Online Appendices to "Intra-Consumer Price Discrimination with Credit Refund Policies"

These appendices provide supplemental materials for the paper and include seven sections:

- Appendix A includes proofs of the lemmas, propositions, and theorems in the main text.
- Appendix B provides a detailed analysis of the firm's problem in each case under an exogenous T and the proof of Theorem 1.
- Appendix C considers the problem with a general distribution of consumer valuations.
- Appendix D explores how the firm's optimal profit and credit refund policy change with the exogenous expiration term.
- Appendix E analyzes the problem when consumers have a deterministic valuation.
- Appendix F investigates the breakage rate, which is a managerially important quantity.
- Appendix G conducts a sensitivity analysis to examine how the optimal profit and expiration term change with the parameters.

Appendix A: Proofs of the Results in the Main Text

Before proving Proposition 1, we introduce two elementary yet useful properties of the bias function $J(\cdot)$ in Lemmas A.1 and A.2.

LEMMA A.1. *The bias function $J(t)$ is increasing and concave in t.*

Intuitively, we expect that an on-hand credit is worth more if it is further from expiration, which implies that $J(t)$ is increasing in t. Lemma A.1 confirms this intuition.

Proof of Lemma A.1

Part 1: We show monotonicity by induction. We first show that $J(1) \geq J(0)$. From (1), we have

$$\begin{aligned}\rho^* + J(1) =& \alpha\lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c, J(0)\} \\ &+ (1-\alpha)\lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v_L) - p + c, J(0)\} \\ &+ (1-\lambda)J(0). \end{aligned} \tag{A.1}$$

Note that the right-hand side of (A.1) is greater than the right-hand side of (2) due to the credit c in the two maximization terms. Therefore, we must have $J(1) \geq J(0)$.

For the inductive step, assume $J(t) \geq J(t-1)$ for $t = 1, \ldots, T-1$. We next show that $J(t+1) \geq J(t)$. From (1), we have

$$\rho^* + J(t+1) = \alpha\lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c, J(t)\}$$

$$\begin{aligned}
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c,J(t)\}\\
&+(1-\lambda)J(t)\\
\geq&\alpha\lambda\max\{\gamma J(T)+(1-\gamma)(J(0)+v_H)-p+c,J(t-1)\}\\
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c,J(t-1)\}\\
&+(1-\lambda)J(t-1)\\
=&\rho^*+J(t).
\end{aligned}$$

In the above, the inequality follows from the inductive assumption that $J(t)\geq J(t-1)$. The last equality follows directly from (1). Hence, $J(t+1)\geq J(t)$.

Part 2: Now, we show the concavity. We expect to show that

$$J(t)-J(t-1)\leq J(t-1)-J(t-2).$$

According to (1), we have

$$\begin{aligned}
\rho^*+J(t)=&\alpha\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-1)\}\\
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-1)\}\\
&+(1-\lambda)J(t-1).
\end{aligned}\tag{A.2}$$

Similarly,

$$\begin{aligned}
\rho^*+J(t-1)=&\alpha\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-2)\}\\
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-2)\}\\
&+(1-\lambda)J(t-2).
\end{aligned}\tag{A.3}$$

Therefore, (A.2) $-$ (A.3) yields

$$\begin{aligned}
&J(t)-J(t-1)\\
=&\alpha\lambda\Big\{\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-1)\big\}\\
&\qquad-\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-2)\big\}\Big\}\\
&+(1-\alpha)\lambda\Big\{\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-1)\big\}\\
&\qquad-\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-2)\big\}\Big\}+(1-\lambda)\Big(J(t-1)-J(t-2)\Big).
\end{aligned}$$

In order to show $J(t)-J(t-1)\leq J(t-1)-J(t-2)$, it suffices to show

$$\begin{aligned}
&\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-1)\big\}\\
&\quad-\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-2)\big\}\\
\leq&J(t-1)-J(t-2).
\end{aligned}\tag{A.4}$$

Next, we consider three cases.

Case 1: Suppose

$$J(t-1) \geq J(t-2) \geq \gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c.$$

In this case, the left hand side of the inequality (A.4) reduces to $J(t-1) - J(t-2)$, which is equal to the right hand side.

Case 2: Suppose

$$J(t-1) \geq \gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c \geq J(t-2).$$

In this case, the left hand side reduces to

$$J(t-1) - \big(\gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c\big) \leq J(t-1) - J(t-2).$$

Case 3: Suppose

$$\gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c \geq J(t-1) \geq J(t-2).$$

In this case, the left hand side reduces to 0, which is clearly smaller than the right hand side. This completes the proof. ■

Lemma A.2 below to bound the difference between $J(t)$ and $J(0)$ for $t \geq 1$. This result is intuitive as we do not expect the value of an on-hand credit with any expiration term to exceed c.

LEMMA A.2. *We have $J(t) - J(0) \leq c$ for all $t = 1, \ldots, T$. That is, the value of a credit is bounded by c.*

Proof of Lemma A.2

According to Lemma A.1, $J(t)$ is increasing in t. Therefore, it suffices to show that $J(T) - J(0) \leq c$. The proof follows from a coupling argument. Consider two copies of the process, one starting in state T and the other one starting in state 0. Let the two processes follow the same decisions over time. Because the two processes are coupled, they will be in state T after a canceled purchase. Prior to matching in the state, the optimal reward differs by c at most. This completes the proof.

■

Proof of Proposition 1

Here, we provide an extended version of Proposition 1, which includes the optimal solution to the optimality equations (1)–(2).

Proposition A.1. *Fix $J(0)=0$. An optimal solution to the optimality equations (1)–(2) can be characterized as follows.*

(a) Suppose $c \le p \le \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L$, an optimal solution is given by

$$J(t) = [1-(1-\lambda)^t]c, \qquad \forall t = 1,\dots,T,$$
$$\rho^* = \lambda\big(\gamma[1-(1-\lambda)^T]c + (1-\gamma)[\alpha v_H + (1-\alpha)v_L] - p\big).$$

Consumers make a purchase in all states;

(b) Suppose $\max\{c, \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L\} < p \le \min\{p^C(T), p^D(T)\}$, an optimal solution is given by

$$J(1) = \frac{\lambda c - (1-\alpha)\lambda p + \lambda(1-\gamma)(1-\alpha)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]},$$
$$J(t) = \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \qquad \forall t = 2,\dots,T,$$
$$\rho^* = \alpha\lambda[\gamma J(T) + (1-\gamma)v_H - p].$$

High-valuation consumers make a purchase in all states and low-valuation consumers make a purchase whenever they have a credit on hand ($t \ge 1$);

(c) Suppose $\max\{c, p^D(\tau+1)\} < p \le \min\{p^C(\tau), p^D(\tau)\}$ for a fixed τ between 1 and $T-1$, an optimal solution is given by

$$J(1) = \lambda\Big(c + (1-\alpha)[\gamma J(T) + (1-\gamma)v_L - p]\Big),$$
$$J(t) = \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \qquad \forall t = 2,\dots,\tau,$$
$$J(t) = c[1-(1-\alpha\lambda)^{t-\tau}] + (1-\alpha\lambda)^{t-\tau}J(\tau), \qquad \forall t = \tau+1,\dots,T-1,$$
$$J(T) = \frac{c[1-(1-\alpha\lambda)^{T-\tau}] + (1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau][c + (1-\alpha)((1-\gamma)v_L - p)]}{1-(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau](1-\alpha)\gamma},$$
$$\rho^* = \alpha\lambda[\gamma J(T) + (1-\gamma)v_H - p].$$

High-valuation consumers make a purchase in all states and low-valuation consumers make a purchase whenever they have a credit within τ periods of expiration.

(d) Suppose $\max\{c, p^D(1)\} < p \le \gamma[1-(1-\alpha\lambda)^T]c + (1-\gamma)v_H$, an optimal solution is given by

$$J(t) = [1-(1-\alpha\lambda)^t]c, \qquad \forall t = 1,\dots,T,$$
$$\rho^* = \alpha\lambda\big(\gamma[1-(1-\alpha\lambda)^T]c + (1-\gamma)v_H - p\big).$$

High-valuation consumers make a purchase in all states and low-valuation consumers never make a purchase.

Proof. Part (a): Assume that a low-valuation consumer makes a purchase without a credit. That is,

$$\gamma J(T)+(1-\gamma)(J(0)+v_L)-p\geq J(0). \tag{A.5}$$

It follows (A.5) immediately that

$$\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c\geq J(0)+c\geq J(t-1),$$

for any $1\leq t\leq T$, where the last inequality holds by Lemma A.2. This implies that the low-valuation consumer makes a purchase in all states. Of course, the high-valuation consumer makes a purchase in all states as well. Therefore, the dynamic programming equations (1) and (2) reduce to

$$\begin{aligned}\rho^*+J(t)=&\alpha\lambda\Big(\gamma J(T)+(1-\gamma)(J(0)+v_H)-p+c\Big)+(1-\alpha)\lambda\Big(\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c\Big)\\&+(1-\lambda)J(t-1),\qquad\forall t=1,\dots,T,\\\rho^*+J(0)=&\alpha\lambda\Big(\gamma J(T)+(1-\gamma)(J(0)+v_H)-p\Big)+(1-\alpha)\lambda\Big(\gamma J(T)+(1-\gamma)(J(0)+v_L)-p\Big)\\&+(1-\lambda)J(0).\end{aligned}$$

Solving the above equations gives the solution of $J(\cdot)$ and ρ^*. The technical condition in the supposition can be obtained by using the expressions for $J(0)$ and $J(T)$ in (A.5). This completes the proof of Part (a).

Parts (b) and (c): Assume that only a high-valuation consumer makes a purchase without a credit.

In this case, only a high-valuation consumer makes a purchase without a credit, but a low-valuation consumer does not make a purchase without a credit. From (2), we must have

$$\gamma J(T)+(1-\gamma)(J(0)+v_H)-p\geq J(0), \tag{A.6}$$

$$\gamma J(T)+(1-\gamma)(J(0)+v_L)-p< J(0). \tag{A.7}$$

From (A.6) and Lemma A.2, we have

$$\gamma J(T)+(1-\gamma)(J(0)+v_H)-p+c\geq c+J(0)\geq J(t),\quad\forall t=1,\dots,T.$$

Using (1), this implies that a high-valuation consumer makes a purchase with a credit in all states. It is not immediately clear whether a low-valuation consumer makes a purchase with a credit. To proceed with our analysis, let

$$\tau=\max\{t\in\{1,\dots,T\}:\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c\geq J(t-1)\}.$$

Note that because $J(t)$ is increasing in t by Lemma A.1, τ is a threshold such that a low-valuation consumer makes a purchase with a credit in state $t \le \tau$, and does not make a purchase with a credit for $t > \tau$. Summarizing the discussion above, we have

$$\rho^* + J(t) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c\Big) + (1-\alpha\lambda)J(t-1), \qquad \forall t = \tau+1,\dots,T, \tag{A.8}$$

$$\begin{aligned}\rho^* + J(t) =& \lambda\Big(\gamma J(T) + (1-\gamma)J(0) - p + c\Big) + \lambda(1-\gamma)\Big(\alpha v_H + (1-\alpha)v_L\Big)\\ &+ (1-\lambda)J(t-1), \qquad \forall t = 1,\dots,\tau,\end{aligned} \tag{A.9}$$

$$\rho^* + J(0) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0)+v_H) - p\Big) + (1-\alpha\lambda)J(0). \tag{A.10}$$

We can solve for ρ^* and $J(t)$ for $t = 0,1,\dots,T$. Note that there are $T+1$ equations and $T+2$ unknowns. Therefore, we fix $J(0) = 0$. The equations above simplify to

$$\rho^* + J(t) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)v_H - p + c\Big) + (1-\alpha\lambda)J(t-1), \qquad \forall t = \tau+1,\dots,T, \tag{A.11}$$

$$\rho^* + J(t) = \lambda\Big(\gamma J(T) - p + c\Big) + \lambda(1-\gamma)\Big(\alpha v_H + (1-\alpha)v_L\Big) + (1-\lambda)J(t-1), \quad \forall t = 1,\dots,\tau, \tag{A.12}$$

$$\rho^* = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)v_H - p\Big). \tag{A.13}$$

We consider two subcases when solving the equations above.

<u>Subcase I: $\tau = T$.</u>

Note that $\tau = T$ means that

$$\gamma J(T) + (1-\gamma)v_L - p + c \ge J(T-1). \tag{A.14}$$

In this case, equation (A.11) vanishes. Using (A.13) in (A.12) leads to

$$J(t) = \lambda c + \lambda(1-\alpha)\Big(\gamma J(T) - p\Big) + \lambda(1-\gamma)(1-\alpha)v_L + (1-\lambda)J(t-1), \quad \forall t = 1,\dots,T. \tag{A.15}$$

Taking the difference in the above equation between t and $t-1$ for $t = 2,\dots,T$, we obtain

$$J(t) - J(t-1) = (1-\lambda)\Big(J(t-1) - J(t-2)\Big).$$

It follows that

$$\begin{aligned}J(t) &= \sum_{i=1}^{t}[J(i) - J(i-1)] = \sum_{i=1}^{t}(1-\lambda)^{i-1}[J(1) - J(0)] = \sum_{i=1}^{t}(1-\lambda)^{i-1}J(1)\\ &= \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \quad \forall t = 2,\dots,T.\end{aligned}$$

Using the expression for $J(T)$ in (A.15) and solving for $J(1)$, we obtain

$$J(1) = \frac{\lambda c - (1-\alpha)\lambda p + \lambda(1-\gamma)(1-\alpha)v_L}{1 - \gamma(1-\alpha)[1-(1-\lambda)^T]}.$$

Putting the expressions for $J(\cdot)$ in (A.6), (A.7), and (A.14) yields $p \le p^C(T)$, $p > (1-\gamma)v_L + \gamma[1-(1-\lambda)^T]c$, and $p \le p^D(T)$, respectively. This completes the proof of Part (b).

Subcase II: $\tau < T$.

Note that $\tau \le T-1$ implies

$$J(\tau-1) \le \gamma J(T) + (1-\gamma)v_L - p + c < J(\tau). \tag{A.16}$$

Using (A.13) in (A.11) and (A.12) leads to

$$J(t) = \alpha\lambda c + (1-\alpha\lambda)J(t-1), \qquad \forall t = \tau+1,\ldots,T, \tag{A.17}$$

$$J(t) = \lambda c + (1-\alpha)\lambda\Big(\gamma J(T) - p\Big) + \lambda(1-\gamma)(1-\alpha)v_L + (1-\lambda)J(t-1), \quad \forall t = 1,\ldots,\tau. \tag{A.18}$$

From (A.17), we have

$$J(t) = c[1-(1-\alpha\lambda)^{t-\tau}] + (1-\alpha\lambda)^{t-\tau}J(\tau), \qquad \forall t = \tau+1,\ldots,T. \tag{A.19}$$

Because $J(0) = 0$, taking $t = 1$ in (A.18) gives

$$\begin{aligned} J(1) &= \lambda c + (1-\alpha)\lambda\Big(\gamma J(T) - p\Big) + \lambda(1-\gamma)(1-\alpha)v_L. \\ &= \lambda\Big(c + (1-\alpha)(\gamma J(T) + (1-\gamma)v_L - p)\Big). \end{aligned} \tag{A.20}$$

Equation (A.18) can be rewritten as

$$J(t) = J(1) + (1-\lambda)J(t-1), \qquad \forall t = 1,\ldots,\tau. \tag{A.21}$$

Equation (A.21) implies that

$$J(t) = \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \qquad \forall t = 1,\ldots,\tau. \tag{A.22}$$

Using (A.22) for $t = \tau$ in (A.19), we obtain

$$J(t) = c[1-(1-\alpha\lambda)^{t-\tau}] + \frac{(1-\alpha\lambda)^{t-\tau}[1-(1-\lambda)^\tau]J(1)}{\lambda}, \qquad \forall t = \tau+1,\ldots,T, \tag{A.23}$$

Taking $t = T$ in (A.23) and using (A.20), we obtain

$$J(T) = c[1-(1-\alpha\lambda)^{T-\tau}] + (1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau][c + (1-\alpha)(\gamma J(T) + (1-\gamma)v_L - p)].$$

Solving the equation above gives

$$J(T) = \frac{c[1-(1-\alpha\lambda)^{T-\tau}] + (1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau][c + (1-\alpha)((1-\gamma)v_L - p)]}{1-(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau](1-\alpha)\gamma}.$$

Putting the expressions for $J(\cdot)$ in (A.6) and (A.16) yields $p \le p^C(\tau)$ and $p^D(\tau+1) \le p \le p^D(\tau)$, respectively. Note that $\gamma J(T) + (1-\gamma)v_L - p + c < J(\tau)$ in (A.16) implies (A.7), so we do not need to derive the condition from constraint (A.7). This completes the proof of Part (c).

Part (d): Assume that a high-valuation consumer makes a purchase without a credit and a low-valuation consumer does not purchase in state 1. That is,

$$\gamma J(T) + (1-\gamma)(J(0)+v_H) - p \ge J(0), \tag{A.24}$$
$$\gamma J(T) + (1-\gamma)(J(0)+v_L) - p + c < J(0). \tag{A.25}$$

It follows (A.24) immediately that

$$\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c \ge J(0) + c \ge J(t-1),$$

for any $1 \le t \le T$, where the last inequality holds by Lemma A.2. This implies that the high-valuation consumer makes a purchase in all states. Moreover, it follows (A.25) immediately that

$$\gamma J(T) + (1-\gamma)(J(0)+v_L) - p < J(0),$$
$$\gamma J(T) + (1-\gamma)(J(0)+v_L) - p + c < J(t-1),$$

for any $1 \le t \le T$, where the last inequality holds by the monotonicity of $J(\cdot)$. This implies that the low-valuation consumer never makes a purchase, with and without a credit. Therefore, the dynamic programming equations (1) and (2) reduce to

$$\rho^* + J(t) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c\Big) + (1-\alpha)\lambda J(t-1) + (1-\lambda)J(t-1), \qquad \forall t = 1,\dots,T,$$
$$\rho^* + J(0) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0)+v_H) - p\Big) + (1-\alpha)\lambda J(0) + (1-\lambda)J(0).$$

Solving the above equations gives the solution of $J(\cdot)$ and ρ^*. The technical condition in the supposition can be obtained by using the expressions for $J(0)$ and $J(T)$ in (A.24) and (A.25). This completes the proof of Part (d). ■

Lemma A.3. *The value T_0 defined in* (5) *exists and is finite.*

Proof of Lemma A.3

Both $A(T)$ and $B(T,T)$ are increasing in T. Moreover, one can verify that $A(1) < B(1,1)$ and $\lim_{T\to\infty} A(T) = \infty > \lim_{T\to\infty} B(T,T)$. Therefore, T_0 exists and is finite. ■

Proof of Theorem 1

The proof of Theorem 1 is relegated to Appendix B. ■

Proof of Lemma 1

Part (a): We show that T_1 exists and is finite. First,

$$\begin{aligned}
&A(1) = (1-\gamma)(v_H - v_L),\\
&B(2,1) = (1-\gamma)\frac{v_H - (v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda)\lambda}{1-\gamma+\gamma(1-\alpha\lambda)(1-\lambda)},\\
&A(1) - B(2,1) = \frac{(1-\gamma)}{1-\gamma+\gamma(1-\alpha\lambda)(1-\lambda)}\Big\{(v_H - v_L)\big(1-\gamma[1-(1-\alpha\lambda)^2]\big) - v_H\Big\} < 0,
\end{aligned}$$

where the last inequality holds because $1-\gamma[1-(1-\alpha\lambda)^2] < 1$.

Second,

$$\begin{aligned}
&\lim_{T\to\infty} A(T-1) = \infty,\\
&\lim_{T\to\infty} B(T,T-1) = v_H - (v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda).
\end{aligned}$$

Clearly,

$$\lim_{T\to\infty} A(T-1) > \lim_{T\to\infty} B(T,T-1).$$

Third,

$$A(T-1) = (1-\gamma)(v_H - v_L)\Big\{\frac{\alpha}{(1-\lambda)^{T-2}+1-\alpha}\Big\}$$

is increasing in T. Let $(1-\lambda)^{T-1} = x$, then

$$B(T,T-1) = \frac{(1-\gamma)v_H - (1-\gamma)(v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda)(1-x)}{1-\gamma+\gamma(1-\alpha\lambda)x}.$$

Taking derivative with respect to x yields

$$\begin{aligned}
&(1-\gamma)(v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda)\Big(1-\gamma+\gamma(1-\alpha\lambda)x\Big)\\
&\quad - \Big\{(1-\gamma)v_H - (1-\gamma)(v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda)(1-x)\Big\}\gamma(1-\alpha\lambda)\\
=&(1-\gamma)\gamma(1-\alpha\lambda)\Big((1-\alpha)(v_H - v_L) - v_H\Big) - (1-\gamma)\gamma(v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda)\alpha\lambda\\
=&(1-\gamma)\gamma(1-\alpha\lambda)\Big((1-\alpha)(v_H - v_L)(1-\alpha\gamma\lambda) - v_H\Big)\\
<&0.
\end{aligned}$$

Therefore, $B(T,T-1)$ is increasing in T.

The above three points imply that T_1 exists and is finite. The proof that T_i ($i \geq 2$) exists and is finite follows a similar approach.

Part (b): We show $T_0 + 1 \leq T_1$. The definition of T_0 implies that

$$A(T_0) \leq B(T_0, T_0). \tag{A.26}$$

According to the definition of T_1, it suffices to show $A(T_0+1-1) \leq B(T_0+1, T_0+1-1)$, that is, $A(T_0) \leq B(T_0+1, T_0)$. Given (A.26), it suffices to show $B(T_0, T_0) \leq B(T_0+1, T_0)$. One can check

$$
\begin{aligned}
&B(T_0,T_0) - B(T_0+1,T_0)\\
=&\frac{v_H-(v_H-v_L)\gamma(1-\alpha)[1-(1-\lambda)^{T_0}]}{1-\gamma[1-(1-\lambda)^{T_0}]} - \frac{v_H-(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T_0}]}{1-\gamma[1-(1-\alpha\lambda)(1-\lambda)^{T_0}]}\\
=&\Big\{v_H-(v_H-v_L)\gamma(1-\alpha)[1-(1-\lambda)^{T_0}]\Big\}\Big\{1-\gamma[1-(1-\alpha\lambda)(1-\lambda)^{T_0}]\Big\}\\
&\quad-\Big\{v_H-(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T_0}]\Big\}\Big\{1-\gamma[1-(1-\lambda)^{T_0}]\Big\}\\
=&-v_H\gamma(1-\lambda)^{T_0}\alpha\lambda-(v_H-v_L)\gamma(1-\alpha)(1-\gamma)[1-(1-\lambda)^{T_0}]\alpha\lambda\\
<&0.
\end{aligned}
$$

Consequently, $T_1 \geq T_0+1$. The proof of $T_i \leq T_{i+1}$ $(i \geq 1)$ follows a similar approach. ■

Proof of Lemma 2

Before proving Lemma 2, we first introduce a lemma.

Lemma A.4. *We have*

$$
\frac{1-\gamma+\gamma(1-\alpha\lambda)^i(1-\lambda)^{T-i}}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}\cdot\frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}]-(1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]}
$$

is decreasing in T.

Proof. Let $(1-\lambda)^{T-i}=x$, then the first term

$$
\frac{1-\gamma+\gamma(1-\alpha\lambda)^i(1-\lambda)^{T-i}}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}=\frac{1-\gamma+\gamma(1-\alpha\lambda)^i x}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i+\gamma(1-\alpha)(1-\alpha\lambda)^i x}.
$$

Taking derivative with respect to x yields

$$
\begin{aligned}
&\frac{1}{\big(1-\gamma(1-\alpha)(1-\alpha\lambda)^i+\gamma(1-\alpha)(1-\alpha\lambda)^i x\big)^2}\cdot\\
&\Big\{\gamma(1-\alpha\lambda)^i\big(1-\gamma(1-\alpha)(1-\alpha\lambda)^i+\gamma(1-\alpha)(1-\alpha\lambda)^i x\big)-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-\gamma+\gamma(1-\alpha\lambda)^i x]\Big\}\\
=&\frac{1}{\big(1-\gamma(1-\alpha)(1-\alpha\lambda)^i+\gamma(1-\alpha)(1-\alpha\lambda)^i x\big)^2}\Big\{\gamma(1-\alpha\lambda)^i\alpha+\gamma^2(1-\alpha)(1-\alpha\lambda)[1-(1-\alpha\lambda)^i]\Big\}\\
>&0.
\end{aligned}
$$

Hence, the first term decreases in T.

While, the second term

$$
\frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}]-(1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]}
$$

$$=\frac{1}{1-\gamma-\frac{(1-\lambda)^{T-i-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}[1-\gamma(1-\alpha\lambda)^{i+1}]}=\frac{1}{1-\gamma-\frac{1-\gamma(1-\alpha\lambda)^{i+1}}{\frac{\alpha}{(1-\lambda)^{T-i-1}}+1-\alpha}},$$

which, clearly, is decreasing in T. This completes the proof. ■

Here, we provide an extended version of Lemma 2 which includes the complete solution to the problem (12)–(16), and then provide the proof of this extended version.

LEMMA A.5. *For each $i\in\{0,1,\ldots,\bar{T}-1\}$, let $p^*_{3,T-i}$, $c^*_{3,T-i}$, and $\pi^*_{3,T-i}$ denote the optimal price, credit, and profit, respectively, to the problem (12)–(16). We have the following results:*

(a) Suppose $T\le T_i$, then

$$p^*_{3,T-i}=\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i}+(1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\Big\}v_H$$
$$+(1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\Big\}v_L,$$
$$\pi^*_{3,T-i}=\lambda(1-\gamma)\alpha v_H+\lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}v_L,$$

and $c^*_{3,T-i}$ takes any value between $A(T-i)$ and $\min\big\{B(T,T-i),A(T-i+1)\big\}$. Moreover, $\pi^*_{3,T-i}$ is increasing in T;

(b) Suppose $T\ge T_i+1$. If $C(T,T-i)\ge 0$, then

$$p^*_{3,T-i}=(1-\gamma)v_L+\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i}$$
$$+\frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c^*_{3,T-i},$$
$$\pi^*_{3,T-i}=\lambda\alpha(1-\gamma)v_L\frac{1-\gamma+\gamma(1-\lambda)^{T-i}(1-\alpha\lambda)^i}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}\cdot$$
$$\frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}]-(1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]},$$
$$c^*_{3,T-i}=C(T,T-i).$$

Moreover, $\pi^*_{3,T-i}$ is decreasing in T. Otherwise, there is no feasible solution to the problem (12)–(16).

Proof. Recall

$$p^C(T-i)=\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c+(1-\gamma)\big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\big\}v_H$$
$$+(1-\gamma)\big\{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\big\}v_L,$$
$$p^D(T-i)=(1-\gamma)v_L+\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c+\frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c.$$

Putting $J(\cdot)$ into the constraints (13) - (15) yields $p \le p^C(T-i)$, $p \le p^D(T-i)$, and $p \ge p^D(T-i+1)$, respectively. Summarizing the constraints gives

$$\max\left\{c, p^D(T-i+1)\right\} \le p \le \min\left\{p^C(T-i), p^D(T-i)\right\}.$$

One can verify that if $c \ge A(T-i)$, then $p^C(T-i) \le p^D(T-i)$, and vice versa. Next, we consider two cases and identify two possible solutions.

Case 1: Note that if $c \le A(T-i+1)$, then $p^C(T-i) \ge p^D(T-i+1)$, and vice versa. Moreover, if $c \le B(T, T-i)$, then $c \le p^C(T-i)$. Therefore, if $A(T-i) \le \min\left\{B(T, T-i), A(T-i+1)\right\}$, then one possible solution to the problem (12)–(16) is

$$\begin{aligned}
p^*_{3,T-i} = p^C(T-i)|_{c=c^*_{3,T-i}} &= \gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i} \\
&+ (1-\gamma)\left\{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\right\}v_H \\
&+ (1-\gamma)\left\{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\right\}v_L, \\
\pi^*_{3,T-i}(p^C(T-i)) &= \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\gamma)^i[1-(1-\lambda)^{T-i}]}{1-(1-\alpha)\gamma(1-\alpha\gamma)^i[1-(1-\lambda)^{T-i}]}v_L,
\end{aligned}$$

where $c^*_{3,T-i}$ takes any value between $A(T-i)$ and $\min\left\{B(T, T-i), A(T-i+1)\right\}$.

Case 2: Note that if $c \le C(T, T-i)$, then $c \le p^D(T-i)$. Moreover, $p^D(T-i) \ge p^D(T-i+1)$ holds automatically. Hence, the other possible solution is

$$\begin{aligned}
c^*_{3,T-i} &= \min\{A(T-i), C(T, T-i)\}, \\
p^*_{3,T-i} &= p^D(T-i)|_{c=c^*_{3,T-i}} = (1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i} \\
&\quad + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c^*_{3,T-i}, \\
\pi^*_{3,T-i}(p^D(T-i)) &= \frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]} \cdot \\
&\quad \left\{(1-\gamma)v_L + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c^*_{3,T-i}\right\}.
\end{aligned}$$

Note that Case 1 requires $A(T-i) \le B(T, T-i)$,[12] hence, for any $T \le T_i$ such that $A(T-i) \le B(T, T-i)$, both Case 1 and Case 2 could occur if the firm sets the appropriate price and credit. However, since $c^*_{3,T-i} \le A(T-i)$ in Case 2, we have

$$\begin{aligned}
&\pi^*_{3,T-i}(p^D(T-i)) \\
\le &\frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]} \cdot
\end{aligned}$$

[12] Note that $A(T-i) \le A(T-i+1)$ holds because of the monotonicity of $A(\cdot)$.

$$\Big\{(1-\gamma)v_L+\frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}A(T-i)\Big\}$$
$$=\lambda\alpha(1-\gamma)v_H+\lambda\alpha(1-\gamma)v_L\frac{(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}$$
$$=\pi^*_{3,T-i}(p^C(T-i)).$$

Therefore, for any $T\le T_i$, Case 1 is optimal. One can verify that $\pi^*_{3,T-i}(p^C(T-i))$ increases in T. This completes the proof of Part (a).

For any $T\ge T_i+1$ such that $A(T-i)>B(T,T-i)$, only Case 2 is possible. Note that if $C(T,T-i)<0$, then $c>C(T,T-i)$, and thus $c>p^D(T-i)$. Therefore, there is no feasible solution to the problem (12)–(16). Hereafter, we assume $C(T,T-i)\ge 0$. Lemma B.5 implies that $A(T-i)>C(T,T-i)$. Hence, $c^*_{3,T-i}=C(T,T-i)$. Putting $C(T,T-i)$ into $\pi^*_{3,T-i}(p^D(T-i))$ yields

$$\pi^*_{3,T-i}(p^D(T-i))=\lambda\alpha(1-\gamma)v_L\frac{1-\gamma+\gamma(1-\lambda)^{T-i}(1-\alpha\lambda)^i}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}\cdot$$
$$\frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}]-(1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]}.$$

Moreover, Lemma A.4 implies that $\pi^*_{3,T-i}(p^D(T-i))$ is decreasing in T. This completes the proof of Part (b). ■

Proof of Theorem 2

Lemma 2 indicates that the optimal profit when $\tau=T-i$ for each fixed i (where $0\le i\le\bar{T}-1$) is first increasing in T when $T\le T_i$ and then decreasing when $T\ge T_i+1$. Consequently,

$$T^*\in\big\{T_0,T_0+1,\ldots,T_{\bar{T}-1},T_{\bar{T}-1}+1,\bar{T}\big\}.$$

Note that for any $i\ge\tilde{i}+1$, we have $T_i>\bar{T}$, so

$$T^*\in\big\{T_0,T_0+1,\ldots,T_{\tilde{i}},T_{\tilde{i}}+1,\bar{T}\big\}.$$

This completes the proof. ■

Proof of Theorem 3

Part (a): Recall that

$$\pi^c=\max\{\lambda(1-\gamma)v_L,\lambda(1-\gamma)\alpha v_H\}.$$

Suppose $T\le T_0$. Theorem 1(a) implies that

$$\pi^*=\max\Big\{\lambda(1-\gamma)v_L,\lambda(1-\gamma)\alpha v_H+\lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}v_L\Big\}.$$

Clearly, $\pi^* \geq \pi^c$, and the strict inequality holds if $v_L/v_H \leq \alpha < 1$.

Suppose $T \geq T_0 + 1$. We know that there exists an $i \in \{1, \ldots, \bar{T} - 1\}$ such that $T_{i-1} < T \leq T_i$. Moreover, Theorem 1(b) implies that

$$\begin{aligned}\pi^* &\geq \max\left\{\lambda(1-\gamma)v_L, \pi_3^{T-i}\right\}\\ &= \max\left\{\lambda(1-\gamma)v_L, \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}v_L\right\},\end{aligned}$$

where the last equality holds because of Lemma 2(a). Clearly, $\pi^* \geq \pi^c$, and the strict inequality holds if $v_L/v_H \leq \alpha < 1$.

Part (b): Note that the optimal profit under the credit refund policy (when setting $T^* = T_0$) can be written as

$$\lambda(1-\gamma)\max\left\{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L, v_L\right\},$$

while the profit of the cash refund policy is

$$\lambda(1-\gamma)\max\{\alpha v_H, v_L\}.$$

When α is sufficiently small such that $\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L < v_L$, the profit ratio is 1.

When $\alpha v_H \leq v_L \leq \alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L$, the profit under the credit refund policy reduces to $\lambda(1-\gamma)\left\{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L\right\}$, which increases in α. While, the profit under the cash refund policy reduces to $\lambda(1-\gamma)v_L$, which is independent of α. Hence, the profit ratio becomes

$$\frac{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L}{v_L},$$

which increases in both α and γ.

When $v_L/v_H \leq \alpha < 1$, the profit ratio is

$$\frac{v_H + \frac{(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L}{v_H}.$$

In order to see how this ratio changes with α and γ, it suffices to investigate how $(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]$ changes with α and γ. According to Proposition G.1, T_0 decreases in α, so $(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]$ decreases in α, and therefore the profit ratio also decreases in α. Similarly, according to Proposition G.1, T_0 increases in γ, so $(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]$ increases in γ, and therefore the profit ratio increases in γ.

Combining the above three cases shows that the profit ratio increases in γ, and increases in α if $\alpha < v_L/v_H$ and decreases otherwise. This completes the proof. ∎

Proof of Proposition 2

Part (a): In a cash refund policy, when the firm sets the effective price to v_H, only high-valuation consumers make a purchase; hence, the aggregate consumers' surplus is 0. In a credit refund policy, the forward-looking consumers will not purchase if the expected surplus is negative; hence, the aggregate consumers' surplus is nonnegative. Therefore, the consumer surplus increases. It follows immediately that the social welfare, which is the sum of the consumer surplus and firm profit, increases as well.

Part (b): In a cash refund policy, when the firm sets the effective price to v_L, both high- and low-valuation consumers make a purchase; hence, the aggregate consumers' surplus is $\lambda(1-\gamma)\alpha(v_H - v_L)$. As for a credit refund policy, the average price paid by the consumer in either case II or III lies between v_L and v_H; hence, the aggregate consumers' surplus is strictly less than $\lambda(1-\gamma)\alpha(v_H - v_L)$.

Note that the firm profit in a cash refund policy is $\lambda(1-\gamma)v_L$, so the social welfare is $\lambda(1-\gamma)(\alpha v_H + (1-\alpha)v_L)$, which is essentially the maximum social welfare possible. Hence, the social welfare also decreases when the firm switches from a cash refund policy with effective price v_L to a credit refund policy. This completes the proof. ■

Proof of Proposition 3

Before proving Proposition 3, we first introduce a lemma.

LEMMA A.6. *If $p \le \gamma \frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c + (1-\gamma)v_L$, then the consumer always makes a purchase when she arrives at the market, independent of her valuation realization.*

Proof. Suppose the low-valuation consumer makes a purchase when she does not have a credit. That is,

$$\gamma\delta J(T) + (1-\gamma)[\delta J(0) + v_L] - p \ge \delta J(0). \tag{A.27}$$

Similar to Lemma A.2, one can show

$$J(t) - J(0) \le c.$$

Hence, it follows immediately that for any $t \ge 1$,

$$\gamma\delta J(T) + (1-\gamma)[\delta J(0) + v_L] - p + c \ge \delta J(t-1),$$

implying that the low-valuation consumer makes a purchase whenever she has a credit. Therefore, the low-valuation consumer always makes a purchase when she arrives at the market, and so does the high-valuation consumer.

Then, the consumer's dynamic programming equations reduce to

$$J(t) = \lambda\Big\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + E[v]) - p + c\Big\} + (1-\lambda)\delta J(t-1),$$
$$J(0) = \lambda\Big\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + E[v]) - p\Big\} + (1-\lambda)\delta J(0).$$

Solving the above set of equations yields

$$J(0) = \frac{\lambda}{1-\delta}[(1-\gamma)E[v] - p] + \frac{\lambda\gamma\delta}{(1-\delta)(1-\delta+\lambda\delta)}[1-(1-\lambda)^T\delta^T]\lambda c,$$
$$J(T) = \frac{\lambda}{1-\delta}[(1-\gamma)E[v] - p] + \frac{1-\delta+\lambda\gamma\delta}{(1-\delta)(1-\delta+\lambda\delta)}[1-(1-\lambda)^T\delta^T]\lambda c.$$

Putting $J(0)$ and $J(T)$ into (A.27) gives

$$p \le \gamma\frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c + (1-\gamma)v_L.$$

This completes the proof. ■

Now, we are ready to prove Proposition 3.

Proof of Proposition 3. Note that the firm's profit π_δ^c in a cash refund policy is

$$\pi_\delta^c = \max\{\lambda(1-\gamma)v_L, \lambda(1-\gamma)\alpha v_H\} = \pi^c.$$

Next, we derive the firm's profit π_δ^* in credit refund policy. To derive the firm's optimal profit, we need to formulate and solve the firm's optimization problem, taking into account the consumer's optimal responses. Similar to the main model, consumers' optimal responses can be divided into four cases:

Case I: Consumers make a purchase in each state, independent of their valuation realization.

Case II: A low-valuation consumer makes a purchase whenever she has a credit on hand.

Case III: A low-valuation consumer makes a purchase when she has a credit and the credit is within τ periods of expiration, where $\tau < T$.

Case IV: A low-valuation consumer never makes a purchase in any state.

We first analyze the firm's profit in Case I. The consumer's state transition is the same as Case I in the main model, as depicted in Figure B.1 in Appendix B.1.1. Therefore, the firm's optimization problem can be written as follows,

$$\begin{aligned} \max_{p,c} \quad & \lambda\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\} \\ s.t. \quad & p \le \gamma\frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c + (1-\gamma)v_L \\ & p \ge c. \end{aligned}$$

Let $p^*_{1,\delta}$, $c^*_{1,\delta}$, and $\pi^*_{1,\delta}$ denote the optimal price, credit, and profit to the above problem. We obtain

$$p^*_{1,\delta} = \gamma\frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c^*_{1,\delta} + (1-\gamma)v_L,$$

$$c^*_{1,\delta} \in \Big[0, \frac{(1-\gamma)v_L}{1-\gamma\frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]}\Big],$$

$$\pi^*_{1,\delta} = \lambda(1-\gamma)v_L + \lambda\gamma c^*_{1,\delta}\Big\{\frac{\lambda\delta}{1-(1-\lambda)\delta}[1-(1-\lambda)^T\delta^T] - [1-(1-\lambda)^T]\Big\}.$$

Note that

$$\begin{aligned}\pi^*_{1,\delta} =& \lambda(1-\gamma)v_L + \lambda\gamma c^*_{1,\delta}\Big\{\lambda\delta[1+(1-\lambda)\delta+\cdots+(1-\lambda)^{T-1}\delta^{T-1}] - \lambda[1+(1-\lambda)+\cdots+(1-\lambda)^{T-1}]\Big\}\\ \le& \lambda(1-\gamma)v_L = \pi^*_1,\end{aligned} \tag{A.28}$$

where the above inequality becomes equality when $\delta = 1$.

Let $\pi^*_{2,\delta}$, $\pi^*_{3,\delta}$, and $\pi^*_{4,\delta}$ denote the firm's profit in Cases II, III, and IV, respectively. Given that consumers discount their future surplus, the expected value of the credit is less than that in the average reward model. Thus,

$$\pi^*_{2,\delta} \le \pi^*_2, \quad \pi^*_{3,\delta} \le \pi^*_3, \quad \pi^*_{4,\delta} \le \pi^*_4. \tag{A.29}$$

Recall that when v_L is sufficiently close to v_H or α is sufficiently small,

$$\max\{\pi^*_2, \pi^*_3, \pi^*_4\} < \pi^*_1 = \lambda(1-\gamma)v_L. \tag{A.30}$$

Combining (A.29) and (A.30) yields

$$\max\{\pi^*_{2,\delta}, \pi^*_{3,\delta}, \pi^*_{4,\delta}\} < \lambda(1-\gamma)v_L.$$

Together with (A.28), we obtain

$$\pi^*_\delta = \max\{\pi^*_{1,\delta}, \pi^*_{2,\delta}, \pi^*_{3,\delta}, \pi^*_{4,\delta}\} < \lambda(1-\gamma)v_L \le \pi^c_\delta.$$

Therefore, if v_L is sufficiently close to v_H or α is sufficiently small, the cash refund policy brings more profit than the credit refund policy. This completes the proof of Part (a).

Note that

$$\pi^*_\delta = \max\{\pi^*_{1,\delta}, \pi^*_{2,\delta}, \pi^*_{3,\delta}, \pi^*_{4,\delta}\}, \quad \pi^* = \max\{\pi^*_1, \pi^*_2, \pi^*_3, \pi^*_4\}.$$

When δ is sufficiently close to 1, $\pi^*_{i,\delta} \approx \pi^*_i$ for each $1 \le i \le 4$. Hence, $\pi^*_\delta \approx \pi^*$. According to Theorem 3, if $v_L/v_H \le \alpha < 1$, then $\pi^* > \pi^c = \pi^c_\delta$. Therefore, $\pi^*_\delta \ge \pi^c_\delta$ if $v_L/v_H \le \alpha < 1$ and δ is sufficiently close to 1. This completes the proof of Part (b). ■

Appendix B: Detailed Analysis for the Firm's Problem under an Exogenous T

B.1 Analysis and Results

B.1.1 Case I: A low-valuation consumer makes a purchase without a credit In order to derive the revenue contribution by each consumer, we first analyze the consumer's state transition. Because low-valuation consumers make a purchase even without a credit, one can verify that they also make a purchase with a credit. Therefore, in this case, consumers make a purchase in each state, independent of their valuation realization. Figure B.1 depicts the consumer's state transition.

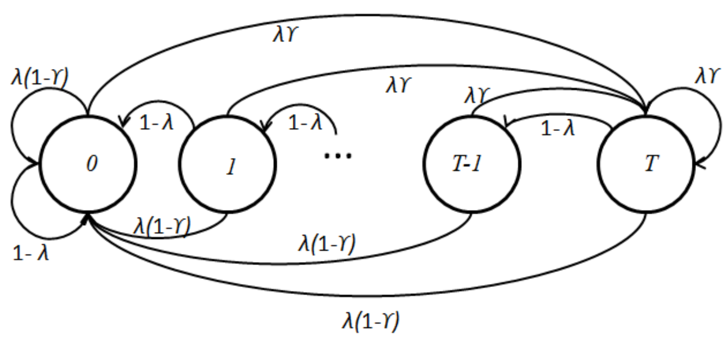


Figure B.1 State transition in Case I.

Lemma B.1 **(Stationary state probabilities in Case I)**. *Let q_i denote the stationary probability of state $i \in \{0,1,\ldots,T\}$. We have*

$$q_0 = 1-\gamma[1-(1-\lambda)^T],$$
$$q_i = (1-\lambda)^{T-i}\lambda\gamma. \qquad \forall i = 1,2\ldots,T.$$

Note that both high- and low-valuation consumers make a purchase in each state, and each consumer pays a price $p-c$ in state $i \neq 0$ and a price p in state 0. Thus, the average profit contribution by each consumer is

$$\lambda\Big\{q_0 p + (1-q_0)(p-c)\Big\} = \lambda\Big\{p-(1-q_0)c\Big\} = \lambda\Big\{p-\gamma[1-(1-\lambda)^T]c\Big\}.$$

Therefore, the firm's optimization problem can be written as follows,

$$\max_{p,c} \quad \lambda\Big\{p-\gamma[1-(1-\lambda)^T]c\Big\} \tag{B.1}$$
$$s.t. \quad p \le \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L \tag{B.2}$$
$$p \ge c. \tag{B.3}$$

Note that (B.2) ensures that a low-valuation consumer makes a purchase, even without a credit on hand. Let p_1^*, c_1^*, and π_1^* denote the optimal price, credit, and corresponding profit to the problem (B.1)–(B.3), respectively. We have the following result.

PROPOSITION B.1. *The optimal solution to the problem* (B.1)–(B.3) *is given by*

$$p_1^* = \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L,$$

$$c_1^* \in \left[0, \frac{(1-\gamma)v_L}{1-\gamma[1-(1-\lambda)^T]}\right],$$

$$\pi_1^* = \lambda(1-\gamma)v_L.$$

Note that the optimal price follows Proposition 1(a). Recall that the first term in the optimal price denotes the expected value of an earned credit upon a product return, and the second term is the utility received by a low-valuation consumer for a product that is not returned. Given the composition of the price, the firm's long-run average profit stems from the utility $(1-\gamma)v_L$ that a low-valuation consumer receives when she does not return the product. Hence, the optimal profit includes only the second term of the optimal price, taking into account the consumer's arrival rate λ.

B.1.2 Case II: A low-valuation consumer makes a purchase whenever she has a credit

This section analyzes the case in which a low-valuation consumer does not make a purchase without a credit but makes a purchase whenever she has a credit on hand.

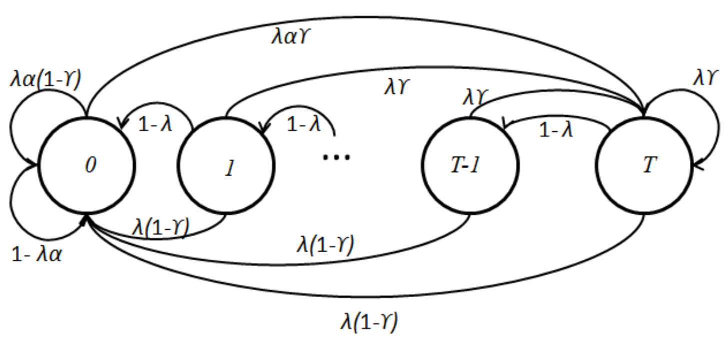


Figure B.2 State transition in Case II.

The state transition in this case is shown in Figure B.2. Notably, the only difference from Case I is that a low-valuation consumer does not make a purchase in state 0; thus, the transition from state 0 to state T occurs with a probability $\lambda\alpha\gamma$ rather than $\lambda\gamma$.

Lemma B.2 **(Stationary state probabilities in Case II)**. *Let q_i denote the stationary probability of state $i \in \{0, 1, \ldots, T\}$. We have*

$$
\begin{aligned}
q_0 &= \frac{1-\gamma[1-(1-\lambda)^T]}{\lambda\gamma\alpha} q_T, \\
q_i &= (1-\lambda)^{T-i} q_T, && \forall i = 1, 2, \ldots, T-1. \\
q_T &= \frac{\lambda\gamma\alpha}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}.
\end{aligned}
$$

To understand the expression of q_T, consider the following event. For both types of consumers, the process to arrive at state T involves "arriving, making a purchase, and returning the product". First, a high-valuation consumer always makes a purchase whenever she arrives to the market, so the probability of "arriving, making a purchase, and returning" is simply $\lambda\gamma$. Hence, the probability that a consumer draws a high valuation and arrives at state T is simply $\alpha\lambda\gamma$. Second, a low-valuation consumer in state T could return to state T by making a purchase before her credit goes expired and then returning the product, the probability of which is $q_T[1-(1-\lambda)^T]\gamma$. Note that a low-valuation consumer who is in state $i \in \{1, \ldots, T-1\}$ could also arrive at state T by making a purchase and then returning, which has been included in $q_T[1-(1-\lambda)^T]\gamma$. Hence, the probability that a consumer draws a low valuation and arrives at state T is $(1-\alpha)q_T[1-(1-\lambda)^T]\gamma$. Because these two events are mutually exclusive, the total probability of both events is the sum, which should be equal to q_T. Therefore, we have the following equation:

$$\alpha\lambda\gamma + (1-\alpha)q_T[1-(1-\lambda)^T]\gamma = q_T.$$

Solving the equations gives the expression for q_T.

Note that both high- and low-valuation consumers make a purchase in state $i \in \{1, 2, \ldots, T\}$ and pay a price $p-c$; however, only a high-valuation consumer makes a purchase in state 0 and pays a price p. Hence, the average profit contribution by each consumer is

$$\lambda\Big\{q_0\alpha p + (1-q_0)(p-c)\Big\} = \frac{\lambda\alpha}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\}.$$

Therefore, the firm's optimization problem is as follows,

$$\max_{p,c} \quad \frac{\lambda\alpha}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\} \tag{B.4}$$

$$s.t. \quad p \le \gamma J(T) + (1-\gamma)v_H \tag{B.5}$$

$$p \le \gamma J(T) + (1-\gamma)v_L + c - J(T-1) \tag{B.6}$$

$$p \ge \gamma J(T) + (1-\gamma)v_L \tag{B.7}$$

$$p \ge c. \tag{B.8}$$

The expressions for $J(T)$ and $J(T-1)$ are given in Proposition A.1. Note that (B.5) implies that a high-valuation consumer makes a purchase without a credit on hand, (B.6) implies that a low-valuation consumer makes a purchase in state T (which implies that she always makes a purchase whenever she has a credit on hand), and (B.7) implies that a low-valuation consumer does not purchase without a credit on hand.

PROPOSITION B.2. *For any fixed T, the optimal solution to the problem* (B.4)–(B.8) *is as follows:*

(a) If $T \le T_0$, then the optimal price is $p^*_{2,a} = p^C(T)|_{c=c^*_{2,a}}$, where $c^*_{2,a}$ can take any value between $A(T)$ and $B(T,T)$. The optimal profit is given by

$$\pi^*_{2,a} = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}v_L;$$

(b) Otherwise, the optimal price is $p^*_{2,b} = p^D(T)|_{c=c^*_{2,b}}$, where $c^*_{2,b} = C(T,T)$. The optimal profit is given by

$$\pi^*_{2,b} = \lambda\alpha(1-\gamma)v_L\frac{1-\gamma[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}\cdot\frac{\alpha+(1-\alpha)(1-\lambda)^{T-1}}{\alpha[1-(1-\lambda)^{T-1}]-\alpha\gamma[1-(1-\lambda)^T]}.$$

Proposition B.2 characterizes the optimal solution in Case II. According to Proposition 1(b), the optimal price takes the minimum between $p^C(T)$ and $p^D(T)$. Recall that $p^C(T)$ is the highest price at which a high-valuation consumer purchases without a credit, while $p^D(T)$ is the highest price at which a low-valuation consumer purchases in state T (which implies that she would also purchase in states $i \in \{1,\dots,T-1\}$). The two parts of Proposition B.2 correspond to situations in which the optimal prices are $p^C(T)$ and $p^D(T)$, respectively.

Part (a) considers the case in which T is small (i.e., $T \le T_0$) and the risk of credit expiration is relatively large. In this case, a low-valuation consumer has a strong incentive to purchase in state T; hence, $p^D(T)$ is higher than $p^C(T)$. Therefore, the optimal price in this case is $p^C(T)$. To understand the expression of the profit $\pi^*_{2,a}$, note that the firm's long-run average profit stems from the utility that a consumer receives when she does not return the product. The probability that a high-valuation consumer shows up, makes a purchase, and then does not return, is $\lambda(1-\gamma)\alpha$. Hence, the expected profit collected from a high-valuation consumer is $\lambda(1-\gamma)\alpha v_H$. Similarly, the expected profit collected from a low-valuation consumer is

$$\lambda(1-q_0)(1-\gamma)(1-\alpha)v_L = \lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}v_L.$$

Combining these two parts gives the optimal profit $\pi^*_{2,a}$.

Part (b) considers the case in which T is large (i.e., $T > T_0$) and the risk of credit expiration is relatively small. In this case, a low-valuation consumer has a lower incentive to purchase; therefore, in order to ensure that a low-valuation consumer purchases in state T, the price $p^D(T)$ must be sufficiently low. Hence, $p^D(T)$ is smaller than $p^C(T)$, and thus the optimal price in this case is $p^D(T)$.

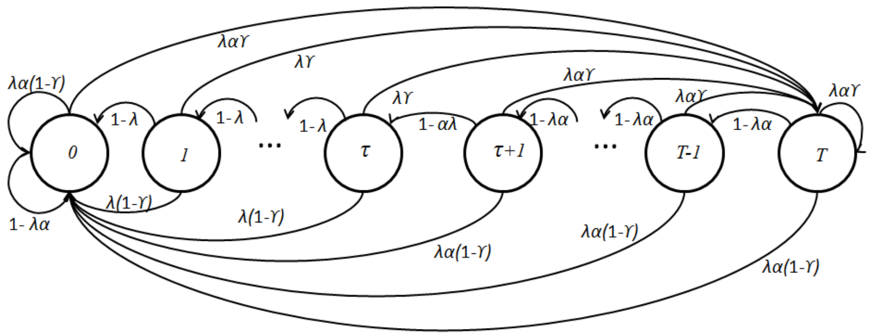


Figure B.3 State transition in Case III.

B.1.3 Case III: A low-valuation consumer makes a purchase when she has a credit and the credit is within τ periods of expiration ($\tau < T$) The state transition in this case is characterized in Figure B.3. The difference with Case II is that a low-valuation consumer does not make a purchase in state $\{\tau+1,\dots,T\}$. Therefore, the flow from state $i \in \{\tau+1,\dots,T\}$ to state T occurs with a probability $\lambda\alpha\gamma$ rather than $\lambda\gamma$, the flow from state $i \in \{\tau+1,\dots,T\}$ to state 0 occurs with a probability $\lambda\alpha(1-\gamma)$ rather than $\lambda(1-\gamma)$, and the flow from state $i \in \{\tau+1,\dots,T\}$ to state $i-1$ occurs with a probability $1-\lambda\alpha$ rather than $1-\lambda$.

Lemma B.3 **(Stationary state probabilities in Case III)**. *Let q_i denote the stationary probability of state $i \in \{0,1,\dots,T\}$. We have*

$$
\begin{aligned}
q_0 &= \frac{(1-\gamma)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^T] + (1-\gamma)[1-(1-\alpha\lambda)^{T-\tau}] + (1-\lambda)^\tau(1-\alpha\lambda)^{T-\tau}}{\lambda\gamma\alpha} q_T, \\
q_i &= (1-\lambda)^{\tau-i}(1-\alpha\lambda)^{T-\tau} q_T, \qquad \forall i = 1,2,\dots,\tau-1. \\
q_i &= (1-\alpha\lambda)^{T-i} q_T, \qquad \forall i = \tau,\dots,T-1. \\
q_T &= \frac{\lambda\gamma\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}.
\end{aligned}
$$

Note that both high- and low-valuation consumers make a purchase in state $i \in \{1,2,\dots,\tau-1,\tau\}$ and pay a price $p-c$. Only high-valuation consumers make a purchase in state $\{0,\tau+1,\dots,T\}$. They pay a price p in state 0 and a price $p-c$ in state $i \in \{\tau+1,\dots,T\}$. Hence, the average profit contribution by each consumer is

$$
\begin{aligned}
&\lambda\Big\{q_0\alpha p + \sum_{i=1}^{\tau}(p-c)q_i + \sum_{i=\tau+1}^{T}\alpha(p-c)q_i\Big\} \\
=&\frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}\Big\{p-\gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c\Big\}.
\end{aligned}
$$

Therefore, the firm's optimization problem becomes

$$
\max_{p,c} \quad \frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}\Big\{p-\gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c\Big\} \tag{B.9}
$$

$$\text{s.t.}\quad p \le \gamma J(T) + (1-\gamma)v_H \tag{B.10}$$

$$p \le \gamma J(T) + (1-\gamma)v_L + c - J(\tau - 1) \tag{B.11}$$

$$p \ge \gamma J(T) + (1-\gamma)v_L + c - J(\tau) \tag{B.12}$$

$$p \ge c. \tag{B.13}$$

The expressions for $J(\cdot)$ are given in Proposition A.1. Note that (B.10) implies that a high-valuation consumer makes a purchase without a credit on hand, (B.11) implies that a low-valuation consumer makes a purchase when the credit is within τ periods of expiration, and (B.12) implies that a low-valuation consumer does not make a purchase when the credit is within $\tau + 1$ periods of expiration. Let p_3^*, c_3^*, and π_3^* denote the optimal price, credit, and corresponding profit to the problem (B.9)–(B.13).

Proposition B.3. *For any fixed T, the optimal solution to the problem* (B.9)–(B.13) *is as follows:*

$$\pi_3^* = \max_{1 \le \tau \le T-1} \pi_3^\tau,$$

$$\tau_3^* = \arg\max_{\tau \in \{1,\dots,T-1\}} \pi_3^\tau,$$

$$p_3^* = p^D(\tau_3^*)|_{c=c_3^*},$$

$$c_3^* = \min\{A(\tau_3^*), C(T, \tau_3^*)\}.$$

In the above,

$$\pi_3^\tau = \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]} + \frac{\lambda\alpha(1-\lambda)^{\tau-1}\min\{A(\tau), C(T,\tau)\}}{\alpha + (1-\alpha)(1-\lambda)^{\tau-1}}, \quad \forall 1 \le \tau \le T.$$

In Proposition B.3, π_3^τ corresponds to the profit for a fixed τ. The key idea of the solution is to compare the profits associated with different values of τ and τ_3^* is the value of τ with the highest profit. At optimality, the firm sets the price p_3^* and credit refund c_3^* such that a low-valuation consumer makes a purchase only in state $\{1, \dots, \tau_3^*\}$.

B.1.4 Case IV: A low-valuation consumer never makes a purchase in any state In order to derive the revenue contribution by each consumer, we first analyze the consumer's state transition. In this case, high-valuation consumers make a purchase in each state, while low-valuation consumers never make a purchase. Figure B.4 depicts the consumer's state transition.

Lemma B.4 **(Stationary state probabilities in Case IV)**. *Let q_i denote the stationary probability of state $i \in \{0, 1, \dots, T\}$. We have*

$$q_0 = 1 - \gamma[1 - (1-\lambda\alpha)^T],$$

$$q_i = (1-\lambda\alpha)^{T-i}\lambda\alpha\gamma. \qquad \forall i = 1, 2 \dots, T.$$

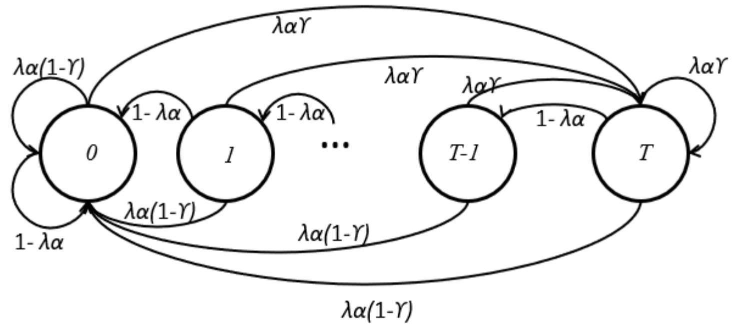


Figure B.4 State transition in Case IV.

Note that only high-valuation consumers make a purchase in each state and pays a price $p-c$ in state $i \neq 0$ and a price p in state 0. Thus, the average profit contribution by each consumer is

$$\lambda\alpha\Big\{q_0 p + (1-q_0)(p-c)\Big\} = \lambda\alpha\Big\{p-(1-q_0)c\Big\} = \lambda\alpha\Big\{p-\gamma[1-(1-\lambda\alpha)^T]c\Big\}.$$

Therefore, the firm's optimization problem can be written as follows,

$$\max_{p,c} \quad \lambda\alpha\Big\{p-\gamma[1-(1-\lambda\alpha)^T]c\Big\} \tag{B.14}$$

$$s.t. \quad p \leq \gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H \tag{B.15}$$

$$p > (1-\gamma)v_L + \gamma[1-(1-\lambda\alpha)^T)]c + c \tag{B.16}$$

$$p \geq c. \tag{B.17}$$

Note that (B.15) ensures a high-valuation consumer makes a purchase even without credit on hand, and (B.16) ensures a low-valuation consumer does not make a purchase without credit on hand. Let p_4^*, c_4^*, and π_4^* denote the optimal price, credit, and the corresponding profit to the problem (B.14)–(B.17), respectively. We have the following result.

Proposition B.4. *The optimal solution to the problem* (B.14)–(B.17) *is given by*

$$p_4^* = \gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H,$$

$$c_4^* \in \Big[0, \frac{(1-\gamma)v_H}{1-\gamma[1-(1-\lambda\alpha)^T]}\Big],$$

$$\pi_4^* = \lambda(1-\gamma)\alpha v_H.$$

To understand the profit expression, note that the optimal price in Case IV is given by $\gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H$. The term $[1-(1-\lambda\alpha)^T]c$ denotes the expected value of an earned credit upon a product return,[13] the probability of which is γ. The term v_H is the utility a high-valuation

[13] In this case, only high-valuation consumers make a purchase, and the purchase is made whenever they arrive at the market. Hence, the probability that a credit will be used before expiration is $1-(1-\lambda\alpha)^T$, different from $1-(1-\lambda)^T$ in Case I.

consumer receives when she does not return the product, the probability of which is $1-\gamma$. Given the composition of the price, the firm's long-run average profit stems from the utility received by a high-valuation consumer when she does not return the product. Hence, the optimal profit includes only the second term of the optimal price, taking into account the arrival rate $\lambda\alpha$ of high-valuation consumers.

B.2 Proofs of Results in Appendix B.1 and Proof of Theorem 1

Proof of Lemma B.1

The balance equations can be written as follows,

$$\begin{aligned}
&\lambda\gamma q_0 = \lambda(1-\gamma)\sum_{i=1}^{T} q_i + (1-\lambda)q_1, && \\
&q_i = (1-\lambda)q_{i+1}, && \forall i = 1, 2\ldots, T-1. \\
&[\lambda(1-\gamma)+(1-\lambda)]q_T = \lambda\gamma\sum_{i=0}^{T-1} q_i, && \\
&\sum_{i=0}^{T} q_i = 1. &&
\end{aligned}$$

Solving the above equations yields the stationary probabilities. ■

Proof of Proposition B.1

Constraint (B.2) indicates that the optimal price

$$p_1^* = \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L.$$

Putting p_1^* into constraint (B.3) yields that

$$c_1^* \leq \frac{(1-\gamma)v_L}{1-\gamma[1-(1-\lambda)^T]}.$$

Putting p_1^* into objective (B.1) gives the optimal profit. ■

Proof of Lemma B.2

The balance equations can be written as follows,

$$\begin{aligned}
&\lambda\alpha\gamma q_0 = \lambda(1-\gamma)\sum_{i=1}^{T} q_i + (1-\lambda)q_1, && \\
&q_i = (1-\lambda)q_{i+1}, && \forall i = 1, 2\ldots, T-1. \\
&[\lambda(1-\gamma)+(1-\lambda)]q_T = \lambda\gamma\sum_{i=1}^{T-1} q_i + \lambda\alpha\gamma q_0, && \\
&\sum_{i=0}^{T} q_i = 1. &&
\end{aligned}$$

Solving the above equations yields the stationary probabilities. ■

Proof of Proposition B.2

Before proving Proposition B.2, we first introduce a lemma.

LEMMA B.5. *Suppose $C(T,k)\geq 0$ for fixed T and k. If $A(k)>(<)B(T,k)$, then $A(k)>(<)C(T,k)$.*

Proof. On the one hand,

$$\begin{aligned}&A(k)-B(T,k)\\=&\frac{1-\gamma}{(1-\lambda)^{k-1}\Big\{1-\gamma[1-(1-\alpha\lambda)^{T-k}(1-\lambda)^{k}]\Big\}}\cdot\\&\Big\{\big(\alpha-\alpha\gamma+(1-\gamma)(1-\alpha)(1-\lambda)^{k-1}+\gamma(1-\alpha\lambda)^{T-k+1}(1-\lambda)^{k-1}\big)(v_H-v_L)-(1-\lambda)^{k-1}v_H\Big\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&A(k)-C(T,k)\\=&\frac{1-\gamma}{(1-\lambda)^{k-1}\Big\{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{k-1}]-(1-\lambda)^{k-1}[1-\gamma(1-\alpha\lambda)^{T-k+1}]\Big\}}\cdot\\&\Big\{\big(\alpha-\alpha\gamma+(1-\gamma)(1-\alpha)(1-\lambda)^{k-1}+\gamma(1-\alpha\lambda)^{T-k+1}(1-\lambda)^{k-1}\big)(v_H-v_L)-(1-\lambda)^{k-1}v_H\Big\}.\end{aligned}$$

Note that the above denominator is positive, indicated by $C(T,k)\geq 0$. This completes the proof. ■

Proof of Proposition B.2. Recall that

$$p^C(T)=\gamma[1-(1-\lambda)^T]c+(1-\gamma)\Big\{1-\gamma(1-\alpha)[1-(1-\lambda)^T]\Big\}v_H+(1-\gamma)\gamma(1-\alpha)[1-(1-\lambda)^T]v_L,$$

$$p^D(T)=(1-\gamma)v_L+\frac{\gamma\alpha[1-(1-\lambda)^T]+(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}}c.$$

Putting $J(\cdot)$ into constraints (B.5) and (B.6) leads to $p\leq p^C(T)$ and $p\leq p^D(T)$, respectively. Note that (B.7) holds automatically if $p=\min\{p^C(T),p^D(T)\}$. Hence, the constraints for this optimization problem can be summarized as

$$c\leq p\leq \min\{p^C(T),p^D(T)\}.$$

One can verify that if $c\geq A(T)$, then $p^C(T)\leq p^D(T)$, and vice versa. We consider two cases and identify two possible solutions.

Case 1: One can check if $c\leq B(T,T)$, then $c\leq p^C(T)$. Therefore, if $A(T)\leq B(T,T)$, then one possible solution is

$$\begin{aligned}p^*_{2,a}=p^C(T)|_{c=c^*_{2,a}}=&\gamma[1-(1-\lambda)^T]c^*_{2,a}+(1-\gamma)\Big\{1-\gamma(1-\alpha)[1-(1-\lambda)^T]\Big\}v_H\\&+(1-\gamma)\gamma(1-\alpha)[1-(1-\lambda)^T]v_L,\end{aligned}$$

$$\pi^*_{2,a} = \pi^*_{2,a}(p^C(T)) = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}v_L,$$

where $c^*_{2,a}$ takes any value between $A(T)$ and $B(T,T)$.

Case 2: One can check if $c \le C(T,T)$, then $c \le p^D(T)$. Hence, the other possible solution is

$$c^*_{2,b} = \min\{A(T), C(T,T)\},$$

$$p^*_{2,b} = p^D(T)|_{c=c^*_{2,b}} = (1-\gamma)v_L + \frac{\gamma\alpha[1-(1-\lambda)^T]+(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}]}c^*_{2,b},$$

$$\pi^*_{2,b} = \pi^*_{2,b}(p^D(T)) = \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} + \frac{\lambda\alpha(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}}c^*_{2,b}.$$

Note that Case 1 requires $A(T) \le B(T,T)$, hence, for any fixed $T \le T_0$ such that $A(T) \le B(T,T)$, both Case 1 and Case 2 could occur if the firm sets the appropriate price and credit. However,

$$\begin{aligned}\pi^*_{2,b}(p^D(T)) &\le \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} + \frac{\lambda\alpha(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}}A(T)\\ &= \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}v_L\\ &= \pi^*_{2,a}(p^C(T)).\end{aligned}$$

Therefore, for any $T \le T_0$, the firm should set the price and credit such that Case 1 occurs.

For any $T \ge T_0+1$, only Case 2 is possible. According to the definition of T_0, we have $A(T) > B(T,T)$. It follows Lemma B.5 that $A(T) > C(T,T)$. Therefore, $c^*_{2,b} = C(T,T)$. Putting $c^*_{2,b}$ into $\pi^*_{2,b}$ yields

$$\pi^*_{2,b} = \lambda\alpha(1-\gamma)v_L\frac{1-\gamma[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}\cdot\frac{\alpha+(1-\alpha)(1-\lambda)^{T-1}}{\alpha[1-(1-\lambda)^{T-1}]-\alpha\gamma[1-(1-\lambda)^T]}.$$

This completes the proof. ■

Proof of Lemma B.3

The balance equations can be written as follows,

$$\lambda\alpha\gamma q_0 = \lambda(1-\gamma)\sum_{i=1}^{\tau}q_i + \lambda\alpha(1-\gamma)\sum_{i=\tau+1}^{T}q_i + (1-\lambda)q_1,$$

$$q_i = (1-\lambda)q_{i+1}, \qquad \forall i = 1,2\ldots,\tau-1.$$

$$q_i = (1-\alpha\lambda)q_{i+1}, \qquad \forall i = \tau,\tau+1,\ldots,T-1.$$

$$[\lambda\alpha(1-\gamma)+(1-\alpha\lambda)]q_T = \lambda\alpha\gamma q_0 + \lambda\gamma\sum_{i=1}^{\tau}q_i + \lambda\alpha\gamma\sum_{i=\tau+1}^{T-1}q_i,$$

$$\sum_{i=0}^{T}q_i = 1.$$

Solving the above equations yields the stationary probabilities. ■

Proof of Proposition B.3

Recall that

$$p^C(\tau) = \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^{\tau}]c + (1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]\Big\}v_H$$
$$+(1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]\Big\}v_L,$$
$$p^D(\tau) = (1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^{\tau}]c + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c.$$

Putting $J(\cdot)$ into constraints (B.10), (B.11), and (B.12) yields $p \le p^C(\tau)$, $p \le p^D(\tau)$, and $p \ge p^D(\tau+1)$, respectively. Hence, the constraints for the problem (B.9)–(B.13) can be summarized as follows,

$$\max\big\{c, p^D(\tau+1)\big\} \le p \le \min\big\{p^C(\tau), p^D(\tau)\big\}.$$

One can verify that if $c \ge A(\tau)$, then $p^C(\tau) \le p^D(\tau)$, and vice versa. We consider two cases regarding the relationship between $p^C(\tau)$ and $p^D(\tau)$.

<u>Case 1:</u> If $c \le A(\tau+1)$, then $p^C(\tau) \ge p^D(\tau+1)$, and vice versa. Moreover, if $c \le B(T,\tau)$, then $c \le p^C(\tau)$. Therefore, for any fixed $\tau \in \{1,\dots,T-1\}$, if $A(\tau) \le \min\big\{B(T,\tau), A(\tau+1)\big\}$, then one possible solution to the problem (7)–(11) is,

$$p_3^\tau = p^C(\tau)|_{c=c_3^\tau} = \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^{\tau}]c_3^\tau + (1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]\Big\}v_H$$
$$+(1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]\Big\}v_L,$$
$$\pi_3^\tau(p^C(\tau)) = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]}v_L,$$

and c_3^τ takes any value between $A(\tau)$ and $\min\big\{B(T,\tau), A(\tau+1)\big\}$.

<u>Case 2:</u> If $c \le C(T,\tau)$, then $c \le p^D(\tau)$. Moreover, $p^D(\tau) \ge p^D(\tau+1)$ holds automatically. Hence, for any fixed $\tau \in \{1,\dots,T-1\}$, the other possible solution is as follows,

$$p_3^\tau = p^D(\tau)|_{c=c_3^\tau} = (1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^{\tau}]c_3^\tau$$
$$+\frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c_3^\tau,$$
$$\pi_3^\tau(p^D(\tau)) = \frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]}\cdot$$
$$\Big\{(1-\gamma)v_L + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c_3^\tau\Big\},$$
$$c_3^\tau = \min\{A(\tau), C(T,\tau)\}.$$

Next, we show that the solution in Case 1 is a special case of that in Case 2. Note that the solution in Case 1 is possible only if $A(\tau) \le B(T,\tau)$. According to Lemma B.5, if $A(\tau) \le B(T,\tau)$, then $A(\tau) \le C(T,\tau)$. One can check that if $c = A(\tau)$, then

$$p^D(\tau) = p^C(\tau),$$

and

$$\pi_3^\tau(p^D(\tau)) = \pi_3^\tau(p^C(\tau)).$$

Hence, we establish that the solution in Case 1 is a special case of Case 2.

Therefore, for any fixed $\tau \in \{1, 2, \ldots, T-1\}$, the optimal price, credit, and profit are $p^D(\tau)|_{c=c_3^\tau}$, c_3^τ, and $\pi_3^\tau(p^D(\tau))$. Comparing the profit for each τ gives the solution to the problem (B.9)–(B.13). This completes the proof. ∎

Proof of Lemma B.4

The balance equations can be written as follows,

$$\begin{aligned}
&\lambda\alpha q_0 = \lambda\alpha(1-\gamma)\sum_{i=0}^{T} q_i + (1-\lambda\alpha)q_1, \\
&q_i = (1-\lambda\alpha)q_{i+1}, && \forall i = 1, 2 \ldots, T-1. \\
&q_T = \lambda\alpha\gamma\sum_{i=0}^{T} q_i, \\
&\sum_{i=0}^{T} q_i = 1.
\end{aligned}$$

Solving the above equations yields the stationary probabilities. ∎

Proof of Proposition B.4

Constraint (B.15) indicates that the optimal price

$$p_4^* = \gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H.$$

Putting p_4^* into constraint (B.17) yields that

$$c_4^* \le \frac{(1-\gamma)v_H}{1-\gamma[1-(1-\lambda\alpha)^T]}.$$

Putting p_4^* into objective (B.14) gives the optimal profit. ∎

Proof of Theorem 1

To derive the firm's optimal decisions, it suffices to compare π_1^*, π_2^*, and π_3^*, because π_4^* is always smaller than π_2^* or π_3^* by Lemma 2(a). Recall that $\pi_1^* = \lambda(1-\gamma)v_L$. Observe that π_1^* is always included in π^*, so to complete the proof, it suffices to investigate/compare π_2^* and π_3^*.

Part (a): Suppose $T \le T_0$. For any $T \le T_0$, according to Proposition B.2(a), we have

$$\pi_{2,a}^* = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha \frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} v_L.$$

Next, we derive π_3^*. If $T \le T_0$, then $T \le T_i$ for each $i \ge 1$, because $T_0 \le T_i$ by Lemma 1(b). Therefore, when $\tau = T - i$, we have $A(\tau) \le B(T,\tau)$. By Lemma B.5, it follows immediately that $A(\tau) \le C(T,\tau)$. Hence, $c_3^\tau = A(\tau)$. Therefore,

$$\begin{aligned}\pi_3^\tau &= \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]} + \frac{\lambda\alpha(1-\lambda)^{\tau-1}A(\tau)}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}} \\ &= \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]} v_L.\end{aligned}$$

Clearly, π_3^τ increases in τ. Thus,

$$\pi_3^* = \max_{\tau\in\{1,\dots,T-1\}} \pi_3^\tau = \pi_3^{T-1} = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha \frac{\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T-1}]}{1-\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T-1}]} v_L.$$

Because $1-(1-\lambda)^T > (1-\alpha\lambda)[1-(1-\lambda)^{T-1}]$, we have $\pi_{2,a}^* > \pi_3^*$. Hence, the optimal solution is either π_1^* or $\pi_{2,a}^*$, whichever is the largest.

Part (b): Suppose $T \ge T_0 + 1$. Here, we do not compare $\pi_{2,b}^*$ and π_3^*, rather, we show

$$\max\{\pi_{2,b}^*, \pi_3^*\} = \max_{\tau\in\{1,\dots,T\}} \pi_3^\tau.$$

Proposition B.3 shows that $\pi_3^* = \max_{\tau\in\{1,\dots,T-1\}} \pi_3^\tau$. To complete the proof, it suffices to show $\pi_{2,b}^*$ is equivalent to π_3^T.

For any $T \ge T_0 + 1$, the definition of T_0 indicates that $A(T) > B(T,T)$ and then it follows Lemma B.5 that $A(T) > C(T,T)$. Consequently, $\min\{A(T), C(T,T)\} = C(T,T)$. We have

$$\begin{aligned}\pi_3^T &= \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-T}[1-(1-\lambda)^T]} + \frac{\lambda\alpha(1-\lambda)^{T-1}C(T,T)}{\alpha+(1-\alpha)(1-\lambda)^{T-1}} \\ &= \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} \cdot \frac{[\alpha+(1-\alpha)(1-\lambda)^{T-1}]\{1-\gamma[1-(1-\lambda)^T]\}}{\alpha[1-(1-\lambda)^{T-1}] - \alpha\gamma[1-(1-\lambda)^T]},\end{aligned}$$

which, according to Proposition B.2(b), is exactly the same as $\pi_{2,b}^*$. Moreover,

$$\begin{aligned}&p^D(T)|_{c=C(T,T)} \\ =&(1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c\end{aligned}$$

$$=(1-\gamma)v_L + \frac{\alpha\gamma[1-(1-\lambda)^T]+(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}c$$
$$=(1-\gamma)v_L + \frac{\alpha\gamma[1-(1-\lambda)^T]+(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}}C(T,T),$$

which is exactly the same as $p^*_{2,b}$.

We have established that for any $T \geq T_0+1$, $\{\pi_3^\tau : 1 \leq \tau \leq T\}$ is a set including the profits in Cases II and III. Thus, $\pi_3^{\hat{\tau}}$, where $\hat{\tau} = \arg\max_{\tau\in\{1,\dots,T\}} \pi_3^\tau$, corresponds to the optimal profit in Cases II and III. Hence, the optimal solution for any $T \geq T_0+1$ is either π_1^* or $\pi_3^{\hat{\tau}}$, whichever is the largest. This completes the proof. ■

Appendix C: General Distribution of Consumer Valuations

Our main analysis assumes that consumers' valuation v for a product follows a two-point distribution. However, we expect that the benefit of a credit refund policy is preserved under more general valuation distributions as long as the firm can still achieve intra-consumer discrimination. Given that explicit characterizations of the optimal credit refund policy is infeasible, we use a numerical study to verify that our main result is robust to the assumption on the valuation distribution.

Suppose a consumer's valuation v for a product follows a general distribution with cdf $F(\cdot)$. Then, a consumer's decision problem can be written as

$$\rho^* + J(t) = \lambda \int_v \max\Big\{\gamma J(T) + (1-\gamma)(J(0)+v) - p + c, J(t-1)\Big\} dF(v) + (1-\lambda)J(t-1), \quad \forall 1 \leq t \leq T, \quad \text{(C.1)}$$
$$\rho^* + J(0) = \lambda \int_v \max\Big\{\gamma J(T) + (1-\gamma)(J(0)+v) - p, J(0)\Big\} dF(v) + (1-\lambda)J(0). \quad \text{(C.2)}$$

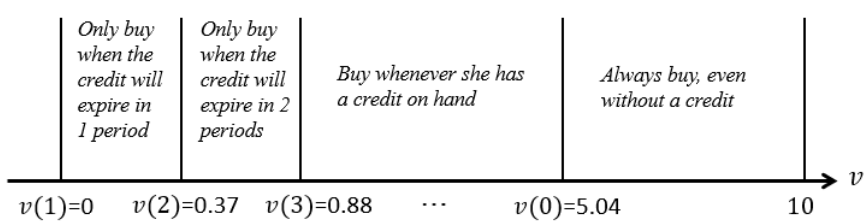


Figure C.1 Consumer behavior when the consumer's valuation has uniform distribution on $[0,10]$.

Intuitively, there exists a threshold $v(t)$ on the consumer's valuation such that, conditional on being in state t, the consumer will make a purchase if $v \geq v(t)$ and will not otherwise. In the numerical example, we assume that the consumer's valuation v follows a uniform distribution $U[0,10]$ and set $(\lambda,\gamma,T) = (0.15, 0.15, 3)$. On the consumer side, the numerical result shows that $v(t)$ is increasing in t for $t \geq 1$; in particular, $v(1)=0$, $v(2)=0.37$, $v(3)=0.88$, and $v(0)=5.04$. The values of $v(t)$ for $t = 0, 1, \ldots, 3$ indicate that, as shown in Figure C.1, a consumer with a valuation realization between $v(0)$ and 10 makes a purchase in each state, even when she does not have a credit on hand; a consumer with a valuation realization between $v(3)$ and $v(0)$ makes a purchase whenever she has a credit; a consumer with a valuation realization between $v(2)$ and $v(3)$ makes a purchase only when the credit will expire within 2 periods; and a consumer with a valuation realization between $v(1)$ and $v(2)$ makes a purchase only when the credit will expire within 1 period. Observe that as the credit is closer to expiration, the consumer is more likely to make a purchase. Moreover, the consumer with no credit makes a purchase only when her valuation realization is above $v(0)$, implying that the consumer without a credit is less likely to make a purchase.

Similar to the base model where the credit earned by the high-valuation consumers sustains the low-valuation consumers who never make a purchase without a credit, a consumer with a valuation realization between $v(0)$ and 10 makes a purchase without a credit, and the earned credit sustains the consumer with a valuation realization below $v(0)$ when she only makes a purchase with a credit. Furthermore, a consumer with a valuation realization between $v(0)$ and 10 pays a full price p^* in state 0 and a price $p^* - c^*$ in other states. A consumer with a valuation realization between $v(1)$ and $v(0)$ pays a price $p^* - c^*$ for each purchase but with different purchase probabilities (a consumer with a valuation realization between $v(1)$ and $v(2)$ has the lowest purchase probability because she makes a purchase only in state 1), leading to distinct effective prices or purchase probabilities for consumers with different valuation realizations. Perhaps unsurprisingly, the intra-consumer discrimination effect persists in such a general setting. Moreover, it can be shown that the firm's profit under the credit refund policy is 0.323, higher than 0.319, the profit under the cash refund policy. To summarize, under a general distribution of consumer valuation, both the demand induction and intra-consumer discrimination effects still exist.

Appendix D: The Effect of T

This section explores how the firm's optimal profit and credit refund policy change with respect to the exogenous expiration term T. Figure D.1 uses the same parameter values as Table 1.

Figure D.1(a) shows the profit of the optimal credit refund policy under exogenous expiration terms between 1 and 20. As a baseline, we also plot the profit of the optimal cash refund policy, which will be discussed in Section 6. Moreover, there exists an intermediate expiration term that maximizes the firm's profit. This observation can be explained as follows. On the one hand, if the expiration term is too long, then according to Theorem 1, a low-valuation consumer would delay her purchase until her credit is close to expiration, which decreases the consumer's purchase probability. On the other hand, if the expiration term is too short, the consumer faces a rather

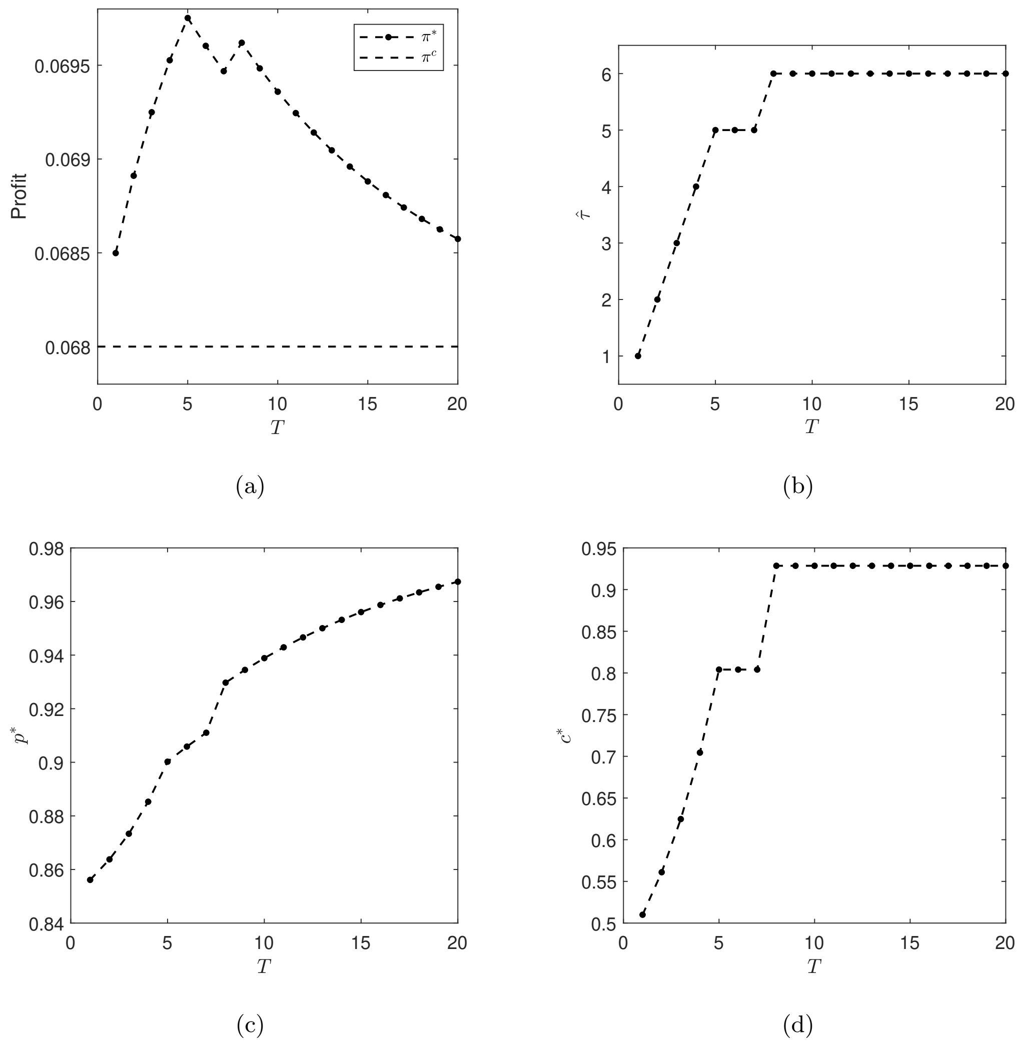


Figure D.1 The firm's optimal decisions vs T

high risk of credit expiration, which would make the credit less valuable to the consumer. Hence, an intermediate expiration term balancing the two effects is optimal.

Figure D.1(b) shows the threshold below which a low-valuation consumer is induced to make a purchase under each exogenous expiration term. For example, when $T = 7$ (8), a low-valuation consumer makes a purchase only when the credit is within 5 (6) periods from expiration, while for all $T \geq 10$, the value of the threshold $\hat{\tau}$ does not change. Notice that the threshold remains relatively small, even under a long expiration term. This is because low-valuation consumers make a purchase only when they face a high risk of credit expiration. Figure D.1(c) shows that the optimal price increases in the expiration term because a longer expiration term increases the value of a credit refund, and allows the firm to charge a higher price. A higher price is often paired with a higher credit, implying that the optimal credit increases in the expiration term as well, which is confirmed in Figure D.1(d).

Appendix E: Deterministic Consumer Valuation

In the main model, we assume that the consumer's valuation is stochastic, i.e., there is a probability α of taking a high valuation, and $1-\alpha$ of taking a low valuation. What if the consumer's valuation is deterministic? Will the credit refund policy still outperform the cash refund policy? To explore this question, we assume that each consumer has a fixed valuation for the product: a fraction α of the consumers takes a high valuation v_H, whereas the rest takes a low valuation v_L. As before, we first analyze the consumers' decision problem and then derive the firm's operational decisions.

The consumers' decision problem can be formulated as an infinite-horizon average reward dynamic program (see, e.g., Puterman 1994). For a generic consumer with a valuation $v \in \{v_H, v_L\}$, the optimality equations are given by

$$\rho^* + J(t) = \lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v) - p + c, J(t-1)\} + (1-\lambda)J(t-1), \quad \forall t = 1, \ldots, T, \tag{E.1}$$

$$\rho^* + J(0) = \lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v) - p, J(0)\} + (1-\lambda)J(0). \qquad t = 0. \tag{E.2}$$

Lemma E.1. *Suppose $\gamma[1-(1-\lambda)^T]c + (1-\gamma)v - p \geq 0$, then it is optimal for the consumer to make a purchase whenever she is in the market, and a solution to the optimality equations* (E.1)-(E.2) *is given by*

$$J(t) = [1-(1-\lambda)^t]c, \qquad \forall t = 0, 1, \ldots, T, \tag{E.3}$$

$$\rho^* = \lambda\big\{\gamma[1-(1-\lambda)^T]c + (1-\gamma)v - p\big\}. \tag{E.4}$$

Note that $[1-(1-\lambda)^T]c$ is the expected utility of an earned credit upon a product return, and v is the utility the consumer collects if she makes a purchase and does not return the product. Hence, $\gamma[1-(1-\lambda)^T]c + (1-\gamma)v$ is the expected utility of a purchase. Lemma E.1 indicates that a consumer always makes a purchase when she arrives to the market if the expected utility of a purchase is greater than the price. Therefore, the state transition, balance equations, and stationary probabilities are exactly the same as those in Case I in Appendix B.1.1.

Because $v \in \{v_H, v_L\}$, the optimal price

$$p^* \in \Big\{\gamma[1-(1-\lambda)^T]c + (1-\gamma)v_H, \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L\Big\}.$$

If $p^* = \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_H$, then only high-valuation consumers make a purchase. Note that each consumer pays a full price in state 0 and a price minus the credit in other states, so the corresponding revenue is

$$\pi = \lambda\alpha\{q_0 p^* + (1-q_0)(p^*-c)\} = \lambda(1-\gamma)\alpha v_H.$$

Otherwise, all consumers make a purchase, and the corresponding revenue is[14]

$$\pi = \lambda\{q_0 p^* + (1-q_0)(p^* - c)\} = \lambda(1-\gamma)v_L.$$

Therefore, the optimal revenue takes $\max\{\lambda(1-\gamma)v_L, \lambda(1-\gamma)\alpha v_H\}$. Clearly, in this deterministic setting, a credit refund policy cannot improve the firm's profit.

PROPOSITION E.1. *If the consumer's valuation is deterministic, then a credit refund policy, compared to a cash refund policy, cannot improve the firm's profit.*

E.1 Proofs

Before proving Lemma E.1, we first introduce a lemma.

LEMMA E.2. *If* $\gamma J(T) + (1-\gamma)[J(0)+v] - p \geq J(0)$, *then*

$$\gamma J(T) + (1-\gamma)[J(0)+v] - p + c \geq J(t-1)$$

for any $1 \leq t \leq T$.

Proof. Due to the monotonicity of $J(\cdot)$, it suffices to show

$$\gamma J(T) + (1-\gamma)[J(0)+v] - p + c \geq J(T-1).$$

According to (E.2),

$$\rho^* + J(0) = \lambda\big\{\gamma J(T) + (1-\gamma)[J(0)+v] - p\big\} + (1-\lambda)J(0) \geq J(0),$$

where the last inequality holds due to the supposition. Hence, we must have

$$\rho^* \geq 0. \tag{E.5}$$

Suppose for a contradiction that

$$\gamma J(T) + (1-\gamma)[J(0)+v] - p + c < J(T-1),$$

then according to (E.1),

$$\rho^* + J(T) = \lambda J(T-1) + (1-\lambda)J(T-1) = J(T-1),$$

implying $\rho^* < 0$ because $J(T) > J(T-1)$, which contradicts (E.5). This completes the proof. ■

[14] In Appendix B.1.1, we show that $q_0 = 1 - \gamma[1-(1-\lambda)^T]$.

Proof of Lemma E.1

According to Lemma E.2, if $\gamma J(T)+(1-\gamma)[J(0)+v]-p \geq J(0)$, then

$$\gamma J(T)+(1-\gamma)[J(0)+v]-p+c \geq J(t-1)$$

for any $1 \leq t \leq T$. That is, if the consumer makes a purchase in state 0, then she always makes a purchase in other states. Hence, the dynamic programming equations (E.1)-(E.2) can be rewritten as follows,

$$\begin{aligned}
\rho^*+J(t) &= \lambda\big\{\gamma J(T)+(1-\gamma)[J(0)+v]-p+c\big\}+(1-\lambda)J(t-1), && \forall t=1,\ldots,T,\\
\rho^*+J(0) &= \lambda\big\{\gamma J(T)+(1-\gamma)[J(0)+v]-p\big\}+(1-\lambda)J(0), && t=0.
\end{aligned}$$

Solving the above set of equations gives

$$\begin{aligned}
J(t) &= [1-(1-\lambda)^t]c,\\
\rho^* &= \lambda\big\{\gamma[1-(1-\lambda)^T]c+(1-\gamma)v-p\big\}.
\end{aligned}$$

Putting $J(0)$ and $J(T)$ into the initial condition $\gamma J(T)+(1-\gamma)[J(0)+v]-p \geq J(0)$ yields

$$\gamma[1-(1-\lambda)^T]c+(1-\gamma)v-p \geq 0.$$

This completes the proof. ■

Appendix F: Breakage Rate

This section takes a closer look at a managerially important quantity, breakage rate, which receives considerable attention from practitioners. The breakage rate measures the proportion of the earned credit lost due to expiration. We let $B(\lambda,\gamma,\alpha,T^*)$ denote the breakage rate under an optimal credit refund policy with an endogenous expiration term.

To better understand the breakage rate[15], consider its complement, the redemption rate denoted by $E(\lambda,\gamma,\alpha,T^*)$, which is the ratio between the credit redeemed by the consumer and the credit issued by the firm. Suppose the optimal expiration term T^* takes T_i or T_i+1 for some $i \geq 0$. At optimality, a high-valuation consumer always makes a purchase, while a low-valuation consumer makes a purchase only in states $\{1,2,\ldots,T^*-i\}$. The firm issues a credit when the consumer makes a purchase and then returns the product. Therefore, the total credit issued by the firm is

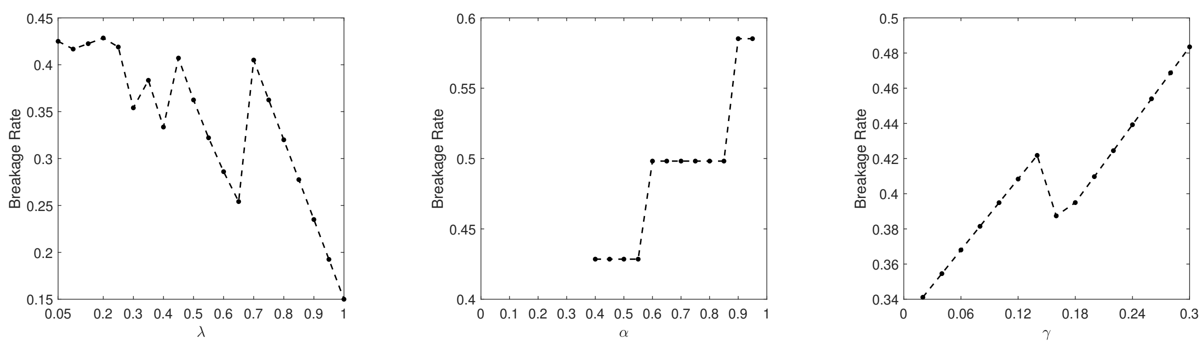
[15] We only discuss the breakage rate for Cases II and III. In Case I, the consumer makes a purchase in each state, regardless of her valuation realization, and thus the breakage rate is $(1-\lambda)^T$. However, the optimal expiration term can take any positive integers, so it is not meaningful to discuss the breakage rate for this case. Case IV is never optimal.

$\lambda\gamma\Big(\alpha q_0+\sum_{j=1}^{T^*-i}q_j+\alpha\sum_{j=T^*-i+1}^{T^*}q_j\Big)$. The consumer utilizes a credit when she makes a purchase with a credit on hand and does not return the product, so the total credit used by the consumer is $\lambda(1-\gamma)\Big(\sum_{j=1}^{T^*-i}q_j+\alpha\sum_{j=T^*-i+1}^{T^*}q_j\Big)$. It follows that

$$E(\lambda,\gamma,\alpha,T^*)=\frac{\lambda(1-\gamma)\Big(\sum_{j=1}^{T^*-i}q_j+\alpha\sum_{j=T^*-i+1}^{T^*}q_j\Big)}{\lambda\gamma\Big(\alpha q_0+\sum_{j=1}^{T^*-i}q_j+\alpha\sum_{j=T^*-i+1}^{T^*}q_j\Big)}$$
$$=\frac{(1-\gamma)[1-(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}]}{1-(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*}+(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}}.$$

Hence,

$$B(\lambda,\gamma,\alpha,T^*)=1-\frac{(1-\gamma)[1-(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}]}{1-(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*}+(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}}.$$



(a) (b) (c)

Figure F.1 Breakage Rate

Figure F.1 uses the same parameter values as in Figure G.1 and depicts how the breakage rate changes with λ, α, and γ, respectively, under an endogenous expiration term. One may expect that as the arrival rate λ increases, it is less likely for the credit to expire, so the breakage rate should decrease. However, Figure F.1(a) shows that the breakage rate is not monotonic with respect to λ. The reason is that as λ increases, the optimal expiration term decreases (as shown in Figure G.1(a)), which increases the chance of credit expiration. These two counteracting effects coexist, and it is not clear which effect dominates as the arrival rate varies.

One may also expect that as the consumer's willingness to pay increases, her purchase probability also increases; thus, the breakage rate should decrease. However, Figure F.1(b) shows that the breakage rate may increase in α. The reason is that, although a consumer's willingness to pay increases, the number of credit issued by the firm also increases. Moreover, as shown in Figure G.1(b), the optimal expiration term decreases in α, leading to an increase in the chance of credit expiration. As discussed in Figure G.1(b), the optimal expiration term is not fixed when α is small and $\alpha = 1$, so we do not discuss the breakage rate for these α values.

Figure F.1(c) shows that the breakage rate is not monotonic with respect to γ. On one hand, as the return rate increases, the probability of utilizing the credit decreases, leading to an increase in the breakage rate. On the other hand, the optimal expiration term may increase in the return rate, resulting in a declining breakage rate. Again, these two counteracting effects coexist, and thus the breakage rate is not monotonic in γ. As discussed in Figure G.1(c), the optimal expiration term is not fixed when $\gamma = 0$, so we do not discuss the breakage rate when $\gamma = 0$.

Appendix G: Sensitivity Analysis

For the firm's problem with endogenous expiration term, we are interested in how the optimal profit and expiration term change with the parameters λ, α, and γ. Our numerical investigation shows that the optimal expiration term often takes the value T_0. Therefore, we first discuss how T_0 changes with the parameters.

PROPOSITION G.1. *The value of T_0 decreases in λ and α, and increases in γ.*

Proof of Proposition G.1

Recall

$$T_0 = \max\{T \in \mathbb{N} : A(T) \leq B(T,T)\},$$

where

$$A(T) = (1-\gamma)(v_H - v_L)\Big\{\frac{\alpha}{(1-\lambda)^{T-1}} + 1 - \alpha\Big\},$$

$$B(T,T) = \frac{(1-\gamma)v_H - (1-\gamma)(v_H - v_L)\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma[1-(1-\lambda)^T]}.$$

One can check

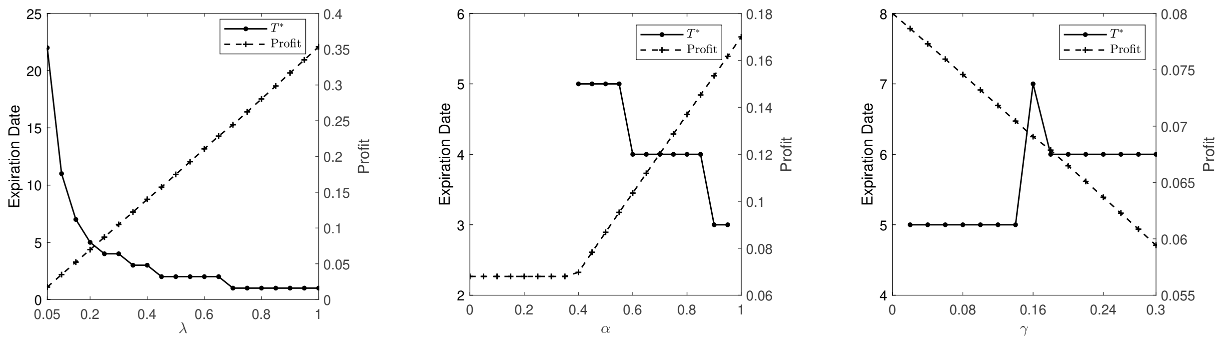
$$A(T) - B(T,T) = \frac{(1-\gamma)}{1-\gamma[1-(1-\lambda)^T]}\Big\{\alpha(1-\gamma)[1-(1-\lambda)^{T-1}](v_H - v_L) - (1-\lambda)^{T-1}v_L\Big\}.$$

Let

$$L = \alpha(1-\gamma)[1-(1-\lambda)^{T-1}](v_H - v_L) - (1-\lambda)^{T-1}v_L.$$

Note that L increases in T and T_0 is the maximum integer that keeps L negative, so in order to show T_0 decreases in λ and α and increases in γ, it suffices to show L increases in λ and α and decreases in γ. One can verify that L indeed increases in λ and α and decreases in γ. This completes the proof. ■

Proposition G.1 verifies that T_0 decreases in λ and α and increases in γ. Therefore, the optimal expiration term may decrease as the arrival rate increases, consumers' willingness to pay increases, or the return rate decreases. Intuitively, as the arrival rate increases, it is less likely for an earned credit to expire, and the firm should decrease the expiration term. A similar argument can be applied to consumers' willingness to pay and the return rate.



(a) (b) (c)

Figure G.1 The sensitivity of profit and expiration date with respect to α, γ, and λ.

We illustrate the sensitivity results using the following numerical example. Figure G.1(a) sets $(\alpha, \gamma, v_H, v_L) = (0.4, 0.15, 1, 0.4)$ and varies λ from 0.05 to 1, with a step size of 0.05. Unsurprisingly, the profit increases in λ. As the arrival rate increases, consumers' purchase probabilities increase, leading to an increase in the profit. The optimal expiration term T^* decreases in λ. We numerically verify that, in this particular example, the optimal expiration term under each λ is T_0, which decreases in λ according to the earlier discussion.

Figure G.1(b) sets $(\lambda, \gamma, v_H, v_L) = (0.2, 0.15, 1, 0.4)$ and varies α from 0 to 1, with a step size of 0.05. Recall that α is the probability that a consumer's valuation is v_H. As α increases, consumers' valuation increases, and thus, the profit increases. Moreover, in this particular example, when $\alpha \leq 0.35$, Case I is optimal and the optimal expiration term can take any positive integers (see Proposition B.1 in Appendix B.1.1). Hence, we do not plot T^* in this range. When $0.4 \leq \alpha \leq 0.95$, Case II is optimal, and the optimal expiration term under each α is T_0. Thus, T^* decreases in α in this range. When $\alpha = 1$, all consumers are high-valuation consumers. The optimal profit is $\lambda(1-\gamma)v_H$ and the optimal expiration term can take any positive integers (see the analysis in Appendix E).

Figure G.1(c) sets $(\lambda, \alpha, v_H, v_L) = (0.2, 0.4, 1, 0.4)$ and varies γ from 0 to 0.3, with a step size of 0.02. The upper bound of 0.3 is informed by empirical evidence indicating that the product return rate is typically less than 30%. Note that the firm earns a lower profit if a consumer returns the product more frequently, so the profit decreases as the return rate increases. Surprisingly, T^* is not monotone in γ. In particular, T^* takes the value of T_1 when $\gamma = 0.16$, but takes the value of T_0 for all other values of γ. When $\gamma = 0$, consumers never return the product. According to Theorem 1, the optimal profit is $\max\{\lambda\alpha v_H, \lambda v_L\}$, and the optimal expiration term can take any positive integer no more than T_0. Hence, we do not plot T^* when $\gamma = 0$.