Electronic Companion for "The Informational Role of Buyback Contracts"

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Appendix A: Proofs in Section 3

To derive the retailer's response, we first examine the retailer's pricing decision after the uncertain baseline demand has realized during the selling season. Conditional on the contract (w,r), the retailer's perceived return risk $\widehat{\theta}$, the realized baseline demand $\alpha_i \in \{\alpha_l, \alpha_h\}$ and the retailer's stocking quantity s, the retailer sets his retail price p_i by maximizing his posterior profit:[16]

$$\Pi_i^R\left(s \,\middle|\, w,r,\widehat{\theta}\right) := \max_{\substack{0\le p_i\le \alpha_i/\beta \\ d_i=\alpha_i-\beta p_i}} p_i \min\{d_i,s\} + \widehat{\theta} r\left[s-\min\{d_i,s\}\right] - ws, \tag{A.1}$$

whose solution, the retailer's price decision, is denoted as $p_i^R = p_i^R\left(s \,\middle|\, w,r,\widehat{\theta}\right)$ for $i=h,l$. We characterize the retailer's optimal pricing decision in the following lemma.

Lemma A.1. *For $i=l,h$, (1) if $s\le \frac{1}{2}\left(\alpha_i-\beta\widehat{\theta}r\right)$, then the retailer sets the price $p_i^R=(\alpha_i-s)/\beta$ to clear the stock; (2) if $s\ge \frac{1}{2}\left(\alpha_i-\beta\widehat{\theta}r\right)$, then the retailer sets the price $p_i^R=(\alpha_i+\beta\widehat{\theta}r)/(2\beta)$ and the unsold inventory at the end of the selling season is $s-\frac{1}{2}\left(\alpha_i-\beta\widehat{\theta}r\right)$. The retailer's optimal posterior profit is given by*

$$\Pi_i^R\left(s \,\middle|\, w,r,\widehat{\theta}\right) = \begin{cases} \left(\frac{\alpha_i-s}{\beta}-w\right)s, & \text{if} \quad s\le \frac{\alpha_i-\beta\widehat{\theta}r}{2}, \\ \frac{\left(\alpha_i-\beta\widehat{\theta}r\right)^2}{4\beta}-\left(w-\widehat{\theta}r\right)s, & \text{if} \quad s\ge\frac{\alpha_i-\beta\widehat{\theta}r}{2}. \end{cases} \tag{A.2}$$

Proof of Lemma A.1. We now rewrite (A.1) as

$$\Pi_i^R\left(s \,\middle|\, w,r,\widehat{\theta}\right) = \max_{0\le p_i\le\frac{\alpha_i}{\beta}} \left(p_i-\widehat{\theta}r\right)\min\{\alpha_i-\beta p_i,s\}-\left(w-\widehat{\theta}r\right)s,$$

where

$$\left(p_i-\widehat{\theta}r\right)\min\{\alpha_i-\beta p_i,s\} = \begin{cases} \left(p_i-\widehat{\theta}r\right)s, & \text{if} \quad p_i\le\frac{\alpha_i-s}{\beta}, \\ \left(p_i-\widehat{\theta}r\right)(\alpha_i-\beta p_i), & \text{if} \quad p_i\ge\frac{\alpha_i-s}{\beta}. \end{cases} \tag{A.3}$$

We note that $\left(p_i-\widehat{\theta}r\right)(\alpha_i-\beta p_i)$ is a quadratic function of p_i that achieves its (unconstrained) maximum at $p_i=(\alpha_i+\beta\widehat{\theta}r)/(2\beta)$. Therefore,

(1) if $\frac{\alpha_i+\beta\widehat{\theta}r}{2\beta}\le\frac{\alpha_i-s}{\beta}$, i.e., $s\le\frac{1}{2}\left(\alpha_i-\beta\widehat{\theta}r\right)$, (A.3) achieves its maximum at $p_i^R=(\alpha_i-s)/\beta$;

(2) if, instead, $\frac{\alpha_i+\beta\widehat{\theta}r}{2\beta}\ge\frac{\alpha_i-s}{\beta}$, i.e., $s\ge\frac{1}{2}\left(\alpha_i-\beta\widehat{\theta}r\right)$, (A.3) achieves its maximum at $p_i^R=(\alpha_i+\beta\widehat{\theta}r)/(2\beta)$.

Substituting the optimal price p_i^R into (A.3) immediately yields (A.2). □

Given his posterior profit function $\Pi_i^R\left(s \,\middle|\, w,r,\widehat{\theta}\right)$, the retailer maximizes the following *ex ante* expected profit by choosing the inventory stocking quantity at the beginning of the selling season:

$$\max_{s\ge0} \widehat{\lambda}\Pi_h^R\left(s \,\middle|\, w,r,\widehat{\theta}\right)+\widehat{\lambda}^c\Pi_l^R\left(s \,\middle|\, w,r,\widehat{\theta}\right). \tag{A.4}$$

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[16] By "posterior", we refer to "after the realization of uncertain baseline demand".

Proof of Lemma 1. For notational efficiency, we denote in this proof that $\widehat{\alpha} := \widehat{\lambda}\alpha_h + \widehat{\lambda}^c\alpha_l$. By Lemma A.1, the objective function in (A.4) reduces to

$$\begin{cases} \left(\frac{\widehat{\alpha}-s}{\beta} - w\right) s, & \text{if} \quad s \leq \frac{\alpha_l - \beta\widehat{\theta}r}{2}, \\ \widehat{\lambda}\frac{(\alpha_h - s)s}{\beta} - \left(w - \widehat{\lambda}^c\widehat{\theta}r\right) s + \widehat{\lambda}^c \frac{\left(\alpha_l - \beta\widehat{\theta}r\right)^2}{4\beta}, & \text{if} \quad \frac{\alpha_l - \beta\widehat{\theta}r}{2} \leq s \leq \frac{\alpha_h - \beta\widehat{\theta}r}{2}, \\ -\left(w - \widehat{\theta}r\right) s + \widehat{\lambda}^c \frac{\left(\alpha_l - \beta\widehat{\theta}r\right)^2}{4\beta} + \widehat{\lambda}\frac{\left(\alpha_h - \beta\widehat{\theta}r\right)^2}{4\beta}, & \text{if} \quad s \geq \frac{\alpha_h - \beta\widehat{\theta}r}{2}, \end{cases} \tag{A.5}$$

where $\left(\frac{\widehat{\alpha}-s}{\beta} - w\right) s$ is a quadratic function achieving its unconstrained maximum at $s_1 := \frac{\widehat{\alpha} - \beta w}{2}$; $\widehat{\lambda}\frac{(\alpha_h - s)s}{\beta} - \left(w - \widehat{\lambda}^c\widehat{\theta}r\right) s + \widehat{\lambda}^c \frac{\left(\alpha_l - \beta\widehat{\theta}r\right)^2}{4\beta}$ is also a quadratic function achieving its unconstrained maximum at $s_2 := \frac{\widehat{\lambda}^c\beta\widehat{\theta}r + \widehat{\lambda}\alpha_h - \beta w}{2\widehat{\lambda}}$; and $-\left(w - \widehat{\theta}r\right) s + \widehat{\lambda}^c \frac{\left(\alpha_l - \beta\widehat{\theta}r\right)^2}{4\beta} + \widehat{\lambda}\frac{\left(\alpha_h - \beta\widehat{\theta}r\right)^2}{4\beta}$ is a linear function of s.

- If $w - \widehat{\theta}r < 0$, obviously, the retailer would stock infinite inventory and earn infinite expected profit. As such, the manufacturer would never offer such a contract.
- If $0 \leq w - \widehat{\theta}r \leq \frac{\widehat{\lambda}\Delta\alpha}{\beta}$, then it is straightforward to verify that

$$\frac{\alpha_l - \beta\widehat{\theta}r}{2} \leq s_1 \leq s_2 \leq \frac{\alpha_h - \beta\widehat{\theta}r}{2}. \tag{A.6}$$

Therefore, $s^R = s_2$. By Lemma A.1, (A.6) implies that all inventory is sold out in the case of high baseline demand α_h realization while there is an excess of inventory $s_2 - \frac{\alpha_l - \beta\widehat{\theta}r}{2} = \frac{1}{2}\left[\Delta\alpha - \frac{\beta}{\widehat{\lambda}}\left(w - \widehat{\theta}r\right)\right]$ in the case of low baseline demand α_l realization.

- If $w - \widehat{\theta}r \geq \frac{\widehat{\lambda}\Delta\alpha}{\beta}$, then it is straightforward to verify that

$$s_2 \leq s_1 \leq \frac{\alpha_l - \beta\widehat{\theta}r}{2} \leq \frac{\alpha_h - \beta\widehat{\theta}r}{2}. \tag{A.7}$$

Therefore, $s^R = s_1$. By Lemma A.1, (A.7) implies that all inventory will be sold out whether the baseline demand is α_h or α_l. $\square$

Appendix B: Proofs in Section 4

Lemma B.1. *Given $\theta \in \left\{\underline{\theta}, \overline{\theta}\right\}$, any contract (w, r) such that $w - \theta r < 0$ is weakly dominated by a contract (w, r') such that $w - \theta r' \geq 0$.*

Proof of Lemma B.1. If $w - \theta r < 0$, because $w \geq 0$, there must exists $r' \in [0, r]$ such that $w - \theta r' \geq 0$. Then, we must have

$$\begin{aligned} \Pi\left(w, r' \,\middle|\, \widehat{\theta}(w, r'), \theta\right) =& \frac{\lambda^c\beta}{2\lambda} \underbrace{(w - \theta r')}_{\geq 0} \left[\frac{\lambda\Delta\alpha}{\beta} - w + \widehat{\theta}(w, r')r'\right]^+ + \frac{1}{2}\left[-\beta w^2 + \alpha w\right] \\ \geq& \frac{1}{2}\left(-\beta w^2 + \alpha w\right) \\ \geq& \frac{\lambda^c\beta}{2\lambda} \underbrace{(w - \theta r)}_{<0} \left[\frac{\lambda\Delta\alpha}{\beta} - w + \widehat{\theta}(w, r)r\right]^+ + \frac{1}{2}\left(-\beta w^2 + \alpha w\right) = \Pi\left(w, r \,\middle|\, \widehat{\theta}(w, r), \theta\right), \end{aligned}$$

which demonstrates the result. $\square$

Proof of Lemma 2. After simple algebraic manipulation, it is straightforward to verify that the manufacturer's expected profit in (2) can be rewritten as

$$\Pi\left(w, r \,\middle|\, \widehat{\theta}, \theta\right) = H\left(w, w - \theta r, w - \widehat{\theta}r\right), \tag{B.1}$$

where

$$H\left(w,u,\widehat{u}\right) := \frac{\lambda^c\beta}{2\lambda}u\left(\frac{\lambda\Delta\alpha}{\beta}-\widehat{u}\right)^{+}+\frac{1}{2}\left(-\beta w^2+\alpha w\right). \tag{B.2}$$

Let (w°, u°) be the solution to the following optimization problem:

$$\max_{w\geq 0, 0\leq u\leq w} H\left(w,u,u\right). \tag{B.3}$$

Then, $w^\circ(\theta)=w^\circ$ and $r^\circ(\theta)=(w^\circ-u^\circ)/\theta$ solves (3). Our general strategy to identify the solution (w°, u°) to (B.3) is to first optimize $H\left(w,u,u\right)$ over $u\in[0,w]$, and then optimize the resulting function that contains only w.

We first note that function $u\left(\lambda\Delta\alpha/\beta-u\right)^{+}$ is increasing in $u\in[0,\lambda\Delta\alpha/(2\beta)]$, decreasing in $u\in[\lambda\Delta\alpha/(2\beta),\lambda\Delta\alpha/\beta]$, and remains a constant zero for $u\geq\lambda\Delta\alpha/\beta$. Therefore, we examine the following two cases.

1. For any $w\leq\lambda\Delta\alpha/(2\beta)$, the maximizing u in (B.3) must equal w, suggesting

$$\max_{0\leq u\leq w} H\left(w,u,u\right)=H\left(w,w,w\right)=\frac{1}{2\lambda}\left[-\beta w^2+\lambda\alpha_h w\right],$$

which is increasing $w\leq\lambda\Delta\alpha/(2\beta)<\alpha/(2\beta)$. As such, we only need to restrict to $w\geq\lambda\Delta\alpha/(2\beta)$.

2. For any $w\geq\lambda\Delta\alpha/(2\beta)$, we have $\max\limits_{0\leq u\leq w} H\left(w,u,u\right)$ is achieved at $u^\circ=\lambda\Delta\alpha/(2\beta)$, suggesting

$$\max_{0\leq u\leq w} H\left(w,u,u\right)=\frac{\lambda^c\left(\lambda\Delta\alpha-\beta c\right)^2}{8\beta\lambda}+\frac{1}{2}\left[-\beta w^2+\alpha w\right],$$

which is quadratic in w and achieves its maximum $\Pi^\circ\left(\theta\right)=\frac{\lambda^c\lambda(\Delta\alpha)^2+\alpha^2}{8\beta}$ at $w^\circ=\alpha/(2\beta)$. Thus, $r^\circ(\theta)=(w^\circ-u^\circ)/\theta=\alpha_l/(2\beta\theta)$. □

Since $w^\circ-\theta r^\circ(\theta)=\frac{\alpha}{2\beta}-\frac{\alpha_l}{2\beta}=\frac{\lambda\Delta\alpha}{2\beta}\leq\frac{\lambda\Delta\alpha}{\beta}$, which, by (1) of Lemma 1, suggests that the retailer orders

$$s^\circ=s^R(w^\circ,r^\circ(\theta),\theta,\lambda)=\frac{\lambda^c\beta\theta r^\circ(\theta)+\lambda\alpha_h-\beta w^\circ}{2\lambda}=\frac{\alpha_h}{4}$$

and that inventory is sold out if the baseline demand is high, but that unsold inventory of an amount $\frac{1}{2}\left[\Delta\alpha-\frac{\beta}{\lambda}\left(w^\circ-\theta r^\circ(\theta)\right)\right]=\frac{\Delta\alpha}{4}$ will be requested by the retailer to return. □

Lemma B.2. *In the returns risk signaling game, the unique equilibrium that survives the intuitive criterion is the most efficient separating equilibrium. In this equilibrium, the riskier manufacturer offers her symmetric information contract* $(w^\circ, \underline{r}^\circ)$.

Proof of Lemma B.2. We first note that $\Pi\left(w,r\,\middle|\,\widehat{\theta},\theta\right)$ is increasing in $\widehat{\theta}$, as direct calculation reveals that

$$\Pi_{\widehat{\theta}}\left(w,r\,\middle|\,\widehat{\theta},\theta\right)=\begin{cases}\frac{\lambda^c\beta}{2\lambda}(w-\theta r)r\geq 0, & \text{if}\quad \beta(w-\widehat{\theta}r)\geq\lambda\Delta\alpha,,\\ 0, & \text{if}\quad \beta(w-\widehat{\theta}r)\leq\lambda\Delta\alpha.\end{cases}$$

By the weakened condition of Cho and Sobel (1990) in Engers (1987), it suffices to show that the marginal rate of substitution (MRS) of *one* of signals (i.e., w or r) for the belief $\widehat{\theta}$ is monotonic in θ. Direct calculation reveals that the MRS of r for $\widehat{\theta}$ is

$$-\frac{\Pi_r\left(w,r\,\middle|\,\widehat{\theta},\theta\right)}{\Pi_{\widehat{\theta}}\left(w,r\,\middle|\,\widehat{\theta},\theta\right)}=\frac{\lambda\Delta\alpha-\beta(w-\widehat{\theta}r)}{\beta r\left[w/\theta-r\right]}-\frac{\widehat{\theta}}{r},\quad \text{for } \beta(w-\widehat{\theta}r)\leq\lambda\Delta\alpha,$$

which is monotonically increasing in θ. (For $\beta(w-\widehat{\theta}r)\geq\lambda\Delta\alpha$, $\Pi\left(w,r\,\middle|\,\widehat{\theta},\theta\right)$ is independent of r, θ, and $\widehat{\theta}$, and hence is irrelevant.)

We now show that the riskier manufacturer must offer the symmetric-information contract terms in any separating equilibrium. By way of contradiction, suppose the manufacturer of type $\underline{\theta}$ offers another contract $(\underline{w},\underline{r})\neq(w^\circ,\underline{r}^\circ)$ in a separating equilibrium and, hence, earns an expected profit of $\Pi\left(\underline{w},\underline{r}\,|\,\underline{\theta},\underline{\theta}\right)$. If manufacturer $\underline{\theta}$ deviates to $(w^\circ,\underline{r}^\circ)$, let $\underline{\widehat{\theta}}\geq\underline{\theta}$ be the retailer's perceived manufacturer's return risk such that $w^\circ-\underline{\widehat{\theta}}\underline{r}^\circ\geq 0$. By the unique optimality of $(w^\circ,\underline{r}^\circ)$ (Lemma 2), we immediately have

$$\Pi\left(\underline{w},\underline{r}\,|\,\underline{\theta},\underline{\theta}\right)<\pi^\circ=\Pi\left(w^\circ,\underline{r}^\circ\,|\,\underline{\theta},\underline{\theta}\right)\leq\Pi\left(w^\circ,\underline{r}^\circ\,\middle|\,\underline{\widehat{\theta}},\underline{\theta}\right),$$

where the last inequality follows from $\Pi_{\widehat{\theta}}\left(w^\circ,\underline{r}^\circ\,\middle|\,\underline{\widehat{\theta}},\underline{\theta}\right)$ is of the sign $w^\circ-\underline{\theta}\underline{r}^\circ=\lambda\Delta\alpha/(2\beta)>0$ and Lemma 2). Therefore, the manufacturer $\underline{\theta}$ can be strictly better off by deviating to $(w^\circ,\underline{r}^\circ)$, which proves that $(\underline{w},\underline{r})$ cannot be played in that separating equilibrium. □

Road map of remaining proofs. The rest of this section is to identify the separating contracts for the less risky manufacturer and establish their properties. As standard in the literature, our general strategy is to first recognize a separating contract as the solution to a constrained optimization problem that maximizes the less risky manufacturer's profit (or equivalently minimize her signaling cost) subject to the riskier manufacturer's non-mimicry incentive constraint (the less risky manufacturer's non-mimicry constraint is always non-binding). Then, we verify that such separating contract can be supported by the most pessimistic off-equilibrium belief (i.e., any deviation away from the separating contract would lead the retailer to believe that the manufacturer is of higher returns risk). More specifically, we first transform the less risky manufacturer's optimization problem in terms of the price decisions to one in terms of the retailer's induced quantity decisions through a change of variable that is essentially equivalent to (5) and (6) but re-centers them at zero stocks (see Lemma B.3). Propositions B.1 and C.1 establish the partial signaling benchmarks formulated in (9) and (10), respectively. In particular, these two problems involves a single decision variable and the optimal solutions can be obtained in closed form, so the supporting off-equilibrium belief can be directly verified. The most efficient separating contract formulated in (4) does not admit closed-form characterization. Despite this challenge, Lemma B.6 identifies its direction of distortion through Lagrangian, and Lemma B.7 shows that the less risky manufacturer has no incentive to deviate from the most efficient separating contract under the pessimistic off-equilibrium belief. All these results culminate in the proofs of Propositions 1 and 2.

LEMMA B.3 **(Change of Variable)**. *For any* $(\overline{w},\overline{r})$ *feasible to* (4), *let* $\widetilde{w}:=\overline{w}-w^\circ=\overline{w}-\frac{\alpha}{2\beta}$ *and* $\widetilde{u}:=\overline{w}-w^\circ-\overline{\theta}\left(\overline{r}-\overline{r}^\circ\right)=\overline{w}-\overline{\theta}\overline{r}-\frac{\lambda\Delta\alpha}{2\beta}$. *Then,* $\overline{w}=w^\circ+\widetilde{w}$, $\overline{r}=\overline{r}^\circ+(\widetilde{w}-\widetilde{u})/\overline{\theta}$, *and it is without loss of generality to restrict to*

$$\widetilde{w}+\frac{\alpha_l}{2\beta}\geq\widetilde{u}\geq-\frac{\lambda\Delta\alpha}{2\beta},\quad \textit{and}\quad \widetilde{u}\leq\frac{\lambda\Delta\alpha}{2\beta}. \tag{B.4}$$

Furthermore, the objective function of (4) *can be written as*

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\overline{\theta},\overline{\theta}\right)=\pi^\circ-\frac{\beta}{2\lambda}\left(\lambda\widetilde{w}^2+\lambda^c\widetilde{u}^2\right), \tag{B.5}$$

the first constraint in (4) *is equivalent to*

$$\lambda\widetilde{w}^2 + \lambda^c\widetilde{u}^2 + \frac{\lambda^c\Delta\theta}{\bar{\theta}}\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\left(\widetilde{u} - \frac{\lambda\Delta\alpha}{2\beta}\right) \geq 0, \tag{B.6}$$

and the second constraint in (4) *is equivalent to*

$$\lambda\widetilde{w}^2 + \lambda^c\widetilde{u}^2 \leq \frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\underline{\theta}}. \tag{B.7}$$

Proof. The first constraint in (B.4) follows from the fact that $\overline{w} \geq \bar{\theta}\overline{r} \geq 0$ by Lemma B.1. By (B.1), it is straightforward to verify that

$$\Pi\left(\overline{w}, \overline{r} \,\middle|\, \bar{\theta}, \bar{\theta}\right) = \begin{cases} \frac{\alpha^2}{8\beta} - \frac{\beta}{2}\widetilde{w}^2, & \text{if } \widetilde{u} \geq \frac{\lambda\Delta\alpha}{2\beta}, \\ \pi^\circ - \frac{\beta}{2\lambda}\left(\lambda\widetilde{w}^2 + \lambda^c\widetilde{u}^2\right), & \text{if } \widetilde{u} \leq \frac{\lambda\Delta\alpha}{2\beta}; \end{cases}$$

$$\Pi\left(\overline{w}, \overline{r} \,\middle|\, \bar{\theta}, \underline{\theta}\right) = \begin{cases} \frac{\alpha^2}{8\beta} - \frac{\beta}{2}\widetilde{w}^2, & \text{if } \widetilde{u} \geq \frac{\lambda\Delta\alpha}{2\beta}, \\ \pi^\circ - \frac{\beta}{2\lambda}\left[\lambda\widetilde{w}^2 + \lambda^c\widetilde{u}^2 + \frac{\lambda^c\Delta\theta}{\bar{\theta}}\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\left(\widetilde{u} - \frac{\lambda\Delta\alpha}{2\beta}\right)\right], & \text{if } \widetilde{u} \leq \frac{\lambda\Delta\alpha}{2\beta}. \end{cases}$$

Also, direct calculation yields

$$\begin{aligned}
\Pi\left(w^\circ, \underline{r}^\circ \,\middle|\, \underline{\theta}, \bar{\theta}\right) =& H\left(w^\circ, w^\circ - \bar{\theta}\underline{r}^\circ, w^\circ - \underline{\theta}\underline{r}^\circ\right) \\
=& \frac{\lambda^c\beta}{2\lambda}\left(w^\circ - \bar{\theta}\underline{r}^\circ\right)\left(\frac{\lambda\Delta\alpha}{\beta} - \left(w^\circ - \underline{\theta}\underline{r}^\circ\right)\right) + \frac{1}{2}\left[-\beta\left(w^\circ\right)^2 + \alpha w^\circ\right] \\
=& \frac{\lambda^c\beta}{2\lambda}\left(w^\circ - \underline{\theta}\underline{r}^\circ\right)\left(\frac{\lambda\Delta\alpha}{\beta} - \left(w^\circ - \underline{\theta}\underline{r}^\circ\right)\right) + \frac{1}{2}\left[-\beta\left(w^\circ\right)^2 + \alpha w^\circ\right] - \frac{\lambda^c\beta}{2\lambda}\Delta\theta\underline{r}^\circ\left(\frac{\lambda\Delta\alpha}{\beta} - \left(w^\circ - \underline{\theta}\underline{r}^\circ\right)\right) \\
=& \pi^\circ - \frac{\lambda^c\Delta\theta\alpha_l\Delta\alpha}{8\beta\underline{\theta}}.
\end{aligned}$$

We claim that we can restrict the search for the optimal solution of (4) within $\widetilde{u} \leq \frac{\lambda\Delta\alpha}{2\beta}$, under which the objective function in (4) is equivalent to (B.5) while the two constraints in (4)reduce to (B.6) and (B.7), respectively. Indeed, for $\widetilde{u} \geq \frac{\lambda\Delta\alpha}{2\beta}$, the first constraint in (4)automatically holds, while the second one reduces to $\widetilde{w}^2 \leq \frac{\lambda^c\Delta\alpha\left(\bar{\theta}\alpha_l - \underline{\theta}\alpha\right)}{4\beta^2\underline{\theta}}$, reducing (4) to

$$\min_{\widetilde{w} + \frac{\alpha_l}{2\beta} \geq \widetilde{u} \geq \frac{\lambda\Delta\alpha}{2\beta}} \widetilde{w}^2, \quad \text{subject to} \quad \widetilde{w}^2 \leq \frac{\lambda^c\Delta\alpha\left(\bar{\theta}\alpha_l - \underline{\theta}\alpha\right)}{4\beta^2\underline{\theta}}.$$

As the decision variable $\widetilde{u}$ is absent from the objective function as well as the other constraint, it can without loss of generality be taken as $\widetilde{u} = \frac{\lambda\Delta\alpha}{2\beta}$, allowing us to focus on $\widetilde{u} \leq \frac{\lambda\Delta\alpha}{2\beta}$. ☐

PROPOSITION B.1. *The solution to* (9) *is given by*

$$\overline{w}^\ddagger = \frac{\alpha}{2\beta} + \frac{1}{4\beta\bar{\bar{\theta}}}\left\{\sqrt{\left(\lambda^c\alpha_l\Delta\theta\right)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta} - \lambda^c\alpha_l\Delta\theta\right\} > w^\circ. \tag{B.8}$$

Contract $(\overline{w}^\ddagger, \bar{r}^\circ)$ *can be sustained as a separating equilibrium of the returns risk signaling game if and only if* $\Delta\theta/\bar{\theta} \leq (1 + \sqrt{\lambda^c})\Delta\alpha/\alpha_l$. *In this equilibrium, the retailer's order quantity and unsold inventory in case of low baseline demand are given by*

$$\bar{s}^\ddagger = \frac{\alpha_h}{4} + \frac{1}{8\lambda\bar{\bar{\theta}}}\left\{\lambda^c\alpha_l\Delta\theta - \sqrt{(\lambda^c\alpha_l\Delta\theta)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta}\right\} < s^\circ, \ \textit{and} \tag{B.9}$$

$$\bar{q}^\ddagger = \frac{\Delta\alpha}{4} + \frac{1}{8\lambda\bar{\bar{\theta}}}\left\{\lambda^c\alpha_l\Delta\theta - \sqrt{(\lambda^c\alpha_l\Delta\theta)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta}\right\} < q^\circ, \ \textit{respectively;} \tag{B.10}$$

no unsold inventory results from high baseline demand realization.

Proof. We solve (9) by first ignoring the second constraint $\Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \bar{\theta},\bar{\theta}\right) \geq \Pi\left(w^\circ,\underline{r}^\circ \,\middle|\, \underline{\theta},\bar{\theta}\right)$ and then verifying that it will be satisfied by the solution to the relaxed problem. Using the change of variable in Lemma B.3, we have the solution to the relaxed problem $\overline{w}^\ddagger = w^\circ + \widetilde{w}^\ddagger$, where $\widetilde{w}^\ddagger$ is the solution to

$$\min_{-\frac{\lambda\Delta\alpha}{2\beta}\leq\widetilde{w}\leq\frac{\lambda\Delta\alpha}{2\beta}} \widetilde{w}^2, \quad \text{subject to } \widetilde{w}^2 + \frac{\lambda^c\alpha_l\Delta\theta}{2\beta\bar{\theta}}\left(\widetilde{w} - \frac{\lambda\Delta\alpha}{2\beta}\right) \geq 0. \tag{B.11}$$

As the quadratic function on the left-hand side of the constraint in (B.11) achieves its minimum at $\widetilde{w} = -\frac{\lambda^c\alpha_l\Delta\theta}{4\beta\bar{\theta}} < 0$, the optimal solution to (B.11) is thus given by its larger (and positive) root

$$\widetilde{w}^\ddagger = -\frac{\lambda^c\alpha_l\Delta\theta}{4\beta\bar{\theta}} + \frac{1}{4\beta\bar{\theta}}\sqrt{\left(\lambda^c\alpha_l\Delta\theta\right)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta} > 0, \tag{B.12}$$

from which (B.8) follows immediately. In particular, we note that

$$\begin{aligned}\widetilde{w}^\ddagger < \frac{\lambda\Delta\alpha}{2\beta} \quad &\Leftrightarrow \quad \frac{4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta}{4\beta\bar{\theta}\left[\lambda^c\alpha_l\Delta\theta + \sqrt{\left(\lambda^c\alpha_l\Delta\theta\right)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta}\right]} < \frac{\lambda\Delta\alpha}{2\beta}\\ &\Leftrightarrow \quad 2\lambda^c\alpha_l\Delta\theta < \lambda^c\alpha_l\Delta\theta + \sqrt{\left(\lambda^c\alpha_l\Delta\theta\right)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta},\end{aligned}$$

which obviously holds.

To verify the ignored constraint $\Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \bar{\theta},\bar{\theta}\right) \geq \Pi\left(w^\circ,\underline{r}^\circ \,\middle|\, \underline{\theta},\bar{\theta}\right)$, it suffices to show, by (B.7), that $\left(\widetilde{w}^\ddagger\right)^2 \leq \frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\underline{\theta}}$. As $\widetilde{w}^\ddagger$ binds the constraint in (B.11), this is equivalent to

$$\left(\widetilde{w}^\ddagger\right)^2 = \frac{\lambda^c\alpha_l\Delta\theta}{2\beta\bar{\theta}}\left(\frac{\lambda\Delta\alpha}{2\beta} - \widetilde{w}^\ddagger\right) \leq \frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\underline{\theta}} \quad \Leftrightarrow \quad \widetilde{w}^\ddagger \geq \frac{\lambda\Delta\alpha}{2\beta}\left(1 - \bar{\theta}/\underline{\theta}\right),$$

which obviously holds by (B.12).

For contract $\left(\overline{w}^\ddagger,\bar{r}^\circ\right)$ to be sustained by some equilibrium belief, we need to show that neither the less risky nor the riskier manufacturer has an incentive to deviate to any off-equilibrium strategy under that belief.

- For the riskier manufacturer, we have shown that her profit of deviating to any $\left(\overline{w},\bar{r}^\circ\right)$ with $\overline{w} \geq \overline{w}^\ddagger$ and hence being mistaken as a less risky type is dominated by her equilibrium profit, i.e., $\Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \bar{\theta},\underline{\theta}\right) \leq \pi^\circ$. Since all other $\left(\underline{w},\underline{r}\right)$ induces a belief that she is the riskier type, the symmetric-information $\left(w^\circ,\underline{r}^\circ\right)$ maximizes her profit $\Pi\left(\underline{w},\underline{r} \,|\, \underline{\theta},\underline{\theta}\right)$ to π°. Therefore, the riskier manufacturer indeed has no incentive to deviate from her symmetric-information $\left(w^\circ,\underline{r}^\circ\right)$.
- For the less risky manufacturer whose return price is restricted to $\bar{r}^\circ$, it suffices to show that she has no incentive to deviate her wholesale price to any $\overline{w} \neq \overline{w}^\ddagger$ and thus to be mistaken as a riskier type $\underline{\theta}$, i.e., $\Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \underline{\theta},\bar{\theta}\right) \leq \Pi\left(\overline{w}^\ddagger,\bar{r}^\circ \,\middle|\, \bar{\theta},\bar{\theta}\right)$ for all $\overline{w}$. Indeed, if this condition fails, no other deviation belief $\widehat{\theta}$ can support $\overline{w}^\ddagger$, because $\Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \widehat{\theta},\bar{\theta}\right)$ is non-decreasing in $\widehat{\theta}$ as pointed out in the proof of Lemma B.2 and hence $\Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \widehat{\theta},\bar{\theta}\right) \geq \Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \underline{\theta},\bar{\theta}\right) > \Pi\left(\overline{w}^\ddagger,\bar{r}^\circ \,\middle|\, \bar{\theta},\bar{\theta}\right)$ for all $\widehat{\theta} \geq \underline{\theta}$. In light of the fact that $\Pi\left(\overline{w}^\ddagger,\bar{r}^\circ \,\middle|\, \bar{\theta},\underline{\theta}\right) = \pi^\circ$, it is equivalent to show that

$$\Pi\left(\overline{w},\bar{r}^\circ \,\middle|\, \underline{\theta},\bar{\theta}\right) - \pi^\circ \leq \Pi\left(\overline{w}^\ddagger,\bar{r}^\circ \,\middle|\, \bar{\theta},\bar{\theta}\right) - \Pi\left(\overline{w}^\ddagger,\bar{r}^\circ \,\middle|\, \bar{\theta},\underline{\theta}\right). \tag{B.13}$$

Direct calculation reveals

$$\Pi\left(\overline{w}^\ddagger,\bar{r}^\circ \,\middle|\, \bar{\theta},\bar{\theta}\right) - \Pi\left(\overline{w}^\ddagger,\bar{r}^\circ \,\middle|\, \bar{\theta},\underline{\theta}\right) = -\frac{\lambda^c\beta}{2\lambda}\Delta\theta\bar{r}^\circ\left(\frac{\lambda\Delta\alpha}{\beta} - \overline{w}^\ddagger + \bar{\theta}\bar{r}^\circ\right)$$

$$=-\frac{\lambda^c\alpha_l\Delta\theta}{8\beta\lambda\bar\theta}\left[\lambda\Delta\alpha+\frac{\lambda^c\alpha_l\Delta\theta}{2\bar\theta}-\sqrt{\frac{\left(\lambda^c\alpha_l\Delta\theta\right)^2}{4\bar\theta^2}+\frac{\lambda\lambda^c\alpha_l\Delta\alpha\Delta\theta}{\bar\theta}}\right], \tag{B.14}$$

and

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}^\circ\,\middle|\,\underline{\theta},\bar\theta\right)-\pi^\circ=&\frac{\lambda^c\beta}{2\lambda}\left(\overline{w}-\bar\theta\overline{r}^\circ\right)\left(\frac{\lambda\Delta\alpha}{\beta}-\overline{w}+\underline{\theta}\overline{r}^\circ\right)^+-\frac{\beta}{2}\left(\overline{w}-\frac{\alpha}{2\beta}\right)^2-\frac{\lambda^c\lambda(\Delta\alpha)^2}{8\beta}\\=&\frac{\lambda^c\beta}{2\lambda}\left(\widetilde{w}+\frac{\lambda\Delta\alpha}{2\beta}\right)\left(\frac{\lambda\Delta\alpha}{2\beta}-\frac{\alpha_l\Delta\theta}{2\beta\bar\theta}-\widetilde{w}\right)^+-\frac{\beta}{2}\widetilde{w}^2-\frac{\lambda^c\lambda(\Delta\alpha)^2}{8\beta}.\end{aligned} \tag{B.15}$$

Therefore, (B.13) is equivalent to showing

$$\begin{aligned}b(\widetilde{w}):=&\ \lambda^c\left(\widetilde{w}+\tfrac{\lambda\Delta\alpha}{2\beta}\right)\left(\tfrac{\lambda\Delta\alpha}{2\beta}-\tfrac{\alpha_l\Delta\theta}{2\beta\bar\theta}-\widetilde{w}\right)^+-\lambda\widetilde{w}^2\\\le&\ \tfrac{\lambda^c(\lambda\Delta\alpha)^2}{4\beta^2}-\tfrac{\lambda^c\alpha_l\Delta\theta}{4\beta^2\bar\theta}\left[\lambda\Delta\alpha+\tfrac{\lambda^c\alpha_l\Delta\theta}{2\bar\theta}-\sqrt{\tfrac{(\lambda^c\alpha_l\Delta\theta)^2}{4\bar\theta^2}+\tfrac{\lambda\lambda^c\alpha_l\Delta\alpha\Delta\theta}{\bar\theta}}\right],\quad\forall\widetilde{w}\ge-\tfrac{\alpha}{2\beta}.\end{aligned} \tag{B.16}$$

We note that the piecewise quadratic equation

$$b(\widetilde{w})=\begin{cases}-\lambda\widetilde{w}^2, & \text{if }\widetilde{w}\ge\frac{1}{2\beta}\left(\lambda\Delta\alpha-\frac{\alpha_l\Delta\theta}{\bar\theta}\right),\\-\widetilde{w}^2-\frac{\lambda^c\alpha_l\Delta\theta}{2\beta\bar\theta}\widetilde{w}+\frac{\lambda^c\lambda\Delta\alpha}{4\beta^2}\left(\lambda\Delta\alpha-\frac{\alpha_l\Delta\theta}{\bar\theta}\right), & \text{if }\widetilde{w}\le\frac{1}{2\beta}\left(\lambda\Delta\alpha-\frac{\alpha_l\Delta\theta}{\bar\theta}\right),\end{cases} \tag{B.17}$$

achieves its maximum

$$\max b(\widetilde{w})=\begin{cases}\left(\frac{\lambda^c\alpha_l\Delta\theta}{4\beta\bar\theta}\right)^2+\frac{\lambda^c\lambda\Delta\alpha}{4\beta^2}\left(\lambda\Delta\alpha-\frac{\alpha_l\Delta\theta}{\bar\theta}\right), & \text{if }\lambda\Delta\alpha\ge\frac{1+\sqrt{\lambda}}{2}\frac{\alpha_l\Delta\theta}{\bar\theta},\\0, & \text{if }\lambda\Delta\alpha\le\frac{1+\sqrt{\lambda}}{2}\frac{\alpha_l\Delta\theta}{\bar\theta},\end{cases} \tag{B.18}$$

at $\widetilde{w}=-\frac{\lambda^c\alpha_l\Delta\theta}{4\beta\bar\theta}>-\frac{\alpha}{2\beta}$ and $\widetilde{w}=0>-\frac{\alpha}{2\beta}$, respectively.

—For $\lambda\Delta\alpha\ge\frac{1+\sqrt{\lambda}}{2}\frac{\alpha_l\Delta\theta}{\bar\theta}$, we have

$$\left(\frac{\lambda^c\alpha_l\Delta\theta}{4\beta\bar\theta}\right)^2+\frac{\lambda^c\lambda\Delta\alpha}{4\beta^2}\left(\lambda\Delta\alpha-\frac{\alpha_l\Delta\theta}{\bar\theta}\right)\le\frac{\lambda^c(\lambda\Delta\alpha)^2}{4\beta^2}-\frac{\lambda^c\alpha_l\Delta\theta}{4\beta^2\bar\theta}\left[\lambda\Delta\alpha+\frac{\lambda^c\alpha_l\Delta\theta}{2\bar\theta}-\sqrt{\frac{\left(\lambda^c\alpha_l\Delta\theta\right)^2}{4\bar\theta^2}+\frac{\lambda\lambda^c\alpha_l\Delta\alpha\Delta\theta}{\bar\theta}}\right]$$

$$\Leftrightarrow\quad\frac{3}{4}\frac{\lambda^c\alpha_l\Delta\theta}{\bar\theta}\le\sqrt{\frac{\left(\lambda^c\alpha_l\Delta\theta\right)^2}{4\bar\theta^2}+\frac{\lambda\lambda^c\alpha_l\Delta\alpha\Delta\theta}{\bar\theta}}\quad\Leftrightarrow\quad\frac{5}{16}\frac{\lambda^c\alpha_l\Delta\theta}{\bar\theta}\le\lambda\Delta\alpha,$$

which holds when $\lambda\Delta\alpha\ge\frac{1+\sqrt{\lambda}}{2}\frac{\alpha_l\Delta\theta}{\bar\theta}$.

—For $\lambda\Delta\alpha\le\frac{1+\sqrt{\lambda}}{2}\frac{\alpha_l\Delta\theta}{\bar\theta}$, we have

$$\begin{aligned}&\frac{\lambda^c(\lambda\Delta\alpha)^2}{4\beta^2}-\frac{\lambda^c\alpha_l\Delta\theta}{4\beta^2\bar\theta}\left[\lambda\Delta\alpha+\frac{\lambda^c\alpha_l\Delta\theta}{2\bar\theta}-\sqrt{\frac{\left(\lambda^c\alpha_l\Delta\theta\right)^2}{4\bar\theta^2}+\frac{\lambda\lambda^c\alpha_l\Delta\alpha\Delta\theta}{\bar\theta}}\right]\ge0\\\Leftrightarrow\quad&\left(\lambda\Delta\alpha\right)^2-2\left(\lambda\Delta\alpha\right)\left(\frac{\alpha_l\Delta\theta}{\bar\theta}\right)+\lambda\left(\frac{\alpha_l\Delta\theta}{\bar\theta}\right)^2\le0,\end{aligned}$$

which holds for $\left(1-\sqrt{\lambda^c}\right)\left(\frac{\alpha_l\Delta\theta}{\bar\theta}\right)\le\lambda\Delta\alpha\le\frac{1+\sqrt{\lambda}}{2}\frac{\alpha_l\Delta\theta}{\bar\theta}$, i.e., if and only if $\Delta\theta/\bar\theta\le(1+\sqrt{\lambda^c})\Delta\alpha/\alpha_l$.

Finally, we determine the retailer's order quantity as well as unsold inventory. By Lemma 1, since we have $\overline{w}^\ddagger-\bar\theta\overline{r}^\circ=w^\circ+\widetilde{w}^\ddagger-\frac{\alpha_l}{2\beta}=\widetilde{w}^\ddagger+\frac{\lambda\Delta\alpha}{2\beta}\le\frac{\lambda\Delta\alpha}{\beta}$ because $\widetilde{w}^\ddagger<\frac{\lambda\Delta\alpha}{2\beta}$, all inventory is sold out in the case of high baseline demand realization and, in particular, (1) implies that the retailer's stocking quantity is

$$\begin{aligned}\bar{s}^\ddagger=\frac{\lambda^c\beta\bar\theta\overline{r}^\circ+\lambda\alpha_h-\beta\overline{w}^\ddagger}{2\lambda}=&\frac{\lambda^c\beta\bar\theta\overline{r}^\circ+\lambda\alpha_h-\beta w^\circ}{2\lambda}+\frac{\beta}{2\lambda}\left(w^\circ-\overline{w}^\ddagger\right)\\=&\frac{\alpha_h}{4}+\frac{1}{8\lambda\bar\theta}\left\{\lambda^c\alpha_l\Delta\theta-\sqrt{(\lambda^c\alpha_l\Delta\theta)^2+4\lambda\lambda^c\alpha_l\Delta\alpha\bar\theta\Delta\theta}\right\},\end{aligned}$$

obtaining (B.9). By Lemma 1, again, the unsold inventory in the case of low baseline demand realization is

$$\bar{q}^\ddagger=\frac{1}{2}\left[\Delta\alpha-\beta/\lambda\left(\overline{w}^\ddagger-\bar\theta\overline{r}^\circ\right)\right]=\frac{1}{2}\left[\Delta\alpha/2-\beta/\lambda\widetilde{w}^\ddagger\right],$$

from which (B.10) follows by (B.12). □

Proposition B.2. *The solution to (10) is given by*

$$\bar{r}^{\dagger}=\frac{\alpha_l}{2\beta\bar{\theta}}+\frac{1}{4\beta\underline{\theta}\bar{\theta}}\left\{\alpha\Delta\theta-\sqrt{(\alpha\Delta\theta)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}\right\}<\bar{r}^{\circ}<\underline{r}^{\circ}. \tag{B.19}$$

Contract $(w^{\circ},\bar{r}^{\dagger})$ can always be sustained as a separating equilibrium of the returns risk signaling game, in which the retailer's order quantity and unsold inventory in case of low baseline demand are given by

$$\bar{s}^{\dagger}=\frac{\alpha_h}{4}+\frac{\lambda^c}{8\lambda\underline{\theta}}\left\{\alpha\Delta\theta-\sqrt{(\alpha\Delta\theta)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}\right\}<s^{\circ},\ \textit{and} \tag{B.20}$$

$$\bar{q}^{\dagger}=\frac{\Delta\alpha}{4}+\frac{1}{8\lambda\underline{\theta}}\left\{\alpha\Delta\theta-\sqrt{(\alpha\Delta\theta)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}\right\}<q^{\circ},\ \textit{respectively;} \tag{B.21}$$

no unsold inventory results from high baseline demand realization.

Proof of Proposition B.2. We solve (10) by first ignoring the second constraint $\Pi\left(w^{\circ},\bar{r}\,\middle|\,\bar{\theta},\bar{\theta}\right)\geq\Pi\left(w^{\circ},\underline{r}^{\circ}\,\middle|\,\underline{\theta},\bar{\theta}\right)$ and then verifying that it will be satisfied by the solution to the relaxed problem. Using the change of variable in Lemma B.3, we have the solution to the relaxed problem $\bar{r}^{\dagger}=\bar{r}^{\circ}-\widetilde{u}^{\dagger}/\bar{\theta}$, where $\widetilde{u}^{\dagger}$ is the solution to

$$\min_{\widetilde{u}\in\left[-\frac{\lambda\Delta\alpha}{2\beta},\frac{\lambda\Delta\alpha}{2\beta}\wedge\frac{\alpha_l}{2\beta}\right]}\widetilde{u}^2,\quad\text{subject to }\widetilde{u}^2+\frac{\Delta\theta}{\bar{\theta}}\left(\frac{\alpha_l}{2\beta}-\widetilde{u}\right)\left(\widetilde{u}-\frac{\lambda\Delta\alpha}{2\beta}\right)\geq 0. \tag{B.22}$$

Straightforward algebra reduces the constraint in (B.22) to

$$\widetilde{u}^2+\frac{\alpha\Delta\theta}{2\beta\underline{\theta}}\widetilde{u}-\frac{\lambda\Delta\alpha\alpha_l\Delta\theta}{4\beta^2\underline{\theta}}\geq 0, \tag{B.23}$$

where the quadratic function on the left-hand side is minimized at $\widetilde{u}=-\frac{\alpha\Delta\theta}{4\beta\underline{\theta}}<0$. Therefore, (B.22) is minimized at its larger (and positive) root

$$\widetilde{u}^{\dagger}=-\frac{\alpha\Delta\theta}{4\beta\underline{\theta}}+\frac{1}{4\beta\underline{\theta}}\sqrt{\left(\alpha\Delta\theta\right)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}>0, \tag{B.24}$$

from which (B.19) follows. In particular, we notice that

$$\begin{aligned}\widetilde{u}^{\dagger}<\frac{\lambda\Delta\alpha}{2\beta}\quad&\Leftrightarrow\quad\frac{4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}{4\beta\underline{\theta}\left[\alpha\Delta\theta+\sqrt{\left(\alpha\Delta\theta\right)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}\right]}<\frac{\lambda\Delta\alpha}{2\beta}\\&\Leftrightarrow\quad 2\alpha_l\Delta\theta<(2\alpha\Delta\theta<)\alpha\Delta\theta+\sqrt{\left(\alpha\Delta\theta\right)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta},\end{aligned}$$

which obviously holds; and that

$$\begin{aligned}\widetilde{u}^{\dagger}<\frac{\alpha_l}{2\beta}\quad&\Leftrightarrow\quad\frac{4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}{4\beta\underline{\theta}\left[\alpha\Delta\theta+\sqrt{\left(\alpha\Delta\theta\right)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}\right]}<\frac{\alpha_l}{2\beta}\\&\Leftrightarrow\quad 2\lambda\Delta\alpha\Delta\theta<(2\alpha\Delta\theta<)\alpha\Delta\theta+\sqrt{\left(\alpha\Delta\theta\right)^2+4\lambda\Delta\alpha\alpha_l\underline{\theta}\Delta\theta},\end{aligned}$$

which again obviously holds.

To verify that $\Pi\left(w^{\circ},\bar{r}^{\dagger}\,\middle|\,\bar{\theta},\bar{\theta}\right)\geq\Pi\left(w^{\circ},\underline{r}^{\circ}\,\middle|\,\underline{\theta},\bar{\theta}\right)$, it suffices to show, by (B.7), that $(\widetilde{u}^{\dagger})^2\leq\frac{\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\underline{\theta}}$. As $\widetilde{u}^{\dagger}$ binds (B.23), this is equivalent to

$$\left(\widetilde{u}^{\dagger}\right)^2=\frac{\lambda\Delta\alpha\alpha_l\Delta\theta}{4\beta^2\underline{\theta}}-\frac{\alpha\Delta\theta}{2\beta\underline{\theta}}\widetilde{u}^{\dagger}\leq\frac{\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\underline{\theta}}\quad\Leftrightarrow\quad\widetilde{u}^{\dagger}\geq 0,$$

which in fact follows from (B.24).

We claim that contract $(w^{\circ},\bar{r}^{\dagger})$ can be sustained by the retailer's posterior belief that the manufacturer offering such a contract is less risky and is otherwise riskier. To that end, we need to show that neither the less risky nor the riskier manufacturer has incentive to deviate to the off-equilibrium strategies under such a posterior belief.

- For the riskier manufacturer, we have shown that her profit of deviating her return price to $\bar{r}^\dagger$ and hence being mistaken as a less risky type is dominated by her equilibrium profit, i.e., $\Pi\left(w^\circ, \bar{r}^\dagger \,\middle|\, \bar{\theta}, \underline{\theta}\right) \le \pi^\circ$. As any other contract $(\underline{w}, \underline{r})$ induces a belief that she is a riskier type, the symmetric-information $(w^\circ, \underline{r}^\circ)$ maximizes her profit $\Pi(\underline{w}, \underline{r} \,|\, \underline{\theta}, \underline{\theta})$ to π°. Therefore, the riskier manufacturer indeed has no incentive to deviate from her symmetric-information contract $(w^\circ, \underline{r}^\circ)$.
- For the less risky manufacturer whose wholesale price is restricted to w°, it suffices to show that she has no incentive to deviate her return price to any $\bar{r} \neq \bar{r}^\dagger$ and thus to be mistaken as a riskier type, i.e., $\Pi\left(w^\circ, \bar{r} \,\middle|\, \underline{\theta}, \bar{\theta}\right) \le \Pi\left(w^\circ, \bar{r}^\dagger \,\middle|\, \bar{\theta}, \bar{\theta}\right)$ for all $\bar{r}$, which is, by (B.1), equivalent to

$$h(\bar{r}) := (w^\circ - \bar{\theta}\bar{r})\left(\frac{\lambda\Delta\alpha}{\beta} - w^\circ + \underline{\theta}\bar{r}\right)^+ \le f(\bar{r}^\dagger) := \left(w^\circ - \bar{\theta}\bar{r}^\dagger\right)\left[\frac{\lambda\Delta\alpha}{\beta} - \left(w^\circ - \bar{\theta}\bar{r}^\dagger\right)\right]^+, \quad \forall \bar{r}. \tag{B.25}$$

We note that $h(\bar{r}) \equiv 0$ for $\bar{r} \le w^\circ/\bar{\theta}$ and hence (B.25) holds, if $w^\circ/\bar{\theta} \le (w^\circ - \lambda\Delta\alpha/\beta)/\underline{\theta}$. Thus, we just need to show (B.25) holds for the parameter range $w^\circ/\bar{\theta} > (w^\circ - \lambda\Delta\alpha/\beta)/\underline{\theta}$ or equivalently

$$\frac{\alpha_l - \lambda\Delta\alpha}{\alpha_l + \lambda\Delta\alpha} \le \underline{\theta}/\bar{\theta}. \tag{B.26}$$

On the other hand, it is straightforward to see that the quadratic function $(w^\circ - \bar{\theta}\bar{r})\left(\frac{\lambda\Delta\alpha}{\beta} - w^\circ + \underline{\theta}\bar{r}\right)$ is maximized at $\bar{r}^\diamond := \frac{(\bar{\theta}+\underline{\theta})\alpha_l - \Delta\theta\lambda\Delta\alpha}{4\beta\bar{\theta}\underline{\theta}}$ and hence $h(\bar{r}) \le h(\bar{r}^\diamond)$. Hence, direct calculation yields

$$\begin{aligned} f(\bar{r}^\dagger) - h(\bar{r}) \ge & f(\bar{r}^\dagger) - h(\bar{r}^\diamond) = \left(\frac{\lambda\Delta\alpha}{2\beta}\right)^2 - \frac{\left(\alpha\Delta\theta - \sqrt{(\alpha\Delta\theta)^2 + 4\lambda\alpha_l\Delta\alpha\underline{\theta}\Delta\theta}\right)^2}{(4\beta\underline{\theta})^2} - \frac{\left[(\underline{\theta}+\bar{\theta})\lambda\Delta\alpha - \Delta\theta\alpha_l\right]^2}{(4\beta)^2\underline{\theta}\bar{\theta}} \\ = & \frac{2\alpha\Delta\theta}{(4\beta\underline{\theta})^2}\left\{\sqrt{(\alpha\Delta\theta)^2 + 4\lambda\alpha_l\Delta\alpha\underline{\theta}\Delta\theta} - \left(1 + \frac{\underline{\theta}}{2\bar{\theta}}\right)\alpha\Delta\theta\right\}, \end{aligned}$$

which is positive because (B.26) implies that

$$\begin{aligned} &(\alpha\Delta\theta)^2 + 4\lambda\alpha_l\Delta\alpha\underline{\theta}\Delta\theta - \left(1 + \frac{\underline{\theta}}{2\bar{\theta}}\right)^2(\alpha\Delta\theta)^2 = \alpha^2\underline{\theta}\Delta\theta\left\{\frac{4\alpha_l\lambda\Delta\alpha}{\alpha^2} - \frac{(4\bar{\theta}+\underline{\theta})\Delta\theta}{4\bar{\theta}^2}\right\} \\ =&\alpha^2\underline{\theta}\Delta\theta\left\{\frac{(3\bar{\theta}+\underline{\theta})\underline{\theta}}{4\bar{\theta}^2} - \left(\frac{\alpha_l - \lambda\Delta\alpha}{\alpha_l + \lambda\Delta\alpha}\right)^2\right\} \ge \alpha^2(\underline{\theta})^2/\bar{\theta}^2\Delta\theta\left\{\frac{3\bar{\theta}+\underline{\theta}}{4} - \underline{\theta}\right\} = \frac{3\alpha^2(\underline{\theta})^2(\Delta\theta)^2}{4\bar{\theta}^2} > 0. \end{aligned}$$

This concludes the verification of the equilibrium belief.

Finally, we determine the retailer's order quantity as well as unsold inventory. Since $w^\circ - \bar{\theta}\bar{r}^\dagger = \widetilde{u}^\dagger + \frac{\lambda\Delta\alpha}{2\beta} < \lambda\Delta\alpha/\beta$ because $\widetilde{u}^\dagger < \frac{\lambda\Delta\alpha}{2\beta}$, Lemma 1 suggests that all inventory is sold out in the case of high baseline demand realization, and in particular, (1) implies that the retailer's stocking quantity $\bar{s}^\dagger = \frac{\lambda^c\beta\bar{\theta}\bar{r}^\dagger + \lambda\alpha_h - \beta w^\circ}{2\lambda}$, which is given by (B.20) and obviously smaller than $s^\circ = \alpha_h/4$. By Lemma 1, again, the unsold inventory in the case of low baseline demand realization is

$$\bar{q}^\dagger = \frac{1}{2}\left[\Delta\alpha - \beta/\lambda\left(w^\circ - \bar{\theta}\bar{r}^\dagger\right)\right] = \frac{1}{2}\left[\Delta\alpha/2 - \beta/\lambda\widetilde{u}^\dagger\right],$$

from which (B.21) follows by (B.24). $\square$

LEMMA B.4. *The solution* $(\widetilde{w}^\star, \widetilde{u}^\star)$ *to* (B.5)-(B.7) *satisfies* $\widetilde{w}^\star > -\frac{\alpha}{2\beta}$ *and* $-\frac{\lambda\Delta\alpha}{2\beta} < \widetilde{u}^\star < \frac{\lambda\Delta\alpha}{2\beta}$.

Proof. Suppose $\widetilde{u}^\star = \pm\frac{\lambda\Delta\alpha}{2\beta}$. Then, $(0, \widetilde{u}^\dagger)$ with $\widetilde{u}^\dagger$ given by (B.24) is a feasible solution to (B.5)-(B.7), but

$$\lambda\left(\widetilde{w}^\star\right)^2 + \lambda^c\left(\widetilde{u}^\star\right)^2 \ge \lambda^c\left(\frac{\lambda\Delta\alpha}{2\beta}\right)^2 > \lambda(0)^2 + \lambda^c(\widetilde{u}^\dagger)^2,$$

contradicting the optimality of $(\widetilde{w}^\star, \widetilde{u}^\star)$. Therefore, we must have $-\frac{\lambda\Delta\alpha}{2\beta} < \widetilde{u}^\star < \frac{\lambda\Delta\alpha}{2\beta}$. Subsequently, we must have $\widetilde{w}^\star \ge \widetilde{u}^\star - \frac{\alpha_l}{2\beta} > -\frac{\lambda\Delta\alpha}{2\beta} - \frac{\alpha_l}{2\beta} = -\frac{\alpha}{2\beta}$. $\square$

LEMMA B.5. *There exists* $(\widetilde{w},\widetilde{u})$ *such that (i)* $-\frac{\lambda\Delta\alpha}{2\beta}\le\widetilde{u}\le 0$, *(ii)* $\widetilde{u}-\frac{\alpha_l}{2\beta}\le\widetilde{w}\le 0$, *(iii)* (B.6) *holds, and (iv)* $\lambda\widetilde{w}^2+\lambda^c\widetilde{u}^2<\frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\overline{\theta}}$. *Therefore, the optimal solution to* (B.5)-(B.6) *automatically satisfies* (B.7).

Proof of Lemma B.5. For $\widetilde{u}=0$, the left-hand side of (B.6) reduces to the quadratic function $\lambda\widetilde{w}^2-\frac{\lambda^c\Delta\theta\lambda\Delta\alpha}{2\beta\overline{\theta}}\left(\widetilde{w}+\frac{\alpha_l}{2\beta}\right)$ in $\widetilde{w}$, which takes a negative value $-\frac{\lambda^c\alpha_l\Delta\theta\lambda\Delta\alpha}{4\beta^2\overline{\theta}}$ at $\widetilde{w}=0$ and a positive value $\frac{\lambda\alpha_l^2}{4\beta^2}$ at $\widetilde{w}=-\frac{\alpha_l}{2\beta}$. Hence, there exists a root $\widetilde{w}^\flat\in\left(-\frac{\alpha_l}{2\beta},0\right)$ of this the quadratic function such that $\widetilde{w}=\widetilde{w}^\flat$ and $\widetilde{u}=0$ satisfies $-\frac{\lambda\Delta\alpha}{2\beta}\le\widetilde{u}\le 0$, $\widetilde{u}-\frac{\alpha_l}{2\beta}\le\widetilde{w}\le 0$ and (in fact binds) (B.6). Straightforward verification reveals that

$$\lambda(\widetilde{w}^\flat)^2+\lambda^c 0^2=\frac{\lambda^c\Delta\theta\lambda\Delta\alpha}{2\beta\overline{\theta}}\left(\widetilde{w}^\flat+\frac{\alpha_l}{2\beta}\right)<\frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\overline{\theta}}<\frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\underline{\theta}}$$

where the first inequality follows from $\widetilde{w}^\flat<0$ and the second inequality indicates that (B.7) will be satisfied by the optimal solution to (B.5)-(B.6). □

LEMMA B.6. *The solution* $(\widetilde{w}^\star,\widetilde{u}^\star)$ *to* (B.5)-(B.7) *is the solution to the following system of equations*

$$\lambda\widetilde{w}^2+\lambda^c\widetilde{u}^2+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\left(\widetilde{w}-\widetilde{u}+\frac{\alpha_l}{2\beta}\right)\left(\widetilde{u}-\frac{\lambda\Delta\alpha}{2\beta}\right)=0, \tag{B.27}$$

$$\lambda^c\widetilde{u}^2-\lambda\widetilde{w}^2+2\lambda\widetilde{w}\widetilde{u}-\frac{\lambda^c\lambda\Delta\alpha}{2\beta}\widetilde{u}-\frac{\lambda\alpha}{2\beta}\widetilde{w}=0, \tag{B.28}$$

such that $\widetilde{w}^\star<0$, $\widetilde{u}^\star>0$, $0<\lambda\widetilde{w}^\star+\lambda^c\widetilde{u}^\star<\lambda^c\widetilde{u}^\dagger$ *and* $\widetilde{w}^\star-\widetilde{u}^\star<-\widetilde{u}^\dagger$, *where* $\widetilde{u}^\dagger$ *is given by* (B.24).

Proof. By Lemma B.5, solving (B.5)-(B.7) is equivalent to solve the relaxed problem (B.5)-(B.6) by ignoring (B.7). By Lemma B.4, we can also ignore the bound constraint $-\frac{\lambda\Delta\alpha}{2\beta}\le\widetilde{u}\le\frac{\lambda\Delta\alpha}{2\beta}$. Furthermore, we are going to ignore the constraint $\widetilde{w}+\frac{\alpha_l}{2\beta}\ge\widetilde{u}$, which will be verified to hold by the optimal $(\widetilde{w}^\star,\widetilde{u}^\star)$ to the relaxed problem.

The necessary condition for $(\widetilde{w}^\star,\widetilde{u}^\star)$ to be the optimal solution to the relaxed problem is that there exists a Lagrangian multiplier $\xi\ge 0$ associated with (B.6) such that

$$2\lambda\widetilde{w}^\star-\xi\left(2\lambda\widetilde{w}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\widetilde{u}^\star-\frac{\lambda^c\Delta\theta\lambda\Delta\alpha}{2\beta\overline{\theta}}\right)=0, \tag{B.29}$$

$$2\lambda^c\widetilde{u}^\star-\xi\left(\frac{2\lambda^c\underline{\theta}}{\overline{\theta}}\widetilde{u}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\widetilde{w}^\star+\frac{\lambda^c\Delta\theta\alpha}{2\beta\overline{\theta}}\right)=0. \tag{B.30}$$

We claim that $\xi>0$. Otherwise, (B.29) and (B.30) immediately imply that $\widetilde{w}^\star=\widetilde{u}^\star=0$, which can be easily verified to violate (B.6). Therefore, (B.6) must be binding, yielding (B.27), which immediately implies that $\widetilde{w}^\star+\frac{\alpha_l}{2\beta}\ge\widetilde{u}^\star$ because $\widetilde{u}^\star<\frac{\lambda\Delta\alpha}{2\beta}$ by Lemma B.4.

Rearranging terms in (B.29), we have

$$2\lambda(1-\xi)\widetilde{w}^\star=\frac{\xi\lambda^c\Delta\theta}{\overline{\theta}}\left(\widetilde{u}^\star-\frac{\lambda\Delta\alpha}{2\beta}\right)<0, \tag{B.31}$$

where the last inequality follows from Lemma B.4; and rearranging terms in (B.30), we have

$$2\left(\overline{\theta}-\underline{\theta}\xi\right)\widetilde{u}^\star=\xi\Delta\theta\left(\widetilde{w}^\star+\frac{\alpha}{2\beta}\right)>0, \tag{B.32}$$

where the last inequality follows again from Lemma B.4. Therefore, we have $\widetilde{w}^\star\neq 0$ and $\widetilde{u}^\star\neq 0$.

By eliminating ξ from (B.29) and (B.30), we obtain

$$\frac{\lambda\widetilde{w}^\star}{\lambda^c\widetilde{u}^\star}=\frac{2\lambda\widetilde{w}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\widetilde{u}^\star-\frac{\lambda^c\Delta\theta\lambda\Delta\alpha}{2\beta\overline{\theta}}}{\frac{2\lambda^c\underline{\theta}}{\overline{\theta}}\widetilde{u}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\widetilde{w}^\star+\frac{\lambda^c\Delta\theta\alpha}{2\beta\overline{\theta}}}=\frac{2\lambda\widetilde{w}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\left(\widetilde{u}^\star-\frac{\lambda\Delta\alpha}{2\beta}\right)}{2\lambda^c\widetilde{u}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\left[\left(\widetilde{w}^\star-\widetilde{u}^\star+\frac{\alpha_l}{2\beta}\right)+\left(\frac{\lambda\Delta\alpha}{2\beta}-\widetilde{u}^\star\right)\right]}. \tag{B.33}$$

Rearranging terms of (B.33) yields

$$\lambda\widetilde{w}^\star\left[\left(\widetilde{w}^\star-\widetilde{u}^\star+\frac{\alpha_l}{2\beta}\right)-\left(\widetilde{u}^\star-\frac{\lambda\Delta\alpha}{2\beta}\right)\right]=\lambda^c\widetilde{u}^\star\left(\widetilde{u}^\star-\frac{\lambda\Delta\alpha}{2\beta}\right),$$

immediately implying (B.28) and

$$\frac{\lambda\widetilde{w}^\star}{\lambda^c\widetilde{u}^\star}=-\frac{\frac{\lambda\Delta\alpha}{2\beta}-\widetilde{u}^\star}{\widetilde{w}^\star-\widetilde{u}^\star+\frac{\alpha_l}{2\beta}+\frac{\lambda\Delta\alpha}{2\beta}-\widetilde{u}^\star}<0, \tag{B.34}$$

where the negativity follows by noting that $\widetilde{u}^\star<\frac{\lambda\Delta\alpha}{2\beta}$ and $\widetilde{w}^\star+\frac{\alpha_l}{2\beta}\geq\widetilde{u}^\star$.

We have the following three possibilities:

1. If $\xi>\overline{\theta}/\underline{\theta}>1$, then (B.31) and (B.32) imply that $\widetilde{w}^\star>0$ and $\widetilde{u}^\star<0$, respectively. However, (B.27) suggests

$$\lambda(\widetilde{w}^\star)^2+\lambda^c(\widetilde{u}^\star)^2=\frac{\lambda^c\Delta\theta}{\overline{\theta}}\underbrace{\left(\widetilde{w}^\star-\widetilde{u}^\star+\frac{\alpha_l}{2\beta}\right)}_{>\frac{\alpha_l}{2\beta}}\underbrace{\left(\frac{\lambda\Delta\alpha}{2\beta}-\widetilde{u}^\star\right)}_{>\frac{\lambda\Delta\alpha}{2\beta}}>\frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2\overline{\theta}},$$

contradicting Lemma B.5. Hence, this case can be ruled out.

2. If $\overline{\theta}/\underline{\theta}>\xi>1$, then (B.31) and (B.32) imply that $\widetilde{w}^\star>0$ and $\widetilde{u}^\star>0$, respectively. However, this contradicts (B.34), ruling out this case as well.

3. As such, we must have $\xi<1<\frac{\overline{\theta}}{\underline{\theta}}$, which implies that $\widetilde{w}^\star<0$ and $\widetilde{u}^\star>0$ according to (B.31) and (B.32), respectively. Together with (B.34), implies that $\frac{\lambda\widetilde{w}^\star}{\lambda^c\widetilde{u}^\star}\in(-1,0)$ and $\lambda\widetilde{w}^\star+\lambda^c\widetilde{u}^\star>0$. Therefore, by (B.33),

$$-1<\frac{2\lambda\widetilde{w}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\left(\widetilde{u}^\star-\frac{\lambda\Delta\alpha}{2\beta}\right)}{2\lambda^c\widetilde{u}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\left[\left(\widetilde{w}^\star-\widetilde{u}^\star+\frac{\alpha_l}{2\beta}\right)+\left(\frac{\lambda\Delta\alpha}{2\beta}-\widetilde{u}^\star\right)\right]}<0, \tag{B.35}$$

which, by multiplying the denominator on both sides of (B.35) and rearranging terms, yields

$$2\lambda\widetilde{w}^\star+2\lambda^c\widetilde{u}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\left(\widetilde{w}^\star-\widetilde{u}^\star+\frac{\alpha_l}{2\beta}\right)>0. \tag{B.36}$$

Finally, we demonstrate $\widetilde{w}^\star-\widetilde{u}^\star<-\widetilde{u}^\dagger$ and $\lambda\widetilde{w}^\star+\lambda^c\widetilde{u}^\star<\lambda^c\widetilde{u}^\dagger$. Indeed, both $(\widetilde{w}^\star,\widetilde{u}^\star)$ and $(0,\widetilde{u}^\dagger)$ satisfy the quadric equation (B.27), which can, via the change of variable $\widetilde{z}=\widetilde{w}-\widetilde{u}$, be rewritten as

$$\frac{\overline{\theta}}{\lambda^c}\widetilde{w}^2-(\overline{\theta}+\underline{\theta})\widetilde{w}\widetilde{z}+\underline{\theta}\widetilde{z}^2+\frac{\Delta\theta\alpha_l}{2\beta}\widetilde{w}-\frac{\alpha\Delta\theta}{2\beta}\widetilde{z}-\frac{\Delta\theta\alpha_l\lambda\Delta\alpha}{4\beta^2}=0. \tag{B.37}$$

Let $\widetilde{z}^\star:=\widetilde{w}^\star-\widetilde{u}^\star$ and $\widetilde{z}^\dagger:=-\widetilde{u}^\dagger$. Then, $(\widetilde{w}^\star,\widetilde{z}^\star)$ and $(0,\widetilde{z}^\dagger)$ satisfy (B.37).

Since $\widetilde{w}^\star\in\left(-\frac{\alpha}{2\beta},0\right)$ and $\widetilde{u}^\star,\widetilde{u}^\dagger\in\left(0,\frac{\lambda\Delta\alpha}{2\beta}\right)$ as shown above and in the proof of Proposition B.2, we focus on examining the quadratic curve (B.37) in the region $\Omega:=\left\{(\widetilde{w},\widetilde{z}):-\frac{\alpha}{2\beta}\leq\widetilde{w}\leq 0\text{ and }0<\widetilde{w}-\widetilde{z}<\frac{\lambda\Delta\alpha}{2\beta}\right\}$.

Total differentiation of (B.37) yields

$$\frac{d\widetilde{z}}{d\widetilde{w}}=\frac{2\frac{\overline{\theta}}{\lambda^c}\widetilde{w}-(\overline{\theta}+\underline{\theta})\widetilde{z}+\frac{\alpha_l\Delta\theta}{2\beta}}{(\overline{\theta}+\underline{\theta})\widetilde{w}-2\underline{\theta}\widetilde{z}+\frac{\alpha\Delta\theta}{2\beta}}, \tag{B.38}$$

where we notice that

$$(\overline{\theta}+\underline{\theta})\widetilde{w}-2\underline{\theta}\widetilde{z}+\frac{\alpha\Delta\theta}{2\beta}=2\underline{\theta}(\widetilde{w}-\widetilde{z})+\Delta\theta\left(\widetilde{w}+\frac{\alpha}{2\beta}\right)>0.$$

Therefore, the quadratic curve in (B.37) is segmented into (at most) two branches in Ω by the straight line $2\frac{\overline{\theta}}{\lambda^c}\widetilde{w}-(\overline{\theta}+\underline{\theta})\widetilde{z}+\frac{\alpha_l\Delta\theta}{2\beta}=0$: in the region where $2\frac{\overline{\theta}}{\lambda^c}\widetilde{w}-(\overline{\theta}+\underline{\theta})\widetilde{z}+\frac{\alpha_l\Delta\theta}{2\beta}>(<)0$, $\widetilde{z}$ is strictly increasing (decreasing) in $\widetilde{w}$. Since

$$2\frac{\overline{\theta}}{\lambda^c}\widetilde{w}^\star-(\overline{\theta}+\underline{\theta})\widetilde{z}^\star+\frac{\alpha_l\Delta\theta}{2\beta}=2\frac{\overline{\theta}}{\lambda^c}\widetilde{w}^\star-(\overline{\theta}+\underline{\theta})\left(\widetilde{w}^\star-\widetilde{u}^\star\right)+\frac{\alpha_l\Delta\theta}{2\beta}$$

$$=\frac{\overline{\theta}}{\lambda^c}\left[2\lambda\widetilde{w}^\star+2\lambda^c\widetilde{u}^\star+\frac{\lambda^c\Delta\theta}{\overline{\theta}}\left(\widetilde{w}^\star-\widetilde{u}^\star+\frac{\alpha_l}{2\beta}\right)\right]>0 \text{ by (B.36)}$$

and

$$2\frac{\overline{\theta}}{\lambda^c}0-(\overline{\theta}+\underline{\theta})\widetilde{z}^\dagger+\frac{\alpha_l\Delta\theta}{2\beta}=(\overline{\theta}+\underline{\theta})\widetilde{u}^\dagger+\frac{\alpha_l\Delta\theta}{2\beta}>0 \text{ because } \widetilde{u}^\dagger>0,$$

$(\widetilde{w}^\star,\widetilde{z}^\star)$ and $(0,\widetilde{z}^\dagger)$ are on the increasing branch of the quadratic curve in (B.37). Therefore, $\widetilde{w}^\star<0$ immediately suggests that

$$\widetilde{w}^\star-\widetilde{u}^\star=\widetilde{z}^\star<\widetilde{z}^\dagger=-\widetilde{u}^\dagger.$$

As $(\widetilde{w}^\star,\widetilde{u}^\star)$ satisfies (B.27), we must have

$$\lambda^c\left[(\widetilde{u}^*)^2+\frac{\Delta\theta}{\overline{\theta}}\left(\frac{\alpha_l}{2\beta}-\widetilde{u}^*\right)\left(\widetilde{u}^*-\frac{\lambda\Delta\alpha}{2\beta}\right)\right]=-\lambda(\widetilde{w}^*)^2-\frac{\lambda^c\Delta\theta}{\overline{\theta}}\widetilde{w}^*\left(\widetilde{u}^*-\frac{\lambda\Delta\alpha}{2\beta}\right)<0,$$

where the inequality follows from $\widetilde{w}^*<0$ and $\widetilde{u}^*<\frac{\lambda\Delta\alpha}{2\beta}$. Namely, $\widetilde{u}^*$ must lie between the negative and positive roots of the quadratic equation on the left-hand side of (B.23). Thus, we have $\widetilde{u}^*<\widetilde{u}^\dagger$, leading to

$$\lambda\widetilde{w}^\star+\lambda^c\widetilde{u}^\star=\lambda(\widetilde{w}^\star-\widetilde{u}^\star)+\widetilde{u}^\star<-\lambda\widetilde{u}^\dagger+\widetilde{u}^\dagger=\lambda^c\widetilde{u}^\dagger,$$

which completes the proof. □

LEMMA B.7. $\Pi\left(\overline{w},\overline{r}\,\middle|\,\underline{\theta},\bar{\theta}\right)\le\Pi\left(\overline{w}^\star,\overline{r}^\star\,\middle|\,\bar{\theta},\bar{\theta}\right)$ *for all* $\overline{w}\ge\bar{\theta}\overline{r}\ge0$.

Proof of Lemma B.7. We first make the following two observations:

Observation 1. The return-price-only signaling strategy $(w^\circ,\overline{r}^\dagger)$ is a feasible solution to (4), implying that $\Pi\left(\overline{w}^\star,\overline{r}^\star\,\middle|\,\bar{\theta},\bar{\theta}\right)>\Pi\left(w^\circ,\overline{r}^\dagger\,\middle|\,\bar{\theta},\bar{\theta}\right)$.

Observation 2. Furthermore, $\Pi\left(\overline{w}^\star,\overline{r}^\star\,\middle|\,\bar{\theta},\bar{\theta}\right)>\Pi\left(w^\circ,\overline{r}^\dagger\,\middle|\,\bar{\theta},\bar{\theta}\right)=\frac{\lambda^c\beta}{2\lambda}f\left(\overline{r}^\dagger\right)+\frac{1}{2}\left[-\beta(w^\circ)^2+\alpha w^\circ\right]>\frac{\alpha^2}{8\beta}\ge\frac{1}{2}\left[-\beta\overline{w}^2+\alpha\overline{w}\right]$ for all $\overline{w}\ge0$, where the function $f(\cdot)$ is defined in the proof of Proposition B.2 showing that $f\left(\overline{r}^\dagger\right)>0$.

We now demonstrate the lemma. For any $\overline{w}-\underline{\theta}\overline{r}\ge\lambda\Delta\alpha/\beta$, (B.1) suggests that $\Pi\left(\overline{w},\overline{r}\,\middle|\,\underline{\theta},\bar{\theta}\right)=\frac{1}{2}\left[-\beta\overline{w}^2+\alpha\overline{w}\right]$ and hence the lemma follows from Observation 2 above. For any $0\le\overline{w}-\underline{\theta}\overline{r}\le\lambda\Delta\alpha/\beta$, (B.1) suggests that

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\underline{\theta},\bar{\theta}\right)=\frac{\lambda^c\beta}{2\lambda}(\overline{w}-\bar{\theta}\overline{r})\left(\frac{\lambda\Delta\alpha}{\beta}-\overline{w}+\underline{\theta}\overline{r}\right)+\frac{1}{2}\left[-\beta\overline{w}^2+\alpha\overline{w}\right], \tag{B.39}$$

whose first term, as a quadratic function of $\overline{r}\in\left[\frac{1}{\underline{\theta}}\left(\overline{w}-\lambda\Delta\alpha/\beta\right)^+,\frac{1}{\underline{\theta}}\overline{w}\right]$, achieves its unconstrained maximum at $\overline{r}=\frac{1}{\underline{\theta}}\left(\frac{\bar{\theta}+\underline{\theta}}{2\bar{\theta}}\overline{w}-\frac{\lambda\Delta\alpha}{2\beta}\right)$. We thus examine the following two cases:

1. If $\overline{w}\ge\frac{\bar{\theta}\lambda\Delta\alpha}{\beta\Delta\theta}$, then we have $0\le\frac{1}{\underline{\theta}}\left(\frac{\bar{\theta}+\underline{\theta}}{2\bar{\theta}}\overline{w}-\frac{\lambda\Delta\alpha}{2\beta}\right)\le\frac{1}{\underline{\theta}}\left(\overline{w}-\lambda\Delta\alpha/\beta\right)$ and hence the first term in (B.39) is decreasing in $\overline{r}\in\left[\frac{1}{\underline{\theta}}\left(\overline{w}-\lambda\Delta\alpha/\beta\right),\frac{1}{\underline{\theta}}\overline{w}\right]$, suggesting

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\underline{\theta},\bar{\theta}\right)\le\Pi\left(\overline{w},\frac{1}{\underline{\theta}}\left(\overline{w}-\lambda\Delta\alpha/\beta\right)\,\middle|\,\underline{\theta},\bar{\theta}\right)=\frac{1}{2}\left[-\beta\overline{w}^2+\alpha\overline{w}\right]<\Pi\left(\overline{w}^\star,\overline{r}^\star\,\middle|\,\bar{\theta},\bar{\theta}\right),$$

where the last inequality follows again from Observation 2 above.

2. If $\overline{w} \le \frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta}$, then $\frac{1}{\underline{\theta}}\left(\overline{w} - \lambda\Delta\alpha/\beta\right) \le \frac{1}{\underline{\theta}}\left(\frac{\bar{\theta}+\underline{\theta}}{2\bar{\theta}}\overline{w} - \frac{\lambda\Delta\alpha}{2\beta}\right) \le 0$ and hence the first term in (B.39) is a quadratic function in r and can straightforwardly be shown to be decreasing in $\bar{r} \in \left[0, \overline{w}/\bar{\theta}\right]$, suggesting that

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, \underline{\theta}, \bar{\theta}\right) \le \Pi\left(\overline{w}, 0 \,\middle|\, \underline{\theta}, \bar{\theta}\right) = \frac{\lambda^c \beta}{2\lambda}\overline{w}\left(\frac{\lambda\Delta\alpha}{\beta} - \overline{w}\right) + \frac{1}{2}\left[-\beta\overline{w}^2 + \alpha\overline{w}\right] = -\frac{\beta}{2\lambda}\overline{w}^2 + \frac{\alpha_h}{2}\overline{w}, \tag{B.40}$$

which is maximized at $\widehat{w} = \frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta}$ if $\alpha_h/\alpha_l \le 2\bar{\theta}/\Delta\theta$ and at $\widehat{w} = \lambda\alpha_h/(2\beta)$ otherwise. In both cases, we claim that $(\widehat{w}, 0)$ satisfies the first constraint in (4), suggesting that

$$\Pi\left(\overline{w}^\star, \bar{r}^\star \,\middle|\, \bar{\theta}, \bar{\theta}\right) \ge \Pi\left(\widehat{w}, 0 \,\middle|\, \bar{\theta}, \bar{\theta}\right) \ge \Pi\left(\widehat{w}, 0 \,\middle|\, \underline{\theta}, \bar{\theta}\right) \ge \Pi\left(\overline{w}, \bar{r} \,\middle|\, \underline{\theta}, \bar{\theta}\right),$$

where the first inequality follows from the optimality of $(\overline{w}^\star, \bar{r}^\star)$ in (4) (with the knowledge that the second constraint in (4) is nonbinding according to Lemma B.5), the second inequality follows from the monotonicity of $\Pi\left(\widehat{w}, 0 \,\middle|\, \widehat{\theta}, \bar{\theta}\right)$ in $\widehat{\theta}$, and the last inequality is because $(\widehat{w}, 0)$ maximizes $\Pi\left(\overline{w}, \bar{r} \,\middle|\, \underline{\theta}, \bar{\theta}\right)$.
Now we verify the claim that $(\widehat{w}, 0)$ satisfies the first constraint in (4), which is, according to Lemma B.3, equivalent to showing that the transformed quantities $\widetilde{w} = \widehat{w} - w^\circ$ and $\widetilde{u} = \bar{\theta}\bar{r}^\circ + \widetilde{w} = \frac{\alpha_l}{2\beta} + \widetilde{w}$ satisfy (B.6), which is immediate by noting that $\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta} = 0$.

3. If $\frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta} \le \overline{w} \le \frac{\bar{\theta}\lambda\Delta\alpha}{\beta\Delta\theta}$, then we have $\frac{1}{\underline{\theta}}\left(\frac{\bar{\theta}+\underline{\theta}}{2\bar{\theta}}\overline{w} - \frac{\lambda\Delta\alpha}{2\beta}\right) \in \left[\frac{1}{\underline{\theta}}\left(\overline{w} - \lambda\Delta\alpha/\beta\right)^+, \frac{1}{\underline{\theta}}\overline{w}\right]$ and hence

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, \underline{\theta}, \bar{\theta}\right) \le \Pi\left(\overline{w}, \frac{1}{\underline{\theta}}\left(\frac{\bar{\theta}+\underline{\theta}}{2\bar{\theta}}\overline{w} - \frac{\lambda\Delta\alpha}{2\beta}\right) \,\middle|\, \underline{\theta}, \bar{\theta}\right) = \frac{\lambda^c\beta}{8\lambda\bar{\theta}\underline{\theta}}\left(\frac{\bar{\theta}\lambda\Delta\alpha}{\beta} - \Delta\theta\overline{w}\right)^2 + \frac{1}{2}\left(-\beta\overline{w}^2 + \alpha\overline{w}\right). \tag{B.41}$$

- When $\lambda^c(\Delta\theta)^2 \ge 4\lambda\bar{\theta}\underline{\theta}$, the quadratic function of $\overline{w}$ on the right-hand side of (B.41) is convex and hence reaches the maximum at either $\overline{w} = \frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta}$ or $\overline{w} = \frac{\bar{\theta}}{\Delta\theta}\frac{\lambda\Delta\alpha}{\beta}$, which correspond to the first two cases, respectively.

- When $\lambda^c(\Delta\theta)^2 < 4\lambda\bar{\theta}\underline{\theta}$ or equivalently $(\Delta\theta)^2 < \lambda\left(\bar{\theta}+\underline{\theta}\right)^2$, the quadratic function of $\overline{w}$ on the right-hand side of (B.41) is concave and achieves its unconstrained maximum at

$$\overline{w} = \frac{\lambda\bar{\theta}\left(2\underline{\theta}\alpha - \lambda^c\Delta\theta\Delta\alpha\right)}{\beta\left[\lambda(\bar{\theta}+\underline{\theta})^2 - (\Delta\theta)^2\right]}, \tag{B.42}$$

which can be verified to be within $\left(\frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta}, \frac{\bar{\theta}}{\Delta\theta}\frac{\lambda\Delta\alpha}{\beta}\right)$ if and only if $\frac{\Delta\theta}{\bar{\theta}+\underline{\theta}} < \frac{\alpha_l}{\Delta\alpha} < \frac{\lambda(\bar{\theta}+\underline{\theta})}{\Delta\theta}$. Therefore, we consider the following three scenarios:

—When $\frac{\alpha_l}{\Delta\alpha} \le \frac{\Delta\theta}{\bar{\theta}+\underline{\theta}}$, the quadratic function of $\overline{w}$ on the right-hand side of (B.41) is decreasing in $\overline{w} \in \left[\frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta}, \frac{\bar{\theta}}{\Delta\theta}\frac{\lambda\Delta\alpha}{\beta}\right]$ and hence achieves its maximum at $\overline{w} = \frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta}$, with the corresponding $\bar{r} = 0$, for which the lemma has been shown in Case 1.

—When $\frac{\alpha_l}{\Delta\alpha} \ge \frac{\bar{\theta}}{\Delta\theta}\frac{\lambda\Delta\alpha}{\beta}$, the quadratic function of $\overline{w}$ on the right-hand side of (B.41) is increasing in $\overline{w} \in \left[\frac{\bar{\theta}}{\bar{\theta}+\underline{\theta}} \frac{\lambda\Delta\alpha}{\beta}, \frac{\bar{\theta}}{\Delta\theta}\frac{\lambda\Delta\alpha}{\beta}\right]$ and hence achieves its maximum at $\overline{w} = \frac{\bar{\theta}}{\Delta\theta}\frac{\lambda\Delta\alpha}{\beta}$, making the first term on the right-hand side of (B.41) zero. Hence, Observation 2 above immediately implies the lemma.

—When $\frac{\Delta\theta}{\bar{\theta}+\underline{\theta}} < \frac{\alpha_l}{\Delta\alpha} < \frac{\lambda(\bar{\theta}+\underline{\theta})}{\Delta\theta}$, the quadratic function of $\overline{w}$ on the right-hand side of (B.41) is maximized at (B.42), yielding the maximum value (after some algebra)

$$\underbrace{\frac{\lambda^c\lambda(\Delta\alpha)^2 + \alpha^2}{8\beta}}_{\pi^\circ} + \frac{\lambda^c}{8\beta}\left\{\frac{\left[(\bar{\theta}+\underline{\theta})\lambda\Delta\alpha - \alpha_l\Delta\theta\right]^2}{\lambda(\bar{\theta}+\underline{\theta})^2 - (\Delta\theta)^2} - \lambda(\Delta\alpha)^2\right\}. \tag{B.43}$$

Therefore, the lemma is equivalent to showing $\Pi\left(\overline{w}^\star,\bar r^\star \,\middle|\, \bar\theta,\bar\theta\right)$ dominates (B.43). By Lemma B.3 and B.5, it suffices to show that there exists $(\widetilde{w},\widetilde{u})$ feasible to (B.4) and (B.6) such that

$$\lambda\widetilde{w}^2+\lambda^c\widetilde{u}^2=\frac{\lambda^c\lambda}{4\beta^2}\left\{\frac{\left[(\bar\theta+\underline\theta)\lambda\Delta\alpha-\alpha_l\Delta\theta\right]^2}{\lambda(\bar\theta+\underline\theta)^2-(\Delta\theta)^2}-\lambda(\Delta\alpha)^2\right\}, \tag{B.44}$$

which can be verified to be greater than zero for $\frac{\Delta\theta}{\bar\theta+\underline\theta}<\frac{\alpha_l}{\Delta\alpha}<\frac{\lambda(\bar\theta+\underline\theta)}{\Delta\theta}$. Making the following change of variable

$$\widetilde{w}=\frac{1}{\sqrt{\lambda}}\left(x\sqrt{\frac{1+\sqrt{\lambda}}{2}}-y\sqrt{\frac{1-\sqrt{\lambda}}{2}}\right),\quad \widetilde{u}=\frac{1}{\sqrt{\lambda^c}}\left(x\sqrt{\frac{1-\sqrt{\lambda}}{2}}+y\sqrt{\frac{1+\sqrt{\lambda}}{2}}\right), \tag{B.45}$$

we then can straightforwardly verify that

$$\lambda\widetilde{w}^2+\lambda^c\widetilde{u}^2=x^2+y^2, \tag{B.46}$$

and (B.6) is equivalent to

$$\left(\bar\theta+\underline\theta+\frac{\Delta\theta}{\sqrt{\lambda}}\right)x^2+\frac{\lambda^c\Delta\theta}{\beta}\frac{\alpha_l-\sqrt{\lambda}\Delta\alpha}{\sqrt{2(1+\sqrt{\lambda})}}x+\left(\bar\theta+\underline\theta-\frac{\Delta\theta}{\sqrt{\lambda}}\right)y^2+\frac{\lambda^c\Delta\theta}{\beta}\frac{\alpha_l+\sqrt{\lambda}\Delta\alpha}{\sqrt{2(1-\sqrt{\lambda})}}y-\frac{\lambda^c\Delta\theta\alpha_l\lambda\Delta\alpha}{2\beta^2}\geq 0. \tag{B.47}$$

Obviously, the $(\widetilde{w},\widetilde{u})$ defined through (B.45) by letting $x=0$ and $y=-\sqrt{\frac{\lambda^c\lambda}{4\beta^2}\left\{\frac{\left[(\bar\theta+\underline\theta)\lambda\Delta\alpha-\alpha_l\Delta\theta\right]^2}{\lambda(\bar\theta+\underline\theta)^2-(\Delta\theta)^2}-\lambda(\Delta\alpha)^2\right\}}$ satisfies (B.44) by virtue of (B.46). It is also straightforward to verify that such $(\widetilde{w},\widetilde{u})$ satisfies (B.4).

We now verify that it also satisfies (B.47), which implies that the corresponding $(\widetilde{w},\widetilde{u})$ must satisfy (B.6). To that end, plugging it to (B.47) renders it to

$$\frac{2(1+\sqrt{\lambda})}{\sqrt{\lambda}(\bar\theta+\underline\theta)-\Delta\theta}\left\{2\lambda\frac{\bar\theta+\underline\theta}{\Delta\theta}\alpha_l\Delta\alpha-\alpha_l^2-\lambda(\Delta\alpha)^2\right\}\geq\frac{\left(\alpha_l+\sqrt{\lambda}\Delta\alpha\right)^2}{\sqrt{\lambda}(\bar\theta+\underline\theta)+\Delta\theta},$$

which is equivalent to

$$\frac{2(1+\sqrt{\lambda})}{\sqrt{\lambda}(\bar\theta+\underline\theta)-\Delta\theta}\left\{2\lambda\frac{\bar\theta+\underline\theta}{\Delta\theta}z-z^2-\lambda\right\}-\frac{\left(z+\sqrt{\lambda}\right)^2}{\sqrt{\lambda}(\bar\theta+\underline\theta)+\Delta\theta}\geq 0, \tag{B.48}$$

with $z:=\alpha_l/\Delta\alpha\in\left(\frac{\Delta\theta}{\bar\theta+\underline\theta},\frac{\lambda(\bar\theta+\underline\theta)}{\Delta\theta}\right)$. As the right-hand side of (B.48) is a concave quadratic function in z, to demonstrate that (B.48) holds for all $z\in\left(\frac{\Delta\theta}{\bar\theta+\underline\theta},\frac{\lambda(\bar\theta+\underline\theta)}{\Delta\theta}\right)$, we just need to show it holds at the ends of the interval. Indeed, when $z=\frac{\Delta\theta}{\bar\theta+\underline\theta}$, (B.48) reduces to

$$\frac{2(1+\sqrt{\lambda})}{\sqrt{\lambda}(\bar\theta+\underline\theta)-\Delta\theta}\left\{\frac{\lambda(\bar\theta+\underline\theta)^2}{(\Delta\theta)^2}-1\right\}-\frac{\sqrt{\lambda}(\bar\theta+\underline\theta)+\Delta\theta}{(\Delta\theta)^2}\geq 0\quad\Leftrightarrow\quad 2(1+\sqrt{\lambda})\geq 1,$$

which obviously holds. When $z=\frac{\lambda(\bar\theta+\underline\theta)}{\Delta\theta}$, (B.48) reduces to

$$\frac{2(1+\sqrt{\lambda})}{\sqrt{\lambda}(\bar\theta+\underline\theta)-\Delta\theta}\left\{1-\frac{(\Delta\theta)^2}{\lambda(\bar\theta+\underline\theta)^2}\right\}-\frac{\sqrt{\lambda}(\bar\theta+\underline\theta)+\Delta\theta}{\lambda(\bar\theta+\underline\theta)^2}\geq 0\quad\Leftrightarrow\quad 2(1+\sqrt{\lambda})\geq 1,$$

which also holds.

Finally, we need to show that $(\widetilde{w},\widetilde{u})$ identified above also satisfies $-\frac{\lambda\Delta\alpha}{2\beta}\leq\widetilde{u}\leq\frac{\lambda\Delta\alpha}{2\beta}$ and $\widetilde{w}-\widetilde{u}\geq-\frac{\alpha_l}{2\beta}$. Under the change of variable in (B.45) with $x=0$, this is equivalent to show $y^2\leq\frac{1-\sqrt{\lambda}}{2\beta^2}\left(\lambda\Delta\alpha\right)^2$ and $\left(\sqrt{\frac{1-\sqrt{\lambda}}{2\lambda}}+\sqrt{\frac{1+\sqrt{\lambda}}{2\lambda^c}}\right)y\leq\frac{\alpha_l}{2\beta}$, which are straightforward to hold by $y=-\sqrt{\frac{\lambda^c\lambda}{4\beta^2}\left\{\frac{\left[(\bar\theta+\underline\theta)\lambda\Delta\alpha-\alpha_l\Delta\theta\right]^2}{\lambda(\bar\theta+\underline\theta)^2-(\Delta\theta)^2}-\lambda(\Delta\alpha)^2\right\}}$.

Therefore, $(\widetilde{w},\widetilde{u})$ defined by (B.45) with $x=0$ and $y=-\sqrt{\frac{\lambda^c\lambda}{4\beta^2}\left\{\frac{\left[(\bar\theta+\underline\theta)\lambda\Delta\alpha-\alpha_l\Delta\theta\right]^2}{\lambda(\bar\theta+\underline\theta)^2-(\Delta\theta)^2}-\lambda(\Delta\alpha)^2\right\}}$ is a feasible solution to (B.5)-(B.6) that satisfies (B.44), completing the proof. ☐

Proof of Proposition 1. By Lemma B.3, the solution $(\overline{w}^\star, \overline{r}^\star)$ to (4) is given by $\overline{w}^\star = w^\circ + \widetilde{w}^\star$ and $\overline{r}^\star = \overline{r}^\circ + (\widetilde{w}^\star - \widetilde{u}^\star)/\overline{\theta}$, where $(\widetilde{w}^\star, \widetilde{u}^\star)$ is the solution to (B.5)-(B.7). Because $\widetilde{w}^\star < 0$ and $\widetilde{u}^\star > 0$ according to Lemma B.6, we thus have $\overline{w}^\star = w^\circ + \widetilde{w}^\star < w^\circ$ and $\overline{r}^\star = \overline{r}^\circ + (\widetilde{w}^\star - \widetilde{u}^\star)/\overline{\theta} < \overline{r}^\circ$.

By (1), the retailer's order quantity under $(\overline{w}^\star, \overline{r}^\star)$ is given by

$$s^\star = s^R\left(\overline{w}^\star, \overline{r}^\star, \overline{\theta}\right) = \frac{\lambda^c \beta \overline{\theta} \overline{r}^\star + \lambda \alpha_h - \beta \overline{w}^\star}{2\lambda} = \underbrace{\frac{\lambda^c \beta \overline{\theta} \overline{r}^\circ + \lambda \alpha_h - \beta \overline{w}^\circ}{2\lambda}}_{\alpha_h/4 = s^\circ} - \frac{\beta}{2\lambda}\left(\lambda \widetilde{w}^\star + \lambda^c \widetilde{u}^\star\right) < s^\circ,$$

where the second equality follows from Lemma B.3 and the last inequality follows from the fact that $\lambda \widetilde{w}^\star + \lambda^c \widetilde{u}^\star > 0$ in Lemma B.6.

Since $\widetilde{u}^\star < \frac{\lambda \Delta\alpha}{2\beta}$, we have $\overline{w}^\star - \overline{\theta}\overline{r}^\star = \widetilde{u}^\star + \frac{\lambda\Delta\alpha}{2\beta} < \frac{\lambda\Delta\alpha}{\beta} = \frac{\alpha_h}{4} = s^\circ$, which, according to Lemma 1, suggests that all inventory is sold out in the case of high baseline demand. Again by Lemma 1 and the fact that $\widetilde{u}^\star = \overline{w}^\star - \overline{\theta}\overline{r}^\star - \frac{\lambda\Delta\alpha}{2\beta}$, we have the unsold inventory in the case of low baseline demand realization to be

$$\bar{q}^\star = \frac{1}{2}\left[\Delta\alpha - \beta/\lambda\left(\overline{w}^\star - \bar{\theta}\overline{r}^\star\right)\right] = \frac{1}{2}\left[\Delta\alpha/2 - \beta/\lambda \widetilde{u}^\star\right] < \frac{\Delta\alpha}{4} = q^\circ,$$

where we use the fact that $\widetilde{u}^\star > 0$ (Lemma B.6) to obtain the inequality.

We now verify that $(\overline{w}^\star, \overline{r}^\star)$ can be sustained as a separating equilibrium by the retailer's posterior belief that the manufacturer is less risky upon contract $(\overline{w}^\star, \overline{r}^\star)$ being offered and is otherwise riskier. To that end, we need to show that neither the less risky nor the riskier manufacturer has incentive to deviate to the off-equilibrium strategies under such a posterior belief.

- The riskier manufacturer's profit of deviating to $(\overline{w}^\star, \overline{r}^\star)$ and hence being mistaken as a less risky type is, by definition, dominated by her equilibrium profit as $(\overline{w}^\star, \overline{r}^\star)$ satisfies the first constraint in (4): $\Pi\left(\overline{w}^\star, \overline{r}^\star \,\middle|\, \bar{\theta}, \underline{\theta}\right) \le \pi^\circ$. Among all $(\underline{w}, \underline{r}) \neq (\overline{w}^\star, \overline{r}^\star)$, under which the manufacturer is believed to be of the riskier type, the symmetric-information $(w^\circ, \underline{r}^\circ)$ maximizes her profit $\Pi(\underline{w}, \underline{r} \,|\, \underline{\theta}, \underline{\theta})$ to π°. Therefore, the riskier manufacturer indeed has no incentive to deviate from her symmetric-information contract terms $(w^\circ, \underline{r}^\circ)$.
- For the less risky manufacturer, we need to show that she has no incentive to deviate to any $(\overline{w}, \bar{r}) \neq (\overline{w}^\star, \overline{r}^\star)$. If the deviation takes place, the manufacturer will be mistaken as a riskier type, earning a profit of $\Pi\left(\overline{w}, \overline{r} \,\middle|\, \underline{\theta}, \bar{\theta}\right)$, which is dominated by her equilibrium profit $\Pi\left(\overline{w}^\star, \overline{r}^\star \,\middle|\, \bar{\theta}, \bar{\theta}\right)$ by Lemma B.7. Thus, the less risky manufacturer has no incentive to deviate from $(\overline{w}^\star, \overline{r}^\star)$. This concludes the verification of the equilibrium belief. □

Proof of Proposition 2. First, $\overline{w}^\star < w^\circ < \overline{w}^\ddagger$ follows from Proposition 1 and (B.8) Proposition B.1.

By Proposition B.2, $\overline{r}^\dagger < \overline{r}^\circ < \underline{r}^\circ$ follows from (B.19). To show $\overline{r}^\star < \overline{r}^\dagger$, we recall that by Lemma B.3, the solution $(\overline{w}^\star, \overline{r}^\star)$ to (4) is given by $\overline{w}^\star = w^\circ + \widetilde{w}^\star$ and $\overline{r}^\star = \overline{r}^\circ + (\widetilde{w}^\star - \widetilde{u}^\star)/\overline{\theta}$, where $(\widetilde{w}^\star, \widetilde{u}^\star)$ is the solution to (B.5)-(B.7). Because $\widetilde{w}^\star < 0$ and $\widetilde{w}^\star - \widetilde{u}^\star < -\widetilde{u}^\dagger$ according to Lemma B.6, we thus have $\overline{w}^\star = w^\circ + \widetilde{w}^\star < w^\circ$ and $\overline{r}^\star = \overline{r}^\circ + (\widetilde{w}^\star - \widetilde{u}^\star)/\overline{\theta} < \overline{r}^\circ - \widetilde{u}^\dagger/\overline{\theta} = \overline{r}^\dagger$.

The profit rank $\overline{\pi}^\dagger < \overline{\pi}^\star < \pi^\circ$ simply follows from the fact that (3) is a relaxed problem of (4), which is in turn a relaxed problem of (10). To show that $\overline{\pi}^\dagger > \overline{\pi}^\ddagger$, we recognize from (B.5) that it is equivalent to show

$$\pi^\circ - \frac{\beta}{2\lambda}\lambda^c\left(\widetilde{u}^\dagger\right)^2 = \overline{\pi}^\dagger > \overline{\pi}^\ddagger = \pi^\circ - \frac{\beta}{2\lambda}\left(\widetilde{w}^\ddagger\right)^2,$$

or equivalently,

$$\sqrt{\lambda^c}\widetilde{u}^\dagger < \widetilde{w}^\ddagger. \tag{B.49}$$

By (B.12) and (B.24), (B.49) is equivalent to

$$\begin{aligned}
&\lambda^c\alpha_l\Delta\theta + \sqrt{(\lambda^c\alpha_l\Delta\theta)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}\Delta\theta} < \sqrt{\lambda^c}\alpha\Delta\theta + \sqrt{\lambda^c(\alpha\Delta\theta)^2 + 4\lambda\lambda^c\Delta\alpha\alpha_l\underline{\theta}\Delta\theta}\\
&\lambda^c\alpha_l + \sqrt{(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\bar{\theta}/\Delta\theta} < \sqrt{\lambda^c}\alpha + \sqrt{\lambda^c\alpha^2 + 4\lambda\lambda^c\Delta\alpha\alpha_l\underline{\theta}/\Delta\theta},
\end{aligned}$$

which always holds if the following function in $x \in [0,\infty)$ is positive:

$$\Upsilon(x) := \sqrt{\lambda^c}\alpha - \lambda^c\alpha_l + \sqrt{\lambda^c\alpha^2 + 4\lambda\lambda^c\Delta\alpha\alpha_l x} - \sqrt{(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha(1+x)} > 0. \tag{B.50}$$

We note that

$$\Upsilon'(x) = \frac{2\lambda\lambda^c\Delta\alpha\alpha_l\left[(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha - \lambda^c\alpha^2\right]}{\sqrt{\lambda^c\alpha^2 + 4\lambda\lambda^c\Delta\alpha\alpha_l x}\sqrt{(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha(1+x)}\left[\sqrt{\lambda^c\alpha^2 + 4\lambda\lambda^c\Delta\alpha\alpha_l x} + \sqrt{(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha(1+x)}\right]},$$

whose sign is given by that of $(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha - \lambda^c\alpha^2$. Namely, $\Upsilon(x)$ is a monotonic function in $x \in [0,\infty)$. Therefore, to show (B.50), it suffices to show

$$\Upsilon(0) = \sqrt{\lambda^c}\alpha - \lambda^c\alpha_l - \sqrt{(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha} > 0, \quad \text{and} \tag{B.51}$$

$$\lim_{x\to\infty}\Upsilon(x) > 0. \tag{B.52}$$

Direct calculation reveals that (B.51) hold because

$$\begin{aligned}
\left(\sqrt{\lambda^c}\alpha - \lambda^c\alpha_l\right)^2 - \left((\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha\right) =& 4\lambda^c\left[\alpha^2 - \sqrt{\lambda^c}\alpha\alpha_l - \lambda\Delta\alpha\alpha_l\right]\\
=& 4\lambda^c\left(1 - \sqrt{\lambda^c}\right)\left[\alpha_l^2 + \lambda\Delta\alpha\alpha_l + \left(1 + \sqrt{\lambda^c}\right)\lambda(\Delta\alpha)^2\right] > 0.
\end{aligned}$$

To see (B.52), we note that

$$\Upsilon(x) = \sqrt{\lambda^c}\alpha - \lambda^c\alpha_l + \frac{\lambda^c\left[\alpha^2 - \lambda^c\alpha_l^2 - 4\lambda\alpha_l\Delta\alpha\right]}{\sqrt{\lambda^c\alpha^2 + 4\lambda\lambda^c\Delta\alpha\alpha_l x} + \sqrt{(\lambda^c\alpha_l)^2 + 4\lambda\lambda^c\alpha_l\Delta\alpha(1+x)}} \to \sqrt{\lambda^c}\alpha - \lambda^c\alpha_l > 0 \quad \text{as } x\to\infty.$$

This completes the proof of (B.49) and hence $\overline{\pi}^\dagger > \overline{\pi}^\ddagger$ holds.

To see that $\bar{s}^\dagger > \bar{s}^\ddagger$, we notice that it is equivalent to

$$s^\circ - \frac{\beta}{2\lambda}\lambda^c\widetilde{u}^\dagger = \bar{s}^\dagger > \bar{s}^\ddagger = s^\circ - \frac{\beta}{2\lambda}\widetilde{w}^\ddagger \quad \Leftrightarrow \quad \lambda^c\widetilde{u}^\dagger < \widetilde{w}^\ddagger,$$

which holds and follows immediately from (B.49).

To see that $s^\star > \bar{s}^\dagger$, we note that

$$\bar{s}^\dagger = \frac{\lambda^c\beta\overline{\theta}\overline{r}^\dagger + \lambda\alpha_h - \beta w^\circ}{2\lambda} = \frac{\lambda^c\beta\overline{\theta}\overline{r}^\circ + \lambda\alpha_h - \beta w^\circ}{2\lambda} - \frac{\beta}{2\lambda}\lambda^c\widetilde{u}^\dagger$$

and hence the result follows from the fact that $\lambda\widetilde{w}^\star + \lambda^c\widetilde{u}^\star < \lambda^c\widetilde{u}^\dagger$ in Lemma B.6. □

Appendix C: Proofs in Section 5

Lemma C.1. *In the demand potential signaling game, the unique equilibrium that survives the intuitive criterion is the most efficient separating equilibrium. In this equilibrium, the low-demand manufacturer offers her symmetric information contract* $(\underline{w}^\circ, r^\circ)$.

Proof of Lemma C.1. We first note that $\Pi\left(w, r \,\middle|\, \widehat{\lambda}, \lambda\right)$ is increasing in $\widehat{\lambda}$, as we compute

$$\Pi_{\widehat{\lambda}}\left(w, r \,\middle|\, \widehat{\lambda}, \lambda\right) = \begin{cases} \frac{1}{2}\widehat{\lambda}^{-2}\beta(w-r)\left(w-\lambda^c r\right), & \text{if} \quad \beta(w-r) \le \widehat{\lambda}\Delta\alpha, \\ \frac{1}{2}\Delta\alpha w, & \text{if} \quad \beta(w-r) \ge \widehat{\lambda}\Delta\alpha. \end{cases}$$

By the weakened condition of Cho and Sobel (1990) in Engers (1987), the selection of the most efficient separating equilibrium hinges on showing that the marginal rate of substitution (MRS) of *one* of signals (i.e., w or r) for the belief $\widehat{\lambda}$ is monotonic in λ. Indeed, direct calculation reveals that the MRS of r for $\widehat{\lambda}$ is given by

$$-\frac{\Pi_r\left(w, r \,\middle|\, \widehat{\lambda}, \lambda\right)}{\Pi_{\widehat{\lambda}}\left(w, r \,\middle|\, \widehat{\lambda}, \lambda\right)} = \frac{\lambda^c\left(\widehat{\lambda}\Delta\alpha - \beta(w-r)\right) + \widehat{\lambda}\beta w}{\widehat{\lambda}^{-1}\beta(w-r)(w-\lambda^c r)} - \frac{\widehat{\lambda}}{w-r}, \quad \text{for } 0 \le \beta(w-r) \le \widehat{\lambda}\Delta\alpha,$$

which is monotonically decreasing in λ. (For $\beta(w-\widehat{\theta}r) \ge \widehat{\lambda}\Delta\alpha$, $\Pi\left(w, r \,\middle|\, \widehat{\theta}, \theta\right)$ is independent of r and λ, and hence is irrelevant.)

The equilibrium strategy for the low-demand manufacturer follows from similar argument as in the proof of Lemma B.2 by recognizing that $\underline{\pi}^\circ = \Pi\left(\underline{w}^\circ, r^\circ \mid \underline{\lambda}, \underline{\lambda}\right) \le \Pi\left(\underline{w}^\circ, r^\circ \,\middle|\, \widehat{\lambda}, \underline{\lambda}\right)$ for any $\widehat{\lambda} \ge \underline{\lambda}$, because of the monotonicity of $\Pi\left(w, r \,\middle|\, \widehat{\lambda}, \lambda\right)$ in $\widehat{\lambda}$. ☐

Lemma C.2. *Any* $(\overline{w}, \overline{r})$ *feasible to* (15) *must satisfy* $\overline{w} - \overline{r} \le \overline{\lambda}\Delta\alpha/\beta$.

Proof of Lemma C.2. If $\overline{w} - \overline{r} > \overline{\lambda}\Delta\alpha/\beta$ on the contrary, then the constraints of (15) imply that

$$\Pi\left(\underline{w}^\circ, r^\circ \mid \underline{\lambda}, \overline{\lambda}\right) \le \Pi\left(\overline{w}, \overline{r} \mid \overline{\lambda}, \overline{\lambda}\right) = \frac{1}{2}\overline{w}(\alpha_l + \bar{\lambda}\Delta\alpha - \beta\overline{w}) = \Pi\left(\overline{w}, \overline{r} \mid \overline{\lambda}, \underline{\lambda}\right) \le \underline{\pi}^\circ = \Pi\left(\underline{w}^\circ, r^\circ \mid \underline{\lambda}, \underline{\lambda}\right),$$

leading to a contradiction because

$$\Pi\left(\underline{w}^\circ, r^\circ \mid \underline{\lambda}, \overline{\lambda}\right) - \Pi\left(\underline{w}^\circ, r^\circ \mid \underline{\lambda}, \underline{\lambda}\right) = \frac{1}{2}\Delta\lambda r^\circ\left[\Delta\alpha - \beta/\underline{\lambda}(\underline{w}^\circ - r^\circ)\right] = \frac{1}{8\beta}\alpha_l\Delta\alpha\Delta\lambda > 0. \quad \square$$

Road map of remaining proofs. Following the solution strategy similar to that in the returns risk case, Lemma C.3 transforms price decisions into retailer's quantity decisions; Propositions C.1 and C.2 establish the two partial signaling benchmarks formulated in (18) and (19), respectively (including the verification of supporting off-equilibrium beliefs). To establish the most efficient separating equilibrium formulated in (15), the proof of Proposition 3 consists of identifying its direction of distortion and verifying that the pessimistic off-equilibrium belief supports it.

Lemma C.3 **(Change of Variable)**. *For any* $(\overline{w}, \overline{r})$ *feasible to* (15), *let* $\widetilde{w} := \overline{w} - \overline{w}^\circ = \overline{w} - \frac{\alpha_l + \bar{\lambda}\Delta\alpha}{2\beta}$ *and* $\widetilde{u} := \overline{w} - \overline{r} - (\overline{w}^\circ - r^\circ) = \overline{w} - \overline{r} - \frac{\bar{\lambda}\Delta\alpha}{2\beta}$. *Then,* $\overline{w} = \frac{\alpha_l + \bar{\lambda}\Delta\alpha}{2\beta} + \widetilde{w}$, $\overline{r} = \widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}$, *and*

$$\widetilde{w} + \frac{\alpha_l}{2\beta} \ge \widetilde{u} \ge -\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \quad \textit{and} \quad \widetilde{u} \le \frac{\bar{\lambda}\Delta\alpha}{2\beta}. \tag{C.1}$$

Furthermore,

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\bar{\lambda}\right)=\overline{\pi}^{\circ}-\frac{\beta}{2\bar{\lambda}}\left(\bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2\right) \tag{C.2}$$

$\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\underline{\lambda}\right)\le\underline{\pi}^{\circ}$ *is equivalent to*

$$\bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2+\Delta\lambda\left(\widetilde{w}-\widetilde{u}+\frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}-\widetilde{u}\right)\ge\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}, \tag{C.3}$$

and $\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\bar{\lambda}\right)\ge\Pi\left(\underline{w}^{\circ},r^{\circ}\,\middle|\,\underline{\lambda},\bar{\lambda}\right)$ *is equivalent to*

$$\bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2\le\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+\alpha_l\Delta\alpha\right]}{4\beta^2}. \tag{C.4}$$

Proof of Lemma C.3. By Lemma C.2, we can restrict to $\overline{w}-\overline{r}\le\bar{\lambda}\Delta\alpha/\beta$. Thus, direct substitution of $\widetilde{w}$ and $\widetilde{u}$ in (12) immediately yields (C.2) and

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\underline{\lambda}\right)=\overline{\pi}^{\circ}-\frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2+\Delta\lambda\left(\widetilde{w}-\widetilde{u}+\frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}-\widetilde{u}\right)\right].$$

Using the symmetric-information expressions in Lemma 3, we thus immediately obtain (C.3) and (C.4). ☐

PROPOSITION C.1. *The solution to (18) is given by*

$$\overline{w}^{\sharp}=\frac{\bar{\alpha}}{2\beta}-\frac{1}{4\beta}\left\{\sqrt{\left(\alpha_l\Delta\lambda\right)^2+4\bar{\lambda}\Delta\lambda\alpha_h\Delta\alpha}-\alpha_l\Delta\lambda\right\}<\underline{w}^{\circ}<\overline{w}^{\circ}. \tag{C.5}$$

Contract $(\overline{w}^{\sharp},r^{\circ})$ *can be sustained as a separating equilibrium of the demand potential signaling game if and only if* $\Delta\lambda/\underline{\lambda}\left[1+\bar{\lambda}/(4\underline{\lambda})\right]\le4\alpha_h\Delta\alpha/\alpha_l^2$. *In this equilibrium, the retailer's order quantity and unsold inventory in case of low baseline demand are given by*

$$\bar{s}^{\sharp}=\frac{\alpha_h}{4}+\frac{1}{8\bar{\lambda}}\left\{\sqrt{\left(\alpha_l\Delta\lambda\right)^2+4\bar{\lambda}\Delta\lambda\alpha_h\Delta\alpha}-\alpha_l\Delta\lambda\right\}>s^{\circ},\ \textit{and} \tag{C.6}$$

$$\bar{q}^{\sharp}=\frac{\Delta\alpha}{4}+\frac{1}{8\bar{\lambda}}\left\{\sqrt{\left(\alpha_l\Delta\lambda\right)^2+4\bar{\lambda}\Delta\lambda\alpha_h\Delta\alpha}-\alpha_l\Delta\lambda\right\}>q^{\circ},\ \textit{respectively;} \tag{C.7}$$

no unsold inventory results from high baseline demand realization.

Proof. We first solve the relaxed problem of (18) by ignoring the second constraint and will then show that the solution to the relaxed problem automatically satisfies the ignored constraint. Using the change of variable specified in Lemma C.3, the solution to the relaxed problem $\overline{w}^{\sharp}=\overline{w}^{\circ}+\widetilde{w}^{\sharp}=\frac{\alpha_l+\bar{\lambda}\Delta\alpha}{2\beta}+\widetilde{w}^{\sharp}$, where $\widetilde{w}^{\sharp}$ is the solution to the following problem

$$\min_{\frac{\bar{\lambda}\Delta\alpha}{2\beta}\ge\widetilde{w}\ge-\frac{\bar{\lambda}\Delta\alpha}{2\beta}}\quad\widetilde{w}^2 \tag{C.8}$$

$$\text{subject to}\quad\widetilde{w}^2+\frac{\alpha_l\Delta\lambda}{2\beta}\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}-\widetilde{w}\right)\ge\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}. \tag{C.9}$$

Straightforward algebra reduces (C.9) to

$$\widetilde{w}^2-\frac{\alpha_l\Delta\lambda}{2\beta}\widetilde{w}-\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+\alpha_l\Delta\alpha\right]}{4\beta^2}\ge0, \tag{C.10}$$

whose left-hand side is a quadratic function of two roots. As this quadratic function is minimized at $\widetilde{w}=\frac{\alpha_l\Delta\lambda}{4\beta}>0$, it immediately follows that the smaller (and negative) root

$$\widetilde{w}^{\sharp}=\frac{\alpha_l\Delta\lambda}{4\beta}-\sqrt{\left(\frac{\alpha_l\Delta\lambda}{4\beta}\right)^2+\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+\alpha_l\Delta\alpha\right]}{4\beta^2}}\in\left(-\frac{\bar{\lambda}\Delta\alpha}{2\beta},0\right) \tag{C.11}$$

minimizes (C.8). In particular, the verification of $\widetilde{w}^\sharp > -\frac{\bar{\lambda}\Delta\alpha}{2\beta}$ is straightforward. Subsequently, we obtain (C.5) by substituting the expression of $\widetilde{w}^\sharp$ into $\overline{w}^\sharp = \overline{w}^\circ + \widetilde{w}^\sharp$.

To see that $\overline{w}^\sharp < \underline{w}^\circ$, we note that

$$\begin{aligned}\underline{w}^\circ - \overline{w}^\sharp =& \frac{1}{4\beta}\sqrt{\left(\alpha_l\Delta\lambda\right)^2 + 4\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]} - \frac{1}{4\beta}\left(2\Delta\lambda\Delta\alpha + \alpha_l\Delta\lambda\right)\\ =& -\frac{\alpha_h\underline{\lambda}\Delta\lambda\Delta\alpha}{\beta\left(\sqrt{\left(\alpha_l\Delta\lambda\right)^2 + 4\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]} + 2\Delta\lambda\Delta\alpha + \alpha_l\Delta\lambda\right)} > 0.\end{aligned}$$

We now verify that the ignored constraint is satisfied. By Lemma C.3, it is equivalent to show that

$$\left(\widetilde{w}^\sharp\right)^2 \le \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2}, \tag{C.12}$$

which is equivalent to (by using the fact that $\widetilde{w}^\sharp$ is the root of the right-hand side of (C.10))

$$\left(\widetilde{w}^\sharp\right)^2 = \frac{\alpha_l\Delta\lambda}{2\beta}\widetilde{w}^\sharp + \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2} \le \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2} \quad \Leftrightarrow \quad \widetilde{w}^\sharp \le 0,$$

and hence obviously holds.

For contract $(\overline{w}^\sharp, r^\circ)$ to be sustained by some equilibrium belief, we need to show that neither high- nor low-demand manufacturer has an incentive to deviate to any off-equilibrium strategy under that belief.

- For the low-demand manufacturer, we have shown that her profit of deviating to $(\overline{w}^\sharp, r^\circ)$ and hence being mistaken as of high demand potential is dominated by her equilibrium profit, i.e., $\Pi\left(\overline{w}^\sharp, r^\circ \,\middle|\, \bar{\lambda}, \underline{\lambda}\right) \le \underline{\pi}^\circ$. As any other contract $(\underline{w}, \underline{r})$ induces a belief that she is of low demand potential, the symmetric-information $(\underline{w}^\circ, r^\circ)$ maximizes her profit $\Pi\left(\underline{w}, \underline{r} \,|\, \underline{\lambda}, \underline{\lambda}\right)$ to $\underline{\pi}^\circ$. Therefore, the low-demand manufacturer indeed has no incentive to deviate away from her symmetric-information contract $(\underline{w}^\circ, r^\circ)$.
- For the high-demand manufacturer whose return price is restricted to r°, it suffices to show that she has no incentive to deviate her wholesale price to any $\overline{w} \neq \overline{w}^\sharp$ and thus to be mistaken as of low demand potential $\underline{\lambda}$, i.e., $\Pi\left(\overline{w}, r^\circ \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) \le \Pi\left(\overline{w}^\sharp, r^\circ \,\middle|\, \bar{\lambda}, \bar{\lambda}\right)$ for all $\overline{w}$. Indeed, if this condition fails, no other deviation belief $\widehat{\lambda}$ can support $\overline{w}^\sharp$, because $\Pi\left(\overline{w}, r^\circ \,\middle|\, \widehat{\lambda}, \bar{\lambda}\right)$ is non-decreasing in $\widehat{\lambda}$ as pointed out in the proof of Lemma C.1 and hence $\Pi\left(\overline{w}, r^\circ \,\middle|\, \widehat{\lambda}, \bar{\lambda}\right) \ge \Pi\left(\overline{w}, r^\circ \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) > \Pi\left(\overline{w}^\sharp, r^\circ \,\middle|\, \bar{\lambda}, \bar{\lambda}\right)$ for all $\widehat{\lambda} \ge \underline{\lambda}$.

—For any $\overline{w} \le r^\circ + \underline{\lambda}\Delta\alpha/\beta = \alpha_l/(2\beta) + \underline{\lambda}\Delta\alpha/\beta$, (12) implies that

$$\begin{aligned}\Pi\left(\overline{w}, r^\circ \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) =& \frac{\beta}{2}\left\{\overline{w}\left(\frac{\alpha_l + \underline{\lambda}\Delta\alpha}{\beta} - \overline{w}\right) + \underline{\lambda}^{-1}\left(\underline{\lambda}^c\overline{w} - \bar{\lambda}^c r^\circ\right)\left(\frac{\underline{\lambda}\Delta\alpha}{\beta} + r^\circ - \overline{w}\right)\right\}\\ =& \frac{\beta}{2}\left\{-\underline{\lambda}^{-1}\overline{w}^2 + \left[\frac{\Delta\alpha}{\beta} + \frac{\alpha_l}{\beta}\left(1 + \frac{\underline{\lambda}^c + \bar{\lambda}^c}{2\underline{\lambda}}\right)\right]\overline{w} - \frac{\bar{\lambda}^c}{\underline{\lambda}}\frac{\alpha_l}{2\beta}\left(\frac{\underline{\lambda}\Delta\alpha}{\beta} + \frac{\alpha_l}{2\beta}\right)\right\},\end{aligned}$$

which reaches its (unconstrained) maximum (calculated below) at $\overline{w} = \frac{\underline{\lambda}}{2}\left[\frac{\Delta\alpha}{\beta} + \frac{\alpha_l}{\beta}\left(1 + \frac{\underline{\lambda}^c + \bar{\lambda}^c}{2\underline{\lambda}}\right)\right] \le \alpha_l/(2\beta) + \underline{\lambda}\Delta\alpha/\beta$:

$$\begin{aligned}&\frac{\beta}{2}\left\{\frac{\underline{\lambda}}{4}\left[\frac{\Delta\alpha}{\beta} + \frac{\alpha_l}{\beta}\left(1 + \frac{\underline{\lambda}^c + \bar{\lambda}^c}{2\underline{\lambda}}\right)\right]^2 - \frac{\bar{\lambda}^c}{\underline{\lambda}}\frac{\alpha_l}{2\beta}\left(\frac{\underline{\lambda}\Delta\alpha}{\beta} + \frac{\alpha_l}{2\beta}\right)\right\}\\ =& \frac{1}{8\beta}\left\{\underline{\lambda}(\Delta\alpha)^2 + (\underline{\lambda} + \bar{\lambda})\alpha_l\Delta\alpha + \left[1 + \frac{(\Delta\lambda)^2}{4\underline{\lambda}}\right]\alpha_l^2\right\}.\end{aligned} \tag{C.13}$$

On the other hand, (C.2) suggests that

$$\Pi\left(\overline{w}^\sharp, r^\circ \,\middle|\, \bar{\lambda}, \bar{\lambda}\right) = \overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left(\widetilde{w}^\sharp\right)^2$$

$$
\begin{aligned}
&=\frac{\bar{\lambda}\left((\Delta\alpha)^2+2\alpha_l\Delta\alpha\right)+\alpha_l^2}{8\beta}-\frac{\beta}{2\bar{\lambda}}\left[\frac{\alpha_l\Delta\lambda}{2\beta}\widetilde{w}^\sharp+\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+\alpha_l\Delta\alpha\right]}{4\beta^2}\right]\\
&=\frac{1}{8\beta}\left\{\underline{\lambda}(\Delta\alpha)^2+(\underline{\lambda}+\bar{\lambda})\alpha_l\Delta\alpha+\alpha_l^2\right\}-\frac{\alpha_l\Delta\lambda}{4\bar{\lambda}}\widetilde{w}^\sharp, \qquad \text{(C.14)}
\end{aligned}
$$

where the second equality follows from (14) and the fact that $\widetilde{w}^\sharp$ is the root of the right-hand side of (C.10).

Therefore, to show that $\Pi\left(\overline{w}, r^\circ \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) \le \Pi\left(\overline{w}^\sharp, r^\circ \,\middle|\, \bar{\lambda}, \bar{\lambda}\right)$ for all $\overline{w} \le \alpha_l/(2\beta)+\underline{\lambda}\Delta\alpha/\beta$, we need, by (C.13) and (C.14),

$$
\frac{1}{8\beta}\left\{\underline{\lambda}(\Delta\alpha)^2+(\underline{\lambda}+\bar{\lambda})\alpha_l\Delta\alpha+\left[1+\frac{(\Delta\lambda)^2}{4\underline{\lambda}}\right]\alpha_l^2\right\} \le \frac{1}{8\beta}\left\{\underline{\lambda}(\Delta\alpha)^2+(\underline{\lambda}+\bar{\lambda})\alpha_l\Delta\alpha+\alpha_l^2\right\}-\frac{\alpha_l\Delta\lambda}{4\bar{\lambda}}\widetilde{w}^\sharp,
$$

or equivalently,

$$
\widetilde{w}^\sharp=\frac{\alpha_l\Delta\lambda}{4\beta}-\sqrt{\left(\frac{\alpha_l\Delta\lambda}{4\beta}\right)^2+\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+\alpha_l\Delta\alpha\right]}{4\beta^2}} \le -\frac{\bar{\lambda}\alpha_l\Delta\lambda}{8\beta\underline{\lambda}},
$$

which holds if and only if $\Delta\lambda/\underline{\lambda}\left[1+\bar{\lambda}/(4\underline{\lambda})\right] \le 4\left[(\Delta\alpha)^2+\alpha_l\Delta\alpha\right]/\alpha_l^2$, the assumption in the proposition.

—For $\overline{w} \ge r^\circ+\underline{\lambda}\Delta\alpha/\beta=\alpha_l/(2\beta)+\underline{\lambda}\Delta\alpha/\beta$, (12) implies that

$$
\Pi\left(\overline{w}, r^\circ \,\middle|\, \underline{\lambda}, \bar{\lambda}\right)=\frac{\beta}{2}\overline{w}\left(\frac{\alpha_l+\underline{\lambda}\Delta\alpha}{\beta}-\overline{w}\right),
$$

whose unconstrained maximum is achieved at $\overline{w}=\frac{\alpha_l+\underline{\lambda}\Delta\alpha}{2\beta}<\alpha_l/(2\beta)+\underline{\lambda}\Delta\alpha/\beta$. Therefore, the maximum of $\Pi\left(\overline{w}, r^\circ \,\middle|\, \underline{\lambda}, \bar{\lambda}\right)$ over $\overline{w} \ge \alpha_l/(2\beta)+\underline{\lambda}\Delta\alpha/\beta$ is achieved at $\overline{w}=\alpha_l/(2\beta)+\underline{\lambda}\Delta\alpha/\beta$, leading us back to the previous case.

Finally, we determine the retailer's order quantity as well as unsold inventory. Since $\overline{w}^\sharp-r^\circ=\overline{w}^\circ+\widetilde{w}^\sharp-\frac{\alpha_l}{2\beta}=\widetilde{w}^\sharp+\frac{\bar{\lambda}\Delta\alpha}{2\beta}\le\frac{\bar{\lambda}\Delta\alpha}{\beta}$ because $\widetilde{w}^\sharp<\frac{\bar{\lambda}\Delta\alpha}{2\beta}$, all inventory is sold out in the case of high baseline demand realization and, in particular, (1) implies that the retailer's order quantity is

$$
\bar{s}^\sharp=s^R\left(\overline{w}^\sharp, r^\circ, 1, \bar{\lambda}\right)=\frac{\bar{\lambda}^c\beta r^\circ+\bar{\lambda}\alpha_h-\beta\overline{w}^\sharp}{2\bar{\lambda}}>\frac{\bar{\lambda}^c\beta r^\circ+\bar{\lambda}\alpha_h-\beta\overline{w}^\circ}{2\bar{\lambda}}=s^\circ,
$$

which yields (C.6) by substituting the expression of $\overline{w}^\sharp$ given in (C.5). By Lemma 1, again, the unsold inventory in the case of low baseline demand realization is

$$
\bar{q}^\sharp=\frac{1}{2}\left[\Delta\alpha-\beta/\bar{\lambda}\left(\overline{w}^\sharp-r^\circ\right)\right]=\frac{1}{2}\left[\Delta\alpha/2-\beta/\bar{\lambda}\widetilde{w}^\sharp\right],
$$

from which (C.7) follows by (C.11). □

PROPOSITION C.2. *The solution to (19) is given by*

$$
\overline{r}^\flat=\frac{\alpha_l}{2\beta}+\frac{1}{4\beta\underline{\lambda}^c}\left\{\sqrt{(\Delta\lambda\overline{\alpha})^2+4\underline{\lambda}^c\bar{\lambda}\Delta\lambda\alpha_h\Delta\alpha}-\Delta\lambda\overline{\alpha}\right\}\in\left(r^\circ, \overline{w}^\circ\right). \qquad \text{(C.15)}
$$

Contract $\left(\overline{w}^\circ, \overline{r}^\flat\right)$ *can always be sustained as a separating equilibrium of the demand potential signaling game, in which the retailer's order quantity and unsold inventory in case of low baseline demand are given by*

$$
\begin{aligned}
\bar{s}^\flat &=\frac{\alpha_h}{4}+\frac{\bar{\lambda}^c}{8\bar{\lambda}\underline{\lambda}^c}\left\{\sqrt{(\Delta\lambda\overline{\alpha})^2+4\underline{\lambda}^c\bar{\lambda}\Delta\lambda\alpha_h\Delta\alpha}-\Delta\lambda\overline{\alpha}\right\}>s^\circ, \ \textit{and} \qquad \text{(C.16)}\\
\bar{q}^\flat &=\frac{\Delta\alpha}{4}+\frac{1}{8\bar{\lambda}\underline{\lambda}^c}\left\{\sqrt{(\Delta\lambda\overline{\alpha})^2+4\underline{\lambda}^c\bar{\lambda}\Delta\lambda\alpha_h\Delta\alpha}-\Delta\lambda\overline{\alpha}\right\}>q^\circ, \ \textit{respectively;} \qquad \text{(C.17)}
\end{aligned}
$$

no unsold inventory results from high baseline demand realization.

Proof. We first solve the relaxed problem of (19) by ignoring the second constraint and will then show that the solution to the relaxed problem automatically satisfies the ignored constraint. Using the change of variable specified in Lemma C.3, the solution to the relaxed problem $\overline{r}^\flat = \frac{\alpha_l}{2\beta} - \widetilde{u}^\flat$, where $\widetilde{u}^\flat$ is the solution to the following problem

$$\min_{\widetilde{u} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta} \wedge \frac{\alpha_l}{2\beta}\right]} \quad \widetilde{u}^2 \tag{C.18}$$

$$\text{subject to} \quad \bar{\lambda}^c \widetilde{u}^2 + \Delta\lambda \left(\frac{\alpha_l}{2\beta} - \widetilde{u}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right) \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}. \tag{C.19}$$

Straightforward algebra reduces (C.19) to

$$\underline{\lambda}^c \widetilde{u}^2 - \frac{\Delta\lambda\overline{\alpha}}{2\beta}\widetilde{u} - \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2} \geq 0, \tag{C.20}$$

whose left-hand side is a quadratic function of two roots. As this quadratic function is minimized at $\widetilde{u} = \frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^c} > 0$, it immediately follows that the smaller (and negative) root

$$\widetilde{u}^\flat = \frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^c} - \sqrt{\left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^c}\right)^2 + \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2\underline{\lambda}^c}} < 0 \tag{C.21}$$

minimizes (C.18). Subsequently, we obtain (C.15) by substituting the expression of $\widetilde{u}^\flat$ into $\overline{r}^\flat = \frac{\alpha_l}{2\beta} - \widetilde{u}^\flat$. In particular, we note that $\overline{r}^\flat = r^\circ - \widetilde{u}^\flat > r^\circ$ and that $\overline{r}^\flat < \overline{w}^\circ$ is equivalent to $\widetilde{u}^\flat > -\frac{\bar{\lambda}\Delta\alpha}{2\beta}$, which indeed holds because

$$\widetilde{u}^\flat > -\frac{\bar{\lambda}\Delta\alpha}{2\beta} \quad \Leftrightarrow \quad \left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^c} + \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)^2 > \left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^c}\right)^2 + \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2\underline{\lambda}^c} \quad \Leftrightarrow \quad \underline{\lambda}^c\bar{\lambda} > \bar{\lambda}^c\Delta\lambda.$$

We now verify that the ignored constraint is satisfied. Since $\widetilde{u}^\flat$ is a root of the right-hand side of (C.20), we have

$$\bar{\lambda}^c\left(\widetilde{u}^\flat\right)^2 < \underline{\lambda}^c\left(\widetilde{u}^\flat\right)^2 = \frac{\Delta\lambda\overline{\alpha}}{2\beta}\widetilde{u}^\flat + \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2} < \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2}, \tag{C.22}$$

which shows that the ignored constraint holds by Lemma C.3.

We claim that contract $\left(\overline{w}^\circ, \overline{r}^\flat\right)$ can be sustained as a separating equilibrium by the retailer's posterior belief that the manufacturer is of high demand potential upon such a contract being offered and is otherwise of low demand potential. To that end, we need to show that neither high- nor low-demand manufacturer has incentive to deviate to any off-equilibrium strategy under such a posterior belief.

- For the low-demand manufacturer, we have shown that her profit of deviating to $\left(\overline{w}^\circ, \overline{r}^\flat\right)$ and hence being mistaken as of high demand potential is dominated by her equilibrium profit, i.e., $\Pi\left(\overline{w}^\circ, \overline{r}^\flat \mid \bar{\lambda}, \underline{\lambda}\right) \leq \underline{\pi}^\circ$. As any other contract $(\underline{w}, \underline{r})$ induces a belief that she is of low demand potential, the symmetric-information $(\underline{w}^\circ, r^\circ)$ maximizes her profit $\Pi(\underline{w}, \underline{r} \mid \underline{\lambda}, \underline{\lambda})$ to $\underline{\pi}^\circ$. Therefore, the low-demand manufacturer indeed has no incentive to deviate away from her symmetric-information contract $(\underline{w}^\circ, r^\circ)$.
- For the high-demand manufacturer whose wholesale price is restricted to $\overline{w}^\circ$, it suffices to show that she has no incentive to deviate her return price to any $\bar{r} \neq \overline{r}^\flat$ and thus to be mistaken as of low demand potential, i.e., $\Pi\left(\overline{w}^\circ, \bar{r} \mid \underline{\lambda}, \bar{\lambda}\right) \leq \Pi\left(\overline{w}^\circ, \overline{r}^\flat \mid \bar{\lambda}, \bar{\lambda}\right)$ for all $\bar{r}$.

—For any $\bar{r}$ such that $\overline{w}^{\circ}-\bar{r}>\underline{\lambda}\Delta\alpha/\beta$, we have, by (C.2),

$$\begin{aligned}\Pi\left(\overline{w}^{\circ},\bar{r}^{\flat}\,\middle|\,\bar{\lambda},\bar{\lambda}\right)-\Pi\left(\overline{w}^{\circ},\bar{r}\,\middle|\,\underline{\lambda},\bar{\lambda}\right)=&\overline{\pi}^{\circ}-\frac{\beta\bar{\lambda}^{c}}{2\bar{\lambda}}\left(\widetilde{u}^{\flat}\right)^{2}-\frac{\beta}{2}\left[\left(\frac{\alpha_{l}+\bar{\lambda}\Delta\alpha}{2\beta}\right)^{2}-\frac{\Delta\lambda\Delta\alpha\left(\alpha_{l}+\bar{\lambda}\Delta\alpha\right)}{2\beta^{2}}\right]\\=&\frac{\beta\bar{\lambda}^{c}}{2\bar{\lambda}}\left[\frac{\bar{\lambda}^{2}(\Delta\alpha)^{2}}{4\beta^{2}}+\frac{\bar{\lambda}\Delta\lambda\left[\bar{\lambda}(\Delta\alpha)^{2}+\alpha_{l}\Delta\alpha\right]}{2\beta^{2}\bar{\lambda}^{c}}-\left(\widetilde{u}^{\flat}\right)^{2}\right]\\>&\frac{\beta\bar{\lambda}^{c}}{2\bar{\lambda}}\left[\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^{2}+\alpha_{l}\Delta\alpha\right]}{4\beta^{2}\underline{\lambda}^{c}}-\left(\widetilde{u}^{\flat}\right)^{2}\right]\geq 0,\end{aligned}$$

where the last inequality follows from (C.22).

—For any $\bar{r}$ such that $0\leq\overline{w}^{\circ}-\bar{r}\leq\underline{\lambda}\Delta\alpha/\beta$, direct calculation from (12) yields

$$\begin{aligned}\Pi\left(\overline{w}^{\circ},\bar{r}\,\middle|\,\underline{\lambda},\bar{\lambda}\right)=&\frac{\overline{\alpha}^{2}}{8\beta}-\frac{\Delta\lambda\Delta\alpha\overline{\alpha}}{4\beta}+\frac{\beta\bar{\lambda}^{c}}{2\underline{\lambda}}\left(\underline{\lambda}^{c}/\bar{\lambda}^{c}\overline{w}^{\circ}-\bar{r}\right)\left(\underline{\lambda}\Delta\alpha/\beta-\overline{w}^{\circ}+\bar{r}\right)\\=&\overline{\pi}^{\circ}-\frac{\bar{\lambda}^{c}\bar{\lambda}(\Delta\alpha)^{2}}{8\beta}-\frac{\Delta\lambda\Delta\alpha\overline{\alpha}}{4\beta}+\frac{\beta\bar{\lambda}^{c}}{2\underline{\lambda}}\left(\underline{\lambda}^{c}/\bar{\lambda}^{c}\overline{w}^{\circ}-\bar{r}\right)\left(\underline{\lambda}\Delta\alpha/\beta-\overline{w}^{\circ}+\bar{r}\right).\end{aligned}$$

Here, the quadratic equation $\left(\underline{\lambda}^{c}/\bar{\lambda}^{c}\overline{w}^{\circ}-\bar{r}\right)\left(\underline{\lambda}\Delta\alpha/\beta-\overline{w}^{\circ}+\bar{r}\right)$ in $\bar{r}$ achieves its unconstrained maximum $\left(\frac{\Delta\lambda}{2\bar{\lambda}^{c}}\overline{w}^{\circ}+\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right)^{2}$ at $\bar{r}=\frac{\underline{\lambda}^{c}+\bar{\lambda}^{c}}{2\bar{\lambda}^{c}}\overline{w}^{\circ}-\frac{\underline{\lambda}\Delta\alpha}{2\beta}$, which surely satisfies $\overline{w}^{\circ}-\bar{r}\leq\underline{\lambda}\Delta\alpha/\beta$ and is smaller than $\overline{w}^{\circ}$ if and only if $\Delta\alpha/\bar{\alpha}\geq\Delta\lambda/(2\bar{\lambda}^{c}\underline{\lambda})$. Thus, we consider the following two cases:

∗ When $\Delta\alpha/\bar{\alpha}\leq\Delta\lambda/(2\bar{\lambda}^{c}\underline{\lambda})$, the quadratic equation $\left(\underline{\lambda}^{c}/\bar{\lambda}^{c}\overline{w}^{\circ}-\bar{r}\right)\left(\underline{\lambda}\Delta\alpha/\beta-\overline{w}^{\circ}+\bar{r}\right)$ reaches it maximum $\frac{\Delta\lambda}{\bar{\lambda}^{c}}\frac{\bar{\alpha}}{2\beta}\frac{\underline{\lambda}\Delta\alpha}{\beta}$ at $\bar{r}=\overline{w}^{\circ}$, leading to

$$\Pi\left(\overline{w}^{\circ},\bar{r}\,\middle|\,\underline{\lambda},\bar{\lambda}\right)\leq\overline{\pi}^{\circ}-\frac{\bar{\lambda}^{c}\bar{\lambda}(\Delta\alpha)^{2}}{8\beta}-\frac{\Delta\lambda\Delta\alpha\overline{\alpha}}{4\beta}+\frac{\beta\bar{\lambda}^{c}}{2\underline{\lambda}}\frac{\Delta\lambda}{\bar{\lambda}^{c}}\frac{\bar{\alpha}}{2\beta}\frac{\underline{\lambda}\Delta\alpha}{\beta}=\overline{\pi}^{\circ}-\frac{\bar{\lambda}^{c}\bar{\lambda}(\Delta\alpha)^{2}}{8\beta}.$$

Therefore, $\Pi\left(\overline{w}^{\circ},\bar{r}\,\middle|\,\underline{\lambda},\bar{\lambda}\right)\leq\Pi\left(\overline{w}^{\circ},\bar{r}^{\flat}\,\middle|\,\bar{\lambda},\bar{\lambda}\right)=\overline{\pi}^{\circ}-\frac{\beta\bar{\lambda}^{c}}{2\bar{\lambda}}\left(\widetilde{u}^{\flat}\right)^{2}$ holds if

$$\left(\widetilde{u}^{\flat}\right)^{2}\leq\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)^{2}\quad\Leftrightarrow\quad\widetilde{u}^{\flat}\geq-\frac{\bar{\lambda}\Delta\alpha}{2\beta},$$

which is, by (C.21), equivalent to

$$\frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^{c}}+\frac{\bar{\lambda}\Delta\alpha}{2\beta}\geq\sqrt{\left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^{c}}\right)^{2}+\frac{\bar{\lambda}\Delta\lambda\Delta\alpha\left(\bar{\lambda}^{c}\Delta\alpha+\bar{\alpha}\right)}{4\beta^{2}\underline{\lambda}^{c}}}\quad\Leftrightarrow\quad\underline{\lambda}^{c}\bar{\lambda}\geq\underline{\lambda}^{c}\Delta\lambda.$$

The last inequality above obviously holds.

∗ When $\Delta\alpha/\bar{\alpha}\geq\Delta\lambda/(2\bar{\lambda}^{c}\underline{\lambda})$, we have

$$\Pi\left(\overline{w}^{\circ},\bar{r}\,\middle|\,\underline{\lambda},\bar{\lambda}\right)\leq\overline{\pi}^{\circ}-\frac{\bar{\lambda}^{c}\bar{\lambda}(\Delta\alpha)^{2}}{8\beta}-\frac{\Delta\lambda\Delta\alpha\overline{\alpha}}{4\beta}+\frac{\beta\bar{\lambda}^{c}}{2\underline{\lambda}}\left(\frac{\Delta\lambda}{2\bar{\lambda}^{c}}\frac{\overline{\alpha}}{2\beta}+\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right)^{2}.$$

Therefore, $\Pi\left(\overline{w}^{\circ},\bar{r}\,\middle|\,\underline{\lambda},\bar{\lambda}\right)\leq\Pi\left(\overline{w}^{\circ},\bar{r}^{\flat}\,\middle|\,\bar{\lambda},\bar{\lambda}\right)=\overline{\pi}^{\circ}-\frac{\beta\bar{\lambda}^{c}}{2\bar{\lambda}}\left(\widetilde{u}^{\flat}\right)^{2}$ holds if

$$\left(\widetilde{u}^{\flat}\right)^{2}\leq\frac{\bar{\lambda}^{2}(\Delta\alpha)^{2}}{4\beta^{2}}+\frac{\bar{\lambda}\Delta\lambda\Delta\alpha\overline{\alpha}}{2\beta^{2}\bar{\lambda}^{c}}-\frac{\bar{\lambda}}{\underline{\lambda}}\left(\frac{\Delta\lambda}{2\bar{\lambda}^{c}}\frac{\overline{\alpha}}{2\beta}+\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right)^{2}=\frac{\bar{\lambda}\Delta\lambda(\Delta\alpha)^{2}}{4\beta^{2}}+\frac{\bar{\lambda}\Delta\lambda\Delta\alpha\overline{\alpha}}{4\beta^{2}\bar{\lambda}^{c}}-\frac{\bar{\lambda}}{\underline{\lambda}}\left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\bar{\lambda}^{c}}\right)^{2}.$$

By noticing that $\widetilde{u}^{\flat}$ binds (C.20), the above inequality can be rewritten as

$$\frac{\Delta\lambda\overline{\alpha}}{2\beta\underline{\lambda}^{c}}\widetilde{u}^{\flat}+\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^{2}+\alpha_{l}\Delta\alpha\right]}{4\beta^{2}\underline{\lambda}^{c}}\leq\frac{\bar{\lambda}\Delta\lambda(\Delta\alpha)^{2}}{4\beta^{2}}+\frac{\bar{\lambda}\Delta\lambda\Delta\alpha\overline{\alpha}}{4\beta^{2}\bar{\lambda}^{c}}-\frac{\bar{\lambda}}{\underline{\lambda}}\left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\bar{\lambda}^{c}}\right)^{2},$$

which is, by (C.21), equivalent to

$$\frac{\Delta\lambda\overline{\alpha}}{2\beta\underline{\lambda}^{c}}\frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^{c}}-\frac{\Delta\lambda\overline{\alpha}}{2\beta\underline{\lambda}^{c}}\sqrt{\left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\underline{\lambda}^{c}}\right)^{2}+\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^{2}+\alpha_{l}\Delta\alpha\right]}{4\beta^{2}\underline{\lambda}^{c}}}\leq\frac{\bar{\lambda}(\Delta\lambda)^{2}(\Delta\alpha)^{2}}{4\beta^{2}\underline{\lambda}^{c}}+\frac{\bar{\lambda}(\Delta\lambda)^{2}\Delta\alpha\overline{\alpha}}{4\beta^{2}\bar{\lambda}^{c}\underline{\lambda}^{c}}-\frac{\bar{\lambda}}{\underline{\lambda}}\left(\frac{\Delta\lambda\overline{\alpha}}{4\beta\bar{\lambda}^{c}}\right)^{2}.$$

Letting $x := 2\Delta\alpha/\bar{\alpha} \ge \Delta\lambda/(\bar{\lambda}^c\underline{\lambda})$, we can reduce the above inequality to

$$2+\frac{\bar{\lambda}}{\underline{\lambda}}\left(\frac{\underline{\lambda}^c}{\bar{\lambda}^c}\right)^2-\underline{\lambda}^c\bar{\lambda}\left(x^2+2/\bar{\lambda}^c x\right)-2\sqrt{1+\frac{\bar{\lambda}^c}{\Delta\lambda}\underline{\lambda}^c\bar{\lambda}\left(x^2+2/\bar{\lambda}^c x\right)}\le 0.$$

The left-hand side of the above inequality is a decreasing function in x, and hence we just need to show it holds for $x := 2\Delta\alpha/\bar{\alpha} = \Delta\lambda/(\bar{\lambda}^c\underline{\lambda})$, which returns to the previous case.

Finally, we determine the retailer's order quantity as well as unsold inventory. Since $\overline{w}^\circ - \overline{r}^\flat = \widetilde{u}^\flat + \frac{\bar{\lambda}\Delta\alpha}{2\beta} < \bar{\lambda}\Delta\alpha/\beta$ because $\widetilde{u}^\flat < \frac{\bar{\lambda}\Delta\alpha}{2\beta}$, Lemma 1 suggests that all inventory is sold out in the case of high baseline demand realization, and in particular, (1) implies that the retailer's order quantity is

$$\overline{s}^\flat = s^R\left(\overline{w}^\circ,\overline{r}^\flat,1,\bar{\lambda}\right)=\frac{\bar{\lambda}^c\beta\overline{r}^\flat+\bar{\lambda}\alpha_h-\beta\overline{w}^\circ}{2\bar{\lambda}}>\frac{\bar{\lambda}^c\beta r^\circ+\bar{\lambda}\alpha_h-\beta\overline{w}^\circ}{2\bar{\lambda}}=s^\circ,$$

which yields (C.16) by substituting the expression of $\overline{r}^\flat$ given in (C.15). Again, by Lemma 1, the unsold inventory in the case of low baseline demand realization is

$$\overline{q}^\flat=\frac{1}{2}\left[\Delta\alpha-\beta/\bar{\lambda}\left(\overline{w}^\circ-\overline{r}^\flat\right)\right]=\frac{1}{2}\left[\Delta\alpha/2-\beta/\bar{\lambda}\widetilde{u}^\flat\right],$$

from which (C.17) follows by (C.21). □

Proof of Proposition 3. We first claim that *we can ignore the second constraint in* (15). Indeed, as the contract $\left(\overline{w}^\circ,\overline{r}^\flat\right)$ identified in Proposition C.2 satisfies the first constraint in (15), the optimal objective value from the relaxed problem must dominate $\Pi\left(\overline{w}^\circ,\overline{r}^\flat \mid \bar{\lambda},\bar{\lambda}\right)\ge\Pi\left(\underline{w}^\circ,r^\circ \mid \underline{\lambda},\bar{\lambda}\right)$, i.e., the ignored constraint must be satisfied by the optimal solution to the relaxed problem.

Therefore, $(\overline{w}^{\star\star},\overline{r}^{\star\star})$ can be identified by ignoring the second constraint in (15). Using the change of variable specified in Lemma C.3, we have $\overline{w}^{\star\star}=\overline{w}^\circ+\widetilde{w}^{\star\star}=\frac{\alpha_l+\bar{\lambda}\Delta\alpha}{2\beta}+\widetilde{w}^{\star\star}$ and $\overline{r}^{\star\star}=\frac{\alpha_l}{2\beta}+\widetilde{w}^{\star\star}-\widetilde{u}^{\star\star}$, where $(\widetilde{w}^{\star\star},\widetilde{u}^{\star\star})$ is the solution to the following problem

$$\min_{\widetilde{w}+\frac{\alpha_l}{2\beta}\ge\widetilde{u}\ge-\frac{\bar{\lambda}\Delta\alpha}{2\beta},\,\widetilde{u}\le\frac{\bar{\lambda}\Delta\alpha}{2\beta}} \quad \bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2,\quad \text{subject to (C.3).} \tag{C.23}$$

We claim that $-\frac{\bar{\lambda}\Delta\alpha}{2\beta}<\widetilde{u}^{\star\star}<\frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and hence the bound constraint on $\widetilde{u}$ can be ignored. Suppose $\widetilde{u}^{\star\star}=\pm\frac{\bar{\lambda}\Delta\alpha}{2\beta}$. Then consider a feasible solution $(0,\widetilde{u}^\flat)$ to the relaxed problem identified in the proof of Proposition C.2. We then have

$$\bar{\lambda}\left(\widetilde{w}^{\star\star}\right)^2+\bar{\lambda}^c\left(\widetilde{u}^{\star\star}\right)^2\ge\bar{\lambda}^c\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)^2>\bar{\lambda}\left(0\right)^2+\bar{\lambda}^c\left(\widetilde{u}^\flat\right)^2,$$

where the strict inequality follows from the fact that $-\frac{\bar{\lambda}\Delta\alpha}{2\beta}<\widetilde{u}^\flat<\frac{\bar{\lambda}\Delta\alpha}{2\beta}$. This immediately contradicts the optimality of $(\widetilde{w}^{\star\star},\widetilde{u}^{\star\star})$, proving that $-\frac{\bar{\lambda}\Delta\alpha}{2\beta}<\widetilde{u}^{\star\star}<\frac{\bar{\lambda}\Delta\alpha}{2\beta}$.

We now solve (C.23) by ignoring the constraint $\widetilde{w}+\frac{\alpha_l}{2\beta}\ge\widetilde{u}$, which will be verified to be satisfied by the optimal solution. The necessary condition for the optimality of $(\widetilde{w}^{\star\star},\widetilde{u}^{\star\star})$ is that there exists a Lagarangian multiplier $\xi\ge 0$ associated with (C.3) such that

$$2\bar{\lambda}\widetilde{w}^{\star\star}-\xi\left[2\bar{\lambda}\widetilde{w}^{\star\star}+\Delta\lambda\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}-\widetilde{u}^{\star\star}\right)\right]=0, \tag{C.24}$$

$$2\bar{\lambda}^c\widetilde{u}^{\star\star}-\xi\left[2\bar{\lambda}^c\widetilde{u}^{\star\star}+\Delta\lambda\left(2\widetilde{u}^{\star\star}-\widetilde{w}^{\star\star}-\frac{\alpha_l+\bar{\lambda}\Delta\alpha}{2\beta}\right)\right]=0. \tag{C.25}$$

We first note that $\xi > 0$. Otherwise, (C.24) and (C.25) suggest that $\widetilde{w}^{\star\star} = \widetilde{u}^{\star\star} = 0$, violating (C.3). Hence, (C.3) must be binding, from which we have

$$\begin{aligned}\Delta\lambda\left(\widetilde{w}^{\star\star} - \widetilde{u}^{\star\star} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star}\right) &= \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \left[\bar{\lambda}\left(\widetilde{w}^{\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star}\right)^2\right] \\ &\geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \left[\bar{\lambda}0^2 + \bar{\lambda}^c\left(\widetilde{u}^{\flat}\right)^2\right] \\ &= \Delta\lambda\left(\frac{\alpha_l}{2\beta} - \widetilde{u}^{\flat}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\flat}\right) > 0\end{aligned}$$

where the inequality follows from the optimality of $(\widetilde{w}^{\star\star}, \widetilde{u}^{\star\star})$, the last equality follows from the fact that $(0, \widetilde{u}^{\flat})$ also binds (C.3), and the last inequality follows from the fact that $\widetilde{u}^{\flat} < 0$. Subsequently, the ignored constraint $\widetilde{w}^{\star\star} + \frac{\alpha_l}{2\beta} > \widetilde{u}^{\star\star}$ is satisfied.

On the one hand, rearranging terms of (C.24) yields

$$2(1-\xi)\bar{\lambda}\widetilde{w}^{\star\star} = \xi\Delta\lambda\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star}\right) > 0, \tag{C.26}$$

where the inequality follows from the fact that $\widetilde{u}^{\star\star} < \frac{\bar{\lambda}\Delta\alpha}{2\beta}$; and rearranging terms of (C.25) yields

$$2(\bar{\lambda}^c - \xi\underline{\lambda}^c)\widetilde{u}^{\star\star} = -\xi\Delta\lambda\left(\widetilde{w}^{\star\star} + \frac{\alpha_l + \bar{\lambda}\Delta\alpha}{2\beta}\right) < 0, \tag{C.27}$$

where the last inequality follows from the fact that $\widetilde{w}^{\star\star} + \frac{\alpha_l}{2\beta} > \widetilde{u}^{\star\star} > -\frac{\bar{\lambda}\Delta\alpha}{2\beta}$. Therefore, we must have $\widetilde{w}^{\star\star} \neq 0$ and $\widetilde{u}^{\star\star} \neq 0$.

On the other hand, eliminating ξ from (C.24) and (C.25) yields

$$\frac{\bar{\lambda}\widetilde{w}^{\star\star}}{\bar{\lambda}^c\widetilde{u}^{\star\star}} = \frac{2\bar{\lambda}\widetilde{w}^{\star\star} + \Delta\lambda\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star}\right)}{2\bar{\lambda}^c\widetilde{u}^{\star\star} + \Delta\lambda\left(2\widetilde{u}^{\star\star} - \widetilde{w}^{\star\star} - \frac{\alpha_l + \bar{\lambda}\Delta\alpha}{2\beta}\right)}, \tag{C.28}$$

or equivalently,

$$\frac{\bar{\lambda}\widetilde{w}^{\star\star}}{\bar{\lambda}^c\widetilde{u}^{\star\star}} = -\frac{\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star}}{\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star} + \widetilde{w}^{\star\star} - \widetilde{u}^{\star\star} + \frac{\alpha_l}{2\beta}} \in (-1, 0), \tag{C.29}$$

where the bound follows from the fact that $\widetilde{u}^{\star\star} < \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and $\widetilde{w}^{\star\star} + \frac{\alpha_l}{2\beta} > \widetilde{u}^{\star\star}$.

Suggested by (C.26) and (C.27), we consider the following three possibilities:

1. If $\xi > 1 > \bar{\lambda}^c/\underline{\lambda}^c$, then (C.26) and (C.27) suggest that $\widetilde{w}^{\star\star} < 0$ and $\widetilde{u}^{\star\star} > 0$, respectively. However, this would imply, by the binding constraint (C.3), that

$$\begin{aligned}\bar{\lambda}\left(\widetilde{w}^{\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star}\right)^2 &= \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \Delta\lambda\left(\widetilde{w}^{\star\star} - \widetilde{u}^{\star\star} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star}\right) \\ &> \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \frac{\bar{\lambda}\Delta\lambda\alpha_l\Delta\alpha}{4\beta^2} = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2},\end{aligned}$$

contradicting the optimality of $(\widetilde{w}^{\star\star}, \widetilde{u}^{\star\star})$, because another feasible solution $(0, \widetilde{u}^{\flat})$ identified in the proof of Proposition C.2 yields an even lower value:

$$\bar{\lambda}0^2 + \bar{\lambda}^c\left(\widetilde{u}^{\flat}\right)^2 \leq \frac{\bar{\lambda}^c}{\underline{\lambda}^c}\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2} < \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{4\beta^2},$$

where the first inequality follows from (C.22). As such, this case can be ruled out.

2. If $1 > \xi > \bar{\lambda}^c/\underline{\lambda}^c$, then (C.26) and (C.27) suggest that $\widetilde{w}^{\star\star} > 0$ and $\widetilde{u}^{\star\star} > 0$, respectively. This, however, contradicts (C.29) and hence can be ruled out.

3. As such, we must have $1 > \bar{\lambda}^c/\underline{\lambda}^c > \xi$, which implies that $\widetilde{w}^{\star\star} > 0$ and $\widetilde{u}^{\star\star} < 0$ according to (C.26) and (C.27), respectively. Therefore, it follows immediately that $\overline{w}^{\star\star} > \overline{w}^\circ$ and $\bar{r}^{\star\star} = \alpha_l/(2\beta) + \widetilde{w}^{\star\star} - \widetilde{u}^{\star\star} > \alpha_l/(2\beta) = r^\circ$. It also follows from (C.29) that

$$\bar{\lambda}\widetilde{w}^{\star\star} + \bar{\lambda}^c\widetilde{u}^{\star\star} < 0 \tag{C.30}$$

and, together with (C.28), that

$$\left(\bar{\lambda} + \underline{\lambda}\right)\widetilde{w}^{\star\star} + \left(\bar{\lambda}^c + \underline{\lambda}^c\right)\widetilde{u}^{\star\star} - \frac{\alpha_l\Delta\lambda}{2\beta} < 0. \tag{C.31}$$

By Lemma 1 and the fact that $\widetilde{u}^{\star\star} = \overline{w}^{\star\star} - \bar{r}^{\star\star} - \frac{\bar{\lambda}\Delta\alpha}{2\beta}$, we have the unsold inventory in the case of low baseline demand realization to be

$$\bar{q}^{\star\star} = \frac{1}{2}\left[\Delta\alpha - \beta/\lambda\left(\overline{w}^{\star\star} - \bar{r}^{\star\star}\right)\right] = \frac{1}{2}\left[\Delta\alpha/2 - \beta/\lambda\widetilde{u}^{\star\star}\right] > \frac{\Delta\alpha}{4} = q^\circ,$$

where the inequality follows from the fact that $\widetilde{u}^{\star\star} < 0$.

Finally, we verify that the equilibrium can be sustained by the retailer's posterior equilibrium belief that the manufacturer is of high demand potential upon contract $(\overline{w}^{\star\star}, \bar{r}^{\star\star})$ being offered and is otherwise of low demand potential. To that end, we need to show that neither high- nor low-demand manufacturer has incentive to deviate to the off-equilibrium strategies under the specified belief.

- The low-demand-potential manufacturer's profit of deviating to $(\overline{w}^{\star\star}, \bar{r}^{\star\star})$ and hence being mistaken as of high demand potential is, by definition, dominated by her equilibrium profit according to the constraints of (15): $\Pi\left(\overline{w}^{\star\star}, \bar{r}^{\star\star} \mid \bar{\lambda}, \underline{\lambda}\right) \le \underline{\pi}^\circ$. Among all $(\underline{w}, \underline{r}) \neq (\overline{w}^{\star\star}, \bar{r}^{\star\star})$, under which the manufacturer is believed to be of low demand potential, the symmetric-information $(\underline{w}^\circ, r^\circ)$ maximizes her profit $\Pi(\underline{w}, \underline{r} \mid \underline{\lambda}, \underline{\lambda})$ to $\underline{\pi}^\circ$. Therefore, the low-demand manufacturer indeed has no incentive to deviate from her symmetric-information contract terms $(\underline{w}^\circ, r^\circ)$.

- For high-demand manufacturer, we need to show that she has no incentive to deviate to any $(\overline{w}, \bar{r}) \neq (\overline{w}^{\star\star}, \bar{r}^{\star\star})$ and hence to be mistaken as of low demand potential, namely

$$\Pi\left(\overline{w}, \bar{r} \mid \underline{\lambda}, \bar{\lambda}\right) \le \Pi\left(\overline{w}^{\star\star}, \bar{r}^{\star\star} \mid \bar{\lambda}, \bar{\lambda}\right). \tag{C.32}$$

For the rest of the proof, we are to establish (C.32) and hence concludes the verification of the equilibrium belief.

1. For $\overline{w} - \bar{r} \ge \underline{\lambda}\Delta\alpha/\beta$, (12) yields

$$\Pi\left(\overline{w}, \bar{r} \mid \underline{\lambda}, \bar{\lambda}\right) = \frac{1}{2}\overline{w}\left(\alpha_l + \underline{\lambda}\Delta\alpha - \beta\overline{w}\right) \le \frac{\left(\alpha_l + \underline{\lambda}\Delta\alpha\right)^2}{8\beta}.$$

On the other hand, the optimality of $(\overline{w}^{\star\star}, \bar{r}^{\star\star})$ suggests

$$\Pi\left(\overline{w}^{\star\star}, \bar{r}^{\star\star} \mid \bar{\lambda}, \bar{\lambda}\right) > \Pi\left(\overline{w}^\circ, \bar{r}^\flat \mid \bar{\lambda}, \bar{\lambda}\right) > \frac{1}{2}\overline{w}^\circ\left(\alpha_l + \bar{\lambda}\Delta\alpha - \beta\overline{w}^\circ\right) = \frac{\left(\alpha_l + \bar{\lambda}\Delta\alpha\right)^2}{8\beta} \tag{C.33}$$

from which and the previous inequality (C.32) then follows.

2. For $0 \le \overline{w} - \bar{r} \le \underline{\lambda}\Delta\alpha/\beta < \bar{\lambda}\Delta\alpha/\beta$, (12) implies

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) = \frac{1}{2}\overline{w}\left(\alpha_l + \underline{\lambda}\Delta\alpha - \beta\overline{w}\right) + \frac{\beta}{2\underline{\lambda}}\left(\underline{\lambda}^c\overline{w} - \bar{\lambda}^c\bar{r}\right)\left(\bar{r} - \overline{w} + \frac{\underline{\lambda}\Delta\alpha}{\beta}\right), \tag{C.34}$$

in which the second term, as a quadratic function of $\bar{r}$, achieves its unconstrained maximum at $\bar{r} = \frac{\bar{\lambda}^c + \underline{\lambda}^c}{2\bar{\lambda}^c}\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta} > \overline{w} - \underline{\lambda}\Delta\alpha/\beta$. Thus, we consider the following two cases.

(a) If $\frac{\bar{\lambda}^c + \underline{\lambda}^c}{2\bar{\lambda}^c}\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta} \ge \overline{w}$, or equivalently, $\overline{w} \ge \frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\Delta\lambda}$, the quadratic function of $\bar{r}$ in the second term of (C.34) is increasing in $\bar{r} \in [\overline{w} - \underline{\lambda}\Delta\alpha/\beta, \overline{w}]$ and thus

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) \le \Pi\left(\overline{w}, \overline{w} \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) = \frac{1}{2}\overline{w}\left(\alpha_l + \underline{\lambda}\Delta\alpha - \beta\overline{w}\right) + \frac{\Delta\lambda\Delta\alpha}{2}\overline{w} = \frac{1}{2}\overline{w}\left(\alpha_l + \bar{\lambda}\Delta\alpha - \beta\overline{w}\right) \le \frac{\left(\alpha_l + \bar{\lambda}\Delta\alpha\right)^2}{8\beta}.$$

Thus, (C.32) again follows from (C.33).

(b) If $\overline{w} - \underline{\lambda}\Delta\alpha/\beta < \frac{\bar{\lambda}^c + \underline{\lambda}^c}{2\bar{\lambda}^c}\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta} \le \overline{w}$, or equivalently, $0 \le \overline{w} \le \frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\Delta\lambda}$, we then have

$$\begin{aligned}\Pi\left(\overline{w}, \bar{r} \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) \le &\Pi\left(\overline{w}, \frac{\bar{\lambda}^c + \underline{\lambda}^c}{2\bar{\lambda}^c}\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta} \,\middle|\, \underline{\lambda}, \bar{\lambda}\right) \\ = &\frac{\beta}{2\underline{\lambda}\bar{\lambda}^c}\left\{\frac{(\underline{\lambda} + \bar{\lambda})^2 - 4\underline{\lambda}}{4}\overline{w}^2 + \frac{\underline{\lambda}\bar{\lambda}^c}{\beta}\left(\alpha_l + \frac{\underline{\lambda} + \bar{\lambda}}{2}\Delta\alpha\right)\overline{w} + \left(\frac{\underline{\lambda}\bar{\lambda}^c\Delta\alpha}{2\beta}\right)^2\right\}.\end{aligned} \tag{C.35}$$

Therefore, if $(\underline{\lambda} + \bar{\lambda})^2 \ge 4\underline{\lambda}$, the quadratic function (C.35) is convex and achieves its maximum at $\overline{w} = \frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\Delta\lambda}$, which falls back to the previous case and hence is proved.

If $(\underline{\lambda} + \bar{\lambda})^2 < 4\underline{\lambda}$, the quadratic function (C.35) is concave and achieves its unconstrained maximum at

$$\overline{w} = \frac{2\underline{\lambda}\bar{\lambda}^c}{\beta\left[4\underline{\lambda} - (\underline{\lambda} + \bar{\lambda})^2\right]}\left(\alpha_l + \frac{\underline{\lambda} + \bar{\lambda}}{2}\Delta\alpha\right) > 0, \tag{C.36}$$

which will be smaller than $\frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\Delta\lambda}$ if and only if

$$\frac{\alpha_l}{\Delta\alpha} + \frac{\underline{\lambda} + \bar{\lambda}}{2} < \frac{4\underline{\lambda} - (\underline{\lambda} + \bar{\lambda})^2}{2\Delta\lambda} \quad \Leftrightarrow \quad 0 \le \frac{\alpha_l}{\Delta\alpha} < \frac{2}{\Delta\lambda}\left(\underline{\lambda} - \frac{\bar{\lambda}(\underline{\lambda} + \bar{\lambda})}{2}\right). \tag{C.37}$$

Therefore, if (C.37) does not hold, the quadratic function (C.35) achieves its maximum at $\overline{w} = \frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\Delta\lambda}$, which again falls back to the previous case and hence is proved.

In the remaining proof, we will work under (C.37), which implies the maximum of the quadratic function (C.35) to be

$$\frac{\underline{\lambda}\bar{\lambda}^c}{2\beta}\left\{\frac{1}{4\underline{\lambda} - (\underline{\lambda} + \bar{\lambda})^2}\left(\alpha_l + \frac{\underline{\lambda} + \bar{\lambda}}{2}\Delta\alpha\right)^2 + \frac{(\Delta\alpha)^2}{4}\right\}.$$

By (C.2), to obtain (C.32), it suffices to show

$$\begin{aligned}\bar{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}\left(\widetilde{w}^{\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star}\right)^2\right] \ge &\frac{\underline{\lambda}\bar{\lambda}^c}{2\beta}\left\{\frac{1}{4\underline{\lambda} - (\underline{\lambda} + \bar{\lambda})^2}\left(\alpha_l + \frac{\underline{\lambda} + \bar{\lambda}}{2}\Delta\alpha\right)^2 + \frac{(\Delta\alpha)^2}{4}\right\} \\ \Leftrightarrow \quad \bar{\lambda}\left(\widetilde{w}^{\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star}\right)^2 \le &\frac{\bar{\lambda}}{4\beta^2}\left\{\bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 - \frac{\left[\left(2\underline{\lambda} - \bar{\lambda}(\bar{\lambda} + \underline{\lambda})\right)\Delta\alpha - \Delta\lambda\alpha_l\right]^2}{4\underline{\lambda} - (\underline{\lambda} + \bar{\lambda})^2}\right\},\end{aligned} \tag{C.38}$$

where the right-hand side of (C.38) is positive and, in fact, greater than

$$\frac{\bar{\lambda}}{4\beta^2}(\Delta\alpha)^2\left\{\bar{\lambda}^c\bar{\lambda} - \frac{\left[2\underline{\lambda} - \bar{\lambda}(\bar{\lambda} + \underline{\lambda})\right]^2}{4\underline{\lambda} - (\underline{\lambda} + \bar{\lambda})^2}\right\} = \frac{\bar{\lambda}(\Delta\alpha)^2}{4\beta^2\left[4\underline{\lambda} - (\underline{\lambda} + \bar{\lambda})^2\right]}\left[\frac{\Delta\lambda\bar{\lambda}^c}{2} + \frac{1}{2}\left(\underline{\lambda} - \frac{\bar{\lambda}(\underline{\lambda} + \bar{\lambda})}{2}\right)\right] > 0,$$

where the inequalities follow from (C.37).

By (C.23), to show (C.38), we just need to show that there exists $(\widetilde{w},\widetilde{u})$ satisfying (C.1) and (C.3) such that

$$\bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2=\frac{\bar{\lambda}}{4\beta^2}\left\{\bar{\lambda}^c\bar{\lambda}\left(\Delta\alpha\right)^2-\frac{\left(B\Delta\alpha-\Delta\lambda\alpha_l\right)^2}{A}\right\},\tag{C.39}$$

where we adopt the abbreviation $A:=4\underline{\lambda}-(\underline{\lambda}+\bar{\lambda})^2$ and $B:=2\underline{\lambda}-\bar{\lambda}(\bar{\lambda}+\underline{\lambda})$ for notational convenience.

To that end, we make the following change of variable

$$\widetilde{w}=\sqrt{\frac{1}{\bar{\lambda}}}\left(x\sqrt{\frac{1+\bar{\lambda}^{\frac{1}{2}}}{2}}-y\sqrt{\frac{1-\bar{\lambda}^{\frac{1}{2}}}{2}}\right),\quad \widetilde{u}=\sqrt{\frac{1}{\bar{\lambda}^c}}\left(x\sqrt{\frac{1-\bar{\lambda}^{\frac{1}{2}}}{2}}+y\sqrt{\frac{1+\bar{\lambda}^{\frac{1}{2}}}{2}}\right),\tag{C.40}$$

we then can straightforwardly verify that

$$\bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2=x^2+y^2,\tag{C.41}$$

and (C.3) can then be written as

$$\left(\bar{\lambda}^c+\underline{\lambda}^c-\frac{\Delta\lambda}{\bar{\lambda}^{\frac{1}{2}}}\right)x^2-\frac{\bar{\lambda}^c\Delta\lambda}{\beta}\frac{\alpha_l-\bar{\lambda}^{\frac{1}{2}}\Delta\alpha}{\sqrt{2(1+\bar{\lambda}^{\frac{1}{2}})}}x+\left(\bar{\lambda}^c+\underline{\lambda}^c+\frac{\Delta\lambda}{\bar{\lambda}^{\frac{1}{2}}}\right)y^2-\frac{\bar{\lambda}^c\Delta\lambda}{\beta}\frac{\alpha_l+\bar{\lambda}^{\frac{1}{2}}\Delta\alpha}{\sqrt{2(1-\bar{\lambda}^{\frac{1}{2}})}}y\geq\frac{\bar{\lambda}^c\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+\alpha_l\Delta\alpha\right]}{2\beta^2}.\tag{C.42}$$

Obviously, the $(\widetilde{w},\widetilde{u})$ defined through (C.40) by letting $x=0$ and $y=-\frac{\bar{\lambda}^{\frac{1}{2}}}{2\beta}\sqrt{\bar{\lambda}^c\bar{\lambda}\left(\Delta\alpha\right)^2-\frac{(B\Delta\alpha-\Delta\lambda\alpha_l)^2}{A}}$ satisfies (C.39) by virtue of (C.41). It is also straightforward to verify that such $(\widetilde{w},\widetilde{u})$ satisfies (C.1).

We now verify that it also satisfies (C.42), which implies that the corresponding $(\widetilde{w},\widetilde{u})$ must satisfy (C.3). To that end, plugging it to (C.42) renders it to

$$\underbrace{\frac{1}{AB}\left\{\left(\bar{\lambda}^{\frac{1}{2}}\Delta\lambda-B\right)\left(B-\Delta\lambda z\right)^2+A\bar{\lambda}^c\bar{\lambda}^{\frac{1}{2}}\left(B\bar{\lambda}^{\frac{1}{2}}-\Delta\lambda z^2\right)\right\}}_{\Psi_1(z)}+\underbrace{\sqrt{2\bar{\lambda}^c\left(1+\bar{\lambda}^{\frac{1}{2}}\right)}(z+\bar{\lambda}^{\frac{1}{2}})\sqrt{\bar{\lambda}^c\bar{\lambda}-\frac{\left(B-\Delta\lambda z\right)^2}{A}}}_{\Psi_2(z)}\geq 0,\tag{C.43}$$

where $z:=\alpha_l/\Delta\alpha<B/\Delta\lambda$ according to (C.37).

When $B\leq\bar{\lambda}^{\frac{1}{2}}\Delta\lambda$ (i.e., $4\underline{\lambda}^2\leq\bar{\lambda}(\bar{\lambda}+\underline{\lambda})^2$), we immediately have the first term on the left-hand side of (C.43) $\Psi_1(z)\geq 0$, and hence (C.43) holds.

When $B>\bar{\lambda}^{\frac{1}{2}}\Delta\lambda$ (i.e., $4\underline{\lambda}^2>\bar{\lambda}(\bar{\lambda}+\underline{\lambda})^2$), we recognize that

$$\Psi_2(z)\geq\Psi_3(z):=\sqrt{2\bar{\lambda}^c\left(1+\bar{\lambda}^{\frac{1}{2}}\right)}\left(z+\bar{\lambda}^{\frac{1}{2}}\right)\left(\left(\bar{\lambda}^c\bar{\lambda}\right)^{\frac{1}{2}}-\frac{B-\Delta\lambda z}{\sqrt{A}}\right),$$

which suggest that it suffices to show

$$\Psi_1(z)+\Psi_3(z)\geq 0,\ \text{ for } z\in\left[0,B/\Delta\lambda\right].\tag{C.44}$$

Direct calculation reveals that

$$\begin{aligned}\Psi_1(z)+\Psi_3(z)=&\underbrace{\bar{\lambda}^c\bar{\lambda}-\frac{B\left(B-\bar{\lambda}^{\frac{1}{2}}\Delta\lambda\right)}{A}+\sqrt{2\bar{\lambda}\bar{\lambda}^c\left(1+\bar{\lambda}^{\frac{1}{2}}\right)}\left(\left(\bar{\lambda}^c\bar{\lambda}\right)^{\frac{1}{2}}-\frac{B}{\sqrt{A}}\right)}_{\text{constant}}\\&+\underbrace{\left[\frac{2\Delta\lambda\left(B-\bar{\lambda}^{\frac{1}{2}}\Delta\lambda\right)}{A}+\sqrt{2\bar{\lambda}^c\left(1+\bar{\lambda}^{\frac{1}{2}}\right)}\left(\left(\bar{\lambda}^c\bar{\lambda}\right)^{\frac{1}{2}}-\frac{B-\bar{\lambda}^{\frac{1}{2}}\Delta\lambda}{\sqrt{A}}\right)\right]}_{>0}z\end{aligned}$$

$$+\Delta\lambda\left(\sqrt{\frac{2\bar{\lambda}^{c}\left(1+\bar{\lambda}^{\frac{1}{2}}\right)}{A}}-\frac{\bar{\lambda}^{c}\bar{\lambda}^{\frac{1}{2}}}{B}-\frac{\Delta\lambda\left(B-\bar{\lambda}^{\frac{1}{2}}\Delta\lambda\right)}{AB}\right)z^{2}.$$

If the coefficient of z^2 is nonnegative, then $\Psi_1(z)+\Psi_3(z)$ is increasing in z and hence (C.44) follows from

$$\Psi_1(0)+\Psi_3(0)>\Psi_1(0)=\frac{\Delta\lambda}{A}\left[\left(1+\bar{\lambda}^{\frac{1}{2}}\right)B+2\underline{\lambda}\bar{\lambda}^{c}\right]>0. \tag{C.45}$$

If the coefficient of z^2 is negative, then $\Psi_1(z)+\Psi_3(z)$ is concave in z, and hence (C.44) follows from (C.45) and

$$\Psi_1\left(B/\Delta\lambda\right)+\Psi_3\left(B/\Delta\lambda\right)=\bar{\lambda}^{c}\bar{\lambda}^{\frac{1}{2}}\left[\sqrt{2\left(1+\bar{\lambda}^{\frac{1}{2}}\right)}\left(B/\Delta\lambda+2\bar{\lambda}^{\frac{1}{2}}\right)-\left(B/\Delta\lambda-\bar{\lambda}^{\frac{1}{2}}\right)\right]>0.$$

This completes the proof. □

Proof of Proposition 4. First, $\overline{w}^{\star\star}>\overline{w}^{\circ}>\underline{w}^{\circ}>\overline{w}^{\sharp}$ follows from Proposition 3 and (C.5) in Proposition C.1.

By Proposition C.2, $\bar{r}^{\flat}>r^{\circ}$ follows from (C.15). To show that $\bar{r}^{\star\star}=\alpha_l/(2\beta)+\widetilde{w}^{\star\star}-\widetilde{u}^{\star\star}>\bar{r}^{\flat}=\alpha_l/(2\beta)-\widetilde{u}^{\flat}$, we now demonstrate that

$$\widetilde{w}^{\star\star}-\widetilde{u}^{\star\star}>-\widetilde{u}^{\flat}, \tag{C.46}$$

where $\widetilde{u}^{\flat}$ is identified in the proof of Proposition C.2. Since both $\left(0,\widetilde{u}^{\flat}\right)$ and $(\widetilde{w}^{\star\star},\widetilde{u}^{\star\star})$ bind the constraint (C.3), $\left(0,\widetilde{z}^{\flat}=-\widetilde{u}^{\flat}\right)$ and $(\widetilde{w}^{\star\star},\widetilde{z}^{\star\star}=\widetilde{w}^{\star\star}-\widetilde{u}^{\star\star})$ lie on the same quadratic curve in the $(\widetilde{w},\widetilde{z})$-space given by

$$\bar{\lambda}\widetilde{w}^{2}+\bar{\lambda}^{c}\left(\widetilde{w}-\widetilde{z}\right)^{2}+\Delta\lambda\left(\widetilde{z}+\frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}+\widetilde{z}-\widetilde{w}\right)=\text{constant},$$

or equivalently,

$$\widetilde{w}^{2}+\underline{\lambda}^{c}\widetilde{z}^{2}-(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{w}\widetilde{z}-\frac{\alpha_l\Delta\lambda}{2\beta}\widetilde{w}+\frac{(\alpha_l+\bar{\lambda}\Delta\alpha)\Delta\lambda}{2\beta}\widetilde{z}=\text{constant}. \tag{C.47}$$

Since $\widetilde{w}^{\star\star}>0$ and $\widetilde{u}^{\star\star},\widetilde{u}^{\flat}\in\left(-\bar{\lambda}\Delta\alpha/(2\beta),0\right)$, we just need to focus on the region $\Omega:=\left\{(\widetilde{w},\widetilde{z}):\widetilde{w}\geq 0\text{ and }\widetilde{z}\in\left[\widetilde{w},\widetilde{w}+\bar{\lambda}\Delta\alpha/(2\beta)\right]\right\}$.

Total differentiation of (C.47) yields

$$\frac{d\widetilde{z}}{d\widetilde{w}}=\frac{2\widetilde{w}-(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{z}-\frac{\alpha_l\Delta\lambda}{2\beta}}{(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{w}-2\underline{\lambda}^{c}\widetilde{z}-\frac{(\alpha_l+\bar{\lambda}\Delta\alpha)\Delta\lambda}{2\beta}}, \tag{C.48}$$

where we note that the denominator $(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{w}-2\underline{\lambda}^{c}\widetilde{z}-\frac{(\alpha_l+\bar{\lambda}\Delta\alpha)\Delta\lambda}{2\beta}=2\underline{\lambda}^{c}(\widetilde{w}-\widetilde{z})-\Delta\lambda\left(\widetilde{w}+\frac{\alpha_l+\bar{\lambda}\Delta\alpha}{2\beta}\right)<0$. Therefore, the region Ω is divided by the straight line $2\widetilde{w}-(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{z}-\frac{\alpha_l\Delta\lambda}{2\beta}=0$ into two segments: in the segment where $2\widetilde{w}-(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{z}-\frac{\alpha_l\Delta\lambda}{2\beta}<(>)0$, $\widetilde{z}$ is strictly increasing (decreasing) in $\widetilde{w}$.

Since

$$2\widetilde{w}^{\star\star}-(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{z}^{\star\star}-\frac{\alpha_l\Delta\lambda}{2\beta}=\left(\bar{\lambda}+\underline{\lambda}\right)\widetilde{w}^{\star\star}+\left(\bar{\lambda}^{c}+\underline{\lambda}^{c}\right)\widetilde{u}^{\star\star}-\frac{\alpha_l\Delta\lambda}{2\beta}<0\quad\text{by (C.31)},$$

and

$$2*0-(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{z}^{\flat}-\frac{\alpha_l\Delta\lambda}{2\beta}=(\bar{\lambda}^{c}+\underline{\lambda}^{c})\widetilde{u}^{\flat}-\frac{\alpha_l\Delta\lambda}{2\beta}<0,$$

as shown in the proof of Proposition C.2, both $\left(0,\widetilde{z}^{\flat}\right)$ and $(\widetilde{w}^{\star\star},\widetilde{z}^{\star\star})$ lie on the increasing branch of the quadratic curve in (C.47). Therefore, $\widetilde{w}^{\star\star}>0$ suggests $\widetilde{w}^{\star\star}-\widetilde{u}^{\star\star}=\widetilde{z}^{\star\star}>\widetilde{z}^{\flat}=-\widetilde{u}^{\flat}$, namely (C.46).

The profit rank $\overline{\pi}^\flat < \overline{\pi}^{\star\star} < \overline{\pi}^\circ$ simply follows from the fact that (3) is a relaxed problem of (15), which is in turn a relaxed problem of (19). To show that $\overline{\pi}^\flat > \overline{\pi}^\sharp$, we recognize from (C.2) that it is equivalent to show

$$\overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\bar{\lambda}^c\left(\widetilde{u}^\flat\right)^2 = \overline{\pi}^\flat > \overline{\pi}^\sharp = \overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left(\widetilde{w}^\sharp\right)^2,$$

or equivalently, because both $\widetilde{w}^\sharp < 0$ and $\widetilde{u}^\flat < 0$,

$$\widetilde{w}^\sharp < \sqrt{\bar{\lambda}^c}\widetilde{u}^\flat. \tag{C.49}$$

By (C.11) and (C.21), (C.49) is equivalent to

$$\begin{aligned}
&\sqrt{(\alpha_l\Delta\lambda)^2 + 4\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]} - \alpha_l\Delta\lambda > \sqrt{\bar{\lambda}^c}\left(\sqrt{\left(\frac{\Delta\lambda\overline{\alpha}}{\underline{\lambda}^c}\right)^2 + \frac{4\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]}{\underline{\lambda}^c}} - \frac{\Delta\lambda\overline{\alpha}}{\underline{\lambda}^c}\right)\\
\Leftrightarrow\quad &\sqrt{(\Delta\lambda\overline{\alpha})^2 + 4\underline{\lambda}^c\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]} + \Delta\lambda\overline{\alpha} > \sqrt{\bar{\lambda}^c(\alpha_l\Delta\lambda)^2 + 4\bar{\lambda}^c\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right]} + \sqrt{\bar{\lambda}^c}\alpha_l\Delta\lambda,
\end{aligned}$$

which obviously holds because $\bar{\alpha} > \alpha_l > \sqrt{\bar{\lambda}^c}\alpha_l$ and $\underline{\lambda}^c > \bar{\lambda}^c$. Therefore, we have shown (C.49) and hence $\overline{\pi}^\flat > \overline{\pi}^\sharp$.

To see that $\bar{s}^\sharp > \bar{s}^\flat$, we notice that it is equivalent to

$$s^\circ - \frac{\beta}{2\bar{\lambda}}\widetilde{w}^\sharp = \bar{s}^\sharp > \bar{s}^\flat = s^\circ - \frac{\beta}{2\bar{\lambda}}\bar{\lambda}^c\widetilde{u}^\flat \quad\Leftrightarrow\quad \widetilde{w}^\sharp < \bar{\lambda}^c\widetilde{u}^\flat,$$

which holds and follows immediately from (C.49).

To show $s^\circ < \bar{s}^{\star\star} < \bar{s}^\flat$, we first claim that $\widetilde{u}^{\star\star} > \widetilde{u}^\flat$. Since both $\left(0, \widetilde{u}^\flat\right)$ and $(\widetilde{w}^{\star\star}, \widetilde{u}^{\star\star})$ bind the constraint (C.3), we have

$$\begin{aligned}
\bar{\lambda}^c\left(\widetilde{u}^\flat\right)^2 + \Delta\lambda\left(\frac{\alpha_l}{2\beta} - \widetilde{u}^\flat\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^\flat\right) =& \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}\\
>& \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \left[\bar{\lambda}\left(\widetilde{w}^{\star\star}\right)^2 + \Delta\lambda\widetilde{w}^{\star\star}\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star}\right)\right]\\
=& \bar{\lambda}^c\left(\widetilde{u}^{\star\star}\right)^2 + \Delta\lambda\left(\frac{\alpha_l}{2\beta} - \widetilde{u}^{\star\star}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star}\right),
\end{aligned}$$

where the inequality follows from the fact that $\widetilde{w}^{\star\star} > 0 > \widetilde{u}^{\star\star}$. That is, the quadratic function

$$\bar{\lambda}^c\widetilde{u}^2 + \Delta\lambda\left(\frac{\alpha_l}{2\beta} - \widetilde{u}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right) - \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2},$$

has $\widetilde{u}^\flat < 0$ as its smaller root while takes negative value at $\widetilde{u}^{\star\star} < 0$, immediately suggesting that $\widetilde{u}^{\star\star} > \widetilde{u}^\flat$. We thus have

$$\bar{\lambda}\widetilde{w}^{\star\star} + \bar{\lambda}^c\widetilde{u}^{\star\star} = \bar{\lambda}\left(\widetilde{w}^{\star\star} - \widetilde{u}^{\star\star}\right) + \widetilde{u}^{\star\star} > -\bar{\lambda}\widetilde{u}^\flat + \widetilde{u}^\flat = \bar{\lambda}^c\widetilde{u}^\flat, \tag{C.50}$$

where we also used (C.46) to obtain the inequality.

To evaluate the retailer's stocking quantity as well as unsold inventory, we first note that $\widetilde{u}^{\star\star} < \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and hence $\overline{w}^{\star\star} - \overline{r}^{\star\star} = \widetilde{u}^{\star\star} + \frac{\bar{\lambda}\Delta\alpha}{2\beta} < \frac{\bar{\lambda}\Delta\alpha}{\beta}$, which, according to Lemma 1, suggests that all inventory is sold out in the case of high baseline demand. In particular, (1) suggests that the retailer orders

$$\begin{aligned}
\bar{s}^\flat =& s^R\left(\overline{w}^\circ, \overline{r}^\flat, 1, \bar{\lambda}\right) = \frac{\bar{\lambda}^c\beta\overline{r}^\flat + \bar{\lambda}\alpha_h - \beta\overline{w}^\circ}{2\bar{\lambda}} = \frac{\bar{\lambda}^c\beta r^\circ + \bar{\lambda}\alpha_h - \beta\overline{w}^\circ}{2\bar{\lambda}} - \frac{\beta}{2\bar{\lambda}}\bar{\lambda}^c\widetilde{u}^\flat,\\
\bar{s}^{\star\star} =& s^R\left(\overline{w}^{\star\star}, \overline{r}^{\star\star}, 1, \bar{\lambda}\right) = \frac{\bar{\lambda}^c\beta\overline{r}^{\star\star} + \bar{\lambda}\alpha_h - \beta\overline{w}^{\star\star}}{2\bar{\lambda}} = \underbrace{\frac{\bar{\lambda}^c\beta r^\circ + \bar{\lambda}\alpha_h - \beta\overline{w}^\circ}{2\bar{\lambda}}}_{s^\circ} - \frac{\beta}{2\bar{\lambda}}\left(\bar{\lambda}\widetilde{w}^{\star\star} + \bar{\lambda}^c\widetilde{u}^{\star\star}\right).
\end{aligned}$$

Therefore, $\bar{s}^{\star\star} > s^\circ$ follows from (C.30) and $\bar{s}^{\star\star} < \bar{s}^\flat$ from (C.50). ☐

Appendix D: Proofs in Section 6

The retailer's prior belief is that the manufacturer is of a low-risk high-demand type $(\bar{\theta}, \bar{\lambda})$ with probability $\upsilon \in (0,1)$ and is of a high-risk low-demand type $(\underline{\theta}, \underline{\lambda})$ with probability $\upsilon^c \in (0,1)$. Under a separating equilibrium, the retailer forms belief $\widehat{\upsilon} = \widehat{\upsilon}(w,r) := \mathbb{P}\left[(\theta,\lambda) = (\bar{\theta},\bar{\lambda}) \,\middle|\, (w,r)\right] \in \{0,1\}$ upon being offered contract (w,r). Accordingly, we denote

$$\widehat{\theta} = \widehat{\theta}(w,r) =: \mathbb{E}\left[\theta \mid (w,r)\right] = \widehat{\upsilon}\bar{\theta} + \widehat{\upsilon}^c\underline{\theta}, \quad \text{and} \quad \widehat{\lambda} = \widehat{\lambda}(w,r) =: \mathbb{E}\left[\lambda \mid (w,r)\right] = \widehat{\upsilon}\bar{\lambda} + \widehat{\upsilon}^c\underline{\lambda}.$$

Consequently, the retailer's ordering strategy is characterized by Lemma 1 and the manufacturer's reduced-form profit function is given by

$$\Pi\left(w, r \mid \widehat{\upsilon}, (\theta,\lambda)\right) = \frac{1}{2}w\left(\alpha_l + \widehat{\lambda}\Delta\alpha - \beta w\right) + \frac{1}{2}\left(\widehat{\lambda}^c w - \lambda^c\theta r\right)\left[\Delta\alpha - \beta/\widehat{\lambda}(w - \widehat{\theta}r)\right]^+. \tag{D.1}$$

Similar to Lemmas 2 and 3, we can specialize (D.3) $\widehat{\upsilon} \in \{0,1\}$ (and hence $\widehat{\lambda} = \lambda$ and $\widehat{\theta}_l = \widehat{\theta}_h = \theta$) to obtain the symmetric-information contract

$$w^\circ(\theta,\lambda) = \frac{\alpha_l + \lambda\Delta\alpha}{2\beta} \quad \text{and} \quad r^\circ(\theta,\lambda) = \frac{\alpha_l}{2\beta\theta}, \tag{D.2}$$

yielding (21) and (22).

Consequently, the retailer's ordering strategy is characterized by Lemma 1 and the manufacturer's reduced-form profit function is given by

$$\Pi\left(w, r \mid \widehat{\upsilon}, (\theta,\lambda)\right) = \frac{1}{2}w\left(\alpha_l + \widehat{\lambda}\Delta\alpha - \beta w\right) + \frac{1}{2}\left(\widehat{\lambda}^c w - \lambda^c\theta r\right)\left[\Delta\alpha - \beta/\widehat{\lambda}(w - \widehat{\theta}r)\right]^+. \tag{D.3}$$

As will be verified later (by identifying the supporting equilibrium belief), manufacturer of type $(\underline{\theta}, \underline{\lambda})$ offers the symmetric-information contract $(\underline{w}^\circ, \underline{r}^\circ)$ and hence earns $\underline{\pi}^\circ$, establishing the first statement of Proposition 5. Thus, manufacturer of type $(\bar{\theta}, \bar{\lambda})$ needs to distinguish herself from type $(\underline{\theta}, \underline{\lambda})$, and, in the most efficient separating equilibrium, offers the buyback contract according to

$$\begin{aligned} &\max_{\overline{w} \geq \bar{\theta}\overline{r} \geq 0} \quad \Pi\left(\overline{w}, \overline{r} \,\middle|\, 1, (\bar{\theta},\bar{\lambda})\right) \\ &\text{subject to} \quad \Pi\left(\overline{w}, \overline{r} \mid 1, (\underline{\theta},\underline{\lambda})\right) \leq \underline{\pi}^\circ \quad \text{and} \quad \Pi\left(\overline{w}, \overline{r} \,\middle|\, 1, (\bar{\theta},\bar{\lambda})\right) \geq \Pi\left(\underline{w}^\circ, \underline{r}^\circ \,\middle|\, 0, (\bar{\theta},\bar{\lambda})\right), \end{aligned} \tag{D.4}$$

where the two IC constraints are the non-mimicry condition for type $(\underline{\theta}, \underline{\lambda})$ and $(\bar{\theta}, \bar{\lambda})$, respectively.

Similar to Lemmas B.3 and C.3, we obtain the following (proof omitted)

Lemma D.1 **(Change of Variable)**. *Let* $\delta := \left(\underline{\theta}\underline{\lambda}^c - \bar{\theta}\bar{\lambda}^c\right)/\bar{\theta}$, $\widetilde{w} := \overline{w} - \frac{\bar{\alpha}}{2\beta}$ *and* $\widetilde{u} := \overline{w} - \bar{\theta}\overline{r} - \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ *(that is,* $\overline{w} = \overline{w}^\circ + \widetilde{w}$ *and* $\bar{r} = \bar{r}^\circ + (\widetilde{w} - \widetilde{u})/\bar{\theta}$*). Then,*

$$\overline{w} \geq \bar{\theta}\overline{r} \geq 0 \quad \Leftrightarrow \quad \widetilde{w} \geq \widetilde{u} - \frac{\alpha_l}{2\beta} \ \textit{and} \ -\frac{\bar{\lambda}\Delta\alpha}{2\beta} \leq \widetilde{u} \leq \frac{\bar{\lambda}\Delta\alpha}{2\beta}, \tag{D.5}$$

$$\Pi\left(\overline{w}, \overline{r} \,\middle|\, 1, (\bar{\theta},\bar{\lambda})\right) = \overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left(\bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c\widetilde{u}^2\right), \tag{D.6}$$

$$\Pi\left(\overline{w}, \overline{r} \mid 1, (\underline{\theta},\underline{\lambda})\right) \leq \underline{\pi}^\circ \quad \Leftrightarrow \quad \bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c\widetilde{u}^2 + \delta\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right) \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}, \tag{D.7}$$

$$\Pi\left(\overline{w}, \overline{r} \,\middle|\, 1, (\bar{\theta},\bar{\lambda})\right) \geq \Pi\left(\underline{w}^\circ, \underline{r}^\circ \,\middle|\, 0, (\bar{\theta},\bar{\lambda})\right) \quad \Leftrightarrow \quad \bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c\widetilde{u}^2 \leq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\frac{\bar{\theta}}{\underline{\theta}}\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2}. \tag{D.8}$$

As before, Lemma D.1 implies that the solution to (D.4) is given by $\overline{w}^{\star\star\star} = \overline{w}^{\circ} + \widetilde{w}^{\star\star\star}$ and $\bar{r}^{\star\star\star} = \bar{r}^{\circ} + \left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star}\right)/\bar{\theta}$, where $\left(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star}\right)$ is the solution to

$$\min_{\widetilde{w} \geq \widetilde{u} - \frac{\alpha_l}{2\beta}, -\frac{\bar{\lambda}\Delta\alpha}{2\beta} \leq \widetilde{u} \leq \frac{\bar{\lambda}\Delta\alpha}{2\beta}} \bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c \widetilde{u}^2, \quad \text{subject to (D.7)}, \tag{D.9}$$

provided that $\left(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star}\right)$ satisfies (D.8), i.e., the optimal objective value of (D.9) is bounded below by the left-hand side of (D.8). We note that if $\left(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star}\right)$ does not satisfy (D.8), then (D.4) is infeasible and hence the most efficient separating equilibrium does not exist.

Straightforward verification yields the following

Lemma D.2. *Parameter* $\delta := \left(\underline{\theta}\underline{\lambda}^c - \bar{\theta}\bar{\lambda}^c\right)/\bar{\theta}$ *satisfies the following properties: i)* $\delta + \bar{\lambda}^c = \underline{\lambda}^c \underline{\theta}/\bar{\theta} > 0$, *ii)* $\delta = \Delta\lambda - \underline{\lambda}^c \Delta\theta/\bar{\theta}$, *iii)* $\delta\bar{\theta}/\underline{\theta} = \Delta\lambda - \bar{\lambda}^c \Delta\theta/\underline{\theta}$, *iv)* $\delta \geq 0 \Leftrightarrow \Delta\theta/\bar{\theta} \leq \Delta\lambda/\underline{\lambda}^c$, *v)* $\frac{\bar{\theta}}{\underline{\theta}}\frac{\delta^2}{4\underline{\lambda}\bar{\lambda}^c} \geq 1 \Leftrightarrow \Delta\theta/\bar{\theta} \leq 1 - \left[\bar{\lambda}^c/\left(1 - \sqrt{\underline{\lambda}}\right)^2\right] \left(< \Delta\lambda/\underline{\lambda}^c\right)$ *or* $\Delta\theta/\bar{\theta} \geq 1 - \left[\bar{\lambda}^c/\left(1 + \sqrt{\underline{\lambda}}\right)^2\right] \left(> \Delta\lambda/\underline{\lambda}^c\right)$.

D.1. Case of $\delta < 0$ (i.e., $\underline{\theta\lambda}^c < \bar{\theta}\bar{\lambda}^c$)

Lemma D.3. *For* $\delta < 0$, *there exists a feasible solution* $(\widetilde{w}, \widetilde{u})$ *to* (D.9) *such that*

$$\bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c \widetilde{u}^2 \leq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l \Delta\alpha\right]}{4\beta^2} - \delta\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2}, \tag{D.10}$$

and hence the optimal solution to (D.9), $\left(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star}\right)$, *must satisfy* (D.8).

Proof. Since the objective of (D.9) is to minimize the left-hand side of (D.8), we can ignore constraint (D.8) once the feasibility of (D.9) is established. Below, we identify such a feasible solution satisfying (D.10), stronger than (D.8).

- If $\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right] - \delta\alpha_l\Delta\alpha \leq \bar{\lambda}(\Delta\alpha)^2$, then

$$\widetilde{w} = \widetilde{u} = \frac{1}{2\beta}\sqrt{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right] - \delta\bar{\lambda}\alpha_l\Delta\alpha} \in \left(0, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$$

binds (D.10) and also satisfies (D.7):

$$\bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c\widetilde{u}^2 + \delta\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right)$$

$$= \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} + \delta\left[\underbrace{\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)}_{=\frac{\alpha_l}{2\beta}}\underbrace{\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right)}_{\leq\frac{\bar{\lambda}\Delta\alpha}{2\beta}} - \frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2}\right] \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}.$$

- Otherwise, $\widetilde{u} = \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and

$$\widetilde{w} = \frac{1}{2\beta}\sqrt{\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right] - \delta\alpha_l\Delta\alpha - \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2} > \widetilde{u} = \frac{\bar{\lambda}\Delta\alpha}{2\beta}$$

bind (D.10) and also satisfy (D.7):

$$\bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c\widetilde{u}^2 + \delta\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right)$$

$$= \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} + \delta\left[\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\underbrace{\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right)}_{=0} - \frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2}\right] \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}. \quad \square$$

Lemma D.4. *For $\delta < 0$, (D.7) must be binding at the optimal solution to (D.9).*

Proof. Suppose $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ satisfy (D.7) with strict inequality. Since $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} = 0$ does not satisfy (D.7), we thus consider the following scenarios:

- If $\widetilde{u}^{\star\star\star} > 0$, there must exist $\epsilon > 0$ such that $\widetilde{u}^{\star\star\star} - \epsilon > 0$ and, by continuity of the left-hand side of (D.7) in $\widetilde{u}$,

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star} - \epsilon\right)^2 + \delta\left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} + \epsilon + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star} + \epsilon\right) \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}.$$

As $\widetilde{u}^{\star\star\star} - \epsilon < \widetilde{u}^{\star\star\star} \leq \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and $\widetilde{w}^{\star\star\star} \geq \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta} > \widetilde{u}^{\star\star\star} - \epsilon - \frac{\alpha_l}{2\beta}$, thus $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star} - \epsilon)$ is a feasible solution to (D.9) but contradicts the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ because $\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star} - \epsilon\right)^2 < \bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2$.

- If $\widetilde{u}^{\star\star\star} < 0$ and $\widetilde{w}^{\star\star\star} > \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$, then there exists $\epsilon > 0$ such that $\widetilde{u}^{\star\star\star} + \epsilon < 0$, $\widetilde{w}^{\star\star\star} \geq \widetilde{u}^{\star\star\star} + \epsilon - \frac{\alpha_l}{2\beta}$ and, by continuity of the left-hand side of (D.7) in $\widetilde{u}$,

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star} + \epsilon\right)^2 + \delta\left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} - \epsilon + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star} - \epsilon\right) \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}.$$

That is, $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star} + \epsilon)$ is a feasible solution to (D.9) but contradicts the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ because $\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star} + \epsilon\right)^2 < \bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2$.

- If $\widetilde{u}^{\star\star\star} < 0$ and $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta} < 0$, then there exists $\epsilon > 0$ such that $\widetilde{u}^{\star\star\star} + \epsilon < 0$, $\widetilde{w}^{\star\star\star} + \epsilon < 0$ and, by continuity of the left-hand side of (D.7) in $(\widetilde{w}, \widetilde{u})$,

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star} + \epsilon\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star} + \epsilon\right)^2 + \delta\left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star} - \epsilon\right) \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}.$$

That is, $(\widetilde{w}^{\star\star\star} + \epsilon, \widetilde{u}^{\star\star\star} + \epsilon)$ is a feasible solution to (D.9) but contradicts the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ because $\bar{\lambda}\left(\widetilde{w}^{\star\star\star} + \epsilon\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star} + \epsilon\right)^2 < \bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2$.

- If $\widetilde{w}^{\star\star\star} < 0$, using arguments similar to the case of $\widetilde{u}^{\star\star\star} > 0$, one can show that there exists $\epsilon > 0$ such that $(\widetilde{w}^{\star\star\star} + \epsilon, \widetilde{u}^{\star\star\star})$ is a feasible solution to (D.9) but contradicts the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$.
- If $\widetilde{w}^{\star\star\star} > 0$ and $\widetilde{w}^{\star\star\star} > \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$, using arguments similar to the case of $\widetilde{u}^{\star\star\star} < 0$ and $\widetilde{w}^{\star\star\star} > \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$, one can show that there exists $\epsilon > 0$ such that $(\widetilde{w}^{\star\star\star} - \epsilon, \widetilde{u}^{\star\star\star})$ is a feasible solution to (D.9) but contradicts the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$.
- If $\widetilde{w}^{\star\star\star} > 0$ and $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$, which implies that $\widetilde{u}^{\star\star\star} > 0$, then using arguments similar to the case of $\widetilde{u}^{\star\star\star} < 0$ and $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$, one can show that there exists $\epsilon > 0$ such that $(\widetilde{w}^{\star\star\star} - \epsilon, \widetilde{u}^{\star\star\star} - \epsilon)$ is a feasible solution to (D.9) but contradicts the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$. □

Lemma D.5. *For $\delta < 0$, the optimal solution to (D.9) must satisfy $\widetilde{u}^{\star\star\star} = \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ or $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta} < 0$ if and only if*

$$\Delta\lambda \geq \frac{\bar{\lambda}^c \min\left\{\alpha_l^2, \bar{\lambda}\left(\Delta\alpha\right)^2\right\}}{\Delta\alpha(\alpha_l + \alpha_h)}. \tag{D.11}$$

Proof. We first claim that $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ satisfies $\widetilde{u}^{\star\star\star} = \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ or $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$ if and only if

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2 = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}. \tag{D.12}$$

The necessity of (D.12) follows from the fact that $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ must bind (D.7) by Lemma D.4 and that $\left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right) = 0$. To see the sufficiency of (D.12), we note, by (D.7), that

$$\begin{aligned}\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2 \geq& \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right)\\ \geq& \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2},\end{aligned}$$

which immediately implies that $\widetilde{u}^{\star\star\star} = \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ or $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$.

It is straightforward to verify that $\widetilde{w}^{\star\star\star}$ exists such that $\widetilde{u}^{\star\star\star} = \frac{\bar{\lambda}\Delta\alpha}{2\beta}$, $\widetilde{w}^{\star\star\star} \geq \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta} = \frac{\bar{\lambda}\Delta\alpha - \alpha_l}{2\beta}$, and (D.12) hold, if and only if

$$\bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 + \left[(\bar{\lambda}\Delta\alpha - \alpha_l)^+\right]^2 \leq \Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right] = \Delta\lambda\Delta\alpha(\alpha_l + \alpha_h). \tag{D.13}$$

There exists $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ such that $\widetilde{u}^{\star\star\star} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$, $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$ and (D.12) hold, if and only if there exists $\widetilde{u} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$ such that $g(\widetilde{u}) = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$, where

$$g(\widetilde{u}) := \bar{\lambda}\left(\widetilde{u} - \frac{\alpha_l}{2\beta}\right)^2 + \bar{\lambda}^c\widetilde{u}^2 = \left(\widetilde{u} - \frac{\bar{\lambda}\alpha_l}{2\beta}\right)^2 + \bar{\lambda}\bar{\lambda}^c\frac{\alpha_l^2}{4\beta^2}. \tag{D.14}$$

It is straightforward to verify that $g\left(-\frac{\bar{\lambda}\Delta\alpha}{2\beta}\right) > \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$.

- If $\Delta\alpha \leq \alpha_l$, then $g(\widetilde{u})$ is monotonically decreasing in $\widetilde{u} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$. Thus, there exists $\widetilde{u} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$ such that $g(\widetilde{u}) = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$ if and only if $g\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}\right) \leq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$, which is equivalent to

$$\bar{\lambda}(\Delta\alpha)^2 - 2\bar{\lambda}\alpha_l\Delta\alpha + \alpha_l^2 \leq \Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]. \tag{D.15}$$

Since $\bar{\lambda}(\Delta\alpha)^2 - 2\bar{\lambda}\alpha_l\Delta\alpha + \alpha_l^2 \geq \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 = \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 + \left[(\bar{\lambda}\Delta\alpha - \alpha_l)^+\right]^2$, (D.15) also implies (D.13), which is equivalent to (D.11).

- If $\Delta\alpha > \alpha_l$, then $g(\widetilde{u})$ reaches its minimum at $\frac{\bar{\lambda}\alpha_l}{2\beta} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\alpha_l}{2\beta}\right] \subset \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$. Thus, there exists $\widetilde{u} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$ such that $g(\widetilde{u}) = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$ if and only if $g\left(\frac{\bar{\lambda}\alpha_l}{2\beta}\right) \leq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$, which is equivalent to

$$\bar{\lambda}^c\alpha_l^2 \leq \Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right] = \Delta\lambda\Delta\alpha(\alpha_l + \alpha_h). \tag{D.16}$$

It is straightforward to verify that

—if $\Delta\alpha > \alpha_l \geq \sqrt{\bar{\lambda}}\Delta\alpha$, then $\bar{\lambda}^c\alpha_l^2 \geq \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 = \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 + \left[(\bar{\lambda}\Delta\alpha - \alpha_l)^+\right]^2$ and hence (D.16) also implies (D.13), which is equivalent to (D.11);

—if $\sqrt{\bar{\lambda}}\Delta\alpha > \alpha_l \geq \bar{\lambda}\Delta\alpha$, then $\bar{\lambda}^c\alpha_l^2 \leq \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 = \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 + \left[(\bar{\lambda}\Delta\alpha - \alpha_l)^+\right]^2$ and hence (D.13) implies (D.16), which is equivalent to (D.11);

—if $\alpha_l < \bar{\lambda}\Delta\alpha$, then $\bar{\lambda}^c\alpha_l^2 \leq \bar{\lambda}(\Delta\alpha)^2 - 2\bar{\lambda}\alpha_l\Delta\alpha + \alpha_l^2 = \bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 + \left[(\bar{\lambda}\Delta\alpha - \alpha_l)^+\right]^2$ and hence (D.13) again implies (D.16), which is equivalent to (D.11). □

Lemma D.6. *For $\delta < 0$, if (D.11) does not hold, the optimal solution to (D.9) must satisfy $0 < \widetilde{u}^{\star\star\star} < \frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and $\widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta} < \widetilde{w}^{\star\star\star} < 0$.*

Proof. If (D.11) does not hold, Lemma D.5 implies that $\widetilde{u}^{\star\star\star} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)$ and $\widetilde{w}^{\star\star\star} > \widetilde{u}^{\star\star\star} - \frac{\alpha_l}{2\beta}$. Hence, the necessary condition for the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ is for there to exist a Lagrangian multiplier $\xi \geq 0$ associated with (D.7) such that

$$2\bar{\lambda}\widetilde{w}^{\star\star\star} - \xi\left[2\bar{\lambda}\widetilde{w}^{\star\star\star} + \delta\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right)\right] = 0, \tag{D.17}$$

$$2\bar{\lambda}^c\widetilde{u}^{\star\star\star} - \xi\left[2\bar{\lambda}^c\widetilde{u}^{\star\star\star} + \delta\left(2\widetilde{u}^{\star\star\star} - \widetilde{w}^{\star\star\star} - \frac{\alpha_l + \bar{\lambda}\Delta\alpha}{2\beta}\right)\right] \geq 0, \text{ with “=” if } \widetilde{u}^{\star\star\star} > -\frac{\bar{\lambda}\Delta\alpha}{2\beta}. \tag{D.18}$$

We first note that $\xi > 0$ and hence (D.7) must be binding. Otherwise, (D.17) and (D.18) suggest that $\widetilde{w}^{\star\star\star} = 0$ and $\widetilde{u}^{\star\star\star} \geq 0$ with “=” if $\widetilde{u}^{\star\star\star} > -\frac{\bar{\lambda}\Delta\alpha}{2\beta}$. Thus, we must have $\widetilde{w}^{\star\star\star} = \widetilde{u}^{\star\star\star} = 0$, which then violates (D.7).

Using the properties in Lemma D.2 and rearranging terms of (D.17) and (D.18) yields

$$2\bar{\lambda}(1-\xi)\widetilde{w}^{\star\star\star} = \xi\delta\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right) < 0, \tag{D.19}$$

$$2(\bar{\theta}\bar{\lambda}^c - \xi\underline{\theta\lambda}^c)/\bar{\theta}\widetilde{u}^{\star\star\star} \geq -\xi\delta\left(\widetilde{w}^{\star\star\star} + \frac{\alpha_l + \bar{\lambda}\Delta\alpha}{2\beta}\right) > 0, \tag{D.20}$$

where the strict inequality follows from the fact that $\widetilde{w}^{\star\star\star} + \frac{\alpha_l}{2\beta} > \widetilde{u}^{\star\star\star} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)$. Therefore, we must have $\widetilde{w}^{\star\star\star} \neq 0$ and $\widetilde{u}^{\star\star\star} \neq 0$. Together with the fact that $\delta < 0$ or equivalently $\bar{\theta}\bar{\lambda}^c > \underline{\theta\lambda}^c$, we consider the following three possibilities:

1. If $\xi > \bar{\theta}\bar{\lambda}^c/(\underline{\theta\lambda}^c) > 1$, then (D.19) and (D.20) suggest that $\widetilde{w}^{\star\star\star} > 0$ and $\widetilde{u}^{\star\star\star} < 0$, respectively. However, as (D.7) is binding, this would imply that

$$\begin{aligned}\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2 =& \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right)\\ >& \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2},\end{aligned}$$

contradicting the optimality of $(\widetilde{w}^{\star\star}, \widetilde{u}^{\star\star})$ by Lemma D.3. Hence, this case can be ruled out.

2. If $\bar{\theta}\bar{\lambda}^c/(\underline{\theta\lambda}^c) > \xi > 1$, (D.19) and (D.20) suggest that $\widetilde{w}^{\star\star\star} > 0$ and $\widetilde{u}^{\star\star\star} > 0$, respectively. However, this leads to contradiction, because eliminating ξ and δ from (D.17) and (D.18) would yield a contradiction

$$\bar{\lambda}^c\underbrace{\widetilde{u}^{\star\star\star}}_{>0}\underbrace{\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right)}_{>0} \leq \bar{\lambda}\underbrace{\widetilde{w}^{\star\star\star}}_{>0}\underbrace{\left(2\widetilde{u}^{\star\star\star} - \widetilde{w}^{\star\star\star} - \frac{\alpha_l + \bar{\lambda}\Delta\alpha}{2\beta}\right)}_{<0}.$$

Hence, this case again can be ruled out.

3. As such, we must have $\bar{\theta}\bar{\lambda}^c/(\underline{\theta\lambda}^c) > 1 > \xi$, which implies that $\widetilde{w}^{\star\star\star} < 0$ and $\widetilde{u}^{\star\star\star} > 0$ according to (D.19) and (D.20), respectively, establishing the lemma. □

Lemma D.7. *For $\delta < 0$, if* (D.11) *does not hold, then*

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2 < \frac{\bar{\lambda}^c\bar{\lambda}^2(\Delta\alpha)^2}{4\beta^2}. \tag{D.21}$$

Proof. If (D.11) does not hold, it is straight forward to verify that the quadratic function in $\widetilde{u}$

$$\bar{\lambda}^c\widetilde{u}^2 + \delta\left(\frac{\alpha_l}{2\beta} - \widetilde{u}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right) = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$$

has a root $\widetilde{u}^\flat \in \left(0, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)$. That is, $\left(0, \widetilde{u}^\flat\right)$ binds (D.7). Thus, the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ must imply that

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2 \leq \bar{\lambda}0^2 + \bar{\lambda}^c\left(\widetilde{u}^\flat\right)^2 < \frac{\bar{\lambda}^c\bar{\lambda}^2\left(\Delta\alpha\right)^2}{4\beta^2},$$

establishing (D.21). □

Proof of Proposition 5.1 and Corollary 1.1. For $\delta < 0$, Lemma D.3 implies that the optimal solution $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ to (D.9) must satisfy (D.8). Thus, the most efficient equilibrium, if exists, must be given by $\overline{w}^{\star\star\star} = \overline{w}^\circ + \widetilde{w}^{\star\star\star}$ and $\bar{r}^{\star\star\star} = \bar{r}^\circ + \left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star}\right)/\bar{\theta}$. By Lemma D.5 and D.6, if $\widetilde{u}^{\star\star\star} = \frac{\bar{\lambda}\Delta\alpha}{2\beta}$, then manufacturer $\left(\bar{\theta}, \bar{\lambda}\right)$ induces the retailer's unsold inventory to be

$$\frac{1}{2}\left[\Delta\alpha - \beta/\bar{\lambda}\left(\overline{w}^{\star\star\star} - \bar{\theta}\bar{r}^{\star\star\star}\right)\right]^+ = \frac{1}{2}\left[\Delta\alpha - \beta/\bar{\lambda}\left(\widetilde{u}^{\star\star\star} + \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)\right]^+ = 0;$$

otherwise, $\widetilde{w}^{\star\star\star} < 0$ and $-\frac{\alpha_l}{2\beta} \leq \widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} < 0$, which implies that $\overline{w}^{\star\star\star} = \overline{w}^\circ + \widetilde{w}^{\star\star\star} < \overline{w}^\circ$ and $\bar{r}^{\star\star\star} = \bar{r}^\circ + \left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star}\right)/\bar{\theta} \in [0, \bar{r}^\circ)$. We now show that if

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) \leq \Pi\left(\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star} \,\middle|\, 1, (\bar{\theta}, \bar{\lambda})\right) = \overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2\right], \tag{D.22}$$

then $(\overline{w}^{\star\star\star}, \overline{w}^{\star\star\star})$ can be sustained as a separating equilibrium by the retailer's posterior belief that the manufacturer is of type $\left(\bar{\theta}, \bar{\lambda}\right)$ upon contract $(\overline{w}^{\star\star\star}, \overline{w}^{\star\star\star})$ being offered and is otherwise of type $(\underline{\theta}, \underline{\lambda})$.

- The manufacturer $(\underline{\theta}, \underline{\lambda})$'s profit of deviating to $(\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star})$ and hence being mistaken as of type $\left(\bar{\theta}, \bar{\lambda}\right)$ is, by definition, dominated by her equilibrium profit according to the constraints of (D.4): $\Pi\left(\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star} \,|\, 1, (\underline{\theta}, \underline{\lambda})\right) \leq \underline{\pi}^\circ$. Among all $(\underline{w}, \underline{r}) \neq (\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star})$, under which the manufacturer is believed to be of type $(\underline{\theta}, \underline{\lambda})$, the symmetric-information contract $(\underline{w}^\circ, \underline{r}^\circ)$ maximizes her profit: $\Pi\left(\underline{w}, \underline{r} \,|\, 0, (\underline{\theta}, \underline{\lambda})\right) < \Pi\left(\underline{w}^\circ, \underline{r}^\circ \,|\, 0, (\underline{\theta}, \underline{\lambda})\right)$. Therefore, the manufacturer $(\underline{\theta}, \underline{\lambda})$ indeed has no incentive to deviate from her symmetric-information contract terms $(\underline{w}^\circ, \underline{r}^\circ)$.
- For manufacturer $\left(\bar{\theta}, \bar{\lambda}\right)$, we need to show that she has no incentive to deviate to any $(\overline{w}, \bar{r}) \neq (\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star})$ and hence to be mistaken as of type $(\underline{\theta}, \underline{\lambda})$, namely the condition (D.22).

Since condition $\Delta\theta/\bar{\theta} > \Delta\lambda/\underline{\lambda}^c \geq \bar{\lambda}^c\left[\alpha_l^2 \wedge \bar{\lambda}\left(\Delta\alpha\right)^2\right]/\left[\underline{\lambda}^c\Delta\alpha(\alpha_l + \alpha_h)\right]$ implies $\delta < 0$ and (D.11), Lemma D.5 implies (D.12) and hence

$$\Pi\left(\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star} \,\middle|\, 1, (\bar{\theta}, \bar{\lambda})\right) = \overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2\right] = \overline{\pi}^\circ - \frac{\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{8\beta} = \underline{\pi}^\circ.$$

On the other hand, (D.3) implies that (after some simple algebra)

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) = \Pi\left(\overline{w}, \bar{r} \,|\, 0, (\underline{\theta}, \underline{\lambda})\right) + \frac{1}{2}\delta\bar{\theta}\bar{r}\left[\Delta\alpha - \beta/\underline{\lambda}(\overline{w} - \underline{\theta}\bar{r})\right]^+ \leq \Pi\left(\overline{w}, \bar{r} \,|\, 0, (\underline{\theta}, \underline{\lambda})\right) \leq \underline{\pi}^\circ,$$

where we used the fact that $\delta < 0$. Therefore, (D.22) holds.

The rest of the proof is to establish (D.22) when $\Delta\theta/\bar{\theta} \geq 1 - \left[\bar{\lambda}^c/\left(1 + \sqrt{\underline{\lambda}}\right)^2\right]$ but (D.11) does not hold, and hence concludes the verification of the equilibrium belief.

1. For $\overline{w} - \underline{\theta}\bar{r} \geq \underline{\lambda}\Delta\alpha/\beta$, (D.3) yields

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) = \frac{1}{2}\overline{w}\left(\underbrace{\alpha_l + \underline{\lambda}\Delta\alpha}_{\underline{\alpha}} - \beta\overline{w}\right) \leq \frac{\underline{\alpha}^2}{8\beta} < \frac{\bar{\alpha}^2}{8\beta}.$$

On the other hand, Lemma D.7 implies that

$$\Pi\left(\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star} \,\middle|\, 1, (\bar{\theta}, \bar{\lambda})\right) = \overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2\right] > \frac{\bar{\lambda}^c\bar{\lambda}(\Delta\alpha)^2 + \bar{\alpha}^2}{8\beta} - \frac{\beta}{2\bar{\lambda}}\frac{\bar{\lambda}^c\bar{\lambda}^2\left(\Delta\alpha\right)^2}{4\beta^2} = \frac{\bar{\alpha}^2}{8\beta},$$

thus establishing (D.22).

2. For $0 \le \overline{w} - \underline{\theta}\bar{r} \le \underline{\lambda}\Delta\alpha/\beta$, (D.3) implies

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) = \frac{1}{2}\overline{w}\left(\alpha_l + \underline{\lambda}\Delta\alpha - \beta\overline{w}\right) + \frac{\beta}{2\underline{\lambda}}\left(\underline{\lambda}^c\overline{w} - \bar{\theta}\bar{\lambda}^c\bar{r}\right)\left(\underline{\theta}\bar{r} - \overline{w} + \frac{\underline{\lambda}\Delta\alpha}{\beta}\right), \tag{D.23}$$

in which the second term, as a quadratic function of $\bar{r} \in \left[1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right)^+, 1/\underline{\theta}\overline{w}\right]$, achieves its unconstrained maximum at

$$\bar{r} = 1/\underline{\theta}\left[\frac{\bar{\theta}\bar{\lambda}^c + \underline{\theta}\underline{\lambda}^c}{2\bar{\theta}\bar{\lambda}^c}\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] = 1/\underline{\theta}\left[\left(1 + \frac{\delta}{2\bar{\lambda}^c}\right)\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] < 1/\underline{\theta}\overline{w}.$$

Thus, we consider the following three cases.

(a) For $\overline{w} \ge -\frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\delta} \ge \underline{\lambda}\Delta\alpha/\beta \ge \left(1 + \frac{\delta}{2\bar{\lambda}^c}\right)^{-1}\frac{\underline{\lambda}\Delta\alpha}{2\beta}$ (by Lemma D.2), we have $0 \le 1/\underline{\theta}\left[\left(1 + \frac{\delta}{2\bar{\lambda}^c}\right)\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] \le 1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right)$ and hence the second term of (D.23) is decreasing in $\bar{r} \in \left[1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right), 1/\underline{\theta}\overline{w}\right]$, implying

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) \le \Pi\left(\overline{w}, 1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right) \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) = \frac{1}{2}\overline{w}\left[\alpha_l + \underline{\lambda}\Delta\alpha - \beta\overline{w}\right].$$

We thus fall back to Case 1, establishing (D.22).

(b) For $\overline{w} \le \left(1 + \frac{\delta}{2\bar{\lambda}^c}\right)^{-1}\frac{\underline{\lambda}\Delta\alpha}{2\beta} \le \underline{\lambda}\Delta\alpha/\beta$, we have $1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right) \le 1/\underline{\theta}\left[\left(1 + \frac{\delta}{2\bar{\lambda}^c}\right)\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] \le 0$ and hence the second term of (D.23) is decreasing in $\bar{r} \in [0, 1/\underline{\theta}\overline{w}]$, implying

$$\Pi\left(\overline{w}, \bar{r} \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) \le \Pi\left(\overline{w}, 0 \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) = -\frac{\beta}{2\underline{\lambda}}\overline{w}^2 + \frac{\alpha_h}{2}\overline{w} \le \frac{\underline{\lambda}\alpha_h^2}{8\beta} = \Pi\left(\frac{\underline{\lambda}\alpha_h}{2\beta}, 0 \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right). \tag{D.24}$$

- If $\underline{\lambda}\alpha_l \le \left(\bar{\lambda} + \Delta\lambda\right)\Delta\alpha$, then $(\overline{w}, \bar{r}) = \left(\frac{\underline{\lambda}\alpha_h}{2\beta}, 0\right)$ is a feasible solution to (D.4), because $\widetilde{w} = \frac{\underline{\lambda}\alpha_h}{2\beta} - \frac{\bar{\alpha}}{2\beta} = -\frac{\underline{\lambda}^c\alpha_l + \Delta\lambda\Delta\alpha}{2\beta}$ and $\widetilde{u} = \frac{\underline{\lambda}\alpha_h}{2\beta} - \frac{\bar{\lambda}\Delta\alpha}{2\beta} = \widetilde{w} + \frac{\alpha_l}{2\beta} = \frac{\underline{\lambda}\alpha_l - \Delta\lambda\Delta\alpha}{2\beta} \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, \frac{\bar{\lambda}\Delta\alpha}{2\beta}\right]$ satisfy (D.7):

$$\begin{aligned}
&\bar{\lambda}\left(-\frac{\underline{\lambda}^c\alpha_l + \Delta\lambda\Delta\alpha}{2\beta}\right)^2 + \bar{\lambda}^c\left(\frac{\underline{\lambda}\alpha_l - \Delta\lambda\Delta\alpha}{2\beta}\right)^2 \\
=&\frac{1}{4\beta^2}\left\{\underbrace{\left[\bar{\lambda}(\underline{\lambda}^c)^2 + \bar{\lambda}^c\underline{\lambda}^2\right]}_{\ge\bar{\lambda}\bar{\lambda}^c}\alpha_l^2 + (\Delta\lambda\Delta\alpha)^2 + 2\left(\bar{\lambda}\underline{\lambda}^c - \bar{\lambda}^c\underline{\lambda}\right)\Delta\lambda\alpha_l\Delta\alpha\right\} \ge \frac{\bar{\lambda}\bar{\lambda}^c}{4\beta^2}\alpha_l^2 > \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2},
\end{aligned}$$

where the last inequality follows from the fact that (D.11) does not hold. Thus, the optimality of $(\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star})$ implies

$$\Pi\left(\overline{w}^{\star\star\star}, \bar{r}^{\star\star\star} \,\middle|\, 1, (\bar{\theta}, \bar{\lambda})\right) \ge \Pi\left(\frac{\underline{\lambda}\alpha_h}{2\beta}, 0 \,\middle|\, 1, (\bar{\theta}, \bar{\lambda})\right). \tag{D.25}$$

Therefore, (D.22) follows from (D.24) and (D.25) by noting that

$$\Pi\left(\frac{\underline{\lambda}\alpha_h}{2\beta}, 0 \,\middle|\, 1, (\bar{\theta}, \bar{\lambda})\right) - \Pi\left(\frac{\underline{\lambda}\alpha_h}{2\beta}, 0 \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) = \frac{\beta}{2}\left(\frac{\underline{\lambda}^c}{\underline{\lambda}} - \frac{\bar{\lambda}^c}{\bar{\lambda}}\right)\left(\frac{\underline{\lambda}\alpha_h}{2\beta}\right)^2 > 0.$$

- If $\underline{\lambda}\alpha_l \ge \left(\bar{\lambda} + \Delta\lambda\right)\Delta\alpha$, to establish (D.22) it suffices to show

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2 \le \frac{2\bar{\lambda}}{\beta}\left[\overline{\pi}^\circ - \frac{\underline{\lambda}\alpha_h^2}{8\beta}\right] = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} + \frac{\bar{\lambda}\underline{\lambda}^c\alpha_l^2}{4\beta^2},$$

which holds by (D.21):

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2 < \frac{\bar{\lambda}^c\bar{\lambda}^2(\Delta\alpha)^2}{4\beta^2} \le \frac{\bar{\lambda}^c\bar{\lambda}^2}{4\beta^2}\left(\frac{\underline{\lambda}\alpha_l}{\bar{\lambda}}\right)^2 \le \frac{\bar{\lambda}\underline{\lambda}^c\alpha_l^2}{4\beta^2}.$$

(c) For $-\frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\delta} \geq \overline{w} \geq \left(1+\frac{\delta}{2\bar{\lambda}^c}\right)^{-1}\frac{\underline{\lambda}\Delta\alpha}{2\beta}$, we have $1/\underline{\theta}\left[\left(1+\frac{\delta}{2\bar{\lambda}^c}\right)\overline{w}-\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] \geq 1/\underline{\theta}\left(\overline{w}-\underline{\lambda}\Delta\alpha/\beta\right)^+$ and hence

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}\,\middle|\,0,(\bar{\theta},\bar{\lambda})\right) \leq& \Pi\left(\overline{w},1/\underline{\theta}\left[\left(1+\frac{\delta}{2\bar{\lambda}^c}\right)\overline{w}-\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right]\,\middle|\,0,(\bar{\theta},\bar{\lambda})\right)\\ =&\frac{1}{2}\overline{w}\left(\alpha_l+\underline{\lambda}\Delta\alpha-\beta\overline{w}\right)+\frac{\beta}{2}\frac{\bar{\theta}\bar{\lambda}^c}{\underline{\theta}\underline{\lambda}}\left[\frac{\delta}{2\bar{\lambda}^c}\overline{w}+\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right]^2\\ =&\frac{1}{2}\left\{\beta\left[\frac{\bar{\theta}}{\underline{\theta}}\frac{\delta^2}{4\underline{\lambda}\bar{\lambda}^c}-1\right]\overline{w}^2+\left[\alpha_l+\left(\underline{\lambda}+\frac{\bar{\theta}}{\underline{\theta}}\frac{\delta}{2}\right)\Delta\alpha\right]\overline{w}+\frac{\bar{\theta}\bar{\lambda}^c\underline{\lambda}\left(\Delta\alpha\right)^2}{4\beta\underline{\theta}}\right\}.\end{aligned}\tag{D.26}$$

By Lemma D.2, $\Delta\theta/\bar{\theta}\geq 1-\left[\bar{\lambda}^c/\left(1+\sqrt{\underline{\lambda}}\right)^2\right]$ implies that both $\delta<0$ and $\frac{\bar{\theta}}{\underline{\theta}}\frac{\delta^2}{4\underline{\lambda}\bar{\lambda}^c}\geq 1$. Hence, the quadratic function of $\overline{w}$ in (D.35) is convex and hence reaches its maximum at either $\overline{w}=-\frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\delta}$ or $\overline{w}=\left(1+\frac{\delta}{2\bar{\lambda}^c}\right)^{-1}\frac{\underline{\lambda}\Delta\alpha}{2\beta}$, corresponding to the above two cases respectively, for which (D.22) has been established. ☐

D.2. Case of $\delta>0$ (i.e., $\underline{\theta\lambda}^c>\bar{\theta}\bar{\lambda}^c$)

Lemma D.8. *For $\delta>0$, the optimal solution to (D.9) that satisfies (D.8) must satisfy $\widetilde{w}^{\star\star\star}>0$ and $\widetilde{u}^{\star\star\star}<0$.*

Proof. First, we claim that $\widetilde{u}^{\star\star\star}<\frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and $\widetilde{w}^{\star\star\star}>\widetilde{u}^{\star\star\star}-\frac{\alpha_l}{2\beta}$. Otherwise, $\widetilde{u}^{\star\star\star}=\frac{\bar{\lambda}\Delta\alpha}{2\beta}$ or $\widetilde{w}^{\star\star\star}=\widetilde{u}^{\star\star\star}-\frac{\alpha_l}{2\beta}$, so (D.7) and (D.8) imply

$$\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}\leq\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2+\bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2\leq\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}-\delta\frac{\bar{\theta}}{\underline{\theta}}\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2},$$

leading to a contradiction. Thus, the necessary condition for the optimality of $(\widetilde{w}^{\star\star\star},\widetilde{u}^{\star\star\star})$ is for there to exist a Lagrangian multiplier $\xi\geq 0$ associated with (D.7) such that

$$2\bar{\lambda}\widetilde{w}^{\star\star\star}-\xi\left[2\bar{\lambda}\widetilde{w}^{\star\star\star}+\delta\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}-\widetilde{u}^{\star\star\star}\right)\right]=0,\tag{D.27}$$

$$2\bar{\lambda}^c\widetilde{u}^{\star\star\star}-\xi\left[2\bar{\lambda}^c\widetilde{u}^{\star\star\star}+\delta\left(2\widetilde{u}^{\star\star\star}-\widetilde{w}^{\star\star\star}-\frac{\alpha_l+\bar{\lambda}\Delta\alpha}{2\beta}\right)\right]\geq 0,\ \text{with “=” if } \widetilde{u}^{\star\star\star}>-\frac{\bar{\lambda}\Delta\alpha}{2\beta}.\tag{D.28}$$

We first note that $\xi>0$ and hence (D.7) must be binding. Otherwise, (D.27) and (D.28) suggest that $\widetilde{w}^{\star\star\star}=0$ and $\widetilde{u}^{\star\star\star}\geq 0$ with “=” if $\widetilde{u}^{\star\star\star}>-\frac{\bar{\lambda}\Delta\alpha}{2\beta}$. Thus, we must have $\widetilde{w}^{\star\star\star}=\widetilde{u}^{\star\star\star}=0$, which then violates (D.7).

Using the properties in Lemma D.2 and rearranging terms of (D.27) and (D.28) yields

$$2\bar{\lambda}(1-\xi)\widetilde{w}^{\star\star\star}=\xi\delta\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}-\widetilde{u}^{\star\star\star}\right)>0,\tag{D.29}$$

$$2(\bar{\theta}\bar{\lambda}^c-\xi\underline{\theta\lambda}^c)/\bar{\theta}\widetilde{u}^{\star\star\star}\geq-\xi\delta\left(\widetilde{w}^{\star\star\star}+\frac{\alpha_l+\bar{\lambda}\Delta\alpha}{2\beta}\right)\in(-\infty,0),\ \text{with “=” if } \widetilde{u}^{\star\star\star}>-\frac{\bar{\lambda}\Delta\alpha}{2\beta},\tag{D.30}$$

where the strict inequality follows from the fact that $\widetilde{w}^{\star\star\star}+\frac{\alpha_l}{2\beta}>\widetilde{u}^{\star\star\star}\in\left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta},\frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)$.

- If $\widetilde{u}^{\star\star\star}=-\frac{\bar{\lambda}\Delta\alpha}{2\beta}<0$, then the binding (D.7) implies

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2+\delta\frac{\bar{\lambda}\Delta\alpha}{\beta}\widetilde{w}^{\star\star\star}=A:=\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}-\bar{\lambda}^c\frac{\left(\bar{\lambda}\Delta\alpha\right)^2}{4\beta^2}-\delta\frac{2\bar{\lambda}\bar{\alpha}\Delta\alpha}{4\beta^2}.\tag{D.31}$$

We claim that $A\geq 0$. Otherwise, it is straightforward to verify that the following quadratic equation

$$\bar{\lambda}^c\widetilde{u}^2+\delta\left(\frac{\alpha_l}{2\beta}-\widetilde{u}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta}-\widetilde{u}\right)=\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}$$

has a unique root $\widetilde{u}^\natural \in \left(-\frac{\bar\lambda\Delta\alpha}{2\beta}, 0\right)$, and hence $(0, \widetilde{u}^\natural)$ binds (D.7). However, this immediately contradicts the optimality of $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ because

$$\bar\lambda\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar\lambda^c\left(\widetilde{u}^{\star\star\star}\right)^2 \geq \bar\lambda^c\left(\frac{\bar\lambda\Delta\alpha}{2\beta}\right)^2 > \bar\lambda\left(0\right)^2 + \bar\lambda^c\left(\widetilde{u}^\natural\right)^2.$$

Therefore, (D.31) immediately implies that $\widetilde{w}^{\star\star\star} = \sqrt{\left(\frac{\delta\Delta\alpha}{2\beta}\right)^2 + A/\bar\lambda} - \frac{\delta\Delta\alpha}{2\beta} \geq 0$. On the other hand, (D.29) implies that $\widetilde{w}^{\star\star\star} \neq 0$. Hence, we must have $\widetilde{w}^{\star\star\star} > 0$, establishing the lemma.

- If $\widetilde{u}^{\star\star\star} > -\frac{\bar\lambda\Delta\alpha}{2\beta}$, then (D.28) and (D.30) are binding. Together with the fact that $\delta > 0$ or equivalently $\bar\theta\bar\lambda^c < \underline{\theta\lambda}^c$, we consider the following three possibilities:

1. If $\xi > 1 > \bar\theta\bar\lambda^c/(\underline{\theta\lambda}^c)$, then (D.29) and binding (D.30) suggest that $\widetilde{w}^{\star\star\star} < 0$ and $\widetilde{u}^{\star\star\star} > 0$, respectively. However, as (D.7) is binding, this would imply that

$$\begin{aligned}\bar\lambda\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar\lambda^c\left(\widetilde{u}^{\star\star\star}\right)^2 =& \frac{\bar\lambda\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar\lambda\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right)\\ >& \frac{\bar\lambda\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\frac{\bar\lambda\alpha_l\Delta\alpha}{4\beta^2} > \frac{\bar\lambda\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\frac{\bar\theta}{\underline{\theta}}\frac{\bar\lambda\alpha_l\Delta\alpha}{4\beta^2},\end{aligned}$$

violating (D.8). Hence, this case can be ruled out.

2. If $1 > \xi > \bar\theta\bar\lambda^c/(\underline{\theta\lambda}^c)$, (D.29) and binding (D.30) suggest that $\widetilde{w}^{\star\star\star} > 0$ and $\widetilde{u}^{\star\star\star} > 0$, respectively. However, this leads to contradiction, because eliminating ξ and δ from (D.27) and binding (D.28) would yield a contradiction

$$\bar\lambda^c\underbrace{\widetilde{u}^{\star\star\star}}_{>0}\underbrace{\left(\frac{\bar\lambda\Delta\alpha}{2\beta} - \widetilde{u}^{\star\star\star}\right)}_{>0} = \bar\lambda\underbrace{\widetilde{w}^{\star\star\star}}_{>0}\underbrace{\left(2\widetilde{u}^{\star\star\star} - \widetilde{w}^{\star\star\star} - \frac{\alpha_l + \bar\lambda\Delta\alpha}{2\beta}\right)}_{<0}.$$

Hence, this case again can be ruled out.

3. As such, we must have $1 > \bar\theta\bar\lambda^c/(\underline{\theta\lambda}^c) > \xi$, which implies that $\widetilde{w}^{\star\star\star} > 0$ and $\widetilde{u}^{\star\star\star} < 0$ according to (D.19) and binding (D.20), respectively, establishing the lemma. □

LEMMA D.9. *For $\delta > 0$, the optimal solution to (D.9) must satisfy (D.8) if $\Delta\theta/\underline{\theta} \leq 2\Delta\alpha/\alpha_l$.*

Proof. Since the objective of (D.9) is to minimize the left-hand side of (D.8), we thus just need to identify a feasible solution to (D.9) that satisfies (D.8).

- For $\Delta\theta/\underline{\theta} \leq \Delta\alpha/\alpha_l$, it is straightforward to verify that $\widetilde{u} = -\bar\lambda\frac{\Delta\theta}{\underline{\theta}}\frac{\alpha_l}{2\beta}$ and

$$\begin{aligned}\widetilde{w} =& \frac{1}{2\beta}\sqrt{\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right] - \delta\frac{\bar\theta}{\underline{\theta}}\alpha_l\Delta\alpha - \bar\lambda\bar\lambda^c\left(\frac{\Delta\theta}{\underline{\theta}}\alpha_l\right)^2}\\ \text{(by Lemma D.2)} \quad =& \frac{1}{2\beta}\sqrt{\left(\bar\lambda^c\frac{\Delta\theta}{\underline{\theta}}\alpha_l\right)^2 + \Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right] + \bar\lambda^c\frac{\Delta\theta}{\underline{\theta}}\alpha_l^2\left(\frac{\Delta\alpha}{\alpha_l} - \frac{\Delta\theta}{\underline{\theta}}\right)}\end{aligned}$$

satisfy $\widetilde{u} \in \left[-\frac{\bar\lambda\Delta\alpha}{2\beta}, 0\right]$, $\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta} \geq \bar\lambda^c\frac{\Delta\theta}{\underline{\theta}}\frac{\alpha_l}{2\beta} + \bar\lambda\frac{\Delta\theta}{\underline{\theta}}\frac{\alpha_l}{2\beta} + \frac{\alpha_l}{2\beta} = \frac{\bar\theta}{\underline{\theta}}\frac{\alpha_l}{2\beta}$, and bind (D.8). Subsequently, $(\widetilde{w}, \widetilde{u})$ also satisfies (D.7):

$$\begin{aligned}&\bar\lambda\widetilde{w}^2 + \bar\lambda^c\widetilde{u}^2 + \delta\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar\lambda\Delta\alpha}{2\beta} - \widetilde{u}\right)\\ =& \frac{\bar\lambda\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\frac{\bar\theta}{\underline{\theta}}\frac{\bar\lambda\alpha_l\Delta\alpha}{4\beta^2} + \delta\underbrace{\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)}_{\geq\frac{\bar\theta}{\underline{\theta}}\frac{\alpha_l}{2\beta}}\underbrace{\left(\frac{\bar\lambda\Delta\alpha}{2\beta} - \widetilde{u}\right)}_{\geq\frac{\bar\lambda\Delta\alpha}{2\beta}} \geq \quad \frac{\bar\lambda\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}.\end{aligned}$$

- For $\Delta\alpha/\alpha_l \leq \Delta\theta/\underline{\theta} \leq 2\Delta\alpha/\alpha_l$, it is straightforward to verify that $\widetilde{u} = -\frac{\bar{\lambda}\Delta\alpha}{2\beta}$ and

$$\begin{aligned} \widetilde{w} &= \frac{1}{2\beta}\sqrt{\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right] - \delta\frac{\bar{\theta}}{\underline{\theta}}\alpha_l\Delta\alpha - \bar{\lambda}\bar{\lambda}^c\left(\Delta\alpha\right)^2} \\ \text{(by Lemma D.2)} \quad &= \frac{1}{2\beta}\sqrt{\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right] + \bar{\lambda}^c\frac{\Delta\theta}{\underline{\theta}}\alpha_l\Delta\alpha - \bar{\lambda}\bar{\lambda}^c\left(\Delta\alpha\right)^2} \\ &\geq \frac{1}{2\beta}\sqrt{\Delta\lambda\left[(\Delta\alpha)^2 + \alpha_l\Delta\alpha\right] + \left(\bar{\lambda}^c\Delta\alpha\right)^2} \geq \frac{\bar{\lambda}^c\Delta\alpha}{2\beta} > 0 > \widetilde{u} - \frac{\alpha_l}{2\beta}, \end{aligned}$$

bind (D.8). Subsequently, $(\widetilde{w}, \widetilde{u})$ also satisfies (D.7):

$$\begin{aligned} &\bar{\lambda}\widetilde{w}^2 + \bar{\lambda}^c\widetilde{u}^2 + \delta\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right) \\ =&\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} - \delta\underbrace{\frac{\bar{\theta}}{\underline{\theta}}}_{\leq 1 + \frac{2\Delta\alpha}{\alpha_l}}\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2} + \delta\underbrace{\left(\widetilde{w} - \widetilde{u} + \frac{\alpha_l}{2\beta}\right)}_{\geq \frac{\Delta\alpha + \alpha_l}{2\beta}}\underbrace{\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right)}_{= \frac{2\bar{\lambda}\Delta\alpha}{2\beta}} \\ \geq&\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} + \delta\left[\frac{2\bar{\lambda}\Delta\alpha\left[\Delta\alpha + \alpha_l\right]}{4\beta^2} - \left(1 + \frac{2\Delta\alpha}{\alpha_l}\right)\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2}\right] \\ =&\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} + \delta\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2} > \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}. \quad \square \end{aligned}$$

Lemma D.10. *Suppose $\delta > 0$. There exists $\widetilde{u}^\diamond \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, 0\right]$ such that $(0, \widetilde{u}^\diamond)$ binds (D.7) if and only if*

$$\Delta\theta/\bar{\theta} \leq \left(\underline{\lambda}^c\bar{\lambda} - \bar{\lambda}^c\Delta\lambda\right)\Delta\alpha/\left(2\underline{\lambda}^c\bar{\alpha}\right). \tag{D.32}$$

Furthermore, (D.32) implies that $\Delta\theta/\underline{\theta} \leq 2\Delta\alpha/\alpha_l$ and hence that the optimal solution to (D.9) must satisfy (D.8).

Proof of Lemma D.10. Substituting $\widetilde{w} = 0$ into the left-hand side of (D.9) results in a quadratic convex function in $\widetilde{u}$,

$$\bar{\lambda}^c\widetilde{u}^2 + \delta\left(\frac{\alpha_l}{2\beta} - \widetilde{u}\right)\left(\frac{\bar{\lambda}\Delta\alpha}{2\beta} - \widetilde{u}\right),$$

which takes value of $\delta\frac{\bar{\lambda}\alpha_l\Delta\alpha}{4\beta^2} < \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2}$ (the right-hand side of (D.9)) at $\widetilde{u} = 0$ and value of $\bar{\lambda}^c\frac{\bar{\lambda}^2(\Delta\alpha)^2}{4\beta^2} + \delta\frac{2\bar{\lambda}\Delta\alpha\left(\alpha_l + \bar{\lambda}\Delta\alpha\right)}{4\beta^2}$ at $\widetilde{u} = -\frac{\bar{\lambda}\Delta\alpha}{2\beta}$. Therefore, there exists $\widetilde{u}^\diamond \in \left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta}, 0\right]$ such that $(0, \widetilde{u}^\diamond)$ binds (D.9) if and only if

$$\bar{\lambda}^c\frac{\bar{\lambda}^2(\Delta\alpha)^2}{4\beta^2} + \delta\frac{2\bar{\lambda}\Delta\alpha\left(\alpha_l + \bar{\lambda}\Delta\alpha\right)}{4\beta^2} \geq \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{4\beta^2} \text{ (the right-hand side of (D.9))},$$

which reduces to (D.32) through straightforward verification. Finally, by direct verification, (D.32) implies that $\Delta\theta/\bar{\theta} \leq 2\Delta\alpha/(2\Delta\alpha + \alpha_l)$, which is equivalent to $\Delta\theta/\underline{\theta} \leq 2\Delta\alpha/\alpha_l$, and hence the last statement in the lemma follows from Lemma D.9. $\square$

Proof of Proposition 5.2 and Corollary 1.2. For $\delta > 0$, Lemma D.8 implies that the most efficient separating equilibrium, if exists, must be given by $\overline{w}^{\star\star\star} = \overline{w}^\circ + \widetilde{w}^{\star\star\star} > \overline{w}^\circ$ and $\overline{r}^{\star\star\star} = \overline{r}^\circ + \left(\widetilde{w}^{\star\star\star} - \widetilde{u}^{\star\star\star}\right)/\bar{\theta} > \overline{r}^\circ$, where $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ is the optimal solution to (D.9) that satisfies (D.8). We now show that if

$$\Pi\left(\overline{w}, \overline{r} \,\middle|\, 0, (\bar{\theta}, \bar{\lambda})\right) \leq \Pi\left(\overline{w}^{\star\star\star}, \overline{r}^{\star\star\star} \,\middle|\, 1, (\bar{\theta}, \bar{\lambda})\right) = \overline{\pi}^\circ - \frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2\right], \tag{D.33}$$

then $(\overline{w}^{\star\star\star}, \overline{w}^{\star\star\star})$ can be sustained as a separating equilibrium by the retailer's posterior belief that the manufacturer is of type $\left(\bar{\theta}, \bar{\lambda}\right)$ upon contract $(\overline{w}^{\star\star\star}, \overline{w}^{\star\star\star})$ being offered and is otherwise of type $(\underline{\theta}, \underline{\lambda})$.

- The manufacturer $(\underline{\theta}, \underline{\lambda})$'s profit of deviating to $(\overline{w}^{\star\star\star}, \overline{r}^{\star\star\star})$ and hence being mistaken as of type $(\bar{\theta}, \bar{\lambda})$ is, by definition, dominated by her equilibrium profit according to the constraints of (D.4): $\Pi\left(\overline{w}^{\star\star\star}, \overline{r}^{\star\star\star} \mid 1, (\underline{\theta}, \underline{\lambda})\right) \leq \underline{\pi}^{\circ}$. Among all $(\underline{w}, \underline{r}) \neq (\overline{w}^{\star\star\star}, \overline{r}^{\star\star\star})$, under which the manufacturer is believed to be of type $(\underline{\theta}, \underline{\lambda})$, the symmetric-information contract $(\underline{w}^{\circ}, \underline{r}^{\circ})$ maximizes her profit: $\Pi\left(\underline{w}, \underline{r} \mid 0, (\underline{\theta}, \underline{\lambda})\right) < \Pi\left(\underline{w}^{\circ}, \underline{r}^{\circ} \mid 0, (\underline{\theta}, \underline{\lambda})\right)$. Therefore, the manufacturer $(\underline{\theta}, \underline{\lambda})$ indeed has no incentive to deviate from her symmetric-information contract terms $(\underline{w}^{\circ}, \underline{r}^{\circ})$.

- For manufacturer $(\bar{\theta}, \bar{\lambda})$, we need to show that she has no incentive to deviate to any $(\overline{w}, \overline{r}) \neq (\overline{w}^{\star\star\star}, \overline{r}^{\star\star\star})$ and hence to be mistaken as of type $(\underline{\theta}, \underline{\lambda})$, namely the condition (D.33). The rest of the proof is to establish (D.33) under condition $\Delta\theta/\bar{\theta} \leq \min\left\{\left(\underline{\lambda}^{c}\bar{\lambda} - \bar{\lambda}^{c}\Delta\lambda\right)\Delta\alpha/\left(2\underline{\lambda}^{c}\bar{\alpha}\right), 1 - \left[\bar{\lambda}^{c}/\left(1-\sqrt{\underline{\lambda}}\right)^{2}\right]\right\}$ and hence concludes the verification of the equilibrium belief.

1. For $\overline{w} - \underline{\theta}\overline{r} \geq \underline{\lambda}\Delta\alpha/\beta$, (D.3) yields

$$\Pi\left(\overline{w}, \overline{r} \mid 0, (\bar{\theta}, \bar{\lambda})\right) = \frac{1}{2}\overline{w}\left(\underbrace{\alpha_{l} + \underline{\lambda}\Delta\alpha}_{\underline{\alpha}} - \beta\overline{w}\right) \leq \frac{\underline{\alpha}^{2}}{8\beta} < \underline{\pi}^{\circ}.$$

Thus, to show (D.33), it suffices to show that

$$\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^{2} + \bar{\lambda}^{c}\left(\widetilde{u}^{\star\star\star}\right)^{2} \leq \frac{2\bar{\lambda}}{\beta}\left(\overline{\pi}^{\circ} - \underline{\pi}^{\circ}\right) = \frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^{2} + 2\alpha_{l}\Delta\alpha\right]}{4\beta^{2}},$$

which indeed holds as $(\widetilde{w}^{\star\star\star}, \widetilde{u}^{\star\star\star})$ must satisfy (D.8) (with $\delta > 0$).

2. For $0 \leq \overline{w} - \underline{\theta}\overline{r} \leq \underline{\lambda}\Delta\alpha/\beta$, (D.3) implies

$$\Pi\left(\overline{w}, \overline{r} \mid 0, (\bar{\theta}, \bar{\lambda})\right) = \frac{1}{2}\overline{w}\left(\alpha_{l} + \underline{\lambda}\Delta\alpha - \beta\overline{w}\right) + \frac{\beta}{2\underline{\lambda}}\left(\underline{\lambda}^{c}\overline{w} - \bar{\theta}\bar{\lambda}^{c}\overline{r}\right)\left(\underline{\theta}\overline{r} - \overline{w} + \frac{\underline{\lambda}\Delta\alpha}{\beta}\right), \tag{D.34}$$

in which the second term, as a quadratic function of $\overline{r} \in \left[1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right)^{+}, 1/\underline{\theta}\overline{w}\right]$, achieves its unconstrained maximum at

$$\overline{r} = 1/\underline{\theta}\left[\frac{\bar{\theta}\bar{\lambda}^{c} + \underline{\theta}\underline{\lambda}^{c}}{2\bar{\theta}\bar{\lambda}^{c}}\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] = 1/\underline{\theta}\left[\left(1 + \frac{\delta}{2\bar{\lambda}^{c}}\right)\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] > 1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right).$$

Thus, we consider the following three cases.

(a) For $\overline{w} \geq \frac{\bar{\lambda}^{c}\underline{\lambda}\Delta\alpha}{\beta\delta}$, we have $1/\underline{\theta}\left[\left(1 + \frac{\delta}{2\bar{\lambda}^{c}}\right)\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] \geq 1/\underline{\theta}\overline{w}$ and hence the second term of (D.34) is increasing in $\overline{r} \in \left[1/\underline{\theta}\left(\overline{w} - \underline{\lambda}\Delta\alpha/\beta\right)^{+}, 1/\underline{\theta}\overline{w}\right]$, implying

$$\Pi\left(\overline{w}, \overline{r} \mid 0, (\bar{\theta}, \bar{\lambda})\right) \leq \Pi\left(\overline{w}, 1/\underline{\theta}\overline{w} \mid 0, (\bar{\theta}, \bar{\lambda})\right) = \frac{1}{2}\overline{w}\left[\alpha_{l} + \left(\underline{\lambda} + \frac{\bar{\theta}}{\underline{\theta}}\delta\right)\Delta\alpha - \beta\overline{w}\right] \leq \frac{1}{8\beta}\left[\alpha_{l} + \left(\underline{\lambda} + \frac{\bar{\theta}}{\underline{\theta}}\delta\right)\Delta\alpha\right]^{2} \leq \frac{\bar{\alpha}^{2}}{8\beta}.$$

On the other hand, condition $\Delta\theta/\bar{\theta} \leq \min\left\{\left(\underline{\lambda}^{c}\bar{\lambda} - \bar{\lambda}^{c}\Delta\lambda\right)\Delta\alpha/\left(2\underline{\lambda}^{c}\bar{\alpha}\right), 1 - \left[\bar{\lambda}^{c}/\left(1-\sqrt{\underline{\lambda}}\right)^{2}\right]\right\}$ implies $\delta > 0$ (by Lemma D.2) and condition (D.32). Thus, Lemma D.10 implies that, by (D.6),

$$\Pi\left(\overline{w}^{\star\star\star}, \overline{r}^{\star\star\star} \mid 1, (\bar{\theta}, \bar{\lambda})\right) = \overline{\pi}^{\circ} - \frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}(\widetilde{w}^{\star\star\star})^{2} + \bar{\lambda}^{c}(\widetilde{u}^{\star\star\star})^{2}\right] \geq \overline{\pi}^{\circ} - \frac{\beta}{2\bar{\lambda}}\bar{\lambda}^{c}(\widetilde{u}^{\diamond})^{2} \geq \overline{\pi}^{\circ} - \frac{\bar{\lambda}^{c}\bar{\lambda}(\Delta\alpha)^{2}}{8\beta} = \frac{\bar{\alpha}^{2}}{8\beta},$$

immediately implying that (D.33) holds.

(b) For $0 \leq \overline{w} \leq \left(1 + \frac{\delta}{2\bar{\lambda}^{c}}\right)^{-1}\frac{\underline{\lambda}\Delta\alpha}{2\beta} < \frac{\bar{\lambda}^{c}\underline{\lambda}\Delta\alpha}{\beta\delta}$, we have $1/\underline{\theta}\left[\left(1 + \frac{\delta}{2\bar{\lambda}^{c}}\right)\overline{w} - \frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] \leq 0$ and hence the second term of (D.34) is decreasing in $\overline{r} \in [0, 1/\underline{\theta}\overline{w}]$, implying

$$\Pi\left(\overline{w}, \overline{r} \mid 0, (\bar{\theta}, \bar{\lambda})\right) \leq \Pi\left(\overline{w}, 0 \mid 0, (\bar{\theta}, \bar{\lambda})\right) == -\frac{\beta}{2\underline{\lambda}}\overline{w}^{2} + \frac{\alpha_{h}}{2}\overline{w} \leq \frac{\lambda\alpha_{h}^{2}}{8\beta} \leq \frac{\bar{\alpha}^{2}}{8\beta} \leq \Pi\left(\overline{w}^{\star\star\star}, \overline{r}^{\star\star\star} \mid 1, (\bar{\theta}, \bar{\lambda})\right),$$

where the last inequality follows from the same argument in part (a). Thus, (D.33) again holds.

(c) For $\left(1+\frac{\delta}{2\bar{\lambda}^c}\right)^{-1}\frac{\underline{\lambda}\Delta\alpha}{2\beta} \le \overline{w}\frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\delta}$, we have $1/\underline{\theta}\left[\left(1+\frac{\delta}{2\bar{\lambda}^c}\right)\overline{w}-\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] \le 1/\underline{\theta}\overline{w}$ and hence

$$
\begin{aligned}
\Pi\left(\overline{w},\overline{r} \,\middle|\, 0,(\bar{\theta},\bar{\lambda})\right) \le & \Pi\left(\overline{w}, 1/\underline{\theta}\left[\left(1+\frac{\delta}{2\bar{\lambda}^c}\right)\overline{w}-\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right] \,\middle|\, 0,(\bar{\theta},\bar{\lambda})\right)\\
= & \frac{1}{2}\overline{w}\left(\alpha_l+\underline{\lambda}\Delta\alpha-\beta\overline{w}\right)+\frac{\beta}{2}\frac{\bar{\theta}\bar{\lambda}^c}{\underline{\theta}\underline{\lambda}}\left[\frac{\delta}{2\bar{\lambda}^c}\overline{w}+\frac{\underline{\lambda}\Delta\alpha}{2\beta}\right]^2\\
= & \frac{1}{2}\left\{\beta\left[\frac{\bar{\theta}}{\underline{\theta}}\frac{\delta^2}{4\underline{\lambda}\bar{\lambda}^c}-1\right]\overline{w}^2+\left[\alpha_l+\left(\underline{\lambda}+\frac{\bar{\theta}}{\underline{\theta}}\frac{\delta}{2}\right)\Delta\alpha\right]\overline{w}+\frac{\bar{\theta}\bar{\lambda}^c\underline{\lambda}\left(\Delta\alpha\right)^2}{4\beta\underline{\theta}}\right\}.
\end{aligned}
\tag{D.35}
$$

Again by Lemma D.2, condition $\Delta\theta/\bar{\theta} \le \min\left\{\left(\underline{\lambda}^c\bar{\lambda}-\bar{\lambda}^c\Delta\lambda\right)\Delta\alpha/\left(2\underline{\lambda}^c\bar{\alpha}\right), 1-\left[\bar{\lambda}^c/\left(1-\sqrt{\underline{\lambda}}\right)^2\right]\right\}$ implies that $\frac{\bar{\theta}}{\underline{\theta}}\frac{\delta^2}{4\underline{\lambda}\bar{\lambda}^c} \ge 1$. Hence, the quadratic function of $\overline{w}$ in (D.35) is convex and hence reaches its maximum at either $\overline{w}=\frac{\bar{\lambda}^c\underline{\lambda}\Delta\alpha}{\beta\delta}$ or $\overline{w}=\left(1+\frac{\delta}{2\bar{\lambda}^c}\right)^{-1}\frac{\underline{\lambda}\Delta\alpha}{2\beta}$, corresponding to the above two cases respectively, for which (D.33) has been established. ☐

D.3. Case of $\delta=0$ (i.e., $\underline{\theta\lambda}^c=\bar{\theta}\bar{\lambda}^c$)

Lemma D.11. *There exist at least two solutions to*

$$
\bar{\lambda}\widetilde{w}^2+\bar{\lambda}^c\widetilde{u}^2=\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2},
\tag{D.36}
$$

both satisfying (D.5), one of which satisfies $\widetilde{w}<0$ and $\widetilde{w}-\widetilde{u}<0$ and the other of which satisfies $\widetilde{w}>0$ and $\widetilde{w}-\widetilde{u}>0$.

Proof. It is straightforward to see that function $g(\widetilde{u}) := \left[\bar{\lambda}\left(1+\frac{\alpha_l}{\bar{\lambda}\Delta\alpha}\right)^2+\bar{\lambda}^c\right]\widetilde{u}^2$ is monotonically decreasing in $\widetilde{u}\in\left[-\frac{\bar{\lambda}\Delta\alpha}{2\beta},0\right]$ with

$$
g\left(-\frac{\bar{\lambda}\Delta\alpha}{2\beta}\right)=\frac{\bar{\lambda}^2\left(\Delta\alpha\right)^2+2\bar{\lambda}^2\alpha_l\Delta\alpha+\bar{\lambda}\alpha_l^2}{4\beta^2}>\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}, \quad \text{and} \quad g(0)=0<\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}.
$$

Thus, the Intermediate Value Theorem implies that there is a unique $\widetilde{u}\in\left(-\frac{\bar{\lambda}\Delta\alpha}{2\beta},0\right)$ such that $g(\widetilde{u})=\frac{\bar{\lambda}\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}{4\beta^2}$. Let $\widetilde{w}=\left(1+\frac{\alpha_l}{\bar{\lambda}\Delta\alpha}\right)\widetilde{u}<\widetilde{u}$. Then, $(\widetilde{w},\widetilde{u})$ specified as such is a solution to (D.36) with $\widetilde{w}<0$ and $\widetilde{w}-\widetilde{u}<0$.

To show the existence of the other solution to (D.36), we simply let $\widetilde{w}=\frac{1}{2\beta}\sqrt{\Delta\lambda\left[(\Delta\alpha)^2+2\alpha_l\Delta\alpha\right]}$ and $\widetilde{u}=0$, which automatically satisfy $\widetilde{w}>0$ and $\widetilde{w}-\widetilde{u}>0$. ☐

Proof of Proposition 5.3 and Corollary 1.3. When $\delta=0$, it is straightforward to see that the optimal solution $(\widetilde{w}^{\star\star\star},\widetilde{u}^{\star\star\star})$ to (D.9) is the solution to (D.36) that satisfies (D.5). By Lemma D.11, at least two of such solutions exist, one with $\widetilde{w}^{\star\star\star}<0$ and $\widetilde{w}^{\star\star\star}-\widetilde{u}^{\star\star\star}<0$ and the other one with $\widetilde{w}^{\star\star\star}>0$ and $\widetilde{w}^{\star\star\star}-\widetilde{u}^{\star\star\star}>0$. Correspondingly, there exist two solutions to (D.4), $\overline{w}^{\star\star\star}=\overline{w}^{\circ}+\widetilde{w}^{\star\star\star}$ and $\overline{r}^{\star\star\star}=\overline{r}^{\circ}+\left(\widetilde{w}^{\star\star\star}-\widetilde{u}^{\star\star\star}\right)/\bar{\theta}$, which satisfies the property described in Proposition 5.1. We now show that $(\overline{w}^{\star\star\star},\overline{w}^{\star\star\star})$ can be sustained as a separating equilibrium by the retailer's posterior belief that the manufacturer is of type $\left(\bar{\theta},\bar{\lambda}\right)$ upon contract $(\overline{w}^{\star\star\star},\overline{w}^{\star\star\star})$ being offered and is otherwise of type $(\underline{\theta},\underline{\lambda})$.

- The manufacturer $(\underline{\theta},\underline{\lambda})$'s profit of deviating to $(\overline{w}^{\star\star\star},\overline{r}^{\star\star\star})$ and hence being mistaken as of type $\left(\bar{\theta},\bar{\lambda}\right)$ is, by definition, dominated by her equilibrium profit according to the constraints of (D.4): $\Pi\left(\overline{w}^{\star\star\star},\overline{r}^{\star\star\star} \mid 1,(\underline{\theta},\underline{\lambda})\right)\le\underline{\pi}^{\circ}$. Among all $(\underline{w},\underline{r})\ne(\overline{w}^{\star\star\star},\overline{r}^{\star\star\star})$, under which the manufacturer is believed to be of type $(\underline{\theta},\underline{\lambda})$, the symmetric-information contract $(\underline{w}^{\circ},\underline{r}^{\circ})$ maximizes her profit: $\Pi\left(\underline{w},\underline{r} \mid 0,(\underline{\theta},\underline{\lambda})\right)<\Pi\left(\underline{w}^{\circ},\underline{r}^{\circ} \mid 0,(\underline{\theta},\underline{\lambda})\right)$. Therefore, the manufacturer $(\underline{\theta},\underline{\lambda})$ indeed has no incentive to deviate from her symmetric-information contract terms $(\underline{w}^{\circ},\underline{r}^{\circ})$.

- For manufacturer $\left(\bar{\theta},\bar{\lambda}\right)$, we need to show that she has no incentive to deviate to any $(\overline{w},\overline{r}) \neq (\overline{w}^{\star\star\star},\overline{r}^{\star\star\star})$ and hence to be mistaken as of type $(\underline{\theta},\underline{\lambda})$, namely

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}\,\middle|\,0,(\bar{\theta},\bar{\lambda})\right) \leq \Pi\left(\overline{w}^{\star\star\star},\overline{r}^{\star\star\star}\,\middle|\,1,(\bar{\theta},\bar{\lambda})\right) =&\overline{\pi}^{\circ} - \frac{\beta}{2\bar{\lambda}}\left[\bar{\lambda}\left(\widetilde{w}^{\star\star\star}\right)^2 + \bar{\lambda}^c\left(\widetilde{u}^{\star\star\star}\right)^2\right]\\ =&\overline{\pi}^{\circ} - \frac{\Delta\lambda\left[(\Delta\alpha)^2 + 2\alpha_l\Delta\alpha\right]}{8\beta} = \underline{\pi}^{\circ},\end{aligned}$$

which holds, because $\delta = \left(\underline{\theta\lambda}^c - \bar{\theta}\bar{\lambda}^c\right)/\bar{\theta} = 0$ and (D.3) imply that

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}\,\middle|\,0,(\bar{\theta},\bar{\lambda})\right) =&\frac{1}{2}w\left(\alpha_l + \underline{\lambda}\Delta\alpha - \beta w\right) + \frac{1}{2}\left(\underline{\lambda}^c w - \bar{\lambda}^c\bar{\theta}r\right)\left[\Delta\alpha - \beta/\underline{\lambda}(w - \underline{\theta}r)\right]^+\\ =&\frac{1}{2}w\left(\alpha_l + \underline{\lambda}\Delta\alpha - \beta w\right) + \frac{1}{2}\left(\underline{\lambda}^c w - \underline{\lambda}^c\underline{\theta}r\right)\left[\Delta\alpha - \beta/\underline{\lambda}(w - \underline{\theta}r)\right]^+\\ =&\Pi\left(\overline{w},\overline{r}\mid(\underline{\theta},\underline{\lambda}),0\right) \leq \underline{\pi}^{\circ}. \quad \square\end{aligned}$$

Appendix E: Positive Marginal Production Cost

In this appendix, we explore the situation with a positive manufacturer's marginal production cost, denoted as $c \geq 0$. We find that a positive marginal cost does not qualitatively impact our insights established in the paper. That is, under asymmetric information about either returns risk or demand potential, signaling requires the separating type (i.e., the less risky or high-demand manufacturer) to suitably distort her returns cost away from the symmetric-information level (via the induced retailer's regular and safety stocks). A positive marginal production only acts to reduce the induced symmetric-information regular and safety stocks, but it does not affect the direction of distortions *relative to the symmetric-information benchmark* (as it does not enter the retailer's quantity decision). As the only nuance, under asymmetric information about returns risk, if the marginal cost is sufficiently high (i.e., $\lambda\Delta\alpha \leq \beta c < \alpha$), then it is no longer profitable for either less risky or riskier manufacturer to induce returns, i.e., to induce the retailer to carry a safety stock and return unsold inventory. As a result, the manufacturer's returns risk is not relevant to the retailer and there is no need for the manufacturer to signal her returns risk.

E.1. Returns Risk.

Given a marginal production cost $c \geq 0$, the manufacturer's expected profit function (2) in the paper needs to be modified as

$$\begin{aligned}\Pi\left(w,r\,\middle|\,\widehat{\theta},\theta\right) :=&(w-c)s^R\left(w,r,\widehat{\theta},\lambda\right) - \frac{1}{2}\lambda^c\theta r\left[\Delta\alpha - \beta/\lambda\left(w - \widehat{\theta}r\right)\right]^+\\ =&\frac{1}{2}(w-c)(\alpha - \beta w) + \frac{\lambda^c}{2}(w - \theta r - c)\left[\Delta\alpha - \beta/\lambda\left(w - \widehat{\theta}r\right)\right]^+.\end{aligned}$$

It is straightforward to verify that the symmetric-information contract is given by

$$w^{\circ}(\theta) \equiv \frac{\alpha + \beta c}{2\beta} \quad \text{and} \quad r^{\circ}(\theta)\begin{cases}\in\left[0, \frac{\alpha_l + \beta c - \lambda\Delta\alpha}{2\beta\theta}\right], & \text{if} \quad \lambda\Delta\alpha \leq \beta c \leq \alpha,\\ = \frac{\alpha_l}{2\beta\theta}, & \text{if} \quad \beta c \leq \lambda\Delta\alpha,\end{cases}$$

where we recall $\alpha = \alpha_l + \lambda\Delta\alpha$ is the average baseline demand. Under the symmetric-information contract, the retailer's induced regular and safety stock are

$$s_r^{\circ} = \frac{(\alpha - \beta c)^+}{4} \quad \text{and} \quad s_s^{\circ} = \frac{\lambda^c}{4}\left(\Delta\alpha - \beta c/\lambda\right)^+, \quad \text{respectively;}$$

and the manufacturer's expected profit is given by

$$\pi^\circ = \frac{\left[(\alpha-\beta c)^+\right]^2}{8\beta} + \frac{\lambda^c\left[(\lambda\Delta\alpha-\beta c)^+\right]^2}{8\beta\lambda} = \frac{2}{\beta\lambda^c}\left\{\lambda^c\left(s_r^\circ\right)^2 + \lambda\left(s_s^\circ\right)^2\right\}.$$

In words, a positive marginal production cost acts to shift the symmetric-information wholesale price upward and the induced regular and safety stocks downward, relative to the case of zero marginal production cost in the paper (see Lemma 2). More specifically, regardless of the manufacturer's type θ,

- for $0 \le \beta c < \lambda\Delta\alpha < \alpha$, the retailer still orders both positive regular and safety stocks (and incurs unsold inventory of amount s_s°/λ^c in case of low baseline demand realization) and the manufacturer earns positive profit;
- for $\lambda\Delta\alpha \le \beta c < \alpha$, the retailer only orders positive regular stock but *no* safety stock (and hence no returns regardless of baseline demand realization) and the manufacturer earns positive profit (from selling regular stock);
- for $\beta c \ge \alpha$, the retailer orders no regular nor safety stocks (and hence no returns regardless of baseline demand realization) and the manufacturer earns *no* profit (i.e., exits the market). Thus, it is meaningful to only focus on the parameter range $\beta c \in [0,\alpha]$.

When returns risk θ becomes the manufacturer's private information, the riskier manufacturer still offers her symmetric-information contract and earns her symmetric-information profit π° and the less risky manufacturer distinguishes herself by offering a contract, say $(\overline{w},\overline{r})$, which may need to be distorted away from her symmetric-information counterpart. Using the same variable transformation as in the paper, we can work with the retailer's induced quantity decision:

$$\begin{aligned}\text{regular stock}\quad & s_r(\overline{w}) := \frac{1}{2}\left(\alpha-\beta\overline{w}\right), \quad \text{and}\\ \text{safety stock}\quad & s_s(\overline{w},\overline{r}) := \frac{\lambda^c}{2}\left[\Delta\alpha - \beta/\lambda\left(\overline{w}-\bar\theta\overline{r}\right)\right].\end{aligned}$$

Notably, the retailer's quantity decision above is independent of the marginal production cost c.

Consequently, the less risky manufacturer's profit from offering contract $(\overline{w},\overline{r})$ can be expressed as

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\overline{\theta},\overline{\theta}\right) = \pi^\circ - \underbrace{\frac{2}{\beta\lambda^c}\left\{\lambda^c\left[s_r^\circ - s_r(\overline{w})\right]^2 + \lambda\left[s_s^\circ - s_s(\overline{w},\overline{r})\right]^2\right\}}_{\text{less risky manufacturer's signaling cost}}, \tag{E.1}$$

and the riskier manufacturer's gain from mimicry can be expressed as

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}\,\middle|\,\overline{\theta},\underline{\theta}\right) - \pi^\circ = & \lambda^c\Delta\theta\cdot\underbrace{\frac{2}{\beta(\lambda^c)^2\bar\theta}\left[\frac{\lambda^c\alpha_l}{2} - \lambda^c s_r(\overline{w}) + \lambda\bar{s}_s(\overline{w},\overline{r})\right]\bar{s}_s(\overline{w},\overline{r})}_{\text{returns cost}}\\ & - \underbrace{\frac{2}{\beta\lambda^c}\left\{\lambda^c\left[s_r^\circ - s_r(\overline{w})\right]^2 + \lambda\left[s_s^\circ - \bar{s}_s(\overline{w},\overline{r})\right]^2\right\}}_{\text{signaling cost}}.\end{aligned} \tag{E.2}$$

We note that (E.1) and (E.2) are exactly the same as (7) and (8) in the paper, respectively. The only difference is that both s_r° and s_s° are lower than their counterparts in (7) and (8). We also note that any $(\overline{w},\overline{r})$ such that $\bar{s}_s(\overline{w},\overline{r}) = 0$ can always make the riskier manufacturer's gain from mimicry in (E.2) non-positive. Thus, separation is always feasible.

- If $s_s^\circ = 0$ (i.e., $\lambda\Delta\alpha \le \beta c < \alpha$), cheap separation is achievable (i.e., the symmetric-information contract is automatically separating and the less risky manufacturer's signaling cost is zero), because $s_r(\overline{w}) = s_r^\circ$ and $\bar{s}_s(\overline{w}, \overline{r}) = s_s^\circ = 0$ makes (E.2) equal to zero.
- Otherwise (i.e., $\beta c < \lambda\Delta\alpha$ and $s_s^\circ > 0$), as in the returns risk case in the paper (see Section 4), the efficient separation must entail *upward* distortion of the regular stock $s_r(\overline{w})$ and *downward* distortion of the safety stock $\bar{s}_s(\overline{w}, \overline{r})$, which translate to *downward* distortion of both the wholesale and returns prices.

E.2. Demand Potential.

Given a marginal production cost $c \ge 0$, the manufacturer's expected profit function (12) in the paper needs to be modified as

$$\begin{aligned}\Pi\left(w, r \,\middle|\, \widehat{\lambda}, \lambda\right) :=& (w-c)s^\star\left(w, r, 1, \widehat{\lambda}\right) - \frac{1}{2}\lambda^c r\left[\Delta\alpha - \beta/\widehat{\lambda}\,(w-r)\right]^+ \\ =& \frac{1}{2}(w-c)(\widehat{\alpha} - \beta w) + \frac{1}{2}\left[\widehat{\lambda}^c(w-c) - \lambda^c r\right]\left[\Delta\alpha - \beta/\widehat{\lambda}\,(w-r)\right]^+,\end{aligned}$$

where $\widehat{\alpha} := \alpha_l + \widehat{\lambda}\Delta\alpha$.

Accordingly, the asymmetric-information contract is given by

$$w^\circ(\lambda) \equiv \frac{\alpha + \beta c}{2\beta} \quad \text{and} \quad r^\circ(\lambda)\begin{cases} \in \left[0, \frac{\alpha_l + \beta c - \lambda\Delta\alpha}{2\beta}\right], & \text{if} \quad \beta c \ge \lambda\Delta\alpha, \\ = \frac{\alpha_l}{2\beta}, & \text{if} \quad \beta c \le \lambda\Delta\alpha, \end{cases}$$

where $\alpha := \alpha_l + \lambda\Delta\alpha$. The corresponding retailer's regular and safety stocks under symmetric information are given by

$$s_r^\circ(\lambda) = \frac{(\alpha - \beta c)^+}{4} \quad \text{and} \quad s_s^\circ(\lambda) = \frac{\lambda^c}{4}\left(\Delta\alpha - \beta c/\lambda\right)^+, \quad \text{respectively;}$$

and the manufacturer's profit is given by

$$\Pi^\circ(\lambda) = \frac{\left[(\alpha - \beta c)^+\right]^2}{8\beta} + \frac{\lambda^c\left[(\lambda\Delta\alpha - \beta c)^+\right]^2}{8\beta\lambda} = \frac{2}{\beta\lambda^c}\left\{\lambda^c\left[s_r^\circ(\lambda)\right]^2 + \lambda\left[s_s^\circ(\lambda)\right]^2\right\}.$$

Again, the effect of a positive marginal production cost is only to shift the symmetric-information wholesale price upward and the induced regular and safety stocks downward, relative to the case of zero marginal production cost in the paper (see Lemma 3). In particular, when $\beta c \ge \lambda\Delta\alpha$, the retailer's safety stock becomes zero (and hence no returns regardless of baseline demand realization).

For subsequent notational convenience, we denote

$$\begin{aligned}\bar{s}_r^\circ =& s_r^\circ(\bar{\lambda}), \quad \bar{s}_s^\circ = s_s^\circ(\bar{\lambda}), \quad \bar{\pi}^\circ = \Pi^\circ\left(\bar{\lambda}\right); \quad \text{and} \\ \underline{s}_r^\circ =& s_r^\circ(\underline{\lambda}), \quad \underline{s}_s^\circ = s_s^\circ(\underline{\lambda}), \quad \underline{\pi}^\circ = \Pi^\circ\left(\underline{\lambda}\right).\end{aligned}$$

When demand potential λ becomes the manufacturer's private information, the low-demand manufacturer still offers her symmetric-information contract and earns her symmetric-information profit $\underline{\pi}^\circ$ and the high-demand manufacturer distinguishes herself by offering a contract, say $(\overline{w}, \overline{r})$, which may need to be distorted away from her symmetric-information counterpart. Using the same change of variables as in the paper, we will work with the retailer's induced quantity decision:

$$\text{regular stock} \quad \bar{s}_r(\overline{w}) := \frac{1}{2}\left(\bar{\alpha} - \beta\overline{w}\right), \quad \text{and}$$

$$\text{safety stock} \quad \bar{s}_s(\overline{w},\overline{r}) := \frac{\bar{\lambda}^c}{2}\left[\Delta\alpha - \beta/\bar{\lambda}\,(\overline{w}-\overline{r})\right].$$

Again, we note that the retailer's quantity decision above is independent of the marginal production cost c.

Subsequently, the high-demand manufacturer's profit from offering contract $(\overline{w},\overline{r})$ can be expressed as

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\bar{\lambda}\right) = \bar{\pi}^\circ - \underbrace{\frac{2}{\beta\bar{\lambda}^c}\left\{\bar{\lambda}^c\left[\bar{s}_r^\circ - \bar{s}_r(\overline{w})\right]^2 + \bar{\lambda}\left[\bar{s}_s^\circ - \bar{s}_s(\overline{w},\overline{r})\right]^2\right\}}_{\text{high-demand manufacturer's signaling cost}}, \tag{E.3}$$

and the low-demand manufacturer's gain from mimicry can be similarly expressed as

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\underline{\lambda}\right) - \underline{\pi}^\circ =& \bar{\pi}^\circ - \underline{\pi}^\circ - \underbrace{\frac{2}{\beta\bar{\lambda}^c}\left\{\bar{\lambda}^c\left[\bar{s}_r^\circ - \bar{s}_r(\overline{w})\right]^2 + \bar{\lambda}\left[\bar{s}_s^\circ - \bar{s}_s(\overline{w},\overline{r})\right]^2\right\}}_{\text{signaling cost}} \\ & - \Delta\lambda\cdot\underbrace{\frac{2}{\beta(\bar{\lambda}^c)^2}\left[\frac{\bar{\lambda}^c\alpha_l}{2} - \bar{\lambda}^c\bar{s}_r(\overline{w}) + \bar{\lambda}\bar{s}_s(\overline{w},\overline{r})\right]\bar{s}_s(\overline{w},\overline{r})}_{\text{returns cost under contract }(\overline{w},\overline{r})}.\end{aligned} \tag{E.4}$$

We note that (E.3) and (E.4) are exactly the same as (16) and (17) in the paper, respectively. The only difference is that both s_r° and s_s° are lower than their counterparts in (16) and (17). We claim that the low-demand manufacturer's gain from mimicry must be non-positive for some $\bar{s}_r(\overline{w}) \in [0, \bar{s}_r^\circ]$ and $\bar{s}_s(\overline{w},\overline{r}) \in [\bar{s}_s^\circ, \frac{\bar{\lambda}^c}{2}\Delta\alpha]$. Namely, separation is always feasible. Indeed, substituting $\bar{s}_r(\overline{w}) = 0$ and $\bar{s}_s(\overline{w},\overline{r}) = \frac{\bar{\lambda}^c}{2}\Delta\alpha$ into (E.4) yields

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\underline{\lambda}\right) - \underline{\pi}^\circ =& \bar{\pi}^\circ - \underline{\pi}^\circ - \frac{2}{\beta\bar{\lambda}^c}\left\{\bar{\lambda}^c(\bar{s}_r^\circ)^2 + \bar{\lambda}\left[\bar{s}_s^\circ - \frac{\bar{\lambda}^c}{2}\Delta\alpha\right]^2 + \frac{\bar{\lambda}^c}{4}\Delta\lambda\Delta\alpha\bar{\alpha}\right\} \\ =& -\underline{\pi}^\circ - \frac{2}{\beta\bar{\lambda}^c}\left\{\bar{\lambda}\left[\bar{s}_s^\circ - \frac{\bar{\lambda}^c}{2}\Delta\alpha\right]^2 - \bar{\lambda}\left[\bar{s}_s^\circ\right]^2 + \frac{\bar{\lambda}^c}{4}\Delta\lambda\Delta\alpha\bar{\alpha}\right\} \le 0,\end{aligned}$$

where we note that (i) if $\bar{s}_s^\circ = 0$, then $\left[\bar{s}_s^\circ - \frac{\bar{\lambda}^c}{2}\Delta\alpha\right]^2 - [\bar{s}_s^\circ]^2 \ge 0$, and (ii) if $\bar{s}_s^\circ > 0$, then

$$\left[\bar{s}_s^\circ - \frac{\bar{\lambda}^c}{2}\Delta\alpha\right]^2 - [\bar{s}_s^\circ]^2 = \left[\frac{\lambda^c}{4}(\Delta\alpha + \beta c/\lambda)\right]^2 - \left[\frac{\lambda^c}{4}(\Delta\alpha - \beta c/\lambda)\right]^2 \ge 0.$$

Therefore, as in the demand potential case in the paper (see Section 5), the efficient separation must entail *downward* distortion of the regular stock $\bar{s}_r(\overline{w})$ and *upward* distortion of the safety stock $\bar{s}_s(\overline{w},\overline{r})$, which translate to *upward* distortion of both the wholesale and returns prices. In particular, as verified below, cheap separation is not achievable.

- If $\bar{s}_s^\circ > 0$ and $\underline{s}_s^\circ > 0$ (i.e., $\beta c < \underline{\lambda}\Delta\alpha < \bar{\lambda}\Delta\alpha$), substituting $\bar{s}_r(\overline{w}) = \bar{s}_r^\circ$ and $\bar{s}_s(\overline{w},\overline{r}) = \bar{s}_s^\circ$ into (E.4) yields positive mimicry gain for the low-demand manufacturer:

$$\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\underline{\lambda}\right) - \underline{\pi}^\circ = \frac{\Delta\lambda}{8\beta}\left[(\Delta\alpha)^2 - \frac{(\beta c)^2}{\bar{\lambda}\underline{\lambda}} + \alpha_l\Delta\alpha + \frac{\beta c}{\bar{\lambda}}\alpha_l\right] > 0.$$

- If $\bar{s}_s^\circ > 0$, $\underline{s}_s^\circ = 0$ and $\underline{s}_r^\circ > 0$ (i.e., $\underline{\lambda}\Delta\alpha \le \beta c < \min\{\underline{\alpha}, \bar{\lambda}\Delta\alpha\}$), substituting $\bar{s}_r(\overline{w}) = \bar{s}_r^\circ$ and $\bar{s}_s(\overline{w},\overline{r}) = \bar{s}_s^\circ$ into (E.4) yields positive mimicry gain for the low-demand manufacturer:

$$\begin{aligned}\Pi\left(\overline{w},\overline{r}\,\middle|\,\bar{\lambda},\underline{\lambda}\right) - \underline{\pi}^\circ =& \frac{1}{8\beta}\left[(\bar{\alpha}-\beta c)^2 + (\bar{\lambda}^c/\bar{\lambda})(\bar{\lambda}\Delta\alpha - \beta c)^2 - (\underline{\alpha}-\beta c)^2 - (\Delta\lambda/\bar{\lambda})\alpha_l(\bar{\lambda}\Delta\alpha - \beta c)\right] \\ =& \frac{1}{8\beta}\left[(1/\bar{\lambda})(\bar{\lambda}\Delta\alpha - \beta c)^2 - (\underline{\lambda}\Delta\alpha - \beta c)^2 + (2 - \Delta\lambda/\bar{\lambda})\alpha_l(\bar{\lambda}\Delta\alpha - \beta c)\right. \\ & \left. + 2\alpha_l(\beta c - \underline{\lambda}\Delta\alpha)\right] > 0.\end{aligned}$$

- If $\bar{s}_s^\circ > 0$, $\underline{s}_s^\circ = 0$ and $\underline{s}_r^\circ = 0$ (i.e., $\underline{\lambda}\Delta\alpha < \underline{\alpha} \le \beta c < \bar{\lambda}\Delta\alpha$ and hence $\alpha_l < \Delta\lambda\Delta\alpha$), substituting $\bar{s}_r(\overline{w}) = \bar{s}_r^\circ$ and $\bar{s}_s(\overline{w},\overline{r}) = \bar{s}_s^\circ$ into (E.4) yields positive mimicry gain for the low-demand manufacturer:

$$\begin{aligned}\Pi\left(\overline{w},\overline{r} \,\middle|\, \bar{\lambda},\underline{\lambda}\right) - \underline{\pi}^\circ =& \frac{1}{8\beta}\left[(\bar{\alpha}-\beta c)^2 + (\bar{\lambda}^c/\bar{\lambda})(\bar{\lambda}\Delta\alpha-\beta c)^2 - (\Delta\lambda/\bar{\lambda})\alpha_l(\bar{\lambda}\Delta\alpha-\beta c)\right]\\ =& \frac{1}{8\beta}\left[\alpha_l^2 + (1/\bar{\lambda})(\bar{\lambda}\Delta\alpha-\beta c)^2 + (2-\Delta\lambda/\bar{\lambda})\alpha_l(\bar{\lambda}\Delta\alpha-\beta c)\right] > 0.\end{aligned}$$

- If $\bar{s}_s^\circ = 0$, $\underline{s}_s^\circ = 0$ and $\bar{s}_r^\circ > \underline{s}_r^\circ > 0$ (i.e., $\underline{\lambda}\Delta\alpha < \bar{\lambda}\Delta\alpha \le \beta c < \underline{\alpha}$ and hence $\alpha_l > \Delta\lambda\Delta\alpha$), substituting $\bar{s}_r(\overline{w}) = \bar{s}_r^\circ$ and $\bar{s}_s(\overline{w},\overline{r}) = \bar{s}_s^\circ$ into (E.4) yields positive mimicry gain for the low-demand manufacturer:

$$\Pi\left(\overline{w},\overline{r} \,\middle|\, \bar{\lambda},\underline{\lambda}\right) - \underline{\pi}^\circ = \frac{1}{8\beta}\left[(\bar{\alpha}-\beta c)^2 - (\underline{\alpha}-\beta c)^2\right] > 0.$$

- If $\bar{s}_r^\circ > 0$, $\bar{s}_s^\circ = 0$, $\underline{s}_s^\circ = 0$ and $\underline{s}_r^\circ = 0$ (i.e., $\underline{\lambda}\Delta\alpha < \max\{\bar{\lambda}\Delta\alpha, \underline{\alpha}\} \le \beta c < \bar{\alpha}$), substituting $\bar{s}_r(\overline{w}) = \bar{s}_r^\circ$ and $\bar{s}_s(\overline{w},\overline{r}) = \bar{s}_s^\circ$ into (E.4) yields positive mimicry gain for the low-demand manufacturer:

$$\Pi\left(\overline{w},\overline{r} \,\middle|\, \bar{\lambda},\underline{\lambda}\right) - \underline{\pi}^\circ = \frac{1}{8\beta}(\bar{\alpha}-\beta c)^2 > 0.$$