Electronic Companion to "Multilocation Newsvendor Problem: Centralization and Inventory Pooling"

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Appendix A: Proofs

Proof of Proposition 1

Part (a). First observe that

$$\Pi^c = \max_{\mathbf{y}, v\in\mathbb{R}} \left\{ v - \frac{1}{\eta}\mathbb{E}[(v - \sum_{j=1}^n \pi_j(y_j, D_j))^+] \right\},$$
$$= \max_{\mathbf{y}} \left\{ -\sum_{j=1}^n c y_j + \max_{u\in\mathbb{R}}\{u - \frac{1}{\eta}\mathbb{E}[(u - \sum_{j=1}^n p(y_j \wedge D_j))^+]\} \right\}.$$

Let

$$f(\mathbf{y}, u) := u - \frac{1}{\eta}\mathbb{E}[(u - \sum_{j=1}^n p(y_j \wedge D_j))^+],$$

and

$$h(\mathbf{y}) := -\sum_{j=1}^n c y_j + \max_{u\in\mathbb{R}}\{f(\mathbf{y}, u)\},\ u(\mathbf{y}) := \arg\max_{u\in\mathbb{R}}\{f(\mathbf{y}, u)\}.$$

The function $f(\mathbf{y}, u)$ can be shown to be jointly concave in $(\mathbf{y}, u)$. Since the term

$$(u - \sum_{j=1}^n p(y_j \wedge D_j))^+ = (u - p\sum_{j=1}^n y_j + \sum_{j=1}^n p(y_j - D_j)^+)^+$$

is a composition of an increasing convex function and a convex function, it is jointly convex in $(\mathbf{y}, u)$. Taking expectation with respect to $\boldsymbol{D}$ preserves convexity and $\mathbb{E}[(u - \sum_{j=1}^n p(y_j \wedge D_j))^+]$ is convex as well. Hence, $h(\mathbf{y})$ is concave in $\mathbf{y}$ as the concavity us preserved under maximization over a convex set (see *e.g.*, Proposition 2.1.15 in Simchi-Levi et al. 2014). We first show that for $u < \sum_{j=1}^n p y_j$, $f(\mathbf{y}, u)$ is differentiable in u and $y_i, 1 \le i \le n$ respectively. We prove the differentiability with respect to u here, and the proof for y_i follows similarly. By concavity, the right and left partial derivatives of $f(\mathbf{y}, u)$, denoted as $\partial_u^+ f(\mathbf{y}, u)$ and $\partial_u^- f(\mathbf{y}, u)$, exist. The right and left derivatives of $(u - \sum_{j=1}^n p(y_j \wedge D_j))^+$ with respect to u are $\mathbf{1}_{\{u \ge \sum_{j=1}^n p(y_j \wedge D_j)\}}$ and $\mathbf{1}_{\{u > \sum_{j=1}^n p(y_j \wedge D_j)\}}$, where $\mathbf{1}_A$ is the indicator function of the set A. Applying differentiation lemma (see, for instance, Theorem 6.28 in Klenke 2013) with respect to the right and left derivatives, we have

$$\partial_u^+ f(\mathbf{y}, u) = 1 - \frac{1}{\eta}\mathbb{P}\left(u \ge \sum_{j=1}^n p(y_j \wedge D_j)\right),\ \partial_u^- f(\mathbf{y}, u) = 1 - \frac{1}{\eta}\mathbb{P}\left(u > \sum_{j=1}^n p(y_j \wedge D_j)\right).$$

Since the set

$$\left\{(D_1, ..., D_n) \middle| u = \sum_{j=1}^n p(y_j \wedge D_j)\right\} = \left\{(D_1, ..., D_n) \middle| \sum_{j=1}^n p(y_j - D_j)^+ = \sum_{j=1}^n p y_j - u\right\}$$

has zero Lebesgue measure in $\mathbb{R}^n$ when $u < \sum_{j=1}^n p y_j$ and the joint probability density of $(D_1, ..., D_n)$ is well defined, we have $\mathbb{P}\left(u = \sum_{j=1}^n p(y_j \wedge D_j)\right) = 0$. Hence, $\partial_u^+ f(\mathbf{y}, u) = \partial_u^- f(\mathbf{y}, u) = \frac{\partial f(\mathbf{y}, u)}{\partial u}$ for $u < \sum_{j=1}^n p y_j$. Note here that if $u = \sum_{j=1}^n p y_j$, then the set $\left\{(D_1, ..., D_n) \middle| \sum_{j=1}^n p(y_j - D_j)^+ = 0\right\} =$

$\{(D_1,...,D_n)\,|D_j \ge y_j, 1\le j\le n\}$ no longer has zero Lebesgue measure and its probability of occurring could be positive.

To show $y_i^{\rm d}\le y_i^{\rm c}$, by concavity of $h(\mathbf{y})$, it is sufficient to show that $\frac{\partial h(\mathbf{y})}{\partial y_i}>0$ for any $y_i<y_i^{\rm d}$ (at the point where $h(\mathbf{y})$ is non-differentiable, we interpret $\frac{\partial h(\mathbf{y})}{\partial y_i}$ as left derivative, which always exists). Following the discussion above, we can write $\frac{\partial f(\mathbf{y},u)}{\partial u}$ (when $u=\sum_{j=1}^n py_j$, $\frac{\partial f(\mathbf{y},u)}{\partial u}$ is interpreted as the right derivative) as

$$\frac{\partial f(\mathbf{y},u)}{\partial u}=\begin{cases}1-\frac{1}{\eta}\mathbb{P}\left(\sum_{j=1}^n p(y_j-D_j)^+>\sum_{j=1}^n py_j-u\right), & \text{if } \sum_{j=1}^n py_j>u;\\ 1-\frac{1}{\eta}, & \text{if } \sum_{j=1}^n py_j\le u.\end{cases}$$

Note that $1-1/\eta<0$, hence $f(y,u)$ is decreasing in u when $u\ge\sum_{j=1}^n py_j$. The optimal solution $u(\mathbf{y})$ then satisfies

$$\begin{cases}\mathbb{P}\left(\sum_{j=1}^n p(y_j-D_j)^+>\sum_{j=1}^n py_j-u(\mathbf{y})\right)=\eta, & \text{if } \sum_{j=1}^n py_j>u(\mathbf{y});\\ u(\mathbf{y})=\sum_{j=1}^n py_j, & \text{otherwise.}\end{cases}$$

If $u(\mathbf{y})<\sum_{j=1}^n py_j$, then by differentiability of $f(\mathbf{y},u)$ in y_i and the concavity of $h(\mathbf{y})$ in y_i, we have $h(\mathbf{y})$ is differentiable in y_i (see Milgrom and Segal 2002). By the envelope theorem

$$\begin{aligned}\frac{\partial h(\mathbf{y})}{\partial y_i}&=-c+\left.\frac{\partial f(\mathbf{y},u)}{\partial y_i}\right|_{u=u(\mathbf{y})}\\ &=-c+\frac{p}{\eta}\mathbb{P}\left(\{\sum_{j=1}^n p(y_j\wedge D_j)<u(\mathbf{y})\}\cap\{D_i>y_i\}\right).\end{aligned}$$

Note that for $y_i<y_i^{\rm d}$

$$\begin{aligned}&\mathbb{P}\left(\{\sum_{j=1}^n p(y_j\wedge D_j)<u(\mathbf{y})\}\cap\{D_i>y_i\}\right)\\ =&\mathbb{P}\left(\sum_{j=1}^n p(y_j\wedge D_j)<u(\mathbf{y})\right)-\mathbb{P}\left(\{\sum_{j=1}^n p(y_j\wedge D_j)<u(\mathbf{y})\}\cap\{D_i\le y_i\}\right)\\ =&\mathbb{P}\left(\sum_{j=1}^n p(y_j-D_j)^+>\sum_{j=1}^n py_j-u(\mathbf{y})\right)-\mathbb{P}\left(\{\sum_{j=1}^n p(y_j\wedge D_j)<u(\mathbf{y})\}\cap\{D_i\le y_i\}\right)\\ \ge&\eta-\mathbb{P}(D_i\le y_i)>\eta-\eta r=\eta\frac{c}{p}.\end{aligned}$$

The last inequality is due to $\mathbb{P}(D_i\le y_i)<\mathbb{P}(D_i\le y_i^{\rm d})=\hat{\eta}_i r\le\eta r$ for $y_i<y_i^{\rm d}$. Therefore, for $y_i<y_i^{\rm d}$,

$$\frac{\partial h(\mathbf{y})}{\partial y_i}>-c+\frac{p}{\eta}\eta\frac{c}{p}=0.$$

If $u(\mathbf{y})=\sum_{j=1}^n py_j$, then

$$f(\mathbf{y},u(\mathbf{y}))=\sum_{j=1}^n py_j-\frac{1}{\eta}\mathbb{E}[(\sum_{j=1}^n py_j-\sum_{j=1}^n p(y_j\wedge D_j))^+]=\sum_{j=1}^n py_j-\frac{1}{\eta}\mathbb{E}[\sum_{j=1}^n p(y_j-D_j)^+].$$

Consequently, for $y_i<y_i^{\rm d}$,

$$\begin{aligned}\frac{\partial h(\mathbf{y})}{\partial y_i}&=-c+p(1-\frac{1}{\eta}\mathbb{P}(D_i\le y_i))\\ &>-c+p(1-\frac{1}{\eta}\eta r)=0.\end{aligned}$$

Similarly, if $\hat{\eta}_i < \eta$, we have if $u(\mathbf{y}) < \sum_{j=1}^n py_j$, then since $\mathbb{P}(D_i \le y_i^{\mathrm{d}}) = \hat{\eta}_i r < \eta r$, we have $\frac{\partial h(\mathbf{y})}{\partial y_i} > 0$ for $y_i = y_i^{\mathrm{d}}$. Similarly, if $u(\mathbf{y}) = \sum_{j=1}^n py_j$, we have $\frac{\partial h(\mathbf{y})}{\partial y_i} > 0$ for $y_i = y_i^{\mathrm{d}}$. That is, y_i^{d} is not optimal, and we must have $y_i^{\mathrm{c}} > y_i^{\mathrm{d}}$ and $\Pi^{\mathrm{d}} < \Pi^{\mathrm{c}}$.

Part (b). Following the proof in part (a), we have already shown that if $u(\mathbf{y}) = \sum_{j=1}^n py_j$, then $h(\mathbf{y}) = \sum_{j=1}^n \{(p-c)y_j - \mathbb{E}[p(y_j - D_j)^+]/\eta\}$. By the first-order condition, it is easy to see that $\mathbf{y}^{\mathrm{d}} = (y_1^{\mathrm{d}}, ..., y_n^{\mathrm{d}})$ maximizes $h(\mathbf{y})$ in this case. It remains to show that indeed we have $u(\mathbf{y}^{\mathrm{d}}) = \sum_{j=1}^n py_j^{\mathrm{d}}$. To see this, it is sufficient to show that $\lim_{u \uparrow \sum_{j=1}^n py_j^{\mathrm{d}}} \frac{\partial f(\mathbf{y}^{\mathrm{d}}, u)}{\partial u} \ge 0$. Equivalently, we need to show that

$$\mathbb{P}\left(\sum_{j=1}^n (y_j^{\mathrm{d}} - D_j)^+ > 0\right) \le \eta.$$

Observe that

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^n (y_j^{\mathrm{d}} - D_j)^+ > 0\right) =& 1 - \mathbb{P}\left(\sum_{j=1}^n (y_j^{\mathrm{d}} - D_j)^+ \le 0\right), \\
=& 1 - \mathbb{P}(D_1 \ge y_1^{\mathrm{d}}, D_2 \ge y_2^{\mathrm{d}}, ..., D_n \ge y_n^{\mathrm{d}}) \\
=& 1 - \prod_{j=1}^n \mathbb{P}(D_j \ge y_j^{\mathrm{d}}) \\
=& 1 - (1 - F_j(y_j^{\mathrm{d}}))^n = 1 - (1 - \eta r)^n.
\end{aligned}$$

The inequality (6) then guarantees $\mathbb{P}(\sum_{j=1}^n (y_j^{\mathrm{d}} - D_j)^+ > 0) \le \eta$. Therefore, $\mathbf{y}^{\mathrm{d}}$ is optimal to problem (4) and hence $\mathbf{y}^{\mathrm{d}} = \mathbf{y}^{\mathrm{c}}, \Pi^{\mathrm{c}} = \Pi^{\mathrm{d}}$.

Proof of Proposition 2

Let $g_i(\mathbf{y}) = \rho_i(\pi(y_i, D_i) + l_i^{\mathrm{w}}(\mathbf{y}, \mathbf{D}))$ be the payoff function for store i. With a slight abuse of notation, for fixed $y_{-i} := (y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$, we will write $g_i(y_i, y_{-i})$ simply as $g_i(y_i)$ and sometimes we will use $g_i(y_i, \hat{\eta}_i)$ to further emphasize the dependence on $\hat{\eta}_i$. By definition,

$$\begin{aligned}
& g_i(\mathbf{y}) \\
=& \mathrm{CVaR}_{\hat{\eta}_i}\left((1-w)\pi(y_i, D_i) + w\gamma_i \pi(\sum_{j\in\mathcal{N}} y_j, \sum_{j\in\mathcal{N}} D_j)\right) \\
=& (1-w)(p-c)y_i + w\gamma_i(p-c)\sum_{j\in\mathcal{N}} y_j + p\mathrm{CVaR}_{\hat{\eta}_i}\left(-[(1-w)(y_i - D_i)^+ + w\gamma_i(\sum_{j\in\mathcal{N}} y_j - \sum_{j\in\mathcal{N}} D_j)^+]\right) \\
=& (1-w)(p-c)y_i + w\gamma_i(p-c)\sum_{j\in\mathcal{N}} y_j \\
& + p\max_v \left\{ v - \frac{1}{\hat{\eta}_i}\mathbb{E}\left[\left(v + (1-w)(y_i - D_i)^+ + w\gamma_i(\sum_{j\in\mathcal{N}} y_j - \sum_{j\in\mathcal{N}} D_j)^+\right)^+\right]\right\}
\end{aligned}$$

Let

$$f_i(\mathbf{y}, v) = v - \frac{1}{\hat{\eta}_i}\mathbb{E}\left[\left(v + (1-w)(y_i - D_i)^+ + w\gamma_i(\sum_{j\in\mathcal{N}} y_j - \sum_{j\in\mathcal{N}} D_j)^+\right)^+\right].$$

Similarly, we will write $f_i(y_i, y_{-i}, v)$ simply as $f_i(y_i, v)$ and use $f_i(y_i, v, \hat{\eta}_i)$ to emphasize the dependence on $\hat{\eta}_i$. Clearly, $f_i(\mathbf{y}, v)$ is continuous and concave in $(\mathbf{y}, v)$. By Berge's Maximum Theorem (see, for example, Ok

2007) and Proposition 2.1.15 in Simchi-Levi et al. (2014), $\max_v f_i(\mathbf{y}, v)$ and hence $g_i(\mathbf{y})$ is also continuous and concave in $\mathbf{y}$. Furthermore, for fixed $y_{-i} := (y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$, as $y_i \to +\infty$, we have $g_i(y_i) \to -\infty$. The existence of Nash equilibrium then follows from Corollary 4.2 in Basar and Jan Olsder (1999).

When $w = 0$, the payoff function of store i becomes $\rho_i(\pi(y_i, D_i))$. Since all store decisions are decoupled, the Nash equilibrium is simply $\mathbf{y}^{\mathrm{NE}} = \mathbf{y}^{\mathrm{d}} = (y_1^{\mathrm{d}}, ..., y_n^{\mathrm{d}})$.

When $w = 1$,

$$\begin{aligned} y_i^{\mathrm{NE}} &= \arg\max_{y_i} \rho_i \left(\gamma_i \pi(y_i + \sum_{j \neq i} y_j^{\mathrm{NE}}, \sum_{i=1}^n D_i) \right) \\ &= \arg\max_{y_i} \gamma_i \mathrm{CVaR}_{\hat{\eta}_i} \left[\pi(y_i + \sum_{j \neq i} y_j^{\mathrm{NE}}, \sum_{i=1}^n D_i) \right]. \end{aligned}$$

The best response function of player i is:

$$y_i(y_{-i}) = (F_N^{-1}(\hat{\eta}_i r) - \sum_{j \neq i} y_j)^+.$$

Without loss of generality, we assume that $\hat{\eta}_1 \leq \hat{\eta}_2 \leq ... \leq \hat{\eta}_n$. Then any Nash equilibrium must satisfy $\sum_{i \in \mathcal{N}} y_i^{\mathrm{NE}} = F_N^{-1}(\hat{\eta}_n r)$. In particular, $\mathbf{y} = (0, ..., 0, F_N^{-1}(\hat{\eta}_n r))$ is a Nash equilibrium. To see this, suppose in some Nash equilibrium, $y_i^{\mathrm{NE}} > 0$ for some $\hat{\eta}_i$ such that $F_N^{-1}(\hat{\eta}_i r) < F_N^{-1}(\hat{\eta}_n r)$. Then, by the best response function, we know $y_i^{\mathrm{NE}} = F_N^{-1}(\hat{\eta}_i r) - \sum_{j \neq i} y_j^{\mathrm{NE}}$ or equivalently $\sum_{j \in \mathcal{N}} y_j^{\mathrm{NE}} = F_N^{-1}(\hat{\eta}_i r)$. Since $y_n^{\mathrm{NE}} \geq 0$, this implies

$$y_n^{\mathrm{NE}} = F_N^{-1}(\hat{\eta}_i r) - \sum_{j < n} y_j^{\mathrm{NE}} \geq 0.$$

Hence, $F_N^{-1}(\hat{\eta}_n r) - \sum_{j<n} y_j^{\mathrm{NE}} > 0$. Then, from the best response function, we know that $y_n(y_{-n}^{\mathrm{NE}}) = (F_N^{-1}(\hat{\eta}_n r) - \sum_{j<n} y_j^{\mathrm{NE}})^+ = F_N^{-1}(\hat{\eta}_n r) - \sum_{j<n} y_j^{\mathrm{NE}}$. That is, $\sum_{j \in \mathcal{N}} y_j^{\mathrm{NE}} = F_N^{-1}(\hat{\eta}_n r)$, which contradicts with $\sum_{j \in \mathcal{N}} y_j^{\mathrm{NE}} = F_N^{-1}(\hat{\eta}_i r) < F_N^{-1}(\hat{\eta}_n r)$. Therefore, we must have $y_i^{\mathrm{NE}} = 0$ for any $\hat{\eta}_i$ such that $F_N^{-1}(\hat{\eta}_i r) < F_N^{-1}(\hat{\eta}_n r)$, which implies that $\sum_{j \in \mathcal{N}} y_j^{\mathrm{NE}} = F_N^{-1}(\hat{\eta}_n r)$. In particular, if $F_N^{-1}(\hat{\eta}_{n-1}) < F_N^{-1}(\hat{\eta}_n)$, then $\mathbf{y} = (0, ..., 0, F_N^{-1}(\hat{\eta}_n r))$ is the unique NE.

When $0 < w < 1$, we show that there exists an increasing function $\bar{y}_i(\hat{\eta}_i)$ with $\lim_{\hat{\eta}_i \to 0} \bar{y}_i(\hat{\eta}_i) = 0$ such that for any fixed y_{-i} and $y_i \geq \bar{y}_i(\hat{\eta}_i)$, we have

$$\frac{1}{1 - w + w\gamma_i} \left. \frac{\partial f_i(y_i, v, \hat{\eta}_i)}{\partial y_i} \right|_{v = v^*(y_i, \hat{\eta}_i)} = -1$$

where $v^*(y_i, \hat{\eta}_i) = \arg\max_v f_i(y_i, v, \hat{\eta}_i)$. It then follows that for any $y_i \geq \bar{y}_i(\hat{\eta}_i)$

$$\begin{aligned} \frac{1}{1 - w + w\gamma_i} \frac{\partial g_i(y_i, \hat{\eta}_i)}{\partial y_i} &= (p - c) + \frac{p}{1 - w + w\gamma_i} \left. \frac{\partial f_i(y_i, v, \hat{\eta}_i)}{\partial y_i} \right|_{v = v^*(y_i, \hat{\eta}_i)} \\ &= (p - c) - p = -c < 0, \end{aligned}$$

which implies the best response $y_i(y_{-i}) \leq \bar{y}_i(\hat{\eta}_i)$ for any y_{-i} and in particular, $y_i^{\mathrm{NE}} = y_i(y_{-i}^{\mathrm{NE}}) \leq \bar{y}_i(\hat{\eta}_i)$.

To show our claim, first note that

$$\frac{\partial f_i(y_i, v, \hat{\eta}_i)}{\partial v} = \begin{cases} 1 - \dfrac{1}{\hat{\eta}_i}, & \text{if } v > 0, \\ 1 - \dfrac{1}{\hat{\eta}_i} \mathbb{P}\left(v + (1 - w)(y_i - D_i)^+ + w\gamma_i (\sum_{j \in \mathcal{N}} y_j - \sum_{j \in \mathcal{N}} D_j)^+ > 0 \right), & \text{if } v \leq 0. \end{cases}$$

It follows that

$$v^*(y_i,\hat{\eta}_i)=\begin{cases}0, \text{ if } \mathbb{P}\left((1-w)(y_i-D_i)^+ + w\gamma_i(\sum_{j\in\mathcal{N}} y_j - \sum_{j\in\mathcal{N}} D_j)^+ > 0\right)\le \hat{\eta}_i,\\ \left\{v \,\middle|\, \mathbb{P}\left(v+(1-w)(y_i-D_i)^+ + w\gamma_i(\sum_{j\in\mathcal{N}} y_j - \sum_{j\in\mathcal{N}} D_j)^+ > 0\right)=\hat{\eta}_i\right\}, \text{ otherwise.}\end{cases}$$

Denote $Y_{-i}=\sum_{j\neq i} y_j$ and

$$y_i^0=\left\{y_i \,\middle|\, \mathbb{P}\left((1-w)(y_i-D_i)^+ + w\gamma_i(y_i+Y_{-i}-D_N)^+ > 0\right)=\hat{\eta}_i\right\},$$

then

$$v^*(y_i,\hat{\eta}_i)=\begin{cases}0, \text{ if } y_i\le y_i^0,\\ \left\{v \,\middle|\, \mathbb{P}\left(v+(1-w)(y_i-D_i)^+ + w\gamma_i(y_i+Y_{-i}-D_N)^+ > 0\right)=\hat{\eta}_i\right\}, \text{ if } y_i>y_i^0.\end{cases}$$

We denote the event

$$\mathcal{A}(y_i)=\left\{v^*(y_i,\hat{\eta}_i)+(1-w)(y_i-D_i)^+ + w\gamma_i(y_i+Y_{-i}-D_N)^+ > 0\right\}.$$

When $y_i>y_i^0$, the definition of $v^*(y_i,\hat{\eta}_i)$ implies $\mathbb{P}(\mathcal{A}(y_i))=\hat{\eta}_i$, and in this case

$$\begin{aligned}&\frac{1}{1-w+w\gamma_i}\left.\frac{\partial f_i(y_i,\hat{\eta}_i,v)}{\partial y_i}\right|_{v=v^*(y_i,\hat{\eta}_i)}\\ =&-\frac{1}{\hat{\eta}_i}\left(\frac{1-w}{1-w+w\gamma_i}\mathbb{P}(\mathcal{A}(y_i)\cap\{D_i\le y_i\})+\frac{w\gamma_i}{1-w+w\gamma_i}\mathbb{P}(\mathcal{A}(y_i)\cap\{D_N\le y_i+Y_{-i}\})\right).\end{aligned}$$

Note that if $\mathcal{A}(y_i)\subseteq\{D_i\le y_i\}$ and $\mathcal{A}(y_i)\subseteq\{D_N\le y_i+Y_{-i}\}$, then we would have

$$\frac{1}{1-w+w\gamma_i}\left.\frac{\partial f_i(y_i,\hat{\eta}_i,v)}{\partial y_i}\right|_{v=v^*(y_i,\hat{\eta}_i)}=-1.$$

We claim below that for any $y_i\ge\psi_i(\hat{\eta}_i,Y_{-i})$, where

$$\psi_i(\hat{\eta}_i,Y_{-i})=\{y_i \,|\, \mathbb{P}(\min\{w\gamma_i(y_i+Y_{-i}),(1-w)y_i\}-(1-w)D_i-w\gamma_iD_N>0)=\hat{\eta}_i\},$$

we will have $\mathcal{A}(y_i)\subseteq\{D_i\le y_i\}$ and $\mathcal{A}(y_i)\subseteq\{D_N\le y_i+Y_{-i}\}$.

We first show that for $y_i\ge\psi_i(\hat{\eta}_i,Y_{-i})$, we must have

$$v^*(y_i,\hat{\eta}_i)\le\min\{-(1-w)y_i,-w\gamma_i(y_i+Y_{-i})\}$$

Indeed, the probability $\mathbb{P}(v+(1-w)(y_i-D_i)^+ + w\gamma_i(y_i+Y_{-i}-D_N)^+>0)$ is increasing in v and when $v=\min\{-(1-w)y_i,-w\gamma_i(y_i+Y_{-i})\}$, the event $\{v+(1-w)(y_i-D_i)^+ + w\gamma_i(y_i+Y_{-i}-D_N)^+>0\}$ implies $\{D_i<y_i\}$ and $\{D_N<y_i+Y_{-i}\}$ (due to $D_i,D_N\ge0$). Hence,

$$\begin{aligned}&\mathbb{P}\left(\min\{-(1-w)y_i,-w\gamma_i(y_i+Y_{-i})\}+(1-w)(y_i-D_i)^+ + w\gamma_i(y_i+Y_{-i}-D_N)^+>0\right)\\ =&\mathbb{P}(\min\{-(1-w)y_i,-w\gamma_i(y_i+Y_{-i})\}+(1-w)(y_i-D_i)+w\gamma_i(y_i+Y_{-i}-D_N)>0)\\ =&\mathbb{P}(\min\{w\gamma_i(y_i+Y_{-i}),(1-w)y_i\}-(1-w)D_i-w\gamma_iD_N>0)\\ \ge&\mathbb{P}(\min\{w\gamma_i(\psi_i(\hat{\eta}_i,Y_{-i})+Y_{-i}),(1-w)\psi_i(\hat{\eta}_i,Y_{-i})\}-(1-w)D_i-w\gamma_iD_N>0)=\hat{\eta}_i,\end{aligned}$$

which implies $v^*(y_i,\hat{\eta}_i)\le\min\{-(1-w)y_i,-w\gamma_i(y_i+Y_{-i})\}$.

It then follows that

$$\begin{aligned}\mathcal{A}(y_i) &= \left\{v^*(y_i,\hat{\eta}_i) + (1-w)(y_i - D_i)^+ + w\gamma_i(y_i + Y_{-i} - D_N)^+ > 0\right\} \\ &\subseteq \left\{\min\{-(1-w)y_i, -w\gamma_i(y_i+Y_{-i})\} + (1-w)(y_i - D_i)^+ + w\gamma_i(y_i + Y_{-i} - D_N)^+ > 0\right\} \\ &\subseteq \{D_i < y_i\}.\end{aligned}$$

Similarly, $\mathcal{A}(y_i) \subseteq \{D_N < y_i + Y_{-i}\}$.

By letting

$$\bar{y}_i(\hat{\eta}_i) = \psi_i(\hat{\eta}_i, 0) = \{y_i \,|\mathbb{P}\left(\min\{w\gamma_i y_i, (1-w)y_i\} - (1-w)D_i - w\gamma_i D_N > 0\right) = \hat{\eta}_i\,\},$$

then we have $\bar{y}_i(\hat{\eta}_i) \geq \psi_i(\hat{\eta}_i, Y_{-i})$ for any $Y_{-i} \geq 0$. Note that $\bar{y}_i(\hat{\eta}_i)$ defined above is increasing in y_i and satisfies $\lim_{\hat{\eta}_i \to 0} \bar{y}_i(\hat{\eta}_i) = 0$. This proves our claim that for $y_i \geq \bar{y}_i(\hat{\eta}_i)$

$$\frac{1}{1-w+w\gamma_i} \left.\frac{\partial f_i(y_i,\hat{\eta}_i,v)}{\partial y_i}\right|_{v=v^*(y_i,\hat{\eta}_i)} = -1,$$

and hence $y_i^{\mathrm{NE}} \leq \bar{y}_i(\hat{\eta}_i)$.

Proof of Theorem 1

Let $\bar{h}(\mathbf{y}) = \max_{v\in\mathbb{R}}\left\{v - \frac{1}{\eta}\mathbb{E}[(v - \pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i))^+]\right\}$ and $\bar{h}_0(y) = \max_{v\in\mathbb{R}}\left\{v - \frac{1}{\eta}\mathbb{E}[(v - \pi(y, \sum_{i\in\mathcal{N}} D_i))^+]\right\}$. Note that $\bar{h}(\mathbf{y}) = \bar{h}_0(\sum_{i\in\mathcal{N}} y_i)$ and recall that $h(\mathbf{y}) = \max_{v\in\mathbb{R}}\left\{v - \frac{1}{\eta}\mathbb{E}[(v - \sum_{i=1}^n \pi_i(y_i, D_i))^+]\right\}$.

We first observe that for any $\mathbf{y}$, we must have $h(\mathbf{y}) \leq \bar{h}(\mathbf{y})$. In particular, $\Pi^{\mathrm{c}} = h(\mathbf{y}^{\mathrm{c}}) \leq \bar{h}(\mathbf{y}^{\mathrm{c}})$. By concavity of $\bar{h}_0(\cdot)$ and $\bar{h}(\cdot)$, the level set $\{y|\bar{h}_0(y) \geq \Pi^{\mathrm{c}}\}$ and $\{\mathbf{y}|\bar{h}(\mathbf{y}) \geq \Pi^{\mathrm{c}}\}$ are both non-empty and convex. That is, there exist $0 < y^L \leq y^U$ such that $\sum_{i\in\mathcal{N}} y_i^{\mathrm{c}} \in [y^L, y^U]$, and for any $\mathbf{y}$ such that $\sum_{i\in\mathcal{N}} y_i \in [y^L, y^U]$, we have $\bar{h}(\mathbf{y}) = \bar{h}_0(\sum_{i\in\mathcal{N}} y_i) \geq \Pi^{\mathrm{c}}$; and for any $\mathbf{y}$ such that $\sum_{i\in\mathcal{N}} y_i < y^L$, we have $\bar{h}(\mathbf{y}) = \bar{h}_0(\sum_{i\in\mathcal{N}} y_i) < \Pi^{\mathrm{c}}$.

Let

$$\hat{\eta}_w = \begin{cases} \sup\left\{\eta' \in (0,\eta] \,\middle|\, \sum_{i\in\mathcal{N}} F_i^{-1}(\eta' r) < y^L\right\}, & \text{if } w = 0, \\ \sup\left\{\eta' \in (0,\eta] \,\middle|\, \sum_{i\in\mathcal{N}} \bar{y}_i(\hat{\eta}_i) < y^L\right\}, & \text{if } 0 < w < 1, \\ \sup\{\eta' \in (0,\eta] | F_N^{-1}(\eta' r) < y^L\}, & \text{if } w = 1. \end{cases}$$

When $w = 0$ and $\hat{\eta}_i \in (0, \hat{\eta}_w)$ for all $i \in \mathcal{N}$, by Proposition 2 part (a), we have $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} = \sum_{i\in\mathcal{N}} y_i^{\mathrm{d}} = \sum_{i\in\mathcal{N}} F_i^{-1}(\hat{\eta}_i r) < \sum_{i\in\mathcal{N}} F_i^{-1}(\hat{\eta}_w r) \leq y^L$. Hence, $\Pi^{\mathrm{dp}} = \bar{h}(\mathbf{y}^{\mathrm{NE}}) < \Pi^{\mathrm{c}}$.

When $0 < w < 1$ and $\hat{\eta}_i \in (0, \hat{\eta}_w)$ for all $i \in \mathcal{N}$, by Proposition 2 part (c), we have $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} \leq \sum_{i\in\mathcal{N}} \bar{y}_i(\hat{\eta}_i)$. Let $\hat{\eta} = \max_{i\in\mathcal{N}} \hat{\eta}_i$ and note that $\hat{\eta} < \hat{\eta}_w$. By definition, we have $\sum_{i\in\mathcal{N}} \bar{y}_i(\hat{\eta}_i) \leq \sum_{i\in\mathcal{N}} \bar{y}_i(\hat{\eta}) < y^L$. Hence, $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} < y^L$ and $\Pi^{\mathrm{dp}} = \bar{h}(\mathbf{y}^{\mathrm{NE}}) < \Pi^{\mathrm{c}}$.

Finally, when $w = 1$ and $\hat{\eta}_i \in (0, \hat{\eta}_w)$ for all $i \in \mathcal{N}$, by Proposition 2 part (b), it holds that $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} = F_N^{-1}(\hat{\eta} r)$, where $\hat{\eta} = \max_{i\in\mathcal{N}} \hat{\eta}_i < \hat{\eta}_w$. As a result, $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} < y^L$ and $\Pi^{\mathrm{dp}} = \bar{h}(\mathbf{y}^{\mathrm{NE}}) < \Pi^{\mathrm{c}}$.

Proof of Theorem 2

Since $\hat{\eta}_i = \hat{\eta}$ for any $i \in \mathcal{N}$ and $D_i, i \in \mathcal{N}$ are identically distributed, we have $y_i^{\mathrm{d}} = y^{\mathrm{d}}$ for $i \in \mathcal{N}$, and for the symmetric equilibrium $y_i^{\mathrm{NE}} = y^{\mathrm{NE}}$ for $i \in \mathcal{N}$.

We first show that when $\alpha = 1$, we must have $\Pi^{\mathrm{dp}} \leq \Pi^{\mathrm{d}} = n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right)$. Indeed, by the superadditivity of CVaR, we have

$$\Pi^{\mathrm{d}} = \rho\left(\sum_{i=1}^{n} \pi\left(y_i^{\mathrm{d}}, D_i\right)\right) \geq \sum_{i=1}^{n} \rho\left(\pi\left(y_i^{\mathrm{d}}, D_i\right)\right) = n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right).$$

On the other hand, when $\alpha = 1$, D_N has the same distribution as nD_i. Therefore,

$$\Pi^{\mathrm{d}} \leq \rho\left(\pi\left(ny^{\mathrm{d}}, \sum_{i=1}^{n} D_i\right)\right) = \rho\left(\pi\left(ny^{\mathrm{d}}, nD_i\right)\right) = \rho\left(n\pi\left(y^{\mathrm{d}}, D_i\right)\right) = n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right),$$

where the first inequality is from $\sum_{i=1}^{n} \pi\left(y_i^{\mathrm{d}}, D_i\right) \leq \pi\left(ny^{\mathrm{d}}, \sum_{i=1}^{n} D_i\right)$, and

$$\Pi^{\mathrm{dp}} = \rho\left(\pi\left(ny^{\mathrm{NE}}, \sum_{i=1}^{n} D_i\right)\right) = \rho\left(\pi\left(ny^{\mathrm{NE}}, nD_i\right)\right) = n\rho\left(\pi\left(y^{\mathrm{NE}}, D_i\right)\right) \leq n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right),$$

where the last inequality is due to $y^{\mathrm{d}} = \arg\max_y \rho\left(\pi\left(y, D_i\right)\right)$. We then have $\Pi^{\mathrm{d}} = n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right)$ and $\Pi^{\mathrm{dp}} \leq n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right) = \Pi^{\mathrm{d}}$.

By Proposition 1 part (c), when $\alpha = 1$, we have $\Pi^{\mathrm{dp}} \leq \Pi^{\mathrm{d}} < \Pi^{\mathrm{c}}$. Let $\alpha_w = \sup\{\alpha \in (1, 2] | \Pi^{\mathrm{dp}} < \Pi^{\mathrm{c}}\}$, then by continuity, we have $\alpha_w > 1$ and for any $\alpha \in (1, \alpha_w]$, $\Pi^{\mathrm{dp}} \leq \Pi^{c}$.

Proof of Proposition 3

For ease of exposition, we define the following notation, for $k = 0, d$,

$$\mu_i^k = \mathbb{E}[\min(y_i^k, D_i)], \quad \bar{\mu}_n^k = \tfrac{1}{n}\textstyle\sum_{j=1}^{n} \mu_j^k,$$
$$(\sigma_i^k)^2 = \mathbb{V}ar[\min(y_i^k, D_i)], \quad (\bar{s}_n^k)^2 = \tfrac{1}{n^2}\textstyle\sum_{j=1}^{n} p_j(\sigma_j^k)^2,$$
$$Z_n^k = \tfrac{1}{n}\textstyle\sum_{j=1}^{n} p_j \min(y_j^k, D_j).$$

Then we have

$$\frac{1}{n}\sum_{j=1}^{n} \pi(y_j^0, D_j) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\pi(y_j^0, D_j)] + (Z_n^0 - \bar{\mu}_n^0), \tag{13}$$
$$\frac{1}{n}\sum_{j=1}^{n} \pi(y_j^{\mathrm{d}}, D_j) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\pi(y_j^{\mathrm{d}}, D_j)] + (Z_n^{\mathrm{d}} - \bar{\mu}_n^{\mathrm{d}}). \tag{14}$$

By the translation invariance property of CVaR,

$$\rho\left(\frac{1}{n}\sum_{j=1}^{n} \pi(y_j^0, D_j)\right) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\pi(y_j^0, D_j)] - \rho(Z_n^0 - \bar{\mu}_n^0), \tag{15}$$
$$\rho\left(\frac{1}{n}\sum_{j=1}^{n} \pi(y_j^{\mathrm{d}}, D_j)\right) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\pi(y_j^{\mathrm{d}}, D_j)] - \rho(Z_n^{\mathrm{d}} - \bar{\mu}_n^{\mathrm{d}}). \tag{16}$$

Thus, we get

$$\begin{aligned}
\frac{1}{n}(\Pi^{\mathrm{c}}-\Pi^{\mathrm{d}}) &= \rho\left(\frac{1}{n}\sum_{j=1}^{n}\pi(y_j^{\mathrm{c}},D_j)\right)-\rho\left(\frac{1}{n}\sum_{j=1}^{n}\pi(y_j^{\mathrm{d}},D_j)\right)\\
&\geq \rho\left(\frac{1}{n}\sum_{j=1}^{n}\pi(y_j^{0},D_j)\right)-\rho\left(\frac{1}{n}\sum_{j=1}^{n}\pi(y_j^{\mathrm{d}},D_j)\right)\\
&= \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\pi(y_j^{0},D_j)]-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\pi(y_j^{\mathrm{d}},D_j)]-\rho(Z_n^0-\bar{\mu}_n^0)+\rho(Z_n^{\mathrm{d}}-\bar{\mu}_n^{\mathrm{d}})\\
&= \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\pi(y_j^{0},D_j)]-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\pi(y_j^{\mathrm{d}},D_j)]-s_n^0\rho\left(\frac{Z_n^0-\bar{\mu}_n^0}{s_n^0}\right)+s_n^{\mathrm{d}}\rho\left(\frac{Z_n^{\mathrm{d}}-\bar{\mu}_n^{\mathrm{d}}}{s_n^{\mathrm{d}}}\right).
\end{aligned}$$

By assumptions (i) and (ii), and the central limit theorem for independent random variables, we know that both $\frac{Z_n^0-\bar{\mu}_n^0}{s_n^0}$ and $\frac{Z_n^{\mathrm{d}}-\bar{\mu}_n^{\mathrm{d}}}{s_n^{\mathrm{d}}}$ converge to the standard normal random variable as the number of stores n goes to infinity. Moreover, we have $\rho(\frac{Z_n^0-\bar{\mu}_n^0}{s_n^0})$ and $\rho(\frac{Z_n^{\mathrm{d}}-\bar{\mu}_n^{\mathrm{d}}}{s_n^{\mathrm{d}}})$ converge to $\rho(\phi)$ as $n\to\infty$ where ϕ represents a standard normal random variable (Choi et al. 2011). Also note that because of assumptions (i) and (ii), s_n^0 and s_n^{d} are of order $1/\sqrt{n}$.

For the expectation terms, we have

$$\begin{aligned}
&\mathbb{E}[\pi(y_j^0,D_j)]-\mathbb{E}[\pi(y_j^{\mathrm{d}},D_j)]\\
&= p\mathbb{E}[\min(y_i^0,D_i)-\min(y_i^{\mathrm{d}},D_i)]-c(y_i^0-y_i^{\mathrm{d}})\\
&= p\int_{x=y_i^{\mathrm{d}}}^{y_i^0}(x-y_i^{\mathrm{d}})f_i(x)dx+p\int_{x=y_i^0}^{\infty}(y_i^0-y_i^{\mathrm{d}})f_i(x)dx-c(y_i^0-y_i^{\mathrm{d}})\\
&= p\int_{x=y_i^{\mathrm{d}}}^{y_i^0}(x-y_i^{\mathrm{d}})f_i(x)dx\\
&\geq p\min_{x\in[y_i^{\mathrm{d}},y_i^0]}\{f_i(x)\}(y_i^0-y_i^{\mathrm{d}})^2/2,
\end{aligned}$$

where the last equality is because $\int_{x=y_i^0}^{\infty}f_i(x)dx=c/p$. Also note that in the above procedure we can replace y_i^0 by any $y\in[y_i^d,y_i^0]$, hence by maximizing over y, we may get a better lower bound.

Then, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\pi(y_j^0,D_j)]-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\pi(y_j^{\mathrm{d}},D_j)]\\
&\geq \frac{p}{2n}\sum_{j=1}^{n}\min_{x\in[y_j^{\mathrm{d}},y_j^0]}\{f_j(x)\}(y_j^0-y_j^{\mathrm{d}})^2/2\\
&\geq \frac{p}{2}\min_{j=1,\ldots,n}\left\{\min_{x\in[y_j^{\mathrm{d}},y_j^0]}\{f_j(x)\}(y_j^0-y_j^{\mathrm{d}})^2\right\}.
\end{aligned}$$

The proof is complete.

Proof of Proposition 4

When $\boldsymbol{D}$ follows a multivariate normal distribution with (μ_i,σ_i) being the mean and standard deviation of D_i and ρ_{ij} being the correlation coefficient between D_i and D_j, we have the total demand at all store

$D_N = \sum_{i=1}^n D_i$ follows normal distribution with mean μ_N and standard deviation $\sigma_N = \sqrt{\sum_{i,j\in\mathcal{N}} \sigma_i\sigma_j\rho_{ij}}$. Under the proportional allocation and inventory decisions, the inventory level for DP system is $Y^{\mathrm{dp}} = F_N^{-1}(\hat{\eta}r)$ where recall that $F_N^{-1}(\cdot)$ is the inverse function of the distribution of D_N. And by the property of the normal distribution, we have

$$Y^{\mathrm{dp}} = \mu_N + \Phi^{-1}(\hat{\eta}r)\cdot\sigma_N,$$

where $\Phi^{-1}(\cdot)$ is the inverse function of the standard normal distribution. Let X be a standard normal random variable, then D_N has the same distribution as $\mu_N + X\cdot\sigma_N$.

Therefore,

$$\begin{aligned}\Pi^{\mathrm{dp}} &= \rho\left[\pi\left(Y^{\mathrm{dp}}, D_N\right)\right] \\ &= \rho\left[\pi\left(\mu_N + \Phi^{-1}(\hat{\eta}r)\cdot\sigma_N, \mu_N + X\cdot\sigma_N\right)\right] \\ &= \rho\left[p\min\left(\mu_N + \Phi^{-1}(\hat{\eta}r)\cdot\sigma_N, \mu_N + X\cdot\sigma_N\right) - c(\mu_N + \Phi^{-1}(\hat{\eta}r)\cdot\sigma_N)\right] \\ &= (p-c)\mu_N + \sigma_N\cdot\rho\left[p\min\left(\Phi^{-1}(\hat{\eta}r), X\right) - c\Phi^{-1}(\hat{\eta}r)\right] \\ &= (p-c)\mu_N - \sigma_N\cdot\left[c\Phi^{-1}(\hat{\eta}r) - \rho\left(p\min\left(\Phi^{-1}(\hat{\eta}r), X\right)\right)\right],\end{aligned}$$

where the second equality is because CVaR is law invariant and the value of $\rho(X)$ only depends on the distribution of X. Let $K := c\Phi^{-1}(\hat{\eta}r) - \rho\left(p\min\left(\Phi^{-1}(\hat{\eta}r), X\right)\right)$, then $\Pi^{\mathrm{dp}} = (p-c)\mu_N - \sigma_N\cdot K$.

Then, similar as above, we can get

$$\rho\left(\pi\left(y_i^{\mathrm{d}}, D_i\right)\right) = (p-c)\mu_i - \sigma_i\cdot\left[c\Phi^{-1}(\hat{\eta}r) - \rho\left(p\min\left(\Phi^{-1}(\hat{\eta}r), X\right)\right)\right]$$

By the superadditivity of CVaR, we have

$$\Pi^{\mathrm{d}} = \rho\left(\sum_{i=1}^n \pi\left(y_i^{\mathrm{d}}, D_i\right)\right) \geq \sum_{i=1}^n \rho\left(\pi\left(y_i^{\mathrm{d}}, D_i\right)\right) = (p-c)\mu_N - \sum_{i=1}^n \sigma_i\cdot K.$$

Therefore, we have

$$\Pi^{\mathrm{dp}} - \Pi^{\mathrm{d}} \leq K\left(\sum_{i=1}^n \sigma_i - \sigma_N\right).$$

The proof is complete.

Proof of Proposition 5

Under the proportional allocation and inventory decisions, the inventory level for DP system is $Y^{\mathrm{dp}} = F_N^{-1}(\hat{\eta}r)$. By Lemma 1 of Bimpikis and Markakis (2015), we know that

$$Y^{\mathrm{dp}} = n^{1/\alpha}F_i^{-1}(\hat{\eta}r) + \mu(n - n^{1/\alpha}) = n^{1/\alpha}y_i^{\mathrm{d}} + \mu(n - n^{1/\alpha}).$$

Therefore,

$$\begin{aligned}\Pi^{\mathrm{dp}} &= \rho\left[\pi\left(Y^{\mathrm{dp}}, D_N\right)\right] \\ &= \rho\left[\pi\left(n^{1/\alpha}y_i^{\mathrm{d}} + \mu(n - n^{1/\alpha}), n^{1/\alpha}D_i + \mu(n - n^{1/\alpha})\right)\right] \\ &= \rho\left[p\min\left(n^{1/\alpha}y_i^{\mathrm{d}} + \mu(n - n^{1/\alpha}), n^{1/\alpha}D_i + \mu(n - n^{1/\alpha})\right) - c(n^{1/\alpha}y_i^{\mathrm{d}} + \mu(n - n^{1/\alpha}))\right] \\ &= (p-c)\mu(n - n^{1/\alpha}) + n^{1/\alpha}\rho\left(p\min\left(y_i^{\mathrm{d}}, D_i\right) - cy_i^{\mathrm{d}}\right) \\ &= (p-c)\mu(n - n^{1/\alpha}) + n^{1/\alpha}\rho\left(\pi(y_i^{\mathrm{d}}, D_i)\right).\end{aligned}$$

By the superadditivity of CVaR, we have

$$\Pi^{\mathrm{d}} = \rho\left(\sum_{i=1}^{n} \pi\left(y_i^{\mathrm{d}}, D_i\right)\right) \geq \sum_{i=1}^{n} \rho\left(\pi\left(y_i^{\mathrm{d}}, D_i\right)\right) = n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right).$$

Therefore, we have

$$\begin{aligned}\Pi^{\mathrm{dp}} - \Pi^{\mathrm{d}} &\leq (p-c)\mu(n - n^{1/\alpha}) + n^{1/\alpha}\rho\left(\pi(y_i^{\mathrm{d}}, D_i)\right) - n\rho\left(\pi\left(y^{\mathrm{d}}, D_i\right)\right) \\ &= (n - n^{1/\alpha})\left[(p-c)\mu - \rho\left(\pi(y_i^{\mathrm{d}}, D_i)\right)\right].\end{aligned}$$

Appendix B: Computational Method and Detailed Numerical Results

Sample-Based Optimization

In the sample-based optimization procedure, we first generate samples $\boldsymbol{D}^1, \boldsymbol{D}^2, \ldots, \boldsymbol{D}^T$ of demand vector, where the vector $\boldsymbol{D}^t = (d_1^t, d_2^t, \ldots, d_n^t),\ t = 1, \ldots, T$ is from the distribution $F(\cdot)$. Then we use the empirical distribution based on the samples to approximate the original demand distribution. In particular, when $T \to \infty$, the optimal value of the problem based on samples approaches the optimal value of the original problem (see Shapiro 2008). In our numerical study, we set $T = 5000$.

With the empirical demand distribution, the optimization problem can be formulated as a linear program. Let $w_j^t,\ j = 1, \ldots, n,\ t = 1, \ldots, T$, denote the leftover stocks of store j when facing demand $\boldsymbol{D}^t$. Let u_t denote the shortfall of the profit to the quantile v when facing demand $\boldsymbol{D}^t$. Then we have the following formulation:

$$
\begin{aligned}
& \max\ v - \frac{1}{\eta} \sum_{t=1}^{T} u_t / T \\
s.t.\ & \sum_{j=1}^{n} [(p-c) y_j - p w_j^t] + u_t \geq v, \quad t = 1, \ldots, T, \\
& y_j - d_j^t \leq w_j^t, \quad j = 1, \ldots, n;\ t = 1, \ldots, T, \\
& w_j^t \geq 0, \quad j = 1, \ldots, n;\ t = 1, \ldots, T, \\
& u_t \geq 0, \quad t = 1, \ldots, T, \\
& y_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

Note that, given the order quantities $y_j,\ j = 1, \ldots, n$, at optimum, $w_j^t = (y_j - d_j^t)^+$ and $u_t = (v - \sum_{i=1}^n \pi(y_i, d_i^t))^+$. Thus, the objective becomes $v - \frac{1}{\eta} \sum_{t=1}^T (v - \sum_{i=1}^n \pi(y_i, d_i^t))^+ / T$.

Numerical Studies: Comparative Statics

In this section, we conduct extensive numerical experiments to study comparative statics of problem parameters. We consider identical stores and the store managers with identical risk preferences. In system DP, we consider the proportional allocation rule. We define the base case as: $n = 2$, $p = 20$, $c = 4$, $\eta = 0.9$, $\hat{\eta}_i = 0.3$, $i = 1, 2$, and independent normally distributed demands with mean $\mu = 10$ and standard deviation $\sigma = 4$.[3] Note that with identical risk preferences and identical demands, we have $y_i^{\mathrm{d}} = y^{\mathrm{d}}$ and $y_i^{\mathrm{c}} = y^{\mathrm{c}}$ for $i = 1, 2$. In all cases, we depict both the optimal order quantity in the left panel and the corresponding CVaR in the middle panel. While both centralization and inventory pooling are effective tools to improve over system D, it is unclear whether the employment of one enlarges (in which case they are strategic complements) or offsets (in which case they are strategic substitutes) the other. To answer this, we measure and compare the benefits from inventory pooling under the centralized system and the decentralized system respectively. That is, if

$$
\Pi^{\mathrm{cp}} - \Pi^{\mathrm{c}} \geq \Pi^{\mathrm{dp}} - \Pi^{\mathrm{d}}
$$

[3] We actually consider truncated normal demand to ensure all demands are non-negative.

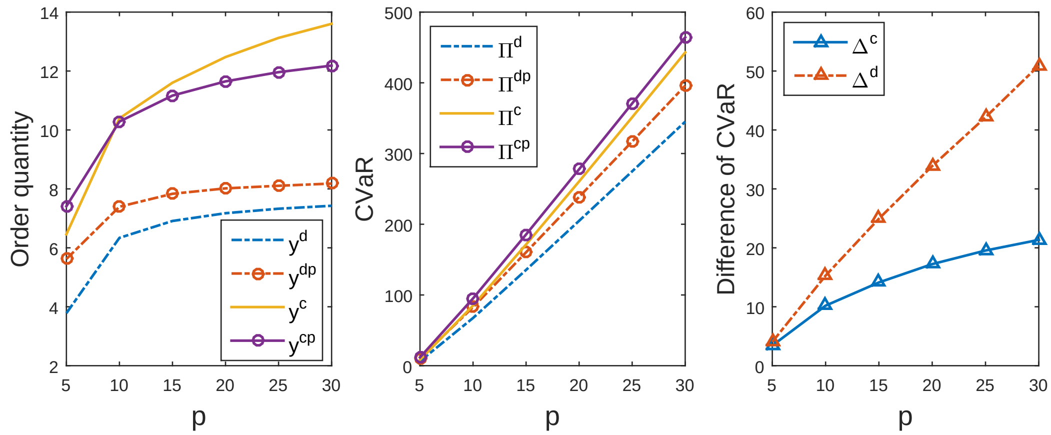

Figure 6 **Effect of price: when the price is low, i.e., the profit margin is low, the optimal order quantity of system CP (note that $y^{\text{cp}} = Y^{\text{cp}}/2$ here) is larger than that of system C, whereas when the price is high, the opposite is true. Moreover, as the price increases, the CVaR increases, and the gaps between the CVaR of system D and the other systems increase.**

holds, then pooling generates higher value under centralized system and we can say that pooling and centralization are strategic complements; otherwise they are strategic substitutes. Here, we numerically investigate how the benefits of pooling under the centralized system and the decentralized system, and their relationship are affected when system parameters change. For ease of exposition, let $\Delta^{\text{c}} := \Pi^{\text{cp}} - \Pi^{\text{c}}$ and $\Delta^{\text{d}} := \Pi^{\text{dp}} - \Pi^{\text{d}}$. In all cases, we depict Δ^{c} and Δ^{d} in the right penal.

The effect of price (profit margin) is shown in Figure 6, where we vary the price p from 5 to 30. We can see that the order quantities increase as the price increases. Furthermore, when the price is low, *i.e.*, the profit margin is low, the optimal order quantity of system CP (note that $y^{\text{cp}} = Y^{\text{cp}}/2$ here) is larger than that of system C, whereas when the price is high, the opposite is true. Intuitively, when the profit margin is low, the central planner tends to order less (than the mean) to protect against the downside (overstocking) risk, and pooling the inventory mitigates this risk. Hence, the order quantity of the system with pooling is larger than the system without pooling; on the other hand, when the profit margin is high, the central planner tends to order more (than the mean) to protect against the upside (understocking) risk, and pooling the inventory mitigates this risk, thus the order quantity of the pooling system is lower than the system without pooling. Moreover, as the price increases, the CVaR increases, and the gaps between the CVaR of system D and the other systems increase. That is, both the benefit of centralization $\Pi^{\text{c}} - \Pi^{\text{d}}$ and the benefit of inventory pooling $\Pi^{\text{dp}} - \Pi^{\text{d}}$ increase as the profit margin increases. For the strategic interplay, we can see that the benefit of pooling under the centralized system is lower than that under the decentralized system, that is, pooling and centralization are strategic substitutes. Moreover, as the price increases, the benefit of pooling under the centralized system grows slower than that under the decentralized system.

The effect of the number of stores n is illustrated in Figure 7. It shows that as the number of stores increases, the order quantity of system CP slightly decreases (here $y^{\text{cp}} = Y^{\text{cp}}/n$), whereas the order quantities of system

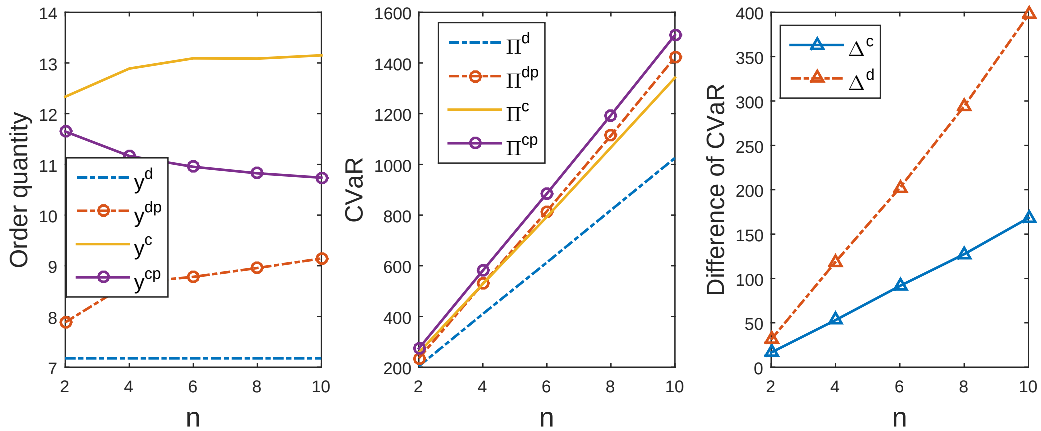

Figure 7 **Effect of number of stores (here $y^{\mathbf{cp}} = Y^{\mathbf{cp}}/n$): As the number of stores increases, the order quantity of system CP (system C and system DP) slightly decreases (increases), the CVaRs linearly increase with the benefits of centralization and pooling both increase, and the benefit of pooling under the centralized system grows slower than that under the decentralized system.**

C and system DP slightly increases with small fluctuations due to random sample errors. Meanwhile, as the number of stores increases, the CVaRs linearly increase with the benefit of centralization and the benefit of pooling both increase as well. Moreover, as the number of stores increases, the benefit of pooling under the centralized system grows slower than that under the decentralized system.

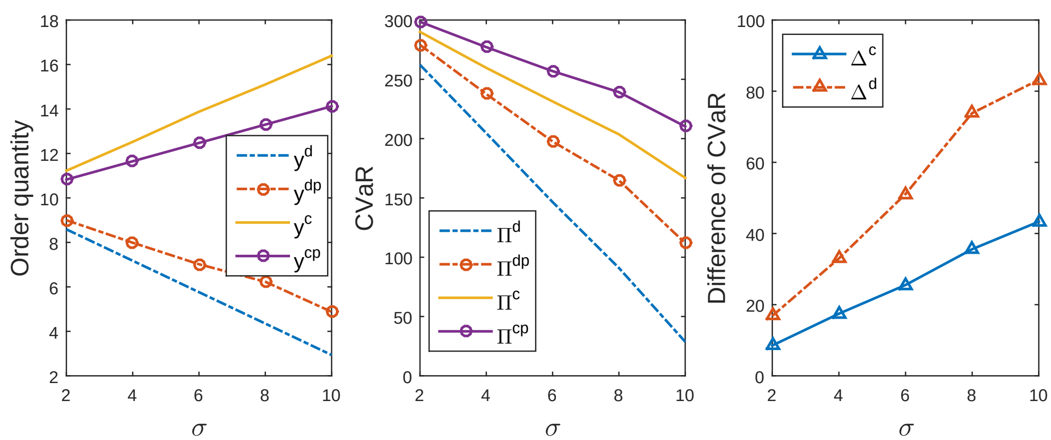

Figure 8 **Effect of demand variance: As the demand variance increases, the order quantity of the decentralized (centralized) system decreases (increases), the CVaRs decrease, and the benefits of pooling increase.**

In Figure 8, we consider the effect of demand variability where the standard deviation σ is varied from 2 to 10. The figure shows that the order quantity of the decentralized system decreases, whereas the order quantities of the centralized systems increase. This is because the stores order less than the mean to protect against the downside risk, and when the demand variance becomes higher, this risk becomes higher, thus the order quantity of the stores becomes lower; in contrast, the central planner orders more than the mean to protect against the upside risk, and when the demand variance is higher, the safety stock is higher. Moreover, the order quantity of system C increases faster than that of system CP because inventory pooling reduces the volatility of the demand. Regarding CVaR, as expected, when the demand variance increases, the CVaRs decrease. Whereas the benefits of inventory pooling under both the decentralized system and the centralized systems increase as the demand variance increases.

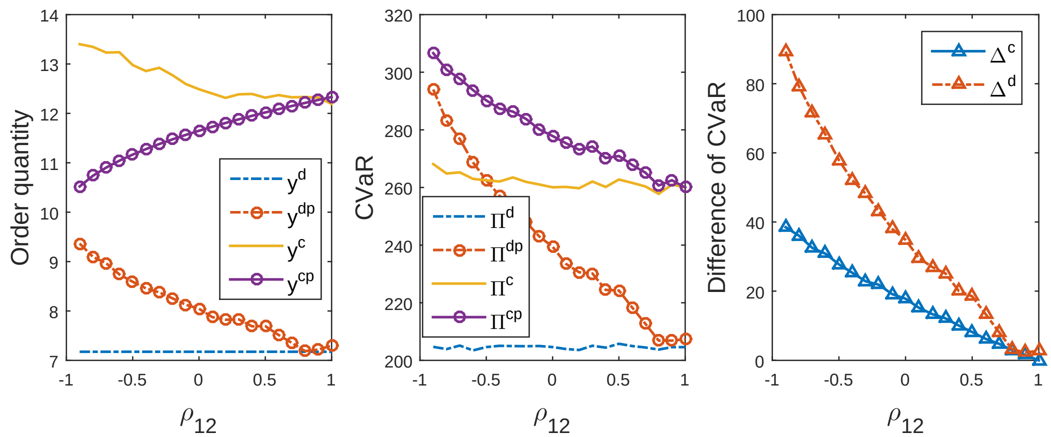

Figure 9 Effect of demand correlation: For system C and system DP, negatively (positively) correlated demand results in a higher (lower) order quantity than independent demand, whereas the opposite is true for system CP. The CVaRs of system D and system C are not sensitive to the demand correlation, but the benefit of pooling declines as the correlation coefficient increases.

In Figure 9, we study the impact of the dependence between the demands of the two retail stores. We vary the correlation coefficient of the demands ρ_{12} from -1 to 1. The numerical result posits that, for system C and system DP, negatively correlated demand results in a higher order quantity than independent demand, and positively correlated demand leads to a lower order quantity, which is consistent with the result of Choi et al. (2011). In contrast, for system CP, the opposite is true, that is, negatively correlated demand results in a lower order quantity than independent demand, and positively correlated demand leads to a higher order quantity. In addition, the CVaRs of system D and system C are not affected much by the correlation coefficient of the demands. Nevertheless, the benefit of inventory pooling declines as the correlation coefficient increases. Therefore, for positively correlated demands, it is more likely that the firm should prioritize decision centralization over inventory pooling. Moreover, as the correlation coefficient increases, the benefit of inventory pooling under the centralized system declines as well.

At last, we study the impact of the tail of the demand distribution on the benefits of centralization and inventory pooling. We extend the demand distribution to the stable distribution. We fix $\beta = 0$, $\sigma = 4$, $\mu = 10$ and vary α from 1 to 2. The result is shown in Figure 10. It is clear that as the tail of the demand distribution becomes heavier (α becomes smaller), the order quantity of system CP increases whereas the order quantity of

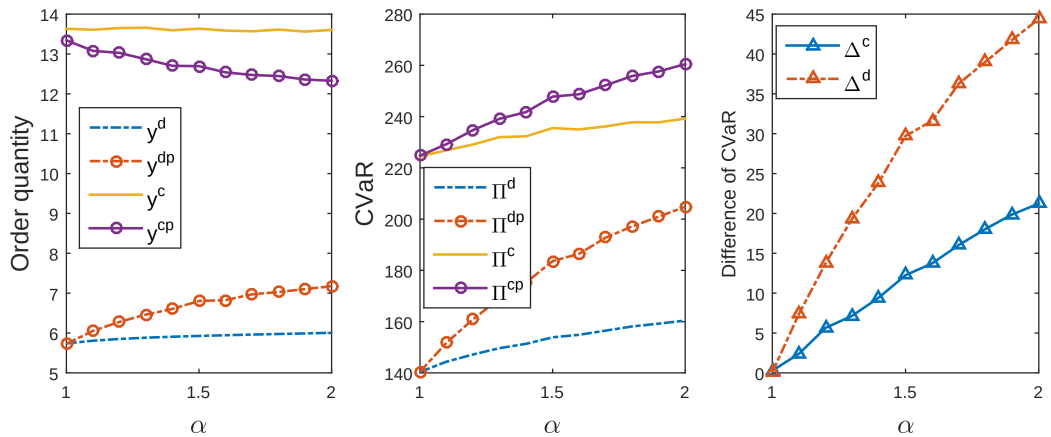

Figure 10 **Effect of tail of demand distribution: As the tail of the demand distribution becomes heavier (α becomes smaller), the order quantity of system CP (system DP) increases (decreases), the CVaRs of all systems decrease, and the benefit of pooling decreases whereas the benefit of centralization is not greatly affected.**

system DP decreases, and the CVaRs of all systems decrease. Moreover, as the tail of the demand distribution becomes heavier, the benefit of inventory pooling decreases, whereas the benefit of centralization is not greatly affected. Therefore, for heavy-tailed demand, it is more likely that the retailer shall prioritize centralization over inventory pooling, which is consistent with our result in Theorem 2. Furthermore, the benefit of inventory pooling under the centralized system is lower than that under the decentralized system, that is, centralization and inventory pooling are still strategic substitutes for heavy-tailed demand.

Appendix C: Additional Results

Analysis of Exponential Utility

In this section, we consider the case when the "risk measure" $\rho(\cdot)$ is specified by the expected value of exponential utility. In particular, given a random profit π, the expected utility to the decision maker is specified by

$$\rho^{\exp}(\pi) := \mathbb{E}[u(\pi)] = \mathbb{E}[(\eta - \eta \exp(-\frac{1}{\eta}\pi))],$$

where $\eta > 0$ measures the degree of risk-aversion—the lower η is, the more risk-averse. In particular, when $\eta \to \infty$, $\rho^{\exp}(\pi) \to \mathbb{E}[\pi]$. The expected utility framework with the exponential utility function introduced above is closely related to the the so-called *entropic* risk measure which is defined by

$$\rho^{\text{ent}}(\pi) := \sup\{m \in \mathbb{R} | \mathbb{E}[u(\pi - m)] \geq 0\} = -\eta \log \mathbb{E}[\exp(-\frac{1}{\eta}\pi)].$$

Alternatively, one can view $\rho^{\text{ent}}(\pi)$ as the certainty equivalence of the random profit π, i.e, $u(\rho^{\text{ent}}(\pi)) = \mathbb{E}[u(\pi)]$.

Given a newsvendor profit $\Pi(y,D)$, it is easy to see that the problems $\max_y \rho^{\exp}(\Pi(y,D))$ and $\max_y \rho^{\text{ent}}(\Pi(y,D))$ are equivalent in the sense that they have the same set of optimal solutions. The only difference is in the interpretation of the optimal value—whether it is an expected utility or a monetary certainty equivalent. Similarly, we can simplify our notation by focusing on the following measure

$$\rho^{\mathrm{e}}(\pi) := \mathbb{E}[u_0(\pi)] = \mathbb{E}[-\eta \exp(-\frac{1}{\eta}\pi)]$$

without changing the optimal solutions to our subsequent problems.

We remark that unlike CVaR, entropic risk measure is not coherent (though it belongs to the family of convex risk measures) and the optimal solution to the problem $\max_y \rho^{\mathrm{ent}}(\Pi(y,D))$ is sensitive to scaling of the profit (see Appendix in Choi et al. 2011).

For a single newsvendor problem, the first order condition is

$$\begin{aligned}&\mathbb{E}[-cu_0'(pD-cy)1_{\{D\le y\}} + (p-c)u_0'((p-c)y)1_{\{D>y\}}]\\ =&-c\mathbb{E}[\alpha e^{-\alpha(pD-cy)}1_{\{D\le y\}}] + (p-c)\mathbb{E}[\alpha e^{-\alpha(p-c)y}1_{\{D>y\}}] = 0.\end{aligned}$$

In general, the above equation does not yield closed-form solution. However, it is well-known that the optimal order quantity as a function of η is strictly increasing in η—an observation consistent with CVaR risk measure (see, for example, Eeckhoudt et al. 1995). When the demand is D, we denote the optimal order quantity as a function of η as $y_D^*(\eta)$,

Now we assume that the degree of risk-aversion of store i's manager is $\hat{\eta}_i$ and that of retailer as η, and we assume $\eta \ge \hat{\eta}_i$ for $i \in \mathcal{N}$. In a centralized system, the retailer solves

$$\Pi^{\mathrm{c}} = \max_{\mathbf{y}} \mathbb{E}\left[-\eta \exp\left(-\frac{1}{\eta}\sum_{i\in\mathcal{N}} \pi(y_i, D_i)\right)\right],$$

while in a decentralized system each store manager solves $\max_{y_i} \mathbb{E}\left[-\hat{\eta}_i \exp\left(-\frac{1}{\hat{\eta}_i}\pi(y_i, D_i)\right)\right]$. We first have the following result about the relationship between the decentralized order quantity y_i^{d} and the centralized order quantity y_i^{c}.

Proposition 6. *Suppose $\eta \ge \hat{\eta}_i$ for $i \in \mathcal{N}$, and demands $D_i, i \in \mathcal{N}$ are independent. Then $y_i^{\mathrm{d}} \le y_i^{\mathrm{c}}$.*

Proof. For the centralized system, we have

$$\begin{aligned}&\max_{\mathbf{y}} \mathbb{E}\left[-\eta \exp\left(-\frac{1}{\eta}\sum_{i\in\mathcal{N}} \pi(y_i, D_i)\right)\right]\\ =&\max_{\mathbf{y}} \mathbb{E}\left[-\eta \prod_{i\in\mathcal{N}} \exp\left(-\frac{1}{\eta}\pi(y_i, D_i)\right)\right]\\ =&\max_{\mathbf{y}} -\eta \prod_{i\in\mathcal{N}} \mathbb{E}\left[\exp\left(-\frac{1}{\eta}\pi(y_i, D_i)\right)\right],\end{aligned}$$

where the last equality is by independence of $D_i, i \in \mathcal{N}$. Clearly, minimizing $\prod_{i\in\mathcal{N}} \mathbb{E}\left[\exp\left(-\frac{1}{\eta}\pi(y_i, D_i)\right)\right]$ is equivalent to minimizing $\mathbb{E}\left[\eta \exp\left(-\frac{1}{\eta}\pi(y_i, D_i)\right)\right]$ for each $i \in \mathcal{N}$. As a result,

$$y_i^{\mathrm{c}} = y_{D_i}^*(\eta) \ge y_{D_i}^*(\hat{\eta}_i) = y_i^{\mathrm{d}}.$$

□

In the following, we characterize the system ordering quantities for the DP system under the proportional allocation rule ($w = 1$)).

LEMMA 1. *Consider the proportional allocation*

$$l_i^{\mathrm{p}}(\mathbf{y},\mathbf{D}) = -\pi(y_i, D_i) + \gamma_i \pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i),$$

with $\gamma_i > 0$ *for any* $i \in \mathcal{N}$. *Let* $\mathbf{y}^{\mathrm{NE}} = (y_1^{\mathrm{NE}}, ..., y_n^{\mathrm{NE}})$ *be any Nash equilibrium of the DP system under the above allocation, and suppose* $\hat{\eta}_1/\gamma_1 \le \hat{\eta}_2/\gamma_2 \cdots \le \hat{\eta}_n/\gamma_n$. *Then, it must hold that* $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} = y^*_{D_N}(\hat{\eta}_n/\gamma_n)$.

Proof. Given $y_{-i} := (y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$, store i seeks to solve

$$\max_{y_i} \mathbb{E}\left[-\hat{\eta}_i \exp\left(-\frac{\gamma_i}{\hat{\eta}_i}\pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i)\right)\right],$$

which is equivalent to

$$\max_{y_i} \mathbb{E}\left[-\hat{\eta}_i/\gamma_i \exp\left(-\frac{1}{\hat{\eta}_i/\gamma_i}\pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i)\right)\right].$$

The best response function of player i is:

$$y_i(y_{-i}) = (y^*_{D_N}(\hat{\eta}_i/\gamma_i) - \sum_{j\neq i} y_j)^+.$$

Similar to the proof of Proposition 4 in the paper, we prove by showing that $y_i^{\mathrm{NE}} = 0$ for $\hat{\eta}_i/\gamma_i < \hat{\eta}_n/\gamma_n$. Suppose on the contrary $y_i^{\mathrm{NE}} > 0$ for $\hat{\eta}_i/\gamma_i < \hat{\eta}_n/\gamma_n$. Then, we know $y_i^{\mathrm{NE}} = y^*_{D_N}(\hat{\eta}_i/\gamma_i) - \sum_{j\neq i} y_j^{\mathrm{NE}}$, which implies

$$y_n^{\mathrm{NE}} = y^*_{D_N}(\hat{\eta}_i/\gamma_i) - \sum_{j<n} y_j^{\mathrm{NE}} \ge 0.$$

By $\hat{\eta}_i/\gamma_i < \hat{\eta}_n/\gamma_n$, $y^*_{D_N}(\hat{\eta}_i/\gamma_i) - \sum_{j<n} y_j^{\mathrm{NE}} > 0$. Again, from the best response function, we know $y_n(y_{-n}^{\mathrm{NE}}) = y^*_{D_N}(\hat{\eta}_n/\gamma_n) - \sum_{j<n} y_j^{\mathrm{NE}}$. That is, $\sum_{j\in\mathcal{N}} y_j^{\mathrm{NE}} = y^*_{D_N}(\hat{\eta}_n/\gamma_n)$, which contradicts with $\sum_{j\in\mathcal{N}} y_j^{\mathrm{NE}} = y^*_{D_N}(\hat{\eta}_i/\gamma_i)$.

Since $y_i^{\mathrm{NE}} > 0$ only for $\hat{\eta}_i/\gamma_i = \hat{\eta}_n/\gamma_n$, from the best response we know $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} = y^*_{D_N}(\hat{\eta}_n/\gamma_n)$. □

We remark that if $\gamma_i = 0$ for some $i \in \mathcal{N}$, the conclusion in the above lemma still holds since store i ordering a risk-neutral order quantity—denoted as $y^*_{D_N}(\infty)$ and the rest of the stores ordering nothing is a Nash equilibrium.

Now we are ready to establish our main result under the exponential utility framework.

THEOREM 3. *Suppose* $\hat{\eta}_i \le \eta$ *for* $i \in \mathcal{N}$ *and the proportional allocation is used. There exist* $\hat{\eta}_l < \hat{\eta}_u$ *with* $\hat{\eta}_l \le \eta \le \hat{\eta}_u$ *such that if* $\hat{\eta}_i < \hat{\eta}_l\gamma_i$ *for all* $i \in \mathcal{N}$ *or* $\hat{\eta}_i > \hat{\eta}_u\gamma_i$ *for some* $i \in \mathcal{N}$, *then* $\Pi^{\mathrm{dp}} < \Pi^{\mathrm{c}}$; *if* $\hat{\eta}_i \le \hat{\eta}_u\gamma_i$ *for all* $i \in \mathcal{N}$ *and* $\hat{\eta}_i \ge \hat{\eta}_l\gamma_i$ *for some* $i \in \mathcal{N}$, *then* $\Pi^{\mathrm{dp}} \ge \Pi^{\mathrm{c}}$.

Proof. Similar to the proof of Theorem 1 in the paper, by concavity of

$$\rho^{\mathrm{e}}(\pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i); \eta) := \mathbb{E}\left[-\eta \exp\left(-\frac{1}{\eta}\pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i)\right)\right]$$

in $\mathbf{y}$, there exist $0 < y^L \le y^U$ such that if $\sum_{i\in\mathcal{N}} y_i \in [y^L, y^U]$, we have

$$\rho^{\mathrm{e}}(\pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i); \eta) \ge \Pi^c;$$

and if $\sum_{i\in\mathcal{N}} y_i < y^L$ or $\sum_{i\in\mathcal{N}} y_i > y^U$, we have

$$\rho^{\mathrm{e}}(\pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i);\eta) < \Pi^{\mathrm{c}}.$$

Clearly, we have $y^*_{D_N}(\eta) \in [y^L, y^U]$. We further define $\hat\eta_l = \inf\{\eta' > 0 | y^*_{D_N}(\eta') > y^L\}$ and $\hat\eta_u = \sup\{\eta' > 0 | y^*_{D_N}(\eta') < y^U\}$. If $\hat\eta_i < \hat\eta_l\gamma_i$ for all $i\in\mathcal{N}$ (or $\hat\eta_i > \hat\eta_u\gamma_i$ for some $i\in\mathcal{N}$), then we must have $\max_{i\in\mathcal{N}}\hat\eta_i/\gamma_i < \hat\eta_l$ (or $\max_{i\in\mathcal{N}}\hat\eta_i/\gamma_i > \hat\eta_u$). From Lemma 1, we know that for any Nash equilibrium $\mathbf{y}^{\mathrm{NE}}$, it holds that $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} = y^*_{D_N}(\max_{i\in\mathcal{N}}\hat\eta_i/\gamma_i)$. As a result, $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} < y^L$ (or $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} > y^U$) and

$$\Pi^{\mathrm{dp}} = \rho^{\mathrm{e}}\left(\pi(\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}}, \sum_{i\in\mathcal{N}} D_i);\eta\right) < \Pi^{\mathrm{c}}.$$

On the other hand, if $\hat\eta_i \le \hat\eta_u\gamma_i$ for all $i\in\mathcal{N}$ and $\hat\eta_i \ge \hat\eta_l\gamma_i$ for some $i\in\mathcal{N}$, then $\hat\eta_l \le \max_{i\in\mathcal{N}}\hat\eta_i/\gamma_i \le \hat\eta_u$ and $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} \in [y^L, y^U]$. Hence, $\Pi^{\mathrm{dp}} \ge \Pi^{\mathrm{c}}$. □

Analysis of More Risk-Averse Retailer

Here, we analyze the case when the retailer is more risk averse than the store managers, *i.e.*, $\eta \le \hat\eta_i$ for $i\in\mathcal{N}$.

Theorem 4. *Suppose $\eta \le \hat\eta_i$ for $i\in\mathcal{N}$. Under the proportional allocation, there exists $\hat\eta_p \ge \eta$ such that if $\hat\eta_i \in (\hat\eta_p, 1]$ for some $i\in\mathcal{N}$, then $\Pi^{\mathrm{dp}} < \Pi^{\mathrm{c}}$.*

Proof. We directly adapt the proof of Theorem 1 in the paper here. Let

$$\bar h(\mathbf{y}) = \max_{v\in\mathbb{R}}\left\{v - \frac{1}{\eta}\mathbb{E}[(v - \pi(\sum_{i\in\mathcal{N}} y_i, \sum_{i\in\mathcal{N}} D_i))^+]\right\}$$

and $\bar h_0(y) = \max_{v\in\mathbb{R}}\left\{v - \frac{1}{\eta}\mathbb{E}[(v-\pi(y,\sum_{i\in\mathcal{N}} D_i))^+]\right\}$. Note that $\bar h(\mathbf{y}) = \bar h_0(\sum_{i\in\mathcal{N}} y_i)$ and recall that $h(\mathbf{y}) = \max_{v\in\mathbb{R}}\left\{v - \frac{1}{\eta}\mathbb{E}[(v - \sum_{i=1}^n \pi_i(y_i, D_i))^+]\right\}$. Observe that for any $\mathbf{y}$, we must have $h(\mathbf{y}) \le \bar h(\mathbf{y})$. In particular, $\Pi^{\mathrm{c}} = h(\mathbf{y}^{\mathrm{c}}) \le \bar h(\mathbf{y}^{\mathrm{c}})$. By concavity of $\bar h_0(\cdot)$ and $\bar h(\cdot)$, the level set $\{y | \bar h_0(y) \ge \Pi^{\mathrm{c}}\}$ and $\{\mathbf{y} | \bar h(\mathbf{y}) \ge \Pi^{\mathrm{c}}\}$ are both non-empty and convex. That is, there exist $0 < y^L \le y^U$ such that $\sum_{i\in\mathcal{N}} y_i^{\mathrm{c}} \in [y^L, y^U]$, and for any $\mathbf{y}$ such that $\sum_{i\in\mathcal{N}} y_i \in [y^L, y^U]$, we have $\bar h(\mathbf{y}) = \bar h_0(\sum_{i\in\mathcal{N}} y_i) \ge \Pi^{\mathrm{c}}$; and for any $\mathbf{y}$ such that $\sum_{i\in\mathcal{N}} y_i \ge y^U$, we have $\bar h(\mathbf{y}) = \bar h_0(\sum_{i\in\mathcal{N}} y_i) \le \Pi^{\mathrm{c}}$.

Let $\hat\eta_p = \inf\{\eta' \ge \eta | F_N^{-1}(\eta' r) > y^U\}$. Then, if $\hat\eta_i \in (\hat\eta_p, 1]$ for some $i\in\mathcal{N}$, we know that for any Nash equilibrium $\mathbf{y}^{\mathrm{NE}}$, it holds that $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} = F_N^{-1}(\hat\eta r)$, where $\hat\eta = \max_{i\in\mathcal{N}}\hat\eta_i \in (\hat\eta_p, 1]$. As a result, $\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}} > y^U$ and

$$\Pi^{\mathrm{dp}} = \rho\left(\pi(\sum_{i\in\mathcal{N}} y_i^{\mathrm{NE}}, \sum_{i\in\mathcal{N}} D_i)\right) < \Pi^{\mathrm{c}}.$$

□