Technical Appendix for the paper "Pollution Regulation of Competitive Markets"

1 Absence of Profit Overshifting

We show that, in our model, profits always fall as the marginal production cost k increases. This rules out the possibility of profit overshifting to explain the increase in firm profits under regulation in our model. (See Seade (1985) and related discussions in Section 2). Firm i's objective, $\forall i$, is

$$\max_{q_i \geq 0} \pi_i \left(q_i \mid Q_{-i}\right) = q_i \cdot \left(p_i \left(q_i ;\ Q_{-i}\right) - k\right)$$

Where the price $p_i \left(q_i ;\ Q_{-i}\right) = a - bq_i - \gamma b Q_{-i}$. The first-order necessary condition is:

$$\frac{\partial \pi_i}{\partial q_i} = a - k - \gamma b Q_{-i} - 2bq_i = 0 \Rightarrow q_i = \frac{a - k - \gamma b Q_{-i}}{2b}$$

Since $Q = Q_{-i} + q_i$, we can rewrite $q_i = \frac{a-k-\gamma bQ}{b(2-\gamma)}$, which does not depend on i. Hence, $\sum_{i=1}^{n} q_i = Q = n \cdot \frac{a-k-\gamma bQ}{b(2-\gamma)}$ from which we derive the equilibrium quantities and prices

$$\begin{aligned} Q^* &= \frac{n\left(a-k\right)}{b\left(2+\left(n-1\right)\gamma\right)} \\ q_i^* &= \frac{a-k}{b\left(2+\left(n-1\right)\gamma\right)} > 0 \\ p_i^* &= \frac{a+k\left(1+\left(n-1\right)\gamma\right)}{2+\left(n-1\right)\gamma} \end{aligned}$$

Since $\frac{\partial^2 \pi_i}{\partial q_i^2} = -2b < 0$, q_i^* is the unique maximizer of π_i. Since $Q_{-i} = \left(n-1\right) q_i$, firm $i's$ equilibrium profits are

$$\begin{aligned} \pi_i^* &= q_i^* \cdot \left(p_i^* - k\right) = q_i^* \left(a - k - b\left(1+\left(n-1\right)\gamma\right) q_i^*\right) \\ &= \left(a-k\right) q_i^* - b\left(1+\left(n-1\right)\gamma\right) q_i^{*2} = \frac{\left(a-k\right)^2}{b\left(2+\left(n-1\right)\gamma\right)^2} \end{aligned}$$

It is easy to see that π_i^* is decreasing in k for $0 < k < a$. Hence, there is no profit overshifting. $\square$

2 Proof of Theorem 1

Proof of Part *(i)*: We derive the unique Nash equilibrium under the Cap-and-Trade mechanism for any S, any γ, and $\forall\ 0 \leq m \leq n$. Let $\underline{S} = \frac{(n-m)a(c_h - c_l)}{c_h\left[b(2+(n-1)\gamma)+2c_l\right]}$, and let $\widehat{s_i} = s_i - t_i$ denote the post-trading emission allowances held by firm i. When $0 < m < n$, the unique equilibrium is:

Case 1, $S > \underline{S}$ (*Interior Solution*–Moderate Regulation) :

$$\begin{aligned}
q_l^{ct} &= q_h^{ct} = \frac{a\left((n-m)\,c_l + mc_h\right) + 2c_l c_h S}{(2+(n-1)\,\gamma)\,b\left((n-m)\,c_l + mc_h\right) + 2nc_l c_h} \\
x_l^{ct} &= c_h \frac{na - b\,(2+(n-1)\,\gamma)\,S}{a\left((n-m)\,c_l + mc_h\right) + 2c_l c_h S};\ x_h^{ct} = c_l \frac{na - b\,(2+(n-1)\,\gamma)\,S}{a\left((n-m)\,c_l + mc_h\right) + 2c_l c_h S} \\
r &= 2c_l c_h \frac{na - b\,(2+(n-1)\,\gamma)\,S}{(2+(n-1)\,\gamma)\,b\left((n-m)\,c_l + mc_h\right) + 2nc_l c_h} \\
\widehat{s_l} &= \frac{c_h\left[b\,(2+(n-1)\,\gamma) + 2c_l\right]S - (n-m)\,a\,(c_h - c_l)}{(2+(n-1)\,\gamma)\,b\left((n-m)\,c_l + mc_h\right) + 2nc_l c_h} \\
\widehat{s_h} &= \frac{c_l\left[b\,(2+(n-1)\,\gamma) + 2c_h\right]S - ma\,(c_h - c_l)}{(2+(n-1)\,\gamma)\,b\left((n-m)\,c_l + mc_h\right) + 2nc_l c_h}
\end{aligned}$$

Case 2, $S \leq \underline{S}$ (*Corner Solution*–Stringent Regulation)

$$\begin{aligned}
q_l^{ct} &= \frac{a\left[b\,(2-\gamma) + 2c_h\right] - 2\gamma b c_h S}{\left[b\,(2+(m-1)\,\gamma) + 2c_l\right]\left[b\,(2+(n-m-1)\,\gamma) + 2c_h\right] - m\,(n-m)\,\gamma^2 b^2} \\
q_h^{ct} &= \frac{a\left[b\,(2-\gamma) + 2c_l\right] + 2c_h\left[b\,(2+(m-1)\,\gamma) + 2c_l\right]S \big/ (n-m)}{\left[b\,(2+(m-1)\,\gamma) + 2c_l\right]\left[b\,(2+(n-m-1)\,\gamma) + 2c_h\right] - m\,(n-m)\,\gamma^2 b^2} \\
x_l^{ct} &= 1;\ x_h^{ct} = \frac{(n-m)\,a\left[b\,(2-\gamma) + 2c_l\right] - b\left[b\,(2-\gamma)\,(2+(n-1)\,\gamma) + 2c_l\,(2+(n-m-1)\,\gamma)\right]S}{(n-m)\,a\left[b\,(2-\gamma) + 2c_l\right] + 2c_h\left[b\,(2+(m-1)\,\gamma) + 2c_l\right]S} \\
r &= 2c_h \frac{(n-m)\,a\left[b\,(2-\gamma) + 2c_l\right] - b\left[(2-\gamma)\,(2+(n-1)\,\gamma)\,b + 2c_l\,(2+(n-m-1)\,\gamma)\right]S}{(n-m)\left[\left[b\,(2+(m-1)\,\gamma) + 2c_l\right]\left[b\,(2+(n-m-1)\,\gamma) + 2c_h\right] - m\,(n-m)\,\gamma^2 b^2\right]} \\
\widehat{s_l} &= 0;\ \widehat{s_h} = \frac{S}{n-m}
\end{aligned}$$

When $m = 0$ (or n), the unique equilibrium $\forall\ S$ is:

$$\begin{aligned}
q_i^{ct} &= \frac{a + 2c_i S/n}{b\,(2+(n-1)\,\gamma) + 2c_i};\ x_i^{ct} = \frac{a - bs\,(2+(n-1)\,\gamma)}{a + 2c_i S/n} \\
r &= \frac{2c_i\left[a - b\,(2+(n-1)\,\gamma)\,S/n\right]}{b\,(2+(n-1)\,\gamma) + 2c_i};\ \widehat{s_i} = \frac{S}{n}
\end{aligned}$$

Proof

Firm i's objective under Cap-and-Trade, $\forall i$, is

$$\max_{q_i>0,\ 0\leq x_i\leq 1,\ t_i\leq s_i} \quad \pi_i\left(q_i,\ x_i,\ t_i \mid Q_{-i},\ r\right) = q_i \cdot \left(a - b\cdot q_i - \gamma\cdot b\cdot Q_{-i}\right) - c_i \cdot \left(q_i \cdot x_i\right)^2 + r\cdot t_i$$

subject to the pollution constraint $q_i \cdot (1 - x_i) \leq s_i - t_i$,

Note that the conditions $q_i > 0,\ x_i \leq 1$, and the pollution constraint jointly guarantee that $t_i \leq s_i$. Hence, we can eliminate the constraint $t_i \leq s_i$.

We analyze the cases $0 < m < n$ and $m = 0$ (or n) separately.

2.1 Case 1: $0 < m < n$.

There are four steps to the proof: *(i)* We analyze the Lagrangian, and derive necessary conditions on q_i, x_i, and t_i based on the Kuhn-Tucker conditions; *(ii)* We show that any equilibrium must be symmetric, meaning that if firms have equal cost coefficients, their production quantities, abatement levels, and trading quantities are identical; *(iii)* We solve for the equilibrium, and prove that it is unique; and *(iv)* We verify the second order sufficient conditions.

2.1.1 Step 1 - Analysis of the Lagrangian

Write the Lagrangian $\mathfrak{L}_i = q_i \cdot (a - b \cdot q_i - \gamma \cdot b \cdot Q_{-i}) - c_i \cdot (q_i \cdot x_i)^2 + r \cdot t_i + \eta_i \cdot x_i - \mu_i \cdot (x_i - 1) - \nu_i \cdot (q_i \cdot (1 - x_i) - s_i + t_i)$. The Kuhn-Tucker necessary conditions are:

$$\frac{\partial \mathfrak{L}_i}{\partial q_i} = a - \gamma b Q_{-i} - 2q_i \cdot \left(b + c_i \cdot x_i^2\right) - \nu_i \cdot (1 - x_i) = 0 \quad (1)$$

$$\frac{\partial \mathfrak{L}_i}{\partial x_i} = q_i \cdot (\nu_i - 2c_i \cdot q_i \cdot x_i) + \eta_i - \mu_i = 0 \quad (2)$$

$$\frac{\partial \mathfrak{L}_i}{\partial t_i} = r - \nu_i = 0 \quad (3)$$

with the complementary slackness conditions $\eta_i \cdot x_i = 0$, $\mu_i \cdot (x_i - 1) = 0$, and $\nu_i \cdot (q_i \cdot (1 - x_i) - s_i + t_i) = 0$, and the feasibility constraints, $q_i > 0;\ \eta_i,\ \mu_i,\ \nu_i \geq 0;\ 0 \leq x_i \leq 1;$ and $q_i \cdot (1 - x_i) \leq s_i - t_i$.

First, we show that $\eta_i = 0$. *Proof.* (By contradiction.) Suppose that $\eta_i > 0$. Then $x_i = 0$, which also implies that $\mu_i = 0$ (by complementary slackness), and from equation (2), $\eta_i = -\nu_i q_i \leq 0$. □

Given that $\eta_i = 0$, we now consider two cases for ν_i:

- If $\nu_i = 0$, then (2) $\Rightarrow \mu_i = -2c_i q_i^2 x_i$. Furthermore, $\mu_i \geq 0 \Rightarrow x_i = 0 \Rightarrow \mu_i = 0$. By equation (1), and the fact that $Q = Q_{-i} + q_i,\ q_i = \frac{a - \gamma b Q_{-i}}{2b} = \frac{a - \gamma b Q}{b(2-\gamma)}$. Also, $t_i \leq s_i - q_i$ (by the pollution constraint).

- If $\nu_i > 0,$ then the pollution constraint binds, i.e.,

$$t_i = s_i - q_i (1 - x_i) \quad (4)$$

  Now, consider the following two sub-cases for μ_i:

1. $\mu_i = 0$: Then, (2) $\Rightarrow \nu_i = 2c_i q_i x_i$. Substituting in (1), we get $q_i = \frac{a - \gamma b Q - \nu_i}{b(2-\gamma)}$. The condition $x_i \leq 1 \Rightarrow \nu_i \leq 2c_i q_i$.

2. $\mu_i > 0$: Then, $x_i = 1$, by complementary slackness. By equation (4), $t_i = s_i$, and by equation (1), and the fact that $Q = Q_{-i} + q_i,\ q_i = \frac{a - \gamma b Q_{-i}}{2(b + c_i)} = \frac{a - \gamma b Q}{b(2-\gamma) + 2c_i}$. By equation (2), $\mu_i = q_i (\nu_i - 2c_i q_i) > 0$, which implies that $\nu_i > 2c_i q_i$.

To summarize: if $q_i > 0$, then $\eta_i = 0$ (always), $\nu_i = r$ by (3), and there are three possible candidate solutions:

$$q_i = \frac{a - \gamma bQ}{b(2-\gamma)};\ x_i = 0;\ t_i \leq s_i - q_i;\ \nu_i = r = \mu_i = 0; \tag{5}$$

$$q_i = \frac{a - \gamma bQ - r}{b(2-\gamma)};\ x_i = \frac{r}{2c_i q_i};\ t_i = s_i - q_i(1 - x_i);\ 0 < \nu_i = r \leq 2c_i q_i;\ \mu_i = 0; \tag{6}$$

$$q_i = \frac{a - \gamma bQ}{b(2-\gamma) + 2c_i};\ x_i = 1;\ t_i = s_i;\ \nu_i = r > 2c_i q_i;\ \mu_i > 0. \tag{7}$$

Consider first the solution candidates given by (5). Since $r = 0$ under this solution, the set of equations represented by the conditions (5) must apply to *all* firms, if they apply to any. By (5), we have $q_l = q_h = \frac{a - b\gamma Q}{b(2-\gamma)}$ and $x_l = x_h = 0$. Since $Q = nq_l$, we have $q_l = q_h = \frac{a}{b(2+(n-1)\gamma)} > 0$. By (5), we also have, $\forall i,\ t_i \leq s_i - q_i$. Hence, $\forall i,\ s_i \geq q_i + t_i$. Thus, $\sum_{i=1}^{n} s_i \geq \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} t_i \Leftrightarrow S \geq \frac{na}{b(2+(n-1)\gamma)} = \overline{S}$ (since $\sum_{i=1}^{n} t_i = 0$ by the market-clearing condition). Thus, the set of conditions given by (5) cannot lead to any feasible equilibrium in the range of interest $S \in \left(0, \overline{S}\right)$. Hence, in the rest of this proof, we focus only the candidate solutions given by conditions (6) and (7).

2.1.2 Step 2 - Proof of Symmetry

Lemma 1 *If $c_i = c_j$, then $\mu_i = \mu_j,\ q_i = q_j,\ x_i = x_j$ and $s_i - t_i = s_j - t_j$.*

Proof. Consider two firms, indexed by i and j, and suppose that $c_i = c_j = c$. By Step 1, we know that the solution candidates must satisfy conditions (6) or (7).

Consider the case ($\mu_i = 0,\ \mu_j > 0$). Conditions (6) apply for firm i and conditions (7) for firm j. From (6), $q_i = \frac{a - \gamma bQ - 2cq_i x_i}{b(2-\gamma)} \Rightarrow q_i = \frac{a - \gamma bQ}{b(2-\gamma) + 2cx_i}$. Thus, $r = 2cq_i x_i > 2cq_j \Rightarrow q_i x_i > q_j \Rightarrow \frac{(a - \gamma bQ)x_i}{b(2-\gamma) + 2cx_i} > \frac{a - \gamma bQ}{b(2-\gamma) + 2c}$, which simplifies to $x_i > 1$, which is infeasible. Therefore you cannot have $\mu_i = 0$ and $\mu_j > 0$ simultaneously. Similarly, you cannot have $\mu_i > 0$ and $\mu_j = 0$ simultaneously. The only two possible cases are:

1. $\mu_i = \mu_j = 0$. In this case, by (6), $q_i = q_j$ which also implies that $x_i = x_j$. Further, $s_i - t_i = q_i(1 - x_i) = q_j(1 - x_j) = s_j - t_j$.

2. $\mu_i > 0$ and $\mu_j > 0$. Then, $x_i = x_j = 1$ and $q_i = q_j = \frac{a - \gamma bQ}{b(2-\gamma) + 2c}$, by (7). From equation (2), $\mu_i = q_i(\nu_i - 2cq_i) = q_i(r - 2cq_i) = q_j(\nu_j - 2cq_j) = \mu_j$. Further, $s_i - t_i = s_j - t_j = 0$. □

2.1.3 Step 3 - Equilibrium Analysis

We know by Lemma 1 that if $c_i = c_j$ then $s_i - t_i = s_j - t_j$, i.e., firms with the same abatement cost coefficient will have the same post-trading caps. Let the post-trading cap of a low-cost firm be $\widehat{s_l}$, and that of a high-cost firm be $\widehat{s_h}$. The market clearing condition stipulates that $\sum_{i=1}^{n} t_i = 0$; hence, $\sum_{i=1}^{n}(s_i - t_i) = S = m\widehat{s_l} + (n-m)\,\widehat{s_h}$, which implies that

$$\widehat{s_h} = \frac{S - m\widehat{s_l}}{n-m} \tag{8}$$

We showed in Step 1 that the equilibrium solution for each firm must satisfy exactly either (6) or (7). Hence, by Lemma 1, firms with identical abatement cost coefficients must satisfy the same set of conditions–(6) or (7).We investigate all possible solutions that can arise from these conditions by varying r.

Case 1: $0 < r \leq \min\{2c_l q_l,\ 2c_h q_h\}$: This is obtained by applying conditions (6) to both types of firms. Using (6) and (8), we have five equations in the five unknowns $q_l = q_h,\ x_l,\ x_h,\ \widehat{s_l}$ and r : (i) $q_l = q_h = \frac{a-\gamma bQ-r}{b(2-\gamma)}$; (ii) $r = 2c_l q_l x_l$; (iii) $r = 2c_h q_h x_h$; (iv) $\widehat{s_l} = q_l\,(1-x_l)$; and (v) $\widehat{s_h} = \frac{S-m\widehat{s_l}}{n-m} = q_h\,(1-x_h)$. By algebraic manipulations, we can prove that $q_l^{ct} = q_h^{ct} = \frac{a((n-m)c_l+mc_h)+2c_lc_hS}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h} > 0$; $x_l^{ct} = c_h\frac{na-b(2+(n-1)\gamma)S}{a((n-m)c_l+mc_h)+2c_lc_hS}$; $x_h^{ct} = \frac{c_l}{c_h}x_l^{ct} = c_l\frac{na-b(2+(n-1)\gamma)S}{a((n-m)c_l+mc_h)+2c_lc_hS}$; $\widehat{s_l} = \frac{c_h[b(2+(n-1)\gamma)+2c_l]S-(n-m)a(c_h-c_l)}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h}$; $\widehat{s_h} = \frac{S-m\widehat{s_l}}{n-m} = \frac{c_l[b(2+(n-1)\gamma)+2c_h]S+ma(c_h-c_l)}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h}$ and $r = \frac{2c_lc_h[na-b(2+(n-1)\gamma)S]}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h}$.

This equilibrium will hold only if $0 < r \leq \min\{2c_lq_l,\ 2c_hq_h\} = 2c_lq_l$, since $q_l = q_h$ and $c_l \leq c_h$. The condition $r \leq 2c_lq_l \iff S \geq \frac{(n-m)a(c_h-c_l)}{c_h[b(2+(n-1)\gamma)+2c_l]} \equiv \underline{S}$. The condition $r > 0 \iff S < \frac{na}{b(2+(n-1)\gamma)} = \overline{S}$. It is easy to check that $\underline{S} \leq \overline{S}$. This candidate solution is an *interior solution* and corresponds precisely to *Case* 1 *(Moderate Regulation)* of Theorem 1, when $0 < m < n$.

Case 2: $\min\{2c_lq_l,\ 2c_hq_h\} < r \leq \max\{2c_lq_l,\ 2c_hq_h\}$. There are two possibles cases: (2.1) $2c_lq_l < r \leq 2c_hq_h$, and (2.2) $2c_hq_h < r \leq 2c_lq_l$. We analyze each case separately.

(2.1) $2c_lq_l < r \leq 2c_hq_h$: This is obtained by applying conditions (7) to low-cost firms, and conditions (6) to high-cost firms. By (7), we have $q_l = \frac{a-\gamma bQ}{b(2-\gamma)+2c_l}$; $x_l^{ct} = 1$; and $\widehat{s_l} = 0$. Hence, $\widehat{s_h} = \frac{S-m\widehat{s_l}}{n-m} = \frac{S}{n-m}$. By (6), $r = 2c_hq_hx_h$ and $\widehat{s_h} = q_h\,(1-x_h) \Rightarrow q_hx_h = q_h - \frac{S}{n-m}$. Also by (6), $q_h = \frac{a-\gamma bQ-2c_hq_hx_h}{b(2-\gamma)} = \frac{a-\gamma bQ+2c_hS/(n-m)}{b(2-\gamma)+2c_h}$. To summarize, we have the following expressions for q_l and q_h : $q_l = \frac{a-\gamma bQ}{b(2-\gamma)+2c_l}$ and $q_h = \frac{a-\gamma bQ+2c_hS/(n-m)}{b(2-\gamma)+2c_h}$, where $Q = mq_l + (n-m)\,q_h$. Solving the above system of linear equations for q_l and q_h, we get the following *corner solution*:

$$q_l^{ct} = \frac{a\,[b\,(2-\gamma)+2c_h] - 2\gamma bc_hS}{[b\,(2+(m-1)\,\gamma)+2c_l]\,[b\,(2+(n-m-1)\,\gamma)+2c_h] - m\,(n-m)\,\gamma^2b^2}$$

$$q_h^{ct} = \frac{a\,[b\,(2-\gamma)+2c_l] + 2c_h\,[b\,(2+(m-1)\,\gamma)+2c_l]\,S/(n-m)}{[b\,(2+(m-1)\,\gamma)+2c_l]\,[b\,(2+(n-m-1)\,\gamma)+2c_h] - m\,(n-m)\,\gamma^2b^2}$$

Grouping the b^2 terms of the denominator and simplifying, we can rewrite the denominators of q_l and q_h as $(2-\gamma)(2+(n-1)\gamma)b^2 + 2b[(2+(n-m-1)\gamma)c_l + (2+(m-1)\gamma)c_h] + 4c_lc_h > 0$. Hence, $q_h > 0$. Further, $q_l = \frac{a-\gamma bQ}{b(2-\gamma)+2c_l} = \frac{a-\gamma bQ-2c_lq_l}{b(2-\gamma)} > q_h = \frac{a-\gamma bQ-2c_hq_hx_h}{b(2-\gamma)}$ only if $2c_lq_l < 2c_hq_hx_h = r$, which is true. Thus, $q_l > 0$ also.

Further, we have $x_h^{ct} = 1 - \frac{S}{(n-m)q_h} = \frac{(n-m)a[b(2-\gamma)+2c_l]-b[b(2-\gamma)(2+(n-1)\gamma)+2c_l(2+(n-m-1)\gamma)]S}{(n-m)a[b(2-\gamma)+2c_l]+2c_h[b(2+(m-1)\gamma)+2c_l]S}$. We also have, $r = 2c_hq_hx_h = 2c_h\left(q_h - \frac{S}{n-m}\right)$, which, after algebraic simplification, yields $r = 2c_h\frac{a[b(2-\gamma)+2c_l]-b[(2-\gamma)(2+(n-1)\gamma)b+2c_l(2+(n-m-1)\gamma)]S/(n-m)}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2}$. The conditions $2c_lq_l < r \leq 2c_hq_h$ imply that $S < \frac{(n-m)a(c_h-c_l)}{c_h[b(2+(n-1)\gamma)+2c_l]} = \underline{S}$, since $r = 2c_hq_hx_h \leq 2c_hq_h$.

(2.2) $2c_hq_h < r \leq 2c_lq_l$: The solution to case (2.2) is obtained from the solution to case (2.1) by substituting c_l and m for c_h and $(n-m)$, respectively, leading to the condition $S < \frac{ma(c_l-c_h)}{c_l[b(2+(n-1)\gamma)+2c_h]} < 0$, which is infeasible.

Case 3: $r > \max\{2c_lq_l,\ 2c_hq_h\}$: This is obtained by applying conditions (7) to both types of firms. By (7), $\widehat{s_l} = \widehat{s_h} = 0$ and, by (8), $\widehat{s_h} = \frac{S-m\widehat{s_l}}{n-m} = 0 \Rightarrow S = 0$, and there is no feasible equilibrium in the range of interest $S \in \left(0, \overline{S}\right)$. □

2.1.4 Step 4 - Second Order Sufficient Conditions

For the interior solution, we show that the Hessian matrix is negative semidefinite, which proves concavity (Theorem M.C.2 of Mas-Colell et al. 1995, p. 933). Since $r = 2c_iq_ix_i$ for the interior solution (by (6)), the Hessian is

$$H = -2\begin{pmatrix} b + c_ix_i^2 & c_iq_ix_i & 0 \\ c_iq_ix_i & c_iq_i^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let H^π denote the matrix in which both the rows and the columns of H are permuted according to some permutation π of the set $\{1,\ 2,\ 3\}$, and let ${}_kH^\pi$ denote the $k \times k$ submatrix consisting of only the first k rows and columns of H^π. H is negative semidefinite iff $(-1)^k \det({}_kH^\pi) \geq 0$ for every permutation π, and for every $k \in \{1,2,3\}$ (Mas-Colell et al. 1995, Theorem M.D.2, p. 936). These conditions are easily verified for H.

For the corner solution, we use an approach from Luenberger and Ye (2008) to show that any nonzero feasible move away from the candidate solution decreases the objective function. For the corner solution, the active constraints are $x \leq 1$ and $q(1-x) \leq s - t$. Let $g_1(q,\ x,\ t) = x - 1$ and $g_2(q,\ x,\ t) = q(1-x) - s + t$. The gradients of g_1 and g_2 at the candidate solution are $\neq 0$, which means that the candidate solution is regular. The gradient of the objective function at this candidate solution is

$$\nabla \pi_i = \begin{pmatrix} 0 & -2c_i q_i^2 & r \end{pmatrix}^T.$$

A nonzero feasible direction, $\underline{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^T$ is defined by $(\nabla g_1)^T \cdot \underline{v} \leq 0$ and $(\nabla g_2)^T \cdot \underline{v} \leq 0$. These conditions are equivalent to $v_2 \leq 0$ and $-v_2 q_i + v_3 \leq 0$. If $\underline{v}$ is a feasible direction, then $\underline{v} \cdot \nabla \pi_i = -2c_i q_i^2 v_2 + r v_3 \leq r(-v_2 q_i + v_3) \leq 0$, because $r > 2c_i q_i$ (by equation (7)).

2.2 Case 2: $m = 0$ (or n).

2.2.1 Step 1 - Analysis of the Lagrangian

This step is identical to Case 1.

2.2.2 Step 2 - Proof of Symmetry

This step is identical to Case 1.

2.2.3 Step 3 - Equilibrium Analysis

First consider the case $m = 0$ ($c_i = c_h$ for all i). By Lemma 1 (of Step 2), all the firms will have the same post-trading cap, $\widehat{s_h}$. The market clearing condition stipulates that $\sum_{i=1}^{n} t_i = 0$; hence, $\sum_{i=1}^{n} (s_i - t_i) = S = n\widehat{s_h}$, which implies that $\widehat{s_h} = \frac{S}{n}$.

We showed in Step 1 that the equilibrium solution for each firm must satisfy exactly either (6) or (7). We investigate the two possible solutions that can arise from these conditions by varying r.

Case 1: $0 < r \leq 2c_h q_h$: Conditions (6) apply to both types of firms. Using (6), we have three equations in the three unknowns q_h, x_h and r : (i) $q_h = \frac{a-\gamma bQ-r}{b(2-\gamma)}$; (ii) $r = 2c_h q_h x_h$; and (iii) $\widehat{s_h} = \frac{S}{n} = q_h(1-x_h)$.

By algebraic manipulations, we can prove that $q_h = \frac{a+2c_h S/n}{b(2+(n-1)\gamma)+2c_h} > 0$; $x_h = \frac{a-b(2+(n-1)\gamma)S/n}{a+2c_h S/n}$ and $r = \frac{2c_h[a-b(2+(n-1)\gamma)S/n]}{b(2+(n-1)\gamma)+2c_h}$.

This equilibrium will hold only if $0 < r \leq 2c_h q_h$. The condition $r \leq 2c_h q_h \iff S \geq 0$. The condition $r > 0 \iff S < \frac{na}{b(2+(n-1)\gamma)} = \overline{S}$.

Case 2: $r > 2c_h q_h$: Conditions (7) apply to both types of firms. By (7), $\widehat{s_h} = 0 = \frac{S}{n} \Rightarrow S = 0$, and there is no feasible equilibrium in the range of interest $S \in (0, \overline{S})$.

Finally, the symmetric case $m = n$ is obtained from the case $m = 0$ by substituting c_l for c_h.

2.2.4 Step 4 - Second Order Sufficient Conditions

This step is identical to Case 1.

By Step 2, the equilibrium is symmetric for any $S,\ \gamma,\ m$ and n.

Proof of Part *(ii)*: The equilibrium solution $\left\{q_l^{ct},\ q_h^{ct}, x_l^{ct},\ x_h^{ct},\ r,\ \widehat{s_l},\ \widehat{s_h}\right\}$ depends on the regulator's total pollution cap S, but is *independent* of the initial allocation $\{s_i\}_{i=1}^{n}$, so long as $\sum_{i=1}^{n} s_i = S$.

This concludes the proof of Theorem 1 for any S, any γ, and $\forall\ 0 \leq m \leq n$. $\square$

3 Proof of Theorem 2

3.1 Proof of Part *(i)*

We solve for the Subgame Perfect Nash equilibrium under Taxes. There are three steps to this derivation: *(i)* Using backward induction, we solve for the firms' Nash Equilibrium given the tax rate τ; *(ii)* We verify the second order sufficient conditions; and *(iii)* We solve for the Subgame Perfect Nash equilibrium by finding the minimum tax rate τ that achieves $P \leq S$, and prove that it is unique.

3.1.1 Step 1 - The firms' Nash Equilibrium given τ

The firm's problem under the Tax mechanism is

$$\max_{q_i>0,\ 0\leq x_i\leq 1} \pi_i\left(q_i, x_i \mid Q_{-i}, \tau\right) = q_i \cdot (a - b\cdot q_i - \gamma \cdot b \cdot Q_{-i}) - c_i \cdot (q_i \cdot x_i)^2 - \tau \cdot q_i \cdot (1 - x_i)$$

Write the Lagrangian for firm i :

$\mathfrak{L}_i = q_i \cdot (a - b\cdot q_i - \gamma \cdot b \cdot Q_{-i}) - c_i \cdot (q_i \cdot x_i)^2 - \tau \cdot q_i \cdot (1 - x_i) + \eta_i \cdot x_i - \mu_i \cdot (x_i - 1)$. The Kuhn-Tucker necessary conditions are:

$$\frac{\partial \mathfrak{L}_i}{\partial q_i} = a - \gamma b Q_{-i} - 2q_i \cdot \left(b + c_i \cdot x_i^2\right) - \tau \cdot (1 - x_i) = 0 \tag{9}$$

$$\frac{\partial \mathfrak{L}_i}{\partial x_i} = q_i \cdot (\tau - 2c_i \cdot q_i \cdot x_i) + \eta_i - \mu_i = 0 \tag{10}$$

with the complementary slackness conditions $\eta_i \cdot x_i = 0$, and $\mu_i \cdot (x_i - 1) = 0$, and the feasibility constraints, $q_i > 0,\ 0 \leq x_i \leq 1$ and $\eta_i,\ \mu_i \geq 0$.

First, we show that $\eta_i = 0$. *Proof.* (By contradiction.) Suppose that $\eta_i > 0$. Then $x_i = 0$, which also implies that $\mu_i = 0$ (by complementary slackness), and from equation (10), $\eta_i = -\tau q_i \leq 0$. $\square$

Given that $\eta_i = 0$, we now consider two cases for μ_i:

- $\mu_i > 0$. Then $x_i = 1$ and by equation (9), $q_i = \frac{a-\gamma b Q_{-i}}{2(b+c_i)} \Rightarrow 2q_i(b+c_i) = a-\gamma b Q_{-i}$. Subtracting $\gamma b q_i$ from both sides of the equation, and since $Q = Q_{-i} + q_i$, we have $q_i = \frac{a-\gamma b Q}{b(2-\gamma)+2c_i}$. The condition $\mu_i > 0 \Rightarrow \tau > 2c_i q_i$ (using equation 10).

- $\mu_i = 0$. Then, by equation (10), $\tau = 2c_i q_i x_i$. After substitution in (9), we get $q_i = \frac{a-\tau-\gamma bQ_{-i}}{2b} = \frac{a-\tau-\gamma bQ}{b(2-\gamma)}$. The solution holds only if $x_i \leq 1 \Rightarrow \tau \leq 2c_i q_i$.

To summarize, there are two possible candidate solutions:

$$q_i = \frac{a-\gamma bQ}{b(2-\gamma)+2c_i};\ x_i = 1;\ \mu_i > 0;\ \tau > 2c_i q_i \tag{11}$$

$$q_i = \frac{a-\tau-\gamma bQ}{b(2-\gamma)};\ x_i = \frac{\tau}{2c_i q_i};\ \mu_i = 0;\ \tau \leq 2c_i q_i \tag{12}$$

Next, we prove that any equilibrium must be symmetric, meaning that if firms have equal cost coefficients, their production quantities and abatement levels are identical.

Lemma 2 *If $c_i = c_j$, then $\mu_i = \mu_j$, $q_i = q_j$ and $x_i = x_j$.*

Proof. Consider two firms, indexed by i and j, and suppose that $c_i = c_j = c$.

Consider the case $(\mu_i = 0,\ \mu_j > 0)$. Conditions (12) apply for firm i and conditions (11) for firm j. From (12), $q_i = \frac{a-\gamma bQ-2cq_i x_i}{b(2-\gamma)} \Rightarrow q_i = \frac{a-\gamma bQ}{b(2-\gamma)+2cx_i}$. Thus, $\tau = 2cq_i x_i > 2cq_j \Rightarrow q_i x_i > q_j \Rightarrow \frac{(a-\gamma bQ)x_i}{b(2-\gamma)+2cx_i} > \frac{a-\gamma bQ}{b(2-\gamma)+2c}$ (also by conditions (11) and (12)), which simplifies to $x_i > 1$, which is infeasible. Therefore you cannot have $\mu_i = 0$ and $\mu_j > 0$ simultaneously. Similarly, you cannot have $\mu_i > 0$ and $\mu_j = 0$ simultaneously. The only two possible cases are:

1. $\mu_i = \mu_j = 0$. In this case, using (12), $a - \tau - \gamma bQ = b(2-\gamma)q_i = b(2-\gamma)q_j \Rightarrow q_i = q_j$. Since $\tau = 2cq_i x_i = 2cq_j x_j$ (again by 12), $q_i = q_j$ also implies that $x_i = x_j$. Thus, Lemma 2 holds.

2. $\mu_i > 0$ and $\mu_j > 0$. Then, $x_i = x_j = 1$ and $q_i = q_j = \frac{a-\gamma bQ}{b(2-\gamma)+2c}$, by (11). From equation (10), $\mu_i = 2q_i(\tau - 2cq_i) = 2q_j(\tau - 2cq_j) = \mu_j$. □

By Lemma 2, firms with identical abatement cost coefficients must satisfy the same set of conditions– (11) or (12). We investigate all possible solutions that can arise from these conditions by varying τ.

Case 1: $0 \leq \tau \leq \min\{2c_l q_l,\ 2c_h q_h\}$. This is obtained by applying conditions (12) apply to both types of firms. By (12), we have $\tau = 2c_l q_l x_l = 2c_h q_h x_h$, $q_l = q_h = \frac{a-\tau-\gamma bQ}{b(2-\gamma)}$. Since $Q = nq_l$, we have $q_l = q_h = \frac{a-\tau}{b(2+(n-1)\gamma)}$. Further, we have $x_l = \frac{b(2+(n-1)\gamma)\tau}{2c_l(a-\tau)} \geq x_h = \frac{b(2+(n-1)\gamma)\tau}{2c_h(a-\tau)}$. The conditions $\tau \leq \min\{2c_l q_l,\ 2c_h q_h\} \Rightarrow \tau \leq \tau_1 \equiv \frac{2ac_l}{(2+(n-1)\gamma)b+2c_l}$. Notice that Case 1 is an *interior solution*.

Case 2: $\min\{2c_l q_l,\ 2c_h q_h\} < \tau \leq \max\{2c_l q_l,\ 2c_h q_h\}$: There are two possible cases: (2.1) $2c_l q_l < \tau \leq 2c_h q_h$, and (2.2) $2c_h q_h < \tau \leq 2c_l q_l$. We analyze each case separately.

(2.1) $2c_l q_l < \tau \leq 2c_h q_h$: This is obtained by applying conditions (11) to low-cost firms, and conditions (12) to high-cost firms. By (11), $x_l = 1$ and $q_l = \frac{a-\gamma bQ}{b(2-\gamma)+2c_l}$. By (12), $x_h = \frac{\tau}{2c_h q_h}$

and $q_h = \frac{a-\tau-\gamma bQ}{b(2-\gamma)}$. Since $Q = mq_l + (n-m)\, q_h$, solving the system of linear equations for q_l and q_h, we get

$$\begin{aligned} q_l &= \frac{a\,(2-\gamma) + (n-m)\,\gamma\tau}{(2-\gamma)\,(2+(n-1)\,\gamma)\,b + 2c_l\,(2+(n-m-1)\,\gamma)} \\ q_h &= \frac{a\,[b\,(2-\gamma)+2c_l] - \tau\,[b\,(2+(m-1)\,\gamma)+2c_l]}{b\,[(2-\gamma)\,(2+(n-1)\,\gamma)\,b + 2c_l\,(2+(n-m-1)\,\gamma)]} \end{aligned}$$

Further, we have $x_h = \frac{b[(2-\gamma)(2+(n-1)\gamma)b+2c_l(2+(n-m-1)\gamma)]\tau}{2c_h[a[b(2-\gamma)+2c_l]-\tau[b(2+(m-1)\gamma)+2c_l]]}$. The conditions $2c_lq_l < \tau \leq\ 2c_hq_h \Rightarrow \tau_1 < \tau \leq \tau_2 \equiv \frac{2ac_h[b(2-\gamma)+2c_l]}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2}$. It is easy to check that $\tau_1 < \tau_2$. Case (2.1) is the *corner solution* with $x_l = 1$.

(2.2) $2c_hq_h < \tau \leq\ 2c_lq_l$. The solution to case (2.2) is obtained from the solution to case (2.1) by substituting c_l and m for c_h and $(n-m)$, respectively, leading to the condition $\frac{2ac_h}{(2+(n-1)\gamma)b+2c_h} < \tau \leq \frac{2ac_l[b(2-\gamma)+2c_h]}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2}$, which is impossible because the LHS is greater than the RHS.

Case 3: $\tau > \max\{2c_lq_l,\ 2c_hq_h\}$: This is obtained by applying conditions (11) to both types of firms. By (11), we have $x_l = x_h = 1$, $q_l = \frac{a-\gamma bQ}{b(2-\gamma)+2c_l}$, and $q_h = \frac{a-\gamma bQ}{b(2-\gamma)+2c_h}$, from which we derive $q_l = \frac{a[b(2-\gamma)+2c_h]}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2}$ and $q_h = \frac{a[b(2-\gamma)+2c_l]}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2}$. To ensure a feasible solution for τ, we need $\tau > \max\{2c_lq_l, 2c_hq_h\} \Rightarrow \tau > \tau_2$.

Finally, we will argue that the regulator will never choose a tax rate $\tau > \tau_2$, and hence, Case 3 will never be an outcome of the Taxes mechanism. To see this, observe that for any $\tau > \tau_2$, the pollution is identically 0. This level of pollution is equally accomplished by setting $\tau = \tau_2$, in which case firms choose the candidate solution (2.1). Since the regulator's objective is to choose the minimum τ that accomplishes $P \leq S$, she will never need to pick $\tau > \tau_2$. Hence, in what follows, we will only focus on Cases 1 and 2.1.

3.1.2 Step 2 - Second Order Sufficient Conditions

For the interior solution, we show that the Hessian matrix is negative semidefinite, which proves concavity (Theorem M.C.2 of Mas-Colell et al. 1995, p. 933). Since $\tau = 2c_iq_ix_i$ for the interior solution (by (12)), the Hessian is

$$H = -2\begin{pmatrix} b + c_ix_i^2 & c_iq_ix_i \\ c_iq_ix_i & c_iq_i^2 \end{pmatrix}$$

Let H^π denote the matrix in which both the rows and the columns of H are permuted according to some permutation π of the set $\{1,\ 2\}$, and let ${}_kH^\pi$ denote the $k\times k$ submatrix consisting of only the first k rows and columns of H^π. H is negative semidefinite iff $(-1)^k \det({}_kH^\pi) \geq 0$ for every permutation π, and for every $k \in \{1, 2\}$ (Mas-Colell et al. 1995, Theorem M.D.2, p. 936). These conditions are easily verified for H.

For the corner solution, we use an approach from Luenberger and Ye (2008) to show that any nonzero feasible move away from the candidate solution decreases the objective function. For the corner solution, the only active constraint is $x \leq 1$. Let $g(q,\ x) = x - 1$. The gradient of g at the candidate solution is $\neq 0$, which means that the candidate solution is regular. The gradient of the objective function at this candidate solution is

$$\bigtriangledown \pi_i = \begin{pmatrix} 0 & q_i (\tau - 2c_i q_i) \end{pmatrix}^T.$$

A nonzero feasible direction, $\underline{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ is defined by $(\bigtriangledown g)^T \cdot \underline{v} \leq 0$. This condition is equivalent to $v_2 \leq 0$. If $\underline{v}$ is a feasible direction, then $\underline{v} \cdot \bigtriangledown \pi_i = v_2 q_i (\tau - 2c_i q_i) \leq 0$, because $\tau > 2c_i q_i$ (by equation (12)).

3.1.3 Step 3 - The Subgame Perfect Nash Equilibrium

Next we solve the regulator's problem which is to choose the smallest τ such that $P \leq S$. Recall that equilibrium E_1 is played when $0 \leq \tau \leq \tau_1$, and equilibrium E_2 is played when $\tau_1 < \tau \leq \tau_2$. We prove the following lemma on the total pollution as a function of the tax rate.

Lemma 3 *The total pollution, $P(\tau)$, generated by firms as a function of the tax rate τ is given by:*

$$P(\tau) = \begin{cases} \frac{n(a-\tau)}{b(2+(n-1)\gamma)} - \frac{m\tau}{2c_l} - \frac{(n-m)\tau}{2c_h}, & \text{if } 0 \leq \tau \leq \tau_1 \\ (n-m)\left(\frac{a[b(2-\gamma)+2c_l]-\tau[b(2+(m-1)\gamma)+2c_l]}{b[(2-\gamma)(2+(n-1)\gamma)b+2c_l(2+(n-m-1)\gamma)]} - \frac{\tau}{2c_h}\right), & \text{if } \tau_1 < \tau \leq \tau_2 \end{cases}$$

Further, $P(\tau)$ is continuous and strictly decreasing in τ.

Proof. $P(\tau) = \sum_{i=1}^{n} [q_i(\tau)(1 - x_i(\tau))] = mq_l(1 - x_l) + (n-m) q_h (1 - x_h)$.

When $0 \leq \tau \leq \tau_1$,

$$P(\tau) = \frac{n(a-\tau)}{b(2+(n-1)\gamma)} - \frac{m\tau}{2c_l} - \frac{(n-m)\tau}{2c_h}$$

When $\tau_1 < \tau \leq \tau_2$, only the high-cost firms pollute and

$$\begin{aligned} P(\tau) &= (n-m) q_h (1 - x_h) \\ &= (n-m)\left[\frac{a[b(2-\gamma)+2c_l] - \tau[b(2+(m-1)\gamma)+2c_l]}{b[(2-\gamma)(2+(n-1)\gamma)b + 2c_l(2+(n-m-1)\gamma)]} - \frac{\tau}{2c_h}\right] \end{aligned}$$

It is easy to see that $P(\tau)$ is linear and strictly decreasing in τ in each of the regions $[0,\ \tau_1]$ and $(\tau_1,\ \tau_2]$. Further, algebraic manipulations establish that $P(\tau_1) = \frac{(n-m)a(c_h-c_l)}{c_h[b(2+(n-1)\gamma)+2c_l]}$ such that $P(\tau)$ is continuous at τ_1. □

We now prove that for any pollution target $S \in (0,\ \overline{S})$, there exists a unique tax rate τ^* such that $P(\tau^*) = S$. Note that $P(0) = \frac{na}{b(2+(n-1)\gamma)} = \overline{S}$, and that $P(\tau_2) = 0$, because $x_l = x_h = 1$. Since by Lemma 3, $P(\tau)$ is continuous and strictly decreasing, the result follows.

Finally, the equilibrium tax rate $\tau(S)$ for any pollution target $S \in \left(0,\ \overline{S}\right)$ is obtained by inverting the expressions for $P(\tau)$, which were derived in Lemma 3. Thus,

$$\tau(S) = P^{-1}(S) = \begin{cases} 2c_h \frac{(n-m)a[b(2-\gamma)+2c_l]-b[(2-\gamma)(2+(n-1)\gamma)b+2c_l(2+(n-m-1)\gamma)]S}{(n-m)[[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2]}, & \text{if } 0 < S < \underline{S} \\ 2c_lc_h \frac{na-b(2+(n-1)\gamma)S}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h}, & \text{if } \underline{S} \leq S < \overline{S} \end{cases} \tag{13}$$

When $0 \leq S < \underline{S}$,

$$\begin{aligned} q_l &= \frac{a(2-\gamma)+(n-m)\gamma\tau(S)}{(2-\gamma)(2+(n-1)\gamma)b+2c_l(2+(n-m-1)\gamma)} \\ &= \frac{a[b(2-\gamma)+2c_h]-2\gamma bc_hS}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2} \\ q_h &= \frac{a[b(2-\gamma)+2c_l]-\tau(S)[b(2+(m-1)\gamma)+2c_l]}{b[(2-\gamma)(2+(n-1)\gamma)b+2c_l(2+(n-m-1)\gamma)]} \\ &= \frac{a[b(2-\gamma)+2c_l]+2c_h[b(2+(m-1)\gamma)+2c_l]S\diagup(n-m)}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2} \\ x_l &= 1 \\ x_h &= \frac{\left(\begin{array}{c} a[b(2-\gamma)+2c_l]- \\ b[b(2-\gamma)(2+(n-1)\gamma)+2c_l(2+(n-m-1)\gamma)]S\diagup(n-m) \end{array}\right)}{a[b(2-\gamma)+2c_l]+2c_h[b(2+(n-m-1)\gamma)+2c_l]S\diagup(n-m)} \end{aligned}$$

When $\underline{S} \leq S \leq \overline{S}$,

$$\begin{aligned} q_l &= q_h = \frac{a-\tau(S)}{b(2+(n-1)\gamma)} = \frac{a((n-m)c_l+mc_h)+2c_lc_hS}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h} \\ x_l &= \frac{\tau(S)}{2c_lq_l^*} = \frac{c_h[na-b(2+(n-1)\gamma)S]}{a((n-m)c_l+mc_h)+2c_lc_hS} \\ x_h &= \frac{c_lx_l}{c_h} = \frac{c_l[na-b(2+(n-1)\gamma)S]}{a((n-m)c_l+mc_h)+2c_lc_hS}. \end{aligned}$$

This equilibrium is identical to the equilibrium under Cap-and-Trade (See proof of Theorem 1), and $\forall S,\ q_i^{tax} = q_i^{ct} > 0,\ x_i^{tax} = x_i^{ct}$ and $\tau(S) = r(S)$, where $r(S)$ is the unique Cap-and-Trade equilibrium (market-clearing) price at the emissions cap S. This completes the proof of Part *(i)*.

3.2 Proof of Part *(ii)*

Part *(ii)* follows directly from Part *(i)*.

3.3 Proof of Part *(iii)*

Since the Cap-and-Trade and Tax equilibria are identical, the difference $\pi_i^{ct} - \pi_i^{tax}$ is equal to the difference between trading revenues (or expenses) under Cap-and-Trade and the tax paid to the regulator under Taxes. Thus, $\pi_i^{ct} - \pi_i^{tax} = rt_i + \tau q_i(1-x_i) = \tau(t_i + q_i(1-x_i)) = \tau s_i$ because under Cap-and-Trade the pollution constraint of each firm is binding $\forall S \in \left(0,\ \overline{S}\right)$, which means that $q_i(1-x_i) = s_i - t_i,\ \forall i$. Equivalently, $\Pi^{ct} - \Pi^{tax} = \sum_{i=1}^{n} \tau s_i = \tau S > 0, \forall S \in \left(0,\ \overline{S}\right)$. □

4 Proof of Proposition 1

Proof of Part *(i)*: By Theorem 1, the total output under Cap-and-Trade, $Q^{ct}(S)$, is piecewise linear and continuous in S, by continuity of q_l and q_h at $\underline{S}$. Straightforward algebraic manipulations show that

$$Q^{ct}(S)=\begin{cases}\frac{a[nb(2-\gamma)+2((n-m)c_l+mc_h)]+2c_h[b(2-\gamma)+2c_l]S}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2}, & \text{for } 0<S<\underline{S}\\ n\frac{a((n-m)c_l+mc_h)+2c_lc_hS}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h}, & \text{for } \underline{S}\le S<\overline{S}\end{cases}$$

Hence, $\forall\gamma,\ m,\ n,$

$$\frac{\partial Q^{ct}(S)}{\partial S}=\begin{cases}\frac{2c_h[b(2-\gamma)+2c_l]}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2}>0, & \text{for } 0<S<\underline{S}\\ \frac{2nc_lc_h}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h}>0, & \text{for } \underline{S}\le S<\overline{S}.\ \square\end{cases}$$

Proof of Part *(ii)*: We show that $\frac{\partial^2Q^{ct}(S)}{\partial S\partial\gamma}<0$. Part (i) of Proposition 1 provides the expressions for $\frac{\partial Q^{ct}(S)}{\partial S}$.

For $0<S<\underline{S}$, $\frac{\partial^2Q^{ct}(S)}{\partial S\partial\gamma}=-\frac{2bc_h\left[(n-1)b^2(2-\gamma)^2+4b[(n-1)(2-\gamma)c_l+m(c_h-c_l)]+4c_l((n-m-1)c_l+mc_h)\right]}{[[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2b^2]^2}<0.$

For $\underline{S}\le S<\overline{S}$, $\frac{\partial^2Q^{ct}(S)}{\partial S\partial\gamma}=-\frac{2(2+(n-1)\gamma)nmbc_lc_h(c_h-c_l)}{[(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h]^2}<0.$ □

5 Proof of Proposition 2

Proof of Part *i(a)*: The firms' joint profits $\Pi=\sum_{i=1}^n\pi_i$ are continuous at $\overline{S}$ by continuity of q_i and x_i at $\overline{S}$ for all i. We show that the firms' joint profits under Cap-and-Trade are always decreasing to the left of $\overline{S}$ iff $\gamma>0$ and $n\ge2$. Formally, $\frac{\partial\Pi}{\partial S}\left(\overline{S}\right)<0$ iff $\gamma>0$ and $n\ge2$.

Rearranging the terms in equation (1) of the proof of Theorem 1, and since $\nu_i=r$, we know that in equilibrium, for all i, $a-bq_i-\gamma bQ_{-i}-c_iq_ix_i^2=bq_i+c_iq_ix_i^2+r(1-x_i)$ and that $\sum_{i=1}^nt_i=0$. Furthermore, the pollution constraints are binding for $S\in\left(0,\ \overline{S}\right)$. Hence, the firms joint profits in equilibrium are

$$\begin{aligned}\Pi &= mq_l(a-bq_l-\gamma bQ_{-l})-mc_l(q_lx_l)^2+(n-m)q_h(a-bq_h-\gamma bQ_{-h})-(n-m)c_h(q_hx_h)^2+r\sum_{i=1}^nt_i\\ &= mq_l\left(a-bq_l-\gamma bQ_{-l}-c_lq_lx_l^2\right)+(n-m)q_h\left(a-bq_h-\gamma bQ_{-h}-c_hq_hx_h^2\right)\\ &= mq_l\left(bq_l+c_lq_lx_l^2+r(1-x_l)\right)+(n-m)q_h\left(bq_h+c_hq_hx_h^2+r(1-x_h)\right)\\ &= mq_l^2\left(b+c_lx_l^2\right)+(n-m)q_h\left(b+c_hx_h^2\right)+r(mq_l(1-x_l)+(n-m)q_h(1-x_h))\\ &= mq_l^2\left(b+c_lx_l^2\right)+(n-m)q_h^2\left(b+c_hx_h^2\right)+rS\end{aligned}$$

On the interval $[\underline{S},\ \overline{S})$, $q_l=q_h$, and $r=2c_lq_lx_l=2c_hq_hx_h$. Thus, we can rewrite $\Pi_1(S)=nbq_l^2+\frac{(n-m)c_l+mc_h}{4c_lc_h}r^2+rS$, where Π_1 is the industry profit on the interval $[\underline{S},\ \overline{S})$, i.e., under moderate regulation. Hence, $\frac{\partial\Pi_1(S)}{\partial S}=2nbq_l\frac{\partial q_l}{\partial S}+\frac{(n-m)c_l+mc_h}{2c_lc_h}r\frac{\partial r}{\partial S}+r+\frac{\partial r}{\partial S}S$. Note from the proof of Theorem 1 that q_l and

r are linear in S. Thus, $\frac{\partial \Pi_1(S)}{\partial S}$ is also linear in S on the interval $[\underline{S},\ \overline{S})$. It is straightforward to show that $\frac{\partial \Pi_1(\overline{S})}{\partial S} = -\frac{2n(n-1)\gamma a c_l c_h}{(2+(n-1)\gamma)[2nc_lc_h+(2+(n-1)\gamma)b((n-m)c_l+mc_h)]} < 0$ iff $\gamma > 0$ and $n \geq 2$, where $\frac{\partial \Pi_1(\overline{S})}{\partial S}$ is the limit of $\frac{\partial \Pi_1(S)}{\partial S}$ as $S \longrightarrow \overline{S}$ with $S < \overline{S}$. Hence, the firms' joint profits are strictly decreasing to the left of $\overline{S}$ under Cap-and-Trade. By continuity, $\exists \hat{S} < \overline{S}$ such that $\Pi|_{S\in(\hat{S},\overline{S})} > \Pi|_{S\geq\overline{S}}$, $\forall\ \gamma > 0$.

Proof of Part *i(b)*: We begin by showing that, on the relevant range, the industry profit under moderate regulation, Π_1, increases as c_l increases. Mathematically, $\exists S_l < \overline{S}$ such that $\left.\frac{\partial \Pi_1}{\partial c_l}\right|_{S\in(S_l,\overline{S})} > 0$. From the proof of Part *i(a)*, we have $\Pi_1(S) = nbq_l^2 + \frac{(n-m)c_l+mc_h}{4c_lc_h}r^2 + rS$. Thus, $\frac{\partial \Pi_1}{\partial c_l} = 2nbq_l\frac{\partial q_l}{\partial c_l} - \frac{m}{4c_l^2}r^2 + \frac{(n-m)c_l+mc_h}{2c_lc_h}r\frac{\partial r}{\partial c_l} + \frac{\partial r}{\partial c_l}S$. From Theorem 1, we have the expressions for q_l and r from which we can calculate $\frac{\partial q_l}{\partial c_l}$ and $\frac{\partial r}{\partial c_l}$, and derive $\frac{\partial \Pi_1}{\partial c_l}$. Algebraic calculations are long but straightforward (and were verified using Mathematica). For brevity, intermediate calculations are omitted. Let $C = (n-m)\,c_l + mc_h$. We get

$$\frac{\partial \Pi_1}{\partial c_l} = \frac{mc_h^2(na-b(2+(n-1)\gamma)S)\left[\left(2nc_lc_h(2+3(n-1)\gamma)+(2+(n-1)\gamma)^2bC\right)bS-na(2nc_lc_h+(2-(n-1)\gamma)bC)\right]}{[2nc_lc_h+(2+(n-1)\gamma)bC]^3}$$

It is easy to see that at $S = \overline{S} = \frac{na}{b(2+(n-1)\gamma)}$, the expression $na - b\,(2+(n-1)\,\gamma)\,S$ in the first parenthesis of the numerator of $\frac{\partial \Pi_1}{\partial c_l}$ equals 0, so that $\left.\frac{\partial \Pi_1}{\partial c_l}\right|_{S=\overline{S}} = 0$. Moreover, $\frac{\partial \Pi_1}{\partial c_l}$ is quadratic in S. Thus, $\frac{\partial}{\partial S}\left(\frac{\partial \Pi_1}{\partial c_l}\right)$ is linear in S. It is straightforward to show that $\left.\frac{\partial}{\partial S}\left(\frac{\partial \Pi_1}{\partial c_l}\right)\right|_{S=\overline{S}} = -\frac{2n(n-1)mab\gamma c_h^2}{[2nc_lc_h+(2+(n-1)\gamma)bC]^2} < 0$ for all $n \geq 2$ and $m > 0$. Hence, $\frac{\partial \Pi_1}{\partial c_l}$ is strictly positive to the left of $\overline{S}$ under Cap-and-Trade (and equal to 0 at $S = \overline{S}$). By continuity, $\exists S_l < \overline{S}$ such that $\left.\frac{\partial \Pi_1}{\partial c_l}\right|_{S\in(S_l,\overline{S})} > 0$.

The proof for $\frac{\partial \Pi_1}{\partial c_h}$ follows the exact same logic. We get $\frac{\partial \Pi_1}{\partial c_h} = \frac{(n-m)c_l^2(na-b(2+(n-1)\gamma)S)\left[\left(2nc_lc_h(2+3(n-1)\gamma)+(2+(n-1)\gamma)^2bC\right)bS-na(2nc_lc_h+(2-(n-1)\gamma)bC)\right]}{[2nc_lc_h+(2+(n-1)\gamma)bC]^3}$, $\left.\frac{\partial \Pi_1}{\partial c_h}\right|_{S=\overline{S}} = 0$ and $\left.\frac{\partial}{\partial S}\left(\frac{\partial \Pi_1}{\partial c_h}\right)\right|_{S=\overline{S}} = -\frac{2n(n-1)(n-m)ab\gamma c_l^2}{[2nc_lc_h+(2+(n-1)\gamma)bC]^2} < 0$ for all $n \geq 2$ and $m < n$. By continuity, $\exists S_h < \overline{S}$ such that $\left.\frac{\partial \Pi_1}{\partial c_h}\right|_{S\in(S_h,\overline{S})} > 0$.

The proof for $\frac{\partial \Pi_1}{\partial m}$ follows the exact same structure again, except that we need to show that $\frac{\partial \Pi_1}{\partial m} < 0$ on the relevant range. We get $\frac{\partial \Pi_1}{\partial m} = \frac{c_lc_h(c_h-c_l)(na-b(2+(n-1)\gamma)S)\left[na(2nc_lc_h+(2-(n-1)\gamma)bC)-\left(2nc_lc_h(2+3(n-1)\gamma)+(2+(n-1)\gamma)^2bC\right)bS\right]}{[2nc_lc_h+(2+(n-1)\gamma)bC]^3}$, $\left.\frac{\partial \Pi_1}{\partial m}\right|_{S=\overline{S}} = 0$ and $\left.\frac{\partial}{\partial S}\left(\frac{\partial \Pi_1}{\partial m}\right)\right|_{S=\overline{S}} = \frac{2n(n-1)ab\gamma c_lc_h(c_h-c_l)}{[2nc_lc_h+(2+(n-1)\gamma)bC]^2} > 0$ for all $n \geq 2$. By continuity, $\exists S_m < \overline{S}$ such that $\left.\frac{\partial \Pi_1}{\partial m}\right|_{S\in(S_m,\overline{S})} > 0$.

Proof of Part *(ii)*: Let $\Pi_0(S)$ denote the firms' joint profits under Cap-and-Trade when $\gamma = 0$. We show that $\Pi_0(S)$ is strictly increasing on $\left(0,\ \overline{S}\right)$.

On $(0,\ \underline{S})$, $\Pi_0(S) = -\frac{bc_hS^2-(n-m)ac_hS}{(n-m)(b+c_h)} + \frac{a^2[nb+(n-m)c_l+mc_h]}{4(b+c_l)(b+c_h)}$, which has a unique maximum at $\frac{(n-m)a}{2b} > \underline{S}$. Thus, $\Pi_0(S)$ is strictly increasing on $(0,\ \underline{S})$.

On $[\underline{S},\ \overline{S})$, $\Pi_0(S) = \frac{-4bc_lc_hS^2+4nac_lc_hS+na^2[(n-m)c_l+mc_h]}{4[b[(n-m)c_l+mc_h]+nc_lc_h]}$, which has a unique maximum at $\overline{S} = \frac{na}{2b}$. Thus, $\Pi_0(S)$ is strictly increasing on $[\underline{S},\ \overline{S})$. $\square$

6 Proof of Proposition 3

Proof of Part *(i)*: The augmented consumer surplus is continuous at $\overline{S}$ by continuity of q_i and x_i at $\overline{S}$ for all i. We show that the augmented consumer surplus under Cap-and-Trade is always decreasing to the left of $\overline{S}$ iff $d > D_\gamma$. The Consumer Surplus is $CS = \sum_{i=1}^{n} \int_0^{q_i} [d_i(q) - p_i(q_i)]\, dq$, where $d_i(q) = a - \gamma b Q_{-i} - bq$ is the demand curve for product i, and $p_i(q_i) = a - \gamma b Q_{-i} - bq_i$, the price charged for product i. Thus, $CS = \sum_{i=1}^{n} \int_0^{q_i} b(q_i - q)\, dq = \frac{b}{2} \sum_{i=1}^{n} q_i^2$. Hence, the augmented consumer surplus is $ACS = CS - D = \frac{b}{2}\left(mq_l^2 + (n-m)\, q_h^2\right) - dS^2$, because the pollution constraints are binding. On the interval $\left[\underline{S},\ \overline{S}\right]$, $q_l = q_h$, which implies that $ACS = n\frac{b}{2}q_l^2 - dS^2$. Hence $ACS'(S) = nbq_l q_l' - 2dS$. Note from the proof of Theorem 1 that q_l is linear in S. Thus, ACS' is also linear in S. It is straightforward to show that $ACS'\left(\overline{S}\right) = \frac{2na}{b(2+(n-1)\gamma)} \left[\frac{bc_lc_h}{2nc_lc_h + (2+(n-1)\gamma)b((n-m)c_l + mc_h)} - d\right] = \frac{2na}{b(2+(n-1)\gamma)}\left[D_\gamma - d\right] < 0$ iff $d > D_\gamma$. By continuity, $\exists \hat{S} < \overline{S}$ such that $ACS|_{S\in(\hat{S},\overline{S})} > ACS|_{S\geq\overline{S}}$, $\forall\ \gamma$ and $d > D_\gamma$.

Proof of Part *(ii)*: We perform a similar analysis for welfare to the left of $\overline{S}$. Recall that $W = \Pi + ACS$. The social welfare is continuous at $\overline{S}$ by continuity of Π and ACS at $\overline{S}$. Using the proof of Proposition 2 and Part *(i)* above, on the interval $\left[\underline{S},\ \overline{S}\right]$, we have $W'\left(\overline{S}\right) = \Pi'\left(\overline{S}\right) + ACS'\left(\overline{S}\right) < 0 \iff d > \frac{bc_lc_h(1-(n-1)\gamma)}{2nc_lc_h + (2+(n-1)\gamma)b((n-m)c_l + mc_h)} = d_\gamma$. By continuity, $\exists \hat{S} < \overline{S}$ such that $W|_{S\in(\hat{S},\overline{S})} > W|_{S\geq\overline{S}}$, $\forall\ \gamma$ and $d > d_\gamma$.

This completes the proof of Proposition 3. □

7 Proof of Corollary 1

Proof of Part *(i)*: If $\gamma \geq \frac{1}{n-1}$, or equivalently $n \geq 1 + \frac{1}{\gamma}$, then $d_\gamma \leq 0$. Hence, by Part *(ii)* of Proposition 3, welfare improves for all $d > 0$ when $\gamma \geq \frac{1}{n-1}$.

Proof of Part *(ii)*: Recall that $D_\gamma = \frac{bc_lc_h}{2nc_lc_h + (2+(n-1)\gamma)b((n-m)c_l + mc_h)}$ and $d_\gamma = \frac{bc_lc_h(1-(n-1)\gamma)}{2nc_lc_h + (2+(n-1)\gamma)b((n-m)c_l + mc_h)}$, and note that D_γ and d_γ are strictly decreasing in γ for $\gamma \in [0, 1]$. Furthermore, at $\gamma = 0$, $D_\gamma = d_\gamma = \frac{bc_lc_h}{2[nc_lc_h + b((n-m)c_l + mc_h)]} \equiv d_0$. Hence, $\forall\ d > d_0$, $d > D_\gamma$ and $d > d_\gamma$, and the result follows directly from Proposition 3. □

8 Proof of Proposition 4

Proof of Part *(i)*: We show that $\frac{\partial^2 Q}{\partial m \partial S} < 0$, which means that increasing the number of low-abatement cost firms mitigates the rate of output reduction in response to regulation. By Proposition 1, we have

$$\frac{\partial Q^{ct}(S)}{\partial S} = \begin{cases} \frac{2c_h[b(2-\gamma)+2c_l]}{[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h] - m(n-m)\gamma^2 b^2} > 0, & \text{for } 0 < S < \underline{S} \\ \frac{2nc_lc_h}{(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_lc_h} > 0, & \text{for } \underline{S} \leq S < \overline{S} \end{cases}$$

Thus,

$$\frac{\partial^2 Q^{ct}(S)}{\partial m \partial S} = \begin{cases} -\frac{4\gamma b c_h (c_h - c_l)(b(2-\gamma)+2c_l)}{[[b(2+(m-1)\gamma)+2c_l][b(2+(n-m-1)\gamma)+2c_h]-m(n-m)\gamma^2 b^2]^2} < 0, & \text{for } 0 < S < \underline{S} \\ -\frac{2nbc_l c_h (c_h - c_l)(2+(n-1)\gamma)}{[(2+(n-1)\gamma)b((n-m)c_l+mc_h)+2nc_l c_h]^2} < 0, & \text{for } \underline{S} \leq S < \overline{S}. \end{cases}$$

Proof of Part *(ii)*: We show that D_γ and d_0 are decreasing in m for any γ, and that d_γ is decreasing in m for $\gamma < \frac{1}{n-1}$. We know by Part *(i)* of Corollary 1 that when $\gamma \geq \frac{1}{n-1}$, $d_\gamma \leq 0$ and Region $R2$ in which moderate regulation improves welfare cannot expand.

From Proposition 3, recall that

$$\begin{aligned} D_\gamma &= \frac{bc_l c_h}{2nc_l c_h + (2+(n-1)\gamma) b((n-m)c_l + mc_h)} \\ d_\gamma &= \frac{bc_l c_h (1-(n-1)\gamma)}{2nc_l c_h + (2+(n-1)\gamma) b((n-m)c_l + mc_h)} \\ d_0 &= \frac{bc_l c_h}{2[nc_l c_h + b((n-m)c_l + mc_h)]} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial D_\gamma}{\partial m} &= -\frac{b^2 c_l c_h (c_h - c_l)(2+(n-1)\gamma)}{[2nc_l c_h + (2+(n-1)\gamma) b((n-m)c_l + mc_h)]^2} < 0 \\ \frac{\partial d_0}{\partial m} &= -\frac{b^2 c_l c_h (c_h - c_l)}{2[nc_l c_h + b((n-m)c_l + mc_h)]^2} < 0 \end{aligned}$$

Furthermore,

$$\frac{\partial d_\gamma}{\partial m} = -\frac{b^2 c_l c_h (c_h - c_l)(2+(n-1)\gamma)(1-(n-1)\gamma)}{[2nc_l c_h + (2+(n-1)\gamma) b((n-m)c_l + mc_h)]^2} < 0 \text{ for } \gamma < \frac{1}{n-1}.$$

Proof of Part *(iii)*: Recall that $\underline{S} = \frac{(n-m)a(c_h - c_l)}{c_h[b(2+(n-1)\gamma)+2c_l]}$, which is clearly decreasing in m and goes to 0 as $m \to n$. □

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