Appendix A: Proofs

Proof of Lemma 1. Fix $X_1,\dots X_N$. Suppose that if the consumer acts optimally, she purchases ℓ products, where $\ell \in \{0,\dots,N\}$. Given that she purchases ℓ products, it is obvious that her utility-maximizing choice of ℓ products are the ℓ products with the highest price-adjusted valuations, that is $\sigma(1),\dots,\sigma(\ell)$. Therefore, the consumer's problem can be written as $\max_{\ell=0,\dots,N}\left\{\sum_{i=1}^{\ell}\left(v(i-1)+X_{\sigma(i)}-p_{\sigma(i)}\right)\right\}$. Define $L := \{1,\dots,\ell\}$, and consider

$$\begin{aligned}
&\sum_{i=1}^{D^1}\left[v(i-1)+X_{\sigma(i)}-p_{\sigma(i)}\right]-\sum_{i=1}^{\ell}\left[v(i-1)+X_{\sigma(i)}-p_{\sigma(i)}\right]\\
&=\sum_{i\in S^f\setminus L}\underbrace{\left[v(i-1)+X_{\sigma(i)}-p_{\sigma(i)}\right]}_{\geq 0}-\sum_{i\in L\setminus S^f}\underbrace{\left[v(i-1)+X_{\sigma(i)}-p_{\sigma(i)}\right]}_{\leq 0}\geq 0,
\end{aligned}$$

implying that her utility is maximized by only purchasing products i that have $\sigma(i)\leq D^1$. Q.E.D.

Proof of Lemma 2. We prove both bounds by induction on n. When $n=0$, because $R_k(0)=0$ for each k, (8) holds trivially noting that $\rho_0^*=0$ by construction. Now suppose that (8) holds for some n and every $k=0,\dots,N-n-1$. We aim to show that (8) holds when there are $n+1$ periods remaining (this is the period when the product with random valuation X_{N-n} is observed). Fix any $k=0,\dots,N-n-2$, and define the following events:

$$\begin{aligned}
A &:= \{X_{N-n}>p_{N-n}-v(k+1)\},\\
B &:= \{p_{N-n}-v(k)<X_{N-n}\leq p_{N-n}-v(k+1)\},\\
C &:= \{X_{N-n}\leq p_{N-n}-v(k)\}.
\end{aligned}$$

Note that $\{A,B,C\}$ form a partition of the probability space. To prove the lower bound,

$$\begin{aligned}
&R_{k+1}(n+1)-R_k(n+1)\\
&=\mathbb{1}_A\left(\underbrace{R_{k+2}(n)-R_{k+1}(n)}_{\leq 0}\right)+\mathbb{1}_B\left(\underbrace{R_{k+1}(n)-R_{k+1}(n)}_{=0}-p_{N-n}\right)+\mathbb{1}_C\left(\underbrace{R_{k+1}(n)-R_k(n)}_{\leq 0}\right)\\
&\leq 0,
\end{aligned}$$

where we have applied the induction hypothesis to establish that the first and third terms are bounded from above by 0. To prove the upper bound,

$$\begin{aligned}
&\rho_{n+1}^*+R_{k+1}(n+1)-R_k(n+1)\\
&=\rho_{n+1}^*+\mathbb{1}_A\left(\underbrace{R_{k+2}(n)-R_{k+1}(n)}_{\geq -\rho_n^*}\right)+\mathbb{1}_B\left(\underbrace{R_{k+1}(n)-R_{k+1}(n)}_{=0}-p_{N-n}\right)+\mathbb{1}_C\left(\underbrace{R_{k+1}(n)-R_k(n)}_{\geq -\rho_n^*}\right)\\
&\geq 0,
\end{aligned}$$

where we have applied the induction hypothesis to establish that the first and third terms are bounded from below by $-\rho_n^*$ with probability 1. The final inequality follows because we may write $\rho_{n+1}^*=\max\{p_{N-n},\rho_n^*\}$, which implies the result. Q.E.D.

Proof of Lemma 3. The expected revenues of the original and interchanged sequences are the same from periods 1 through $j-1$, and therefore, it suffices to compare the expected residual revenue of each sequence from periods j through N. For notational convenience, throughout the proof, define $n := N-j-1$. Also, denote expected revenue for periods j through N as $\mathbb{E}\left[R_k^{orig}(n+2)\right]$ and $\mathbb{E}\left[R_k^{swap}(n+2)\right]$ for the original and interchanged product sequences.

For notational brevity, for $i = j, j+1$, define

$$\alpha_i := \mathbb{P}\left(X_i > p_i - v(k)\right) \qquad \text{and} \qquad \widetilde{\alpha}_i := \mathbb{P}\left(X_i > p_i - v(k+1)\right)$$

By the assumption that v is decreasing, we have $0 \leq \widetilde{\alpha}_i \leq \alpha_i \leq 1$. Note that by construction, we have $\beta_i \alpha_i = \widetilde{\alpha}_i$, even if $\alpha_i = 0$.

For the original sequence, expanding the expression for the retailer's expected revenue by considering cases, we have

$$\begin{aligned}\mathbb{E}\left[R_k^{orig}(n+2)\right] :=\ & \alpha_j p_j + \alpha_j\left(\widetilde{\alpha}_{j+1} p_{j+1} + \widetilde{\alpha}_{j+1}\mathbb{E}\left[R_{k+2}(n)\right] + (1-\widetilde{\alpha}_{j+1})\mathbb{E}\left[R_{k+1}(n)\right]\right) \\ & + (1-\alpha_j)\left(\alpha_{j+1} p_{j+1} + \alpha_{j+1}\mathbb{E}\left[R_{k+1}(n)\right] + (1-\alpha_{j+1})\mathbb{E}\left[R_k(n)\right]\right)\end{aligned}$$

The first term is the retailer's expected revenue from the j^{th} item; the second term is the expected revenue from the $(j+1)^{th}$ item onwards conditional on the customer purchasing item j; the third and final term is the corresponding expected revenue conditional on the customer not purchasing item j.

In the interchanged sequence, we have a similar expression

$$\begin{aligned}\mathbb{E}\left[R_k^{swap}(n+2)\right] :=\ & \alpha_{j+1} p_{j+1} + \alpha_{j+1}\left(\widetilde{\alpha}_j p_j + \widetilde{\alpha}_j\mathbb{E}\left[R_{k+2}(n)\right] + (1-\widetilde{\alpha}_j)\mathbb{E}\left[R_{k+1}(n)\right]\right) \\ & + (1-\alpha_{j+1})\left(\alpha_j p_j + \alpha_j\mathbb{E}\left[R_{k+1}(n)\right] + (1-\alpha_j)\mathbb{E}\left[R_k(n)\right]\right)\end{aligned}$$

We note that interchanging j and $j+1$ have no effect on the distributions of $R_k(n), R_{k+1}(n)$ or $R_{k+2}(n)$. Therefore, the difference $\mathbb{E}\left[R_k^{orig}(n+2)\right] - \mathbb{E}\left[R_k^{swap}(n+2)\right]$ can be obtained by taking the difference and eliminating common terms, to get

$$\begin{aligned}&\mathbb{E}\left[R_k^{orig}(n+2)\right] - \mathbb{E}\left[R_k^{swap}(n+2)\right] \\ &\quad = \alpha_j \widetilde{\alpha}_{j+1}\left(p_{j+1} + \mathbb{E}\left[R_{k+2}(n)\right] - \mathbb{E}\left[R_{k+1}(n)\right]\right) - \widetilde{\alpha}_j \alpha_{j+1}\left(p_j + \mathbb{E}\left[R_{k+2}(n)\right] - \mathbb{E}\left[R_{k+1}(n)\right]\right) \\ &\qquad - \alpha_j \alpha_{j+1}(p_{j+1} - p_j) \\ &\quad = \alpha_j \alpha_{j+1}\beta_{j+1}\left(p_{j+1} + \Delta_{k+1}(n)\right) - \alpha_j \alpha_{j+1}\beta_j\left(p_j + \Delta_{k+1}(n)\right) + \alpha_j\alpha_{j+1}\left(p_j - p_{j+1}\right) \\ &\quad = \alpha_j\alpha_{j+1}\left[(1-\beta_j)p_j - \beta_j\Delta_{k+1}(n) - (1-\beta_{j+1})p_{j+1} + \beta_{j+1}\Delta_{k+1}(n)\right],\end{aligned}$$

where the second equality is because $\widetilde{\alpha}_i = \beta_i\alpha_i, \forall i = j, j+1$. Parts (a) and (b) follow directly from the expression above. Q.E.D.

Proof of Corollary 1. The proof will proceed by first showing that such a sequence of products satisfies condition (11) in Lemma 3(b). Fix an arbitrary period j, and assume that for some $k \leq j-1$, the consumer had purchased k products before that period. We assume WLOG that $\mathbb{P}\left(X_i > p - v(k)\right) > 0, i = j, j+1$; otherwise, there is nothing to prove.

By assumption, we have $X_j \leq_{hr} X_{j+1}$, which, by (10), implies that $\beta_j \leq \beta_{j+1}$. Further,

$$\Delta_{k+1}(N-j-1) + p = \mathbb{E}\left[R_{k+2}(N-j-1)\right] - \mathbb{E}\left[R_{k+1}(N-j-1)\right] + p \geq 0, \tag{22}$$

where the equality is by definition (9) and the inequality is by Lemma 2, using the fact that all prices are equal to p. Thus, considering the LHS of (11),

$$\begin{aligned} & p_j(1-\beta_j) - p_{j+1}(1-\beta_{j+1}) + (\beta_{j+1}-\beta_j)\Delta_{k+1}(N-j-1) \\ &= p(\beta_{j+1}-\beta_j) + (\beta_{j+1}-\beta_j)\Delta_{k+1}(N-j-1) \qquad [p_i = p, \forall i] \\ &= (\beta_{j+1}-\beta_j)\left(\Delta_{k+1}(N-j-1)+p\right) \\ &\geq 0, \end{aligned}$$

where the final inequality is by $\beta_j \leq \beta_{j+1}$ and (22).

Given that such a sequence of products satisfies condition (11) in Lemma 3(b), we can conclude that interchanging any two consecutive products leads to a suboptimal sequence. It remains to be shown that interchanging any two non-consecutive products also leads to a suboptimal sequence. Consider products in the original sequence j and $j' > j+1$. We will show that by swapping j and j', condition (11) in Lemma 3(b) no longer holds for period j (now occupied by product j'), implying that it is a suboptimal sequence.

The LHS of (11) for period j after the swap becomes

$$\begin{aligned} & p(1-\beta_{j'}) - p(1-\beta_{j+1}) + (\beta_{j+1}-\beta_{j'})\Delta_{k+1}(N-j-1) \\ &= p(\beta_{j+1}-\beta_{j'}) + (\beta_{j+1}-\beta_{j'})\Delta_{k+1}(N-j-1) \\ &= (\beta_{j+1}-\beta_{j'})\left(\Delta_{k+1}(N-j-1)+p\right) \\ &\leq 0, \end{aligned}$$

where the inequality follows from $\beta_{j+1} \leq \beta_{j'}$ (transitive property of hazard rate ordering) and $\Delta_{k+1}(N-j-1)+p \geq 0$ from (22). Q.E.D.

Proof of Corollary 2. We prove this by induction on N. It suffices to prove the inductive step. For some $N \geq 1$, inductively assume that the statement is true for $N-1$. Specifically, we inductively assume that products $2, 3, \ldots, N$ are ordered such that $p_2 \geq p_3 \geq \ldots \geq p_N$, and this is an optimal order for those sequence of products, regardless of whether the consumer has purchased one or zero products by period 2. Now, the upper bound of Lemma 2 implies that

$$\Delta_1(N-2) + p_2 \geq 0. \tag{23}$$

Now suppose that $p_1 \geq p_2$. We aim to show that the current product ordering for all N products is optimal. Because X_i are identically distributed according to an IFR distribution, we note that Shaked and Shanthikumar (2007, Theorem 1.A.30) implies $\frac{\mathbb{P}(X_1 > p_1 - v(1))}{\mathbb{P}(X_1 > p_1 - v(0))} \leq \frac{\mathbb{P}(X_2 > p_2 - v(1))}{\mathbb{P}(X_2 > p_2 - v(0))}$. That is, $\beta_1 \leq \beta_2$.

Consider the LHS of (11) for $j = 1$, which, after rearranging, is $(p_1 - p_2)(1-\beta_1) + (\beta_2 - \beta_1)(\Delta_1(N-2) + p_2) \geq 0$, where the inequality holds as a consequence of (23), $\beta_1 \leq \beta_2$, and the assumption that $p_1 \geq p_2$.

Given that such a sequence of products satisfies condition (11) in Lemma 3(b), we can conclude that interchanging any two consecutive products leads to a suboptimal sequence. It remains to be shown that interchanging any two non-consecutive products also leads to a suboptimal sequence. Consider products in the original sequence j and $j' > j+1$. We will show that by swapping j and j', condition (11) in Lemma 3(b) no longer holds for period j (now occupied by product j'), implying that it is a suboptimal sequence.

The LHS of (11) for period j after the swap becomes

$$\begin{aligned}
&p_{j'}(1-\beta_{j'}) - p_{j+1}(1-\beta_{j+1}) + (\beta_{j+1}-\beta_{j'})\Delta_{k+1}(N-j-1)\\
&\le p_{j+1}(\beta_{j+1}-\beta_{j'}) + (\beta_{j+1}-\beta_{j'})\Delta_{k+1}(N-j-1)\\
&= (\beta_{j+1}-\beta_{j'})\left(\Delta_{k+1}(N-j-1)+p_{j+1}\right)\\
&\le 0,
\end{aligned}$$

where the first inequality is because $p_{j'} \le p_{j+1}$ and the final inequality follows from $\beta_{j+1} \le \beta_{j'}$ (property of IFR distribution; Shaked and Shanthikumar (2007, Theorem 1.A.30)) and $\Delta_{k+1}(N-j-1)+p_{j+1} \ge 0$ from the upper bound in Lemma 2. Q.E.D.

Before proving Theorem 1, we first state an intermediate result that is used in the proof:

LEMMA 4. *Suppose that $p_1 \ge \max\{p_2,\dots,p_N\}$. Let R represent the retailer's revenue under the original sequence of product prices and valuations $(p_1,X_1),(p_2,X_2),\dots,(p_N,X_N)$ and $\hat{R}$ the revenue that interchanges both the prices and realized valuations of products 1 and 2 in the event that $X_1-p_1 \le X_2-p_2$. Then, $R \ge \hat{R}$.*

Proof of Lemma 4. Define the following events

$$\begin{aligned}
A_{11} &:= \{X_2-p_2 > v(0), X_1-p_1 > v(1)\}, \; A_{10} := \{X_2-p_2 > v(0), X_1-p_1 \le v(1)\},\\
A_{01} &:= \{X_2-p_2 \le v(0), X_1-p_1 > v(0)\}, \; A_{00} := \{X_2-p_2 \le v(0), X_1-p_1 \le v(0)\},\\
B &:= \{X_2-p_2 \ge X_1-p_1\}
\end{aligned}$$

Note that $A_{11}, A_{10}, A_{01}, A_{00}$ form a partition of the probability space. By construction,

$$R-\hat{R} = \left(R-\hat{R}\right)\mathbb{1}_B. \tag{24}$$

Furthermore, by inspection, we have

$$\mathbb{P}(A_{01}\cap B)=0, \quad \left(R-\hat{R}\right)\mathbb{1}_{A_{11}\cap B}=0, \quad \text{and} \quad \left(R-\hat{R}\right)\mathbb{1}_{A_{00}\cap B}=0 \tag{25}$$

Therefore, from (24) and (25), we have $R-\hat{R} = \left(R-\hat{R}\right)\mathbb{1}_{A_{10}\cap B}$.

Recall that $R_k(n)$ denotes the revenue from the purchases in the last n remaining periods with k products already purchased. For the interchanged sequence, we have $\hat{R}\mathbb{1}_{A_{10}\cap B} = (p_2+R_1(N-2))\,\mathbb{1}_{A_{10}\cap B}$. Define the events $C := \{X_1-p_1 > v(0)\}$ and $D := \{X_2-p_2 > v(1)\}$. For the original sequence, by considering cases, we get:

$$\begin{aligned}
R\mathbb{1}_{A_{10}\cap B} &= (p_1+R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D^c} + (p_1+p_2+R_2(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D}\\
&\quad + (p_2+R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C^c}
\end{aligned}$$

Therefore, putting these together, we have

$$\begin{aligned}
R-\hat{R} &= (p_1+R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D^c} + (p_1+p_2+R_2(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D}\\
&\qquad - (p_2+R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C}\\
&= (p_1+R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D^c} + (p_1+R_2(N-2)-R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D}\\
&\qquad - (p_2+R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D^c}\\
&= (p_1-p_2)\,\mathbb{1}_{A_{10}\cap B\cap C\cap D^c} + (p_1+R_2(N-2)-R_1(N-2))\,\mathbb{1}_{A_{10}\cap B\cap C\cap D}\\
&\ge 0,
\end{aligned}$$

where the final inequality is by Lemma 2, and the assumption that $p_1 \ge \max\{p_2,\dots,p_N\}$. Q.E.D.

Proof of Theorem 1. To prove (a), let $X_1, \ldots, X_M$ represent the realized valuation shocks from the M products. With a slight abuse of notation, let $\sigma(\cdot)$ be a permutation of $\{1, \ldots, M\}$ so that

$$X_{\sigma(1)} - p_{\sigma(1)} \geq X_{\sigma(2)} - p_{\sigma(2)} \geq \ldots \geq X_{\sigma(M)} - p_{\sigma(M)}.$$

It follows from an immediate generalization of Lemma 1 that consumers, when presented with a fixed assortment, will purchase products in the order specified by σ. Therefore, part (a) will hold if there exists a set of successive pairwise interchanges of products (both prices and valuations), starting from the original sequence of products (under which the retailer earns revenue R), that satisfies two conditions: (i) the set of successive interchanges terminates with the permutation $(\sigma(1), \ldots, \sigma(M))$, and (ii) the retailer's revenue decreases (with probability 1) on each interchange. If these hold, then at the end of this chain of interchanges, the retailer's revenue would coincide with R^f, and we have shown $R \geq R^f$.

We will show that such a sequence of interchanges exists by induction on the last m products, for $m = 2, \ldots, M$. First consider the case that $m = 2$. Then, the only interchange that is potentially necessary to satisfy (i) is to swap the two products, in which case, (ii) follows from Lemma 4 (stated immediately preceding this proof).

Now, inductively assume that for some $m-1$, there exists a set of successive interchanges made such that these last $m-1$ products are sorted according to the permutation σ, and each interchange decreases the retailer's revenue. Assume therefore that this set of interchanges has been performed. Notice, however, that these interchanges (to the last $m-1$ products) do not change the position of the mth product from the end, which, by assumption, also has the highest price of all the last m products. For brevity, let us call this the "critical product". Since the last $m-1$ products have already been sorted, it suffices to additionally perform a finite number (at most m) additional pairwise interchanges to ensure that all m products are sorted by (i). Each interchange involves this critical product, which has the highest price of the last m products. Hence, each of these interchanges satisfies the conditions of Lemma 4, which implies that the retailer's revenue decreases with each interchange, thus satisfying (ii). Therefore, by induction, the proof of (a) is complete.

To prove part (b), we observe that $\mathbb{E}[R^s_*] \geq \mathbb{E}[R] \geq \mathbb{E}[R^f_*]$, where the first inequality is because R^s_* represents the expected revenue from an optimized sequential assortment with N products, which, by definition, exceeds the expected revenue from a suboptimal sequential assortment with $M \leq N$ products. The second inequality is a consequence of part (a). Q.E.D.

Proof of Theorem 2. For part (a), suppose that $v(k) = v, k = 0, \ldots, N-1$. We first show that if the retailer chooses the fixed assortment, it is optimal to also make all N products available. Suppose WLOG that the retailer chooses the first $M < N$ products to be sold in the fixed assortment. By (4), the retailer's revenue from including these M products is $R^f(M) := \sum_{i=1}^M p_{\sigma(i)} \mathbb{1}_{\{v + X_{\sigma(i)} - p_{\sigma(i)} \geq 0\}} = \sum_{i=1}^M p_i \mathbb{1}_{\{v + X_i - p_i \geq 0\}}$. From this expression, we observe that the retailer's incremental revenue from including product $M+1$ would be $p_{M+1} \mathbb{1}_{\{v + X_{M+1} - p_{M+1} \geq 0\}} \geq 0$, and thus it would do so optimally do so, until all N products are included. For the sequential assortment, let $\widehat{\sigma}$ be an arbitrary permutation of $\{1, \ldots, N\}$ representing the retailer's product sequence. By (6) and Remark 1, and using $v(k) = v$,

$$R^s = \sum_{i=1}^N p_{\widehat{\sigma}(i)} \mathbb{1}_{\{v + X_{\widehat{\sigma}(i)} - p_{\widehat{\sigma}(i)} \geq 0\}} = \sum_{i=1}^N p_i \mathbb{1}_{\{v + X_i - p_i \geq 0\}} = R^f(N).$$

For part (b), suppose that $X_i = x$ with probability 1 and $p_i = p$ for all $i = 1, \dots, N$. As before, we first show that if the retailer chooses the fixed assortment, it is optimal to also make all N products available. Suppose WLOG that the retailer chooses the first $M < N$ products to be sold in the fixed assortment. By (4), the retailer's revenue from including these M products can be expressed as $R^f = \sum_{i=1}^M p \mathbb{1}_{\{v(i-1)+x-p\geq 0\}}$. From this expression, it is clear that the retailer gains nonnegative revenue from including product $M+1$ and so on until all N products are included. Next, recall the definition of D^f from Lemma 1. By construction, $v(i-1)+x-p \geq 0$ for all $i \leq D^f$ and $v(D^f)+x-p < 0$. Recall the definition of $\{W_i, i = 0, \dots, N\}$ via (6). We claim that $W_i = \min\{i, D^f\}$. This holds by definition for $i = 0$. Suppose it holds for some $i-1$, then, for $i \leq D^f$, we have $W_i = W_{i-1} + \mathbb{1}_{\{v(W_{i-1})+x-p\geq 0\}} = i - 1 + \mathbb{1}_{\{v(i-1)+x-p\geq 0\}} = i$; for $i > D^f$, we have $W_i = W_{i-1} + \mathbb{1}_{\{v(W_{i-1})+x-p\geq 0\}} = D^f + \mathbb{1}_{\{v(D^f)+x-p\geq 0\}} = D^f$, establishing the claim for all i. To conclude the proof, we note that in this case, (6) implies that we have $R^s = pW_N = p\min\{N, D^f\} = pD^f = R^f$. Q.E.D.

Proof of Theorem 3. For this proof, denote $v(\infty) := \lim_{k\to\infty} v(k)$, and note that $v(\infty)$ is finite because v is assumed to be decreasing and bounded from below. Furthermore, note that although we do not denote this explicitly, R^t for each $t \in \{f, s\}$ all depend on N. In addition, under the assumption of equal prices and a common valuation distribution of every product, it is easy to see that (i) it is optimal in the fixed assortment to make all products available, and (ii) any product sequence is optimal in the sequential assortment.

First, we prove part (a) for $t = s$. Construct the random variable B_N as $B_N := \sum_{i=1}^N \mathbb{1}_{\{v(\infty)+X_i-p\geq 0\}}$, where we have used the assumption that $p_i = p$ for all i. Furthermore, recalling the definition of the process $\{W_i\}_{i=0}^N$ from the construction (6), for an arbitrary K, construct the random variable B_N^K as $B_N^K := \sum_{i=1}^N \mathbb{1}_{\{v(\min\{W_{i-1},K\})+X_i-p\geq 0\}}$. Since $v(\min\{W_i, K\}) \geq v(W_i) \geq v(\infty)$, we have $B_N \leq W_N \leq B_N^K$, which, by (6), under the assumption that $p_i = p$, implies that

$$pB_N \leq R^s \leq pB_N^K. \tag{26}$$

Define the (integer-valued) random variable L_N^K as $L_N^K := \sup\{i \in \{1, \dots, N+1\} : W_{i-1} < K\}$, so that we can decompose B_N^K as

$$B_N^K = \sum_{i=1}^{L_N^K} \mathbb{1}_{\{v(W_{i-1})+X_i-p\geq 0\}} + \sum_{i=L_N^K+1}^{N} \mathbb{1}_{\{v(K)+X_i-p\geq 0\}}.$$

Since we have assumed that $X_i \overset{d}{=} X$, by the Strong Law of Large Numbers, we have $B_N/N \overset{a.s.}{\to} \delta$. From the lower bound of (26), we necessarily have $\lim_{i\to\infty} W_i = \infty$ with probability 1, which implies that $\lim_{N\to\infty} L_N^K$ is bounded almost surely. Hence,

$$\begin{aligned}
\lim_{N\to\infty} \frac{B_N^K}{N} &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{L_N^K} \mathbb{1}_{\{v(W_{i-1})+X_i-p\geq 0\}} + \lim_{N\to\infty} \frac{1}{N} \sum_{i=L_N^K+1}^{N} \mathbb{1}_{\{v(K)+X_i-p\geq 0\}} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{i=L_N^K+1}^{N} \mathbb{1}_{\{v(K)+X_i-p\geq 0\}} \qquad [\lim_{N\to\infty} L_N^K \text{ is a.s. bounded}] \\
&\leq \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{1}_{\{v(K)+X_{i+1}-p\geq 0\}} \\
&= \mathbb{P}(v(K) + X - p \geq 0)
\end{aligned}$$

where the final equality is again by the assumption that X_i have a common distribution and by the Strong Law of Large Numbers. Therefore, we have shown that with probability 1,

$$p\delta \le \lim_{N\to\infty} R^s/N \le p\mathbb{P}\left(v(k)+X-p\ge 0\right).$$

We complete the proof by sending K to $+\infty$ on the RHS.

To prove (a) for $t=f$, we note that B_N can be equivalently written as $B_N=\sum_{i=1}^N \mathbb{1}_{\left\{v(\infty)+X_{\sigma(i)}-p\ge 0\right\}}$, and we therefore have $pB_N\le R^f\le R^s$, where the upper bound follows from Theorem 1. Hence, the result follows from the case of $t=s$ by a sandwich argument.

To prove (b), consider $\hat{v}$ that is constructed as $\hat{v}(k):=\max\{v(k),v(N_0-1)\}$ for all k, and for each $t\in\{f,s\}$ let $\hat{R}^t$ represent the corresponding revenue when $\hat{v}$ is used instead of v. Since $\hat{v}\ge v$, we necessarily have $R^t\le\hat{R}^t$, and in particular, $\mathbb{1}_{\left\{\hat{R}^t\le pN_0\right\}}R^t=\mathbb{1}_{\left\{\hat{R}^t\le N_0\right\}}\hat{R}^t$. Therefore, noting that $R^t\le pN_0$ almost surely, we may write

$$R^t=\min\left\{\hat{R}^t,pN_0\right\}. \tag{27}$$

Notice that by construction of $\hat{v}$, we necessarily have $\lim_{N\to\infty}\mathbb{P}\left(X\ge p-\hat{v}(N-1)\right)=\lim_{N\to\infty}\mathbb{P}\left(X\ge p-v(N_0-1)\right)>0$, where the final inequality is by construction of N_0. This implies that we can apply part (a) for $\hat{R}^t$, which in turn implies that with probability 1, $\lim_{N\to\infty}\hat{R}^t=+\infty$. Therefore, using this with (27) implies that with probability 1, $\lim_{N\to\infty}R^t=pN_0$. Q.E.D.

Proof of Proposition 1. Throughout this proof, we will use the more compact notation $s_i^k:=s_i(k)$. We first prove a recursive expression for s_i^k. Note that we can rewrite the Bellman equation (12) as

$$U_i(k)=U_{i+1}(k)+\mathbb{E}\left[\left(X_i-s_i^k\right)_+\right]=U_{i+1}(k)+\int_{s_i^k}^{\infty}\overline{F}(u)du, \tag{28}$$

where $(z)_+:=\max\{z,0\}$, and the final equality is a result of the identity $\mathbb{E}\left[(X-x)_+\right]\equiv\int_x^\infty\overline{F}(u)du$. Now consider

$$\begin{aligned}
s_{i-1}^k-s_i^k &= U_i(k)-U_i(k+1)-U_{i+1}(k)+U_{i+1}(k+1) && \text{[By (14)]}\\
&= \int_{s_i^k}^{\infty}\overline{F}(u)du-\int_{s_i^{k+1}}^{\infty}\overline{F}(u)du && \text{[By (28)]}\\
&= \int_{s_i^k}^{s_i^{k+1}}\overline{F}(u)du
\end{aligned}$$

Therefore,

$$s_{i-1}^k=s_i^k+\int_{s_i^k}^{s_i^{k+1}}\overline{F}(u)du. \tag{29}$$

We prove part (a) by backward induction on i. For $i=N$, we have $s_N^k=s_N(k)=p-v(k)$, which is increasing in k. Inductively suppose that for some i, s_i^k is increasing in k for every k. Then, from (29), we have

$$\begin{aligned}
s_{i-1}^{k+1}-s_{i-1}^k &= \left(s_i^{k+1}-s_i^k\right)+\int_{s_i^{k+1}}^{s_i^{k+2}}\overline{F}(u)du-\int_{s_i^k}^{s_i^{k+1}}\overline{F}(u)du \quad \text{[By (29)]}\\
&= \int_{s_i^{k+1}}^{s_i^{k+2}}\overline{F}(u)du+\int_{s_i^k}^{s_i^{k+1}}(1-\overline{F}(u))du\\
&= \int_{s_i^{k+1}}^{s_i^{k+2}}\overline{F}(u)du+\int_{s_i^k}^{s_i^{k+1}}F(u)du\\
&\ge 0. && \text{[Induction hypothesis]}
\end{aligned}$$

Part (b) follows from (a) and expression (29).

For part (c), consider some valuation distribution G such that $\overline{G} \geq \overline{F}$; that is, G is stochastically larger than F. Denote $s_i^k(G)$ and $s_i^k(F)$ as the thresholds under valuation distributions G and F, respectively. Our goal is to show that for every i, k,

$$s_i^k(G) \geq s_i^k(F). \tag{30}$$

We prove (30) by induction on i. For $i = N$, $s_i^k(G) = s_i^k(F) = p - v(k)$, and (30) holds trivially. Inductively assume that this holds for some i and for every k. We have

$$\begin{aligned}
&s_{i-1}^k(G) - s_{i-1}^k(F) \\
&= s_i^k(G) - s_i^k(F) + \int_{s_i^k(G)}^{s_i^{k+1}(G)} \overline{G}(u)du - \int_{s_i^k(F)}^{s_i^{k+1}(F)} \overline{F}(u)du && \text{[By (29)]} \\
&\geq s_i^k(G) - s_i^k(F) + \int_{s_i^k(G)}^{s_i^{k+1}(F)} \overline{G}(u)du - \int_{s_i^k(F)}^{s_i^{k+1}(F)} \overline{F}(u)du && \text{[Induction Hypothesis, } s_i^{k+1}(G) \geq s_i^{k+1}(F)\text{]} \\
&= s_i^k(G) - s_i^k(F) - \int_{s_i^k(F)}^{s_i^k(G)} \overline{G}(u)du + \int_{s_i^k(F)}^{s_i^{k+1}(F)} \left[\overline{G}(u) - \overline{F}(u)\right] du \\
&\geq s_i^k(G) - s_i^k(F) - \int_{s_i^k(F)}^{s_i^k(G)} \overline{G}(u)du && \text{[By part (a) and } \overline{G} \geq \overline{F}\text{]} \\
&= \int_{s_i^k(F)}^{s_i^k(G)} G(u)du \\
&\geq 0 && \text{[Induction hypothesis, } s_i^k(G) \geq s_i^k(F)\text{]}
\end{aligned}$$

Finally, we prove (d) by induction on i. This holds trivially when $i = N$. Suppose that it holds for some i. Then we have

$$\begin{aligned}
s_{i-1}^k &= s_i^k + \int_{s_i^k}^{s_i^{k+1}} \overline{F}(u)du && \text{[By (29)]} \\
&\leq s_i^k + \left(s_i^{k+1} - s_i^k\right) \overline{F}(s_i^k) && \text{[By part (a), } \overline{F} \text{ decreasing]} \\
&= s_i^k F(s_i^k) + s_i^{k+1} \overline{F}(s_i^k) \\
&\leq (p - v(k+N-i))F(s_i^k) + (p - v(k+1+N-i))\overline{F}(s_i^k) && \text{[Induction hypothesis]} \\
&\leq (p - v(k+N-i+1))F(s_i^k) + (p - v(k+N-i+1))\overline{F}(s_i^k) && [v(\cdot) \text{ decreasing]} \\
&= p - v(k+N-(i-1)).
\end{aligned}$$

Q.E.D.

Proof of Theorem 4 Let $D^f (D^s)$ represent the number of products the retailer sells when it follows the fixed (sequential) assortment strategy. Note that $R_*^f = pD^f$ and $\widetilde{R}_*^s = pD^s$. Note that $\mathbb{E}[D^f] = \mathbb{P}(D^f \geq 1) + \mathbb{P}(D^f \geq 2)$, and similarly for D^s. We have

$$\begin{aligned}
\mathbb{P}\left(D^f \geq 1\right) &= \mathbb{P}(\max\{X_1, X_2\} > p - v(0)) = 1 - F^2(p - v(0)) = 1 - F^2(a) \\
\mathbb{P}\left(D^f \geq 2\right) &= \mathbb{P}(\min\{X_1, X_2\} > p - v(1)) = \overline{F}^2(p - v(1)) = \overline{F}^2(c).
\end{aligned}$$

For D^s, we first note that $b = a + \int_a^c \overline{F}(u)du = p - v(0) - U_2(1) + U_2(0)$; we have

$$\begin{aligned}
\mathbb{P}(D^s \geq 1) &= \mathbb{P}(X_1 > b) + \mathbb{P}(X_1 \leq b, X_2 > a) = \overline{F}(b) + F(b)\overline{F}(a) \\
\mathbb{P}(D^s \geq 2) &= \mathbb{P}(X_1 > b, X_2 > c) = \overline{F}(b)\overline{F}(c).
\end{aligned}$$

Hence, $\mathbb{E}[D^s] \geq \mathbb{E}[D^f]$ and therefore $\mathbb{E}\left[\widetilde{R}^s_*\right] \geq \mathbb{E}[R^f_*]$ if and only if

$$\begin{aligned} &\overline{F}(b) + F(b)\overline{F}(a) + \overline{F}(b)\overline{F}(c) \geq 1 - F^2(a) + \overline{F}^2(c) \\ \iff &\overline{F}(c)(\overline{F}(b) - \overline{F}(c)) \geq F(b)(1 - \overline{F}(a)) - F^2(a) \\ \iff &\overline{F}(c)(\overline{F}(b) - \overline{F}(c)) \geq F(a)(F(b) - F(a)). \end{aligned}$$

Q.E.D.

Proof of Theorem 5. Throughout this proof, for notational brevity, we write $s_i^k := s_i(k)$. Again, let D^f (D^s) represent the number of products the retailer sells when it follows the fixed (sequential) assortment strategy. Let $D_k(n)$ represent the number of products purchased under the sequential assortment strategy by the forward-looking consumer in the last n assortments of the season, if she has already purchased k products. Note that $D^s := D_0(N)$. By an identical argument to the proof of Lemma 2, one can show that in this setting of forward-looking consumers, we have $D_{k+1}(n) \leq D_k(n) \leq 1 + D_{k+1}(n)$ with probability one for $n = 0, ..., N$ and $k = 0, ..., N - n - 1$. Finally, note that if the consumer has n assortments left to view, then in the dynamic programming formulation (12), she is in period $N - n + 1$.

To prove (a), we begin with a definition and a preliminary result. For $n = 0, \dots, N$ and $k = 0, \dots, N - n - 1$, define

$$\delta_n^k := 1 + \mathbb{E}[D_{k+1}(n)] - \mathbb{E}[D_k(n)]. \tag{31}$$

Note that Lemma 2 implies that $\delta_n^k \geq 0$ for each n, k.

We can rearrange the definition of δ_n^k, and by conditioning on whether a purchase was made or not in period $N - n$, we get

$$\mathbb{E}[D_k(n+1)] = \mathbb{E}[D_k(n)] + \overline{F}(s_{N-n}^k)\delta_n^k. \tag{32}$$

We claim that

$$\delta_{n+1}^k \geq \overline{F}(s_{N-n}^{k+1})\delta_n^{k+1}. \tag{33}$$

To see this, consider

$$\begin{aligned} \delta_{n+1}^k &= 1 + \mathbb{E}[D_{k+1}(n+1)] - \mathbb{E}[D_k(n+1)] && \text{[By (31)]} \\ &= 1 + \mathbb{E}[D_{k+1}(n)] + \overline{F}(s_{N-n}^{k+1})\delta_n^{k+1} - \mathbb{E}[D_k(n)] - \overline{F}(s_{N-n}^k)\delta_n^k && \text{[By (32)]} \\ &= \delta_n^k + \overline{F}(s_{N-n}^{k+1})\delta_n^{k+1} - \overline{F}(s_{N-n}^k)\delta_n^k && \text{[By (31)]} \\ &= \overline{F}(s_{N-n}^{k+1})\delta_n^{k+1} + F(s_{N-n}^k)\delta_n^k \\ &\geq \overline{F}(s_{N-n}^{k+1})\delta_n^{k+1}, && [\delta_n^k \geq 0 \text{ by Lemma 2}] \end{aligned}$$

thus establishing (33).

Recalling that $D^s = D_0(N)$, we apply (32) for the case $k = 0$, $n = N - 1$ to get

$$\mathbb{E}[D_0(N)] = \mathbb{E}[D_0(N-1)] + \overline{F}(s_1^0)\delta_{N-1}^0.$$

Next, we claim that under the assumption that $X_i \geq p - v(N-2)$ a.s., we necessarily have $D_0(N-1) = N-1$ with probability 1. This is because we can equivalently interpret $D_0(N-1)$ as the demand from a consumer who only views $N-1$ assortments. Hence, we can use representation (15) for $D_0(N-1)$, which yields

$$D_0(N-1) = \sum_{i=1}^{N-1} \mathbb{1}_{\left\{X_i \geq s_i(\widetilde{W}_{i-1})\right\}},$$

where $\widetilde{W}_j \leq j$ for each $j = 0, \ldots, N-1$. Consider

$$\begin{aligned} s_i(\widetilde{W}_{i-1}) &\leq p - v(\widetilde{W}_{i-1} + N - 1 - i) && \text{[Proposition 1(d) for } N-1\text{]} \\ &\leq p - v(N-2) && [\widetilde{W}_{i-1} \leq i-1,\ v(\cdot) \text{ decreasing}] \end{aligned}$$

Therefore, $D_0(N-1) = N-1$ with probability 1.

For the fixed assortment, apply (4) and the condition that $X_i \geq p - v(N-2)$ with probability 1 to get

$$\mathbb{E}\left[D^f\right] = \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}_{\left\{X_{\sigma(i)} \geq p - v(i-1)\right\}}\right] = (N-1) + \mathbb{P}\left(X_{\sigma(N)} \geq p - v(N-1)\right) = (N-1) + \overline{F}^N(p - v(N-1)).$$

Comparing the expressions for $\mathbb{E}[D^s]$ and $\mathbb{E}[D^f]$, and noting that from Proposition 1(d), we have $s_1(0) \leq p - v(N-1)$, and since $\overline{F}$ is decreasing, $\overline{F}(p - v(N-1)) \leq \overline{F}(s_1(0))$. Therefore, to complete the proof, it suffices to show that

$$\delta^0_{N-1} \geq \overline{F}^{N-1}(p - v(N-1)). \tag{34}$$

We will do this by proving that for every $n = 0, \ldots, N-1$ and $k = N-1-n$, we have

$$\delta^k_n \geq \overline{F}^n\left(p - v(N-1)\right), \tag{35}$$

of which (34) is a special case.

We prove (35) by induction on n. The case of $n = 0$ holds trivially. Suppose that (35) holds for some n, and we aim to show it for $n+1$. Then,

$$\begin{aligned} \delta^{k-1}_{n+1} &\geq \overline{F}(s^k_{N-n})\delta^k_n && \text{[By (33)]} \\ &\geq \overline{F}(s^k_{N-n})\overline{F}^n(p - v(N-1)) && \text{[Induction hypothesis]} \\ &\geq \overline{F}(p - v(k+n))\overline{F}^n(p - v(N-1)) && \text{[Proposition 1(d) and } \overline{F} \text{ decreasing]} \\ &= \overline{F}^{n+1}(p - v(N-1)) && [k = N-1-n \text{ by assumption}] \end{aligned}$$

Therefore, (34) holds, and hence, $\mathbb{E}[D^s] \geq \mathbb{E}[D^f]$ and $\mathbb{E}\left[\widetilde{R}^s_*\right] \geq \mathbb{E}[R^f_*]$.

To prove (b), we first note that in this setting,

$$\mathbb{E}\left[D^f\right] = 1 - F^N(p - v(0)).$$

Furthermore, for any $i \geq 1, k \geq 1$ we have $s^k_i \geq s^k_N = p - v(k) \geq p - v(1)$, where the first inequality is by Proposition 1(b), and the second because $v(\cdot)$ is decreasing, which implies that $\overline{F}(s^k_i) \leq \overline{F}(p - v(1))$. Because we have assumed that $X_i \leq p - v(1)$ a.s., this implies that for $k \geq 1$, we necessarily have $\overline{F}(s^k_i) = 0$ for any $i \geq 1$. In particular, using (32), this implies that for $k \geq 1$, and any $n = 0, \ldots, N-1$, we have

$$\mathbb{E}\left[D_k(n+1)\right] = \mathbb{E}\left[D_k(n)\right] = \ldots = \mathbb{E}\left[D_k(0)\right] = 0.$$

Now we use the definition of δ_n^k from (31) to expand (32) for the case of $k=0$ to get

$$\mathbb{E}\left[D_0(n+1)\right] = \overline{F}(s_{N-n}^0) + F(s_{N-n}^0)\mathbb{E}\left[D_0(n)\right].$$

Defining $Q_n := 1 - \mathbb{E}\left[D_0(n)\right]$, the above may be written as $Q_{n+1} = F(s_{N-n}^0)Q_n$, for $n = 0, \ldots, N-1$. Since $Q_0 := 1$, we have

$$Q_N = \prod_{n=0}^{N-1} F(s_{N-n}^0) \geq \prod_{n=0}^{N-1} F(p - v(0)) = F^N(p - v(0)),$$

where the inequality is by Proposition 1(b) and because F is increasing. Therefore, we conclude that

$$\mathbb{E}\left[D^s\right] = \mathbb{E}\left[D_0(N)\right] = 1 - Q_N \leq 1 - F^N(p - v(0)) = \mathbb{E}\left[D^f\right],$$

and therefore $\mathbb{E}\left[\widetilde{R}_*^s\right] \leq \mathbb{E}\left[R_*^f\right]$. Q.E.D.

Appendix B: Supplementary Material

B.1. MILP formulation of SAA for optimal assortment composition

Let N_S represent the total number of i.i.d. SAA samples drawn, and let $\left(X_1^{(n)}, X_2^{(n)}, \ldots, X_N^{(n)}\right)$ represent the realized values of demand shocks on the nth random draw, $n = 1, \ldots, N_S$. We use $\sigma_n(\cdot)$ to denote an ordering of $X_i^{(n)}$ such that $X_{\sigma_n(1)}^{(n)} - p_{\sigma_n(1)} \geq X_{\sigma_n(2)}^{(n)} - p_{\sigma_n(2)} \geq \ldots \geq X_{\sigma_n(N)}^{(n)} - p_{\sigma_n(N)}$. For each $i = 1, \ldots, N$, $k = 0, \ldots, N-1$, and $n = 1, \ldots, N_S$, we define $U_{i,k,n} := X_{\sigma_n(i)}^{(n)} - p_{\sigma_n(i)} + v(k)$ and further define $B_{i,k,n}$ as a binary indicator that $U_{i,k,n}$ is positive, i.e., $B_{i,k,n} = 1$ if $U_{i,k,n} > 0$, and $B_{i,k,n} = 0$ otherwise. We note that for a given draw of SAA samples, both $U_{i,k,n}$ and $B_{i,k,n}$ are deterministic functions of the sample and can be pre-computed before the MILP is constructed and solved.

The MILP formulation uses two key decision variables: $z_i = 1$ is interpreted to mean that product i is excluded from the assortment and $y_{i,k,n}$ is an assignment variable that means that product i is assigned to position k for the random draw n. The key idea in the MILP is that for a given choice of $z_1, \ldots, z_N$, there is a unique value of $y_{i,k,n}$ that satisfies the constraints, and this value corresponds to choosing products in decreasing order of $X_i^{(n)} - p_i$ if they are included in the assortment, which matches the utility-maximizing decisions of the consumer in our model when she is faced with a fixed assortment. In the MILP below, for brevity of presentation, when the ranges of indices are omitted, they are to be interpreted as being over their full range, i.e, $\forall i$ means $\forall i = 1, \ldots, N$, $\forall k$ means $k = 0, \ldots, N-1$, and $\forall n$ means $n = 1, \ldots, N_S$.

$$\max_{y,z,w} \quad \frac{1}{N_S} \sum_{n=1}^{N_S} \sum_{k=0}^{N-1} \sum_{i=1}^{N} y_{i,k,n} p_i \tag{36a}$$

$$\text{s.t.} \quad y_{i,k,n} \leq B_{i,k,n} \qquad \forall i; \forall k; \forall n \tag{36b}$$

$$\sum_{k=0}^{N-1} y_{i,k,n} \leq 1 - z_i \qquad \forall i; \forall n \tag{36c}$$

$$\sum_{i=1}^{N} y_{i,k,n} \leq 1 \qquad \forall k; \forall n \tag{36d}$$

$$w_{i,k,n} = 1 - \sum_{j=i}^{N} y_{\sigma_n(j),k,n} \qquad \forall i; \forall k; \forall n \tag{36e}$$

$$w_{i,k-1,n} \leq w_{i,k,n} \qquad \forall i; \forall k = 1, \ldots, N-1; \forall n \tag{36f}$$

$$\sum_{i=1}^{N} w_{i,k,n} \leq k + \sum_{j=1}^{k+1} z_{\sigma_n(j)} + N\left(1 - \sum_{i=1}^{N} y_{i,k,n}\right) \qquad \forall k; \forall n \tag{36g}$$

$$y_{i,k,n}, z_i \in \{0,1\} \qquad \forall i; \forall k; \forall n \tag{36h}$$

The objective (36a) aims to maximize the retailer's average revenue (taken over the SAA samples). Constraints (36b), (36c), and (36d) represent assignment constraints: (36b) states that only assignments with nonnegative utility for the consumer can be chosen; (36c) states that at most one product can be assigned to the kth position if that product is included in the assortment; (36d) states that each position can be filled by at most one product. Constraints (36f) and (36g) enforce the purchase of products in descending

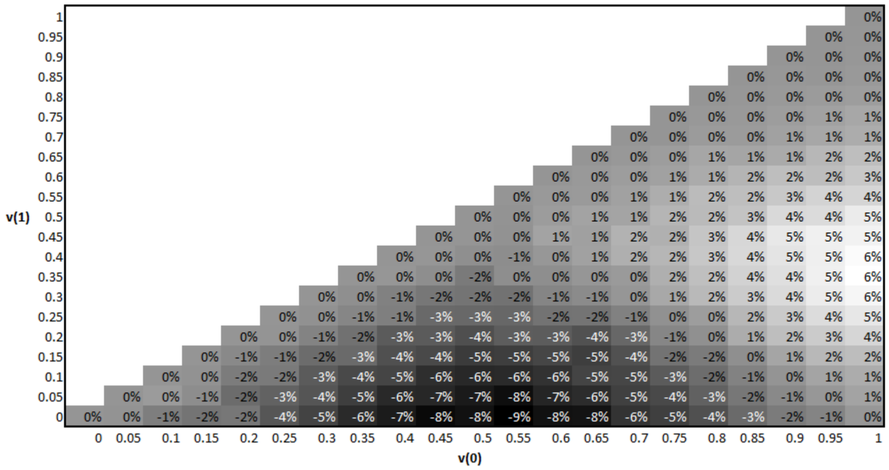

Figure 5 **Value of concealment as a percentage of expected revenue from the fixed assortment strategy:** $p=1$ **and** $X \sim Normal(\mu=0.5, \sigma^2=\frac{1}{12})$

order of $X_i^{(n)} - p_i$ in order to match the decisions of a utility-maximizing consumer. Namely, (36f) states that the product in position $k-1$ has to have a higher value of $X_i^{(n)} - p_i$ compared to the product in position k. Finally,(36g) is a "big-M" style constraint that ensures that if we have two products i and i', such that $X_i^{(n)} - p_i > X_{i'}^{(n)} - p_{i'}$, and i' is assigned to a position, then product i can only be "skipped" from being assigned if, and only if, product i is permanently excluded from the assortment (i.e., from every SAA sample).

B.2. Numerical Results for forward-looking consumers when $N=2$, $X_i \sim Normal(\mu=0.5, \sigma^2=\frac{1}{12})$

Here we present numerical results when $X_i \sim Normal(\mu=0.5, \sigma^2=\frac{1}{12})$; note that the mean and variance of this distribution are identical to the mean and variance of the $Uniform(0,1)$ distribution used in Figure 1. Figure 5 shows a heat map of the value of concealment, expressed as a percentage of the retailer's expected revenue from following the fixed assortment strategy, for forward-looking consumers when $p=1, X \sim Normal(\mu=0.5, \sigma^2=\frac{1}{12})$, and for different values of $v(0) \in [0,1], v(1) \in [0, v(0)]$. These numerical results are averages of 10,000 simulation runs per set of parameters.

As in Figure 1, the right-most region of the heat map in Figure 5 represents values of $v(0)$ and $v(1)$ where the value of concealment is positive, and the bottom-most region of the heat map represents values of $v(0)$ and $v(1)$ where the value of concealment is negative. The line $v(0)+v(1)=1$ separates the two regions as it did in Figure 1. Comparing Figures 1 and 5 suggests that our intuition extends beyond the case where $X_i \sim Uniform(\alpha,\beta)$ to the case where $X_i \sim Normal(\mu,\sigma^2)$, and we conjecture that these insights are broadly applicable to many sets of parameters over the forward-looking consumer's valuations.

B.3. Value of concealment for forward-looking consumers, special case of $N=3$

Consider the special case where $X_i \sim Uniform(0,1)$. The following proposition establishes necessary and sufficient conditions on the parameters $(x,y,z) := (p-v(0), p-v(1), p-v(2))$ to establish $\mathbb{E}\left[\widetilde{R}_*^s\right] \geq \mathbb{E}\left[R_*^f\right]$ and vice versa.

Proposition 2. *Let $N = 3$ and $X_i \sim Uniform(0,1)$. For a given trivariate, 7th degree, polynomial, $g(x,y,z)$, the value of concealment is positive if and only if $g(x,y,z) \geq 0$.*

Proof of Proposition 2. As in the proof of Theorem 4, let $D^f (D^s)$ represent the number of products the retailer sells when it follows the fixed (sequential) assortment strategy. We calculate $\mathbb{E}[D^f]$ similarly to Theorem 4, by using $\mathbb{E}[D^f] = \sum_{k=1}^{3} \mathbb{P}(D^f \geq k)$ and evaluating each term of the sum directly. For $\mathbb{E}[D^s]$, we first calculate the thresholds by applying (29) repeatedly, and subsequently calculate $\mathbb{E}[D^s]$ by recursively evaluating the expected number of items purchased in each period, starting from the last period. We then evaluate g as $\mathbb{E}\left[\widetilde{R}^s_*\right] - \mathbb{E}[R^f_*] = p(\mathbb{E}[D^s] - \mathbb{E}[D^f]) = Kg(x,y,z)$, where K is a scaling constant. This gives us the following expression for g:

$$
\begin{aligned}
g(x,y,z) := {} & x^7 - x^6y + x^6 - 3x^5y^2 + 6x^5y + 2x^4y^3 + x^4y^2z - 10x^4y^2 + x^4yz^2 - 2x^4yz + 8x^4y - x^4z^3 \\
& + 3x^4z^2 - 4x^4z + 2x^4 + 2x^3y^4 - 12x^3y^3 + 2x^3y^2z^2 - 4x^3y^2z + 16x^3y^2 - x^3z^4 \\
& + 4x^3z^3 - 8x^3z^2 + 8x^3z - 16x^3 - 2x^2y^4z + 12x^2y^4 - 4x^2y^3z^2 + 12x^2y^3z - 36x^2y^3 \\
& + 2x^2y^2z^3 - 4x^2y^2z + 20x^2y^2 + x^2yz^4 - 8x^2yz^3 + 20x^2yz^2 - 24x^2yz + 8x^2y - x^2z^4 + \\
& + 4x^2z^3 - 8x^2z^2 + 8x^2z + 4xy^5 - 2xy^4z^2 + 4xy^4z - 16xy^4 + 4xy^3z^2 - 8xy^3z + 16xy^3 + xy^2z^4 \\
& - 4xy^2z^3 + 8xy^2z^2 - 8xy^2z - 2xyz^4 + 8xyz^3 - 16xyz^2 + 16xyz - 4y^5z + 16y^5 + 2y^4z^3 - 16y^4z^2 \\
& + 32y^4z - 48y^4 + 2y^3z^4 - 4y^3z^3 + 12y^3z^2 - 16y^3z + 64y^3 - 3y^2z^5 + 18y^2z^4 - 52y^2z^3 + 84y^2z^2 \\
& - 80y^2z - 24y^2 - yz^6 + 6yz^5 - 20yz^4 + 40yz^3 - 56yz^2 + 48yz + z^7 - 7z^6 + 24z^5 \\
& - 50z^4 + 56z^3 - 24z^2,
\end{aligned}
$$

which can be seen to be a 7th degree polynomial in the three variables (x,y,z). Q.E.D.

Figure 6 illustrates the boundary constructed from g where the expected demand under the fixed and sequential assortment strategies are equivalent. Our analysis suggests that it is unlikely that there are simple necessary and sufficient conditions to characterize the sign of the value of concealment.

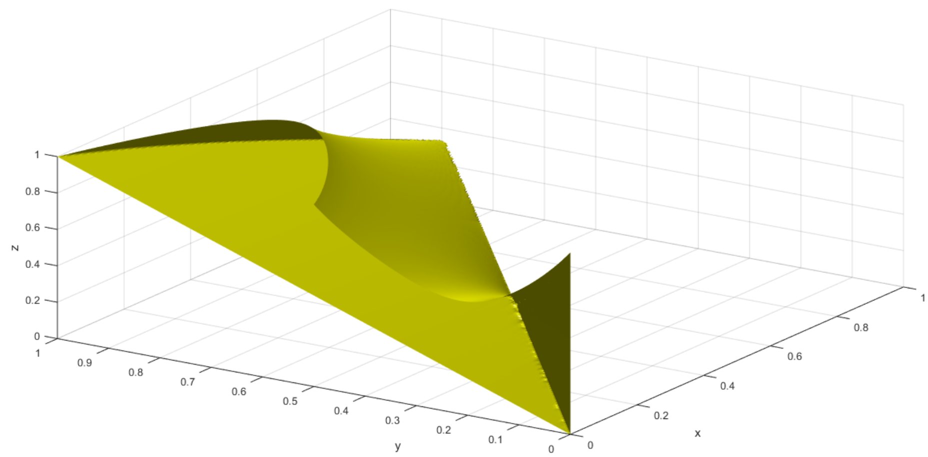

Figure 6 3D plot of boundary on which the value of concealment is zero: $N = 3$ **and** $X_i \sim Uniform(0, 1)$