# Online Appendix for: Surge Pricing and its Spatial Supply Response

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This appendix is divided into five sub-appendices, Appendix A covers the proofs in Section 4 and Section 5; Appendix B covers the proofs in Section 6; Appendix C covers the proofs in Section 7; Appendix D covers the proofs in Section 8; Appendix E provides additional numerical results for section 8.3.

## A Proofs for Section 4 and Section 5

***Proof of Lemma 1***. Consider any $z, y \in \mathcal{C}$. Then, for essentially any $w \in \mathcal{B}$, we have

$$V_{\mathcal{B}}(y) \geq U(w) - \|w-y\| \;=\; U(w) - \|z-w\| + \|z-w\| - \|w-y\| \;\geq\; U(w) - \|z-w\| - \|z-y\|,$$

where the second inequality follows from the triangular inequality. This implies, by the definition of the essential supremum, that

$$V_{\mathcal{B}}(y) + \|z-y\| \geq V_{\mathcal{B}}(z).$$

Next, we would like to subtract $V_{\mathcal{B}}(y)$ from both sides of the previous inequality. This operation can be done only if $V_{\mathcal{B}}(y)$ is finite for any $y$ in $\mathcal{C}$, but this is guaranteed by Lemma A-1 (stated and proved right after this proof). Hence, we obtain $V_{\mathcal{B}}(z) - V_{\mathcal{B}}(y) \leq \|z-y\|$. Since we can interchange the roles of $z$ and $y$, we have proved that $|V_{\mathcal{B}}(z) - V_{\mathcal{B}}(y)| \leq \|z-y\|$, for all $z, y \in \mathcal{C}$. □

**Lemma A-1.** *Consider a measurable set $\mathcal{B} \subseteq \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$, let $p$ be a measurable mapping $p : \mathcal{B} \to \mathbb{R}_+$, and let $\mathcal{T} \in \mathcal{F}(\Theta)$. Then, $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \in \left[-H, \alpha \cdot \overline{V}\right]$ for all $x \in \mathcal{C}$, where $H = \max_{x,y\in\mathcal{C}} \|x-y\|$. Furthermore, $V(x|\, p, \mathcal{T}) \geq 0$ for all $x \in \operatorname{supp}(\Gamma)$.*

*Proof.* Fix $x \in \mathcal{C}$, we show that $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \in \left[-H, \alpha \cdot \overline{V}\right]$. For the lower bound, note that for any $y \in \mathcal{B}$, we have $U(y) - \|y-x\| \geq -H$. Since $\Gamma(\mathcal{B}) > 0$, the definition of essential supremum implies that $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \geq -H$. Similarly, for the upper bound, note that for any $y \in \mathcal{B}$, $\alpha \cdot \overline{V} \geq U(y) - \|y-x\|$ and hence the definition of essential supremum yields $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \leq \alpha \cdot \overline{V}$.

Finally, we show that $V(x|p, \mathcal{T}) \geq 0$ for all $x \in \operatorname{supp}(\Gamma)$. Since $x \in \operatorname{supp}(\Gamma)$ we have that $\Gamma(B(x,\delta)) > 0$ for all $\delta > 0$, where $B(x,\delta)$ is an open ball of radius $\delta$. For any $y \in B(x,\delta)$ we have $U(y) - \|y-x\| > -\delta$, and since $\Gamma(B(x,\delta)) > 0$ we deduce that $V_{B(x,\delta)}(x|\, p, \mathcal{T}) > -\delta$ for all $\delta > 0$. In turn, we have $V(x|\, p, \mathcal{T}) \geq V_{B(x,\delta)}(x|\, p, \mathcal{T}) > -\delta$ for all $\delta > 0$ and, therefore, $V(x|\, p, \mathcal{T}) \geq 0$. □

***Proof of Proposition 1***. We show how to reformulate the platform's objective as in the statement of the proposition. The key step is to establish that

$$U\left(x, p(x), s^{\mathcal{T}}(x)\right) \;=\; V(x|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\; x \in \mathcal{C}, \tag{A-1}$$

namely, whenever there is post-relocation supply at a given location in equilibrium, the drivers originating at such a location can achieve maximum utility by staying at that location. We state and prove this result in Lemma A-2 (stated and proved following this proof). Note that this result holds $\mathcal{T}_2 - a.e$ so if we want to interchange $U\left(x, p(x), s^{\mathcal{T}}(x)\right)$ with $V(x|\, p, \mathcal{T})$ we have to do it under the measure $\mathcal{T}_2$. We next analyze the main term in the platform's objective function.

$$\begin{aligned}
\int_{\mathcal{C}} p(y)\cdot \min\left\{s^{\mathcal{T}}(y), \overline{F}_y(p(y))\lambda(y)\right\} d\Gamma(y) &\overset{(a)}{=} \int_{\mathcal{C}} p(y)\cdot \min\left\{s^{\mathcal{T}}(y), \overline{F}_y(p(y))\lambda(y)\right\} \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\Gamma(y) \\
&= \frac{1}{\alpha}\int_{\mathcal{C}} \alpha\, p(y)\cdot \min\left\{1, \frac{\overline{F}_y(p(y))\lambda(y)}{s^{\mathcal{T}}(y)}\right\} \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} s^{\mathcal{T}}(y) d\Gamma(y) \\
&= \frac{1}{\alpha}\int_{\mathcal{C}} U(y, p(y), s^{\mathcal{T}}(y)) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} s^{\mathcal{T}}(y) d\Gamma(y) \\
&\overset{(b)}{=} \frac{1}{\alpha}\int_{\mathcal{C}} U(y, p(y), s^{\mathcal{T}}(y)) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\mathcal{T}_2(y) \\
&\overset{(c)}{=} \frac{1}{\alpha}\int_{\mathcal{C}} V(y) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\mathcal{T}_2(y),
\end{aligned}$$

where (a) holds because whenever $s^{\mathcal{T}}(y) = 0$, the minimum term in the integral becomes zero; ($b$) follows from the fact that $U(y, p(y), s^{\mathcal{T}}(y))\mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}}$ is a measurable function with values in $[0, \alpha\cdot\overline{V}]$ and from recalling that $s^{\mathcal{T}} = d\mathcal{T}_2/d\Gamma$; and ($c$) is a consequence of Eq. (A-1) since we are integrating over the measure $\mathcal{T}_2$. In turn, focusing on the platform's objective function, this yields

$$\begin{aligned}
(1-\alpha)\int_{\mathcal{C}} p(y)\cdot \min\left\{s^{\mathcal{T}}(y), \overline{F}_y(p(y))\lambda(y)\right\} d\Gamma(y) &= \gamma \int_{\mathcal{C}} V(y)\mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\mathcal{T}_2(y) \\
&\overset{(a)}{=} \gamma \int_{\mathcal{C}} V(y)\mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} s^{\mathcal{T}}(y) d\Gamma(y) \\
&= \gamma \int_{\mathcal{C}} V(y) s^{\mathcal{T}}(y) d\Gamma(y),
\end{aligned}$$

where (a) holds because $V(y)\mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}}$ is measurable with values in $[0, \alpha\cdot\overline{V}]$ and we recall again that $s^{\mathcal{T}} = d\mathcal{T}_2/d\Gamma$. This completes the proof. □

**Lemma A-2** (Equilibrium Utilities)**.** *For any price mapping $p$ and corresponding equilibrium $\mathcal{T}$, let $\mathcal{B}\subseteq\mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$, then*

$$U\left(y, p(y), s^{\mathcal{T}}(y)\right) \;=\; V_{\mathcal{B}}(y|\, p, \mathcal{T}) = V(y|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\ y\in\mathcal{B}.$$

*Furthermore,*

$$U\left(y, p(y), s^{\mathcal{T}}(y)\right) \;\leq\; V_{\mathcal{B}}(y|\, p, \mathcal{T}) \qquad \Gamma - a.e.\ y\in\mathcal{B}.$$

*Proof.* We prove that

$$U\left(y, p(y), s^{\mathcal{T}}(y)\right) \;=\; V_{\mathcal{B}}(y|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\ y\in\mathcal{B}.$$

The proof for $V(y|\, p, \mathcal{T})$ instead of $V_{\mathcal{B}}(y|\, p, \mathcal{T})$ follows the same steps and is, thus, omitted. Let $A\subseteq\mathcal{B}$ be a set defined by

$$A \triangleq \{y\in\mathcal{B} : U(y) = V_{\mathcal{B}}(y)\}. \tag{A-2}$$

We want to prove $\mathcal{T}_2(A^c) = 0$, where the complement is taken with respect to $\mathcal{B}$. Consider the sets

$$A^- \triangleq \{y\in\mathcal{B} : U(y) < V_{\mathcal{B}}(y)\}, \quad A^+ \triangleq \{y\in\mathcal{B} : U(y) > V_{\mathcal{B}}(y)\}.$$

We will establish that $\mathcal{T}_2(A^-) = 0$ and $\mathcal{T}_2(A^+) = 0$. We begin with $A^-$ and note that

$$\begin{aligned}\mathcal{T}_2(A^-) &= \mathcal{T}(\mathcal{C} \times A^-) \\ &\overset{(a)}{=} \mathcal{T}(\{(x,y) \in \mathcal{C} \times A^- : U(y) - \|y - x\| = V(x)\}) \\ &\overset{(b)}{\leq} \mathcal{T}(\{(x,y) \in \mathcal{C} \times A^- : U(y) \geq V(y)\}) \\ &\overset{(c)}{\leq} \mathcal{T}(\{(x,y) \in \mathcal{C} \times A^- : U(y) \geq V_{\mathcal{B}}(y)\}) \\ &\overset{(d)}{\leq} \mathcal{T}(\{(x,y) \in \mathcal{C} \times \mathcal{B} : V_{\mathcal{B}}(y) > U(y) \geq V_{\mathcal{B}}(y)\}) \\ &= 0,\end{aligned}$$

where $(a)$ follows from the equilibrium definition, and $(b)$ from the fact that $V(x) + \|x - y\| \geq V(y)$ (see Lemma 1). In $(c)$ we have used $V(y) \geq V_{\mathcal{B}}(y)$, while $(d)$ follows from $y \in A^-$ and $A^- \subseteq \mathcal{B}$.

To show that $\mathcal{T}_2(A^+) = 0$, it suffices to show that show $\Gamma(A^+) = 0$ (this will also show the last statement of the lemma). For any $n \in \mathbb{N}$ define the set $A_n^+ \triangleq \{y \in \mathcal{B} : U(y) \geq V_{\mathcal{B}}(y) + \frac{1}{n}\}$, and note that $A^+ = \bigcup_{n\in\mathbb{N}} A_n^+$. It is enough to show that $\Gamma(A_n^+) = 0$ for all $n \in \mathbb{N}$. We proceed by contradiction. Suppose there exists $n \in \mathbb{N}$ such that $\Gamma(A_n^+) > 0$. Let $\epsilon > 0$ be such that $\epsilon < \frac{1}{2n}$, and consider a finite partition $\{I_i^\epsilon\}_{i=1}^{K(\epsilon)}$ of $\mathcal{C}$, where for any $x, y \in I_i^\epsilon$ we have $\|x - y\| \leq \epsilon$. Observe that

$$0 < \Gamma(A_n^+) = \Gamma(A_n^+ \cap \bigcup_{i=1}^{K(\epsilon)} I_i^\epsilon) = \sum_{i=1}^{K(\epsilon)} \Gamma(A_n^+ \cap I_i^\epsilon),$$

therefore, there exists $i \in \{1, \ldots, K(\epsilon)\}$ such that $\Gamma(A_n^+ \cap I_i^\epsilon) > 0$. Take $x \in I_i^\epsilon$ , then for any $y \in A_n^+ \cap I_i^\epsilon$

$$U(y) \geq V_{\mathcal{B}}(y) + \frac{1}{n} \geq V_{\mathcal{B}}(x) - \|y - x\| + \frac{1}{n} > V_{\mathcal{B}}(x) - \|y - x\| + 2\epsilon \geq V_{\mathcal{B}}(x) + \|y - x\|,$$

where the second inequality comes from the Lipschitz property (see Lemma 1). The last two inequalities hold because of our choice of $\epsilon$ and $x, y \in I_i^\epsilon$. We conclude that

$$A_n^+ \cap I_i^\epsilon \subseteq \{y \in \mathcal{B} : \Pi(x,y) > V_{\mathcal{B}}(x)\}.$$

This would therefore imply that $\Gamma(\{y \in \mathcal{B} : \Pi(x,y) > V_{\mathcal{B}}(x)\}) > 0$. However, this contradicts the definition of $V_{\mathcal{B}}(x)$. Hence we must have $\Gamma(A_n^+) = 0$ for all $n \in \mathbb{N}$, and in turn $\Gamma(A^+) = 0$. □

**Lemma A-3.** *The congestion function $\psi_x(\cdot)$ is a strictly decreasing function $\Gamma - a.e.\ x$ in $\mathcal{C}$.*

***Proof of Lemma A-3.*** Recall that $\lambda(x) > 0$ $\Gamma - a.e.\ x$ in $\mathcal{C}$ and that the price achieving the maximum in the definition of $R_x^{loc}(s)$ is $\rho_x^{loc}(s) = \max\{\rho_x^{bal}(s), \rho_x^u\}$. Let $s^u$ be equal to $\lambda(x) \cdot \overline{F}_x(\rho_x^u)$, that is, $\rho_x^{bal}(s^u) = \rho_x^u$ (here we are using that $q \mapsto q \cdot \overline{F}_y(q)$ is continuous and unimodular in $q$). Then, since $\rho_x^{bal}(\cdot)$ is decreasing we have that $\rho_x^{loc}(s) = \rho_x^{bal}(s)$ for all $0 < s \leq s^u$ and, therefore,

$$\frac{R_x^{loc}(s)}{s} = \rho_x^{bal}(s) = F^{-1}(1 - \frac{s}{\lambda(x)}), \quad \text{for all } 0 < s \leq s^u.$$

Since $F$ is strictly increasing, the quotient above is strictly decreasing for $s \in (0, s^u]$. Moreover, since $F^{-1}(1) = \overline{V}$, the point just made also includes $s = 0$. Now, for $s > s^u$ we have $\rho_x^{loc}(s) = \rho_x^u$, thus

$$\frac{R_x^{loc}(s)}{s} = \rho_x^u \cdot \frac{\lambda(x) \cdot \overline{F}_x(\rho_x^u)}{s},$$

which is strictly decreasing. In any case, we conclude that $\psi_x(\cdot)$ is strictly decreasing $\Gamma - a.e.\ x$ in $\mathcal{C}$. □

***Proof of Proposition 2.*** Define the set $B \triangleq \{x \in C : V(x) > \psi_x(s^{\mathcal{T}}(x))\}$. We want to show that $\Gamma(B) = 0$. First we argue that $B \subseteq \{x \in \mathcal{C} : U(x) \neq V(x)\}$, indeed, let $x \in B$ then

$$V(x) > \psi_x(s^{\mathcal{T}}(x)) \geq U\Big(x, p(x), s^{\mathcal{T}}(x)\Big),$$

that is, $V(x) > U(x)$ as desired. By Lemma A-2 we know that $\mathcal{T}_2(\{x \in \mathcal{C} : U(x) \neq V(x)\}) = 0$ and, therefore, $\mathcal{T}_2(B) = 0$. This yields,

$$0 = \mathcal{T}_2(B) = \int_B s^{\mathcal{T}}(x)\, d\Gamma(x). \tag{A-3}$$

If $\Gamma(B) = 0$ then we are done. Suppose $\Gamma(B) > 0$, from equation (A-3) we can conclude that $s^{\mathcal{T}}(x) = 0$, $\Gamma - a.e.$ $x \in B$. The definition of $\psi_x(s^{\mathcal{T}}(x))$ implies that $\Gamma - a.e$ in $B$ we have that $\psi_x(s^{\mathcal{T}}(x))$ equals $\alpha \cdot \overline{V}$. Because $\alpha \cdot \overline{V}$ is the maximum value that $V(\cdot)$ can attain (see Lemma A-1), we conclude that

$$\alpha \cdot \overline{V} \geq V(x) > \psi_x(s^{\mathcal{T}}(x)) = \alpha \cdot \overline{V} \qquad \Gamma - a.e.\ x \in B.$$

But since we are assuming that $\Gamma(B) > 0$, this yields a contradiction. $\square$

## B Proofs for Section 6

When $z$ is a sink, we represent the endpoints of its attraction region along a ray $a \in R_z$ by

$$X_a(z|\, p, \mathcal{T}) \triangleq \sup\{x \in A_a(z|\, p, \mathcal{T})\},$$

where $A_a(z|\, p, \mathcal{T})$ is the restriction of $A(z|\, p, \mathcal{T})$ in the direction of ray $a$. Note that in the definition of $X_a(z|\, p, \mathcal{T})$, as $x$ moves away from $z$ along $A_a(z|\, p, \mathcal{T})$, $x$ increases. In turn, for every ray $a$ the segment $[z, X_a(z|\, p, \mathcal{T})]$ represents the contribution of $A_a(z|\, p, \mathcal{T})$ to $A(z|\, p, \mathcal{T})$. The next result formalizes this and characterizes the shape of attraction regions.

**Lemma B-1** (Attraction Region)**.** *Let $(p, \mathcal{T})$ be a feasible solution of $(\mathcal{P}_2)$. For any sink $z \in \mathcal{C}$, its attraction region $A(z|\, p, \mathcal{T})$ is a closed set containing $z$, $A_a(z|\, p, \mathcal{T}) = [z, X_a(z|\, p, \mathcal{T})]$ and*

$$A(z|\, p, \mathcal{T}) = \bigcup_{a \in R_z} A_a(z|\, p, \mathcal{T}).$$

***Proof of Lemma B-1***. For ease of notation let us use $x_a$ to denote $X_a(z|\, p, \mathcal{T})$. We also denote $A(z|p, \mathcal{T})$ by $A(z)$.

**Closure:** Let the sequence $\{x^n\}_{n\in\mathbb{N}} \subset A(z)$ be such that $x^n \to x$. We show that $x \in A(z)$, that is, $V(x) = V(z) - \|x - z\|$. Indeed, since $x_n \in A(z)$ we have that $V(x_n) = V(z) - \|x_n - z\|$. Because $V(\cdot)$ is Lipschitz (see Lemma 1), it is continuous, and the desired conclusion follows.

**Interval:** We show that $A_a(z) = [z, x_a]$. The definition of $x_a$ immediately implies that $A_a(z) \subseteq [z, x_a]$, so we only need to prove the reverse inclusion. First, since we can always construct a sequence $\{x^n\}_{n\in\mathbb{N}} \subset A(z)$, with $x^n \to x_a$, the closure property implies that $x_a \in A(z)$. Second, we make use of Lemma B-2 (stated and proved right after this proof). Consider $x \in [z, x_a]$ then Lemma B-2 implies that $z \in \mathcal{IR}(x|\, p, \mathcal{T})$ or, equivalently, $x \in A_a(z)$.

**Union:** Since for every $a \in R_z$ we have $A_a(z) \subset A(z)$, the same is true for the union. In the opposite direction, if we take $x \in A(z)$ then there exists $a \in R_z$ such that $x \in [z, x_a] = A_a(z)$.

□

**Lemma B-2.** *For any price mapping $p$ and corresponding equilibrium $\mathcal{T}$, if $y \in \mathcal{IR}(x|\, p, \mathcal{T})$ then $y \in \mathcal{IR}(z|\, p, \mathcal{T})$ for all $z \in [x \wedge y, x \vee y]$.*

*Proof.* Let $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. If $x = y$ there is nothing to prove. Without loss of generality suppose $x < y$, where we use the natural order in the segment $[x, y]$. Fix $z \in [x, y]$, we want to prove that $y \in \mathcal{IR}(z|\, p, \mathcal{T})$, i.e., $V(z) = V(y) - |z - y|$. Note that

$$\begin{aligned}\|z - y\| &\overset{(a)}{\geq} V(y) - V(z) = V(y) - V(x) + V(x) - V(z) \overset{(b)}{=} \|x - y\| + V(x) - V(z) \\ &\overset{(c)}{\geq} \|x - y\| - \|x - z\| \\ &\overset{(d)}{=} \|z - y\|,\end{aligned}$$

where $(a)$ and $(c)$ come from the Lipschitz property (see Lemma 1), $(b)$ follows from $y \in \mathcal{IR}(x|\, p, \mathcal{T})$, and $(d)$ holds because $x, z, y$ are collinear points.

□

***Proof of Proposition 3***. Consider the segment $[x, y]$ (with the standard order) and define the set

$$L \triangleq \{y' \in \mathcal{C} : \exists t \geq 0 \text{ such that } y' = x + t \cdot (y - x)\},$$

that is $L$ is the set of point along the ray that starts at $x$ and contains the segment $[x, y]$. Since $y \in L$ and $y \in \mathcal{IR}(x|\, p, \mathcal{T})$ the following quantity is well defined

$$z \triangleq \sup\{y' \in L : y' \in \mathcal{IR}(x|\, p, \mathcal{T})\}.$$

We prove that $z$ is a sink location such that $x, y \in A(z|\, p, \mathcal{T})$. First, we show that $z \in \mathcal{IR}(x|\, p, \mathcal{T})$. Consider a sequence $\{z_n\} \subset L$ such that $z_n \in \mathcal{IR}(x|p, \mathcal{T})$ and $z_n \to z$. Then, $V(z_n) - \|z_n - x\| = V(x)$, we can take the limit as $n \uparrow \infty$ and use the Lipschitz property (see Lemma 1) to conclude that $V(z) - \|z - x\| = V(x)$. That is, $z \in \mathcal{IR}(x|\, p, \mathcal{T})$ which also shows that $A(z) \neq \emptyset$.

Next, to show that $z$ is a sink location we argue that we cannot have $z \in A(z')$ for some $z' \neq z$. If we did then $z' \in \mathcal{IR}(z|p, \mathcal{T})$ for some $z' \neq z$. First suppose that $z' \in L$. If $z' > z$ this would contradict the definition of $z$ as being maximal. If $z' < z$ then by the definition of $\mathcal{IR}(z|\, p, \mathcal{T})$ the function $V(\cdot)$ would be decreasing in $(z', z)$, but since $z \in \mathcal{IR}(x|\, p, \mathcal{T})$ we have that $V(\cdot)$ is increasing in $(x, z)$. This is a contradiction.

Second, suppose that $z' \notin L$. That is the vectors $z'-x$ and $z-x$ are not collinear. Because $z' \in \mathcal{IR}(z|p, \mathcal{T})$ and $z \in \mathcal{IR}(x|\, p, \mathcal{T})$ we have

$$V(z') - \|z' - z\| = V(z) \text{ and } V(z) - \|z - x\| = V(x).$$

Combining these expression yields

$$V(z') - V(x) = \|z' - z\| + \|z - x\| > \|z' - x\|,$$

where the strict inequality comes from the fact that $z' - x$ and $z - x$ are not collinear. But this contradict the fact that $V(\cdot)$ is Lipschitz (see Lemma 1). We conclude that $z$ is a sink location. Moreover, because $x \in A(z)$ ($z \in \mathcal{IR}(x|\, p, \mathcal{T})$) and $x < y \leq z$ (recall these three points are collinear) Lemma B-2 guarantees that $y \in A(z)$. □

**Proposition B-1** (Flow Separation)**.** *Let $(p, \mathcal{T})$ be a feasible solution of $(\mathcal{P}_2)$, and let $z \in \mathcal{C}$ be a sink. Then, there is no flow crossing the endpoints of the attraction region, and there is no flow crossing the sink, $z$. Formally, with some abuse of notation, let $L(z|\, p, \mathcal{T})$ denote $\bigcup_{a \in R_z} \{X_a(z|\, p, \mathcal{T})\}$ then*

*(i)* $\mathcal{T}(A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})) = 0$ *and* $\mathcal{T}(\bigcup_{a \in R_z} [z, X_a(z|\, p, \mathcal{T}) \times \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)) = 0.$

*(ii) Let $R_1, R_2 \subset R_z$ with $R_1 \cap R_2 = \emptyset$ then* $\mathcal{T}\Big(\bigcup_{a \in R_1} (z, X_a(z|\, p, \mathcal{T})] \times \bigcup_{a \in R_2} (z, X_a(z|\, p, \mathcal{T})]\Big) = 0.$

***Proof of Proposition B-1***. With some abuse of notation let

$$A^{\circ}(z|\, p, \mathcal{T}) = \bigcup_{a \in R_z} (z, X_a(z|\, p, \mathcal{T}).$$

This result is based on the following properties:

a) For all $(x, y) \in A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})$, $y \notin \mathcal{IR}(x|\, p, \mathcal{T})$.

b) For all $(x, y) \in (A^{\circ}(z|\, p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)$, $y \notin \mathcal{IR}(x|\, p, \mathcal{T})$.

Before we provide a formal proof of these properties, we use them to show the statement of the proposition. We will also make use of Lemma B-3 which we prove and state after the present proof.

We begin with the first part of *(i)*, that is, we show that $\mathcal{T}(A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})) = 0$. If this is not true then by Lemma B-3 we can find $(x, y) \in A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. We obtain a contradiction with property a) above. Therefore it must be the case that $\mathcal{T}(A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})) = 0$.

Next, we show the second part of *(i)*, namely, $\mathcal{T}((A^{\circ}(z|p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|p, \mathcal{T})^c \cup L(z|p, \mathcal{T}) \setminus \{z\}\Big)) = 0$. If this is not true then by Lemma B-3 we can find $(x, y) \in (A^{\circ}(z|p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|p, \mathcal{T})^c \cup L(z|p, \mathcal{T}) \setminus \{z\}\Big)$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$ but this contradicts property b) above. Therefore it must be the case that $\mathcal{T}((A^{\circ}(z|\, p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)) = 0$.

Now we provide a proof for *(ii)*. Let $R_1, R_2 \subset R_z$ with $R_1 \cap R_2 = \emptyset$ we show that

$$\mathcal{T}\Big(\bigcup_{a \in R_1} (z, X_a(z|\, p, \mathcal{T})] \times \bigcup_{a \in R_2} (z, X_a(z|\, p, \mathcal{T})]\Big) = 0.$$

Suppose by contradiction that this is not true then by Lemma B-3 we can find $(x, y) \in \bigcup_{a\in R_1}(z, X_a(z|p, \mathcal{T})] \times \bigcup_{a\in R_2}(z, X_a(z|\, p, \mathcal{T})]$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. This implies that $x \in A(y|\, p, \mathcal{T})$. Moreover, since $z$ is a sink location we have $x \in A(z|\, p, \mathcal{T})$ and $y \in A(z|\, p, \mathcal{T})$ thus

$$V(x) = V(y) - \|y - x\|, \quad V(x) = V(z) - \|z - x\|, \quad \text{and} \quad V(y) = V(z) - \|z - y\|.$$

In turn, we can use the first to equalities to obtain $V(y) = V(z)+\|y-x\|-\|z-x\|$. Plugging this into the last equality yields $\|z - x\| = \|y - x\| + \|z - y\|$; however, because $R_1 \cap R_2 = \emptyset$ we have that $x \in (z, X_{a_1}(z|\, p, \mathcal{T})]$ and $y \in (z, X_{a_2}(z|p, \mathcal{T})]$ with $a_1 \neq a_2$. In other words, $x$ and $y$ belong to different rays around $z$. In turn, the latter equality cannot hold and we must have that $\mathcal{T}\Big(\bigcup_{a\in R_1}(z, X_a(z|\, p, \mathcal{T})] \times \bigcup_{a\in R_2}(z, X_a(z|\, p, \mathcal{T})]\Big) = 0$.

Next we verify properties a) and b). We start with a). We argue by contradiction. Suppose there exists $x \in A(z|\, p, \mathcal{T})^c$ and $y \in A(z|\, p, \mathcal{T})$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. Let $a$ index the ray that contains the vector $(x - z)$. Recall that by Lemma B-1 we have that $A_a(z|p, \mathcal{T}) = [z, X_a(z|p, \mathcal{T})]$. Since $x \in A(z|\, p, \mathcal{T})^c$ we must have that $x \notin [z, X_a(z|p, \mathcal{T})]$. In particular $\|x - z\| > |X_a(z|p, \mathcal{T}) - z|$. Hence if we show that $z \in \mathcal{IR}(x|p, \mathcal{T})$ we would contradict the maximality of $X_a(z|\, p, \mathcal{T})$. Indeed,

$$\begin{aligned} V(z) - \|x - z\| = V(z) - V(y) + V(y) - \|x - z\| &\overset{(a)}{=} \|z - y\| + V(y) - \|x - z\| \\ &= \|z - y\| + V(y) - \|x - z\| - \|y - x\| + \|y - x\| \\ &\overset{(b)}{=} \|z - y\| + V(x) - \|y - x\| + \|y - x\| \\ &\overset{(c)}{\geq} V(x), \end{aligned}$$

where $(a)$ follows from $y \in A(z|\, p, \mathcal{T})$, $(b)$ from $y \in \mathcal{IR}(x|\, p, \mathcal{T})$ and $(c)$ from the triangular inequality. Using the Lipchitz property of V (see Lemma 1) we conclude that $V(z) - \|x - z\| = V(x)$, that is, $z \in \mathcal{IR}(x|\, p, \mathcal{T})$.

Now we show b). Let $x \in (A^{\circ}(z|\, p, \mathcal{T}) \cup \{z\})$ and $y \in \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)$. We look into two cases: $x \neq z$ and $x = z$. In both cases we proceed by contradiction assuming that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. Let us start with $x \neq z$. Let $a$ index the ray that contains the vector $(x - z)$. Recall that by Lemma B-1 we have that $A_a(z|\, p, \mathcal{T}) = [z, X_a(z|\, p, \mathcal{T})]$. Since, $x \in A^{\circ}(z|\, p, \mathcal{T})$ and $x \neq z$ we must have that $x \in (z, X_a(z|\, p, \mathcal{T}))$, hence

$$V(x) = V(y) - \|y - x\| \text{ and } V(X_a(z|\, p, \mathcal{T})) = V(x) - \|x - X_a(z|\, p, \mathcal{T})\|,$$

that is,

$$V(y) - V(X_a(z|\, p, \mathcal{T})) = \|y - x\| + \|x - X_a(z|\, p, \mathcal{T})\|. \tag{B-1}$$

If $y = X_a(z|\, p, \mathcal{T})$ the previous equality implies $x = X_a(z|\, p, \mathcal{T})$, but since $x \in (z, X_a(z|\, p, \mathcal{T}))$ this is not possible. If $y \neq X_a(z|\, p, \mathcal{T})$ then since $y \in \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)$ we must have that $y \notin (z, X_a(z|\, p, \mathcal{T}))$. Also, $y$ cannot be equal to some point $x + t(z - x)$ for some $t > 1$ because that would contradict the fact that $z$ is a sink location. Therefore, Eq. (B-1) together with the triangular inequality deliver $V(y) - V(X_a(z|\, p, \mathcal{T})) > |y - X_a(z|\, p, \mathcal{T})|$, but this contradicts the Lipschitz property of $V(\cdot)$.

To conclude, consider the case $x = z$. In this case we would have $z \in A(y|\, p, \mathcal{T})$ but this contradicts the fact that $z$ is a sink location. □

**Lemma B-3.** *Let $\mathcal{L} \subset \mathcal{C} \times \mathcal{C}$. If $\mathcal{T}(\mathcal{L}) > 0$ then there exists $(x, y) \in \mathcal{L}$ such that $y \in \mathcal{IR}(x|p, \mathcal{T})$.*

*Proof.* Suppose $\mathcal{T}(\mathcal{L}) > 0$. We first argue that there exists a pair $(x, y) \in \mathcal{L}$ such that for all $\delta > 0$

$$\mathcal{T}(B(x, \delta) \times B(y, \delta)) > 0, \tag{B-2}$$

where $B(x, \delta)$ is an open ball of radius $\delta$. If this is not true then for any $(x, y) \in \mathcal{L}$ we can find $\delta_{x,y} > 0$ such that Eq. (B-2) does not hold when we replace $\delta$ with $\delta_{x,y}$, that is, $\mathcal{T}(B(x, \delta_{x,y}) \times B(y, \delta_{x,y})) = 0$ for all $(x, y) \in \mathcal{L}$. The collection $\mathcal{I}$ defined by

$$\mathcal{I} = \{B(x, \delta_{x,y}) \times B(y, \delta_{x,y})\}_{(x,y)\in\mathcal{L}}$$

is an open cover of $\mathcal{L}$. Moreover the set $\mathcal{L}$ is separable because $\mathcal{C}\times\mathcal{C}$ is separable. This implies that we can find a countable sub-cover of $\mathcal{L}$ in $\mathcal{I}$, that is, there exists $\{B(x_n,\delta_{x_n,y_n})\times B(y_n,\delta_{x_n,y_n})\}_{n\in\mathbb{N}}$ such that

$$\mathcal{L}\subset\bigcup_{n\in\mathbb{N}}B(x_n,\delta_{x_n,y_n})\times B(y_n,\delta_{x_n,y_n}).$$

The existence of the sub-cover is guaranteed by the Lindelöf property of separable metric spaces, see e.g., Sierpiński & Krieger (1952) Theorem 69, p. 116. Since $\mathcal{T}$ is a measure we have

$$\mathcal{T}(\mathcal{L})\le\mathcal{T}\Big(\bigcup_{n\in\mathbb{N}}B(x_n,\delta_{x_n,y_n})\times B(y_n,\delta_{x_n,y_n})\Big)\le\sum_{n\in\mathbb{N}}\mathcal{T}(B(x_n,\delta_{x_n,y_n})\times B(y_n,\delta_{x_n,y_n}))=0,$$

a contradiction. Therefore, for some $(x,y)\in\mathcal{L}$, Eq. (B-2) holds for any $\delta>0$.

We next show that $y\in\mathcal{IR}(x|\,p,\mathcal{T})$. First we prove that

$$\forall\epsilon>0,\exists\delta_0>0\ \text{ such that }\ \forall\delta<\delta_0\quad \epsilon+V_{B(y,\delta)}(x)\ge V(x). \tag{B-3}$$

Let $\epsilon>0$ and let $\delta_0=\frac{\epsilon}{2}$. Consider $\delta<\delta_0$, from Eq. (B-2) and the equilibrium definition we have

$$\begin{aligned}0&<\mathcal{T}(B(x,\delta)\times B(y,\delta))\\&=\mathcal{T}\Big(\Big\{(x',y')\in B(x,\delta)\times B(y,\delta):\Pi(x',y')=V(x')\Big\}\Big)\\&\le\mathcal{T}_2\Big(\underbrace{\Big\{y'\in B(y,\delta):\exists x'\in B(x,\delta)\ \text{ such that }\ \Pi(x',y')=V(x')\Big\}}_{\triangleq R^{x,y,\delta}}\Big),\end{aligned}$$

since $\mathcal{T}_2\ll\Gamma$ this implies that $\Gamma(R^{x,y,\delta})>0$. Now we argue that $R^{x,y,\delta}\subset\{y'\in B(y,\delta):\Pi(x,y')\ge V(x)-\epsilon\}$. Indeed, let $y'\in R^{x,y,\delta}$ then there exists $x'\in B(x,\delta)$ for which

$$\begin{aligned}U(y')&=V(x')+\|y'-x'\|\\&\ge V(x)-\|x'-x\|+\|y'-x'\|\\&=V(x)-\|x'-x\|+\|y'-x'\|-\|y'-x\|+\|y'-x\|\\&\ge V(x)-\|x'-x\|-\|x'-x\|+\|y'-x\|,\end{aligned}$$

where in the first inequality we used the Lipchitz property of V (see Lemma 1), and in the second we use triangular inequality. Since $\|x'-x\|\le\delta_0=\frac{\epsilon}{2}$ we have that $U(y')\ge V(x)-\epsilon+\|y'-x\|$, that is, $R^{x,y,\delta}\subset\{y'\in B(y,\delta):\Pi(x,y')\ge V(x)-\epsilon\}$. Therefore, $\Gamma(\{y'\in B(y,\delta):\Pi(x,y')\ge V(x)-\epsilon\})>0$, which implies that $V_{B(y,\delta)}(x)\ge V(x)-\epsilon$.

Next we argue that

$$V(y)-\|y-x\|+2\delta\ge V_{B(y,\delta)}(x),\quad\forall\delta>0. \tag{B-4}$$

Indeed, the following holds $\Gamma-a.e.\ y'$ in $B(y,\delta)$

$$\begin{aligned}V(y)-\|y-x\|+2\delta&\ge U(y')-\|y'-y\|-\|y-x\|+2\delta\\&=U(y')-\|y'-x\|+\|y'-x\|-\|y'-y\|-\|y-x\|+2\delta\\&\ge U(y')-\|y'-x\|+\|y-x\|-\|y'-y\|-\|y'-y\|-\|y-x\|+2\delta\\&\ge U(y')-\|y'-x\|,\end{aligned}$$

the first inequality comes from the definition of $V(y)$, the second from the triangular inequality and the last inequality follows from $\|y'-y\|\le\delta$.

Finally, Eq. (B-3) and Eq. (B-4) together yields that for any $\epsilon>0$ we can find $\delta(\epsilon)>0$ such that for all $\delta\in(0,\delta(\epsilon))$ we have $V(y)-\|y-x\|+2\delta\ge V_{B(y,\delta)}(x)\ge V(x)-\epsilon$. We can take $\delta\downarrow 0$, and then $\epsilon\downarrow 0$ to conclude that $V(y)-\|y-x\|\ge V(x)$. But by the Lipschitz property (see Lemma 1) we have $V(y)-\|y-x\|\le V(x)$. Therefore, $y\in\mathcal{IR}(x|\,p,\mathcal{T})$. □

## B.1 Local Equilibria and Pasting

The flow separation result in Proposition B-1 will enable us to geographically decompose the platform's problem into multiple weakly coupled local problems. To that end, we introduce some additional notation that will allow us to "localize the analysis". Formally, for any measurable $\mathcal{B} \subset \mathcal{C}$ and measure $\tilde{\Theta} \in \mathcal{M}(\mathcal{B})$, we define the set of feasible flows restricted to $\mathcal{B}$ to be

$$\mathcal{F}_{\mathcal{B}}(\tilde{\Theta}) = \{\mathcal{T} \in \mathcal{M}(\mathcal{B} \times \mathcal{B}) : \mathcal{T}_1 = \tilde{\Theta}, \quad \mathcal{T}_2 \ll \Gamma|_{\mathcal{B}}\},$$

where for any measure $\mathcal{V}$ we denote its restriction to a set $\mathcal{B}$ by $\mathcal{V}|_{\mathcal{B}}$. In addition, we define local equilibria as follows.

**Definition 3** (Local Equilibrium)**.** *For any $\mathcal{B} \subset \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$ and $\tilde{\Theta} \in \mathcal{M}(\mathcal{B})$, a flow $\mathcal{T} \in \mathcal{F}_{\mathcal{B}}(\tilde{\Theta})$ is a local equilibrium in $\mathcal{B}$ if it satisfies*

$$\mathcal{T}\left(\left\{(x,y) \in \mathcal{B} \times \mathcal{B} : \Pi(x,y,p(y),s^{\mathcal{T}}(y)) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x,\cdot,p(\cdot),s^{\mathcal{T}}(\cdot)\Big)\right\}\right) = \tilde{\Theta}(\mathcal{B}).$$

That is, a local equilibrium in $\mathcal{B}$ is a feasible flow such that no driver wishes to unilaterally change his destination when restricting attention to the set $\mathcal{B}$. With this definition in hand, we may now state our next result. Informally, this result states the following "pasting" property. Suppose we start from a price-equilibrium pair $(p, \mathcal{T})$ and a sink $z$ and its attraction region $A(z|\, p, \mathcal{T})$. Then, we can replace the flow that occurs within $A(z|\, p, \mathcal{T})$ with any other local equilibrium within that attraction region as long as certain properties of $V(x|\, p, \mathcal{T})$ are maintained in $A(z|\, p, \mathcal{T})$.

**Proposition B-2.** *(Pasting) Let $(p, \mathcal{T})$ be a feasible solution of ($\mathcal{P}_2$), and let $z \in \mathcal{C}$ be a sink. Denote $\mathcal{A} = A(z|\, p, \mathcal{T})$ and $\mathcal{L} = \bigcup_{a \in R_z} \{X_a(z|\, p, \mathcal{T})\}$. Let $\tilde{\Theta} \in \mathcal{M}(\mathcal{A})$ be the measure representing drivers that stay within $\mathcal{A}$ according to flow $\mathcal{T}$, i.e., $\tilde{\Theta}(\mathcal{B}) \triangleq \mathcal{T}(\mathcal{B} \times \mathcal{A})$ for any measurable set $\mathcal{B} \subseteq \mathcal{A}$. Suppose there exists a measurable price mapping $\tilde{p} : \mathcal{A} \to [0, \overline{V}]$ and a flow $\tilde{\mathcal{T}} \in \mathcal{F}_{\mathcal{A}}(\tilde{\Theta})$ such that $\tilde{\mathcal{T}}$ is a local equilibrium in $\mathcal{A}$ under pricing $\tilde{p}$. Furthermore, suppose $V_{\mathcal{A}}(\cdot|\, \tilde{p}, \tilde{\mathcal{T}})$ is lower or equal than $V(\cdot|\, p, \mathcal{T})$ in $\mathcal{A}$, and that $\tilde{\Theta}(\{x \in \mathcal{A} : V_{\mathcal{A}}(x|\, \tilde{p}, \tilde{\mathcal{T}}) = V(x|\, p, \mathcal{T})\}) = \tilde{\Theta}(\mathcal{A})$. Define the pasted pricing function $\hat{p} : \mathcal{C} \to [0, \overline{V}]$,*

$$\hat{p}(x) \triangleq \begin{cases} \tilde{p}(x) & \text{if } x \in \mathcal{A}; \\ p(x) & \text{if } x \in \mathcal{A}^c, \end{cases}$$

*and the pasted flow $\hat{\mathcal{T}} \in \mathcal{F}(\Theta)$, where for any measurable $\mathcal{B} \subseteq \mathcal{C} \times \mathcal{C}$*

$$\hat{\mathcal{T}}(\mathcal{B}) \triangleq \mathcal{T}(\mathcal{B} \cap ((\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c)) + \tilde{\mathcal{T}}(\mathcal{B} \cap (\mathcal{A} \times \mathcal{A})).$$

*Then, the pasted solution $(\hat{p}, \hat{\mathcal{T}})$ is a feasible solution of problem ($\mathcal{P}_2$) such that*

$$s^{\hat{\mathcal{T}}} = \begin{cases} s^{\tilde{\mathcal{T}}}(x) & \text{if } x \in \mathcal{A}; \\ s^{\mathcal{T}}(x) & \text{if } x \in \mathcal{A}^c. \end{cases}$$

***Proof of Proposition B-2.*** For ease of notation we use $X_a$ to denote $X_a(z|p, \mathcal{T})$. We show that $\hat{\mathcal{T}}$ belongs to $\mathcal{F}_{\mathcal{C}}(\Theta)$ and that it is an equilibrium in $\mathcal{C}$. First we argue that $\hat{\mathcal{T}} \in \mathcal{F}_{\mathcal{C}}(\Theta)$. Since $\hat{\mathcal{T}}$ is the sum of two non-negative measures we have that $\hat{\mathcal{T}} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. In order see why $\hat{\mathcal{T}}_1$ coincides with $\Theta$, let $B$ be a measurable subset of $\mathcal{C}$ then

$$\begin{aligned}\hat{\mathcal{T}}_1(B) &= \hat{\mathcal{T}}(B\times\mathcal{C})\\ &= \mathcal{T}((B\cap(\mathcal{A}^c\cup\mathcal{L}))\times\mathcal{A}^c)+\tilde{\mathcal{T}}((B\cap\mathcal{A})\times\mathcal{A})\\ &\overset{(a)}{=} \mathcal{T}((B\cap(\mathcal{A}^c\cup\mathcal{L}))\times\mathcal{A}^c)+\tilde{\Theta}(B\cap\mathcal{A})\\ &= \mathcal{T}((B\cap(\mathcal{A}^c\cup\mathcal{L}))\times\mathcal{A}^c)+\mathcal{T}((B\cap\mathcal{A})\times\mathcal{A})\\ &\overset{(b)}{=} \mathcal{T}((B\cap\mathcal{A}^c)\times\mathcal{A}^c)+\mathcal{T}((B\cap\mathcal{L})\times\mathcal{A}^c)+\mathcal{T}((B\cap\mathcal{A})\times\mathcal{A})\\ &\overset{(c)}{=} \mathcal{T}((B\cap\mathcal{A}^c)\times\mathcal{C})+\mathcal{T}((B\cap\mathcal{A})\times\mathcal{C})\\ &= \Theta(B),\end{aligned}$$

where $(a)$ comes from the fact that $\tilde{\mathcal{T}}$ belongs to $\mathcal{F}_{\mathcal{A}}(\tilde{\Theta})$. In $(b)$ we use the fact that $\mathcal{A}$ is a closed set. Equality $(c)$ comes from Proposition B-1 part *(i)*. That is, $\hat{\mathcal{T}}_1$ coincides with $\Theta$. Now, we show that $\hat{\mathcal{T}}_2 \ll \Gamma$. Let $B$ be as before and suppose $\Gamma(B)=0$ then

$$\hat{\mathcal{T}}_2(B)=\hat{\mathcal{T}}(\mathcal{C}\times B)=\mathcal{T}((\mathcal{A}^c\cup\mathcal{L})\times(B\cap\mathcal{A}^c))+\tilde{\mathcal{T}}(\mathcal{A}\times(B\cap\mathcal{A}))\le\mathcal{T}_2(B\cap\mathcal{A}^c)+\tilde{\mathcal{T}}_2(B\cap\mathcal{A})=0,$$

where the last equality holds because $\mathcal{T}_2\ll\Gamma$ and $\tilde{\mathcal{T}}_2\ll\Gamma|_{\mathcal{A}}$. Now we show that $\hat{\mathcal{T}}$ is an equilibrium. We need to verify that $\hat{\mathcal{T}}(\hat{\mathcal{E}})$ equals $\Theta(\mathcal{C})$, where

$$\hat{\mathcal{E}}\triangleq\left\{(x,y)\in\mathcal{C}\times\mathcal{C}:\Pi(x,y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y))=\operatorname*{ess\,sup}_{\mathcal{C}}\Pi\Big(x,\cdot,\hat{p}(\cdot),s^{\hat{\mathcal{T}}}(\cdot)\Big)\right\}.$$

In order to verify this we compute first $s^{\hat{\mathcal{T}}}$ and $V(x|\,\hat{p},\hat{\mathcal{T}})$. First we show that $\Gamma-$a.e we have

$$s^{\hat{\mathcal{T}}}(x)=\begin{cases}s^{\mathcal{T}}(x) & \text{if } x\in\mathcal{A}^c\\ s^{\tilde{\mathcal{T}}}(x) & \text{if } x\in\mathcal{A}.\end{cases}$$

Let $B$ be a measurable subset of $\mathcal{A}^c$ then

$$\hat{\mathcal{T}}_2(B)=\mathcal{T}((\mathcal{C}\times B)\cap((\mathcal{A}^c\cup\mathcal{L})\times\mathcal{A}^c))=\mathcal{T}((\mathcal{A}^c\cup\mathcal{L})\times B)\overset{(a)}{=}\mathcal{T}(\mathcal{C}\times B)=\mathcal{T}_2(B),$$

where $(a)$ comes from Proposition B-1 part *(i)*. Therefore, $s^{\hat{\mathcal{T}}}(x)$ equals $s^{\mathcal{T}}(x)$ $\Gamma-a.e.\ x$ in $\mathcal{A}^c$. Similarly, for $B$ a measurable subset of $\mathcal{A}$ we have

$$\hat{\mathcal{T}}_2(B)=\tilde{\mathcal{T}}(\mathcal{A}\times B)=\tilde{\mathcal{T}}_2(B),$$

where the second equality holds because from Proposition B-1 we have $\mathcal{T}(\mathcal{A}^c\times\mathcal{A})=0$, and also because $\tilde{\mathcal{T}}$ is an equilibrium in $\mathcal{A}$.

We next show that for $\Theta_m(\mathcal{B})$ defined by $\tau(\mathcal{B}\times\mathcal{A}^c)$ for $\mathcal{B}\subset\mathcal{A}^c\cup\mathcal{L}$, $V(x|\,\hat{p},\hat{\mathcal{T}})$ satisfies

$$\Theta_m(\{x\in\mathcal{A}^c\cup\mathcal{L}:V(x|\,\hat{p},\hat{\mathcal{T}})=V(x|\,p,\mathcal{T})\})=\mathcal{T}((\mathcal{A}^c\cup\mathcal{L})\times\mathcal{A}^c).\tag{B-5}$$

First, we argue that $V(x|\,p,\mathcal{T})\ge V(x|\,\hat{p},\hat{\mathcal{T}})$ for all $x\in\mathcal{A}^c\cup\mathcal{L}$. We proceed by contradiction. Let $x\in\mathcal{A}^c\cup\mathcal{L}$ and suppose that $V(x|\,p,\mathcal{T})<V(x|\,\hat{p},\hat{\mathcal{T}})$ then we must have that

$$\begin{aligned}&0<\Gamma(y\in\mathcal{C}:\Pi(x,\hat{p}(y),s^{\hat{\mathcal{T}}}(y),y)>V(x|\,p,\mathcal{T}))\\ &\quad=\Gamma(y\in\mathcal{A}:\Pi(x,\tilde{p}(y),s^{\tilde{\mathcal{T}}}(y),y)>V(x|\,p,\mathcal{T}))+\Gamma(y\in\mathcal{A}^c:\Pi(x,p(y),s^{\mathcal{T}}(y),y)>V(x|\,p,\mathcal{T}))\\ &\quad\overset{(a)}{=}\Gamma(y\in\mathcal{A}:U(\tilde{p}(y),s^{\tilde{\mathcal{T}}}(y),y)-\|x-y\|>V(x|\,p,\mathcal{T}))\\ &\quad\overset{(b)}{\le}\Gamma(y\in\mathcal{A}:U(\tilde{p}(y),s^{\tilde{\mathcal{T}}}(y),y)-\|x-y\|>V_{\mathcal{A}}(x_y|\,\tilde{p},\tilde{\mathcal{T}})-\|x_y-x\|)\\ &\quad\overset{(c)}{=}\Gamma(y\in\mathcal{A}:U(\tilde{p}(y),s^{\tilde{\mathcal{T}}}(y),y)>V_{\mathcal{A}}(x_y|\,\tilde{p},\tilde{\mathcal{T}})+\|x_y-y\|)\\ &\quad\overset{(d)}{\le}\Gamma(y\in\mathcal{A}:U(\tilde{p}(y),s^{\tilde{\mathcal{T}}}(y),y)>V_{\mathcal{A}}(y|\,\tilde{p},\tilde{\mathcal{T}}))\\ &\quad\overset{(e)}{=}0,\end{aligned}$$

where $(a)$ follows from that the definition of $V(x|\, p, \mathcal{T})$ implies that the second term in the previous line is zero. For any $y \in \mathcal{A}$ we can consider the segment $[x, y]$ that passes through the boundary of $\mathcal{A}$ (because $x \in \mathcal{A}^c \cup \mathcal{L}$), thus, in $(b)$ we take $x_y \in [x, y] \cap \partial\mathcal{A}$ and then apply the Lipschitz property together with the assumption that $V_{\mathcal{A}}(x_y|\, \tilde{p}, \tilde{\mathcal{T}}) \leq V(x_y|\, p, \mathcal{T})$. In $(c)$ we made use of the collinearity of $x, y$ and $x_y$, and in $(d)$ we applied once again the Lipschitz property. The last line $(e)$ follows from Lemma A-2.

Thus to prove Eq. (B-5) we need to show that $\Theta_m(\{x \in \mathcal{A}^c \cup \mathcal{L} : V(x|\, \hat{p}, \hat{\mathcal{T}}) < V(x|\, p, \mathcal{T})\}) = 0$. If this is not true then since $V(x|\, \hat{p}, \hat{\mathcal{T}}) \geq V_{\mathcal{A}^c}(x|\, p, \mathcal{T})$ we have

$$\begin{aligned}
0 &< \Theta_m(\{x \in \mathcal{A}^c \cup \mathcal{L} : V_{\mathcal{A}^c}(x|\, p, \mathcal{T}) < V(x|\, p, \mathcal{T})\}) \\
&= \mathcal{T}\Big((x, y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : V_{\mathcal{A}^c}(x|\, p, \mathcal{T}) < V(x|\, p, \mathcal{T})\Big) \\
&\overset{(a)}{=} \mathcal{T}\Big((x, y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : V_{\mathcal{A}^c}(x|\, p, \mathcal{T}) < V(x|\, p, \mathcal{T}), U(p(y), s^{\mathcal{T}}(y), y) - \|x - y\| = V(x|\, p, \mathcal{T})\Big) \\
&\leq \mathcal{T}\Big((x, y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : U(p(y), s^{\mathcal{T}}(y), y) - \|x - y\| > V_{\mathcal{A}^c}(x|\, p, \mathcal{T})\Big) \\
&\leq \mathcal{T}_2\Big(y \in \mathcal{A}^c : \exists x \in (\mathcal{A}^c \cup \mathcal{L}), U(p(y), s^{\mathcal{T}}(y), y) - \|x - y\| > V_{\mathcal{A}^c}(x|\, p, \mathcal{T})\Big) \\
&\overset{(b)}{=} \mathcal{T}_2\Big(y \in \mathcal{A}^c : \exists x \in (\mathcal{A}^c \cup \mathcal{L}), V_{\mathcal{A}^c}(y|\, p, \mathcal{T}) - \|x - y\| > V_{\mathcal{A}^c}(x|\, p, \mathcal{T})\Big) \\
&\overset{(c)}{=} 0,
\end{aligned}$$

where $(a)$ is true because $\mathcal{T}$ is supported in a set where $U(p(y), s^{\mathcal{T}}(y), y) - \|x - y\| = V(x|\, p, \mathcal{T})$, $(b)$ comes from Lemma A-2 and $(c)$ from the Lipschitz property. This proves Eq. (B-5) .

Lastly, we verify that $\hat{\mathcal{T}}(\hat{\mathcal{E}})$ equals $\Theta(\mathcal{C})$. Define the sets

$$\begin{aligned}
\mathcal{E}_1 &\triangleq \Big\{(x, y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : \Pi(x, y, \hat{p}(y), s^{\hat{\mathcal{T}}}(y)) = V(x|\, \hat{p}, \hat{\mathcal{T}})\Big\} \\
\mathcal{E}_2 &\triangleq \Big\{(x, y) \in \mathcal{A} \times \mathcal{A} : \Pi(x, y, \hat{p}(y), s^{\hat{\mathcal{T}}}(y)) = V(x|\, \hat{p}, \hat{\mathcal{T}})\Big\}
\end{aligned}$$

then $\hat{\mathcal{T}}(\hat{\mathcal{E}}) = \mathcal{T}(\mathcal{E}_1) + \tilde{\mathcal{T}}(\mathcal{E}_2)$. We can replace the definition of $\hat{p}$ and what we have proved about $s^{\hat{\mathcal{T}}}$ in the expressions above to obtain

$$\begin{aligned}
\mathcal{T}(\mathcal{E}_1) &= \mathcal{T}\Big(\Big\{(x, y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = V(x|\, \hat{p}, \hat{\mathcal{T}})\Big\}\Big), \\
\tilde{\mathcal{T}}(\mathcal{E}_2) &= \tilde{\mathcal{T}}\Big(\Big\{(x, y) \in \mathcal{A} \times \mathcal{A} : \Pi(x, y, \tilde{p}(y), s^{\tilde{\mathcal{T}}}(y)) = V(x|\, \hat{p}, \hat{\mathcal{T}})\Big\}\Big).
\end{aligned}$$

From Eq. (B-5) we deduce that

$$\mathcal{T}(\mathcal{E}_1) = \mathcal{T}\Big(\Big\{(x, y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = V(x|\, p, \mathcal{T})\Big\}\Big).$$

Since $V_{\mathcal{A}}(x|\tilde{p}, \tilde{\mathcal{T}}) \leq V(x|p, \mathcal{T})\})$ for $x \in \mathcal{A}$ we deduce that $V(x|\hat{p}, \hat{\mathcal{T}}) \leq V(x|p, \mathcal{T})$ for $x \in \mathcal{A}$. Hence, because $\tilde{\Theta}(\{x \in \mathcal{A} : V_{\mathcal{A}}(x|\, \tilde{p}, \tilde{\mathcal{T}}) = V(x|\, p, \mathcal{T})\}) = \tilde{\Theta}(\mathcal{A})$ and $V(x|\, \hat{p}, \hat{\mathcal{T}}) \geq V_{\mathcal{A}}(x|\, \tilde{p}, \tilde{\mathcal{T}})$ we must have

$$\tilde{\Theta}(\{x \in \mathcal{A} : V_{\mathcal{A}}(x|\, \tilde{p}, \tilde{\mathcal{T}}) = V(x|\, \hat{p}, \hat{\mathcal{T}})\}) = \tilde{\Theta}(\mathcal{A}),$$

in turn,

$$\tilde{\mathcal{T}}(\mathcal{E}_2) = \tilde{\mathcal{T}}\Big(\Big\{(x, y) \in \mathcal{A} \times \mathcal{A} : \Pi(x, y, \tilde{p}(y), s^{\tilde{\mathcal{T}}}(y)) = V_{\mathcal{A}}(x|\, \tilde{p}, \tilde{\mathcal{T}})\Big\}\Big) = \tilde{\Theta}(\mathcal{A}),$$

where the second line comes from the fact that $\tilde{\mathcal{T}}$ is an equilibrium in $\mathcal{A}$. Let $\mathcal{E}$ be defined analogously to $\hat{\mathcal{E}}$ but with $(\hat{p}, \hat{\mathcal{T}})$ replaced by $(p, \mathcal{T})$, then

$$\begin{aligned}
\hat{\mathcal{T}}(\hat{\mathcal{E}}) = \mathcal{T}(\mathcal{E}_1) + \tilde{\Theta}(\mathcal{A}) \overset{(a)}{=} \mathcal{T}(\mathcal{E}_1) + \mathcal{T}(\mathcal{A} \times \mathcal{A}) \overset{(b)}{=} \mathcal{T}(\mathcal{E} \cap ((\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c)) + \mathcal{T}(\mathcal{E} \cap (\mathcal{A} \times \mathcal{A})) &\overset{(c)}{=} \mathcal{T}(\mathcal{E}) \\
&\overset{(d)}{=} \Theta(\mathcal{C}),
\end{aligned}$$

where in $(a)$ we use the definition of $\tilde{\Theta}$. In $(b)$ and $(d)$ we use the fact that $\mathcal{T}$ only puts mass in $\mathcal{E}$, and in $(c)$ we use Proposition B-1 part *(i)*.

$\square$

# C Proofs for Section 7

***Proof of Theorem 1***. The proof of this theorem consists of several parts. In the first part perform a disintegration step that will enable us to analyze the problem along rays stemming from $z$. Then we pose an optimization problem which is a relaxation of platform's optimization problem restricted to the attraction region $A(z)$. Then we introduce some notation. Given this, the relaxation has a similar structure to a continuous bounded knapsack problem, and we characterize the structure of the optimal solution as stated in the statement of the theorem. Next we construct a local price-equilibrium pair $(\hat{p}, \hat{\mathcal{T}})$ in $A(z)$ that implements the relaxation's solution. We conclude by applying the pasting result of Proposition B-2 to globally extend our price-equilibrium pair $(\hat{p}, \hat{\mathcal{T}})$ in $\mathcal{C}$ as in the statement of the theorem. In summary the parts of the proof are: Disintegration, Relaxation, Notation, Knapsack, Implementation and Conclusion. We enumerate all these parts from 0 to 5, and present them in boldface to make the presentation clearer.

**Part 0: Disintegration.** Recall that we use $R_z$ to denote the set of all rays originating from $z$ (excluding $z$) and index the elements of $R_z$ by $a$. The advantage of this is that now we can disintegrate the city measure into a family of measures concentrated along the rays, $\{\Gamma_a\}$, which we can integrate with respect to another measure $\Gamma^{\mathrm{p}}$ in $R_z$ to obtain $\Gamma$. That is, for any measurable set $\mathcal{B}$ we have

$$\Gamma(\mathcal{B}) = \Gamma(\{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z} \Gamma_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a). \tag{C-1}$$

Any measure $\Gamma_a$ can be thought of as a conditional probability, and we can use it to measure quantities such as supply and demand along ray $a \in R_z$. See Ambrosio & Pratelli (2003) for a formal statement about disintegration of measures.

**Part 1: Relaxation.** We consider the attraction region $A(z)$. In it, the congestion bound must be satisfied. Moreover, due to our flow separation result in Proposition B-1 part *(i)* we have $\mathcal{T}_2(A(z)) = \mathcal{T}(A(z) \times A(z))$. Also, since flow is not transported across rays (see Proposition B-1 part *(ii)*), the total supply in the ray $(z, X_a]$ cannot be larger than its initial supply. Therefore, in $A(z)$ the platform's problem is bounded above by

$$\max_{s(\cdot)} \quad \int_{A(z)} V(x) \cdot s(x)\, d\Gamma(x) \tag{$\mathcal{P}_{KP}(z)$}$$

$$\text{s.t} \quad s(x) \leq H_x(V(x|\, p, \mathcal{T})), \quad \Gamma - a.e.\ x \text{ in } A(z) \tag{CB}$$

$$\int_{A(z)} s(x)\, d\Gamma(x) = \mathcal{T}(A(z) \times A(z)) \tag{FC}$$

$$\int_{(z,X_a]} s(x)\, d\Gamma_a(x) \leq \int_{(z,X_a]} s^{\mathcal{T}}(x)\, d\Gamma_a(x), \quad \Gamma^{\mathrm{p}} - a.e.\ a \in R_z, \tag{FR$_a$}$$

where $H_x(V) = \psi_x^{-1}(V)$. Observe that $s^{\mathcal{T}}$ (which defines $\mathcal{T}_2$) is a feasible solution for $(\mathcal{P}_{KP}(z))$. The supply density $s^{\hat{\mathcal{T}}}$ (as in the statement of the present theorem) will be shown to be an optimal solution for this relaxation.

**Part 2: Notation.**

1. Next we rename the quantities on the RHS of equations (FC) and (FR$_a$).

$$\mathcal{T}_{\text{total}} = \mathcal{T}(A(z) \times A(z)),\ \ \mathcal{T}_a = \int_{(z,X_a]} s^{\mathcal{T}}(x)\, d\Gamma_a(x),\ \ \mathcal{T}_c = \mathcal{T}(A(z) \times \{z\}).$$

2. For any measurable set $B \subseteq Az$ we define the measure

$$S^H(B) \triangleq \int_B H_x(V(x))\, d\Gamma(x),$$

$S^H(\cdot)$ is the measure with density $H_x(V(x))$ with respect to the $\Gamma$ measure. Moving forward we will use $s^H(x)$ to denote its density.

**Part 3: Knapsack.** We show that any optimal solution to ($\mathcal{P}_{KP}(z)$) is as $s^{\hat{\mathcal{T}}}$ in the statement of the theorem. There are two cases.

**Case 1.** First suppose that $0 < \mathcal{T}_{\text{total}} \le S^H(\{z\})$ (so that there is an atom at $z$). Then, we define $r_a = z$ for all $a \in R_z$, and let the solution to ($\mathcal{P}_{KP}(z)$) be

$$s^\star(x) = \frac{\mathcal{T}_{\text{total}}}{\Gamma(\{z\})} \cdot \mathbf{1}_{\{x=z\}},$$

which is feasible, and optimal because for any feasible $s$ we have

$$\int_{A(z)} V(x) \cdot s(x) d\Gamma(x) \le V(z) \cdot \int_{A(z)} s(x)\, d\Gamma(x) = V(z) \cdot \mathcal{T}_{\text{total}},$$

which is equal to the objective function at $s^\star$. So in this case the optimal solution coincides with the description of $s^{\hat{\mathcal{T}}}$ as in the statement of the theorem.

**Case 2.** Now let us assume that $\mathcal{T}_{\text{total}} > S^H(\{z\})$. We start by showing that in this case we have $s^\star(z) = s^H(z)$. If $z$ is not a point with positive $\Gamma$- mass then setting $s^\star(z)$ in this way is without loss of generality. If the point $z$ has positive mass then we argue by contradiction that $s^\star(z)$ must be choose in this way. Let $s^\star$ be an optimal solution to ($\mathcal{P}_{KP}(z)$) such that $s^\star(z) < s^H(z)$. Then,

$$\mathcal{T}_{\text{total}} = \int_{A(z)\setminus\{z\}} s^\star d\Gamma + s^\star(z) \cdot \Gamma(\{z\}) < \underbrace{\int_{A(z)\setminus\{z\}} s^\star d\Gamma}_{K} + s^H(z) \cdot \Gamma(\{z\}). \tag{C-2}$$

Let $\epsilon \in (0, 1)$ be such that $(\mathcal{T}_{\text{total}} - \epsilon \cdot K)/\Gamma(\{z\}) = s^H(z)$, this is well defined because we are assuming $\mathcal{T}_{\text{total}} > S^H(\{z\})$. Next define a new solution $\bar{s}$ by

$$\bar{s}(x) = \begin{cases} s^H(z) & \text{if } x = z, \\ \epsilon \cdot s^\star(x) & \text{if } x \neq z. \end{cases}$$

Note that $\bar{s}$ is feasible: it satisfies (FR$_a$) for all $a \in R_z$ and (CB), and for (FC) we have

$$\int_{A(z)} \bar{s} d\Gamma = \epsilon \cdot K + s^H(z) \cdot \Gamma(\{z\}) = \mathcal{T}_{\text{total}}.$$

Furthermore, $\bar{s}$ yields an strictly larger objective than $s^\star$,

$$\begin{aligned}
\int_{A(z)} V(x) \cdot s^\star(x)\, d\Gamma(x) &= \int_{A(z)\setminus\{z\}} V(x) \cdot s^\star(x)\, d\Gamma(x) + V(z) \cdot s^\star(z) \cdot \Gamma(\{z\}) \\
&= \epsilon \cdot \int_{A(z)\setminus\{z\}} V(x) \cdot s^\star(x)\, d\Gamma(x) + (1-\epsilon) \cdot \int_{A(z)\setminus\{z\}} V(x) \cdot s^\star(x)\, d\Gamma(x) \\
&+ V(z) \cdot s^\star(z) \cdot \Gamma(\{z\}) \\
&\overset{(a)}{<} \int_{A_{\text{p}}(z)\setminus\{z\}} V(x) \cdot \bar{s}(x)\, d\Gamma(x) + (1-\epsilon) \cdot V(z) \cdot K \\
&+ V(z) \cdot s^\star(z) \cdot \Gamma(\{z\}) \\
&= \int_{A(z)\setminus\{z\}} V(x) \cdot \bar{s}(x)\, d\Gamma(x) + V(z) \cdot (\mathcal{T}_{\text{total}} - \epsilon \cdot K) \\
&\overset{(b)}{=} \int_{A(z)} V(x) \cdot \bar{s}(x)\, d\Gamma(x),
\end{aligned}$$

where $(a)$ comes from Eq. (C-2), and $(b)$ holds because $(\mathcal{T}_{\text{total}} - \epsilon \cdot K)/\Gamma(\{z\}) = s^H(z)$. Hence, whenever $\mathcal{T}_{\text{total}} > S^H(\{z\})$, we can assume that $s^\star(z) = s^H(z)$. We assume this for the reminder of the proof.

Let $s^\star(z)$ be an optimal solution. We show how to build $\hat{s}$ with the properties described in the theorem's statement. Next we construct $r_a$. First, note that

$$\int_{A(z)\setminus\{z\}} s^\star(x)d\mathit{\Gamma} = \int_{A(z)\setminus\{z\}} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}d\mathit{\Gamma} = \int_{R_z} \underbrace{\int_{(z,X_a]} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}d\mathit{\Gamma}_a(x)}_{q_a} d\mathit{\Gamma}^{\mathrm{p}}(a)$$

define $r_a$ by

$$r_a \triangleq \inf\Big\{r\in(z,X_a] : \int_{(z,r]} s^H(x)\mathbf{1}_{\{s^\star\le s^H\}}d\mathit{\Gamma}_a(x)\ge q_a\Big\}.$$

Observe that for $r=X_a$ the integral in the definition of $r_a$ is larger or equal than $q_a$. Therefore, $r_a$ is well defined. Let us define (with some abuse of notation)

$$A_r(z)\triangleq\bigcup_{a\in R_z}[z,r_a],\quad \text{and}\quad L_r(z)\triangleq\bigcup_{a\in R_z}\{r_a\}.$$

Let's define a new solution $\hat{s}$ by

$$\hat{s}(x)\triangleq\begin{cases} s^\star(z)=s^H(z) & \text{if } x=z\\ s^H(x) & \text{if } x\in A_r(z)\setminus(L_r(z)\cup\{z\}),\\ \mathbf{1}_{\{\mathit{\Gamma}_a(\{x\})>0\}}\dfrac{\left(q_a-\int_{(0,x)}s^H(y)\mathbf{1}_{\{s^\star\le s^H\}}d\mathit{\Gamma}_a(y)\right)}{\mathit{\Gamma}_a(\{x\})} & \text{if } x\in L_r(z),\end{cases}$$

and $\hat{s}(x)=0$ otherwise. We show that $\hat{s}$ weakly revenue dominates $s^\star$ and that is feasible. Let us do first the <u>revenue dominance</u>. Note that the objective in $\{z\}$ of both solutions coincide; thus, we only need to compare the objective in the set $Q\triangleq A(z)\setminus\{z\}$. Note that $A_r(z)\setminus\{z\}\subset Q$, then

$$\begin{aligned}\int_Q V(x)\cdot s^\star(x)\,d\mathit{\Gamma}(x) &= \int_{A_r(z)\setminus\{z\}} V(x)\cdot s^\star(x)\,d\mathit{\Gamma}(x)+\int_{Q\setminus(A_r(z)\setminus\{z\})}V(x)\cdot s^\star(x)\,d\mathit{\Gamma}(x)\\ &= \int_{A_r(z)\setminus\{z\}} V(x)\cdot \hat{s}(x)\,d\mathit{\Gamma}(x)+\int_{A_r(z)\setminus\{z\}}V(x)\cdot (s^\star-\hat{s})(x)\,d\mathit{\Gamma}(x)\\ &+ \underbrace{\int_{Q\setminus(A_r(z)\setminus\{z\})} V(x)\cdot s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\mathit{\Gamma}(x),}_{I}\end{aligned}$$

for the last term above we have

$$\begin{aligned} I &\le \int_{R_z} V(r_a)\Big[\int_{(r_a,X_a]} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\mathit{\Gamma}_a(x)\Big]d\mathit{\Gamma}^{\mathrm{p}}(a)\\ &= \int_{R_z} V(r_a)\Big[q_a-\int_{(z,r_a]} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\mathit{\Gamma}_a(x)\Big]d\mathit{\Gamma}^{\mathrm{p}}(a)\\ &= \int_{R_z} V(r_a)\Big[\int_{(z,r_a)} (s^H-s^\star)(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\mathit{\Gamma}_a(x)+(\hat{s}-s^\star\mathbf{1}_{\{s^\star\le s^H\}})(r_a)\mathit{\Gamma}_a(r_a)\Big]d\mathit{\Gamma}^{\mathrm{p}}(a)\\ &\le \int_{R_z} \Big[\int_{(z,r_a)} V(x)(s^H-s^\star)(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\mathit{\Gamma}_a(x)+V(r_a)(\hat{s}-s^\star\mathbf{1}_{\{s^\star\le s^H\}})(r_a)\mathit{\Gamma}_a(r_a)\Big]d\mathit{\Gamma}^{\mathrm{p}}(a)\\ &= \int_{A_r(z)\setminus\{z\}} V(x)(\hat{s}-s^\star)(x)d\mathit{\Gamma}(x),\end{aligned}$$

hence

$$\int_Q V(x)\cdot s^\star(x)\,d\mathit{\Gamma}(x)\le\int_{A_r(z)\setminus\{z\}} V(x)\cdot\hat{s}(x)\,d\mathit{\Gamma}(x).$$

Since the right hand side above equals the objective under $\hat{s}$ in $A_r(z)$ we conclude that $\hat{s}$ is an optimal solution.

For the feasibility of $\hat{s}$, by construction and the definition of $r_a$ we have that $\hat{s}$ satisfies (CB). Furthermore, because $s^\star$ satisfies (FR$_a$) and since $\hat{s}$ only redistributes the mass of $s^\star$ across rays but no between rays that originate in $z$, $\hat{s}$ also satisfies (FR$_a$). In order to verify (FC) note that

$$\begin{aligned}
\int_{A(z)} \hat{s}(x)\, d\Gamma(x) &= \hat{s}(z) \cdot \Gamma(\{z\}) + \int_{A_r(z)\setminus\{z\}} \hat{s}(x)\, d\Gamma(x) \\
&= s^\star(z) \cdot \Gamma(\{z\}) + \int_{A_r(z)\setminus\{z\}} \hat{s}(x)\, d\Gamma(x) \\
&= s^\star(z) \cdot \Gamma(\{z\}) + \int_{R_z} \Big[ \int_{(z,r_a)} s^H(z) \mathbf{1}_{\{s^\star \le s^H\}}\, d\Gamma_a(x) + \hat{s}(r_a)\Gamma_a(\{r_a\}) \Big] d\Gamma^{\mathrm{p}}(a) \\
&= s^\star(z) \cdot \Gamma^{\mathrm{p}}(\{z\}) + \int_{R_z} \Big[ q_a \Big] d\Gamma^{\mathrm{p}}(a) \\
&= s^\star(z) \cdot \Gamma^{\mathrm{p}}(\{z\}) + \int_{R_z} \Big[ \int_{(z,X_a]} s^\star(x) \mathbf{1}_{\{s^\star \le s^H\}} d\Gamma_a(x) \Big] d\Gamma^{\mathrm{p}}(a) \\
&= s^\star(z) \cdot \Gamma^{\mathrm{p}}(\{z\}) + \int_{A(z)\setminus\{z\}} s^\star(x)\, d\Gamma(x) \\
&= \mathcal{T}_{\text{total}}.
\end{aligned}$$

In conclusion, the solution $\hat{s}$ constructed is as defined in the statement of the theorem. Next, we use this solution to define prices and flows. We use $\hat{S}$ to denote the measure induced by $\hat{s}$. Observe that $\hat{S}$ has support in $A_r(z)$.

**Part 4: Implementation.** We construct a price-equilibrium pair $(\hat{p}, \hat{\mathcal{T}})$ in $A(z)$ with $\hat{\mathcal{T}} \in \mathcal{F}_{A(z)}(\tilde{\Theta})$ and

$$\tilde{\Theta}(\mathcal{B}) \triangleq \mathcal{T}((\mathcal{B} \cap A(z)) \times A(z)), \quad \mathcal{B} \subseteq \mathcal{C} \text{ measurable.}$$

- **Prices.** Define $\hat{p} : A(z) \to [0, \overline{V}]$ by

$$\hat{p}(x) = \begin{cases} \rho_x^{loc}(\hat{s}(x)) & \text{if } x \in A_r(z) \setminus L_r(z); \\ p_a & \text{if } x = r_a,\ a \in R_z; \\ \overline{V} & \text{otherwise,} \end{cases}$$

  where $p_a$ is such that $U(r_a, p_a, \hat{s}(r_a)) = V(r_a |\, p, \mathcal{T})$ for $a \in R_z$. By the way we constructed $\hat{s}(r_a)$, it is bounded by $H_{r_a}(V(r_a))$ and, therefore, the value $p_a$ is always well defined ($\Gamma$-a.e).

- **Flows:** We define $\hat{\mathcal{T}}$ as a transport plan between $\tilde{\Theta}$ and $\hat{S}$. We start by defining the flow that $\hat{\mathcal{T}}$ sends to $z$ and then the flow along rays.

  **Flow to the center.** Next we define the flow that $\hat{\mathcal{T}}$ sends to $\{z\}$. We define

$$\tilde{\Theta}_a(\mathcal{B}) \triangleq \int_{\mathcal{B}\cap(z,X_a]} \frac{d\tilde{\Theta}}{d\Gamma}(y) d\Gamma_a(y) \text{ and } \hat{S}_a(\mathcal{B}) \triangleq \int_{\mathcal{B}\cap(z,X_a]} \frac{d\hat{S}}{d\Gamma}(y) d\Gamma_a(y).$$

  Then,

$$\tilde{\Theta}(\mathcal{B}) = \mathcal{T}(\{z\} \times \{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z} \tilde{\Theta}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \text{ and } \hat{S}(\mathcal{B}) = \hat{S}(\{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a).$$

  We define the quantities

$$\varDelta_a \triangleq \tilde{\Theta}_a((z, X_a]) - \hat{S}_a((z, X_a]),$$

  note that because of (FR$_a$), $\varDelta_a \ge 0$, $\Gamma^{\mathrm{p}} - a.e$ $a$ in $R_z$. Further define

$$h_a \triangleq z + \inf\{\delta \ge 0 : \tilde{\Theta}_a((z, z+\delta]) \ge \varDelta_a\}.$$

For any set $\mathcal{B} \subseteq A(z)$ we define the mass going to the center from ray $a \in R_z$ by the measures

$$\Theta^c_a(\mathcal{B}) \triangleq \tilde{\Theta}_a(\mathcal{B} \cap (z, h_a)) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\Delta_a - \tilde{\Theta}_a(z, h_a)),$$

observe that by the definition $h_a$, the atoms above have non-negative mass, $\Gamma^{\mathrm{p}} - a.e$ $a$ in $R_z$ . Let $\mathcal{Q}_z \triangleq \{z\} \times \{z\}$. For any measurable set $\mathcal{R} \subseteq A(z) \times A(z)$, the measure that sends flow to the origin is defined by

$$\mathcal{T}^c(\mathcal{R}) \triangleq \mathcal{T}(\mathcal{R} \cap \mathcal{Q}_z) + \int_{R_z} \Theta^c_a(\pi_1(\mathcal{R} \cap A(z) \times \{z\}))d\Gamma^{\mathrm{p}}(a),$$

where $\pi_1$ is the mapping that to each pair $(x, y)$ assigns the first component $x$. Using Lemma C-1 (which we state and prove after the present proof) we can verify that $\mathcal{T}^c \in \mathcal{M}(A(z) \times A(z))$.

**Flow along rays.** For any ray $a \in R_z$ define the flow $\tilde{\gamma}_a$ along that ray to be the solution to the following optimal transport problem:

$$\begin{aligned} \min & \int_{(z,X_a] \times (z,X_a]} \|x - y\| \, d\gamma_a(x, y) \\ \text{s.t } & \gamma_a \in \Pi(\tilde{\Theta}^{\mathrm{r}}_a, \hat{S}_a), \end{aligned}$$

where

$$\tilde{\Theta}^{\mathrm{r}}_a(\mathcal{B}) \triangleq \tilde{\Theta}_a(\mathcal{B} \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\tilde{\Theta}_a(z, h_a] - \Delta_a),$$

where $\Pi(\tilde{\Theta}^{\mathrm{r}}_a, \hat{S}_a)$ is the set of transport plans between $\tilde{\Theta}^{\mathrm{r}}_a$ and $\hat{S}_a$. Any solution to this problem satisfies:

$$\tilde{\gamma}_a\Big(\{(x, y) \in (z, X_a] \times (z, X_a] : y > x\}\Big) = 0, \quad \Gamma^{\mathrm{p}} - a.e. \ a \in R_z. \tag{C-3}$$

We provide a proof Eq. (C-3) after **Part 5**.

We will argue that $\hat{\mathcal{T}}$ defined by

$$\hat{\mathcal{T}}(\mathcal{R}) = \mathcal{T}^c(\mathcal{R}) + \int_{R_z} \tilde{\gamma}_a(\mathcal{R})d\Gamma^{\mathrm{p}}(a)$$

yields an equilibrium, that is, for the set

$$\tilde{\mathcal{E}} \triangleq \Big\{(x, y) \in A(z) \times A(z) : U(y, \hat{p}(y), s^{\hat{\mathcal{T}}}(y)) - |y - x| = V_{A(z)}(x| \, \hat{p}, \hat{\mathcal{T}})\Big\},$$

we have that $\hat{\mathcal{T}}(\tilde{\mathcal{E}})$ equals $\tilde{\Theta}(A(z))$. Note that with this definition of $\hat{\mathcal{T}}$ there is not flow being transported across rays but only within rays. Before verifying the equilibrium condition we check that $\hat{\mathcal{T}} \in \mathcal{F}_{A(z)}(\tilde{\Theta})$. Clearly $\hat{\mathcal{T}}$ is a non-negative measure in $A(z) \times A(z)$ because is the sum of non-negative measures. Now we check that $\hat{\mathcal{T}}_1 = \tilde{\Theta}$. Consider a measurable set $\mathcal{B} \subseteq A(z)$ then

$$\begin{aligned} \tilde{\mathcal{T}}_1(\mathcal{B}) &= \tilde{\mathcal{T}}(\mathcal{B} \times A(z)) \\ &= \mathcal{T}^c(\mathcal{B} \times A(z)) + \int_{R_z} \tilde{\gamma}_a(\mathcal{B} \times A(z))d\Gamma^{\mathrm{p}}(a) \\ &= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z \in \mathcal{B}\}} + \int_{R_z} \Theta^c_a(\mathcal{B})d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \tilde{\Theta}^{\mathrm{r}}_a(\mathcal{B} \cap (z, X_a])d\Gamma^{\mathrm{p}}(a) \\ &= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z \in \mathcal{B}\}} + \int_{R_z} \Big[\tilde{\Theta}_a(\mathcal{B} \cap (z, h_a)) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\Delta_a - \tilde{\Theta}_a(z, h_a)) \\ &+ \tilde{\Theta}_a(\mathcal{B} \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\tilde{\Theta}_a(z, h_a] - \Delta_a)\Big]d\Gamma^{\mathrm{p}}(a) \\ &= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z \in \mathcal{B}\}} + \int_{R_z} \tilde{\Theta}_a(\mathcal{B})d\Gamma^{\mathrm{p}}(a) \\ &= \tilde{\Theta}(\mathcal{B}) \end{aligned}$$

and from the definition of $\tilde{\Theta}$ we also have $\hat{\mathcal{T}}_1 \ll \Gamma$. For the second marginal of $\hat{\mathcal{T}}$ we have

$$\begin{aligned}
\hat{\mathcal{T}}_2(\mathcal{B}) &= \tilde{\mathcal{T}}(A(z)\times\mathcal{B})\\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z} \Theta_a^c(A(z))\mathbf{1}_{\{z\in\mathcal{B}\}} d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a)\\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} \Big[\tilde{\Theta}_a((z,h_a)) + \mathbf{1}_{\{h_a\in(z,X_a]\}}\cdot(\Delta_a - \tilde{\Theta}_a(z,h_a))\Big] d\Gamma^{\mathrm{p}}(a)\\
&+ \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a)\\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} \Delta_a d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a)\\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} [\tilde{\Theta}_a((z,X_a]) - \hat{S}_a((z,X_a])] d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a)\\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} [\tilde{\Theta}_a(A(z)) - \hat{S}_a(A(z))] d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a)\\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \Big[\tilde{\Theta}(A(z)) - \mathcal{T}(\mathcal{Q}_z) - \hat{S}(A(z)) + \hat{S}(\{z\})\Big] + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a)\\
&= \hat{S}(\{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \underbrace{\Big[\tilde{\Theta}(A(z)) - \hat{S}(A(z))\Big]}_{=0} + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a)\\
&= \hat{S}(\mathcal{B}).
\end{aligned}$$

Since $\hat{S}$ is such that $\hat{S} \ll \Gamma$, we conclude that $\hat{\mathcal{T}} \in \mathcal{F}_{A(z)}(\tilde{\Theta})$. Also, $s^{\hat{\mathcal{T}}}$ coincides with $\hat{s}$ $\Gamma$ almost everywhere. Before we move to verify that $\hat{\mathcal{T}}$ is an equilibrium, we next compute $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})$ and $U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y))$.

- **Equilibrium utilities:** From the definition of $\hat{p}$ and the value of $s^{\hat{\mathcal{T}}}$ we have that $\Gamma - a.e.$ $y$ in $A(z)$

$$U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y)) = \begin{cases} V(z|\,p,\mathcal{T}) - |z-y| & \text{if } y\in A_r(z),\\ 0 & \text{if } y \in A(z)\setminus A_r(z).\end{cases}$$

Next we verify that $V_{A(z)}(\cdot|\,\hat{p},\hat{\mathcal{T}})$ satisfies the hypothesis in Proposition B-2, the pasting result. First, for any $x\in A(z)$ we argue that $V(x|\,p,\mathcal{T}) = V(z|\,p,\mathcal{T}) - |z-x| \geq V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})$. It is enough to show that

$$\Gamma(y\in A(z) : U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y)) - |y-x| > V(x|\,p,\mathcal{T})) = 0.$$

Suppose this is not true. Lemma A-1 implies that $V(y|\,p,\mathcal{T})$ is non-negative $\Gamma - a.e.$ Also $V(y|\,p,\mathcal{T})$ equals $V(z|\,p,\mathcal{T}) - |z-y|$ for any $y\in A(z)$. Hence it must be true that $V(y|\,p,\mathcal{T})$ is larger or equal than $U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y))$ $\Gamma - a.e$ (see the value of this expression above). Thus our current assumption implies

$$\Gamma(y\in A(z) : V(y|\,p,\mathcal{T}) - |y-x| > V(x|\,p,\mathcal{T})) > 0,$$

but this contradicts the Lipschitz property of $V(\cdot|\,p,\mathcal{T})$. Second, we show that

$$\tilde{\Theta}(\{x\in A(z) : V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) = V(x|\,p,\mathcal{T})\}) = \tilde{\Theta}(A(z)). \tag{C-4}$$

If this is not true then $\tilde{\Theta}(\mathcal{X}) > 0$ where $\mathcal{X} = \{x\in A(z) : V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) < V(x|\,p,\mathcal{T})\}$. In turn this implies that $\hat{\mathcal{T}}(\mathcal{X}\times A(z)) > 0$. Then as in the proof of Lemma B-3 (see Eq. (B-2)) we have that there exists $(x,y)\in\mathcal{X}\times A(z)$ such that $\hat{\mathcal{T}}(B(x,\delta)\times B(y,\delta)) > 0$ for all $\delta > 0$,where $B(x,\delta)$ is an open ball of radius $\delta$. Since $x\in\mathcal{X}$ we have that $x\in A(z)$ and $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) < V(x|\,p,\mathcal{T})$. Thus we can find $\delta > 0$ such that $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) + 2\delta < V(x|\,p,\mathcal{T})$. Moreover since $\hat{\mathcal{T}}$ only send flows along rays originating from $z$ we must have that $\hat{\mathcal{T}}(B(x,\delta)\times B(y,\delta)) = \hat{\mathcal{T}}(\{(\bar{x},\bar{y})\in B(x,\delta)\times B(y,\delta) : \|z-x\|+2\delta \geq \|\bar{y}-z\|+\|\bar{y}-x\|\})>0$. This in turn implies that

$$\begin{aligned}
0 &< \Gamma(\bar{y}\in B(y,\delta)\cap A_r(z) :\ 2\delta+\|z-x\| \geq \|\bar{y}-x\|+\|\bar{y}-z\|)\\
&= \Gamma(\bar{y}\in B(y,\delta)\cap A_r(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y}))+\ 2\delta+\|z-x\| \geq V(\bar{y}|\,p,\mathcal{T})+\|\bar{y}-x\|+\|\bar{y}-z\|)\\
&= \Gamma(\bar{y}\in B(y,\delta)\cap A_r(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y}))-\|\bar{y}-x\| \geq V(x|\,p,\mathcal{T})-2\delta)\\
&\leq \Gamma(\bar{y}\in B(y,\delta)\cap A_r(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y}))-\|\bar{y}-x\| > V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}))\\
&\leq \Gamma(\bar{y}\in A(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y}))-\|\bar{y}-x\| > V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})),
\end{aligned}$$

which contradicts the definition of $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})$. This shows that Eq. (C-4) holds.

- **Equilibrium condition:** Consider the equilibrium set

$$\tilde{\mathcal{E}} \triangleq \Big\{(x,y)\in A(z)\times A(z) : U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y))-\|y-x\| = V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})\Big\},$$

we need to verify that $\hat{\mathcal{T}}(\tilde{\mathcal{E}})$ equals $\tilde{\Theta}(A(z))$. First, for $\hat{\mathcal{T}}(\tilde{\mathcal{E}})$ we have,

$$\hat{\mathcal{T}}(\tilde{\mathcal{E}}) \overset{(a)}{=} \hat{\mathcal{T}}\Big(\Big\{(x,y)\in A(z)\times A_r(z) :\ \|z-y\|+\|y-x\| = \|z-x\|\Big\}\Big)$$

In $(a)$ we use Eq. (C-4), that $V(x|\,p,\mathcal{T}) = V(z|\,p,\mathcal{T})-\|x-z\|$ and that $\hat{\mathcal{T}}_2$ only puts mass in $A_r(z)$. Consider the sets

$$\tilde{\mathcal{E}}_c \triangleq A(z)\times\{z\},\ \text{ and }\ \tilde{\mathcal{E}}_a \triangleq \Big\{(x,y)\in (z,X_a]\times(z,r_a] :\ y\leq x\Big\}.$$

Then,

$$\hat{\mathcal{T}}(\tilde{\mathcal{E}}) = \hat{\mathcal{T}}(\tilde{\mathcal{E}}_c)+\int_{R_z}\tilde{\gamma}_a(\tilde{\mathcal{E}}_a)d\Gamma^{\mathrm{p}}(a)+Z.$$

For the first term we have $\hat{\mathcal{T}}(\tilde{\mathcal{E}}_c) = \hat{\mathcal{T}}_2(\{z\})$. For the second term we have that for any ray $a$, $\tilde{\gamma}_a(\tilde{\mathcal{E}}_a)$ equals $\hat{S}_a((z,r_a])$. This is true because the plan $\tilde{\gamma}_a$ only sends mass to $(z,r_a]$(this is the support of $\hat{S}_a$) and it does not send mass in the opposite direction of $z$, see Eq. (C-3). Therefore,

$$\hat{\mathcal{T}}(\tilde{\mathcal{E}}) = \hat{\mathcal{T}}_2(\{z\})+\int_{R_z}\hat{S}_a((z,r_a])d\Gamma^{\mathrm{p}}(a) = \hat{S}(A_r(z))$$

Now, recall that $\tilde{\Theta}(A(z)) = \hat{S}(A_r(z))$ and, therefore, $\hat{\mathcal{T}}(\tilde{\mathcal{E}}) = \tilde{\Theta}(A(z))$, as desired.

**Part 5: Conclusion.** We conclude by applying Proposition B-2. The price-equilibrium pair $(\hat{p},\hat{\mathcal{T}})$ satisfies the hypothesis in Proposition B-2, so we can create a global price-equilibrium pair which we still denote by $(\hat{p},\hat{\mathcal{T}})$ in $\mathcal{C}$. This new solution has the same objective that $(p,\mathcal{T})$ in $A(z)^c$, but it dominates the platform revenue in $A(z)$. Therefore, $(\hat{p},\hat{\mathcal{T}})$ revenue dominates $(p,\mathcal{T})$.

**Proof of Eq.** (C-3): We show that

$$\tilde{\gamma}_a\Big(\{(x,y)\in(z,X_a]\times(z,X_a] : y>x\}\Big) = 0,\quad \Gamma^{\mathrm{p}}-a.e.\ a\in R_z.$$

First we show that both measures $\tilde{\Theta}^{\mathrm{r}}_a$ and $\hat{S}_a$ satisfy:

$$\tilde{\Theta}^{\mathrm{r}}_a((z,b_a]) \leq \hat{S}_a((z,c_a])\quad \forall b_a,c_a\in(z,X_a],\ b_a\leq c_a,\quad \Gamma^{\mathrm{p}}-a.e.\ a\in R_z, \tag{C-5}$$

where $b_a$ and $c_a$ lie in the ray indexed by $a\in R_z$. To see why this is true let us proceed by contradiction. Let us denote by $Q$ de set where Eq. (C-5) is not satisfied, we have that $\Gamma^{\mathrm{p}}(Q)>0$. Note that for any $a\in Q$ we can find $b_a$ and $c_a$ for which the inequality in Eq. (C-5) is not satisfied, so let us thus fix such collection of $b_a$ and $c_a$. Moreover, from the definition of $\tilde{\Theta}^{\mathtt{r}}_a$ we deduce that for any $a \in Q$ we have $h_a \le b_a$ (otherwise $\tilde{\Theta}^{\mathtt{r}}_a((z, b_a]) = 0$ and, as a consequence, $a$ could not belong to $Q$). Then,

$$\begin{aligned}
\int_Q \tilde{\Theta}^{\mathtt{r}}_a((z,b_a])d\Gamma^{\mathtt{p}}(a) &= \int_Q \tilde{\Theta}_a((z,b_a] \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \in (z,b_a]\}} \cdot (\tilde{\Theta}_a(z,h_a] - \Delta_a)d\Gamma^{\mathtt{p}}(a) \\
&\overset{(a)}{\le} \int_Q \tilde{\Theta}_a((z,b_a] \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \le b_a\}} \cdot (\tilde{\Theta}_a(z,h_a] - \Delta_a)d\Gamma^{\mathtt{p}}(a) \\
&= \int_Q (\tilde{\Theta}_a((h_a,b_a]) + \tilde{\Theta}_a((z,h_a]) - \Delta_a)d\Gamma^{\mathtt{p}}(a) \\
&= \int_Q (\tilde{\Theta}_a((z,b_a]) - \Delta_a)d\Gamma^{\mathtt{p}}(a) \\
&= \underbrace{\int_{Q \cap \{a : r_a \le b_a\}} (\tilde{\Theta}_a((z,b_a]) - \Delta_a)d\Gamma^{\mathtt{p}}(a)}_{(*)} \\
&\quad + \underbrace{\int_{Q \cap \{a : r_a > b_a\}} (\tilde{\Theta}_a((z,b_a]) - \Delta_a)d\Gamma^{\mathtt{p}}(a)}_{(**)},
\end{aligned}$$

where $(a)$ follows from $\tilde{\Theta}_a(z, h_a] \ge \Delta_a$. For $(*)$ we have

$$\begin{aligned}
(*) &= \int_{Q \cap \{a : r_a \le b_a\}} (\tilde{\Theta}_a((z,b_a]) - \tilde{\Theta}((z,X_a]) + \hat{S}_a((z,X_a]))d\Gamma^{\mathtt{p}}(a) \\
&= \int_{Q \cap \{a : r_a \le b_a\}} (-\tilde{\Theta}((b_a,X_a]) + \hat{S}_a((z,X_a]))d\Gamma^{\mathtt{p}}(a) \\
&\le \int_{Q \cap \{a : r_a \le b_a\}} \hat{S}_a((z,X_a])d\Gamma^{\mathtt{p}}(a) \\
&= \int_{Q \cap \{a : r_a \le b_a\}} \hat{S}_a((z,c_a])d\Gamma^{\mathtt{p}}(a),
\end{aligned}$$

the last inequality holds because

Now we analyze $(**)$. Denote by $Q^r$ the set of rays $a \in R_z$ such that $r_a > b_a$ and $a \in Q$. Then

$$\begin{aligned}
\int_{Q^r} \tilde{\Theta}_a((z,b_a])d\Gamma^{\mathtt{p}}(a) &= \tilde{\Theta}\Big(\bigcup_{a \in Q^r} (z,b_a]\Big) \\
&= \mathcal{T}\Big(\bigcup_{a \in Q^r} (z,b_a] \times \bigcup_{a \in Q^r} (z,X_a]\Big) + \underbrace{\mathcal{T}\Big(\bigcup_{a \in Q^r} (z,b_a] \times \{z\}\Big)}_{\triangleq \ell_r} \\
&= \mathcal{T}\Big(\bigcup_{a \in Q^r} (z,b_a] \times \bigcup_{a \in Q^r} (z,b_a]\Big) + \mathcal{T}\Big(\bigcup_{a \in Q^r} (z,b_a] \times \bigcup_{a \in Q^r} (b_a,X_a]\Big) + \ell_r \\
&= \mathcal{T}\Big(\bigcup_{a \in Q^r} (z,b_a] \times \bigcup_{a \in Q^r} (z,b_a]\Big) + \ell_r \\
&\le \mathcal{T}_2\Big(\bigcup_{a \in Q^r} (z,b_a]\Big) + \ell_r \\
&\le \hat{S}\Big(\bigcup_{a \in Q^r} (z,b_a]\Big) + \ell_r \\
&= \int_{Q^r} \hat{S}_a((z,c_a])d\Gamma^{\mathtt{p}}(a) + \ell_r,
\end{aligned}$$

the first equality comes from the definition of $\tilde{\Theta}_a$ and then integrating this disintegration of measures. The second and fourth equality come from Proposition B-1 part *(ii)*. The last inequality comes from the congestion bound. For $\varDelta_a$ we have

$$\begin{aligned}\int_{Q^r} \varDelta_a d\varGamma^{\mathrm{p}}(a) &= \tilde{\Theta}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big) - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &= \mathcal{T}\Big(\bigcup_{a\in Q^r}(z,X_a]\times\bigcup_{a\in Q^r}(z,X_a]\Big) + \mathcal{T}\Big(\bigcup_{a\in Q^r}(z,X_a]\times\{z\}\Big) - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &\geq \mathcal{T}\Big(\bigcup_{a\in Q^r}(z,X_a]\times\bigcup_{a\in Q^r}(z,X_a]\Big) + \ell_r - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &= \mathcal{T}_2\Big(\bigcup_{a\in Q^r}(z,X_a]\Big) + \ell_r - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &\geq \ell_r,\end{aligned}$$

where the last inequality comes from Eq. ($\mathrm{FR}_a$). As a consequence we deduce that

$$(**) = \int_{Q\cap\{a:r_a>b_a\}} (\tilde{\Theta}_a((z,b_a]) - \varDelta_a)d\varGamma^{\mathrm{p}}(a) \leq \int_{Q\cap\{a:r_a>b_a\}} \hat{S}_a((z,b_a])d\varGamma^{\mathrm{p}}(a)$$

Putting together the bounds for $(*)$ and $(**)$ we deduce that

$$\int_Q \tilde{\Theta}^{\mathrm{r}}_a((z,b_a])d\varGamma^{\mathrm{p}}(a) \leq \int_Q \hat{S}_a((z,c_a])d\varGamma^{\mathrm{p}}(a),$$

since $\varGamma^{\mathrm{p}}(Q) > 0$ the previous inequality yields a contradiction. We conclude that Eq. (C-5) holds.

To finalize the proof of Eq. (C-3). Consider the set where Eq. (C-5) holds (the complement of this set has $\varGamma^{\mathrm{p}}$ measure equal to zero). for any ray $a$ in this suppose that

$$\tilde{\gamma}_a\Big(\{(x,y)\in(z,X_a]\times(z,X_a] : y > x\}\Big) > 0.$$

From the proof of Lemma B-3 we deduce that there exists $(x,y)\in(z,X_a]\times(z,X_a]$ such that $y > x$ and $\tilde{\gamma}_a((z,x+\delta]\times(y-\delta,X_a)) > 0$, where $\delta > 0$ can be taken small enough such that $x+\delta < y-\delta$. Then,

$$\begin{aligned}\hat{S}_a((z,x+\delta]) &\geq \tilde{\Theta}^{\mathrm{r}}_a((z,x+\delta])\\ &= \tilde{\gamma}_a((z,x+\delta]\times(z,X_a])\\ &= \tilde{\gamma}_a((z,x+\delta]\times(z,x+\delta]) + \tilde{\gamma}_a((z,x+\delta]\times(x+\delta,X_a])\\ &> \tilde{\gamma}_a((z,x+\delta]\times(z,x+\delta])\\ &= \hat{S}_a((z,x+\delta]) - \tilde{\gamma}_a((x+\delta,X_a]\times(z,x+\delta]),\end{aligned}$$

Thus,

$$\tilde{\gamma}_a((x+\delta,X_a]\times(z,x+\delta]) > 0, \text{ and we also have } \tilde{\gamma}_a((z,x+\delta]\times(y-\delta,X_a)) > 0,$$

but this is not possible because $\tilde{\gamma}_a$ is an optimal transport and, therefore, it is concentrated on a $c$-cyclically monotone set where $c(x,y) = \|x-y\|$, see Villani (2008). This concludes the proof of Eq. (C-3). □

**Lemma C-1.** *Let $\nu$ be a non-negative measure in $\mathcal{C}$, and $\pi_1$ a mapping be such that $\pi_1(x,y) = x$. Consider any measurable subset $K$ of $\mathcal{C}$ and some $z\in\mathcal{C}$ then the mappings $\nu(\pi_1(\cdot\cap\mathcal{D})\cap K)$ and $\nu(\pi_1(\cdot\cap(K\times\{z\})))$, defined on the Borel sets of $\mathcal{C}\times\mathcal{C}$, belong to $\mathcal{M}(\mathcal{C}\times\mathcal{C})$.*

*Proof.* For any Borel set $\mathcal{L} \subset \mathcal{C} \times \mathcal{C}$ define

$$\mathcal{T}_a(\mathcal{L}) \triangleq \nu(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap K) \text{ and } \mathcal{T}_b(\mathcal{L}) \triangleq \nu(\pi_1(\mathcal{L} \cap (K \times \{z\}))).$$

We show that $\mathcal{T}_a, \mathcal{T}_b \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. Note that because $\nu \in \mathcal{M}(\mathcal{C})$ for $i \in \{a, b\}$ we have that $\mathcal{T}_i(\emptyset) = 0$, and for any Borel set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ that $\mathcal{T}_i(\mathcal{L}) \in [0, \infty)$. To verify $\sigma-$ additivity consider a countable partition $\{\mathcal{L}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C} \times \mathcal{C}$, we need to show that

$$\mathcal{T}_i\Big(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n\Big) = \sum_{n \in \mathbb{N}} \mathcal{T}_i(\mathcal{L}_n).$$

Note that from the definition of $\mathcal{D}$ and the fact the set $K \times \{z\}$ has second component equal to 0, both collections $\{\pi_1(\mathcal{L}_n \cap \mathcal{D})\}_{n \in \mathbb{N}}$ and $\{\pi_1(\mathcal{L}_n \cap (K \times \{z\}))\}_{n \in \mathbb{N}}$ form a partition. Given this we can verify $\sigma-$ additivity, we do it for both $\mathcal{T}_a$ and $\mathcal{T}_b$ at the same time

$$\begin{aligned}
\mathcal{T}_a\Big(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n\Big) + \mathcal{T}_b\Big(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n\Big) &= \nu(\pi_1(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n \cap \mathcal{D}) \cap K) + \nu(\pi_1(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n \cap K \times \{z\})) \\
&= \nu(\bigcup_{n \in \mathbb{N}} \pi_1(\mathcal{L}_n \cap \mathcal{D}) \cap K) + \nu(\bigcup_{n \in \mathbb{N}} \pi_1(\mathcal{L}_n \cap K \times \{z\})) \\
&= \sum_{n \in \mathbb{N}} \nu(\pi_1(\mathcal{L}_n \cap \mathcal{D}) \cap K) + \sum_{n \in \mathbb{N}} \nu(\pi_1(\mathcal{L}_n \cap K \times \{z\})) \\
&= \sum_{n \in \mathbb{N}} \mathcal{T}_a(\mathcal{L}_n) + \sum_{n \in \mathbb{N}} \mathcal{T}_b(\mathcal{L}_n),
\end{aligned}$$

where the third line comes from the $\sigma-$additivity of the $\nu$ measure. Thus $\mathcal{T} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. □

# D Proofs for Section 8

**Preliminaries.** We use $m \in \mathcal{M}(\mathcal{C})$ denotes the Lebesgue measure in $\mathcal{C}$. We use $\mathcal{D}$ to denote the subset of $\mathcal{C} \times \mathcal{C}$ with equal first and second components, that is, $\mathcal{D} = \{(x, y) \in \mathcal{C} \times \mathcal{C} : x = y\}$. For any measurable set $\mathcal{B} \subseteq \mathcal{C}$ and a price-equilibrium pair $(p, \mathcal{T})$ we denote the platform's revenue in $\mathcal{B}$ under $(p, \mathcal{T})$ by $\mathbf{Rev}_{\mathcal{B}}(p, \mathcal{T})$. In case that $\mathcal{B}$ is $\mathcal{C}$ we simply use $\mathbf{Rev}(p, \mathcal{T})$. For any measure $\mathcal{V}$ we denote its restriction to a set $\mathcal{B}$ by $\mathcal{V}|_{\mathcal{B}}$, and for any set $\mathcal{E}$ we denote its restriction to a set $\mathcal{B}$ by $\mathcal{E}|_{\mathcal{B}}$. Let $B(x, \delta)$ be an open ball of radius $\delta$ around $x$.

**Proposition D-1** (Pre-demand Shock Environment)**.** *Suppose $\lambda_0 = 0$. Then, the optimal policy and corresponding supply equilibrium and flows can be characterized as follows.*

*(i) (Prices) The optimal pricing policy is given by $p(x) = \rho_1$, for all $x$ in $\mathcal{C}$.*

*(ii) (Flow) All supply units stay at their original locations.*

*Furthermore, the optimal revenue equals $\gamma \cdot \psi_1 \cdot \theta_1 \cdot 2H$.*

***Proof of Proposition D-1***. Let $(p, \mathcal{T})$ be any feasible price-equilibrium pair by Lemma D-1 (which state and prove after this proof) we have $V(x|\, p, \mathcal{T}) \leq \psi_1$, $\Gamma$ almost everywhere in $\mathcal{C}_\lambda = \mathcal{C} \setminus \{0\}$. This yields the following upper bound for the platform's objective

$$\int_{\mathcal{C}_\lambda} V(x|\, p, \mathcal{T}) \cdot s^{\mathcal{T}}(x)\, dx \leq \psi_1 \cdot \int_{\mathcal{C}_\lambda} s^{\mathcal{T}}(x)\, dx \leq \psi_1 \cdot \theta_1 \cdot m(\mathcal{C}).$$

The maximum revenue the platform can achieve in this case is bounded above by $\gamma \cdot \psi_1 \cdot \theta_1 \cdot m(\mathcal{C})$. Next, we show that the solution given in the statement of the lemma is feasible and achieves the upper bound.

**Flow feasibility.** We show that $\mathcal{T} \in \mathcal{F}(\Theta)$. A complete definition of the measure $\mathcal{T}$ is $\mathcal{T}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D})$. From the definition of $\mathcal{T}$ it is clear that $\mathcal{T} \in \mathcal{M}(\mathcal{C})$. Furthermore, $\mathcal{T}_1$ coincides with $\Theta$ and so does $\mathcal{T}_2$. Since $\Theta$ is the Lebesgue measure times a constant and $\Gamma$ is the Lebesgue measure plus an atom, we have $\mathcal{T}_1, \mathcal{T}_2 \ll \Gamma$. From this we can deduce that $m-$ a.e in $\mathcal{C}_\lambda$, $s^{\mathcal{T}}(x)$ equals $\theta_1$.

**Equilibrium utilities.** We show that $V(x|\, p, \mathcal{T})$ equals $\psi_1$. Note that

$$U(y, p(y), s^{\mathcal{T}}(y)) = \psi_1, \quad \Gamma - a.e. \ y \ \text{in} \ \mathcal{C}_\lambda.$$

Fix $x \in \mathcal{C}$, we have that

$$\Gamma(\{y \in \mathcal{C} : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| > \psi_1\}) = \mathbf{1}_{\{0 - |0 - x| > \psi_1\}} + \Gamma(\{y \in \mathcal{C} \setminus \{0\} : -|y - x| > 0\}) = 0.$$

Moreover, for any $\epsilon > 0$

$$\Gamma(\{y \in \mathcal{C} : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| > \psi_1 - \epsilon\}) \geq \Gamma(\{y \in \mathcal{C}_\lambda : -|y - x| > \epsilon\}) > 0,$$

where the last inequality comes from the fact that $\Gamma$ corresponds to the Lebesgue measure (plus an atom). That is, $V(x|\, p, \mathcal{T})$ equals $\psi_1$.

**Equilibrium condition.** Consider the equilibrium set

$$\mathcal{E} \triangleq \Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| = V(x|\, p, \mathcal{T})\Big\}.$$

Then,

$$\mathcal{T}(\mathcal{E}) = \mathcal{T}\Big(\Big\{(x, y) \in \mathcal{C} \times \{0\} : -|y - x| = \psi_1\Big\}\Big) + \mathcal{T}\Big(\Big\{(x, y) \in \mathcal{C} \times \mathcal{C}_\lambda : -|y - x| = 0\Big\}\Big) = \Theta(\mathcal{C}).$$

We have proven that the solution is the statement is feasible, and because of the values of $V(\cdot|\, p, \mathcal{T})$ and $s^{\mathcal{T}}(\cdot)$ we conclude that this solution achieves the upper bound. □

**Lemma D-1.** *Let $p$ be any price mapping and $\mathcal{T}$ a corresponding equilibrium flow. Then for any measurable set $\mathcal{B} \subseteq \mathcal{C}_\lambda$ such that $0 \notin \mathcal{B}$ and $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ we have*

$$V(x|p,\mathcal{T}) \;\leq\; \psi_1, \quad \mathit{\Gamma} - a.e. \;\; x \;\; in \;\; \mathcal{B}.$$

*Furthermore, in the pre-shock environment we can replace $\mathcal{B}$ with $\mathcal{C}_\lambda$ in the inequality above.*

*Proof.* Define the set

$$\mathcal{L} \triangleq \{x \in \mathcal{B} : V(x|p,\mathcal{T}) \;\leq\; \psi_1\}.$$

We would like to show that $\mathit{\Gamma}(\mathcal{L}^c) = 0$ where the complement is taken with respect to $\mathcal{B}$. Suppose this is not the case, and note that

$$\theta_1 \cdot m(\mathcal{L}^c) = \mathit{\Theta}(\mathcal{L}^c) = \mathcal{T}(\mathcal{L}^c \times \mathcal{C}) = \mathcal{T}(\mathcal{L}^c \times \mathcal{B}) + \mathcal{T}(\mathcal{L}^c \times \mathcal{B}^c),$$

since $\mathcal{L}^c \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$, the second term in the expression above is zero. This yields,

$$\begin{aligned}
\theta_1 \cdot m(\mathcal{L}^c) &= \mathcal{T}(\mathcal{L}^c \times \mathcal{B}) \\
&= \mathcal{T}(\mathcal{L}^c \times \mathcal{B} \cap \mathcal{L}^c) + \mathcal{T}(\mathcal{L}^c \times \mathcal{B} \cap \mathcal{L}) \\
&= \mathcal{T}(\mathcal{L}^c \times \mathcal{L}^c) + \mathcal{T}(\mathcal{L}^c \times \mathcal{L})
\end{aligned}$$

There are two cases. First, if $\mathcal{T}(\mathcal{L}^c \times \mathcal{L}) > 0$ then by Lemma B-3 there exists a pair $(x,y) \in \mathcal{L}^c \times \mathcal{L}$ such that $y \in \mathcal{IR}(x|p,\mathcal{T})$ thus

$$V(y|p,\mathcal{T}) = V(x|p,\mathcal{T}) + |x-y|.$$

However, since $(x,y) \in \mathcal{L}^c \times \mathcal{L}$

$$V(y|p,\mathcal{T}) \leq \psi_1 \text{ and } V(x|p,\mathcal{T}) > \psi_1.$$

Using the previous equation we can deduce that $\psi_1 > \psi_1$, which is not possible. The second case is $\mathcal{T}(\mathcal{L}^c \times \mathcal{L}) = 0$. Note that

$$\mathcal{T}_2(\mathcal{L}^c) = \mathcal{T}(\mathcal{C} \times \mathcal{L}^c) \geq \mathcal{T}(\mathcal{L}^c \times \mathcal{L}^c) = \theta_1 \cdot m(\mathcal{L}^c).$$

We also have that

$$\mathcal{T}_2(\mathcal{L}^c) = \int_{\mathcal{L}^c} s^{\mathcal{T}}(x) d\mathit{\Gamma}(x) \leq \int_{\mathcal{L}^c} \psi_x^{-1}(V(x|\, p,\mathcal{T})) d\mathit{\Gamma}(x) < \theta_1 \cdot \mathit{\Gamma}(\mathcal{L}^c),$$

where the first inequality comes from Proposition 2, and the second from the fact that $\psi_x(\cdot)$ is a strictly decreasing function, the definition of $\mathcal{L}^c$ and $\mathit{\Gamma}(\mathcal{L}^c) > 0$. Note that this inequality holds in both of the cases in the statement of the lemma. In both cases we have $0 \notin \mathcal{B}$ so $\mathit{\Gamma}(\mathcal{L}^c)$ equals $m(\mathcal{L}^c)$, yielding

$$\theta_1 \cdot m(\mathcal{L}^c) \leq \mathcal{T}_2(\mathcal{L}^c) < \theta_1 \cdot \mathit{\Gamma}(\mathcal{L}^c) = \theta_1 \cdot m(\mathcal{L}^c).$$

□

## D.1 Proofs for Section 8.1

***Proof of Proposition 4***. The proof of this proposition consists of several steps. In the first step we establish that the origin is an attraction region, characterize some properties of it and compute the value of the equilibrium utility function outside the attraction region. After this step, the drivers utility function will be pinned down in the entire city as a function of its value in the origin, $V(0|\, p,\mathcal{T})$. The second step supplies us with a full characterization, up to $V(0|\, p,\mathcal{T})$, of the post-relocation supply $\mathcal{T}_2$ in the entire city. Finally, in step three we show how to solve for the optimal value of $V(0|\, p,\mathcal{T})$ and, therefore, we pin down both $V(\cdot|\, p,\mathcal{T})$ and $\mathcal{T}_2$. We further show how to find the optimal $p(0)$ and the corresponding optimal flow $\mathcal{T}$.

**Step 1:** We show that we can restrict attention to solutions $(p,\mathcal{T})$ such that $X_l < 0 < X_r$, $X_r = V(0)-\psi_1$ and $X_l = -X_r$. Furthermore, such solutions have $V(x|\, p,\mathcal{T}) = \psi_1$ for all $x \in \mathcal{C} \setminus [X_l, X_r]$.

**Proof of Step 1:** Let $(p,\mathcal{T})$ be a feasible solution. First, we show that at any optimal solution we must have $X_l < 0 < X_r$. By Lemma D-2 (which we state and prove after the proof of the present proposition) we have that if either of the sets $\{x \in (0,H] : 0 \in \mathcal{IR}(x|p,\mathcal{T})\}$ or $\{x \in [-H,0) : 0 \in \mathcal{IR}(x|p,\mathcal{T})\}$ is empty then the revenue the platform makes satisfies

$$\frac{1}{\gamma}\cdot \mathbf{Rev}(p,\mathcal{T}) \le \psi_1 \cdot \theta_1 \cdot 2 \cdot H.$$

Now we construct a new feasible solution $(\tilde{p},\tilde{\mathcal{T}})$ for which both sets are non-empty and such that

$$\frac{1}{\gamma}\cdot \mathbf{Rev}(\tilde{p},\tilde{\mathcal{T}}) > \psi_1 \cdot \theta_1 \cdot 2 \cdot H, \tag{D-1}$$

where $\tilde{p}$ equals $\rho_1$ in $\mathcal{C} \setminus \{0\}$ and $p(0)$ is appropriately chosen. This will imply that any optimal solution must satisfy $\{x \in (0,H] : 0 \in \mathcal{IR}(x|p,\mathcal{T})\} \neq \emptyset$ and $\{x \in [-H,0) : 0 \in \mathcal{IR}(x|p,\mathcal{T})\} \neq \emptyset$ and, therefore, $X_l < 0 < X_r$. This also implies that the optimal revenue in this case is strictly larger than the one in the pre-shock environment.

Our solution will send flow in $[-h,h]$ to the origin, where $h > 0$ is to be determined. Inside this interval, all the flow in the subinterval $[-\bar{h}(h),\bar{h}(h)]$ goes to the origin where $0 \le \bar{h}(h) \le h$. The rest of the flow in $[-h,h]$ partially stays at its original position and partially goes to the origin. We now show how to determine $\bar{h}(h)$ and $h$. For any given $h > 0$ we define

$$\bar{h}(h) \triangleq (\psi_1 + h - \alpha \cdot \rho_1)^+,$$

note that when $\psi_1$ equals $\alpha \cdot \rho_1$ we have that $\bar{h}(h)$ equals $h$, and we will send all the flow in $[-h,h]$ to the origin. However, when $\psi_1 < \alpha \cdot \rho_1$ not all the flow will be sent to the origin. Define

$$\theta_1(x) \triangleq \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \overline{F}(\rho_1)}{\psi_1 + h - |x|},$$

then

$$\frac{\lambda_1 \overline{F}(\rho_1)}{\theta_1(x)} \le 1, \quad x \in [-h,h] \setminus [-\bar{h}(h),\bar{h}(h)].$$

The idea is that for every location $x \in K(h) \triangleq [-h,h] \setminus [-\bar{h}(h),\bar{h}(h)]$ we will leave a density $\theta_1(x)$ of flow there and send $\theta_1 - \theta_1(x)$ (note that this difference is non-negative) to the origin. In order to make this possible, we need to chose $h$ appropriately. Observe that the total supply we will send to the origin is

$$S_T(h) = 2\bar{h}(h)\theta_1 + 2\int_{\bar{h}(h)}^{h} (\theta_1 - \theta_1(x))\, dm(x),$$

where $\lim_{h\to 0} S_T(h) = 0$. Hence, since $\psi_1 < \overline{\alpha} \cdot \overline{V}$, we can always find $h > 0$ such that

$$\alpha \cdot \overline{V} - h \ge \alpha \cdot F^{-1}\Big(1 - \frac{S_T(h)}{\lambda_0}\Big) - h \ge \psi_1. \tag{D-2}$$

This yields

$$\overline{F}\Big(\frac{\psi_1 + h}{\alpha}\Big) \ge \frac{S_T(h)}{\lambda_0}.$$

Now we construct the solution $(\tilde{p},\tilde{\mathcal{T}})$. Fix any $h$ satisfying Eq. (D-2) and consider prices defined by

$$\tilde{p}(x) = \begin{cases} \frac{\psi_1 + h}{\alpha} & \text{if } x = 0 \\ \rho_1 & \text{if } x \in \mathcal{C} \setminus \{0\}, \end{cases}$$

and flows for any measurable set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ defined by

$$\begin{aligned}\tilde{\mathcal{T}}(\mathcal{L}) &= \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap [-h,h]^c) + \Theta(\pi_1(\mathcal{L} \cap [-\bar{h}(h), \bar{h}(h)] \times \{0\})) \\ &\quad + G_0(\pi_1(\mathcal{L} \cap K(h) \times \{0\})) + G_1(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap K(h)),\end{aligned}$$

where $G_0, G_1$ are measures defined for any measurable set $\mathcal{B} \subseteq K(h)$ by

$$G_0(\mathcal{B}) \triangleq \int_{\mathcal{B}} (\theta_1 - \theta_1(x))\, dm(x), \quad G_1(\mathcal{B}) \triangleq \int_{\mathcal{B}} \theta_1(x)\, dm(x).$$

We argue that $(\tilde{p}, \tilde{\mathcal{T}})$ is a feasible solution that complies with Eq. (D-1). From Lemma C-1 we have that $\tilde{\mathcal{T}} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$, also note that for any measurable set $\mathcal{B} \subseteq \mathcal{C}$ the first marginal of $\tilde{\mathcal{T}}$ satisfies

$$\tilde{\mathcal{T}}_1(\mathcal{B}) = \Theta(\mathcal{B} \cap [-h,h]^c) + \Theta(\mathcal{B} \cap [-\bar{h}(h), \bar{h}(h)]) + G_0(\mathcal{B} \cap K(h)) + G_1(\mathcal{B} \cap K(h)) = \Theta(\mathcal{B}).$$

The post-relocation supply measure is

$$\tilde{\mathcal{T}}_2(\mathcal{B}) = \Theta(\mathcal{B} \cap [-h,h]^c) + S_T(h) \cdot \mathbf{1}_{\{0 \in \mathcal{B}\}} + G_1(\mathcal{B} \cap K(h)),$$

clearly $\tilde{\mathcal{T}}_2 \ll \Gamma$. Therefore, $\tilde{\mathcal{T}} \in \mathcal{F}(\Theta)$. Next, we need to show that $\tilde{\mathcal{T}}$ is a supply equilibrium. The Radon-Nikodym derivative of $\tilde{\mathcal{T}}_2$ with respect the city measure is ($\Gamma$-a.e)

$$s(x) = \begin{cases} S_T(h) & \text{if } x = 0 \\ 0 & \text{if } x \in [-\bar{h}(h), \bar{h}(h)] \setminus \{0\} \\ \theta_1(x) & \text{if } x \in K(h) \\ \theta_1 & \text{if } x \in [-h,h]^c. \end{cases}$$

Indeed,

$$\int_{\mathcal{L}} s(x)\, d\Gamma(x) = S_T(h) \mathbf{1}_{\{0 \in \mathcal{L}\}} + \int_{\mathcal{L} \cap [-h,h]^c} \theta_1\, dm(x) + \int_{\mathcal{L} \cap K(h)} \theta_1(x)\, dm(x) = \tilde{\mathcal{T}}_2(\mathcal{L}),$$

that is, $\frac{d\tilde{\mathcal{T}}_2}{d\Gamma}(\cdot)$ equals $s(\cdot)$ $\Gamma$-a.e. From this we can compute $V(\cdot|\tilde{p}, \tilde{\mathcal{T}})$. Note that ($\Gamma$-a.e)

$$\tilde{U}(y) = U\Big(y, \tilde{p}(y), \frac{d\tilde{\mathcal{T}}_2}{d\Gamma}(y)\Big) = \begin{cases} \psi_1 + h & \text{if } y = 0; \\ \alpha \cdot \rho_1 & \text{if } y \in [-\bar{h}(h), \bar{h}(h)] \setminus \{0\}; \\ \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \overline{F}(\rho_1)}{\theta_1(x)} & \text{if } y \in K(h); \\ \psi_1 & \text{if } y \in [-h,h]^c. \end{cases}$$

Let $a(x)$ be defined by

$$a(x) \triangleq \begin{cases} \psi_1 + h - |x| & \text{if } x \in [-h,h], \\ \psi_1 & \text{if } x \in [-h,h]^c. \end{cases}$$

We argue that $V(\cdot|\tilde{p}, \tilde{\mathcal{T}}) \equiv a(\cdot)$. Fix $x \in \mathcal{C}$, it is not hard to verify that

$$\Gamma(y \in \mathcal{C} : \tilde{U}(y) - |y - x| > a(x)) = 0,$$

and, thus, $a(x) \geq V(x|\, \tilde{p}, \tilde{\mathcal{T}})$. Suppose that $x \in [-h,h]$ and $a(x) > V(x|\, \tilde{p}, \tilde{\mathcal{T}})$ then, because $\Gamma(\{0\}) > 0$, we have that

$$\psi_1 + h - |x| = a(x) > V(x|\, \tilde{p}, \tilde{\mathcal{T}}) \geq \Pi(x, 0) = \psi_1 + h - |0 - x|,$$

a contradiction. Thus, for $x \in [-h,h]$ we have $a(x) = V(x|\tilde{p}, \tilde{\mathcal{T}})$. For any other $x$ we can use a similar argument to conclude that $a(x) = V(x|\tilde{p}, \tilde{\mathcal{T}})$.

Now we are ready to verify the equilibrium condition. Observe that

$$\mathcal{E} = \Big\{(x, y) \in \mathcal{C}\times\mathcal{C} : \Pi(x, y) = V(x|\tilde{p}, \tilde{\mathcal{T}})\Big)\Big\} = ([-h, h]\times\{0\})\cup([-h, h]^c\times[-h, h]^c\cap\mathcal{D})\cup(K(h)\times K(h)\cap\mathcal{D}),$$

then

$$\begin{aligned}\tilde{\mathcal{T}}(\mathcal{E}) &= \Theta(\pi_1(\mathcal{E}\cap\mathcal{D})\cap[-h, h]^c) + \Theta(\pi_1(\mathcal{E}\cap[-\bar{h}(h), \bar{h}(h)]\times\{0\})) \\ &\quad + G_1(\pi_1(\mathcal{E}\cap\mathcal{D})\cap K(h)) + G_0(\pi_1(\mathcal{E}\cap K(h)\times\{0\})) \\ &= \Theta([-h, h]^c) + \Theta([-\bar{h}(h), \bar{h}(h)]) + G_1(K(h)) + G_0(K(h)) \\ &= \Theta(\mathcal{C}).\end{aligned}$$

This proves that $\tilde{\mathcal{T}}$ is an equilibrium. Next we need to show $(\tilde{p}, \tilde{\mathcal{T}})$ satisfies Eq. (D-1). From Proposition 1 we have

$$\begin{aligned}\gamma\mathbf{Rev}(\tilde{p}, \tilde{\mathcal{T}}) &= \int_{\mathcal{C}} V(x)\cdot\frac{d\tilde{\mathcal{T}}_2}{d\Gamma}(x)d\Gamma(x) \\ &= (\psi_1 + h)\cdot S_T(h) + 2\int_{\bar{h}(h)}^{h}(\psi_1 + h - x)\theta_1(x)\, dm(x) + \psi_1\cdot\theta_1\cdot 2(H - h) \\ &\geq h\cdot S_T(h) + \psi_1\Big(S_T(h) + 2\int_{\bar{h}(h)}^{h}\theta_1(x)dx\Big) + \psi_1\cdot\theta_1\cdot 2(H - h) \\ &= h\cdot S_T(h) + \psi_1\Big(2\bar{h}(h)\theta_1 + 2\int_{\bar{h}(h)}^{h}(\theta_1 - \theta_1(x))dx + 2\int_{\bar{h}(h)}^{h}\theta_1(x)dx\Big) + \psi_1\cdot\theta_1\cdot 2(H - h) \\ &= h\cdot S_T(h) + \psi_1\cdot\theta_1\cdot 2\cdot H.\end{aligned}$$

Since $h\cdot S_T(h) > 0$, Eq. (D-1) obtains. This proves that $X_l < 0 < X_r$ in any optimal solution.

The next step of the proof of Step 1 consists on arguing that given $V(0)$, $X_r = V(0) - \psi_1$ and $X_l = -(V(0) - \psi_1)$. Consider a feasible solution $(p, \mathcal{T})$ where $p(\cdot)$ equals $\rho_1$ everywhere but at the origin, and $X_l < 0 < X_r$. From Proposition B-1 and the fact that $\Theta(\{X_r\}) = 0$ we have that

$$\mathcal{T}([X_r, H]\times[X_r, H]^c) \leq \Theta(\{X_r\}) + \mathcal{T}((X_r, H]\times[X_r, H]^c) = 0.$$

Then by Lemma D-1 we have that $V(x) \leq \psi_1$, $\Gamma - a.e.\ x$ in $[X_r, H]$. This, together with the continuity of $V(\cdot)$ imply that $V(x) \leq \psi_1$ for all $x \in [X_r, H]$.

Suppose first that $X_r < V(0) - \psi_1$ then

$$V(X_r|\, p, \mathcal{T}) = V(0) - X_r > \psi_1,$$

but this violates the continuity of $V$ to the right of $X_r$. Thus $X_r \geq V(0) - \psi_1$. On the other hand, suppose $X_r > V(0) - \psi_1$ then we must have that $\psi_1 > V(x|\, p, \mathcal{T}) = V(0) - x$ for all $x \in (V(0) - \psi_1, X_r]$. Observe that

$$\Theta([V(0) - \psi_1, X_r]) \geq \mathcal{T}_2([V(0) - \psi_1, X_r]) = \int_{[V(0)-\psi_1, X_r]} s^{\mathcal{T}}(x)\, d\Gamma(x). \qquad \text{(D-3)}$$

Define the set

$$K \triangleq \{y \in [V(0) - \psi_1, X_r] : s^{\mathcal{T}}(y) \leq \theta_1\},$$

it must be that $\Gamma(K) = 0$; otherwise, from the definition of $V(X_r|\, p, \mathcal{T})$ we have

$$\begin{aligned}V(0) - X_r = V(X_r) &\geq U(y, \rho_1, s^{\mathcal{T}}(y)) - |y - X_r|, \quad \Gamma - a.e.\ y \text{ in } K \\ &\geq U(y, \rho_1, \theta_1) - |y - X_r|, \quad \Gamma - a.e.\ y \text{ in } K \\ &= \psi_1 - (X_r - y), \quad \Gamma - a.e.\ y \text{ in } K,\end{aligned}$$

and $\Gamma(K) > 0$ implies that $V(0)-y \geq \psi_1$ for some $y \in (V(0)-\psi_1, X_r]$. However, we know that $\psi_1 > V(0)-y$ for $y \in (V(0)-\psi_1, X_r]$ and, therefore, we must have $\Gamma(K) = 0$. Using this in Eq. (D-3) yields

$$\Theta([V(0)-\psi_1, X_r]) > \theta_1 \cdot \Gamma([V(0)-\psi_1, X_r]) = \Theta([V(0)-\psi_1, X_r]),$$

which is not possible. Hence, $X_r = V(0) - \psi_1$ and the same arguments applies to $X_l$, yielding $X_l = -(V(0) - \psi_1)$.

In order to conclude the proof for Step 1 we show that we can restrict attention to solutions $(p, \mathcal{T})$ such that $V(x|\, p, \mathcal{T})$ equals $\psi_1$ for all $x \in [X_l, X_r]^c$. In turn, this will show that $s^{\mathcal{T}}(x)$ equals $\theta_1$, $\Gamma - a.e.\ x$ in $[X_l, X_r]^c$. We base the proof of the latter statements in Lemma D-3 (which we state and prove after the proof of the present result), this lemma enables us to separate the city into two regions $[X_l, X_r]$ and $[X_l, X_r]^c$. For each region we can modify the prices and equilibria, and then paste them together to obtain a new solution that is an equilibrium for the entire city.

Consider a feasible solution $(p, \mathcal{T})$ such that $X_l < 0 < X_r$, $X_r = V(0) - \psi_1$ and $X_l = -X_r$. Since $\mathcal{T}([X_l, X_r] \times [X_l, X_r]^c) = 0$ and $0 \notin [X_l, X_r]^c$, Lemma D-1 delivers

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T}) \leq \frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_l, X_r]}(p, \mathcal{T}) + 2 \cdot \theta_1 \cdot \psi_1 \cdot (H - X_r). \tag{D-4}$$

We show that we can always modify $(p, \mathcal{T})$ so that the previous upper bound is achieve. Let $\mathcal{B} = [X_l, X_r]$, since $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$, Lemma D-3 ensures that $(p, \mathcal{T})|_{\mathcal{B}}$ is a price equilibrium pair in $\mathcal{B}$. Such equilibrium satisfies $V_{\mathcal{B}}(x) = \psi_1$ for $x \in \partial\mathcal{B}$.

Now, we choose prices $p^{\mathcal{B}^c}(x)$ equal to $\rho_1$ for all $x \in \mathcal{B}^c$ and a flow $\mathcal{T}^{\mathcal{B}^c}$ defines by for any measurable set $\mathcal{L}_1 \times \mathcal{L}_2 \subseteq \mathcal{B}^c \times \mathcal{B}^c$

$$\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}_1 \times \mathcal{L}_2) = \Theta(\mathcal{L}_1 \cap \mathcal{L}_2).$$

Then, it is easy to verify (as we did in the pre-shock environment, see Proposition D-1) that $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ forms a price-equilibrium pair in $\mathcal{B}^c$. This solution satisfy that $V_{\mathcal{B}^c}(x) = \psi_1$ for $x \in \mathcal{B}^c$, and that $s^{\mathcal{T}^{\mathcal{B}^c}}(x)$ equals $\theta_1$, $\Gamma - a.e.\ x$ in $\mathcal{B}^c$.

Lemma D-3 enables us to paste the solutions $(p, \mathcal{T})|_{\mathcal{B}}$ and $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$, and generate a new solution in the entire city. Such solution preserve the prices and flows in both $\mathcal{B}$ and $\mathcal{B}^c$ and, therefore, the upper bound in Eq. (D-4) is achieved. In conclusion, we can restrict attention to solutions $(p, \mathcal{T})$ such that $V(x|\, p, \mathcal{T})$ equals $\psi_1$ for all $x \in [X_l, X_r]^c$, and that $s^{\mathcal{T}}(x)$ equals $\theta_1$, $\Gamma - a.e.\ x$ in $[X_l, X_r]^c$.

**Step 2:** We characterize $s^{\mathcal{T}}(\cdot)$ (this completely characterizes $\mathcal{T}_2$). Let

$$X_r^0 = (V(0) - \alpha \cdot \rho_1))^+ \quad \text{and} \quad X_l^0 = -X_r^0,$$

and

$$\theta_1(y) \triangleq \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \cdot \overline{F}(\rho_1)}{V(0) - |y|}, \quad S_T = 2 \cdot \theta_1 \cdot X_r^0 + 2 \int_{X_r^0}^{X_r} (\theta_1 - \theta_1(x)) dx.$$

In this step we show that $(\Gamma - a.e)$

$$s^{\mathcal{T}}(y) = \begin{cases} S_T & \text{if } y = 0 \\ 0 & \text{if } y \in [X_l^0, X_r^0] \setminus \{0\} \\ \theta_1(y) & \text{if } y \in [X_l, X_r] \setminus [X_l^0, X_r^0] \\ \theta_1 & \text{if } y \in [X_l, X_r]^c. \end{cases}$$

**Proof of Step 2:** Note that at the end of the previous step we showed the result for $y \in [X_l, X_r]^c$. So first we show

$$s^{\mathcal{T}}(y) = 0, \quad \Gamma - a.e.\ x \text{ in } [X_l^0, X_r^0] \setminus \{0\}.$$

Define the set $K_1 \triangleq \{y \in [X_l^0, X_r^0] \setminus \{0\} : s^{\mathcal{T}}(y) > 0\}$. We argue that $\Gamma(K_1) = 0$. If this is not the case then $\Gamma(K_1) > 0$ and, therefore,

$$\mathcal{T}_2(K_1) = \int_{K_1} s^{\mathcal{T}}(x)\, d\Gamma(x) > 0.$$

Then Lemma A-2 ensures that

$$U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big) \;=\; V(x|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\ x \in K_1, \tag{D-5}$$

but for $x \in K_1 \subseteq [X_l^0, X_r^0] \setminus \{0\}$ we have $V(x|\, p, \mathcal{T}) = V(0) - |x|$ and $V(0) - |x| \geq \alpha \cdot \rho_1$. Then Eq. (D-5) implies the existence of $x \in (X_l^0, X_r^0) \setminus \{0\}$ such that $\alpha \cdot \rho_1 < U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big) \leq \alpha \cdot \rho_1$, yielding a contradiction. Next we show that

$$s^{\mathcal{T}}(y) = \theta_1(y), \quad \Gamma - a.e.\ y \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0].$$

By Lemma A-2 we have that

$$U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big) = V(x) = V(0) - |x|, \quad \Gamma - a.e.\ x \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0], \tag{D-6}$$

but for any $x \in [X_l, X_r] \setminus [X_l^0, X_r^0]$ the definition of $X_l^0$ and $X_r^0$ imply that $V(0) - |x| < \alpha \cdot \rho_1$. Thus Eq. (D-6) and the definition of $U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big)$ deliver

$$\lambda_1 \cdot \overline{F}(\rho_1)/s^{\mathcal{T}}(x) < 1, \quad \Gamma - a.e.\ x \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0].$$

Using the again Eq. (D-6) and the definition of $U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big)$ we conclude that

$$s^{\mathcal{T}}(x) = \alpha \cdot \rho_1 \cdot \frac{\overline{F}(\rho_1)}{V(0) - |x|}, \quad \Gamma - a.e.\ x \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0],$$

as needed. Next we compute $s^{\mathcal{T}}(0)$,

$$\begin{aligned} s^{\mathcal{T}}(0) \cdot \Gamma(\{0\}) = \int_{\{0\}} s^{\mathcal{T}}(x)\, d\Gamma &= \mathcal{T}_2(\{0\}) \\ &= \mathcal{T}(\mathcal{C} \times \{0\}) \\ &= \mathcal{T}([X_l, X_r] \times \{0\}) \\ &= \underbrace{\mathcal{T}([X_l^0, X_r^0] \times \{0\})}_{(1)} + \underbrace{\mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\})}_{(2)}, \end{aligned}$$

for (1) we have

$$\begin{aligned} \mathcal{T}([X_l^0, X_r^0] \times \{0\}) &= \Theta([X_l^0, X_r^0]) - \mathcal{T}([X_l^0, X_r^0] \times \mathcal{C} \setminus \{0\}) \\ &= 2\theta_1 \cdot X_r^0 - \mathcal{T}([X_l^0, X_r^0] \times [X_l^0, X_r^0] \setminus \{0\}) \\ &\overset{(a)}{=} 2\theta_1 \cdot X_r^0, \end{aligned}$$

in $(a)$ we use $s^{\mathcal{T}}(x) = 0$, $\Gamma - a.e.\ x$ in $[X_l^0, X_r^0] \setminus \{0\}$. For (2) we have

$$\begin{aligned} \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\}) &= \Theta([X_l, X_r] \setminus [X_l^0, X_r^0]) - \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l, X_r] \setminus \{0\}) \\ &= 2\theta_1 \cdot (X_r - X_r^0) - \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l^0, X_r^0] \setminus \{0\}) \\ &\quad - \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l, X_r] \setminus [X_l^0, X_r^0]) \\ &= 2\theta_1 \cdot (X_r - X_r^0) - 0 - \mathcal{T}_2([X_l, X_r] \setminus [X_l^0, X_r^0]) \\ &= 2\theta_1 \cdot (X_r - X_r^0) - \int_{[X_l, X_r] \setminus [X_l^0, X_r^0]} \theta_1(x)\, d\Gamma, \end{aligned}$$

from this we conclude that

$$s^{\mathcal{T}}(0) = 2 \cdot \theta_1 \cdot X_r^0 + 2\int_{X_r^0}^{X_r} (\theta_1 - \theta_1(x))\, dx.$$

**Step 3:** Now we can provide a full solution for the optimization problem. Recall that we are only optimizing over $p(0)$ or, equivalently, over $V(0)$. By our congestion bound (see Proposition 2), any solution has to satisfy $V(0|\, p, \mathcal{T}) \leq \psi_0(s^{\mathcal{T}}(0))$. Moreover, Step 2 characterizes the supply-demand ratio at every location as a function of $V(0)$. Thus, the following formulation is a natural relaxation for the platform's problem

$$\begin{aligned} \max_{V(0)} \quad & V(0) \cdot S_T + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - X_r^0) & (\mathcal{P}_{loc-reac}) \\ \text{s.t} \quad & X_r^0 = (V(0) - \alpha \cdot \rho_1))^+, \quad X_r = V(0) - \psi_1 \\ & S_T = 2X_r^0\theta_1 + 2\int_{X_r^0}^{X_r} (\theta_1 - \theta_1(x))dx, \qquad \psi_1 < V(0) \leq \psi_0(S_T). \end{aligned}$$

We show that the optimal $V^\star(0)$ in $(\mathcal{P}_{loc-reac})$ is the unique solution to

$$V^\star(0) = \psi_0(S_T(V^\star(0))).$$

The optimal solution to the platform's problem set price at the origin $p^\star(0) = \rho_0^{loc}(S_T(V^\star(0)))$ such that $p^\star(0) \geq \rho_1$, and flows for any measurable set $\mathcal{B} \subset \mathcal{C} \times \mathcal{C}$ given by

$$\begin{aligned} \mathcal{T}(\mathcal{B}) &= \Theta(\pi_1(\mathcal{B} \cap \mathcal{D}) \cap [X_l, X_r]^c) + \Theta(\pi_1(\mathcal{B} \cap [X_l^0, X_r^0] \times \{0\})) \\ &\quad + G_1(\pi_1(\mathcal{B} \cap \mathcal{D}) \cap [X_l, X_r] \setminus [X_l^0, X_r^0]) + G_0(\pi_1(\mathcal{B} \cap [X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\})), \end{aligned}$$

where $G_0, G_1$ are measures defined for any measurable set $\mathcal{L} \subset [X_l, X_r] \setminus [X_l^0, X_r^0]$ by

$$G_0(\mathcal{L}) \triangleq \int_{\mathcal{L}} (\theta_1 - \theta_1(x))\, dm(x), \quad G_1(\mathcal{L}) \triangleq \int_{\mathcal{L}} \theta_1(x)\, dm(x).$$

**Proof of Step 3:** The proof consists of two parts. First, we show that $V^\star(0)$ as stated above is an optimal solution for $(\mathcal{P}_{loc-reac})$. To do this we prove that $S_T(V(0))$ is increasing for $V(0) > \psi_1$, with $S_T(\psi_1) = 0$. This implies that $\psi_0(S_T(V(0)))$ is decreasing and, therefore, it crosses with $V(0)$ at only one point. Then, we show the objective function increases with $V(0)$. These two facts imply the optimality of $V^\star(0)$. Second, we show that $(p, \mathcal{T})$ with $p(0) = p^\star(0)$ (and equal to $\rho_1$ for $x \neq 0$) and and $\mathcal{T}$ as stated above, are a feasible price-equilibrium pair that achieve the same revenue than the optimal solution of $(\mathcal{P}_{loc-reac})$. Since this problem is a relaxation to our original optimization problem we have optimality.

We begin with the first part. Note that

$$S_T(V(0)) = 2\theta_1 \cdot (V(0) - \psi_1) + 2\psi_1 \cdot \theta_1 \cdot \log\Big(\frac{\psi_1}{V(0) - (V(0) - \alpha\rho_1)^+}\Big).$$

From this it follows that $S_T(\psi_1) = 0$. If $V(0) \geq \alpha\rho_1$ then $S_T(V(0))$ is clearly increasing. If $V(0) \in (\psi_1, \alpha\rho_1)$ then the derivative of $S_T(V(0))$ with respect to $V(0)$ equals

$$2\theta_1 - 2\psi_1 \cdot \theta_1 \cdot \frac{V(0)}{\psi_1} \cdot \frac{\psi_1}{V(0)^2} = 2\theta_1 - 2\psi_1 \cdot \theta_1 \cdot \frac{1}{V(0)},$$

which is nonnegative if and only if $V(0) \geq \psi_1$. Since this is in our domain, we conclude that $S_T(\cdot)$ is increasing in $(\psi_1, \alpha\rho_1)$ and, therefore, is increasing for all $V(0) > \psi_1$.

Next, we show the objective is increasing in $V(0)$, the objective function is

$$V(0) \cdot S_T(V(0)) + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - (V(0) - \alpha\rho_1)^+),$$

when $V(0) \geq \alpha \cdot \rho_1$, the objective becomes

$$2\theta_1 \cdot V(0) \cdot (V(0) - \psi_1) + 2\psi_1 \cdot \theta_1 \cdot V(0) \cdot \log\left(\frac{\psi_1}{\alpha\rho_1}\right) + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - V(0) + \alpha\rho_1).$$

Its derivative is non-negative if and only if $2\frac{V(0)}{\psi_1} \geq 2 + \log\left(\frac{\alpha\rho_1}{\psi_1}\right)$, but from $V(0) \geq \alpha \cdot \rho_1$ and that the logarithm is a concave function the latter inequality is always true. Similarly, for $V(0) \in (\psi_1, \alpha \cdot \rho_1)$ the objective's derivative is non-negative if and only if $2\frac{V(0)}{\psi_1} \geq 2 + \log\left(\frac{V(0)}{\psi_1}\right)$, which, since $V(0) > \psi_1$, is always true. Observe that in both cases the inequalities for the sign of the objective's derivative is strict except when $V(0) = \psi_1$. Thus, the objective is strictly increasing in the domain.

For the second part we need to show that $(p, \mathcal{T})$ with $p(0) = p^\star(0)$ (and equal to $\rho_1$ for $x \neq 0$) and and $\mathcal{T}$, implement the solution of $(\mathcal{P}_{loc-reac})$. To do this we first need to argue that this solution is feasible. It can be easily seen that this flow yields the exact same flows as in Step 2, only this time we replace $V^\star(0)$ in all the quantities that depend on $V(0)$. Given the value of $s^{\mathcal{T}}$ and the fact that under $p^\star(0)$ we have $U(0, p(0), s^{\mathcal{T}}(0)) = V(0|\, p, \mathcal{T}) = V^\star(0)$, we can do the same as we did in Step 1(to show that $\tilde{\mathcal{T}}$ is an equilibrium) and show that $\mathcal{T}$ is an equilibrium. Since we have pinned the value of $V(0|\, p, \mathcal{T})$ (and thus the value of $V(|\, p, \mathcal{T})$ in the entire city) and the value of $s^{\mathcal{T}}(\cdot)$, it is easy to see (using Proposition 1) that $\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T})$ coincides with the optimal value of $(\mathcal{P}_{loc-reac})$. Therefore, $(p, \mathcal{T})$ is the optimal solution.

To conclude we argue that $p^\star(0) \geq \rho_1$. There are two cases. If $\theta_1 \leq \lambda_1 \cdot \overline{F}(\rho_1)$ then $\psi_1$ equals $\alpha \cdot \rho_1$. Since $V^\star(0) > \psi$ and $V^\star(0) = \psi_0(S_T(V^\star(0)))$ we have have that

$$\alpha \cdot \rho_1 = \psi_1 < V^\star(0) = \psi_0(S_T(V^\star(0))) \leq \alpha \cdot \rho_0^{loc}(S_T(V^\star(0))) = \alpha \cdot p^\star(0),$$

that is, $\rho_1 < p^\star(0)$. The second case is $\theta_1 > \lambda_1 \cdot \overline{F}(\rho_1)$. Here $\rho_1$ equals $\rho^u$ and, since $\rho_0^{loc}(S_T(V^\star(0)))$ equals $\max\{\rho_0^{bal}, \rho^u\}$, we have that $\rho_1 \leq p^\star(0)$. □

**Lemma D-2.** *Let $(p, \mathcal{T})$ be a feasible price-equilibrium pair for either the myopic price response environment (Section 8.1) or the global price response environment (Section 8.2). If either $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} = \emptyset$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} = \emptyset$, then the platform's objective satisfies*

$$\gamma \cdot \mathbf{Rev}(p, \mathcal{T}) \leq \psi_1 \cdot \theta_1 \cdot 2 \cdot H.$$

*Proof.* WLOG let us just assume that $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} = \emptyset$. That is, for all $x \in (0, H]$ we have $0 \notin \mathcal{IR}(x|p, \mathcal{T})$. In turn, this implies that $\mathcal{T}((0, H] \times [-H, 0]) = 0$ and, therefore, by Lemma D-1 we conclude that

$$V(x|p, \mathcal{T}) \leq \psi_1 \qquad \Gamma - a.e. \text{ in } \ (0, H],$$

which, from the continuity of $V(\cdot|p, \mathcal{T})$, implies that $V(x|p, \mathcal{T}) \leq \psi_1$ for all $x \in [0, H]$. Now, we show that the same bound holds for $x \in [-H, 0)$. If $\mathcal{T}([-H, 0) \times \mathcal{B}) = 0$ for any $\mathcal{B} \subset [0, H]$, we can use Lemma D-1 to obtain the upper bound. On the other hand, if there exists $\mathcal{B} \subset [0, H]$ such that $\mathcal{T}([-H, 0) \times \mathcal{B}) > 0$ then by Lemma B-3 we know there exists a pair $(x, y) \in [-H, 0) \times \mathcal{B}$ for which $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. Thus, we can define

$$\underline{x} = \inf\{z \in [-H, 0) : y \in \mathcal{IR}(z|\, p, \mathcal{T})\},$$

and by Proposition B-1, $y \in \mathcal{IR}(\underline{x}|\, p, \mathcal{T})$. Also, by definition of indifference region and Lemma B-2 we have

$$V(z|p, \mathcal{T}) = V(\underline{x}|p, \mathcal{T}) + z - \underline{x}, \quad \forall\, z \in [\underline{x}, y].$$

This implies $V(z|p, \mathcal{T}) \leq V(y|p, \mathcal{T})$ for all $z \in [\underline{x}, y]$, and because $y \in \mathcal{B} \subset [0, H]$ we have $V(y|p, \mathcal{T}) \leq \psi_1$, yielding $V(z|p, \mathcal{T}) \leq \psi_1$ for all $z \in [\underline{x}, y]$. Furthermore, from Lemma B-3 and the definition of $\underline{x}$ we can conclude that $\mathcal{T}([-H, \underline{x}] \times (\underline{x}, H]) = 0$ which together with Lemma D-1 and the continuity of $V$ imply that $V(x|p, \mathcal{T}) \leq \psi_1$ for all $x \in [-H, \underline{x}]$. Completing the argument for the upper bound.

In order to bound the revenue, simply note that

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T}) = \int_{\mathcal{C}} V(x) s^{\mathcal{T}}(x)\, d\Gamma(x) \leq \psi_1 \cdot \int_{\mathcal{C}} s^{\mathcal{T}}(x)\, d\Gamma(x) = \psi_1 \cdot \theta_1 \cdot 2 \cdot H.$$

□

**Lemma D-3.** *(Equilibria Separation and Pasting) Consider a set $\mathcal{B} \subset \mathcal{C}$ such that both $\mathcal{B}$ and $\mathcal{B}^c$ are intervals or union of intervals with $\Gamma(\partial\mathcal{B}) = 0$, where $\partial\mathcal{B}$ is the boundary of $\mathcal{B}$.*

1. *(Separation) Let $(p, \mathcal{T})$ be a price-equilibrium in $\mathcal{C}$, if $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$ then $(p|_{\mathcal{B}}, \mathcal{T}|_{\mathcal{B}\times\mathcal{B}})$ and $(p|_{\mathcal{B}^c}, \mathcal{T}|_{\mathcal{B}^c\times\mathcal{B}^c})$ are price-equilibrium pairs in $\mathcal{B}$ and $\mathcal{B}^c$, respectively. Moreover, $V(\cdot|\, p|_{\mathcal{B}}, \mathcal{T}|_{\mathcal{B}\times\mathcal{B}})$ equals $V(\cdot|\, p|_{\mathcal{B}^c}, \mathcal{T}|_{\mathcal{B}^c\times\mathcal{B}^c})$ in $\partial\mathcal{B}$, $V(\cdot|\, p|_{\mathcal{B}}, \mathcal{T}|_{\mathcal{B}\times\mathcal{B}})$ coincides with $V(\cdot|\, p, \mathcal{T})|_{\mathcal{B}}$ and the same holds for $\mathcal{B}^c$.*
2. *(Pasting) Suppose we have two price-equilibrium pairs $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ and $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ in $\mathcal{B}$ and $\mathcal{B}^c$ such that $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\Theta|_{\mathcal{B}})$ and $\mathcal{T}^{\mathcal{B}^c} \in \mathcal{F}_{\mathcal{B}^c}(\Theta|_{\mathcal{B}^c})$, respectively. If $V(\cdot|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ equals $V(\cdot|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ in $\partial\mathcal{B}$ then the flow $\mathcal{T}$ defined by for any measurable set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$*

$$\mathcal{T}(\mathcal{L}) = \mathcal{T}^{\mathcal{B}}(\mathcal{L} \cap \mathcal{B} \times \mathcal{B}) + \mathcal{T}^{\mathcal{B}^c}(\mathcal{L} \cap \mathcal{B}^c \times \mathcal{B}^c),$$

*belongs to $\mathcal{F}(\Theta)$ and is an equilibrium in $\mathcal{C}$ for a price $p$ equal to $p^{\mathcal{B}}$ in $\mathcal{B}$ and equal to $p^{\mathcal{B}^c}$ in $\mathcal{B}^c$. Moreover, $V(x|p, \mathcal{T}) = V(x|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ in $\mathcal{B}$ and $V(x|p, \mathcal{T}) = V(x|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ in $\mathcal{B}^c$.*

*Proof.* **Separation.** Suppose that $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$. Let $\mathcal{T}^{\mathcal{B}} = \mathcal{T}|_{\mathcal{B}\times\mathcal{B}}$ and $p^{\mathcal{B}} = p|_{\mathcal{B}}$, we show that $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ is a price-equilibrium pair. The proof for $(p|_{\mathcal{B}^c}, \mathcal{T}|_{\mathcal{B}^c\times\mathcal{B}^c})$ is analogous and, thus, omitted. We need to prove that $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\Theta^{\mathcal{B}})$, where $\Theta^{\mathcal{B}}$ coincides with $\Theta|_{\mathcal{B}}$, and that the set

$$\mathcal{E}|_{\mathcal{B}} \triangleq \Big\{(x, y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p^{\mathcal{B}}(y), \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(y)) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi(x, \cdot, p^{\mathcal{B}}(\cdot), \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(\cdot)\Big)\Big\},$$

satisfies $\mathcal{T}_{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}) = \Theta|_{\mathcal{B}}(\mathcal{B})$.

First we verify that $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\Theta^{\mathcal{B}})$. Since $\mathcal{T}^{\mathcal{B}}$ is the restriction of $\mathcal{T}$ to $\mathcal{B} \times \mathcal{B}$ it clearly belongs to $\mathcal{M}(\mathcal{B} \times \mathcal{B})$. Also, for any $\mathcal{L}_1$ measurable subset of $\mathcal{B}$ we have that $\mathcal{T}_1^{\mathcal{B}}(\mathcal{L}_1)$ equals

$$\mathcal{T}^{\mathcal{B}}(\mathcal{L}_1 \times \mathcal{B}) = \mathcal{T}((\mathcal{L}_1 \times \mathcal{B}) \cap (\mathcal{B} \times \mathcal{B})) = \mathcal{T}(\mathcal{L}_1 \times \mathcal{B}) = \mathcal{T}(\mathcal{L}_1 \times \mathcal{C}) = \mathcal{T}_1(\mathcal{L}_1) = \Theta(\mathcal{L}_1).$$

Thus, $\mathcal{T}_1^{\mathcal{B}} = \Theta|_{\mathcal{B}}$. Now we need to prove that $\mathcal{T}_2^{\mathcal{B}} \ll \Gamma|_{\mathcal{B}}$. Observe that for any $\mathcal{L}_2$ measurable subset of $\mathcal{B}$ we have that $\mathcal{T}_2^{\mathcal{B}}(\mathcal{L}_2)$ equals

$$\mathcal{T}_{\mathcal{B}}(\mathcal{B} \times \mathcal{L}_2) = \mathcal{T}((\mathcal{B} \times \mathcal{L}_2) \cap (\mathcal{B} \times \mathcal{B})) = \mathcal{T}(\mathcal{B} \times \mathcal{L}_2) = \mathcal{T}(\mathcal{C} \times \mathcal{L}_2) = \mathcal{T}_2(\mathcal{L}_2),$$

that is, $\mathcal{T}_2^{\mathcal{B}} = \mathcal{T}_2|_{\mathcal{B}}$. Therefore, since $\mathcal{T}_2 \ll \Gamma$, we have that $\mathcal{T}_2^{\mathcal{B}} \ll \Gamma|_{\mathcal{B}}$. In turn, $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}$.

Now we show $\mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}) = \Theta|_{\mathcal{B}}(\mathcal{B})$. It suffices to prove that $\mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) = 0$ where the complement is taken with respect to $\mathcal{B} \times \mathcal{B}$, we do this by contradiction. Assume that $\mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) > 0$, this implies that $0 < \mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) = \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c)$, and we must have that $\mathcal{T}_2(\mathcal{B}) > 0$, indeed

$$0 < \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c) \leq \mathcal{T}(\mathcal{C} \times \mathcal{B}) = \mathcal{T}_2(\mathcal{B}).$$

Next, observe that for any $\mathcal{L}_2$ measurable subset of $\mathcal{B}$

$$\mathcal{T}_2^{\mathcal{B}}(\mathcal{L}_2) = \mathcal{T}_2(\mathcal{L}_2) = \int_{\mathcal{L}_2} s^{\mathcal{T}}(x)\, d\Gamma(x) = \int_{\mathcal{L}_2} s^{\mathcal{T}}(x)\, d\Gamma|_{\mathcal{B}}(x),$$

therefore,

$$\frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(x) \;=\; s^{\mathcal{T}}(x), \qquad \Gamma - a.e.\ x \text{ in } \mathcal{B}. \tag{D-7}$$

This implies that

$$V(x|p^{\mathcal{B}},\mathcal{T}^{\mathcal{B}}) = \operatorname*{ess\,sup}_{\mathcal{B}} \varPi(x,\cdot,p^{\mathcal{B}}(\cdot),\frac{d\mathcal{T}_2^{\mathcal{B}}}{d\varGamma|_{\mathcal{B}}}(\cdot)\Big) = \operatorname*{ess\,sup}_{\mathcal{B}} \varPi(x,\cdot,p(\cdot),\frac{d\mathcal{T}_2}{d\varGamma}(\cdot)\Big) = V_{\mathcal{B}}(x|p,\mathcal{T}). \tag{D-8}$$

Consider the set $\mathcal{G} \triangleq \{y \in \mathcal{B} : \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\varGamma|_{\mathcal{B}}}(y) = s^{\mathcal{T}}(y)\}$. Then, by Eq. (D-7) we have

$$\mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c \cap (\mathcal{B}\times\mathcal{G}^c)) \leq \mathcal{T}(\mathcal{C}\times\mathcal{G}^c) = \mathcal{T}_2(\mathcal{G}^c) = 0,$$

where the complement is take with respect to $\mathcal{B}$. Therefore, $0 < \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c) = \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c \cap (\mathcal{B}\times\mathcal{G}))$ and we can conclude that

$$\mathcal{T}\Big(\underbrace{\{(x,y)\in\mathcal{B}\times\mathcal{B} : \varPi(x,y,p(y),\frac{d\mathcal{T}_2}{d\varGamma}(y)) \neq V_{\mathcal{B}}(x|p,\mathcal{T})\}}_{\triangleq R}\Big) > 0.$$

Define the sets $R^-$ and $R^+$ by

$$\begin{aligned} R^- &= \{(x,y)\in\mathcal{B}\times\mathcal{B} : \varPi(x,y,p(y),\frac{d\mathcal{T}_2}{d\varGamma}(y)) > V_{\mathcal{B}}(x|p,\mathcal{T})\} \\ R^+ &= \{(x,y)\in\mathcal{B}\times\mathcal{B} : \varPi(x,y,p(y),\frac{d\mathcal{T}_2}{d\varGamma}(y)) < V_{\mathcal{B}}(x|p,\mathcal{T})\}, \end{aligned}$$

and note that $R = R^- \cup R^+$. To obtain a contradiction we argue that $\mathcal{T}(R^-\cup R^+) = 0$. Consider first the set $R^+$, and note that $\mathcal{T}(R^+) = \mathcal{T}(R^+\cap\mathcal{E})$. However, any $(x,y)\in R^+\cap\mathcal{E}$ satisfies

$$\varPi(x,y,(p(y),\frac{d\mathcal{T}_2}{d\varGamma}(y)) < V_{\mathcal{B}}(x|p,\mathcal{T}) \text{ and } \varPi(x,y,(p(y),\frac{d\mathcal{T}_2}{d\varGamma}(y)) = V(x|p,\mathcal{T}),$$

but $V(x) \geq V_{\mathcal{B}}(x)$ implies that $R^+\cap\mathcal{E} = \emptyset$ and, therefore, $\mathcal{T}(R^+) = 0$.

Consider $R^-$. Define $A \triangleq \{y \in \mathcal{B} : U(y) = V_{\mathcal{B}}(y|p,\mathcal{T})\}$, then by Lemma A-2 we have $\mathcal{T}(R^-) = \mathcal{T}(R^-\cap(\mathcal{B}\times A))$. Take any $(x,y)\in R^-\cap(\mathcal{B}\times A)$ then $V_{\mathcal{B}}(y|p,\mathcal{T}) - |y-x| > V_{\mathcal{B}}(x|p,\mathcal{T})$, which, because of the Lipchitz property (see Lemma 1), is not possible. Thus, $R^-\cap(\mathcal{B}\times A)) = \emptyset$ and we have that $\mathcal{T}(R^-) = 0$. This proves that $\mathcal{T}^{\mathcal{B}}$ is an equilibrium in $\mathcal{B}$.

Now we show that $V(x|p^{\mathcal{B}},\mathcal{T}^{\mathcal{B}}) = V(x|p^{\mathcal{B}^c},\mathcal{T}^{\mathcal{B}^c})$ for all $x\in\partial\mathcal{B}$. From equation (D-8) we have

$$V(x|p^{\mathcal{B}},\mathcal{T}^{\mathcal{B}}) = V_{\mathcal{B}}(x|p,\mathcal{T}) \quad \text{and} \quad V(x|p^{\mathcal{B}^c},\mathcal{T}^{\mathcal{B}^c}) = V_{\mathcal{B}^c}(x|p,\mathcal{T}),$$

so we just need to show $V_{\mathcal{B}}(x|p,\mathcal{T})$ equals $V_{\mathcal{B}^c}(x|p,\mathcal{T})$ for all $x\in\partial B$. We first show that $V_{\mathcal{B}}(x|p,\mathcal{T}) = V(x|p,\mathcal{T})$ for all $x\in\mathcal{B}$. Let $x\in\mathcal{B}$, since $\mathcal{B}$ is an interval or a union of intervals we must have $\varTheta(B(x,\frac{1}{n})\cap\mathcal{B}) > 0$ for all $n\in\mathbb{N}$. In turn, this implies

$$0 < \mathcal{T}(B(x,\frac{1}{n})\cap\mathcal{B}\times\mathcal{B}) = \mathcal{T}(B(x,\frac{1}{n})\cap\mathcal{B}\times(\mathcal{B}\cap A)),$$

where $A \triangleq \{y\in\mathcal{B} : V(y|p,\mathcal{T}) = V_{\mathcal{B}}(y|p,\mathcal{T})\}$, and we have used Lemma A-2. From Lemma B-3 there exists $(z_n,y_n)\in B(x,\frac{1}{n})\cap\mathcal{B}\times(\mathcal{B}\cap A)$ such that $y_n\in\mathcal{IR}(z_n|p,\mathcal{T})$, that is, $V(y_n)-\|z_n-y_n\| = V(z_n)$. Since $y_n\in A$ we have $V_{\mathcal{B}}(y_n)-\|z_n-y_n\| = V(z_n)$. From the Lipschitz property we have $V_{\mathcal{B}}(y_n)-\|z_n-y_n\| \leq V_{\mathcal{B}}(z_n)$. Thus, $V(z_n)\leq V_{\mathcal{B}}(z_n)$. Taking limit $n\uparrow\infty$ and noting that $z_n\to x$ we deduce that $V(x)\leq V_{\mathcal{B}}(x)$ (recall that both $V(\cdot)$ and $V_{\mathcal{B}}(\cdot)$ are continuous functions). But we always have that $V(x)\geq V_{\mathcal{B}}(x)$ and, therefore, $V(x) = V_{\mathcal{B}}(x)$. The same argument shows that $V(x) = V_{\mathcal{B}^c}(x)$ for all $x\in\mathcal{B}^c$.

To conclude we need to prove that $V_{\mathcal{B}}(x|p,\mathcal{T})$ equals $V_{\mathcal{B}^c}(x|p,\mathcal{T})$ for all $x\in\partial B$. Consider $x\in\partial\mathcal{B}$. Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{B}$ be a sequence converging to $x$. Then the continuity of $V_{\mathcal{B}}$ implies $V_{\mathcal{B}}(x_n)\to V_{\mathcal{B}}(x)$. At the same time, since $x_n\in\mathcal{B}$ we have $V_{\mathcal{B}}(x_n) = V(x_n)$ and by continuity $V(x_n)\to V(x)$. Then $V_{\mathcal{B}}(x) = V(x)$ and the same is true for $\mathcal{B}^c$, which implies $V_{\mathcal{B}}(x|p,\mathcal{T}) = V_{\mathcal{B}^c}(x|p,\mathcal{T})$ for all $x\in\partial B$.

**Pasting.** First we check that $\mathcal{T} \in \mathcal{F}(\Theta)$. Let $\mathcal{L}_1$ be any measurable subset of $\mathcal{C}$ we have that

$$\begin{aligned}
\mathcal{T}_1(\mathcal{L}_1) &= \mathcal{T}(\mathcal{L}_1 \times \mathcal{C}) \\
&= \mathcal{T}^{\mathcal{B}}((\mathcal{L}_1 \times \mathcal{C}) \cap (\mathcal{B} \times \mathcal{B})) + \mathcal{T}^{\mathcal{B}^c}((\mathcal{L}_1 \times \mathcal{C}) \cap (\mathcal{B}^c \times \mathcal{B}^c)) \\
&= \mathcal{T}^{\mathcal{B}}((\mathcal{L}_1 \cap \mathcal{B}) \times \mathcal{B}) + \mathcal{T}^{\mathcal{B}^c}((\mathcal{L}_1 \cap \mathcal{B}^c) \times \mathcal{B}^c) \\
&= \Theta|_{\mathcal{B}}(\mathcal{L}_1 \cap \mathcal{B}) + \Theta|_{\mathcal{B}^c}(\mathcal{L}_1 \cap \mathcal{B}^c) \\
&= \Theta(\mathcal{L}_1).
\end{aligned}$$

Also, if $\Gamma(\mathcal{L}_1) = 0$ then $\Gamma|_{\mathcal{B}}(\mathcal{L}_1) = \Gamma|_{\mathcal{B}^c}(\mathcal{L}_1) = 0$. Therefore, $\mathcal{T}_2^{\mathcal{B}}(\mathcal{L}_1) = \mathcal{T}_2^{\mathcal{B}^c}(\mathcal{L}_1) = 0$, which in turn implies $\mathcal{T}_2 \ll \Gamma$. Hence $\mathcal{T} \in \mathcal{F}(\Theta)$.

Now we show the set

$$\mathcal{E} \triangleq \Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = \operatorname*{ess\,sup}_{\mathcal{C}} \Pi(x, \cdot, p(\cdot), s^{\mathcal{T}}(\cdot)\Big)\Big\},$$

satisfies $\mathcal{T}(\mathcal{E}) = \Theta(\mathcal{C})$. Note that

$$\mathcal{E} \cap \mathcal{B} \times \mathcal{B} = \Big\{(x, y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = V(x|p, \mathcal{T})\Big\}.$$

It is enough to prove that $\mathcal{T}^{\mathcal{B}}(\mathcal{E} \cap \mathcal{B} \times \mathcal{B}) = \Theta(\mathcal{B})$. As we did in the first part of the proof (see Eq. (D-7)) we can show that

$$\frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(x) \;=\; s^{\mathcal{T}}(x), \qquad \Gamma - a.e.\ x \ \text{in}\ \mathcal{B},$$

so if we prove that $V(\cdot|p, \mathcal{T})|_{\mathcal{B}} \equiv V(\cdot|p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ we will be done (the proof for $\mathcal{B}^c$ is analogous). Fix $x \in \mathcal{B}$, as in Eq. (D-8) we have

$$V(x|p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi(x, \cdot, p^{\mathcal{B}}(\cdot), \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(\cdot)\Big) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi(x, \cdot, p(\cdot), \frac{d\mathcal{T}_2}{d\Gamma}(\cdot)\Big) = V_{\mathcal{B}}(x|p, \mathcal{T}).$$

So we just need to verify that $V(x|p, \mathcal{T}) = V_{\mathcal{B}}(x|p, \mathcal{T})$. We show that $V(x|p, \mathcal{T}) \leq V_{\mathcal{B}}(x|p, \mathcal{T})$, the other inequality always holds. Let $I(x)$ be the interval in $\mathcal{B}$ to which $x$ belongs to. Let $y_L = \inf I(x)$ and $y_U = \sup I(x)$, note that $y_L$ and $y_U$ do not necessarily belong to $\mathcal{B}$ but they do belong to $\partial\mathcal{B}$. By assumption $V(y|\, p^{\mathcal{B}}, \mathcal{T}_{\mathcal{B}}) = V(y|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ for $y \in \{y_L, y_U\}$, in turn this implies that $V_{\mathcal{B}}(y|p, \mathcal{T})$ equals $V_{\mathcal{B}^c}(y|p, \mathcal{T})$ for $y \in \{y_L, y_U\}$. Consider the sets $\mathcal{B}_L^c = [H, y_L] \cap \mathcal{B}^c$ and $\mathcal{B}_U^c = [y_U, H] \cap \mathcal{B}^c$ then

$$\begin{aligned}
V_{\mathcal{B}}(x|p, \mathcal{T}) &\overset{(a)}{\geq} V_{\mathcal{B}}(y_U|p, \mathcal{T}) - |x - y_U| \\
&= V_{\mathcal{B}}(y_U|p, \mathcal{T}) - (y_U - x) \\
&\overset{(b)}{\geq} U(w, s^{\mathcal{T}}(w)) - |y_U - w| - (y_U - x), \quad \Gamma - a.e.\ w \ \text{in}\ \mathcal{B}_U^c \\
&\overset{(c)}{\geq} U(w, s^{\mathcal{T}}(w)) - (w - y_U) - (y_U - x), \quad \Gamma - a.e.\ w \ \text{in}\ \mathcal{B}_U^c \\
&\overset{(d)}{\geq} U(w, s^{\mathcal{T}}(w)) - |w - x|, \quad \Gamma - a.e.\ w \ \text{in}\ \mathcal{B}_U^c,
\end{aligned}$$

where $(a)$ follows from the Lipschitz property (see Lemma 1), and $(b)$ from the definition of $V_{\mathcal{B}}(y_U|p, \mathcal{T})$ and $\Gamma(\mathcal{B}_U^c) > 0$; $(c), (d)$ hold because for $w \in \mathcal{B}_U^c$ we have $x \leq y_U \leq w$. Similarly,

$$\begin{aligned}
V_{\mathcal{B}}(x|p, \mathcal{T}) &\geq V_{\mathcal{B}}(y_L|p, \mathcal{T}) - |x - y_L| \\
&= V_{\mathcal{B}}(y_L|p, \mathcal{T}) - (x - y_L) \\
&\geq U(w, s^{\mathcal{T}}(w)) - |y_L - w| - (x - y_L), \quad \Gamma - a.e.\ w \ \text{in}\ \mathcal{B}_L^c \\
&= U(w, s^{\mathcal{T}}(w)) - (y_L - w) - (x - y_L), \quad \Gamma - a.e.\ w \ \text{in}\ \mathcal{B}_L^c \\
&= U(w, s^{\mathcal{T}}(w)) - |w - x|, \quad \Gamma - a.e.\ w \ \text{in}\ \mathcal{B}_L^c.
\end{aligned}$$

Since $\mathcal{B}_L^c \cup \mathcal{B}_U^c = \mathcal{B}^c$ this implies that $V_{\mathcal{B}}(x|p, \mathcal{T}) \geq V(x|p, \mathcal{T})$. This concludes the proof. □

## D.2 Proofs for Section 8.2

***Proof of Lemma 2.*** Let $(p, \mathcal{T})$ be a feasible solution. We show that at any optimal solution we must have $X_l < 0 < X_r$, in turn this implies that 0 is a sink. By Lemma D-2 we have that if either of the sets $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\}$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\}$ is empty then the revenue the platform makes satisfies $\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T}) \leq \psi_1 \cdot \theta_1 \cdot 2 \cdot H$. However, the solution $(p, \mathcal{T})$ given in Proposition 4 has both sets non-empty because $0 \in \mathcal{IR}(X_r|\, p, \mathcal{T})$ and $0 \in \mathcal{IR}(-X_r|\, p, \mathcal{T})$ with $X_r > 0$. Furthermore, $\mathbf{Rev}(p, \mathcal{T})$ is strictly large than the revenue of the pre-demand shock environment or, equivalently, strictly larger than $\psi_1 \cdot \theta_1 \cdot 2 \cdot H$. This implies that any optimal solution must satisfy $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} \neq \emptyset$ and $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} \neq \emptyset$ and, therefore, $X_l < 0 < X_r$. □

**Lemma D-4.** *(Upper bound) An optimal price-equilibrium pair $(p, \mathcal{T})$ satisfies*

$$V(x) \leq \min\{V(X_r) + x - X_r, \psi_1\}, \quad \text{for all } x \in (X_r, H]. \tag{D-9}$$

***Proof of Lemma D-4.*** If $X_r = H$ there is nothing to prove, so let's assume $X_r < H$. Fix $x \in [X_r, H]$. From the Lipschitz property (see Lemma 1) we have that $V(x|\, p, \mathcal{T}) \leq V(X_r|\, p, \mathcal{T}) + (x - X_r)$. Moreover, Proposition B-1 ensures that $\mathcal{T}([X_r, H] \times [X_r, H]^c) = 0$ and, hence, because $0 \notin [X_r, H]$ we can apply Lemma D-1 to deduce that

$$V(x|p, \mathcal{T}) \;\leq\; \psi_1, \quad \Gamma - a.e.\ x \ \text{ in } [X_r, H]. \tag{D-10}$$

To show that the previous inequality holds everywhere, notice that if $V(x|\, p, \mathcal{T}) > \psi_1$ the from the Lipschitz continuity property of $V(\cdot|\, p, \mathcal{T})$ we could find a subset of of $[X_r, H]$ with positive $\Gamma$ measure (in this set $\Gamma$ coincides with the Lebesgue measure) in which $V(\cdot|\, p, \mathcal{T})$ is strictly larger than $\psi_1$. This is not possible because it would contradict Eq. (D-10). Putting together both upper bounds yields the desire result. □

**Proposition D-2.** *(Monotonicity in the periphery) Without loss of optimality, we can focus on price-equilibrium pairs $(p, \mathcal{T})$ such that $V(\cdot)$ is non-decreasing in $(X_r, H]$. Furthermore, if $V(X_r) = \psi_1$, then $V(x) = \psi_1$ for all $x \geq X_r$.*

***Proof of Proposition D-2.*** Let $(p, \mathcal{T})$ be optimal for problem $(\mathcal{P}_2)$ as in Lemma 2 so we have $0 < X_r$. Note that is $X_r = H$ then the result trivially holds, so let's assume $X_r < H$. Before we begin note that for any $x \geq X_r$, by Lemma D-4 and the Lipschitz continuity property of $V(\cdot|\, p, \mathcal{T})$ (see Lemma 1), we must have $V(x) \leq \psi_1$.

We first prove the second statement of the proposition. Suppose $V(X_r) = \psi_1$ and define the set $R \triangleq \{x \in [X_r, H] : V(x) = \psi_1\}$. We show by contradiction that we cannot have $\mathcal{T}_2(R^c) > 0$ (the complement is taken with respect to $[X_r, H]$). If $\mathcal{T}_2(R^c) > 0$, because $\psi_1$ is an upper bound from Proposition 1 we have the following

$$\begin{aligned}
\frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_r,H]}(p, \mathcal{T}) &= \int_{[X_r,H]} V(x)\, d\mathcal{T}_2(x) \\
&= \int_R V(x)\, d\mathcal{T}_2(x) + \int_{R^c} V(x)\, d\mathcal{T}_2(x) \\
&< \int_R V(x)\, d\mathcal{T}_2(x) + \int_{R^c} \psi_1\, d\mathcal{T}_2(x) \\
&\leq \psi_1 \cdot \mathcal{T}_2([X_r, H]) \\
&= \psi_1 \cdot \theta_1 \cdot (H - X_r),
\end{aligned}$$

where the last line comes Proposition B-1. Thus, the quantity $\mathbf{Rev}_{[-H,X_r]}(p, \mathcal{T}) + \gamma \cdot \psi_1 \cdot \theta_1 \cdot (H - X_r)$, strictly upper bounds the platform's objective. So if we are able to construct a solution such that attains the upper bound, we will contradict the optimality of $(p, \mathcal{T})$. Observe that Lemma D-3 enables us to separate the solution $(p, \mathcal{T})$ in $[-H, X_r]$ and $(X_r, H]$. The separated solution $(p^{[-H,X_r]}, \mathcal{T}^{[-H,X_r]})$ (see Lemma D-3 for notation) in $[-H, X_r]$ has revenue equal to $\mathbf{Rev}_{[-H,X_r]}(p, \mathcal{T})$, and $V(X_r|\, p^{[-H,X_r]}, \mathcal{T}^{[-H,X_r]})$ coincides with $V(X_r|\, p, \mathcal{T})$ which equals $\psi_1$. For $(X_r, H]$ consider prices $\tilde{p}(x) = \rho_1$ for all $x \in (X_r, H]^c$, and flows $\tilde{\mathcal{T}}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for any measurable set $\mathcal{L} \subset (X_r, H] \times (X_r, H]$. The pair $(\tilde{p}, \tilde{\mathcal{T}})$ is the same solution as in Proposition D-1 with the sole difference that we have changed the city to be $(X_r, H]$ instead of $\mathcal{C}$. Therefore, $(\tilde{p}, \tilde{\mathcal{T}})$ is a feasible price-equilibrium in $(X_r, H]$ with revenue equal to $\gamma \cdot \psi_1 \cdot \theta_1 \cdot (H - X_r)$, and such that $V(x|\, \tilde{p}, \tilde{\mathcal{T}})$ equal to $\psi_1$ for all $x \in (X_r, H]$. Thus we can use Lemma D-3 to paste both solution and obtain an equilibrium in the entire city. This new equilibrium achieves the upper bound.

Suppose that $\mathcal{T}_2(R^c) = 0$ and define the sets

$$L_+ \triangleq \{x : \theta_1 > s^{\mathcal{T}}(x)\}, \quad L_0 \triangleq \{x : \theta_1 = s^{\mathcal{T}}(x)\}, \quad L_- \triangleq \{x : \theta_1 < s^{\mathcal{T}}(x)\}.$$

Then by Lemma 2 it holds that $\Gamma(R \cap L_-) = 0$. Moreover, if $\Gamma(R \cap L_+) > 0$ we have

$$\Theta([X_r, H]) = \mathcal{T}_2([X_r, H]) \overset{(a)}{=} \mathcal{T}_2(R) = \int_{R \cap L_+} s^{\mathcal{T}}(x)\, d\Gamma(x) + \int_{R \cap L_0} s^{\mathcal{T}}(x)\, d\Gamma(x) < \theta_1 \Gamma(R) \leq \Theta([X_r, H]),$$

not possible, where $(a)$ comes from Proposition B-1. Thus $\Gamma(R \cap L_+) = 0$. This implies that $\Gamma(R \cap L_0) = \Gamma(R)$ and

$$\theta_1 \Gamma([X_r, H]) = \Theta([X_r, H]) = \int_{R \cap L_0} s^{\mathcal{T}}(x)\, d\Gamma(x) = \theta_1 \Gamma(R),$$

that is, $\Gamma(R) = \Gamma([X_r, H])$ or $\Gamma(R^c) = 0$. In turn, $\Gamma - a.e.\ x \in [X_r, H]$ we have that $V(x)$ equals $\psi_1$. Since, $V(\cdot)$ is continuous and $\Gamma|_{[X_r,H]}$ has full support in $[X_r, H]$ which has non-empty interior we conclude that $V(x) = \psi_1$ for all $x \in [X_r, H]$.

For the reminder of the proof we assume $V(X_r) < \psi_1$. We show that if $V(\cdot)$ is not non-decreasing in $[X_r, H]$ then there is an strict objective improvement. In the proof we define several critical points in the interval $[X_r, H]$ which will help us to create a flow separated region (no flow leaves this region). Then we show the objective strict improvement in this region. In Figure 11 we provide a graphical representation of the points just mentioned.

So assume that $V(x)$ is not non-decreasing in $[X_r, H]$, then there exists $\hat{x} > \hat{y} \geq X_r$ such that $V(\hat{x}) < V(\hat{y})$. Let,

$$\bar{y} \triangleq \sup\{z \in [\hat{y}, \hat{x}] : V(z) = V(\hat{y})\},$$

note that since for $z = \hat{y}$, $V(z) = V(\hat{y})$ thus the set over which we take the supremum above is both bounded and non-empty. Hence, $\bar{y}$ is well defined and it corresponds to the last point $z$ in $[\hat{y}, \hat{x}]$ such that $V(z)$ equals $V(\hat{y})$. Moreover, because $V(\cdot)$ is continuous $\bar{y} < \hat{x}$, and for all $z \in (\bar{y}, \hat{x}]$ we have $V(z) < V(\hat{y}) = V(\bar{y})$. Let

$$y_0 \triangleq \inf\{z \in [X_r, \bar{y}] : \exists x \in (\bar{y}, H] \quad \text{such that} \quad z \in \mathcal{IR}(x)\},$$

if for all $z \in [X_r, \bar{y}]$ and for all $x \in (\bar{y}, H]$ we have $z \notin \mathcal{IR}(x)$, we let $y_0 = \bar{y}$. That is, $y_0$ is the smallest $z$ in $[X_r, \bar{y}]$ to which some location in $(\bar{y}, H]$ is indifferent to travel to. Note that for all $z \in (y_0, \hat{x}]$ we have $V(z) < V(y_0)$. Also, the definition of $y_0$ and Lemma B-3 imply that $\mathcal{T}([-H, y_0] \times (y_0, H]) = 0$ and $\mathcal{T}((y_0, H] \times [-H, y_0]) = 0$. Let

$$y_1 \triangleq \inf\{z \in [\hat{x}, H] : V(z) > V(y_0)\},$$

that is, $y_1$ is the first value after $\hat{x}$ for which $V(\cdot)$ hits $V(y_0)$. Note that when well defined $y_1$ satisfies that $\mathcal{T}([y_1, H] \times [-H, y_1]) = 0$. If this is not the case then since points do not have measure(different from the origin) we would have $\mathcal{T}((y_1, H] \times [-H, y_1)) > 0$ and, therefore, by Lemma B-3 we can find $(x, y) \in (y_1, H] \times [-H, y_1)$ such that $y \in \mathcal{IR}(x)$. But this would contradict the definition of $y_1$.

There are two cases:

1. $y_1$ is not well defined: In this case we have that for all $z \in [\hat{x}, H]$, $V(z) \leq V(y_0)$. Recall that from our previous discussion we have that $V(z) < V(y_0)$ for all $z \in (y_0, \hat{x}]$. Also, Property 1 (which we prove at

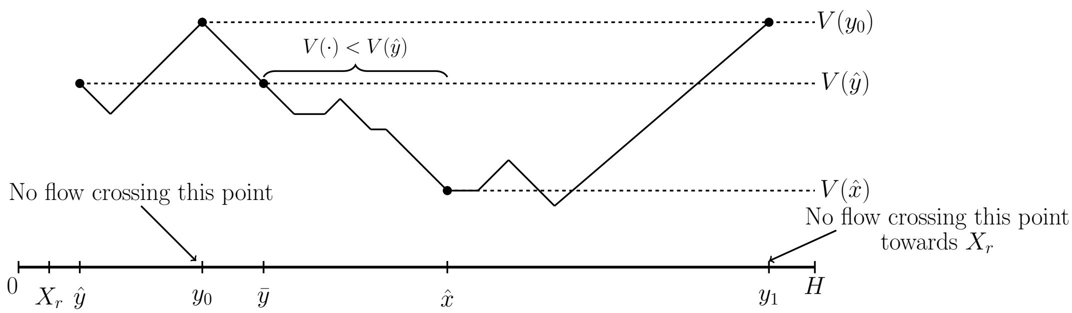


Figure 11: Graphical representation of $\hat{y}, \hat{x}, \bar{y}, y_0$ and $y_1$.

the end of the present proof) establishes that $\mathcal{T}_2((y_0, \hat{x}]) > 0$. Using this observations we create a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ with revenue strictly larger than that of $(p, \mathcal{T})$.

Let $\mathcal{B} = [-H, y_0]$ and note that we have both $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$, so we can use the separation result in Lemma D-3. Hence $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ (see Lemma D-3 for notation) is a price-equilibrium pair in $\mathcal{B}$. Its revenue equals the revenue of $(p, \mathcal{T})$ in $\mathcal{B}$, and $V(y_0 | \, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$.

For $\mathcal{B}^c$ we choose flows $\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for all $\mathcal{L} \subset \mathcal{B}^c \times \mathcal{B}^c$. That is all drivers stay at their initial location. It is not hard to see that $s^{\mathcal{T}^{\mathcal{B}^c}}(x)$ equals $\theta_1$, $\Gamma - a.e.\ x$ in $\mathcal{B}^c$. We choose prices $p^{\mathcal{B}^c}(x) = p_0$ for all $x \in \mathcal{B}^c$, where $p_0$ is such that

$$\alpha \cdot p_0 \cdot \min\{1, \frac{\lambda_1 \cdot \overline{F}(p_0)}{\theta_1}\} = V(y_0), \tag{D-11}$$

note that since $V(y_0) \leq \psi_1$, $p_0$ is well defined. That is, the solution $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ is the same solution as in pre-demand shock environment but in smaller city, $\mathcal{B}^c$ and with a larger price across all locations. Using Proposition 1 it is not hard to see that the revenue associated with this solution is $\gamma \cdot V(y_0) \cdot \theta_1 \cdot (H - y_0)$.

By Lemma D-3, we can paste the two previous solutions to create a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ in entire city. This new solution yields a strict objective improvement. Indeed,

$$\begin{aligned}
\mathbf{Rev}_{[y_0,H]}(p, \mathcal{T}) &= \int_{[y_0,H]} V(x)\, d\mathcal{T}_2(x) \\
&= \int_{(y_0,\hat{x}]} V(x)\, d\mathcal{T}_2(x) + \int_{(\hat{x},H]} V(x)\, d\mathcal{T}_2(x) \\
&\overset{(a)}{<} V(y_0) \cdot \mathcal{T}_2((y_0, \hat{x}]) + \int_{(\hat{x},H]} V(x)\, d\mathcal{T}_2(x) \\
&\leq V(y_0) \cdot \mathcal{T}_2((y_0, \hat{x}]) + V(y_0) \cdot \mathcal{T}_2((\hat{x}, H]) \\
&\overset{(b)}{=} V(y_0) \cdot \Theta([y_0, H]) \\
&= V(y_0) \cdot \theta_1 \cdot (H - y_0) \\
&= \mathbf{Rev}_{[y_0,H]}(\tilde{p}, \tilde{\mathcal{T}}),
\end{aligned}$$

where $(a)$ comes from $\mathcal{T}_2((y_0, \hat{x}]) > 0$, $(b)$ comes from the fact that under $\mathcal{T}$ no flow leaves or enters $[y_0, H]$, and the last two lines from the definition of $(\tilde{p}, \tilde{\mathcal{T}})$ restricted to $[y_0, H]$.

2. $y_1$ is well defined: In this case there exists $z \in [\hat{x}, H]$ such that $V(z) > V(y_0)$. Also, we must have $y_1 > \hat{x}$, and we already argued that $\mathcal{T}([y_1, H] \times [-H, y_1]) = 0$. There are two more cases.

   a) $\forall y \in (y_0, y_1]$, $\forall x > y_1$, $x \notin \mathcal{IR}(y)$: This together with Lemma B-3 imply that $\mathcal{T}([y_0, y_1] \times ([-H, y_0] \cup [y_1, H])) = 0$, and we also have $\mathcal{T}(([-H, y_0] \cup [y_1, H]) \times [y_0, y_1]) = 0$. From this we can construct a new feasible solution $(\tilde{p}, \tilde{\mathcal{T}})$ with revenue strictly larger than that of $(p, \mathcal{T})$.

Let $\mathcal{B} = [-H, y_0) \cup (y_1, H]$ and note that we have both $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$, so we can use the separation result in Lemma D-3. Thus $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ (see Lemma D-3 for notation) is a price-equilibrium pair in $\mathcal{B}$. Its revenue equals the revenue of $(p, \mathcal{T})$ in $\mathcal{B}$, and $V(y_0|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$ and $V(y_1|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$.

For $\mathcal{B}^c$ we choose flows $\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for all $\mathcal{L} \subset \mathcal{B}^c \times \mathcal{B}^c$. We choose prices $p^{\mathcal{B}^c}(x) = p_0$ for all $x \in \mathcal{B}^c$, where $p_0$ is as in Eq. (D-11). As we argued before this solution forms an price-equilibrium pair with revenue equal to $V(y_0) \cdot \theta_1 \cdot (y_1 - y_0)$.

We can then paste both solutions (see Lemma D-3) to obtain a solution $(\tilde{p}, \tilde{\mathcal{T}})$ in the entire city. As before, it yields a strict revenue improvement.

b) $\exists\, y \in (y_0, y_1]$, $\exists\, x > y_1$ such that $x \in \mathcal{IR}(y)$: Then the following points are well defined

$$\overline{y}_1 \triangleq \sup\{x \in [y_1, H] : \exists\, y \in [y_0, y_1] \text{ such that } x \in \mathcal{IR}(y)\},$$
$$\underline{y}_1 \triangleq \inf\{y \in [y_0, y_1] : \exists\, x \in [y_1, H] \text{ such that } x \in \mathcal{IR}(y)\}.$$

That is, $\overline{y}_1$ is largest point after $y_1$ for which some location in $[y_0, y_1]$ has drivers indifferent to travel to it. As for $\underline{y}_1$, it corresponds to the smallest point in $[y_0, y_1]$ that has drivers willing to travel to some location in $[y_1, H]$. Note that from the definition of $\overline{y}_1$ and Lemma B-3 we can deduce that there is no flow crossing $\overline{y}_1$ in any direction, that is, $\mathcal{T}([-H, \overline{y}_1] \times [\overline{y}_1, H]) = 0$. Also, from Property 2 (which we prove at the end of the present proof) for any $z \in [\underline{y}_1, \overline{y}_1]$, $\overline{y}_1 \in \mathcal{IR}(z)$. That is, for any $z \in [\underline{y}_1, \overline{y}_1]$, $V(z|\, p, \mathcal{T}) = V(\overline{y}_1) - |\overline{y}_1 - z|$.

The idea is to again construct an strict objective improvement. First, define $y^c$ to be such that $V(y_0) + (y^c - y_0) = V(\overline{y}_1)$, that is, $y^c = V(\overline{y}_1) - V(y_0) + y_0$. Next we argue that $y_c \in (y_0, \overline{y}_1)$. In fact, by the definition of $\overline{y}_1$ we must have $V(\overline{y}_1) > V(y_0)$ thus $y_c > y_0$. Also, if $y_c \geq \overline{y}_1$ then

$$V(y_0) + (y^c - y_0) \geq V(y_0) + (\overline{y}_1 - y_0) \Leftrightarrow V(\overline{y}_1) \geq V(y_0) + (\overline{y}_1 - y_0),$$

and since $V(\overline{y}_1) = V(y_1) + (\overline{y}_1 - y_1)$ we would have

$$V(y_1) + (\overline{y}_1 - y_1) \geq V(y_0) + (\overline{y}_1 - y_0) \Leftrightarrow V(y_1) - V(y_0) \geq y_1 - y_0,$$

which, since $y_1 > y_0$, implies that $V(y_1) > V(y_0)$, contradicting the definition of $y_1$. From this we can also infer that $y^c - y_0 = \overline{y}_1 - y_1$.

Second, let $h \triangleq \overline{y}_1 - y^c$ and for any set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ define the set

$$\mathcal{L}_h \triangleq \{(x + h, y + h) \in \mathcal{C} \times \mathcal{C} : (x, y) \in \mathcal{L}\}.$$

We now construct a new solution $(\tilde{p}, \tilde{\mathcal{T}})$. Let $\mathcal{B} = [-H, y_0) \cup (\overline{y}_1, H]$, so that $\mathcal{B}^c = [y_0, \overline{y}_1]$. Following our previous scheme of proof we construct two price-equilibrium pairs one in $\mathcal{B}$ and another in $\mathcal{B}^c$, and then we paste them to create $(\tilde{p}, \tilde{\mathcal{T}})$. As we did before we can use the separation result (see Lemma D-3) to obtain a solution $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ in $\mathcal{B}$ such that $V(y_0|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$ and $V(\overline{y}_1|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(\overline{y}_1)$.

For $\mathcal{B}^c$ define the flow $\mathcal{T}^{\mathcal{B}^c}$ for any $\mathcal{L} \subseteq \mathcal{B}^c \times \mathcal{B}^c$ by

$$\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}) = \mathcal{T}\Big((\mathcal{L} \cap ([y_0, y^c] \times [y_0, \overline{y}_1]))_h\Big) + \Theta(\pi_1(\mathcal{L} \cap ([y^c, \overline{y}_1] \times [y_0, \overline{y}_1]) \cap \mathcal{D})), \tag{D-12}$$

We next show that this flow belongs to $\mathcal{F}_{\mathcal{B}^c}(\Theta|_{\mathcal{B}^c})$ and that it is an equilibrium for some prices $p^{\mathcal{B}^c}$ yet to be defined. Indeed, for any measurable subset $K$ of $\mathcal{B}^c$ we have

$$\begin{aligned}
\mathcal{T}_1^{\mathcal{B}^c}(K) &= \mathcal{T}\Big(((K\times\mathcal{B}^c)\cap([y_0,y^c]\times[y_0,\overline{y}_1]))_h\Big)+\Theta(\pi_1((K\times\mathcal{B}^c)\cap([y^c,\overline{y}_1]\times[y_0,\overline{y}_1])\cap\mathcal{D}))\\
&= \mathcal{T}\Big(((K\cap[y_0,y^c])\times[y_0,\overline{y}_1])_h\Big)+\Theta(K\cap[y^c,\overline{y}_1])\\
&= \mathcal{T}\Big(((K+h)\cap[y_0+h,y^c+h])\times[y_0+h,\overline{y}_1+h]\Big)+\Theta(K\cap[y^c,\overline{y}_1])\\
&= \mathcal{T}\Big(((K+h)\cap[y_1,\overline{y}_1])\times[y_1,\overline{y}_1+h]\Big)+\Theta(K\cap[y^c,\overline{y}_1])\\
&\overset{(a)}{=} \mathcal{T}\Big(((K+h)\cap[y_1,\overline{y}_1])\times\mathcal{C}\Big)+\Theta(K\cap[y^c,\overline{y}_1])\\
&= \Theta((K+h)\cap[y_1,\overline{y}_1])+\Theta(K\cap[y^c,\overline{y}_1])\\
&= \Theta((K\cap[y_0,y^c])+h)+\Theta(K\cap[y^c,\overline{y}_1])\\
&\overset{(b)}{=} \Theta(K\cap[y_0,y_c])+\Theta(K\cap[y^c,\overline{y}_1])\\
&= \Theta(K),
\end{aligned}$$

where $(a)$ holds because by construction in $[y_1,\overline{y}_1]$ the flow there can be transported only inside the same set and, therefore, $\mathcal{T}([y_1,\overline{y}_1]\times[y_1,\overline{y}_1+h]^c)$ equals zero. Equality $(b)$ comes from the fact that $\Theta$ is invariant under translation (it is a multiple of the Lebesgue measure). Therefore, $\mathcal{T}_1^{\mathcal{B}^c}$ coincides with $\Theta|_{\mathcal{B}^c}$. Also, it is clear from the definition of $\mathcal{T}^{\mathcal{B}^c}$ that $\mathcal{T}_2^{\mathcal{B}^c}\ll\Gamma$. Hence, $\mathcal{T}^{\mathcal{B}^c}$ belongs to $\mathcal{F}_{\mathcal{B}^c}(\Theta|_{\mathcal{B}^c})$. Furthermore, Property 3 (which we prove at the end of the present proof) ensures that

$$\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)\le\frac{d\mathcal{T}_2}{d\Gamma}(x+h)\ \ \Gamma-a.e.\ x\ \text{ in }[y_0,y^c],\quad\text{and}\quad\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)=\theta_1\ \ \Gamma-a.e.\ x\ \text{ in }[y^c,\overline{y}_1].\tag{D-13}$$

We choose the prices $p^{\mathcal{B}^c}$ as follows. In $[y^c,\overline{y}_1]$ we set constant prices equal to $p_1$ such that

$$\alpha\cdot p_1\cdot\min\{1,\frac{\lambda_1\cdot\overline{F}(p_1)}{\theta_1}\}=V(\overline{y}_1),$$

this price is well defined because $V(\overline{y}_1)\le\psi_1$. For locations in $[y_0,y^c]$ consider the set

$$K\triangleq\Big\{x\in[y_0,y^c]:\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)\le\frac{d\mathcal{T}_2}{d\Gamma}(x+h)\Big\},\tag{D-14}$$

note from Eq. D-13 we have $\Gamma(K^c)=0$. We set prices for $x\in K$ to be such that

$$U\Big(x,p^{\mathcal{B}^c}(x),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)\Big)=U\Big(x+h,p(x+h),s^{\mathcal{T}}(x+h)\Big),\tag{D-15}$$

such prices are well defined because the new Radon-Nikodym is smaller than the old one (shifted by $h$) in $K$. For $x\in K^c$ we set the prices equal to zero. Now we need to verify that this selection of prices and flows yields an equilibrium. That is, we need show that the set

$$\mathcal{E}_{\mathcal{B}^c}=\Big\{(x,y)\in\mathcal{B}^c\times\mathcal{B}^c:\Pi(x,y,p^{\mathcal{B}^c}(y),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(y))=\operatorname*{ess\,sup}_{\mathcal{B}^c}\Pi\Big(x,\cdot,p^{\mathcal{B}^c}(\cdot),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(\cdot)\Big)\Big\},$$

has $\mathcal{T}^{\mathcal{B}^c}$ measure equal to $\Theta(\mathcal{B}^c)$. First, from Property 3 we have

$$V(x|\,p^{\mathcal{B}^c},\mathcal{T}^{\mathcal{B}^c})=\operatorname*{ess\,sup}_{\mathcal{B}^c}\Pi\Big(x,\cdot,p^{\mathcal{B}^c}(\cdot),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(\cdot)\Big)=\begin{cases}V(y_1)+(x-y_0)&\text{if }x\in[y_0,y^c]\\V(\overline{y}_1)&\text{if }[y^c,\overline{y}_1].\end{cases}\tag{D-16}$$

For the first term in Eq. (D-12) observe that $\mathcal{T}((\mathcal{E}_{\mathcal{B}^c}\cap[y_0,y^c]\times[y_0,\overline{y}_1])_h)$ equals

$$\mathcal{T}\Big(\Big\{(x,y)\in[y_1,\overline{y}_1]\times[y_1,\overline{y}_1]:\Pi(x-h,y-h,p^{\mathcal{B}^c}(y-h),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(y-h))=V(y_1)+(x-y_1)\Big\}\Big),$$

using that $\Gamma(K^c) = 0$ and Eq. (D-24) one can verify that this expression equals

$$\mathcal{T}\Big(\Big\{(x,y) \in [y_1, \overline{y}_1] \times [y_1, \overline{y}_1] : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = V(x|\, p, \mathcal{T})\Big\}\Big).$$

In turn, from the definition of $\underline{y}_1$ and $\overline{y}_1$, and the fact that $\mathcal{T}$ is an equilibrium flow this last expression equals $\Theta([y_1, \overline{y}_1])$. For the second term in Eq. (D-12) we have

$$\mathcal{E}_{\mathcal{B}^c} \cap [y^c, \overline{y}_1] \times [y_0, \overline{y}_1] \cap \mathcal{D} = \Big\{(x,y) \in [y^c, \overline{y}_1] \times [y_0, \overline{y}_1] : \Pi(x, y, p^{\mathcal{B}^c}(y), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(y)) = V(\overline{y}_1)\Big\} \cap \mathcal{D},$$

Thus the second term in Eq. (D-12) equals

$$\Theta\Big(\Big\{x \in [y^c, \overline{y}_1] : U(x, p^{\mathcal{B}^c}(x), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)) = V(\overline{y}_1)\Big\}\Big) = \Theta([y^c, \overline{y}_1]) = \Theta([y_0, y_1]),$$

where the first equality comes from Eq. (D-13) and the discussion that it follows it. The second equality comes from $\Theta$ being invariant under translation and $y^c - y_0 = \overline{y}_1 - y_1$. Putting all these together yields

$$\mathcal{T}^{\mathcal{B}^c}(\mathcal{E}_{\mathcal{B}^c}) = \Theta([\overline{y}_1, y_1]) + \Theta([y_0, y_1]) = \Theta([y_0, \overline{y}_1]) = \Theta(\mathcal{B}^c).$$

In order to create the new solution $(\tilde{p}, \tilde{\mathcal{T}})$ we just use Lemma D-3 to paste the two solutions we constructed in $\mathcal{B}$ and $\mathcal{B}^c$. Note that the pasting is allowed because $V(y_0|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c}) = V(y_0)$ and $V(\overline{y}_1|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c}) = V(\overline{y}_1)$.

We now finally show the objective improvement. It is sufficient to prove that $\mathbf{Rev}_{[y_0, \overline{y}_1]}(\tilde{p}, \tilde{\mathcal{T}}) > \mathbf{Rev}_{[y_0, \overline{y}_1]}(p, \mathcal{T})$,

$$\begin{aligned}
\mathbf{Rev}_{[y_0,\overline{y}_1]}(p, \mathcal{T}) = \int_{[y_0,\overline{y}_1]} V(x)\, d\mathcal{T}_2(x) &\overset{(a)}{<} \int_{[y_0,\overline{y}_1]} V(y_0)\, d\mathcal{T}_2(x) \\
&\overset{(b)}{=} \int_{[y_0,\overline{y}_1]} V(y_0)\, d\mathcal{T}_2^{\mathcal{B}^c}(x) \\
&\overset{(c)}{\leq} \int_{[y_0,\overline{y}_1]} V(x|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})\, d\mathcal{T}_2^{\mathcal{B}^c}(x) \\
&= \mathbf{Rev}_{[y_0,\overline{y}_1]}(\tilde{p}, \tilde{\mathcal{T}}),
\end{aligned}$$

where in $(a)$ use Property 1. In $(b)$ we use that under $\mathcal{T}$ no flow leaves or enters $\mathcal{B}^c$ and, thus,

$$\mathcal{T}_2^{\mathcal{B}^c}(\mathcal{B}^c) = \mathcal{T}^{\mathcal{B}^c}(\mathcal{B}^c \times \mathcal{B}^c) = \Theta(\mathcal{B}^c) = \mathcal{T}(\mathcal{B}^c \times \mathcal{C}) = \mathcal{T}(\mathcal{B}^c \times \mathcal{B}^c) = \mathcal{T}(\mathcal{C} \times \mathcal{B}^c) = \mathcal{T}_2(\mathcal{B}^c).$$

In $(c)$ we simply use Eq. (D-16).

In what follows we provide a complete proof of the three properties that we use to obtain the result.

**Property 1.** $\mathcal{T}_2((y_0, \hat{x}]) > 0$.

**Proof of Property 1.** First we show that $\exists\, h \in (0, \hat{x} - y_0)$ such that $\mathcal{T}((y_0, y_0 + h) \times [\hat{x}, y_1]) = 0$. Suppose this is not true then for all $n \in \mathbb{N}$ large enough we have that $\mathcal{T}((y_0, y_0 + \frac{1}{n}) \times [\hat{x}, y_1]) > 0$, which thanks to Lemma B-3 implies that for all $n \in \mathbb{N}$ large enough there exists $(x_n, y_n) \in (y_0, y_0 + \frac{1}{n}) \times [\hat{x}, y_1]$ such that $y_n \in \mathcal{IR}(x_n)$, that is, $V(x_n) = V(y_n) - |y_n - x_n|$. Since $y_n \in [\hat{x}, y_1]$ we must have $V(y_n) \leq V(y_0)$ for all $n \in \mathbb{N}$ large (when $y_1$ is not well defined we replaced by $H$ and the argument still goes through). Furthermore, $x_n$ converges to $y_0$ so the continuity of $V(\cdot)$ yields

$$V(y_0) = \lim_{n\to\infty} V(x_n) = \lim_{n\to\infty} V(y_n) - |y_n - x_n| \leq V(y_0) - \lim_{n\to\infty}(y_n - x_n) < V(y_0),$$

not possible. We conclude that $\exists\, h \in (0, \hat{x} - y_0)$ such that $\mathcal{T}((y_0, y_0 + h) \times [\hat{x}, y_1]) = 0$. Note that the same must be true for some $h \in (0, (\hat{x} - y_0) \wedge \frac{(y_1 - y_0)}{2})$. We fix $h$ in this interval with the property we just proved.

Next, note we also have that $\mathcal{T}((y_0, y_0+h)\times(y_1,H]) = 0$; otherwise, by Lemma B-3 we can find $(x,y) \in (y_0, y_0+h)\times(y_1,H]$ such that $y \in \mathcal{IR}(x)$, which implies that $y \in \mathcal{IR}(y_1)$, that is, $V(y_1) = V(y) - |y-y_1|$ and $V(x) = V(y) - |y-x|$. Since $V(y_1) = V(y_0)$ we have $(y_1 - x) = V(y_0) - V(x)$, but our choice of $h$ implies that $y_1 - x > h$ thus

$$h < (y_1 - x) = V(y_0) - V(x) \le |y_0 - x| \le h,$$

again a contradiction. The last inequality comes from the Lipschitz property (see Lemma 1). In summary, we have that there exists $h \in (0, (\hat{x} - y_0) \wedge \frac{(y_1-y_0)}{2})$ such that $\mathcal{T}((y_0, y_0+h)\times[\hat{x}, H]) = 0$. To conclude the proof note the following

$$\begin{aligned}
0 &\overset{(a)}{<} \Theta((y_0, y_0+h)) \\
&= \mathcal{T}((y_0, y_0+h)\times \mathcal{C}) \\
&\overset{(b)}{=} \mathcal{T}((y_0, y_0+h)\times[y_0, H]) \\
&= \mathcal{T}((y_0, y_0+h)\times[y_0, \hat{x})) + \mathcal{T}((y_0, y_0+h)\times[\hat{x}, H]) \\
&= \mathcal{T}((y_0, y_0+h)\times[y_0, \hat{x})) \\
&\le \mathcal{T}_2([y_0, \hat{x}]) \\
&\overset{(c)}{=} \mathcal{T}_2((y_0, \hat{x}]),
\end{aligned}$$

where $(a)$ comes from the fact that the measure $\Theta$ has full support in $\mathcal{C}$. The equality $(b)$ holds because by construction no flow leaves $[y_0, H]$, and $(c)$ is true because $\mathcal{T}_2 \ll \Gamma$ and $\Gamma$ does not have atoms in $[y_0, \hat{x}]$. This concludes the proof of Property 1.

**Property 2.** Both $\overline{y}_1$ and $\underline{y}_1$ are achieved in the set where they are defined. Furthermore, for any $z \in [\underline{y}_1, \overline{y}_1]$, $\overline{y}_1 \in \mathcal{IR}(z)$.

**Proof of Property 2.** First we show both

$$\exists\, y_q \in [y_0, y_1] \text{ such that } \overline{y}_1 \in \mathcal{IR}(y_q) \quad \text{and} \quad \exists\, x_q \in [y_1, H] \text{ such that } x_q \in \mathcal{IR}(\underline{y}_1). \tag{D-17}$$

Let us begin with the first statement. Let $x^n$ be a sequence in $A$ converging to $\overline{y}_1$, where

$$A = \{x \in [y_1, H] : \exists\, y \in [y_0, y_1] \text{ such that } x \in \mathcal{IR}(y)\}.$$

Then there exists a sequence $\{y^n\} \subset [y_0, y_1]$ such that $x^n \in \mathcal{IR}(y^n)$. Note that since $\{y^n\} \subset [y_0, y_1]$ and $x^n \in [y_1, H]$, Lemma B-2 implies that $x^n \in \mathcal{IR}(y_1)$. Hence, $V(x^n) - \|x^n - y_1\| = V(y_1)$, taking limit yields $V(\overline{y}_1) - \|\overline{y}_1 - y_1\| = V(y_1)$, that is, $\overline{y}_1 \in \mathcal{IR}(y_1)$.

Now we prove that $\underline{y}_1 \in A$ where

$$A = \{y \in [y_0, y_1] : \exists\, x \in [y_1, H] \text{ such that } x \in \mathcal{IR}(y)\}.$$

By the definition of $\underline{y}_1$ we can always construct a sequence $\{y^n\} \subset A$ converging to $\underline{y}_1$. From the definition of $A$ there exists another sequence $\{x^n\} \subset [y_1, H]$ such that $x^n \in \mathcal{IR}(y^n)$ for all $n$. We can extract a subsequence $\{x^{n_k}\}$ from $\{x^n\}$ that converges to some point $x_q \in [y_1, H]$. Then we have $V(x^{n_k}) - \|x^{n_k} - y^{n_k}\| = V(y^{n_k})$. Taking limits and using the continuity of $V(\cdot)$ yields $V(x_q) - \|x_q - \underline{y}_1\| = V(\underline{y}_1)$, that is, $x_q \in \mathcal{IR}(\underline{y}_1)$. This concludes the proof for Eq. (D-17).

Next, we show that for all $z \in [\underline{y}_1, \overline{y}_1]$, $\overline{y}_1 \in \mathcal{IR}(z)$. First, from our previos argument we know there exists $y_q$ and $x_q$ as in Eq. (D-17). Then Lemma B-2 implies $\overline{y}_1 \in \mathcal{IR}(z)$ for all $z \in [y_q, \overline{y}_1]$. Observe that this yields $\overline{y}_1 \in \mathcal{IR}(x_q)$ because $x_q \in [y_q, \overline{y}_1]$. Take $z \in [\underline{y}_1, y_q]$ then since $x_q \in \mathcal{IR}(\underline{y}_1)$ from Lemma B-2 we conclude that $x_q \in \mathcal{IR}(z)$. Hence, we have

$$V(z) = V(x_q) - (x_q - z) = V(\overline{y}_1) - (\overline{y}_1 - x_q) - (x_q - z) = V(\overline{y}_1) - (\overline{y}_1 - z).$$

which implies that $\overline{y}_1 \in \mathcal{IR}(z)$. This concludes the proof of Property 2.

**Property 3.** Both Eq. (D-13) and Eq. (D-16) hold.

**Proof of Property 3.** Let us star with Eq. (D-13). In order to prove the first part in Eq. (D-13) consider the following set

$$K = \Big\{x \in [y_0, y^c] : \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x) \leq \frac{d\mathcal{T}_2}{d\Gamma}(x+h)\Big\}.$$

We want to show that $\Gamma(K^c) = 0$ (the complement is taken with respect to $[y_0, y^c]$). If this is not true then $\Gamma(K^c) > 0$ and we have

$$\mathcal{T}_2^{\mathcal{B}^c}(K^c) = \int_{K^c} \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)\, d\Gamma(x) > \int_{K^c} \frac{d\mathcal{T}_2}{d\Gamma}(x+h)\, d\Gamma(x) = \mathcal{T}_2(K^c + h). \tag{D-18}$$

However,

$$\begin{aligned}
\mathcal{T}_2^{\mathcal{B}^c}(K^c) &= \mathcal{T}\Big(([y_0, y^c] \times K^c)_h\Big) + \Theta(\pi_1(([y^c, \overline{y}_1] \times K^c) \cap \mathcal{D})) \\
&= \mathcal{T}\Big(([y_0, y^c] \times K^c)_h\Big) \\
&= \mathcal{T}\Big([y_0 + h, y^c + h] \times (K^c + h)\Big) \\
&\leq \mathcal{T}\Big(\mathcal{C} \times (K^c + h)\Big) \\
&= \mathcal{T}_2(K^c + h).
\end{aligned}$$

This together with Eq. D-18 yield a contradiction. To prove the second part of Eq. (D-13) consider any $\mathcal{R} \subset [y^c, \overline{y}_1]$, and observe that

$$\mathcal{T}_2^{\mathcal{B}^c}(\mathcal{R}) = \mathcal{T}\Big([y_1, \overline{y}_1] \times (\mathcal{R} + h)\Big) + \Theta(\mathcal{R}) = \Theta(\mathcal{R}) = \int_{\mathcal{R}} \theta_1 d\Gamma(x),$$

where the second equality comes from $\mathcal{R} + h \subset [\overline{y}_1, \overline{y}_1 + h]$ and $\mathcal{T}([y_1, \overline{y}_1] \times [\overline{y}_1, \overline{y}_1 + h]) = 0$.

Finally, we provide a proof for Eq. (D-16). Let $Z(x) \triangleq \min\{V(y_0) + (x - y_0), V(\overline{y}_1)\}$. We verify that for all $x \in \mathcal{B}^c$

$$Z(x) \geq U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c, \tag{D-19}$$

and that $Z(x)$ is the smallest with such property. First, fix $x \in [y^c, \overline{y}_1]$ so $Z(x) = V(\overline{y}_1)$. Note that from our choice of prices in $[y^c, \overline{y}_1]$ we have

$$Z(x) = V(\overline{y}_1) \geq V(\overline{y}_1) - |w - x| = U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } [y^c, \overline{y}_1].$$

So we only need to show the same inequality but this time for $[y_0, y^c]$. From the definition of $\underline{y}_1$ and $\overline{y}_1$ and Lemma B-2 we have that $V(\overline{y}_1) - |\overline{y}_1 - y_1|$ equals $V(y_1 |\, p, \mathcal{T})$ and, therefore,

$$\begin{aligned}
V(\overline{y}_1) &\geq U(w, p(w), s^{\mathcal{T}}(w)) - |w - y_1| + |\overline{y}_1 - y_1|, \quad \Gamma - a.e.\ w \text{ in } [y_1, \overline{y}_1] \\
&\geq U(w, p(w), s^{\mathcal{T}}(w)), \quad \Gamma - a.e.\ w \text{ in } [y_1, \overline{y}_1].
\end{aligned}$$

We can use this together with the fact that $[y_0, y^c] + h = [y_1, \overline{y}_1]$ to obtain

$$\begin{aligned}
Z(x) = V(\overline{y}_1) &\overset{(a)}{\geq} U\Big(w + h, p(w + h), s^{\mathcal{T}}(w + h)\Big), \quad \Gamma - a.e.\ w \text{ in } [y_0, y^c] \\
&\geq U\Big(w + h, p(w + h), s^{\mathcal{T}}(w + h)\Big) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } [y_0, y^c] \\
&\overset{(b)}{=} U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } [y_0, y^c],
\end{aligned}$$

Inequality $(a)$ comes from the fact that $\Gamma$ in the interval under consideration is invariant under a shift; $(b)$ comes from Eq. (D-24). That is, for $x \in [y^c, \overline{y}_1]$ Eq. (D-19) is satisfied. It is left to verify that $Z(x)$ is the smallest value satisfying Eq. (D-19). For any $\epsilon > 0$, since $x \in [y^c, \overline{y}_1]$ we have

$$\begin{aligned} 0 &< \Gamma(B(x,\epsilon) \cap [y^c, \overline{y}_1]) \\ &= \Gamma\Big(w \in [y^c, \overline{y}_1] : V(\overline{y}_1) - |w-x| > V(\overline{y}_1) - \epsilon\Big) \\ &= \Gamma\Big(w \in [y^c, \overline{y}_1] : U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w-x| > V(\overline{y}_1) - \epsilon\Big), \end{aligned}$$

hence $V(\overline{y}_1)$ is the smallest value satisfying Eq. (D-19).

Now we show Eq. (D-19) for $x \in [y_0, y^c]$. Fix $x \in [y_0, y^c]$ so $Z(x) = V(y_0) + (x - y_0)$. Note that $V(y_0)$ equals $V(y_1)$, and from the definition of $\overline{y}_1$ and the envelope result we have that $V(y_1)$ equals $V(\overline{y}_1) - (\overline{y}_1 - y_1)$. Therefore,

$$\begin{aligned} Z(x) &= V(\overline{y}_1) - (\overline{y}_1 - y_1) + (x - y_0) \\ &\overset{(a)}{\geq} V(\overline{y}_1) - (w - x), \quad \Gamma - a.e. \ w \ \text{in} \ [y^c, \overline{y}_1] \\ &\overset{(b)}{=} U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w-x|, \quad \Gamma - a.e. \ w \ \text{in} \ [y^c, \overline{y}_1], \end{aligned}$$

where $(a)$ follows from $w \geq y_c$ and $y^c - y_0 = \overline{y}_1 - y_1$. Line $(b)$ holds from our choice of prices in $[y^c, \overline{y}_1]$. Hence, $Z(x)$ upper bounds (almost surely) the desire quantity in $[y^c, \overline{y}_1]$, so we just need to prove the same bound for $[y_0, y^c]$. Note that from the definition of $\underline{y}_1$ and $\overline{y}_1$ we have that

$$V(x+h) = V(y_1) + (x + h - y_1) = V(y_1) + (x - y_0) = Z(x),$$

and thus

$$\begin{aligned} Z(x) &= V(x+h|\, p, \mathcal{T}) \\ &\overset{(a)}{\geq} U(w, p(w), s^{\mathcal{T}}(w)) - |w - (x+h)|, \quad \Gamma - a.e. \ w \ \text{in} \ [y_1, \overline{y}_1] \\ &\overset{(b)}{=} U(w+h, p(w+h), s^{\mathcal{T}}(w+h)) - |w + h - (x+h)|, \quad \Gamma - a.e. \ w \ \text{in} \ [y_0, y^c] \\ &\overset{(c)}{=} U(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)) - |w - x)|, \quad \Gamma - a.e. \ w \ \text{in} \ [y_0, y^c], \end{aligned}$$

where $(a)$ comes from the definition of $V(x+h|\, p, \mathcal{T})$, $(b)$ from the invariance under translation of $\Gamma$. Line $(c)$ follows from Eq. (D-24). Therefore, $Z(x)$ satisfies Eq. (D-19). To see why $Z(x)$ is the smallest value satisfying this equation observe that

$$\begin{aligned} 0 &< \Gamma(B(y^c,\epsilon) \cap [y^c, \overline{y}_1]) \\ &\overset{(a)}{=} \Gamma\Big(w \in [y^c, \overline{y}_1] : V(\overline{y}_1) - (w - x) > V(\overline{y}_1) - (\overline{y}_1 - y_1) + (x - y_0) - \epsilon\Big) \\ &= \Gamma\Big(w \in [y^c, \overline{y}_1] : U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w-x| > Z(x) - \epsilon\Big), \end{aligned}$$

where in $(a)$ we use that $y^c - y_0 = \overline{y}_1 - y_1$. This implies that $Z(x)$ is the smallest value satisfying Eq. (D-19), completing the proof. ☐

**Proposition D-3.** *(Tight upper bound) Without loss of optimality, we can focus on price-equilibrium pairs $(p, \mathcal{T})$ such that the upper bound in Eq. (7) is tight.*

***Proof of Proposition D-3***. If $X_r = H$ there is nothing to prove, so assume $X_r < H$. Let $(p, \mathcal{T})$ be a feasible solution such that $V(\cdot|p, \mathcal{T})$ is non-decreasing. Due to Proposition D-2 we can always restrict attention to this type of solution. We proceed by contradiction. Assume that there exists $\tilde{x} \in (X_r, H]$ such that

$$V(\tilde{x}) < \min\{V(X_r) + (\tilde{x} - X_r), \psi_1\} \triangleq Z(\tilde{x}). \tag{D-20}$$

First, we construct an interval $\tilde{I}$ such that $\mathcal{T}_2(\tilde{I}) > 0$ and $V(x) < Z(x)$ for all $x \in \tilde{I}$. Then, we show that $Z(x)$ can be achieved in a feasible manner by appropriately creating a price-equilibrium pair $(\tilde{p}, \tilde{\mathcal{T}})$ that mimics the flow generated by $\mathcal{T}$ in $(X_r, H]$. The final step of the proof is to use the interval $\tilde{I}$ and the flow $\tilde{\mathcal{T}}$ to show an strict objective improvement.

**Interval construction**. From Eq. (D-20) and the continuity of $V(\cdot)$ we can deduce the existence of an interval $[\tilde{a}, \tilde{b}] \subset (X_r, H]$ such $V(x) < Z(x)$ for all $x \in [\tilde{a}, \tilde{b}]$. Furthermore, the Lipchitz property (see Lemma 1) and Lemma D-4 imply that $V(x) < Z(x)$ for all $x \in [\tilde{a}, \tilde{c}]$ where $\tilde{c}$ is the minimum between $H$ and the value $c$ such that $V(\tilde{a}) + (c - \tilde{a}) = \psi_1$. Also, Proposition D-2 and Lemma B-3 imply that $\mathcal{T}([\tilde{a}, \tilde{c}] \times \mathcal{C}) = \mathcal{T}([\tilde{a}, \tilde{c}] \times [\tilde{a}, \tilde{c}])$. Putting all of this together we conclude that there exists an interval $\tilde{I} = (\tilde{a}, \tilde{c})$ such that $\mathcal{T}_2(\tilde{I}) > 0$ and $V(x) < Z(x)$ for all $x \in \tilde{I}$.

**Flow mimicking.** Define the collection of intervals

$$\mathcal{I} \triangleq \{I \subset (X_r, H] : I = [a, b], a < b, b \in \mathcal{IR}(a), a \text{ is minimal and } b \text{ is maximal}\}.$$

There are two cases: $\mathcal{I} = \emptyset$ and $\mathcal{I} \neq \emptyset$. We only do the latter because its treatment contains the former.

Suppose $\mathcal{I} \neq \emptyset$, then there exists $X_r < a < b$ such that $b \in \mathcal{IR}(a)$, where $a$ and $b$ are minimal and maximal with this property, respectively. We first look at some properties of the equilibrium in each element of $\mathcal{I}$ and then we look at its complement.

Note that from the minimality of $a$ we have that for any $x < a$, $a \notin \mathcal{IR}(x)$. Similarly, for any $x > b$ we have $x \notin \mathcal{IR}(b)$. This, together with Proposition D-2 and Lemma B-3 imply that $[a, b]$ is a flow-separated region, that is, there is no flow coming in nor flow going out of $[a, b]$, $\mathcal{T}([a, b] \times [a, b]^c) = 0$ and $\mathcal{T}([a, b]^c \times [a, b]) = 0$. Observe that our flow separation result in Lemma D-3 implies that in each interval $I \in \mathcal{I}$ we have an equilibrium. Furthermore, from the definition of $\mathcal{IR}$ we must have

$$V(x) = V(a) + (x - a), \quad \forall x \in [a, b].$$

From the previous discussion we infer that the elements in the collection $\mathcal{I}$ are disjoint intervals and, since $V$ is non-decreasing, the collection is at most countable.

For any $a, b$ such that $[a, b] \in \mathcal{I}$ we define

$$t(a) \triangleq V(a) - V(X_r) + X_r, \quad \text{and} \quad t(b) \triangleq V(b) - V(X_r) + X_r.$$

Note that since $V$ is non-decreasing we have $V(a) \geq V(X_r)$ and, therefore, $t(b) > t(a) \geq X_r$. Also, for any

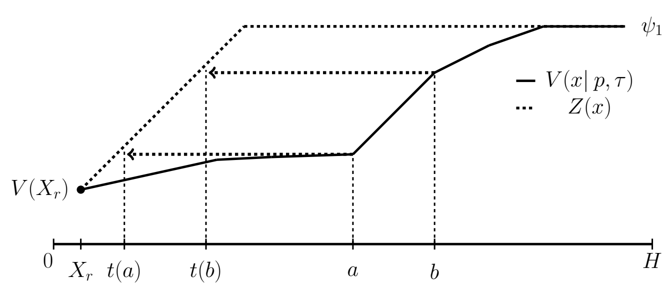


Figure 12: Graphical representation of $t(a)$ and $t(b)$.

such $b$ we have $t(b) < Y_r$. The points $t(a), t(b)$ are the corresponding points to $a, b$ in the interval $[X_r, Y_r]$ (see Figure 12). Furthermore, $t(\cdot)$ is a non-decreasing mapping.

We denote by $\mathcal{I}^c$ the collection of intervals whose elements are the intervals that do not belong to $\mathcal{I}$. Observe that the elements in $\mathcal{I}$ and $\mathcal{I}^c$ alternate in a consecutive manner. That is, if we have an interval

$(c,d) \in \mathcal{I}^c$ then it can only be followed by and interval $[a,b] \in \mathcal{I}$ with $a = d$. In the case that $I = (c,d) \in \mathcal{I}^c$ is not followed by an interval in $\mathcal{I}$ then $I$ equals $(c,H]$. Define the sets

$$\mathcal{K} \triangleq \bigcup_{I\in\mathcal{I}} I \text{ and } \mathcal{K}^c \triangleq \bigcup_{I\in\mathcal{I}^c} I.$$

Note that $(X_r, H] = \mathcal{K} \cup \mathcal{K}^c$ up to a set of $\Gamma$ measure zero. Also, for each interval $I \in \mathcal{I}^c$ we must have that for all measurable sets $A \subset I$, $\mathcal{T}(A \times A) = \Theta(A) = \mathcal{T}_2(A)$; otherwise, by Lemma B-3 we would get a contradiction with the definition of $\mathcal{I}$. In turn, this implies that $\frac{d\mathcal{T}_2}{d\Gamma}(x) = \theta_1$, $\Gamma - a.e.\ x$ in $\mathcal{K}^c$.

We denote by $\mathcal{I}_t$ the collection of intervals $\{[t(a), t(b)]\}_{[a,b]\in\mathcal{I}}$, and $\mathcal{I}_t^c$ is defined in analogous manner. Also, $\mathcal{K}_t$ and $\mathcal{K}_t^c$ are defined similarly to $\mathcal{K}$ and $\mathcal{K}^c$ replacing $\mathcal{I}$ with $\mathcal{I}_t$ and $\mathcal{I}^c$ with $\mathcal{I}_t^c$, respectively.

The idea now is to construct a solution $(\tilde{p}, \tilde{\mathcal{T}})$ in $(X_r, H]$ and then paste it with the old solution $(p, \mathcal{T})$ restricted to $[-H, X_r)$. To construct $(\tilde{p}, \tilde{\mathcal{T}})$ we will make use of the collections $\mathcal{I}_t$ and $\mathcal{I}_t^c$. For each element in these collections we will create a price-equilibrium. For intervals $[t(a), t(b)] \in \mathcal{I}_t$ the idea is that the solution $(\tilde{p}, \tilde{\mathcal{T}})$ has the same equilibrium than $(p, \mathcal{T})$ in $[a, b]$. For the interval in $\mathcal{I}_t^c$ we choose prices such that no drivers will have an incentive to move. Finally, using Lemma D-3 we will paste the equilibria generated in all the intervals.

First, we show how to construct prices and an equilibrium in some $[t(a), t(b)]$. Fix $[a, b] = I \in \mathcal{I}$ and denote the mimicking set $[t(a), t(b)]$ by $I_t$. Choose prices $p^{I_t}(x)$ equal to $p(x + (a - t(a)))$ for all $x \in I_t$. For the flows, we define $\mathcal{T}^{I_t}$ for any $\mathcal{L} \subseteq I_t \times I_t$ by

$$\mathcal{T}^{I_t}(\mathcal{L}) = \mathcal{T}\Big(\mathcal{L} + (a - t(a), a - t(a))\Big),$$

that is, $\mathcal{T}^{I_t}$ mimics $\mathcal{T}$ in $I \times I$. It can be shown that (see Property 1 at the end of this proof) $(p^{I_t}, \mathcal{T}^{I_t})$ forms a price-equilibrium pair in $I_t$ such that $\mathcal{T}^{I_t} \in \mathcal{F}_{I_t}(\Theta|_{I_t})$. Also, $V(x|\, p^{I_t}, \mathcal{T}^{I_t})$ equals $V(x + a - t(a)|\, p, \mathcal{T})$ for all $x \in I_t$, and

$$\frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(x) = \frac{d\mathcal{T}_2}{d\Gamma}(x + a - t(a)), \quad \Gamma - a.e.\ x \text{ in } I_t. \tag{D-21}$$

Furthermore, because $I \in \mathcal{I}$ we have

$$V(x|\, p^{I_t}, \mathcal{T}^{I_t}) = V(x + a - t(a)|\, p, \mathcal{T}) = V(a) + (x - t(a)) = V(X_r) + (x - X_r) = Z(x), \quad \forall x \in I_t,$$

that is, for all intervals $I_t$ the associated solution $(p^{I_t}, \mathcal{T}^{I_t})$ achieves the upper bound $Z(x)$.

Second, we show how to set the prices and construct an equilibrium everywhere else. Consider any two consecutive sets in $\mathcal{I}$, $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. The corresponding mimicking sets are $[t(a_1), t(b_1)]$ and $[t(a_2), t(b_2)]$. We need to set prices and define the flow in the interval $J_t = (t(b_1), t(a_2))$. We choose the prices $p^{J_t}$ to be such that

$$U\Big(x, p^{J_t}(x), \theta_1\Big) = Z(x), \quad \forall x \in J_t.$$

Since $Z(x) \le \psi_1$ these prices are guaranteed to exist. We define the measure $\mathcal{T}^{J_t}$ for any measurable set $\mathcal{L} \subseteq J_t \times J_t$ by

$$\mathcal{T}^{J_t}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D})).$$

This measure has $d\mathcal{T}_2^{J_t}/d\Gamma = \theta_1$, $\Gamma - a.e$ in $J_t$. It can be shown that (see Property 2 at the end of this proof) $(p^{J_t}, \mathcal{T}^{J_t})$ forms a price-equilibrium pair in $J_t$ such that $\mathcal{T}^{J_t} \in \mathcal{F}_{J_t}(\Theta|_{J_t})$ and $V(x|\, p^{J_t}, \mathcal{T}^{J_t})$ equals $Z(x)$ for all $x \in J_t$.

Third, the solutions $\{(p^{I_t}, \mathcal{T}^{I_t})\}_{I_t\in\mathcal{I}_t}$ and $\{(p^{J_t}, \mathcal{T}^{J_t})\}_{J_t\in\mathcal{I}_t^c}$ cover the whole interval $(X_r, H]$. Moreover they are defined in disjoint interval, and are such that the respective $V(\cdot)$ functions coincide at the boundaries of the interval (these functions coincide with $Z(\cdot)$). Thus, we can apply Lemma D-3 to paste all these solutions and obtain a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ in $(X_r, H]$. As mentioned before we can use the same lemma to paste this solution with the old solution restricted to $[-H, X_r]$. This would yield a solution in the entire city.

**Objective improvement**. Consider the revenue under $(p, \mathcal{T})$ in $(X_r, H]$, it easy to observe that

$$\begin{aligned}\mathbf{Rev}_{(X_r,H]}(p,\mathcal{T}) &= \int_{(X_r,H]} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) \\ &= \int_{\mathcal{K}} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) + \int_{\mathcal{K}^c} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) \\ &= \underbrace{\sum_{I\in\mathcal{I}} \int_I V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x)}_{=(a)} + \underbrace{\int_{\mathcal{K}^c} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x)}_{=(b)}.\end{aligned}$$

Let us develop the integral of the term $(a)$. Let $I$ be equal to $[a, b]$ and $I_t$ equal to $[t(a), t(b)]$ then

$$\begin{aligned}\int_{[a,b]} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) &= \int_{[t(a),t(b)]} V(x+a-t(a)|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x+a-t(a))\, d\Gamma(x) \\ &= \int_{[t(a),t(b)]} V(x|p^{I_t},\mathcal{T}^{I_t}) \cdot s^{\mathcal{T}^{I_t}}(x)\, d\Gamma(x),\end{aligned}$$

where in the first line we use the invariance under translation of $\Gamma$, and in the second line we use that $V(x|\, p^{I_t}, \mathcal{T}^{I_t})$ equals $V(x + a - t(a)|\, p, \mathcal{T})$ for all $x \in I_t$ and Eq. (D-21). Thus,

$$\begin{aligned}\mathbf{Rev}_{(X_r,H]}(p,\mathcal{T}) &= \sum_{I_t\in\mathcal{I}_t} \int_{I_t} V(x|p^{I_t},\mathcal{T}^{I_t}) \cdot s^{\mathcal{T}^{I_t}}(x)\, d\Gamma(x) + (b) \\ &= \int_{\mathcal{K}_t} Z(x) \cdot s^{\tilde{\mathcal{T}}}(x)\, d\Gamma(x) + (b).\end{aligned}$$

Thus, to conclude the proof we only need to show that

$$(b) = \int_{\mathcal{K}^c} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) < \int_{\mathcal{K}_t^c} Z(x) \cdot s^{\tilde{\mathcal{T}}}(x)\, d\Gamma(x). \tag{D-22}$$

Define the following functions

$$V_e(x) = \begin{cases} V(x|p,\mathcal{T}) & \text{if } x \in \mathcal{K}^c, \\ V(a|p,\mathcal{T}) & \text{if } x \in [a,b], \text{ some } [a,b] \in \mathcal{I}, \end{cases}$$

$$Z_e(x) = \begin{cases} Z(x) & \text{if } x \in \mathcal{K}_t^c, \\ Z(t(a)) & \text{if } x \in [t(a),t(b)], \text{ some } [t(a),t(b)] \in \mathcal{I}_t. \end{cases}$$

We verify that $V_e(x) \leq Z_e(x)$ for all $x \in (X_r, H]$, and the we use this inequality to prove the objective improvement. Let $x \in \mathcal{K}^c$ then there exists an interval $(c, d) \in \mathcal{I}^c$ with $x \in (c, d)$. If $x \in \mathcal{K}_t^c$ then the upper bound is trivial. If $x \notin \mathcal{K}_t^c$ then $x \in [t(a), t(b)]$ for some $[t(a), t(b)] \in \mathcal{I}_t$. We must have that $a \geq d$; otherwise, since $(c, d) \in \mathcal{I}$, it must be the case that $b \leq c$. In turn, this implies that $[t(a), t(b)] \cap (c, d) = \emptyset$ which contradiction our current assumption. Therefore,

$$V_e(x) = V(x|p,\mathcal{T}) \leq V(d|p,\mathcal{T}) \leq V(a|p,\mathcal{T}) = Z(t(a)) = Z_e(x).$$

Let $x \in [a, b]$ for some $[a, b] \in \mathcal{I}$. If $x \in \mathcal{K}_t^c$, $t(b) < x$ otherwise we would have that $t(a) \leq a \leq a \leq t(b)$, that is, $x \in [t(a), t(b)] \in \mathcal{I}_t$. Under our current assumption this is not possible. Then,

$$V_e(x) = V(a|p,\mathcal{T}) < V(b|p,\mathcal{T}) = Z(t(b)) \leq Z(x) = Z_e(x), \tag{D-23}$$

that is, when $x \in \mathcal{K} \cap \mathcal{K}_t^c$ we have $V_e(x) < Z_e(x)$. If $x \in [t(\hat{a}), t(\hat{b})]$ for some $[t(\hat{a}), t(\hat{b})] \in \mathcal{I}_t$. Using similar arguments as before we can show that $\hat{a} \geq a$ and, therefore,

$$V_e(x) = V(a|p,\mathcal{T}) = Z(t(a)) \leq Z(t(\hat{a})) = Z_e(x).$$

Now, recall that in the **Interval construction** part of the proof we defined an interval $\tilde{I} = [\tilde{a}, \tilde{c}]$ in which the function $V(\cdot|p, \mathcal{T})$ is uniformly strictly bounded by $Z(\cdot)$. Now we relate this interval to $\mathcal{K}_t^c$ by showing that there exists $\epsilon > 0$ such that $(\tilde{c} - \epsilon, \tilde{c}) \subseteq I_t^c$ with $I_t^c \in \mathcal{I}_t^c$. The idea is to use that $(\tilde{c} - \epsilon, \tilde{c}) \subset \tilde{I}$ and $(\tilde{c} - \epsilon, \tilde{c}) \subset \mathcal{K}_t^c$ together with Eq. (D-23) to show an strict objective improvement.

Note that if $\tilde{c} = H$ then

$$\begin{aligned}
\sup_{[t(a),t(b)] \in \mathcal{I}_t} t(b) &\overset{(1)}{\leq} t(\tilde{c}) \\
&= V(\tilde{c}) - V(X_r) + X_r \\
&= (V(\tilde{c}) - V(\tilde{a})) + (V(\tilde{a}) - V(X_r)) + X_r \\
&\overset{(2)}{<} (V(\tilde{c}) - V(\tilde{a})) + (Z(\tilde{a}) - Z(X_r)) + X_r \\
&\overset{(3)}{\leq} (\tilde{c} - \tilde{a}) + (\tilde{a} - X_r) + X_r \\
&= \tilde{c},
\end{aligned}$$

where (1) comes from the fact that $t(\cdot)$ is non-decreasing and $\tilde{c} = H$, line (2) follows from the $V(\tilde{a}) < Z(\tilde{a})$ and $V(X_r) = Z(X_r)$. Inequality, (3) holds because both $V$ and $Z$ are $1-$Lipschitz functions. In the case that $\tilde{c} < H$ we have $V(\tilde{a}) + (\tilde{c} - \tilde{a}) = \psi_1$. Also, we always have that $t(b) \leq Y_r$ where $Y_r$ is such that $V(X_r) + (Y_r - X_r) = \psi_1$. From this we deduce that $Y_r < \tilde{c}$ and, therefore, we have that $\sup_{[t(a),t(b)] \in \mathcal{I}_t} t(b) < \tilde{c}$. Either way we can always find $\epsilon \in (0, \tilde{c} - \tilde{a})$ such that the interval $(\tilde{c} - \epsilon, \tilde{c})$ does not intersect with any interval in $\mathcal{I}_t$. Hence, since $\mathcal{I}_t^c$ are all the intervals that do not belong to $\mathcal{I}_t$ we must have that $(\tilde{c} - \epsilon, \tilde{c}) \subseteq I_t^c$ for some $I_t^c \in \mathcal{I}_t^c$.

Because $(\tilde{c} - \epsilon, \tilde{c})$ is a subset of both $\mathcal{K}_t^c$ and $(\tilde{a}, \tilde{c})$, for $x \in (\tilde{c} - \epsilon, \tilde{c}) \cap \mathcal{K}^c$ we have $V_e(x) < Z_e(x)$. Also, for $x \in (\tilde{c} - \epsilon, \tilde{c}) \cap \mathcal{K}$ from equation Eq. (D-23) we have $V_e(x) < Z_e(x)$. That is, $V_e(x) < Z_e(x)$ for all $x \in (\tilde{c} - \epsilon, \tilde{c})$ and, therefore,

$$\begin{aligned}
\int_{\mathcal{K}^c} V(x|p, \mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) &= \int_{(X_r,H]} V_e(x|p, \mathcal{T}) \cdot \theta_1 d\Gamma(x) - \sum_{[a,b] \in \mathcal{I}} \int_{[a,b]} V(a|p, \mathcal{T}) \cdot \theta_1 d\Gamma(x) \\
&< \int_{(X_r,H]} Z_e(x) \cdot \theta_1 d\Gamma(x) - \sum_{[a,b] \in \mathcal{I}} \int_{[a,b]} V(a|p, \mathcal{T}) \cdot \theta_1 d\Gamma(x) \\
&= \int_{(X_r,H]} Z_e(x) \cdot \theta_1 d\Gamma(x) - \sum_{[a,b] \in \mathcal{I}} V(a|p, \mathcal{T}) \Theta([a,b]) \\
&= \int_{(X_r,H]} Z_e(x) \cdot \theta_1 d\Gamma(x) - \sum_{[t(a),t(b)] \in \mathcal{I}_t} Z(t(a)) \Theta([t(a), t(b)]) \\
&= \int_{\mathcal{K}_t^c} Z(x) \cdot \theta_1 d\Gamma(x),
\end{aligned}$$

which proves Eq. (D-22). To conclude, we provide a proof for both Property 1 and Property 2.

**Property 1.** $(p^{I_t}, \mathcal{T}^{I_t})$ forms a price-equilibrium pair in $I_t$ such that $\mathcal{T}^{I_t} \in \mathcal{F}_{I_t}(\Theta|_{I_t})$. Also, $V(x|p^{I_t}, \mathcal{T}^{I_t})$ equals $V(x + a - t(a)|\, p, \mathcal{T})$ for all $x \in I_t$, and

$$\frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(x) = \frac{d\mathcal{T}_2}{d\Gamma}(x + a - t(a)), \quad \Gamma - a.e.\ x \text{ in } I_t.$$

**Proof of Property 1.** We first show that $\mathcal{T}^{I_t} \in \mathcal{F}_{I_t}(\Theta|_{I_t})$. It is clear that $\mathcal{T}^{I_t} \in \mathcal{M}(I_t \times I_t)$, and that

$\mathcal{T}_2^{I_t} \ll \Gamma$. To see why $\mathcal{T}_1^{I_t}$ coincides with $\theta_{I_t}$ consider a set $K \subset I_t$ then $\mathcal{T}_1^{I_t}(K)$ equals

$$\begin{aligned}\mathcal{T}_1^{I_t}(K \times I_t) = \mathcal{T}((K+a-t(a)) \times (I_t+a-t(a))) &= \mathcal{T}((K+a-t(a)) \times [a,b]) \\ &= \mathcal{T}((K+a-t(a)) \times \mathcal{C}) \\ &= \Theta(K+a-t(a)) \\ &= \Theta(K),\end{aligned}$$

where the fourth line holds because the set $K+a-t(a)$ is contained in $[a,b]$, and we know there is no flow leaving this interval. Next, using a similar argument we show the property for $d\mathcal{T}_2^{I_t}/d\Gamma$, let $K$ be a measurable subset of $I_t$ then

$$\begin{aligned}\int_K \frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(x)\, d\Gamma(x) &= \mathcal{T}^{I_t}(I_t \times K) \\ &= \mathcal{T}([a,b] \times (K+a-t(a))) \\ &= \int_{(K+a-t(a))} \frac{d\mathcal{T}_2}{d\Gamma}(x)\, d\Gamma(x) \\ &= \int_K \frac{d\mathcal{T}_2}{d\Gamma}(x+a-t(a))\, d\Gamma(x).\end{aligned}$$

Using this last property and the prices definition is easy to see that

$$\begin{aligned}V(x|\, p^{I_t}, \mathcal{T}^{I_t}) &= \inf\{u \in \mathbb{R} : \Gamma(y \in I_t : U(y, p^{I_t}(y), \frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(y)) - |y-x| > u) = 0\} \\ &= \inf\{u \in \mathbb{R} : \Gamma(y \in I_t : U(y, p(y+a-t(a)), \frac{d\mathcal{T}_2}{d\Gamma}(y+a-t(a))) - |y-x| > u) = 0\} \\ &= \inf\{u \in \mathbb{R} : \Gamma(y \in I : U(y, p(y), \frac{d\mathcal{T}_2}{d\Gamma}(y)) - |y-(x+a-t(a))| > u) = 0\} \\ &= V_I(x+a-t(a)|\, p, \mathcal{T}),\end{aligned}$$

but from out flow separation result (see Lemma D-3) we have that $V_I(x+a-t(a)|p,\mathcal{T}) = V(x+a-t(a)|p,\mathcal{T})$. Using this same approach, the definition of $\mathcal{T}^{I_t}$ and the fact that $\mathcal{T}$ is an equilibrium in $[a,b]$ it is easy to verify the equilibrium condition.

**Property 2.** The pair $(p^{J_t}, \mathcal{T}^{J_t})$ forms a price-equilibrium pair in $J_t$ such that $\mathcal{T}^{J_t} \in \mathcal{F}_{J_t}(\Theta|_{J_t})$ and $V(x|\, p^{J_t}, \mathcal{T}^{J_t})$ equals $Z(x)$ for all $x \in J_t$.

**Proof of Property 2.** From the definition of $\mathcal{T}^{J_t}$ it is clear that $\mathcal{T}^{J_t} \in \mathcal{F}_{J_t}(\Theta|_{J_t})$. Also, $d\mathcal{T}_2^{J_t}/d\Gamma = \theta_1$, $\Gamma - a.e$ in $J_t$. To see why $V(x|\, p^{J_t}, \mathcal{T}^{J_t})$ equals $Z(x)$ for all $x \in J_t$, note that for fixed $x \in J_t$

$$\Gamma(y \in J_t : U(y, p^{J_t}(y), \frac{d\mathcal{T}_2^{J_t}}{d\Gamma}(y)) - |y-x| > Z(x)) = \Gamma(y \in J_t : Z(y) - |x-y| > Z(x)) = 0,$$

where in the first equality we use the definition of $p^{J(t)}$ together with $d\mathcal{T}_2^{J_t}/d\Gamma = \theta_1$, $\Gamma - a.e$ in $J_t$. In the second equality we use the Lipschitz property of the function $Z(\cdot)$. That is, $Z(x) \geq V(x|\, p^{J_t}, \mathcal{T}^{J_t})$. This upper bound ($\Gamma-$a.e) is tight. Let $\epsilon > 0$ then

$$\begin{aligned}0 &< \Gamma(B(x,\epsilon/2) \cap J_t) \\ &\leq \Gamma(y \in B(x,\epsilon/2) \cap J_t) : \epsilon > |x-y| + (Z(x) - Z(y))) \\ &= \Gamma(y \in B(x,\epsilon/2) \cap J_t) : Z(y) - |y-x| > Z(x) - \epsilon) \\ &= \Gamma(y \in B(x,\epsilon/2) \cap J_t) : U(y, p^{J_t}(y), \frac{d\mathcal{T}_2^{J_t}}{d\Gamma}(y)) - |y-x| > Z(x) - \epsilon),\end{aligned}$$

thus $Z(x)$ is the smallest upper bound ($\Gamma-$a.e) and we have $Z(x) = V(x|\, p^{J_t}, \mathcal{T}^{J_t})$. It is not hard to verify that the equilibrium condition reduces to

$$\mathcal{T}^{J_t}((x,y) \in J_t \times J_t : Z(y) - |y-x| = Z(x)) = \Theta(J_t),$$

and by the definition of $\mathcal{T}^{J_t}$ this is immediately satisfied. $\square$

***Proof of Theorem 2.*** The result follows directly from Proposition D-3, and the fact that $[X_l, X_r]$ is an attraction region where $V(\cdot)$ is pinned down. $\square$

***Proof of Theorem 3.*** We separate the proof in several steps. First, we argue that there are at most three attraction regions in the any optimal solution. Then we show that any optimal solution does not have drivers moving to the interval $[W_r, X_r]$ and $[X_l, W_l]$; otherwise, the platform can incentivize the movement of a positive fraction of drivers outside of the center and make strictly larger revenue. After this we put into practice Theorem 1 which prescribes what are the optimal prices and post-relocation supply in each attraction region. In the final main step of the proof we argue that the optimal solution has to be symmetric. We present the proof of two properties that we will use during the main arguments, Property 1 and Property 2, after the main proof.

**Attraction regions identification:** Lemma 2 establishes that at an optimal solution the attraction region of the origin is well defined with $X_l < 0 < X_r$. So Our first attraction region is the interval $[X_l, X_r]$.

The second and third attraction regions correspond to the intervals $[Y_l, X_l]$ and $[X_r, Y_r]$ with $Y_l$ and $Y_r$ being sinks. WLOG consider only the right interval, if $Y_r = X_r$ we do not identify any attraction region to the right of $X_r$. Assume that $X_r < Y_r$, we will show that $A(Y_r) = [X_r, Y_r]$ and $Y_r \notin A(z)$ for any $z \neq Y_r$. In order to show this we first show that $Y_r \in \mathcal{IR}(X_r|\, p, \mathcal{T})$. From Theorem 2 we know that $V(x)$ equals $V(X_r) + (x - X_r)$ for all $x \in [X_r, Y_r]$. In particular, $V(X_r) + (Y_r - X_r)$. In other words, $Y_r \in \mathcal{IR}(X_r|\, p, \mathcal{T})$. Now, $Y_r$ cannot belong to any other attraction region; otherwise, the value function would not be as in Theorem 2. Therefore, $Y_r$ is a sink and $[X_r, Y_r] \subseteq A(Y_r)$. If there existed $x \in A(Y_r)$ but $x \notin [X_r, Y_r]$, the value function would not be as in Theorem 2. In conclusion, $A(Y_r) = [X_r, Y_r]$ and $Y_r \notin A(z)$ for any $z \neq Y_r$.

**No supply in $[W_r, X_r]$:** Next we argue that at an optimal solution $(p, \mathcal{T})$ we must have that $\mathcal{T}_2([W_r, X_r]) = 0$, the same is true for the left side. Suppose by contradiction that $\mathcal{T}_2([W_r, X_r]) > 0$ and denote this amount of supply by $q_r$, we construct a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ that yields an strict objective improvement. Observe that,

$$0 < q_r = \mathcal{T}(\mathcal{C} \times [W_r, X_r]) = \mathcal{T}([W_r, X_r] \times [W_r, X_r]) \leq \Theta([W_r, X_r]) = \theta_1 \cdot (X_r - W_r).$$

That is, from the total amount of initial supply in $[W_r, X_r]$ we have that $q_r$ units stay within $[W_r, X_r]$ and a total of $\theta_1 \cdot (X_r - W_r) - q_r$ units travel to $[0, W_r]$. Note that for this $q_r$ units of mass their $V$ is bounded by $\psi_1$ and, therefore, what the platform can make from them is strictly bounded by $\psi_1 \cdot q_r$ (times a scaling factor). Let $\tilde{X}_r \in [W_r, X_r)$ be such that $q_r = \theta_1 \cdot (X_r - \tilde{X}_r)$. In the new solution, we will modify the attraction region $[X_l, X_r]$ to be $[X_l, \tilde{X}_r]$. We will maintain the same prices and post-relocation supply in the origin's attraction region. However, to the right side of $\tilde{X}_r$ we will set new prices that will be consistent with a new value function and flows that upper bound those of the old solution, see Figure 13.

We begin our construction of $(\tilde{p}, \tilde{\mathcal{T}})$ with the interval $I_r^1 = [\tilde{X}_r, \tilde{Y}_r]$, where $\tilde{Y}_r$ is such that $\psi_1 = V(\tilde{X}_r) + (\tilde{Y}_r - \tilde{X}_r)$. Let $h \triangleq 2 \cdot (X_r - \tilde{X}_r)$, we define flows for any $\mathcal{L} \subseteq I_r^1 \times I_r^1$ by

$$\mathcal{T}^{I_r^1}(\mathcal{L}) = \mathcal{T}\Big(\mathcal{L} + (h, h)\Big).$$

Consider the set $K \triangleq \{x \in I_r^1 : \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(x) \leq \frac{d\mathcal{T}_2}{d\Gamma}(x + h)\}$. We set prices to be such that

$$U\Big(x, p^{I_r^1}(x), \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(x)\Big) = U\Big(x + h, p(x + h), s^{\mathcal{T}}(x + h)\Big), \quad \forall x \in K, \tag{D-24}$$

and zero otherwise. We prove, in Property 1 (see end of present proof), that $(p^{I_r^1}, \mathcal{T}^{I_r^1})$ is a price-equilibrium pair in $I_r^1$ such that $V(x|\, p^{I_r^1}, \mathcal{T}^{I_r^1}) = V(\tilde{X}_r) + (x - \tilde{X}_r)$ and $\Gamma(K^c) = 0$.

In the interval $I_r^2 = (\tilde{Y}_r, H]$ we can achieve the optimal solution when there is no demand shock. As in the optimal solution in the pre-demand shock environment (see Proposition D-1) we set prices equal to $\rho_1$ and the flows are such that $d\mathcal{T}^{I_r^2}/d\Gamma$ equals $\theta_1$, $\Gamma - a.e$ in $I_r^2$.

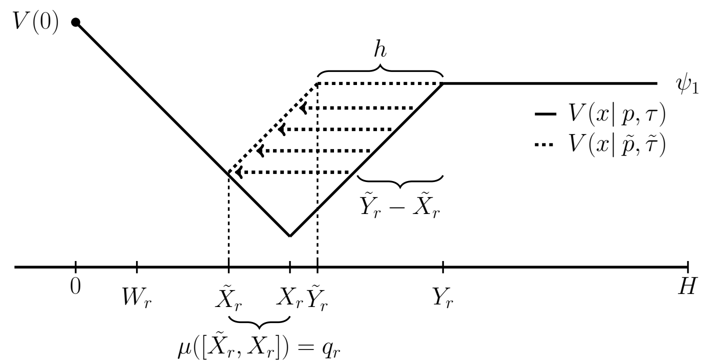


Figure 13: **No supply in** $[W_r, X_r]$. The new solution moves the right end of the attraction region from $X_r$ to $\tilde{X}_r$, so now a mass $q_r$ of drivers can travel towards the periphery. From this mass the platform now makes $\psi_1$ instead of $V(x)$ with $V(x) < \psi_1$.

The interval $I_r^0 = [X_l, \tilde{X}_r]$ is more involved. Observe that all the initial flow to the right of the origin that we have to allocate in $[0, \tilde{X}_r]$ equals $\theta_1 \cdot X_r - q_r$. This is exactly the same amount of drivers in $[0, X_r]$ that travels to $[0, W_r]$ according to $\mathcal{T}$. Our new solution will generate the same post-relocation supply than $\mathcal{T}$ in $[0, W_r]$ but this time only using drivers from $[0, \tilde{X}_r]$.

We use the same prices, that is $p^{I_r^0}(x) = p(x)$ for all $x \in [X_l, \tilde{X}_r]$. For the flows we define them through two measures: the flow that goes from $[X_l, 0]$ to $[X_l, 0]$ and the flow that goes from $[0, \tilde{X}_r]$ to $[0, \tilde{X}_r]$. For the first flow we use $\mathcal{T}^\ell = \mathcal{T}|_{[X_l,0]}$, for the second measure $\mathcal{T}^r$ we will use a monotone coupling as in the proof of Theorem 1 (see e.g, Santambrogio (2015) for details). Define the initial supply to the right measure $\Theta^r$ to be equal to $\Theta|_{[0,\tilde{X}_r]}$, and the final supply $S^r$ to be

$$S^r(\mathcal{B}) \triangleq \mathcal{T}([0, X_r] \times \mathcal{B}), \quad \text{for any measurable set } \mathcal{B} \subseteq [0, \tilde{X}_r].$$

Note that $S^r([0, W_r])$ equals $\Theta^r([0, \tilde{X}_r])$. Given this we define $\mathcal{T}^r$ by

$$\mathcal{T}^r(\mathcal{L}) \triangleq (F_{\Theta^r}^{[-1]}, F_{S^r}^{[-1]})_{\#} m(\mathcal{L}), \quad \text{for any measurable set } \mathcal{L} \subseteq [0, \tilde{X}_r] \times [0, \tilde{X}_r],$$

where $\#$ correspond to the push-forward operator. For any measure $\nu$ defined in $[0, \tilde{X}_r]$ we define its cumulative function and pseudo-inverse by

$$F_\nu(y) \triangleq \nu([0, y]), \quad \forall\, y \geq 0 \quad \text{and} \quad F_\nu^{[-1]}(t) \triangleq \inf\{y \geq 0 : F_\nu(y) \geq t\}, \quad \forall t \in [0, \Theta^r([0, \tilde{X}_r])].$$

Effectively, $\mathcal{T}^r$ transports the initial mass in $[0, \tilde{X}_r]$ to the final supply distribution (considering only drivers that come from the right) in $[0, W_r]$ as prescribed by $\mathcal{T}$. The final flow measure $\mathcal{T}^{I_r^0}$ correspond to $\mathcal{T}^\ell + \mathcal{T}^r|_{[0,\tilde{X}_r]}$. In Property 2 below we show that $(p^{I_r^0}, \mathcal{T}^{I_r^0})$ is a price-equilibrium pair such that $\mathbf{Rev}_{[X_l,W_r]}(p^{I_r^0}, \mathcal{T}^{I_r^0}) = \mathbf{Rev}_{[X_l,W_r]}(p, \mathcal{T})$.

The solution $(\tilde{p}, \tilde{\mathcal{T}})$ is constructed by pasting (see Lemma D-3) the old solution is $[-H, X_l)$ with the new solution in $I^0, I_r^1$ and $I_r^2$. The pasting is possible because the equilibrium utility function coincide in the boundaries of these intervals. This new solution preserves the platform's revenue in $[-H, W_r] \cup [Y_r, H]$ but it strictly improves it in $[W_r, Y_r]$. Indeed, note that

$$q_r = \int_{[X_r,Y_r]} s^{\mathcal{T}}(x)\, dx - \int_{[\tilde{X}_r,\tilde{Y}_r]} s^{\tilde{\mathcal{T}}}(x)\, dx = \int_{[\tilde{X}_r,\tilde{Y}_r]} \underbrace{(s^{\mathcal{T}}(x+h) - s^{\tilde{\mathcal{T}}}(x))}_{\geq 0 \ \Gamma-a.e}\, dx + \int_{[X_r, X_r+(X_r-\tilde{X}_r)]} s^{\mathcal{T}}(x)\, dx, \tag{D-25}$$

thus

$$
\begin{aligned}
\frac{1}{\gamma}\cdot \mathbf{Rev}_{[W_r,Y_r]}(\tilde p,\tilde{\mathcal{T}}) &= \int_{[W_r,\tilde X_r]} V(x|\tilde p,\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx + \int_{[\tilde X_r,Y_r]} V(x|\tilde p,\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx\\
&\overset{(a)}{=} \int_{[\tilde X_r,Y_r]} V(x|\tilde p,\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx\\
&\overset{(b)}{=} \int_{[\tilde X_r,\tilde Y_r]} V(x|\tilde p,\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx + \psi_1\cdot 2\cdot q_r\\
&\overset{(c)}{>} \int_{[\tilde X_r,\tilde Y_r]} V(x|\tilde p,\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx + \psi_1\cdot q_r + \int_{[W_r,X_r]} V(x)\cdot s^{\mathcal{T}}(x)\,dx\\
&\overset{(d)}{\geq} \int_{[\tilde X_r,\tilde Y_r]} V(x|\tilde p,\tilde{\mathcal{T}})\cdot s^{\mathcal{T}}(x+h)\,dx + \int_{[W_r,X_r+(X_r-\tilde X_r)]} V(x)\cdot s^{\mathcal{T}}(x)\,dx\\
&\overset{(e)}{=} \int_{[W_r,Y_r]} V(x)\cdot s^{\mathcal{T}}(x)\,dx = \frac{1}{\gamma}\cdot \mathbf{Rev}_{[W_r,Y_r]}(p,\mathcal{T}),
\end{aligned}
$$

where $(a)$ follows because $\tilde{\mathcal{T}}$ does not put mass in $[W_r,\tilde X_r]$, $(b)$ because $Y_r-\tilde Y_r$ equals $2\cdot(X_r-\tilde X_r)$. Using the fact that $\mathcal{T}_2([W_r,X_r])=q_r$ we obtain $(c)$, while $(d)$ follows from Eq. (D-25) and $(e)$ from $V(x|\,\tilde p,\tilde{\mathcal{T}})$ being equal to $V(x+h)$ for all $x\in[\tilde X_r,\tilde Y_r]$.

In conclusion, any optimal solution must satisfy both $\mathcal{T}_2([W_r,X_r])=0$ and $\mathcal{T}_2([X_l,W_l])=0$.

**Using Theorem 1:** All the conditions in Theorem 1 are met. So, for any of the three attraction regions if $(p,\mathcal{T})$ is not already as in the statement of the theorem we can find at least a weak improvement. That is, we can restrict to solution as in Theorem 1. Therefore, the prices are as stated in the present theorem, and there exists $\beta_c^l\in[W_l,0]$, $\beta_c^r\in[0,W_r]$, $\beta_p^l\in[Y_l,X_l]$ and $\beta_p^r\in[X_r,Y_r]$ such that

$$
s^{\mathcal{T}}(x)=\begin{cases}0 & \text{if } x\in(\beta_c^r,\beta_p^r)\cup(\beta_p^l,\beta_c^l),\\ \psi_x^{-1}(V(x|\,p,\mathcal{T})) & \text{otherwise,}\end{cases}
$$

with

$$
\int_{\beta_c^l}^{\beta_c^r}\psi_x^{-1}(V(x|\,p,\mathcal{T}))\,d\Gamma(x)=\theta_1\cdot(X_r-X_l)
$$

and

$$
\int_{\beta_p^r}^{Y_r}\psi_x^{-1}(V(x|\,p,\mathcal{T}))\,d\Gamma(x)=\theta_1\cdot(Y_r-X_r),\qquad \int_{Y_l}^{\beta_p^l}\psi_x^{-1}(V(x|\,p,\mathcal{T}))\,d\Gamma(x)=\theta_1\cdot(X_l-Y_l).
$$

Note that the fact that $\beta_c^l\in[W_l,0]$ and $\beta_c^r\in[0,W_r]$, does not come directly from Theorem 1 but rather is a consequence of that any optimal solution must satisfy both $\mathcal{T}_2([W_r,X_r])=0$ and $\mathcal{T}_2([X_l,W_l])=0$. Also, observe that Theorem 1 only gives us a solution in each attraction but above we have stated the solution for the entire city. The only missing interval are $[-H,Y_l]$ and $[Y_r,H]$. In this intervals, as in the pre-shock environment, the solution set prices equal to $\rho_1$ and the supply at every location is $\theta_1$, in turn, the $V$ equals $\psi_1$ in this region. This gives a complete solution to the platform's problem up to three values: $V(0),X_l,X_r$.

**Symmetry:** In the last main step of the proof we argue that the solution is symmetric. After proving this, the solution will take the exact form in the statement of the present theorem.

Note that given a value for $V(0)$ and an central attraction region characterize by $X_l$ and $X_r$ we can characterize the optimal solution as we did in **Using Theorem 1**. So fix these three values and the optimal solution associated to them. We now proceed to construct a new solution that yields a strict objective improvement when the solution is not symmetric. WLOG assume that $|X_l|>X_r$ and let $\delta=(|X_l|-X_r)/2$. Consider the solution $(\tilde p,\tilde{\mathcal{T}})$ associated to the values

$$
\tilde V(0)=V(0),\quad \tilde X_l=X_l+\delta,\quad \tilde X_r=X_r+\delta.
$$

Note that with this values we have $|\tilde{X}_l|, \tilde{W}_r \geq W_r$ and $\tilde{Y}_i = Y_i + 2 \cdot \delta$ for $i \in \{l, r\}$. We next show that this new solution yields a weak objective improvement in the center, and a strict objective improvement in the periphery.

Note that given $\tilde{V}(0), \tilde{X}_l$ and $\tilde{X}_r$ Theorem 2 characterizes $V(\cdot|\tilde{p}, \tilde{\mathcal{T}})$. It has the same shape than $V(\cdot|p, \mathcal{T})$ except that now the dip in $[\tilde{Y}_l, W_l]$ is smaller, while the dip in $[W_r, Y_r]$ is larger. See Figure 14 for a graphical representation. Consider first the solution in the center, $[\tilde{X}_l, \tilde{X}_r]$. This interval contains the same amount

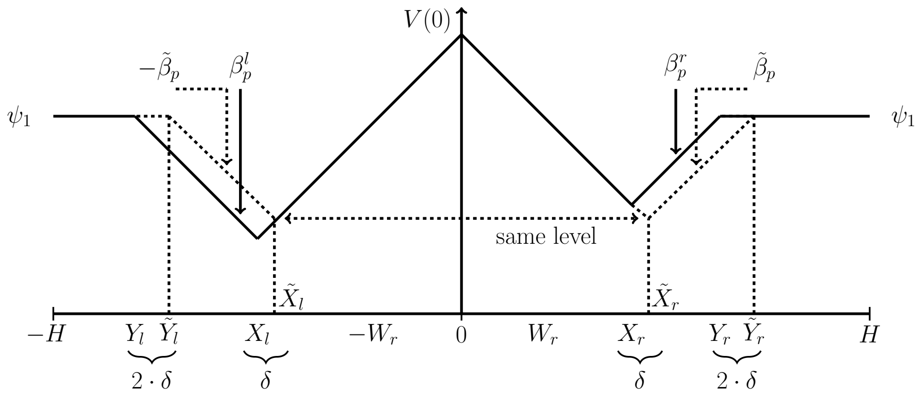


Figure 14: **Symmetry** argument.

of drivers that the old attraction region. The difference is that it lost a mass of $\theta_1 \cdot \delta$ drivers to the left and gain the same mass to the right. As in the discussion that follows Theorem 1 the optimal solution in $[\tilde{X}_l, \tilde{X}_r]$ can be obtained using a knapsack argument. This new attraction region is symmetric, $|\tilde{X}_l| = \tilde{X}_r$, with equal mass of drivers at both sides of the origin. Therefore the knapsack solution must be symmetric, with $\tilde{\beta}_c \in [0, W_r]$ such that

$$s^{\tilde{\mathcal{T}}}(x) = \psi_x^{-1}(V(x|\,\tilde{p}, \tilde{\mathcal{T}})) = \psi_x^{-1}(V(x|\,p, \mathcal{T})), \quad \forall x \in [-\tilde{\beta}_c, \tilde{\beta}_c],$$

and equals zero otherwise, and

$$\int_{-\tilde{\beta}_c}^{\tilde{\beta}_c} \psi_x^{-1}(V(x|\,\tilde{p}, \tilde{\mathcal{T}})\, d\Gamma(x) = \theta_1 \cdot (\tilde{X}_r - \tilde{X}_l) = \theta_1 \cdot (X_r - X_l).$$

Note that $\tilde{\beta}_c \in [0, W_r]$ is a consequence of the having $\beta_c^l \in [W_l, 0]$ and $\beta_c^r \in [0, W_r]$ in the old solution. Theorem 1 prescribes how to formally implement this solution through prices and flows. We omit the details of how to construct the flows, but we note that the optimal prices are given $\tilde{p}(x) = \rho_x^{loc}(s^{\tilde{\mathcal{T}}}(x))$. In the case that $\tilde{\beta} = 0$ then $s^{\tilde{\mathcal{T}}}(0) = \theta_1 \cdot (X_r - X_l)$ and $\tilde{p}(0)$ is such that $U(0, p(0), s^{\tilde{\mathcal{T}}}(0)) = V(0)$. The platform's revenue in the new center is then

$$\frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l, \tilde{X}_r]}(\tilde{p}, \tilde{\mathcal{T}}) = \int_{\tilde{X}_l}^{\tilde{X}_r} V(x|\,\tilde{p}, \tilde{\mathcal{T}}) \cdot s^{\tilde{\mathcal{T}}}(x)\, dx = \int_{-\tilde{\beta}_c}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx.$$

This expression is an upper bound for the platform's revenue under $(p, \mathcal{T})$ in $[X_l, X_r]$. In fact, WLOG assume

$\beta_c^r \geq |\beta_c^l|$ which implies that $\tilde{\beta}_c \in [|\beta_c^l|, \beta_c^r]$ and we must have

$$\begin{aligned}
\frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_l,X_r]}(p,\mathcal{T}) &= \int_{\beta_c^l}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \int_{\beta_c^l}^{|\beta_c^l|} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{|\beta_c^l|}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}) - 2 \cdot \int_{|\beta_c^l|}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{|\beta_c^l|}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}) - \int_{|\beta_c^l|}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{\tilde{\beta}_c}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&\leq \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}) + V(\tilde{\beta}_c) \cdot \Big( - \int_{|\beta_c^l|}^{\tilde{\beta}_c} \psi_x^{-1}(V(x))\, dx + \int_{\tilde{\beta}_c}^{\beta_c^r} \psi_x^{-1}(V(x))\, dx \Big) \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}).
\end{aligned}$$

That is, the new solution in the center is a weakly improvement over the old solution.

Now let us consider the periphery. Since $|\tilde{X}_l| = \tilde{X}_r$ both right and left periphery are symmetric. Thus the optimal solution as given by Theorem 1 is the symmetric at both sides. The post-relocation supply is characterize by $\tilde{\beta}_p \in [\tilde{X}_r, \tilde{Y}_r]$ such that

$$s^{\tilde{\mathcal{T}}}(x) = \psi_x^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}})) = \psi_x^{-1}(V(X_r) + (x - X_r) - 2 \cdot \delta), \quad \forall x \in [\tilde{\beta}_p, \tilde{Y}_r],$$

and equals zero otherwise, and

$$\int_{\tilde{\beta}_p}^{\tilde{Y}_r} \psi_x^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}})\, d\Gamma(x) = \theta_1 \cdot (\tilde{Y}_r - \tilde{X}_r) = \theta_1 \cdot (Y_r - X_r) + \theta_1 \cdot \delta.$$

The optimal prices are $\tilde{p}(x) = \rho_x^{loc}(s^{\tilde{\mathcal{T}}}(x))$. As before we omit the characterization of the equilibrium flow as their existence is guaranteed by Theorem 1. The platforms revenue in the periphery is

$$\frac{1}{\gamma} \cdot \mathbf{Rev}_{[-H,\tilde{X}_l]\cup[\tilde{X}_r,H]}(\tilde{p},\tilde{\mathcal{T}}) = 2 \cdot \int_{\tilde{\beta}_p}^{\tilde{Y}_r} V(x|\,\tilde{p},\tilde{\mathcal{T}}) \cdot \psi^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}}))\, dx + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - \tilde{Y}_r),$$

where we have dropped the subindex $x$ from $\psi_x^{-1}$ to stress the fact that in this part of the city this subindex does not change the congestion function. We need to compare this revenue with the revenue of the old solution in the periphery. Not that since $|X_l| > X_r$ we must have

$$Y_r - \beta_p^r < \tilde{Y}_r - \tilde{\beta}_p < \beta_p^l - Y_l.$$

Thus,

$$
\begin{aligned}
\frac{1}{\gamma}\cdot\mathbf{Rev}_{[-H,X_l]\cup[X_r,H]}(p,\mathcal{T}) &= \int_{Y_l}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + \int_{\beta_p^r}^{Y_r} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&\quad + \psi_1\cdot\theta_1\cdot(H - Y_r + Y_l + H) \\
&= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + 2\cdot\int_{\beta_p^r}^{Y_r} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&\quad + \psi_1\cdot\theta_1\cdot(H - Y_r + Y_l + H) \\
&= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + 2\cdot\int_{\beta_p^r+2\cdot\delta}^{\tilde{Y}_r} V(x|\,\tilde{p},\tilde{\mathcal{T}})\cdot\psi^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}}))\,dx \\
&\quad + 2\cdot\psi_1\cdot\theta_1\cdot(H-\tilde{Y}_r) \\
&= \underbrace{\int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - 2\cdot\int_{\tilde{\beta}_p}^{\beta_p^r+2\cdot\delta} V(x|\,\tilde{p},\tilde{\mathcal{T}})\cdot\psi^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}}))\,dx}_{(a)} \\
&\quad + \frac{1}{\gamma}\cdot\mathbf{Rev}_{[-H,\tilde{X}_l]\cup[\tilde{X}_r,H]}(\tilde{p},\tilde{\mathcal{T}}),
\end{aligned}
$$

So if we show that the term $(a)$ is strictly negative we will be done. Not that

$$
\begin{aligned}
(a) \quad &= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - 2\cdot\int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&= \int_{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - \int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&< V(Y_l+(\tilde{Y}_r-\tilde{\beta}_p))\cdot\Big(\int_{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)}^{\beta_p^l} \psi^{-1}(V(x))\,dx - \int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} \psi^{-1}(V(x))\,dx\Big) \\
&= 0.
\end{aligned}
$$

In conclusion, we have constructed a new symmetric solution that yields an strict revenue improvement over the old solution. Therefore, any optimal solution ought to be symmetric.

**Property 1.** $(p^{I_r^1},\mathcal{T}^{I_r^1})$ forms a price-equilibrium pair in $I_r^1$ such that $V(x|\,p^{I_r^1},\mathcal{T}^{I_r^1})$ equals $V(\tilde{X}_r) + (x-\tilde{X}_r)$ and $\Gamma(K^c)=0$.

**Proof of Property 1.** We first show that $\mathcal{T}^{I_r^1}\in\mathcal{F}_{I_r^1}(\Theta|_{I_r^1})$. It is clear that $\mathcal{T}^{I_r^1}\in\mathcal{M}(I_r^1\times I_r^1)$, and that $\mathcal{T}_2^{I_r^1}\ll\Gamma$. To see why $\mathcal{T}_1^{I_r^1}$ coincides with $\theta_{I_r^1}$ consider a set $I\subset I_r^1$ then $\mathcal{T}_1^{I_r^1}(K)$ equals

$$
\mathcal{T}_1^{I_r^1}(K\times I_r^1) = \mathcal{T}((I+h)\times(I_r^1+h)) = \mathcal{T}((I+h)\times[\tilde{X}_r+h,Y_r]) = \mathcal{T}((I+h)\times\mathcal{C}) = \Theta(I+h) = \Theta(I),
$$

where the fourth line holds because the set $I+h$ is contain in $[\tilde{X}_r+h,Y_r]$, and we know there is no flow leaving this interval. Next, using a similar argument we show the property for $d\mathcal{T}_2^{I_r^1}/d\Gamma$, let $I$ be a measurable subset of $I_r^1$ then

$$
\begin{aligned}
\int_I \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(x)\,d\Gamma(x) = \mathcal{T}^{I_r^1}(I_r^1\times I) &= \mathcal{T}([\tilde{X}_r+h,Y_r]\times(I+h)) \\
&\le \mathcal{T}([X_r,Y_r]\times(I+h)) \\
&= \int_{(I+h)} \frac{d\mathcal{T}_2}{d\Gamma}(x)\,d\Gamma(x) \\
&= \int_I \frac{d\mathcal{T}_2}{d\Gamma}(x+h)\,d\Gamma(x),
\end{aligned}
$$

that is, $\Gamma(K^c) = 0$. As for the equilibrium utility function let $x \in [\tilde{X}_r, \tilde{Y}_r)$ we have

$$
\begin{aligned}
V(x|\, p^{I_r^1}, \mathcal{T}^{I_r^1}) &= \inf\{u \in \mathbb{R} : \Gamma(y \in I_r^1 : U(y, p^{I_r^1}(y), \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y)) - |y - x| > u) = 0\} \\
&= \inf\{u \in \mathbb{R} : \Gamma(y \in I_r^1 : U(y, p(y+h), \frac{d\mathcal{T}_2}{d\Gamma}(y+h)) - |y - x| > u) = 0\} \\
&= \inf\{u \in \mathbb{R} : \Gamma(y \in [\tilde{X}_r + h, Y_r] : U(y, p(y), \frac{d\mathcal{T}_2}{d\Gamma}(y)) - |y - (x+h)| > u) = 0\} \\
&\leq V(x+h|\, p, \mathcal{T}).
\end{aligned}
$$

Actually this upper bound is tight. Indeed, Fix any $\epsilon > 0$ and consider $\delta > 0$ small enough such that $(x+h) \notin B(Y_r, \delta)$. We have $\mathcal{T}_2(\{y \in B(y,\delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y)\}) > 0$ which implies that $\Gamma(\{y \in B(Y_r,\delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y)\}) > 0$ and, therefore,

$$
\begin{aligned}
0 &< \Gamma(\{y \in B(Y_r,\delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y), \epsilon + y - (x+h) > |y - (x+h)|\}) \\
&= \Gamma(\{y \in B(Y_r,\delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y), U(y) - |y - (x+h)| > V(x+h) - \epsilon\}) \\
&\leq \Gamma(\{y \in [\tilde{X}_r + h, Y_r] : U(y) - |y - (x+h)| > V(x+h) - \epsilon\}) \\
&= \Gamma(\{y \in I_r^1 : U(y, p^{I_r^1}(y), \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y)) - |y - x| > V(x+h) - \epsilon\}),
\end{aligned}
$$

therefore $V(x|\, p^{I_r^1}, \mathcal{T}^{I_r^1})$ equals $V(x+h)$ for all $x \in [\tilde{X}_r, \tilde{Y}_r)$, and by continuity for all $x \in I_r^1$. Since $V(x+h)$ equals $V(\tilde{X}_r) + (x - \tilde{X}_r)$ we obtain the desired result.

Now we need to verify that this selection of prices and flows yields an equilibrium. That is, we need show that the set

$$
\mathcal{E}_{I_r^1} = \Big\{(x,y) \in I_r^1 \times I_r^1 : \Pi(x, y, p^{I_r^1}(y), \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y)) = V(x|\, p^{I_r^1}, \mathcal{T}^{I_r^1})\Big\},
$$

has $\mathcal{T}^{I_r^1}$ measure equal to $\Theta(I_r^1)$. Observe that $\mathcal{T}(\mathcal{E}_{I_r^1})$ equals

$$
\mathcal{T}\Big(\Big\{(x,y) \in [\tilde{X}_r + h, Y_r] \times [\tilde{X}_r + h, Y_r] : \Pi(x-h, y-h, p^{I_r^1}(y-h), \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y-h)) = V(x)\Big\}\Big),
$$

using that $\Gamma(K^c) = 0$ and the way we chose the prices one can verify that this expression equals

$$
\mathcal{T}\Big(\Big\{(x,y) \in [\tilde{X}_r + h, Y_r] \times [\tilde{X}_r + h, Y_r] : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = V(x|\, p, \mathcal{T})\Big\}\Big).
$$

There is no $\mathcal{T}$ flow of drivers leaving $[\tilde{X}_r + h, Y_r]$ so the fact that $\mathcal{T}$ is an equilibrium flow implies that this last expression equals $\Theta([\tilde{X}_r + h, Y_r])$, which equals $\Theta(I_r^1)$.

**Property 2.** $(p^{I_r^0}, \mathcal{T}^{I_r^0})$ is a price-equilibrium pair such that $\mathbf{Rev}_{[X_l,W_r]}(p^{I_r^0}, \mathcal{T}^{I_r^0}) = \mathbf{Rev}_{[X_l,W_r]}(p, \mathcal{T})$.

**Proof of Property 2.** First a couple of observations, note that for any $y \in [0, \tilde{X}_r]$ and the set $[0,y]$ then

$$
\begin{aligned}
\mathcal{T}_1^r([0,y]) = \mathcal{T}^r([0,y] \times [0,\tilde{X}_r]) &= m\Big(t \in [0, \Theta^r([0,\tilde{X}_r])] : F_{\Theta^r}^{[-1]}(t) \in [0,y]\Big) \\
&= m\Big(t \in [0, \Theta^r([0,\tilde{X}_r])] : 0 \leq t \leq F_{\Theta^r}(y)\Big) \\
&= F_{\Theta^r}(y),
\end{aligned}
$$

and the same argument holds for $\mathcal{T}_2^r$ and $S^r$, this characterizes the first and second marginals of $\mathcal{T}^r$. Furthermore, it's not difficult to see that for $y_1, y_2 \in [0, \tilde{X}_r]$ we have

$$
\mathcal{T}^r([0,y_1] \times [0,y_2]) = m\Big(t \in [0, \Theta^r([0,\tilde{X}_r])] : t \leq F_{\Theta^r}(y_1),\ t \leq F_{S^r}(y_2)\Big) = F_{\Theta^r}(y_1) \wedge F_{S^r}(y_2). \tag{D-26}
$$

Next, we show that $\mathcal{T}^{I_r^0} \in \mathcal{F}_{I_r^0}(\Theta|_{I_r^0})$ is an equilibrium in $I_r^0$. In order to do so we first show that$\mathcal{T}^{I_r^0} \in \mathcal{F}_{I_r^0}(\Theta|_{I_r^0})$. Second, we compute the supply density of $\mathcal{T}_2^{I_r^0}$ and corroborate they coincide with $s^{\mathcal{T}}$. Third, we compute $V_{I_r^0}(\cdot|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$ and verify is coincides with $V(\cdot|\, p, \mathcal{T})$ in $I_r^0$. Finally, we check the equilibrium condition.

Clearly $\mathcal{T}^{I_r^0}$ is a non-negative measure in $I_r^0 \times I_r^0$ because is the sum of non-negative measures. Now we check that $\mathcal{T}_1^{I_r^0} = \Theta|_{I_r^0}$. Consider a measurable set $\mathcal{B} \subseteq I_r^0$ then

$$\begin{aligned}
\mathcal{T}_1^{I_r^0}(\mathcal{B}) &= \mathcal{T}((\mathcal{B} \cap [X_l, 0]) \times [X_l, 0]) + \mathcal{T}^r((\mathcal{B} \cap [0, \tilde{X}_r]) \times [0, \tilde{X}_r]) \\
&= \mathcal{T}((\mathcal{B} \cap [X_l, 0]) \times \mathcal{C}) + \Theta^r(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \Theta(\mathcal{B} \cap [X_l, 0]) + \Theta(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \Theta|_{I_r^0}(\mathcal{B})
\end{aligned}$$

and thus we also have $\mathcal{T}_1^{I_r^0} \ll \Gamma$. For the second marginal of $\mathcal{T}^{I_r^0}$ we have

$$\begin{aligned}
\mathcal{T}_2^{I_r^0}(\mathcal{B}) &= \mathcal{T}([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + \mathcal{T}^r([0, \tilde{X}_r] \times (\mathcal{B} \cap [0, \tilde{X}_r])) \\
&= \mathcal{T}([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + S^r(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \mathcal{T}([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + \mathcal{T}([0, X_r] \times (\mathcal{B} \cap [0, \tilde{X}_r])) \\
&= \mathcal{T}_2(\mathcal{B} \cap [X_l, 0)) + \mathcal{T}_2(\mathcal{B} \cap (0, \tilde{X}_r]) + \mathcal{T}_2(\mathcal{B} \cap \{0\}) \\
&= \mathcal{T}_2|_{I_r^0}(\mathcal{B}),
\end{aligned}$$

and thus $\mathcal{T}_2^{I_r^0} \ll \Gamma$. We conclude that $\mathcal{T}^{I_r^0} \in \mathcal{F}_{I_r^0}(\Theta|_{I_r^0})$. From this we can also conclude that

$$\frac{d\mathcal{T}_2^{I_r^0}}{d\Gamma}(x) = s^{\mathcal{T}}(x), \quad \Gamma - a.e.\ x \text{ in } I_r^0.$$

Next we compute the equilibrium utilities. We show that $V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$ equals $V(x|\, p, \mathcal{T})$ for all $x \in I_r^0$. Observe that $\Gamma - a.e.\ y$ in $I_r^0$ we have $U(y, p^{I_r^0}(y), s^{\mathcal{T}^{I_r^0}}(y)) = U(y, p(y), s^{\mathcal{T}}(y))$, and, therefore, $V(x|\, p, \mathcal{T}) \geq V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$. Using the same argument that we used for the proof of Property 1 we can argue that this upper bound is tight, that is, $V(x|\, p, \mathcal{T}) = V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$.

Now the equilibrium condition. Consider the equilibrium set

$$\mathcal{E}_{I_r^0} \triangleq \left\{(x, y) \in I_r^0 \times I_r^0 : U(y, p^{I_r^0}(y), s^{\mathcal{T}^{I_r^0}}(y)) - |y - x| = V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})\right\},$$

we need to verify that $\mathcal{T}^{I_r^0}(\mathcal{E}_{I_r^0})$ equals $\Theta(I_r^0)$. First, for $\mathcal{T}^l(\mathcal{E}_{I_r^0})$ we have

$$\begin{aligned}
\mathcal{T}^l(\mathcal{E}_{I_r^0}) &= \mathcal{T}\Big(\Big\{(x, y) \in [X_l, 0] \times [X_l, 0] : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| = V(x|\, p, \mathcal{T})\Big\}\Big) \\
&= \mathcal{T}([X_l, 0] \times [X_l, 0]) \\
&= \mathcal{T}([X_l, 0] \times \mathcal{C}) \\
&= \Theta([X_l, 0])
\end{aligned}$$

where we have used our choice of prices, the relation between $d\mathcal{T}_2^{I_r^0}/d\Gamma$ and $s^{\mathcal{T}}$, and the fact that $\mathcal{T}$ is an equilibrium flow that does not setnd flow out of $[X_l, 0]$. For $\mathcal{T}^r|_{[0,\tilde{X}_r]}$, note that its second marginal is $S^r$ and, therefore, Lemma A-2 implies that

$$\mathcal{T}^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0}) = \mathcal{T}^r\Big(\Big\{(x, y) \in [0, \tilde{X}_r] \times [0, \tilde{X}_r] : V(y|\, p, \mathcal{T}) - |y - x| = V(x|\, p, \mathcal{T})\Big\}\Big),$$

and because $V(z|\, p, \mathcal{T})$ equals $V(0) - z$ for any $z \in [0, \tilde{X}_r]$ we have

$$\begin{aligned}
\mathcal{T}^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0}) &= \mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : -y - |y-x| = -x\Big\}\Big) \\
&= \mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x \geq y\Big\}\Big) \\
&= \Theta^r([0,\tilde{X}_r]) - \mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x < y\Big\}\Big),
\end{aligned}$$

but

$$\begin{aligned}
\mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x < y\Big\}\Big) &\leq \sum_{q \in \cap [0,\tilde{X}_r]} \mathcal{T}^r([0,q] \times (q,\tilde{X}_r]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \mathcal{T}^r([0,q] \times [0,\tilde{X}_r]) - \mathcal{T}^r([0,q] \times [0,q]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \Theta^r([0,q]) \wedge S^r([0,\tilde{X}_r]) - \Theta^r([0,q]) \wedge S^r([0,q]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \Theta^r([0,q]) \wedge S^r([0,\tilde{X}_r]) - \Theta^r([0,q]) \wedge S^r([0,q]) = 0,
\end{aligned}$$

where in the last line we used that $\Theta^r([0,q]) \leq S^r([0,q])$. Adding up $\mathcal{T}^l(\mathcal{E}_{I_r^0})$ with $\mathcal{T}^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0})$, yields that $\mathcal{T}^{I_r^0}(\mathcal{E}_{I_r^0})$ equals $\Theta(I_r^0)$, and the equilibrium condition is satisfied. Finally, the revenue condition in the statement of the Property is immediately satisfied as $d\mathcal{T}_2^{I_r^0}/d\Gamma$ coincide with $s^{\mathcal{T}}$ in $I_0^r$, and the same is true for the equilibrium utilities. $\square$

# E Additional Numerical Results for Section 8.3

**Policy structure.** Figure 15 depicts the core spatial thresholds characterizing the optimal pricing policy and the myopic price response as the supply conditions $\theta_1$ change (on the $y$-axis). In particular, we track the changes in $X_r, \beta_p, \beta_c$ and $Y_r$ for the optimal solution (cf. Theorem 3) and the changes in $X_r$ and $X_r^0$ for the local price response (cf. Proposition 4).

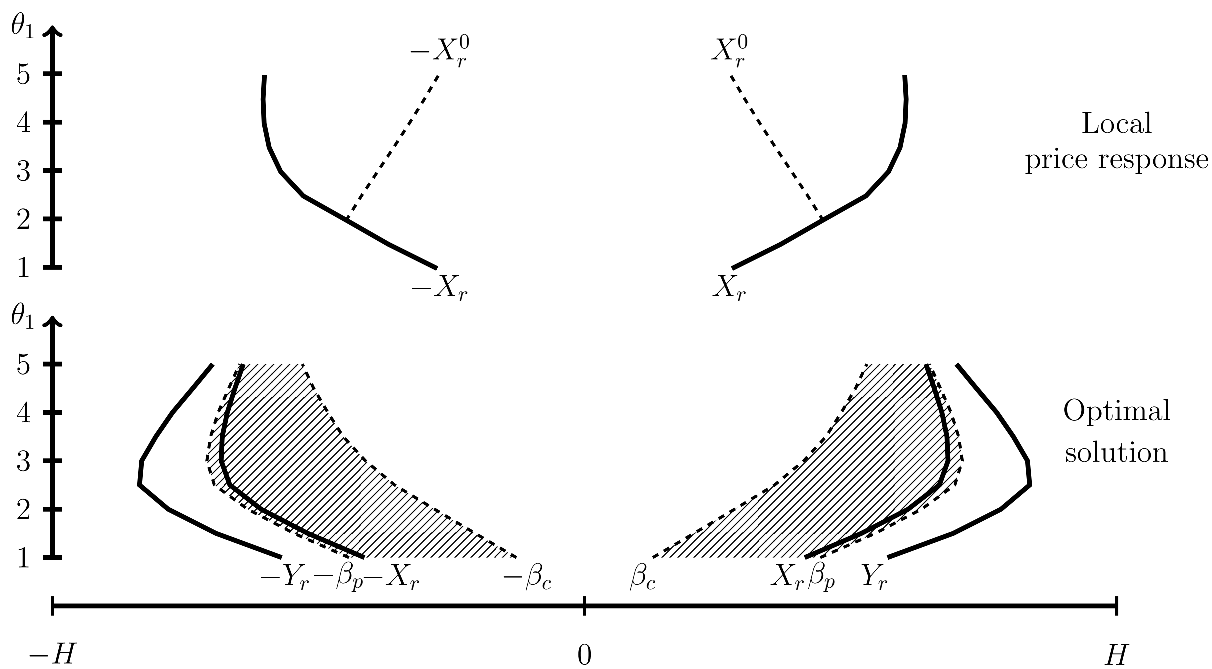


Figure 15: **Policy structure.** Spatial thresholds characterizing the optimal pricing policy and the local price response as the the supply conditions change. The shaded regions have no supply in equilibrium. The figure assumes $\lambda_0 = 9$ and $\lambda_1 = 4$.

The first thing to note is that the structure of supply in the attraction region of 0 differs significantly between the local price response and the optimal policy. In the local price response, there are no drivers who stay put around the origin; and post-relocation, drivers are either at the origin or in $[X_r^{0,\mathrm{lr}}, X_r^{\mathtt{my}}]$. In contrast for the the optimal policy, there are no drivers in a region separated from the origin $[\beta_c, X_r^{\mathtt{opt}}]$ but there are drivers in $[0, \beta_c]$. This contrast can be better understood through the reformulation of the objective in Proposition 1, in conjunction with the shape of the equilibrium utility function in the attraction region of 0 . Given the objective, the platform would ideally like to have supply as close to the origin as possible (subject to the congestion bound constraint) as it maximizes the integral of $V(x) \cdot s^{\mathcal{T}}(x)$. With a local price response, as a result of the lack of flexibility in setting prices throughout the city, the platform is unable to "optimize" the supply in the attraction region and ends up with drivers at locations with low $V$ in $[X_r^{0,\mathrm{lr}}, X_r^{\mathtt{my}}]$ while locations with higher $V$'s have no drivers in $(0, X_r^{\mathtt{my}}]$. Meanwhile, the optimal policy is able to set prices so as to induces the best possible distribution of supply in the attraction region.

In the periphery of the optimal solution, which is outside the origin's attraction region under the optimal pricing policy, the local price response behaves exactly as in the pre-demand shock environment. In stark contrast, the optimal solution incentivizes movement of drivers from the periphery away from the demand shock. In particular, the region $[X_r, Y_r]$, which has a non-trivial size, is artificially damaged. This region is needed for the optimal solution to steer more drivers towards the origin.

**Welfare Implications.** The revenue improvement of the optimal solutions relies on creating a special region in which drivers' utilities are below of what they could earn if the platform responded only locally to the demand shock. This raises the question of whether revenue-optimal pricing leads to lower or higher surpluses for drivers and consumers compared to the benchmark solution.

The social welfare ($SW$) equals the sum of the platform's revenue, and the driver ($DS$) and consumer surpluses ($CS$), as given by

$$DS = \int_{\mathcal{C}_{\text{diff}}} V(x)\, d\Theta(x), \quad CS = \int_{\mathcal{C}_{\text{diff}}} \mathbb{E}[(v - p(x))|v \geq p(x)] \cdot \min\left\{s^{\mathcal{T}}(x), \lambda_x \cdot \overline{F}(p(x))\right\} d\Gamma(x).$$

Driver surplus corresponds to nothing more than the integral of driver equilibrium utilities across all locations in $\mathcal{C}_{\text{diff}}$. Similarly, consumer surplus corresponds to the gains enjoyed across $\mathcal{C}_{\text{diff}}$ by all those consumers who are willing to pay and are matched to some driver.

In Table 3 we display the percentage differences of driver and consumer surpluses, as well as social welfare between the optimal and benchmark solutions. We note that there are instances where the optimal solution is a Pareto improvement over the myopic price response, in the sense that it is better for the platform, drivers and consumers. There are also instances where the platform's revenue gain is at the expense of both drivers and consumers.

| $\theta_1$ | | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\lambda_0 = 3$ | -0.67 | 3.09 | 11.3 | 13.64 | 14.6 | 12.44 | 10.00 | 7.53 | 4.92 |
| $DS$ | $\lambda_0 = 6$ | -4.15 | -3.99 | -1.62 | -2.01 | -0.82 | 0.74 | 3.00 | 5.35 | 7.80 |
| | $\lambda_0 = 9$ | -6.22 | -7.35 | -7.48 | -9.45 | -9.72 | -9.02 | -8.14 | -6.36 | -4.32 |
| | $\lambda_0 = 3$ | -10.96 | -14.1 | -18.48 | -7.24 | -3.15 | -0.44 | 1.01 | 1.57 | 1.58 |
| $CS$ | $\lambda_0 = 6$ | -12.03 | -10.58 | -17.15 | -6.32 | 1.18 | 4.18 | 4.24 | 2.85 | 0.69 |
| | $\lambda_0 = 9$ | -14.33 | -11.94 | -22.43 | -12.58 | -1.39 | 5.77 | 9.73 | 10.98 | 10.44 |
| | $\lambda_0 = 3$ | -1.04 | 0.81 | 4.26 | 8.28 | 9.70 | 8.83 | 7.44 | 5.8 | 3.96 |
| $SW$ | $\lambda_0 = 6$ | -3.60 | -3.56 | -3.49 | -1.05 | 1.50 | 3.16 | 4.43 | 5.29 | 5.87 |
| | $\lambda_0 = 9$ | -5.24 | -5.95 | -8.16 | -6.84 | -4.40 | -2.32 | -0.86 | 0.51 | 1.58 |

Table 3: Driver surplus, consumer surplus and social welfare difference (in %) of optimal solution over myopic price response in $\mathcal{C}_{\text{diff}}$.

For a given level of supply, the driver surplus degrades with respect to the benchmark as the demand shock becomes more intense. We also find that, independently of the size of the demand shock, the optimal solution performs better than the benchmark in terms of consumer surplus when the supply level is high. More drivers in the city imply more matches and lower prices and, thus, higher consumer surplus.