hat{\sigma}_n + {x_{B,n}}^\top \Sigma_n x_{B,n}$. Hence, we have

$$\begin{aligned}
\Pr\left(\hat{\eta}_{n+1} = \hat{\eta} \mid S_n, x_n, \kappa_{n+1} = k\right) &= \int_0^\infty \frac{\sqrt{r}}{k\sqrt{2\pi\hat{\sigma}}} e^{-\frac{r\left(k\sqrt{\frac{b_n\hat{\sigma}}{a_n}}\hat{\eta}\right)^2}{2k^2\hat{\sigma}}} \times \frac{b_n^{a_n}}{\Gamma(a_n)} r^{a_n-1} e^{-b_n r} dr \times k\sqrt{\frac{b_n\hat{\sigma}}{a_n}} \\
&= \frac{1}{\Gamma(a_n)\sqrt{2a_n\pi}} \int_0^\infty e^{-rb_n\left(1+\frac{\hat{\eta}^2}{2a_n}\right)} b_n^{a_n+\frac{1}{2}} r^{a_n-\frac{1}{2}} dr \\
&= \frac{1}{\Gamma(a_n)\sqrt{2a_n\pi}} \left(1 + \frac{\hat{\eta}^2}{2a_n}\right)^{-a_n-\frac{1}{2}} \int_0^\infty e^{-x} x^{a_n-\frac{1}{2}} dx \\
&= \frac{1}{\Gamma(a_n)\sqrt{2a_n\pi}} \left(1 + \frac{\hat{\eta}^2}{2a_n}\right)^{-a_n-\frac{1}{2}} \Gamma\left(a_n + \frac{1}{2}\right).
\end{aligned}$$

The last line is the probability density function of the standard Student's t-distribution with $2a_n$ degrees of freedom. Hence, $\hat{\eta}_{n+1} | S_n, x_n, \kappa_{n+1} = k > 0$ follows a standard Student's t-distribution with $2a_n$ degrees of freedom. This completes the proof of (12). □

EC.4.4. Proof of Proposition 4

Proof. We proceed by backward induction on n. Problem (14) is reduced to

$$\sup_{\pi\in\Pi} \mathbb{E}^\pi\left[\max_{x\in\mathcal{X}} \lambda^{\mathscr{M}}(x)\left(\mu_Z^\top x_Z + \theta_N^\top x_B\right)\right]. \tag{EC.27}$$

Recall that $\theta_N := \mathbb{E}[\mu_B | S_N]$. For $n = N-1$, for any $x_{N-1} \in \mathcal{X}$,

$$\begin{aligned}
Q_{N-1}(S_{N-1}, x_{N-1}) &= \mathbb{E}\left[V_N\left(S^{\mathscr{M}}\left(S_{N-1}, x_{N-1}, (\kappa_N, \eta_N)\right)\right)\right] \\
&= \mathbb{E}\left[\max_{y\in\mathcal{X}} \lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z + \theta_N^\top y_B\right)\right] \\
&= \mathbb{E}\left[\max_{y\in\mathcal{X}} \lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z + \theta_{N-1}^\top y_B + \sqrt{\frac{b_{N-1}}{a_{N-1}\hat{\sigma}}} y_B^\top \Sigma_{N-1} x_{B,N-1} T_{2a_{N-1}}\right)\right] \\
&\geq \max_{y\in\mathcal{X}} \mathbb{E}\left[\lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z + \theta_{N-1}^\top y_B + \sqrt{\frac{b_{N-1}}{a_{N-1}\hat{\sigma}}} y_B^\top \Sigma_{N-1} x_{B,N-1} T_{2a_{N-1}}\right)\right] \\
&= \max_{y\in\mathcal{X}} \left\{\lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z + \theta_{N-1}^\top y_B + \sqrt{\frac{b_{N-1}}{a_{N-1}\hat{\sigma}}} y_B^\top \Sigma_{N-1} x_{B,N-1} \mathbb{E}[T_{2a_{N-1}}]\right)\right\} \\
&= \max_{y\in\mathcal{X}} \lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z + \theta_{N-1}^\top y_B\right) = V_N(S_{N-1}).
\end{aligned}$$

Here, we used the notation $\hat{\sigma}$ defined as $\hat{\sigma} = \hat{\sigma}_{N-1}(x_{Z,N-1}, x_{B,N-1}) + x_{B,N-1}^\top \Sigma_{N-1} x_{B,N-1}$. In the last equation, we used the fact that $\mathbb{E}[T_{2a_{N-1}}] = 0$.

Next, we prove the induction step. Suppose the statement is true for $n+1$, we prove it for n. For any $x_n \in \mathcal{X}$,

$$\begin{aligned}
Q_n(S_n, x_n) &= \mathbb{E}\left[V_{n+1}\left(S^{\mathscr{M}}\left(S_n, x_n, (\kappa_{n+1}, \eta_{n+1})\right)\right)\right], \\
&= \mathbb{E}\left[\max_{y\in\mathcal{X}} \mathbb{E}\left[V_{n+2}\left(S^{\mathscr{M}}\left(S^{\mathscr{M}}\left(S_n, x_n, (\kappa_{n+1}, \eta_{n+1})\right), y, (\kappa_y, \eta_y)\right)\right)\right]\right] && \text{(EC.28)} \\
&\geq \max_{y\in\mathcal{X}} \mathbb{E}\left[\mathbb{E}\left[V_{n+2}\left(S^{\mathscr{M}}\left(S^{\mathscr{M}}\left(S_n, x_n, (\kappa_{n+1}, \eta_{n+1})\right), y, (\kappa_y, \eta_y)\right)\right)\right]\right] && \text{(EC.29)} \\
&= \max_{y\in\mathcal{X}} \mathbb{E}\left[\mathbb{E}\left[V_{n+2}\left(S^{\mathscr{M}}\left(S^{\mathscr{M}}(S_n, y, (\kappa_y, \eta_y)), x_n, (\kappa_{n+1}, \eta_{n+1})\right)\right)\right]\right] && \text{(EC.30)} \\
&= \max_{y\in\mathcal{X}} \mathbb{E}\left[Q_{n+1}\left(S^{\mathscr{M}}\left(S_n, y, (\kappa_y, \eta_y)\right), x_n\right)\right], && \text{(EC.31)}
\end{aligned}$$

where (κ_y, η_y) is the observed outcome of the marketing campaign characterized by the input feature vector y. Equality (EC.28) comes from the optimality equation. In equation (EC.30), we use the fact that the belief model and the induced transition function $S^{\mathscr{M}}$ satisfy

$$S^{\mathscr{M}}\left(S^{\mathscr{M}}\left(S_n, y, (\kappa_y, \eta_y)\right), x, (\kappa_x, \eta_x)\right) = S^{\mathscr{M}}\left(S^{\mathscr{M}}\left(S_n, x, (\kappa_x, \eta_x)\right), y, (\kappa_y, \eta_y)\right), \ \forall\, y, x \in \mathcal{X}.$$

The induction hypothesis implies that

$$Q_{n+1}\left(S^{\mathscr{M}}\left(S_n, y, (\kappa_y, \eta_y)\right), x_n\right) \geq V_{n+2}\left(S^{\mathscr{M}}\left(S_n, y, (\kappa_y, \eta_y)\right)\right), \quad \text{almost surely}, \quad \forall\, y \in \mathcal{X}.$$

By applying this inequality in (EC.31), we arrive at

$$\begin{aligned}
Q_n(S_n, x_n) &\geq \max_{y\in\mathcal{X}} \mathbb{E}\left[Q_{n+1}\left(S^{\mathscr{M}}\left(S_n, y, (\kappa_y, \eta_y)\right), x_n\right)\right] \\
&\geq \max_{y\in\mathcal{X}} \mathbb{E}\left[V_{n+2}(S^{\mathscr{M}}\left(S_n, y, (\kappa_y, \eta_y)\right))\right] = V_{n+1}(S_n), \quad \forall\, x_n \in \mathcal{X}.
\end{aligned}$$

This shows the validity of the statement for n, and completes the proof of the first part. When the additional marketing campaign can be measured according to the optimal policy π^*, we have $Q_n(S_n, x_n^*) = V_n(S_n)$. This, along with the inequality $Q_n(S_n, x_n) \geq V_{n+1}(S_n)$ for any $x_n \in \mathcal{X}$, implies that $V_n(S_n) \geq V_{n+1}(S_n)$. □

EC.4.5. Proof of Proposition 5

Proof. It follows from $V_N^\pi(S_n) := U(\mathbb{E}[\mu_B | S_N = S_n]) = U(\theta_n)$, that

$$\begin{aligned}
v_x^{\mathrm{KG},n} &= \mathbb{E}\left[V_N\left(S^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) - V_N(S_n) \mid S_n, x\right] \\
&= \mathbb{E}\left[V_N\left(S^{\mathscr{M}}\left(S_n, x, (\kappa_{n+1}, \eta_{n+1})\right)\right) - U(\theta_n) \mid S_n, x\right] \\
&= \mathbb{E}\left[U\left(\theta^{\mathscr{M}}\left(S_n, x, (\kappa_{n+1}, \eta_{n+1})\right)\right) \mid S_n, x\right] - U(\theta_n), && \text{(EC.32)}
\end{aligned}$$

where $\theta^{\mathscr{M}}(S_n, x, (\eta_{n+1}, \kappa_{n+1})) = \mathbb{E}\left[\mu_B \mid S_N = S^{\mathscr{M}}(S_n, x, (\eta_{n+1}, \kappa_{n+1}))\right]$. The first term is then equal to

$$\begin{aligned}
&\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) \mid S_n, x\right] \\
&= \mathbb{E}_{\kappa_{n+1}}\left[\mathbb{E}_{\eta_{n+1}}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) \mid S_n, x, \kappa_{n+1}\right]\right] \\
&= \Pr(\kappa_{n+1} = 0|x) \times \mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) \mid S_n, x, \kappa_{n+1} = 0\right] \\
&\quad + \Pr(\kappa_{n+1} > 0|x) \times \mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) \mid S_n, x, \kappa_{n+1} > 0\right] \\
&= e^{-\lambda^{\mathscr{M}}(x)} U(\theta_n) + (1 - e^{-\lambda^{\mathscr{M}}(x)})\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) \mid S_n, x, \kappa_{n+1} > 0\right]. \quad \text{(EC.33)}
\end{aligned}$$

Here, we used the observation that $\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1})) = \theta_n$, given $\kappa_{n+1} = 0$. Consequently $\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) \mid S_n, x, \kappa_{n+1} = 0\right] = U(\theta_n)$.

From Proposition 3 and given $\kappa_{n+1} > 0$, η_{n+1} follows the Student's t-distribution. In particular, using equation (13), we have

$$\theta^{\mathscr{M}}(S_n, x, (\eta_{n+1}, \kappa_{n+1})) = \theta_{n+1} = \theta_n + \sqrt{\frac{b_n}{a_n(\hat{\sigma}_n(x) + x_B^\top \Sigma_n x_B)}} \Sigma_n x_B T_{2a_n}.$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))\right) \mid S_n, x, \kappa_{n+1} > 0\right] \\
&= \mathbb{E}\left[U\left(\theta_n + \sqrt{\frac{b_n}{a_n(\hat{\sigma}_n(x) + x_B^\top \Sigma_n x_B)}} \Sigma_n x_B T_{2a_n}\right) \mid S_n, x, \kappa_{n+1} > 0\right] \\
&= \mathbb{E}\left[\max_{y \in \mathcal{X}} \lambda(y)\left(\mu_Z^\top y_Z + \left(\theta_n + \sqrt{\frac{b_n}{a_n(\hat{\sigma}_n(x) + x_B^\top \Sigma_n x_B)}} \Sigma_n x_B T_{2a_n}\right)^\top y_B\right) \mid S_n, x, \kappa_{n+1} > 0\right] \\
&= \mathbb{E}\left[\max_{y \in \mathcal{X}} \left(p_y^n + q_y^n(x) T_{2a_n}\right) \mid S_n, x, \kappa_{n+1} > 0\right], \qquad \text{(EC.34)}
\end{aligned}$$

where p_y^n and $q_y^n(x_n)$ are given in equations (18) and (19). Applying equation (EC.34) in (EC.33), and finally in equation (EC.32) completes the derivation of (17). □

EC.4.6. Proof of Proposition 6

Proof. Using the expression (21) and the notation ν in (23),

$$\max_{x \in \mathcal{X}} v_n^{\mathrm{KG}}(x) = \max_{x, y^1, \cdots, y^J \in \mathcal{X}} \left(1 - e^{-\lambda^{\mathscr{M}}(x)}\right)\left(\sum_{j=1}^J w_j p_{y^j}^n + \sum_{j=1}^J w_j q_{y^j}^n(x) t_j - \nu\right), \qquad \text{(EC.35)}$$

where $p_{y^j}^n$ and $q_{y^j}^n(x)$ are given in equations (18) and (19). Given the description of $\mathcal{X}$ in equation (1), for every feasible $x \in \mathcal{X}$, $h^\top h = (Ax)^\top (Ax) = x^\top A^\top A x$. It then follows from $\|h\|_2 \neq 0$, that $1 = x^\top \frac{A^\top A}{h^\top h} x$. Using this equality and the definition of the matrix P_n in (22), we get $\|P_n^{1/2} x\|_2^2 = x^\top P_n x = a_n\left(1 + x_Z^\top \Sigma_Z x_Z + x_B^\top (\Sigma_n + \Sigma_B) x_B\right)/b_n$ and consequently from (19), $q_{y^j}^n(x) =$

$\lambda^{\mathscr{M}}(y^j)\frac{x_B{}^\top \Sigma_n y_B^j}{\|P_n^{1/2}x\|_2}$. Replacing this equation and the expression of $p_{y^j}^n$ in equation (EC.35) implies that $\max_{x\in\mathcal{X}} v_n^{\mathrm{KG}}(x)$ equals

$$\max_{x,y^1,\cdots,y^J\in\mathcal{X}} \left(1-e^{-\lambda^{\mathscr{M}}(x)}\right)\left(\sum_{j=1}^J w_j\lambda^{\mathscr{M}}(y^j)\left(\mu_Z^\top y_Z^j+\theta_n^\top y_B^j\right)+\sum_{j=1}^J w_j\lambda^{\mathscr{M}}(y^j)\frac{x_B{}^\top\Sigma_n y_B^j}{\|P_n^{1/2}x\|_2}t_j-\nu\right), \quad \text{(EC.36)}$$

or equivalently

$$\max_{\substack{x,y^1,\cdots,y^J\in\mathcal{X},\\ \tau\le\sum_{j=1}^J w_j\lambda^{\mathscr{M}}(y^j)\frac{x_B{}^\top\Sigma_n y_B^j}{\|P_n^{1/2}x\|_2}t_j}} \left(1-e^{-\lambda^{\mathscr{M}}(x)}\right)\left(\sum_{j=1}^J w_j\lambda^{\mathscr{M}}(y^j)\left(\mu_Z^\top y_Z^j+\theta_n^\top y_B^j\right)+\tau-\nu\right). \quad \text{(EC.37)}$$

For the maximization problem, at the optimal solution, we have $\tau=\sum_{j=1}^J w_j\lambda^{\mathscr{M}}(y^j)\frac{x_B{}^\top\Sigma_n y_B^j}{\|P_n^{1/2}x\|_2}t_j$. This constraint can be written as $\sum_{j=1}^J w_j t_j\lambda^{\mathscr{M}}(y^j)x_B{}^\top\Sigma_n y_B^j\ge\tau\|P_n^{1/2}x\|_2=\|P_n^{1/2}\hat{x}\|_2$ where $\hat{x}=\tau x$. For $x\in\mathcal{X}$, the nonlinear constraint $\hat{x}=\tau x$ can be expressed by the linear constraints $(\hat{x},\tau)\in\mathcal{X}^+$ and $\tau\cdot\mathbf{1}_m-M(\mathbf{1}_m-x)\le\hat{x}\le Mx$ and $k\le M$ for a large constant M. □

EC.4.7. Proof of Proposition 7

Proof. For any feasible solution of problem $(\mathcal{P}_\alpha)$, the constraints $\tau\cdot\mathbf{1}_m-M(\mathbf{1}_m-x)\le\hat{x}\le Mx$, $\tau\le M$, and $(\hat{x},\tau)\in\mathcal{X}^+$ collectively imply that $\hat{x}=\tau x$. Constraints $0\le u_{ii'}\le Mx_i$, $0\le u_{ii'}\le M\hat{x}_{i'}$, and $\hat{x}_{i'}-M(1-x_i)\le u_{ii'}\le\hat{x}_{i'}$ for all $i,i'=1,\ldots,m$, yield $u_{ii'}=x_i\hat{x}_{i'}$. It thus follows from the constraint $vec(P_n)^\top vec(u)=1$ that $x^\top P_n\hat{x}=1$ or equivalently $x^\top P_n x=\frac{1}{\tau}$ and $\|P_n^{1/2}x\|_2=\frac{1}{\sqrt{\tau}}$. The constraints $0\le(z_B^j)_{i,i'}\le(y_B^j)_i$, $0\le(z_B^j)_{i,i'}\le(x_B)_{i'}$, and $(z_B^j)_{i,i'}\ge(y_B^j)_i+(x_B)_{i'}-1$, for all $i,i'=1,\ldots,r$, yield $z_B^j=y_B^j x_B^\top$. It thus follows from $\sqrt{\tau}\|P_n^{1/2}x\|_2=1$ that $\mathrm{vec}(\Sigma_n)^\top\mathrm{vec}(z_B^j)=x_B^\top\Sigma_n y_B^j=\frac{x_B^\top\Sigma_n y_B^j}{\sqrt{\tau}\|P_n^{1/2}x\|_2}$. Therefore,

$$\alpha\sum_{j=1}^J\lambda w_j t_j\mathrm{vec}(\Sigma_n)^\top\mathrm{vec}(z_B^j)=\alpha\sum_{j=1}^J\lambda w_j t_j\frac{x_B^\top\Sigma_n y_B^j}{\sqrt{\tau}\|P_n^{1/2}x\|_2}=\frac{\alpha}{\sqrt{\tau}}\sum_{j=1}^J\lambda w_j t_j\frac{x_B^\top\Sigma_n y_B^j}{\|P_n^{1/2}x\|_2}.$$

Hence, at any feasible solution of problem $(\mathcal{P}_\alpha)$, the objective function of this problem equals

$$\begin{aligned}&\sum_{j=1}^J\lambda w_j\left(\mu_Z^\top y_Z^j+\theta_n^\top y_B^j\right)+\alpha\sum_{j=1}^J\lambda w_j t_j\mathrm{vec}(\Sigma_n)^\top\mathrm{vec}\left(z_B^j\right)-\nu\\ &=\sum_{j=1}^J\lambda w_j\left(\mu_Z^\top y_Z^j+\theta_n^\top y_B^j\right)+\frac{\alpha}{\sqrt{\tau}}\sum_{j=1}^J\lambda w_j t_j\frac{x_B^\top\Sigma_n y_B^j}{\|P_n^{1/2}x\|_2}-\nu.\end{aligned}$$

This equality implies that the objective function of problem (24) (reproduced as equation (EC.36) and under the assumption $\lambda^{\mathscr{M}}(x)=\lambda$),

$$\max_{x,y^1,\cdots,y^J\in\mathcal{X}}\left\{\sum_{j=1}^J w_j\lambda^{\mathscr{M}}(y^j)\left(\mu_Z^\top y_Z^j+\theta_n^\top y_B^j\right)+\sum_{j=1}^J w_j\lambda^{\mathscr{M}}(y^j)\frac{x_B{}^\top\Sigma_n y_B^j}{\|P_n^{1/2}x\|_2}t_j-\nu\right\}, \quad \text{(EC.38)}$$

and the objective function of problem $(\mathcal{P}_\alpha)$ when $\tau = \alpha^2$ coincide. This completes the proof of the first part.

The set of constraints of Problem $(\mathcal{P}_\alpha)$ imply that it is equivalent to the following problem:

$$\max_{x,y^1,\cdots,y^J\in\mathcal{X},\tau} \sum_{j=1}^{J}\lambda w_j\left(\mu_Z^\top y_Z^j + \theta_n^\top y_B^j\right) + \frac{\alpha}{\sqrt{\tau}}\sum_{j=1}^{J}\lambda w_j t_j \frac{x_B^\top \Sigma_n y_B^j}{\|P_n^{1/2}x\|_2} - \nu \text{ s.t. } \|P_n^{1/2}x\|_2 = \frac{1}{\sqrt{\tau}}. \quad \text{(EC.39)}$$

Let α be such that $\tau_x = \alpha^2$, where $(x, y^1, \cdots, y^J, \tau_x)$ is the solution of Problem $(\mathcal{P}_\alpha)$. In particular, we have $\tau_x = \frac{1}{\|P_n^{1/2}x\|_2^2}$. For such α's, the point $(x, y^1, \cdots, y^J)$ is an optimal solution of the following parametric optimization problem implying the same optimal objective value:

$$\Lambda(\alpha) \stackrel{\text{def}}{=} \max_{x,y^1,\cdots,y^J\in\mathcal{X}} \sum_{j=1}^{J}\lambda w_j\left(\mu_Z^\top y_Z^j + \theta_n^\top y_B^j\right) + \sum_{j=1}^{J}\lambda w_j t_j \frac{x_B^\top \Sigma_n y_B^j}{\|P_n^{1/2}x\|_2} - \nu \text{ s.t. } \|P_n^{1/2}x\|_2 = \frac{1}{\alpha}. \quad \text{(EC.40)}$$

Problem (EC.38) can be expressed as $\max_{\alpha\geq 0} \Lambda(\alpha)$. Let $(x^*, (y^1)^*, \cdots, (y^J)^*)$ constitute an optimal solution of problem (EC.38), resulting in the level $\alpha^* := \frac{1}{\|P_n^{1/2}x^*\|_2}$ and the optimal objective value $\Lambda(\alpha^*)$. Therefore, an optimal solution of problem $(\mathcal{P}_\alpha)$ is an exact optimal solution of problem (EC.38) ($x_n^{\text{KG}} = x$), when $\tau^* = \alpha^2$ and $\alpha \in \arg\max_{\alpha\geq 0}\Lambda(\alpha)$, i.e., problems (EC.38) and $(\mathcal{P}_\alpha)$ have identical optimal objective values. □

The iterative process, outlined in Figure 2, aims to find the solution of the equation $\frac{1}{\|P_n^{1/2}x_\alpha\|_2} = \alpha$, where x_α is an optimal solution of problem $(\mathcal{P}_\alpha)$. Our computational experiments in Section 9 indicate that this iterative procedure converges. In addition, Section 9 reports numerical assessments of a computed solution from this method.

EC.4.8. Proof of Proposition 8

Proof. Since (μ_B, ρ) follows a multivariate normal-gamma distribution with parameters $S_n = (\theta_n, \Sigma_n, a_n, b_n)$, the joint density is given by

$$\begin{aligned}\Pr(\mu_B, \rho|S_n) &= \Pr(\mu_B|\rho, \theta_n, \Sigma_n)\ \Pr(\rho|a_n, b_n)\\ &= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_n|^{-\frac{1}{2}} e^{-\frac{\rho}{2}(\mu_B-\theta_n)^\top\Sigma_n^{-1}(\mu_B-\theta_n)} \times \frac{b_n^{a_n}}{\Gamma(a_n)}\rho^{a_n-1}e^{-b_n\rho},\end{aligned} \quad \text{(EC.41)}$$

where Γ is the gamma function.

Let η_x and κ_x be the observations corresponding to the measurement x. It follows from

$$\zeta_k \sim \mathcal{N}\left(\mu_Z, \frac{1}{\rho}\Sigma_Z\right), \qquad \beta_k \sim \mathcal{N}\left(\mu_B, \frac{1}{\rho}\Sigma_B\right), \qquad \text{for } k = 1, \cdots, \kappa_x,$$

and the assumption that ζ_k, β_k, and ϵ_k are independent, that for every measurement $x \in \mathcal{X}$,

$$\left(\zeta_k^\top x_Z + \beta_k^\top x_B + \epsilon_k\right)\ |\ \mu_B, \rho\ \sim \mathcal{N}\left(\mu_Z^\top x_Z + \mu_B^\top x_B\ ,\ \frac{1}{\rho}\left(1 + x_Z^\top\Sigma_Z x_Z + x_B^\top \Sigma_B x_B\right)\right).$$

Hence, $\{\eta_x \mid \mu_B, \rho, x, \kappa_x\} \sim \mathcal{N}\left(\kappa_x\left(\mu_Z^\top x_Z + \mu_B^\top x_B\right), \frac{\kappa_x}{\hat{\rho}}\right)$, where $\hat{\rho} \stackrel{\text{def}}{=} \frac{1}{1+x_Z^\top \Sigma_Z x_Z + x_B{}^\top \Sigma_B x_B}\rho$. Thus,

$$\Pr(\eta_x|\mu_B, \rho, x, \kappa_x) = \frac{\sqrt{\hat{\rho}}}{\sqrt{2\pi\kappa_x}} \exp\left(-\frac{\hat{\rho}}{2\kappa_x}\left(\eta_x - \kappa_x(\mu_Z^\top x_Z + \mu_B^\top x_B)\right)^2\right) \tag{EC.42}$$

Bayes' rule implies that

$$\begin{aligned}\Pr(\mu_B, \rho|S_n, x, \kappa_x, \eta_x) &\propto \Pr(\mu_B, \rho|S_n, x, \kappa_x) \times \Pr(\eta_x|\mu_B, \rho, S_n, x_n, \kappa_x)\\ &= \Pr(\mu_B, \rho|S_n) \times \Pr(\eta_x|\mu_B, \rho, x, \kappa_x),\end{aligned}$$

where the equality comes from the fact that the realized frequency κ_x does not impact the distribution of (μ_B, ρ), i.e., $\Pr(\mu_B, \rho|S_n, x, \kappa_x) = \Pr(\mu_B, \rho|S_n)$. In addition, given μ_B and ρ, the description of η_x implies that $\Pr(\eta_x|\mu_B, \rho, S_n, x, \kappa_x) = \Pr(\eta_x|\mu_B, \rho, x, \kappa_x)$.

Using equations (EC.41) and (EC.42),

$$\begin{aligned}\Pr(\mu_B, \rho|S_n) \times \Pr(\eta_x|\mu_B, \rho, x, \kappa_x) = &\left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_n|^{-1/2} e^{-\frac{\rho}{2}(\mu_B-\theta_n)^\top \Sigma_n^{-1}(\mu_B-\theta_n)} \times \frac{b_n^{a_n}}{\Gamma(a_n)}\rho^{a_n-1}e^{-b_n\rho}\\ &\times \frac{\sqrt{\hat{\rho}}}{\sqrt{2\pi\kappa_x}} e^{-\frac{\hat{\rho}\kappa_x}{2}\left(\frac{\eta_x}{\kappa_x} - (\mu_Z^\top x_{Z,n} + \mu_B^\top x_{B,n})\right)^2}\end{aligned} \tag{EC.43}$$

It follows from equation (31) and the matrix inversion lemma that, $\Sigma_{n+1}^{-1} = \Sigma_n^{-1} + \alpha x_B x_B{}^\top$, where $\alpha \stackrel{\text{def}}{=} \kappa_x\left(1 + x_Z{}^\top \Sigma_Z x_Z + x_B{}^\top \Sigma_B x_B\right)^{-1} = \kappa_x \frac{\hat{\rho}}{\rho}$. Therefore,

$$|\Sigma_{n+1}|^{-1} = |\Sigma_{n+1}^{-1}| = |\Sigma_n^{-1} + \alpha x_B x_B{}^\top| = |\Sigma_n^{-1}||1 + \alpha x_B{}^\top \Sigma_n x_B| = |\Sigma_n|^{-1}(1 + \alpha x_B{}^\top \Sigma_n x_B),$$

where the last equality comes from the positive semidefiniteness of Σ_Z, Σ_B, and Σ_n. Applying this along with equations (30) and (33), we can show that

$$\begin{aligned}&-\frac{\rho}{2}(\mu_B - \theta_n)^\top \Sigma_n^{-1}(\mu_B - \theta_n) - b_n\rho - \frac{\hat{\rho}\kappa_x}{2}\left(\frac{\eta_x}{\kappa_x} - (\mu_Z^\top x_Z + \mu_B^\top x_B)\right)^2\\ &= -\frac{\rho}{2}(\mu_B - \theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B - \theta_{n+1}) - b_{n+1}\rho.\end{aligned} \tag{EC.44}$$

Using equation (EC.44), the right-hand side of equation (EC.43) will be equal to

$$\left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_n|^{-\frac{1}{2}} e^{-\frac{\rho}{2}(\mu_B-\theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B-\theta_{n+1})} \times \frac{b_n^{a_n}\sqrt{\hat{\rho}}}{\sqrt{2\pi\kappa_x}\Gamma(a_n)}\rho^{a_n-1}e^{-b_{n+1}\rho}$$

$$= \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_n|^{-\frac{1}{2}} e^{-\frac{\rho}{2}(\mu_B-\theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B-\theta_{n+1})} \times \frac{b_n^{a_n}\sqrt{\hat{\rho}}}{\sqrt{2\rho\kappa_x}B(a_n, \frac{1}{2})\Gamma(a_{n+1})}\rho^{a_{n+1}-1}e^{-b_{n+1}\rho} \tag{EC.45}$$

$$= c \times \left(\frac{\rho}{2\pi}\right)^{\frac{r}{2}} |\Sigma_{n+1}|^{-\frac{1}{2}} e^{-\frac{\rho}{2}(\mu_B-\theta_{n+1})^\top \Sigma_{n+1}^{-1}(\mu_B-\theta_{n+1})} \times \frac{b_{n+1}^{a_{n+1}}}{\Gamma(a_{n+1})}\rho^{a_{n+1}-1}e^{-b_{n+1}\rho}, \tag{EC.46}$$

where equality (EC.45) comes from equation (32) and the description of the beta function $B(a_n, \frac{1}{2}) = \frac{\Gamma(a_n)\Gamma(\frac{1}{2})}{\Gamma(a_n+\frac{1}{2})} = \frac{\sqrt{\pi}\Gamma(a_n)}{\Gamma(a_{n+1})}$. In equation (EC.46), we have

$$c \stackrel{\text{def}}{=} \sqrt{\frac{\kappa_x}{1 + x_Z{}^\top \Sigma_Z x_Z + x_B{}^\top \Sigma_B x_B + \kappa_x x_B{}^\top \Sigma_n x_B}} \times \frac{b_n^{a_n}}{\kappa_x\sqrt{2}B(a_n, \frac{1}{2})b_{n+1}^{a_{n+1}}}$$

which is constant with respect to ρ and μ_B. Thus, $\Pr(\mu_B, \rho|S_n, x, \kappa_x, \eta_x)$ is proportional to the normal-gamma density with parameters $(\theta_{n+1}, \Sigma_{n+1}, a_{n+1}, b_{n+1})$ calculated according to the desired updating equations. □

EC.4.9. Proof of Proposition 9

Proof. It follows from the model for η_x in (29) that

$$\eta_x \mid S_n, x, \kappa_x, \mu_B, \rho \sim \mathcal{N}\left(\kappa_x(\mu_Z^\top x_Z + \mu_B^\top x_B), \frac{\kappa_x}{\rho}(x_Z{}^\top \Sigma_Z x_Z + x_B{}^\top \Sigma_B x_B + 1)\right),$$

where $S_n \stackrel{\text{def}}{=} (\theta_n, \Sigma_n, a_n, b_n)$. Since $\mu_B|\rho$ follows $\mathcal{N}(\theta_n, \frac{1}{\rho}\Sigma_n)$, we have

$$J \stackrel{\text{def}}{=} \left(\eta_x - \sum_{k=1}^{\kappa_x}\left(\mu_Z^\top x_Z + \mu_B^\top x_B\right) \mid S_n, \rho, x, \mu_B, \kappa_x\right) \sim \mathcal{N}\left(0, \frac{\kappa_x}{\rho}(x_Z{}^\top \Sigma_Z x_Z + x_B{}^\top \Sigma_B x_B + 1)\right).$$

Hence, the random variable J is independent of μ_B, and thus independent of $\sum_{k=1}^{\kappa_x}(\mu_Z^\top x_Z + \mu^\top x_B)$. Therefore,

$$\begin{aligned}
\eta_x \mid S, \rho, x, \kappa_x &= J + \kappa_x(\mu_Z^\top x_Z + \mu^\top x_B) \\
&= \mathcal{N}\left(0, \frac{\kappa_x}{\rho}\left(x_Z{}^\top \Sigma_Z x_Z + x_B{}^\top \Sigma_B x_B + 1\right)\right) + \mathcal{N}\left(\kappa_x\left(\mu_Z^\top x_Z + \theta_n^\top x_B\right), \frac{\kappa_x^2}{\rho} x_B{}^\top \Sigma_n x_B\right) \\
&= \mathcal{N}\left(\kappa_x(\mu_Z^\top x_Z + \theta_n^\top x_B), \frac{\kappa_x}{\rho}(x_Z{}^\top \Sigma_Z x_Z + x_B{}^\top \Sigma_B x_B + 1 + \kappa_x x_B{}^\top \Sigma_n x_B)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\Pr(\eta_x = \eta \mid S_n, x, \kappa_x) &= \int_0^\infty \Pr(\eta_x = \eta \mid \rho = r, S_n, x, \kappa_x) \Pr(\rho = r) dr \\
&= \int_0^\infty \frac{\sqrt{r}}{\sqrt{2\pi k_x \hat{\sigma}}} e^{-\frac{r\left(\eta - k_x(\mu_Z^\top x_Z + \theta_n^\top x_B)\right)^2}{2k_x\hat{\sigma}}} \times \frac{b_n^{a_n}}{\Gamma(a_n)} r^{a_n - 1} e^{-b_n r} dr.
\end{aligned}$$

where $\hat{\sigma} \stackrel{\text{def}}{=} 1 + x_Z{}^\top \Sigma_Z x_Z + x_B{}^\top(\Sigma_B + \kappa_x \Sigma_n) x_B$. Define, $\hat{\eta}_x \stackrel{\text{def}}{=} \sqrt{\frac{a_n}{b_n k_x \hat{\sigma}}}(\eta_x - k_x(\mu_Z^\top x_Z + \theta_n^\top x_B))$. Thus, $\Pr(\hat{\eta}_x = \hat{\eta} \mid S_n, x, \kappa_x) = \Pr(\eta_x = g(\hat{\eta}) | S_n, x, \kappa_x) \times g'(\hat{\eta})$, where $g(\hat{\eta}) \stackrel{\text{def}}{=} k_x(\mu_Z^\top x_Z + \theta_n^\top x_B) + \sqrt{\frac{b_n \kappa_x \hat{\sigma}}{a_n}}\hat{\eta}$. Hence,

$$\begin{aligned}
\Pr(\hat{\eta}_x = \hat{\eta} \mid S_n, x, \kappa_x = k > 0) &= \int_0^\infty \frac{\sqrt{r}}{\sqrt{2\pi k\hat{\sigma}}} e^{-\frac{r\left(\sqrt{\frac{b_n k\hat{\sigma}}{a_n}}\hat{\eta}\right)^2}{2k\hat{\sigma}}} \times \frac{b_n^{a_n}}{\Gamma(a_n)} r^{a_n - 1} e^{-b_n r} dr \times \sqrt{\frac{b_n k\hat{\sigma}}{a_n}} \\
&= \frac{1}{\Gamma(a_n)\sqrt{2a_n\pi}} \int_0^\infty e^{-rb_n\left(1 + \frac{\hat{\eta}^2}{2a_n}\right)} b_n^{a_n + \frac{1}{2}} r^{a_n - \frac{1}{2}} dr \\
&= \frac{1}{\Gamma(a_n)\sqrt{2a_n\pi}} \left(1 + \frac{\hat{\eta}^2}{2a_n}\right)^{-a_n - \frac{1}{2}} \Gamma\left(a_n + \frac{1}{2}\right).
\end{aligned}$$

The last line is the pdf of the standard Student's t-distribution with $2a_n$ degrees of freedom. Hence, $\hat{\eta}_x \mid S_n, x, \kappa_x = k > 0$ follows a standard Student's t-distribution with $2a_n$ degrees of freedom. □

EC.4.10. Proof of Proposition 10

Proof. It follows from $V_N^\pi(S_n) = U(\theta_n)$ that

$$\begin{aligned}
v_n^{\text{KG}}(x) &= \mathbb{E}\left[V_N\left(S^{\mathscr{M}}(S_n, x_n, (\eta_{n+1}, \kappa_{n+1}))\right) - V_N(S_n) \mid S_n = s, x_n = x\right] \\
&= \mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\eta_x, \kappa_x))\right) \mid S_n = s\right] - U(\theta_n).
\end{aligned} \tag{EC.47}$$

where $\theta^{\mathscr{M}}(S_n, x, (\eta_x, \kappa_x)) \stackrel{\text{def}}{=} \theta_{S^{\mathscr{M}}(S_n,x,(\eta_x,\kappa_x))}$. From Proposition 9, for all $\kappa_x \geq 0$, θ_{n+1} in equation (30) can be expressed as

$$\theta_{n+1} = \theta_n + \frac{\sqrt{\kappa_x}}{\sqrt{\frac{a_n}{b_n}\left(1 + (x_Z)^\top \Sigma_Z x_Z + x_B{}^\top(\Sigma_B + \kappa_x \Sigma_n) x_B\right)}} \Sigma_n x_B T_{2a_n}. \tag{EC.48}$$

The term θ_{n+1} can be expressed as $\theta^{\mathscr{M}}(S_n, x, (\eta_x, \kappa_x)) = \theta_n + \sqrt{\frac{\kappa_x}{\frac{a_n}{b_n}\hat{\sigma}}} \Sigma_n x_B T_{2a_n}$, where $\hat{\sigma} := 1 + (x_Z)^\top \Sigma_Z x_Z + x_B{}^\top(\Sigma_B + \kappa_x \Sigma_n) x_B$. Therefore

$$\mathbb{E}_{\kappa_x,\eta_x}\left[U\left(\theta^{\mathscr{M}}(S_n, x, (\eta_x, \kappa_x))\right) \mid S_n = s\right] = \mathbb{E}_{\kappa_x, T_{2a_n}}\left[U\left(\theta_n + \sqrt{\frac{\kappa_x b_n}{a_n \hat{\sigma}}} \Sigma_n x_B T_{2a_n}\right) \mid S_n = s\right],$$

which then equals

$$\begin{aligned}
&\mathbb{E}_{\kappa_x, T_{2a_n}}\left[\max_{y\in\mathcal{X}} \lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z + \left(\theta_n + \sqrt{\frac{\kappa_x b_n}{a_n\hat{\sigma}}}\Sigma_n x_B T_{2a_n}\right)^\top y_B\right) \mid S_n = s\right] \\
&= \mathbb{E}_{\kappa_x, T_{2a_n}}\left[\max_{y\in\mathcal{X}} \left(p_y^n + q_y^n(x,\kappa_x) T_{2a_n}\right) \mid S_n = s\right].
\end{aligned} \tag{EC.49}$$

Here, p_y^n and $q_y^n(x,\kappa_x)$ are defined as in equations (18) and

$$q_y^n(x,\kappa_x) = \lambda^{\mathscr{M}}(y)\sqrt{\frac{\kappa_x}{\frac{a_n}{b_n}\left(1 + x_Z^\top \Sigma_Z x_Z + x_B{}^\top\left(\kappa_x\Sigma_n + \Sigma_B\right) x_B\right)}}\, x_B{}^\top \Sigma_n y_B. \tag{EC.50}$$

Applying the equation (EC.49) to (EC.47) completes the proof. □

EC.4.11. Proof of Proposition 11

Proof. It follows from $V_N^\pi(S_n) := U(\mathbb{E}[\mu_B | S_N = S_n]) = U(\theta_n)$ that

$$\begin{aligned}
v_n^{\mathrm{KG}}(\{x\}_{r=1}^R) &= \mathbb{E}\left[V_N\left(S^{\mathscr{M}}\left(S_n, \{x^r\}_{r=1}^R, \{(\kappa_{n+1}, \eta_{n+1})\}_{r=1}^R\right)\right) - V_N(S_n) \mid S_n, \{x^r\}_{r=1}^R\right] \\
&= \mathbb{E}\left[U\left(\theta^{\mathscr{M}}\left(S_n, \{x^r\}_{r=1}^R, \{(\kappa_{n+1}^r, \eta_{n+1}^r)\}_{r=1}^R\right)\right) \mid S_n, \{x^r\}_{r=1}^R\right] - U(\theta_n),
\end{aligned} \tag{EC.51}$$

where $\theta^{\mathscr{M}}\left(S_n, \{x^r\}_{r=1}^R, \{(\eta_{n+1}^r, \kappa_{n+1}^r)\}_{r=1}^R\right) = \mathbb{E}\left[\mu_B \mid S^{\mathscr{M}}\left(S_n, \{x^r\}_{r=1}^R, \{(\eta_{n+1}^r, \kappa_{n+1}^r)\}_{r=1}^R\right)\right]$. Suppose $R = 2$. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[U\left(\theta^{\mathscr{M}}\left(S_n, \{x^r\}_{r=1}^R, \{(\kappa_{n+1}^r, \eta_{n+1}^r)\}_{r=1}^R\right)\right) \mid S_n, \{x^r\}_{r=1}^R\right] \\
&= \Pr(\kappa_{n+1}^1 = 0, \kappa_{n+1}^2 = 0 | x) \times \mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, \{x^r\}_{r=1}^R, \{(\kappa_{n+1}^r, \eta_{n+1}^r)\}_{r=1}^R)\right) \mid S_n, \{x^r\}_{r=1}^R, \kappa_{n+1}^1 = 0, \kappa_{n+1}^2 = 0\right] \\
&\quad + \Pr(\kappa_{n+1}^1 > 0, \kappa_{n+1}^2 = 0 | x) \times \mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, \{x^r\}_{r=1}^R, \{(\kappa_{n+1}^r, \eta_{n+1}^r)\}_{r=1}^R)\right) \mid S_n, \{x^r\}_{r=1}^R, \kappa_{n+1}^1 > 0, \kappa_{n+1}^2 = 0\right] \\
&\quad + \Pr(\kappa_{n+1}^1 = 0, \kappa_{n+1}^2 > 0 | x) \times \mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, \{x^r\}_{r=1}^R, \{(\kappa_{n+1}^r, \eta_{n+1}^r)\}_{r=1}^R)\right) \mid S_n, \{x^r\}_{r=1}^R, \kappa_{n+1}^1 = 0, \kappa_{n+1}^2 > 0\right] \\
&\quad + \Pr(\kappa_{n+1}^1 > 0, \kappa_{n+1}^2 > 0 | x) \times \mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n, \{x^r\}_{r=1}^R, \{(\kappa_{n+1}^r, \eta_{n+1}^r)\}_{r=1}^R)\right) \mid S_n, \{x^r\}_{r=1}^R, \kappa_{n+1}^1 > 0, \kappa_{n+1}^2 > 0\right].
\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left[U\left(\theta^{\mathscr{M}}\left(S_n,\{x^r\}_{r=1}^R,\{(\kappa_{n+1}^r,\eta_{n+1}^r)\}_{r=1}^R\right)\right) \mid S_n,\{x^r\}_{r=1}^R\right] = e^{-\sum_{r=1}^R \lambda^{\mathscr{M}}(x^r)}U(\theta_n)$$
$$+e^{-\lambda^{\mathscr{M}}(x^2)}\left(1-e^{-\lambda^{\mathscr{M}}(x^1)}\right)\times\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n,x^1,(\kappa_{n+1}^1,\eta_{n+1}^1))\right) \mid S_n,\{x^r\}_{r=1}^R,\kappa_{n+1}^1>0\right]$$
$$+e^{-\lambda^{\mathscr{M}}(x^1)}\left(1-e^{-\lambda^{\mathscr{M}}(x^2)}\right)\times\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n,x^2,(\kappa_{n+1}^2,\eta_{n+1}^2))\right) \mid S_n,\{x^r\}_{r=1}^R,\kappa_{n+1}^2>0\right]$$
$$+\prod_{r=1}^R\left(1-e^{-\lambda^{\mathscr{M}}(x^r)}\right)\times\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n,\{x^r\}_{r=1}^R,\{(\kappa_{n+1}^r,\eta_{n+1}^r)\}_{r=1}^R)\right) \mid S_n,\{x^r\}_{r=1}^R,\kappa_{n+1}^1>0,\kappa_{n+1}^2>0\right].$$

From equation (EC.34), we have

$$\mathbb{E}\left[U\left(\theta^{\mathscr{M}}\left(S_n,x^1,(\kappa_{n+1}^1,\eta_{n+1}^1)\right)\right) \mid S_n,x^1,\kappa_{n+1}^1>0\right]=\mathbb{E}\left[\max_{y\in\mathcal{X}}\left(p_y^n+q_y^n(x^1)T_{2a_n}\right) \mid S_n,\kappa_{n+1}^1>0\right], \quad \text{(EC.52)}$$
$$\mathbb{E}\left[U\left(\theta^{\mathscr{M}}\left(S_n,x^2,(\kappa_{n+1}^2,\eta_{n+1}^2)\right)\right) \mid S_n,x^2,\kappa_{n+1}^2>0\right]=\mathbb{E}\left[\max_{y\in\mathcal{X}}\left(p_y^n+q_y^n(x^2)T_{2a_n}\right) \mid S_n,\kappa_{n+1}^2>0\right]. \quad \text{(EC.53)}$$

where p_y^n and $q_y^n(x)$ are as in the equations (18) and (19), i.e.,

$$p_y^n \stackrel{\text{def}}{=} \lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z+\theta_n^\top y_B\right),$$
$$q_y^n(x) \stackrel{\text{def}}{=} \lambda^{\mathscr{M}}(y)\sqrt{\frac{b_n}{a_n\left(1+x_Z^\top\Sigma_Z x_Z+x_B^\top(\Sigma_B+\Sigma_n)x_B\right)}}{x_B}^\top\Sigma_n y_B.$$

Given $\kappa_{n+1}^1,\kappa_{n+1}^2>0$, using equation (13) twice, it holds that $\theta^{\mathscr{M}}\left(S_n,\{x^r\}_{r=1}^R,\{(\eta_{n+1}^r,\kappa_{n+1}^r)\}_{r=1}^R\right)$ equals

$$\theta_{n+1}=\theta_n+\sqrt{\frac{b_n}{a_n\left(\hat\sigma(x^1)+(x_B^1)^\top\Sigma_n(x_B^1)\right)}}\Sigma_n(x_B^1)T_{2a_n}+\sqrt{\frac{b_n'}{a_n'\left(\hat\sigma(x^2)+(x_B^2)^\top\Sigma_n'(x_B^2)\right)}}\Sigma_n'(x_B^2)T_{2a_n'}. \quad \text{(EC.54)}$$

Recall that $\hat\sigma(x) \stackrel{\text{def}}{=} 1+x_Z^\top\Sigma_Z x_Z+x_B^\top\Sigma_B x_B$. Here, Σ_n',a_n',b_n' are updated state variables using equations (9), (10), (11) and observations $(\kappa_{n+1}^1,\eta_{n+1}^1)$ corresponding to x^1. Hence,

$$\mathbb{E}\left[U\left(\theta^{\mathscr{M}}(S_n,\{x^r\}_{r=1}^R,\{(\kappa_{n+1}^r,\eta_{n+1}^r)\}_{r=1}^R)\right) \mid S_n,\{x^r\}_{r=1}^R,\kappa_{n+1}^1>0,\kappa_{n+1}^2>0\right]$$
$$=\mathbb{E}\left[U\left(\theta_{n+1}\right) \mid S_n,x,\kappa_{n+1}>0\right]$$
$$=\mathbb{E}\left[\max_{y\in\mathcal{X}}\lambda^{\mathscr{M}}(y)\left(\mu_Z^\top y_Z+\theta_{n+1}^\top y_B\right) \mid S_n,x,\kappa_{n+1}>0\right]$$
$$=\mathbb{E}\left[\max_{y\in\mathcal{X}}\left(p_y^n+q_y^n(x^1)T_{2a_n}+q_y^n(x^1,x^2)T_{2a_n+1}\right) \mid S_n,x,\kappa_{n+1}>0\right], \quad \text{(EC.55)}$$

where θ_{n+1} is as in equation (EC.54), and p_y^n and $q_y^n(x_n)$ are given in equations (18) and (19). In the last term, $q_y^n(x^1,x^2)$ is defined as

$$q_y^n(x^1,x^2) \stackrel{\text{def}}{=} \lambda^{\mathscr{M}}(y)\sqrt{\frac{b_n'}{a_n'\left(1+(x_Z^2)^\top\Sigma_Z(x_Z^2)+(x_B^2)^\top(\Sigma_B+\Sigma_n')(x_B^2)\right)}}(x_B^2)^\top\Sigma_n' y_B.$$

By applying equations (EC.52), (EC.53), and (EC.55) in (EC.51), we get

$$
\begin{aligned}
v_n^{\mathrm{KG}}(x^1,x^2) &= \mathbb{E}\left[V_N\left(S^{\mathscr{M}}(S_n,\{x^r\}_{r=1}^R,\{(\kappa_{n+1},\eta_{n+1})\}_{r=1}^R\right)-V_N(S_n)\mid S_n,\{x^r\}_{r=1}^R\right]\\
&= \mathbb{E}\left[U\left(\theta^{\mathscr{M}}\left(S_n,\{x^r\}_{r=1}^R,\{(\kappa_{n+1}^r,\eta_{n+1}^r)\}_{r=1}^R\right)\right)\mid S_n,\{x^r\}_{r=1}^R\right]-U(\theta_n)\\
&= -\left(1-e^{-\sum_{r=1}^R \lambda^{\mathscr{M}}(x^r)}\right)\max_{y\in\mathcal{X}} p_y^n\\
&\quad + e^{-\lambda^{\mathscr{M}}(x^2)}\left(1-e^{-\lambda^{\mathscr{M}}(x^1)}\right)\times\mathbb{E}\left[\max_{y\in\mathcal{X}}\left(p_y^n+q_y^n(x^1)T_{2a_n}\right)\mid S_n,\kappa_{n+1}^1>0\right]\\
&\quad + e^{-\lambda^{\mathscr{M}}(x^1)}\left(1-e^{-\lambda^{\mathscr{M}}(x^2)}\right)\times\mathbb{E}\left[\max_{y\in\mathcal{X}}\left(p_y^n+q_y^n(x^2)T_{2a_n}\right)\mid S_n,\kappa_{n+1}^2>0\right]\\
&\quad + \prod_{r=1}^2\left(1-e^{-\lambda^{\mathscr{M}}(x^r)}\right)\mathbb{E}\left[\max_{y\in\mathcal{X}}\left(p_y^n+q_y^n(x^1)T_{2a_n}+q_y^n(x^1,x^2)T_{2a_n+1}\right)\mid S_n,\kappa_{n+1}^1,\kappa_{n+1}^2>0\right].
\end{aligned}
$$

Note that T_{2a_n} and T_{2a_n+1} are two independent random variables. □

EC.5. Benchmark Policies

This section provides an overview of some of the methods involving the idea of adaptive sequential learning to improve our belief about the output. These policies serve as benchmark in Section 9.

EC.5.1. Optimal Design Policies

Optimal design policies are derived using principles from the *optimal design of experiments*, see e.g. Boyd and Vandenberghe (2004), Bechhofer et al. (1995). The *A-Optimal Design* policy minimizes the trace of the covariance matrix, that is $\min_{x\in\mathcal{X}} \operatorname{tr}(\Sigma_{n+1})$, where $\operatorname{tr}(\Sigma_{n+1})$ is computed as in equation (EC.57). The *D-Optimal Design* policy minimizes the log-determinant of the covariance matrix, that is $\min_{x\in\mathcal{X}} \log\det(\Sigma_{n+1})$. These problems can be cast as mixed integer second-order cone optimization problems. Below, Proposition EC.2 explains this model for A-Optimal Design. Next, Proposition EC.3 presents the second-order cone optimization to compute the decision of the *D-Optimal Design* policy. We use these policies in a sequential framework, in which each step aims to find an alternative x_n minimizing the scalarization metric of the updated covariance matrix Σ_{n+1}, given Σ_n.

PROPOSITION EC.2. *Let* $Q_n \stackrel{\text{def}}{=} \frac{A^\top A}{h^\top h} + \begin{pmatrix} \Sigma_Z & 0 \\ 0 & \Sigma_n + \Sigma_B \end{pmatrix}$. *The A-optimal design policy* $\min\limits_{x_n\in\mathcal{X}} \operatorname{tr}(\Sigma_{n+1})$ *can be computed by solving the mixed-integer second-order cone problem*

$$\begin{aligned} \min_{x_n\in\mathcal{X},\ V\geq 0,\ z} \quad & z \\ s.t \quad & \left\| \begin{matrix} 2Q_n^{1/2}x_n \\ \operatorname{tr}(\Sigma_n^2 V) - z \end{matrix} \right\|_2 \leq \operatorname{tr}(\Sigma_n^2 V) + z, \\ & (x_{B,n})_k + (x_{B,n})_l - 1 \leq V_{kl} \leq \min\{(x_{B,n})_k, (x_{B,n})_l\}, \quad k,l = 1,\ldots,r. \end{aligned} \tag{EC.56}$$

Proof of Proposition EC.2. From equation (9) and linearity of the trace we have

$$\begin{aligned} \operatorname{tr}(\Sigma_{n+1}) &= \operatorname{tr}(\Sigma_n) - \frac{1}{1 + x_{B,n}{}^\top \Sigma_n x_{B,n} + x_{B,n}{}^\top \Sigma_B x_{B,n} + x_{Z,n}^\top \Sigma_Z x_{Z,n}} \operatorname{tr}(\Sigma_n x_{B,n} x_{B,n}{}^\top \Sigma_n) \\ &= \operatorname{tr}(\Sigma_n) - \frac{1}{x_n^\top Q_n x_n} x_{B,n}{}^\top \Sigma_n^2 x_{B,n}. \end{aligned} \tag{EC.57}$$

Since at stage n, Σ_n is a constant, to minimize $\operatorname{tr}(\Sigma_{n+1})$ it is sufficient to maximize $\frac{x_{B,n}{}^\top \Sigma_n^2 x_{B,n}}{x_n^\top Q_n x_n}$. By introducing $V = x_{B,n} x_{B,n}{}^\top$, we have $x_{B,n}{}^\top \Sigma_n^2 x_{B,n} = \operatorname{tr}(\Sigma_n^2 x_{B,n} x_{B,n}^\top) = \operatorname{tr}(\Sigma_n^2 V)$. Hence, we arrive at minimizing z subject to $\frac{x_n^\top Q_n x_n}{\operatorname{tr}(\Sigma_n^2 V)} \leq z$. This contraint is equivalent to the second-order cone constraint $\left\| \begin{matrix} 2Q_n^{1/2}x_n \\ \operatorname{tr}(\Sigma_n^2 V) - z \end{matrix} \right\|_2 \leq \operatorname{tr}(\Sigma_n^2 V) + z$. $\square$

PROPOSITION EC.3. *Let* $Q_n \stackrel{\text{def}}{=} \frac{A^\top A}{h^\top h} + \begin{pmatrix} \Sigma_Z & 0 \\ 0 & \Sigma_n + \Sigma_B \end{pmatrix}$. *The D-optimal design policy* $\min\limits_{x_n\in\mathcal{X}} \log\det(\Sigma_{n+1})$ *can be computed by solving the mixed-integer second-order cone problem*

$$\min_{x_n\in\mathcal{X},\ V\geq 0,\ z} \quad z \tag{EC.58}$$

$$
\begin{aligned}
s.t \qquad & \left\| \begin{matrix} 2Q_n^{1/2}x_n \\ \operatorname{tr}(\Sigma_n V) - z \end{matrix} \right\|_2 \leq \operatorname{tr}(\Sigma_n V) + z, \\
& (x_{B,n})_k + (x_{B,n})_l - 1 \leq V_{kl} \leq \min\{(x_{B,n})_k, (x_{B,n})_l\}, \quad k, l = 1, \ldots, r.
\end{aligned}
$$

Proof. For Σ_{n+1} as in equation (9), and under the assumption that Σ_n is symmetric positive definite, we have

$$
\begin{aligned}
\log\det(\Sigma_{n+1}) &= \log\det\left(\Sigma_n - \frac{\Sigma_n x_{B,n} {x_{B,n}}^\top \Sigma_n}{x_n^\top Q_n x_n}\right) \\
&= \log\det\left(\Sigma_n^{1/2}\left(I - \frac{1}{x_n^\top Q_n x_n}\Sigma_n^{-1/2}\Sigma_n x_{B,n} {x_{B,n}}^\top \Sigma_n \Sigma_n^{-1/2}\right)\Sigma_n^{1/2}\right) \\
&= \log\left(1 - \frac{1}{x_n^\top Q_n x_n} {x_{B,n}}^\top \Sigma_n x_{B,n}\right) + \log\det(\Sigma_n),
\end{aligned}
$$

where the last equality follows from the property $\det(I + cc^\top) = 1 + c^\top c$ for any column vector c, thus in particular for $c = \Sigma_n^{1/2} x_{B,n}$. Since at stage n, Σ_n is a constant and log is an increasing function, to minimize $\log\det(\Sigma_{n+1})$, it is sufficient to maximize $\frac{{x_{B,n}}^\top \Sigma_n x_{B,n}}{x_n^\top Q_n x_n} = \frac{\operatorname{tr}(\Sigma_n V)}{x_n^\top Q_n x_n}$, where $V = x_{B,n} {x_{B,n}}^\top$. This is equivalent to minimizing z such that $\frac{x_n^\top Q_n x_n}{\operatorname{tr}(\Sigma_n V)} \leq z$. This last constraint is equivalent to the second-order cone constraint $\left\| \begin{matrix} 2Q_n^{1/2}x_n \\ \operatorname{tr}(\Sigma_n V) - z \end{matrix} \right\|_2 \leq \operatorname{tr}(\Sigma_n V) + z$. $\square$

EC.5.2. Optimal Computing Budget Allocation (OCBA)

The Optimal Computing Budget Allocation (OCBA) is an approach to allocate a given simulation computing budget in order to find the best design by resampling a fixed number of alternatives, see Chen (1995), Fu et al. (2007), Chen et al. (2008a), Chen and Lee (2011). The OCBA procedure allocates a fixed simulation budget N among a finite number of designs $|\mathcal{X}|$ in order to maximize some measure of the probability of correct selection (P_{CS}), i.e.,

$$
\max_{N_x,\ \forall x \in \mathcal{X}} P_{\text{CS}} \quad \text{subject to} \quad \sum_{x \in \mathcal{X}} N_x = N. \tag{EC.59}
$$

Here, the decision variable N_x denotes the number of simulation replications allocated to design x. Picking the best design, i.e., the design with the maximum true mean, can serve as a measure for the "correct selection", P_{CS}. Hence, $P_{\text{CS}} := \Pr(\bar{\eta}_{x^*} \geq \bar{\eta}_x,\ \forall\, x \in \mathcal{X})$, where x^* is the true best design, and $\bar{\eta}_x$ is the sample average for design x over N_x replications. Various approximations for this objective function, based on the concept of indifference-zone representing the smallest difference in values that is significant, are typically considered. For most measures P_{CS}, solving (EC.59) remains computationally complex and thus approximate or heuristic solutions for the problem are employed.

The standard OCBA algorithm is an iterative two-stage allocation process which involves estimating the means and standard deviations for outcome of each alternative, and the second stage calculates $\{N_x^n\}$ based on these estimates to allocate the remaining computational budget. It starts by either initial beliefs $\eta_x|S_0$, or sample averages and variances obtained by taking an initial sample N_x^0 of each alternative x. In the latter case, $\sum_{x\in\mathcal{X}} N_x^0$ measurements from the learning budget B are consumed. Suppose that $\hat{\eta}_x^n$ denotes the updated outcome mean for alternative x at stage n, and $\hat{\sigma}_x^n$ is the standard deviation of alternative x after stage n. Let $\hat{x} := \arg\max_{x\in\mathcal{X}} \hat{\eta}_x^n$. For a given increment parameter Δ, calculate new allocations $\{N_x^{n+1}\}_{x\in\mathcal{X}}$ by rounding up a solution to the system of equations

$$N_{\hat{x}}^{n+1} = \hat{\sigma}_{\hat{x}}^n \sqrt{\sum_{x\in\mathcal{X}\setminus\{\hat{x}\}} \left(\frac{N_x^{n+1}}{\hat{\sigma}_x^n}\right)^2}, \tag{EC.60}$$

$$\frac{N_x^{n+1}}{N_y^{n+1}} = \left(\frac{\hat{\sigma}_x^n/(\hat{\eta}_{\hat{x}}^n - \hat{\eta}_x^n)}{\hat{\sigma}_y^n/(\hat{\eta}_{\hat{x}}^n - \hat{\eta}_y^n)}\right)^2, \quad \forall\, x,y \in \mathcal{X}\setminus\{\hat{x}\}, \quad x \neq y, \tag{EC.61}$$

$$\sum_{x\in\mathcal{X}} N_x^{n+1} = \Delta. \tag{EC.62}$$

Then, perform additional $\max\{0, N_x^{n+1} - N_x^n\}$ replications for alternative x, for all $x \in \mathcal{X}$, and update the estimates $\hat{\eta}_x^{n+1}$. If the sum of the used simulations is still smaller than N, using updated estimates $\hat{\eta}_x^{n+1}$ and $\hat{\sigma}_x^{n+1}$ and equations (EC.60)-(EC.62), calculate new allocations N_x^{n+2}. Otherwise, return $\arg\max_{x\in\mathcal{X}} \hat{\eta}_x^{n+1}$. Equations (EC.60)-(EC.62) can be solved by expressing each N_x^{n+1} in terms of some fixed alternative (other than $\hat{x}$), using equations (EC.60) and (EC.61). Then, use equation (EC.62) to determine N_x^{n+1} for the fixed alternative and consequently others. We consider two implementations of this algorithm:

- **Sequential OCBA Policy**: This policy starts with the initial belief model, and selects only one alternative to be measured per iteration, i.e., $\Delta = 1$.
- **Batch OCBA Policy**: This policy starts with the initial belief model, and selects a subset of alternatives to be measured all in one iteration, i.e., $\Delta = N$.

In the computational results in Section 9, the revenue or regret reported for the Batch OCBA policy at stage n corresponds to the Batch OCBA for $\Delta = N$ where $N = 4, 5, 6$.

Theoretical results on the optimality of the OCBA selection algorithm are limited to the normally distributed samples. Branke et al. (2007) studies the performance of the OCBA when compared to other ranking and selection algorithms. Extensions of the OCBA policy, to select the best m out of $|\mathcal{X}|$ alternatives, are studied in Chen et al. (2008b), LaPorte et al. (2012).

EC.5.3. Linear Loss (LL) Policy

At phase n, let $\hat{x} = \arg\max_{x\in\mathcal{X}} \mathbb{E}[\eta_x|S_n]$ and $\hat{\sigma}_x^2 = \text{var}[\mathbb{E}[\eta_x|\mu_B]|S_n]$. Denote $\lambda_{x,y} = \sqrt{\frac{1}{\hat{\sigma}_x^2} + \frac{1}{1+\hat{\sigma}_y^2}}$ and $\ell_x = \sqrt{\frac{1}{\hat{\sigma}_x^2} + \frac{1}{\hat{\sigma}_{\hat{x}}^2}}$. Define

$$D_x := \begin{cases} \lambda_{\hat{x},x} f\left(\frac{\mathbb{E}[\eta_{\hat{x}}|S_n] - \mathbb{E}[\eta_x|S_n]}{\lambda_{\hat{x},x}}\right) - \ell_x f\left(\frac{\mathbb{E}[\eta_{\hat{x}}|S_n] - \mathbb{E}[\eta_x|S_n]}{\ell_x}\right) & \text{if } x \neq \hat{x} \\ \sum_{y\neq\hat{x}} \lambda_{y,\hat{x}} f\left(\frac{\mathbb{E}[\eta_{\hat{x}}|S_n] - \mathbb{E}[\eta_y|S_n]}{\lambda_{y,\hat{x}}}\right) - \ell_y f\left(\frac{\mathbb{E}[\eta_{\hat{x}}|S_n] - \mathbb{E}[\eta_y|S_n]}{\ell_y}\right) & \text{if } x = \hat{x}. \end{cases}$$

Here, $f(z) \stackrel{\text{def}}{=} z\Phi(z) + \phi(z)$, where $\Phi(z)$ and $\phi(z)$ are the cumulative standard normal distribution and the standard normal density, respectively. The LL policy thus allocates one sample for the alternative with the lowest value of D_x. For further discussion on this policy see Frazier and Powell (2011) and Ryzhov and Powell (2012). The above mentioned method assumes that η_x is normally distributed with known variance. For a constant exposure factor $\kappa = k$ and given the precision ρ, this policy can be adopted, where $\mathbb{E}[\eta_x|S_n] = k\,(\mu_Z^\top x_Z + \theta_0^\top x_B)$ and $\text{var}[\mathbb{E}[\eta_x|\mu_B]|S_n] = \text{var}[k\,(\mu_Z^\top x_Z + \mu_B^\top x_B)\,|S_n] = \frac{k^2}{\rho} x_B^\top \Sigma_n x_B$.

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EC.6. The Bayesian Multi-Armed Bandit Model

An extension of the problem (14) is to modify the objective function to take into account the outcome of each individual test. This can be expressed as

$$\sup_{\pi\in\Pi}\ \mathbb{E}^{\pi}\left[\sum_{n=0}^{N}\gamma^{n}U(\theta_n)\right], \tag{EC.63}$$

where $U(\cdot)$ is as in equation (4) and $\gamma\in(0,1]$ is a discount factor. Problem (EC.63) lies in the class of *multi-armed bandit* (MAB) problems. For a review on multi-armed bandit models, see Berry and Fristedt (1985), Scott (2010), Gittins et al. (2011), Bubeck and Cesa-Bianchi (2012).

The literature on the analysis of MAB problems and solution strategies has typically focused on settings where the arm payoffs are statistically independent. While the formulation in Chapter 2 of Berry and Fristedt (1985) allows for correlated arm payoffs, the analysis and solution methods for bandit problems with dependent arms is relatively scarce. Pandey et al. (2007) study bandit problems where the dependence of the arm rewards is represented via a hierarchical model. Mersereau et al. (2009) investigate the optimality of the myopic policy for a bandit problem with a special reward structure.

EC.6.1. Multi-Armed Bandit Policies

For a class of discounted infinite-horizon MAB problems ($N\to\infty$) with independent alternative priors ($\Sigma_0=I$), Gittins and Jones (1974) characterized its optimal policy. However, the optimality of the Gittins index policy does not generalize to most other bandit problems (Gittins et al. 2011). This has led to the development of policies to minimize the learning loss or regret, see e.g. Bubeck and Cesa-Bianchi (2012). Next, we discuss these two classes of policies.

EC.6.1.1. Gittins Index Policy: In this policy, a *Gittins index* for each feasible alternative $x\in\mathcal{X}$ is computed independently at each phase, and the alternative with the highest index is measured next. Formally, at phase n and state S_n, the Gittins index policy selects an alternative $x\in\mathcal{X}$ with the highest Gittin's index $\nu_x^{\mathrm{Gitt,n}}$, defined as (see e.g. equation (2.6) in Gittins et al. (2011))

$$\nu_x^{\mathrm{Gitt,n}}(s):=\sup_{\tau>0}\ \frac{\mathbb{E}\left[\sum_{t=0}^{\tau-1}\gamma^t r_x(S_{n+t})\mid S_n=s\right]}{\mathbb{E}\left[\sum_{t=0}^{\tau-1}\gamma^t\mid S_n=s\right]}. \tag{EC.64}$$

Here, τ is an adapted stopping time, and $r_x(S_n)$ is the reward of measuring x at state S_n.

In our problem, $r_x(S_n)=\eta_x|S_n$, where η_x is as in equation (2). Suppose that the exposures are deterministic, $\kappa_t\equiv\lambda_x$, and the precision parameter ρ is known. Every observation η_x^{n+1} is normal with mean $\mu_x\overset{\text{def}}{=}\lambda_x(\mu_Z^\top x_Z+\mu_B^\top x_B)$ and known variance $\sigma_W^2\overset{\text{def}}{=}\frac{\lambda_x^2}{\rho}\,(x_B^\top\Sigma_B x_B+x_Z^\top\Sigma_Z x_Z+1)$. Every unknown mean μ_x is normally distributed with prior mean $\theta_x^n\overset{\text{def}}{=}\lambda_x(\mu_Z^\top x_Z+\theta_n^\top x_B)$ and prior variance $\sigma_{x,n}^2 \stackrel{\text{def}}{=} \frac{\lambda_x^2}{\rho}\left(x_B^\top \Sigma_n x_B\right)$. Under these assumptions, the Gittins index of the alternative x at phase n can be written as (see e.g. Theorem 7.13 in Gittins et al. (2011))

$$\nu_x^{\text{Gitt,n}}(\theta_n, \Sigma_n, \sigma_W, \gamma) = \theta_x^n + \sigma_W \nu_x^{\text{Gitt,n}}\left(0, \frac{\sigma_{x,n}}{\sigma_W}, 1, \gamma\right). \tag{EC.65}$$

The term $\nu_x^{\text{Gitt,n}}\left(0, \frac{\sigma_{x,n}}{\sigma_W}, 1, \gamma\right)$ can be approximated by $\sqrt{-\log(\gamma)}\,\tilde{b}\left(-\frac{\sigma_{x,n}^2}{\sigma_W^2 \log(\gamma)}\right)$ where

$$\tilde{b}(s) \stackrel{\text{def}}{=} \begin{cases} \frac{s}{\sqrt{2}} & s \le 0.2 \\ 0.49\sqrt{s} - 0.11 & 0.2 \le s \le 1 \\ 0.63\sqrt{s} - 0.26 & 1 \le s \le 5 \\ 0.77\sqrt{s} - 0.58 & 5 \le s \le 15 \\ \sqrt{s}\left(2\log s - \log\log s - \log 16\pi\right)^{1/2} & s > 15. \end{cases} \tag{EC.66}$$

For this approximation, see Brezzi and Lai (2002). Chick and Gans (2008) provide a simpler approximation as follows (see Appendix E in Chick and Gans (2008)),

$$\tilde{b}(s) = \begin{cases} \frac{s}{\sqrt{2}} & s \le \frac{1}{7} \\ \exp(-0.02645(\log s)^2 + 0.89106\log s - 0.4873) & \frac{1}{7} < s \le 100 \\ \sqrt{s}\left(2\log s - \log\log s - \log 16\pi\right)^{1/2} & s > 100. \end{cases} \tag{EC.67}$$

For a review on various algorithms to compute the Gittins index and in other settings, particularly for Markovian bandit processes, and their computational details, see Chakravorty and Mahajan (2014).

The Gittins index policies are effective strategies for Markovian bandit problems. Karoui and Karatzas (1993) prove the optimality of Gittins index processes in non-Markovian MAB problems. For analysis on the Gittins index policy, see Mahajan and Teneketzis (2008). For Markovian bandits, see Gittins et al. (2011). For bandit models with finite state spaces, see Tsitsiklis (1994). For bandit models with countable state spaces, see Frostig and Weiss (1999). For a counterpart of the Gittins index policy for a finite-horizon one-armed bandit problem with a discrete (finite or countably infinite) state space, see Nino-Mora (2011).

When following the Gittins index policy, incomplete learning may occur (Brezzi and Lai 2000), in the sense that there is a positive probability that the chosen process does not have the maximum reward (the chosen arm is sub-optimal), i.e., the Gittins index is an inconsistent estimator of the location of the optimal arm.

A generalization within the MAB framework is the *branching bandit* (Weiss 1988). In such models, new bandits arrive over time, see Chapter 4 of Gittins et al. (2011). Index policies remain optimal for these bandit problems. For a proof of the index policy for branching bandits see Tsitsiklis (1994).

EC.6.1.2. Upper Confidence Bounding (UCB) Policy: For stochastic bandit problems, an UCB index policy plays the arm I_n^{UCB} with the highest an upper confidence bound at the n^{th} phase. When the rewards are normally distributed with unknown means and known variances, the UCB index is expressed as (see e.g. Bubeck (2010), Bubeck and Cesa-Bianchi (2012))

$$I_n^{\text{UCB}} = \arg\max_{x \in \mathcal{X}} \left(\hat{\mu}_{x,n-1} + \sqrt{\frac{6\sigma_x^2 \log n}{T_x(n-1)}} \right), \tag{EC.68}$$

where $\hat{\mu}_x$ is the average reward obtained from machine x in the first $n-1$ rounds, number of plays $T_x(n-1) = \sum_{s=1}^{n-1} 1[I_s^{\text{UCB}} = x]$, and σ_x is the standard deviation of the reward distribution of alternative x. For further discussion on derivations of the upper confidence bounds, see Lai and Robbins (1985) and Agrawal (1995). Lai and Robbins (1985) compute such bounds by maximizing the expected reward when the parameters vary within appropriate confidence sets. Agrawal (1995) introduce a class of deterministic UCB policies where the index is expressed by sample-means of the arms.

Policies based on UCB are easy to implement, efficient, and effective (Audibert et al. 2007). These policies can achieve a logarithmic regret for some settings, see e.g. Auer (2002), Auer et al. (2002a), Auer et al. (2002b), Audibert et al. (2007), Bubeck and Cesa-Bianchi (2012).

EC.6.2. The Knowledge Gradient for Bandit Problems (Online KG policy)

The KG quantity for the bandit problem is given by

$$\begin{aligned} v_n^{\text{KG-MAB}}(x) &= \mathbb{E}[\eta_x|S_n] + \gamma \mathbb{E}\left[V_{n+1}\left(S^{\mathscr{M}}\left(S_n, x_n, (\kappa_{n+1}, \eta_{n+1})\right)\right) - V_{n+1}\left(S_n\right)\right] \\ &= \mathbb{E}[\eta_x|S_n] + \gamma\alpha(N-n)\mathbb{E}\left[U(\theta^{\mathscr{M}}(S_n, x, (\kappa_{n+1}, \eta_{n+1}))) - U(\theta_n)|S_n, x\right] \\ &= \mathbb{E}[\eta_x|S_n] + \gamma\alpha(N-n)\mathbb{E}[\max_{x'} \theta_{x'}^{n+1} - \max_{x'} \theta_{x'}^n | S_n, x_n = x] \\ &= \mathbb{E}[\eta_x|S_n] + \gamma\alpha(N-n) v_n^{\text{KG}}(x), \end{aligned} \tag{EC.69}$$

where $v_n^{\text{KG}}(x)$ in the last term is as in (17). Here,

$$\alpha \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } N \text{ is finite and } \gamma = 1 \\ \frac{\gamma}{1-\gamma}(1-\gamma^{N-n}) & \text{if } N < \infty \text{ and } \gamma < 1. \end{cases}$$

For infinite-horizon bandit problems ($N \to \infty$), $\alpha = \frac{\gamma}{1-\gamma}$. Compared to the typical MAB policies, the KG policy is not an index policy, in the sense that it depends not only on the reward at x, but also on other rewards at $x' \neq x$. Thus, it can handle correlated measurements. Unlike the offline KG policy, the decisions made by the online KG policy depends on both n and S_n.

The decision from the KG policy is easier to compute than the decision from a Gittins index policy. In addition, the KG policy can be defined for both finite- and infinite-horizon problems, while the Gittins index policies are only designed for infinite-horizon problems. The KG policy is not an optimal policy, but it can be used in many settings. The performance of the KG policy for multi-armed bandit problems with normally distributed rewards in comparison to some of the popular policies including the the Gittins index policy and UCB is investigated in Ryzhov et al. (2012).

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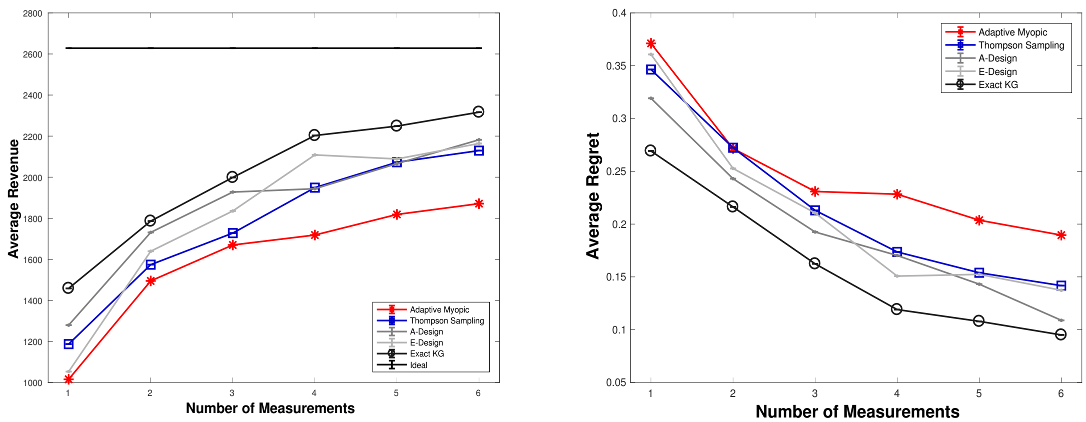
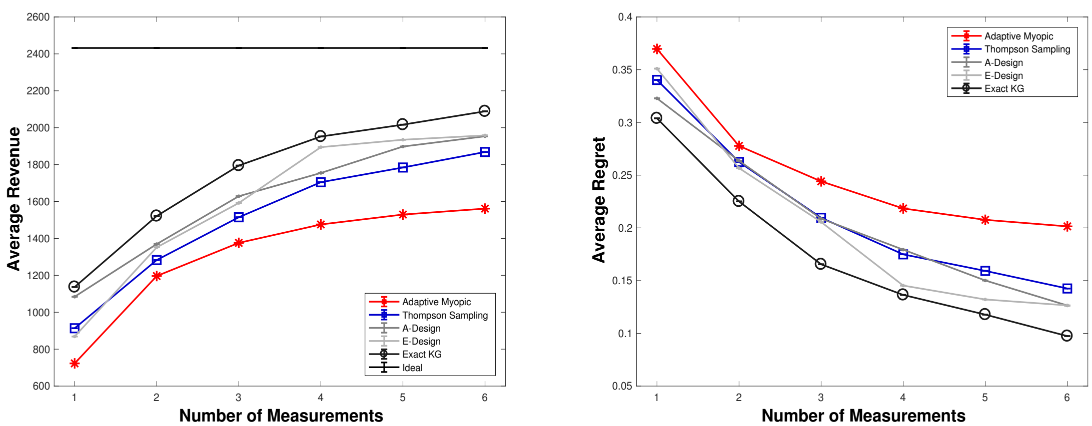
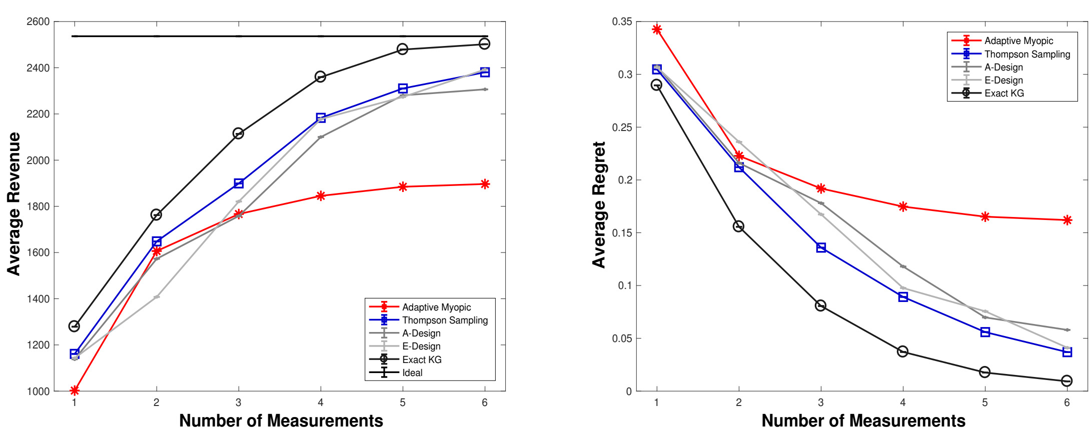
EC.7. Computational Results: Plots



(a) $K = 34560$, $\Sigma_Z = \Sigma_Z^{\mathrm{b}}$, $\Sigma_B = \Sigma_B^{\mathrm{b}}$

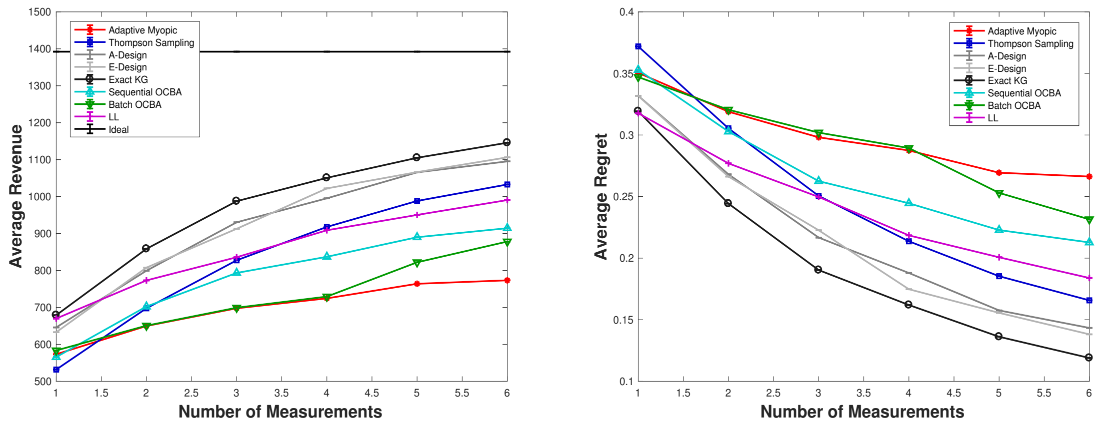


(b) $K = 56$, $\Sigma_Z = \Sigma_Z^{\mathrm{b}}$, $\Sigma_B = \Sigma_B^{\mathrm{b}}$



(c) $K = 56$, $\Sigma_Z = 0$, $\Sigma_B = 0$

Figure EC.1 **Average revenue R_N^π and average regret ε_N^π**



(a) $K = 56$, $\Sigma_Z = \Sigma_Z^{\mathrm{b}}$, $\Sigma_B = \Sigma_B^{\mathrm{b}}$

Figure EC.2 Average revenue R_N^π and average regret ε_N^π for all benchmark policies

N	Revenue Improvement				Regret Improvement			
	KG vs. Myopic	Thompson vs. Myopic	A-Design vs. Myopic	D-Design vs. Myopic	KG vs. Myopic	Thompson vs. Myopic	A-Design vs. Myopic	D-Design vs. Myopic
4	28.21	13.43	13.12	22.71	47.86	23.95	25.42	33.98
5	23.66	13.98	13.61	14.86	47.03	24.43	29.74	25.18
6	23.81	13.80	16.62	15.66	49.89	25.24	42.57	27.56

(a) $K = 34560$, $\Sigma_Z = \Sigma_Z^{\mathrm{b}}$, $\Sigma_B = \Sigma_B^{\mathrm{b}}$.

N	Revenue Improvement				Regret Improvement			
	KG vs. Myopic	Thompson vs. Myopic	A-Design vs. Myopic	D-Design vs. Myopic	KG vs. Myopic	Thompson vs. Myopic	A-Design vs. Myopic	D-Design vs. Myopic
4	32.29	15.51	18.91	28.38	37.54	19.93	17.89	33.47
5	31.85	16.67	24.09	26.50	43.28	23.31	27.69	36.37
6	33.76	19.65	25.10	25.40	51.65	29.24	37.24	37.19

(b) $K = 56$, $\Sigma_Z = \Sigma_Z^{\mathrm{b}}$, $\Sigma_B = \Sigma_B^{\mathrm{b}}$.

N	Revenue Improvement				Regret Improvement			
	KG vs. Myopic	Thompson vs. Myopic	A-Design vs. Myopic	D-Design vs. Myopic	KG vs. Myopic	Thompson vs. Myopic	A-Design vs. Myopic	D-Design vs. Myopic
4	27.85	18.30	13.80	18.01	78.78	48.96	32.49	44.13
5	31.52	22.58	21.00	20.62	89.38	66.17	57.77	54.25
6	31.90	25.51	21.61	26.12	94.29	77.22	64.21	74.64

(c) $K = 56$, $\Sigma_Z = 0$, $\Sigma_B = 0$.

Table EC.1 Relative Improvement in average revenue R_N^π and Average regret ε_N^π