Online Appendix to "Benefits of Customer Loyalty in Markets with Endogenous Search Costs"

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Contents

Derivation of Equilibrium Price Distributions in the Pricing Subgame

As discussed in the main text, given store search costs, s_1 and s_2, we derive the stores' equilibrium price distributions $F_1(p)$ and $F_2(p)$ from the indifference equations on the stores' pricing, taking into account the optimal consumer behavior. The various possibilities of the costly-searcher reservation prices discussed in the main text can be put together with the possibilities of whether a store wants to price above the reservation price (which is when its profit is not strictly above what it can obtain from the loyal consumers by pricing at their valuation) and the potential mass points (noting that both stores cannot in equilibrium have a mass point at the same price due to the competition for the shopper segment), to obtain the possible equilibrium cases as listed in Table A1. The four possible types of price distributions are illustrated in Figure 1. This table, together with Figure A1, are color coded based on the relative values of reservation prices. Below, we first explain these cases and then proceed to fully characterize each case in the next section.

In each of the cases, we use indifference equations to solve for unknown points of price distributions (p_0, r, F_{ri}, etc.) and we derive costly-searchers' equilibrium search and purchase behavior. In particular, we obtain, γ_0, the portion of costly-searchers who do not search/purchase, and γ_i, the portion of costly-searchers who start their visit from Store i, for $i \in \{1,2\}$. We should always have

$$\gamma_0 + \gamma_1 + \gamma_2 = \gamma = 1 - \alpha_1 - \alpha_2 - \beta. \quad (1)$$

In Case 1, search costs are high (see Figure A1) such that reservation prices become higher than v, which results in all costly-searchers staying out (see Table A1, first row). When one of the store's search costs becomes smaller, we enter the region of Case 2. In this case, $r_i = v < r_j$, and so a positive portion of costly-searchers (but not all of them) visit Store i. When this search cost decreases even further, we enter region of Case 3, in which all costly-searchers visit and purchase from one of the stores. In this case, $r_i < v < r_j$, and thus $\gamma_i = \gamma$, whereas $\gamma_0 = \gamma_j = 0$. In Case 4, where both search costs are in the intermediate-to-high range, we have $r_i = r_j = v$, resulting in $\gamma_0 > 0$, $\gamma_j > 0$ and $\gamma_i > 0$. In Cases 1-4, the price distribution is of Type A.

In Case 5, 7, 9, and 11, we have $r_i = r_j < v$, which implies all costly-searchers enter the market and split between stores, that is, $\gamma_i > 0$, $\gamma_j > 0$, and $\gamma_0 = 0$. The difference between these cases (for which we use yellow both in Table A1 and Figure A1) is in the type of price distributions. These cases occur when search costs are in the intermediate range and close to each other. Finally, in Cases 6, 8, 10, and 12, marked by purple, we have $r_i < r_j < v$. Therefore, *all* costly-searchers start their search from Store i, that is, $\gamma_i = \gamma$, $\gamma_j = 0$, and $\gamma_0 = 0$. These four cases happen when one or both search costs are low.

We note that stores are asymmetric, because we assumed $\alpha_1 > \alpha_2$. Therefore, which stores take the role of Store i (defined as the one with $r_i < r_j$ when $r_i \neq r_j$) is important. Thus, a case is further labeled "a" if $i = 1$ and $j = 2$, and is labeled "b" for the reverse situation ($i = 2$ and $j = 1$). For this reason, we see two regions, for example, for Case 2, in Figure A1 (labeled Case $2a$ and Case $2b$). These regions ("a" and "b" for the same case number) are *not* symmetric in Figure A1, because we have asymmetric stores ($\alpha_1 > \alpha_2$).

In each case, given the pattern of the costly-searcher behavior (whether $\gamma_0, \gamma_i, \gamma_j$ are zero or positive), and the shape of the price distribution (Type A, B, C, or D), we write indifference equations and use store search equations

$$s_i = \int_{p_0}^{r_i} (r_i - p)\, dF_i(p). \tag{2}$$

Solving these equations together yields (1) $\gamma_0, \gamma_i, \gamma_j$, (2) reservation prices, and (3) the unknown values in the price distributions (p_0, r, F_{ri}, etc.). To shorten equations, we use the notation $m_x = 1 - F_x$ throughout the Appendix.

Type A Price Distributions (Cases 1-4)

Case 1

In this case, s_1 and s_2 are both high. All of the costly-searchers stay out of the market ($\gamma_0 = \gamma, \gamma_1 = 0$, and $\gamma_2 = 0$). Store 1, with a larger loyal segment, has a mass point at v (the other way around does not occur because $\alpha_1 > \alpha_2$; thus,

$$\pi_1 = (\alpha_1 + \beta)p_0 = \alpha_1 p + \beta p\big(1 - F_2(p)\big) = \alpha_1 v,$$

$$\pi_2 = (\alpha_2 + \beta)p_0 = \alpha_2 p + \beta p\big(1 - F_1(p)\big) = \alpha_2 v + \beta v m_{1v}.$$

Solving them, we get $p_0 = \frac{\alpha_1 v}{\alpha_1 + \beta}$ and $m_{1v} = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \beta}$. Moreover, price distributions are $F_2(p) = \frac{(\alpha_1+\beta)(p-p_0)}{\beta p}$ and $F_1(p) = \frac{(\alpha_2+\beta)(p-p_0)}{\beta p}$. The costly-searchers stay out of the market as $r_1 > v$ and $r_2 > v$. We plug $F_1(p)$ and $F_2(p)$ into the search equation and take the integral to obtain

$$s_i = \int_{p_0}^{r_i} (r_i - p)\, dF_i(p) = \int_{p_0}^{v} (v - p)\, dF_i(p) + (r_i - v) =$$

$$\int_{p_0}^{v} (v - p)\, d\,\frac{(\alpha_j+\beta)(p-p_0)}{\beta p} + (r_i - v) = \frac{\alpha_j+\beta}{\beta} p_0 \left(\frac{v}{p_0} - Log\,\frac{v}{p_0} - 1\right) + (r_i - v) \Longrightarrow$$

$$r_1 = v + s_1 - \frac{\alpha_1 v}{\beta}\frac{\alpha_2+\beta}{\alpha_1+\beta}\left(\frac{\alpha_1+\beta}{\alpha_1} - Log\,\frac{\alpha_1+\beta}{\alpha_1} - 1\right) \text{ and } r_2 = v + s_2 - \frac{\alpha_1 v}{\beta}\left(\frac{\alpha_1+\beta}{\alpha_1} - Log\,\frac{\alpha_1+\beta}{\alpha_1} - 1\right).$$

Thus, the region of this case ($r_1 > v$ and $r_2 > v$), illustrated in Figure A1, is

$$s_1 > \bar{s}_1 \equiv \frac{\alpha_1 v}{\beta}\frac{\alpha_2+\beta}{\alpha_1+\beta}\left(\frac{\alpha_1+\beta}{\alpha_1} - Log\,\frac{\alpha_1+\beta}{\alpha_1} - 1\right),\ s_2 > \bar{s}_2 \equiv \frac{\alpha_1 v}{\beta}\left(\frac{\alpha_1+\beta}{\alpha_1} - Log\,\frac{\alpha_1+\beta}{\alpha_1} - 1\right).$$

In Case 1, store profits are $\pi_1 = \alpha_1 v$ and $\pi_2 = \frac{\alpha_2+\beta}{\alpha_1+\beta}\alpha_1 v$.

Case 2

In this case, a fraction of costly-searchers stays out and the remaining visit and purchase from Store i ($\gamma_0 > 0, \gamma_i > 0$, and $\gamma_j = 0$). In terms of reservation prices, we have $r_i = v < r_j$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma_i + \beta)p_0 = (\alpha_i + \gamma_i)p + \beta p(1 - F_j(p)) = (\alpha_i + \gamma_i)v$$

$$\pi_j = (\alpha_j + \beta)p_0 = (\alpha_j)p + \beta p(1 - F_i(p)) = \alpha_j v + \beta v m_{iv}.$$

Using (2), we get $s_i = \frac{vp_0(1-m_{iv})}{v-p_0}\left(\frac{v}{p_0} - Log\frac{v}{p_0} - 1\right)$ and $s_j > \frac{vp_0}{v-p_0}\left(\frac{v}{p_0} - Log\frac{v}{p_0} - 1\right)$. We solve these equations to find p_0, γ_i, and m_{iv}. Plugging in m_{iv} from π_j equation, we obtain

$$s_i = \frac{\alpha_j+\beta}{\beta} p_0 \left(\frac{v}{p_0} - Log\frac{v}{p_0} - 1\right). \tag{3}$$

Equation (3) has a unique solution for p_0. Moreover, p_0 decreases with s_i:

$$\frac{\partial s_i}{\partial p_0} = \frac{\alpha_j+\beta}{\beta}\frac{\partial(v + p_0 Log\, p_0 - p_0 Log\, v - p_0)}{\partial p_0} = Log\, p_0 - Log\, v < 0.$$

Also, from indifference equations for Store i, we have $\gamma_i = \frac{\beta p_0}{v-p_0} - \alpha_i$. Hence, γ_i increases with p_0, and so it decreases with s_i. We conclude that in his case, $\frac{\partial \pi_i}{\partial s_i} < 0$, whereas $\frac{\partial \pi_j}{\partial s_j} = 0$.

The region of Case 2 consists of $0 < \gamma_i < \gamma$, $m_{iv} > 0$, and $v < r_j$. We have $\gamma_i > 0 \Leftrightarrow p_0 > \frac{\alpha_i v}{\alpha_i+\beta}$, $\gamma_i < \gamma \Leftrightarrow p_0 < \frac{\alpha_i+\gamma}{\alpha_i+\gamma+\beta}v$, and $m_{iv} > 0 \Leftrightarrow (\alpha_j + \beta)p_0 > \alpha_j v \Leftrightarrow p_0 > \frac{\alpha_j v}{\alpha_j+\beta}$. Therefore, we have $\max\left\{\frac{\alpha_1 v}{\alpha_1+\beta}, \frac{\alpha_2 v}{\alpha_2+\beta}\right\} = \frac{\alpha_1 v}{\alpha_1+\beta} < p_0 < \frac{\alpha_i+\gamma}{\alpha_i+\gamma+\beta}v$. Thus, plugging in (3), we get the region of Case 2,

$$\underline{s_i} = \frac{(\alpha_i+\gamma)(\alpha_j+\beta)}{\beta(\alpha_i+\gamma+\beta)}v\left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right) < s_i < \overline{s}_i = \frac{\alpha_1(\alpha_j+\beta)}{\beta(\alpha_1+\beta)}v\left(\frac{\alpha_1+\beta}{\alpha_1} - Log\frac{\alpha_1+\beta}{\alpha_1} - 1\right).$$

Moreover, $< r_j \Leftrightarrow s_j > \frac{vp_0}{v-p_0}\left(\frac{v}{p_0} - Log\frac{v}{p_0} - 1\right)$, where p_0 solves equation (3). By canceling p_0, we get $s_i > \frac{(\alpha_j+\beta)s_j - \beta s_i}{\beta s_j}v\left(\frac{(\alpha_j+\beta)s_j}{(\alpha_j+\beta)s_j-\beta s_i} - Log\frac{(\alpha_j+\beta)s_j}{(\alpha_j+\beta)s_j-\beta s_i} - 1\right)$.

Note that in Case $2a$, $\frac{\alpha_1 v}{\alpha_1+\beta} < p_0 < \frac{\alpha_1+\gamma}{\alpha_1+\gamma+\beta}v \to \frac{\alpha_1}{\alpha_1+\beta} < \frac{\alpha_1+\gamma}{\alpha_1+\gamma+\beta}$ which is always correct. In Case $2b$, $\frac{\alpha_1 v}{\alpha_1+\beta} < p_0 < \frac{\alpha_2+\gamma}{\alpha_2+\gamma+\beta}v$ which requires $\alpha_1 < \alpha_2 + \gamma$. If $\alpha_1 \geq \alpha_2 + \gamma$, then only the region of Case $2a$ exists.

Case 3

In this case, one of the stores has a low search cost and the other has a high search cost, which results in $\gamma_0 = 0, \gamma_i = \gamma$, and $\gamma_j = 0$. In terms of reservation prices, we have $r_i < v < r_j$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma + \beta)p_0 = (\alpha_i + \gamma)v \ , \ \pi_j = (\alpha_j + \beta)p_0 = \alpha_j v + \beta v m_{iv}.$$

Thus, we obtain $p_0 = \frac{(\alpha_i+\gamma)v}{\alpha_i+\gamma+\beta}$ and $m_{iv} = \frac{\alpha_i-\alpha_j+\gamma}{\alpha_i+\beta+\gamma}$. Store profits are $\pi_i = (\alpha_i + \gamma)v$ and $\pi_j = \frac{(\alpha_i+\gamma)(\alpha_j+\beta)v}{\alpha_i+\gamma+\beta}$.

The region of Case 3 is obtained from $r_i < v < r_j \Leftrightarrow s_i < \underline{s}_i$ & $s_j > s_{jv}$, where,

$$\underline{s}_i = \frac{(\alpha_i+\gamma)(\alpha_j+\beta)}{\beta(\alpha_i+\gamma+\beta)} v\left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right) \text{ and } s_{jv} = \frac{(\alpha_i+\gamma)}{\beta} v\left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right).$$

Moreover, $m_{iv} > 0$ requires $\alpha_i - \alpha_j + \gamma > 0$ which is always true in Case $3a$. In Case $3b$, we should have $\alpha_2 - \alpha_1 + \gamma > 0$. Otherwise, Store 1 (instead of Store 2) will have a mass point at v. In this case, we have,

$$\pi_1 = (\alpha_1 + \beta)p_0 = \alpha_1 v, \quad \pi_2 = (\alpha_2 + \gamma + \beta)p_0 = (\alpha_2 + \gamma)v + \beta v m_{1v}.$$

Thus, we obtain $p_0 = \frac{\alpha_1 v}{\alpha_1+\beta}$ and $m_{1v} = \frac{\alpha_1-\alpha_2-\gamma}{\alpha_1+\beta}$. Store profits are $\pi_1 = \alpha_1 v$ and $\pi_2 = \frac{\alpha_1(\alpha_2+\gamma+\beta)v}{\alpha_1+\beta}$.

Case 4

In this case, search costs are relatively high, and so some of the costly-searchers stay out while the remaining are divided between the two stores ($\gamma_0 > 0$, $\gamma_i > 0$, and $\gamma_j > 0$). In terms of reservation prices, we have $r_i = r_j = v$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma_i + \beta)p_0 = (\alpha_i + \gamma_i)v \ \text{ and } \ \pi_j = \left(\alpha_j + \gamma_j + \beta\right)p_0 = \left(\alpha_j + \gamma_j\right)v + \beta v m_{iv}.$$

We plug $r_i = r_j = v$ into equation (2) to obtain

$$s_i = \frac{v p_0(1-m_{iv})}{v-p_0}\left(\frac{v}{p_0} - Log\frac{v}{p_0} - 1\right) \text{ and } s_j = \frac{v p_0}{v-p_0}\left(\frac{v}{p_0} - Log\frac{v}{p_0} - 1\right).$$

Using these two search equations, we get $1 - m_{iv} = s_i/s_j$. Therefore, $m_{iv} > 0$ and $m_{jv} = 0$ if and only if $s_i < s_j$. Moreover, p_0 uniquely solves Store j's search equation, and it is a decreasing function of s_j. However, p_0 does not change with s_i. Therefore, γ_i and π_i do not change with s_i. Plugging p_0 and m_{iv} into indifference equations gives us $\gamma_i = \frac{\beta p_0}{v-p_0} - \alpha_i$ and $\gamma_j = \frac{\beta v}{v-p_0}\frac{s_i}{s_j} - \alpha_j - \beta$. Moreover, $\gamma_0 = \gamma - \gamma_i - \gamma_j = 1 - \frac{\beta}{v-p_0}\left(p_0 + v\frac{s_i}{s_j}\right)$. Both π_j and γ_j decrease with s_j.

The region of Case 4 consists of $\gamma_0 > 0$, $\gamma_i > 0$, $\gamma_j > 0$, and $m_{iv} > 0$. These imply $s_i < \frac{(\alpha_j+\beta)s_j-\beta s_i}{\beta s_j} v\left(\frac{(\alpha_j+\beta)s_j}{(\alpha_j+\beta)s_j-\beta s_i} - Log\frac{(\alpha_j+\beta)s_j}{(\alpha_j+\beta)s_j-\beta s_i} - 1\right)$, $\frac{s_j-\beta s_i}{\beta(s_i+s_j)} v\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right) < s_j$, and $s_i < s_j$.

Type B Price Distributions (Cases 5-6)

Case 5

In this case, search costs are lower and hence all costly-searchers visit and purchase at one of the stores. Hence, $\gamma_0 = 0$, $\gamma_i > 0$, and $\gamma_j > 0$. In terms of reservation prices, we have $r_i = r_j < v$. Price distributions are of Type B. Indifference equations are

$$\pi_i = (\alpha_i + \gamma_i + \beta)p_0 = (\alpha_i + \gamma_i)r \quad \text{and} \quad \pi_j = (\alpha_j + \gamma_j + \beta)p_0 = (\alpha_j + \gamma_j)r + \beta r m_{ir}.$$

Search equations are $s_i = \frac{\alpha_j+\gamma_j+\beta}{\beta} p_0 \left(\frac{r}{p_0} - Log\frac{r}{p_0} - 1\right)$, $s_j = \frac{\alpha_i+\gamma_i+\beta}{\beta} p_0 \left(\frac{r}{p_0} - Log\frac{r}{p_0} - 1\right)$.

From the search equations, we have $\frac{s_i}{s_j} = \frac{\alpha_j+\gamma_j+\beta}{\alpha_i+\gamma_i+\beta}$. Combining with $\gamma_i + \gamma_j = \gamma$, we get $\gamma_i = \frac{(1-\alpha_i)s_j-(\alpha_i+\beta)s_i}{s_i+s_j}$, *and* $\gamma_j = \frac{(1-\alpha_j)s_i-(\alpha_j+\beta)s_j}{s_i+s_j}$. Moreover, $p_0 = \frac{\beta(s_i+s_j)}{1+\beta}\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right)^{-1}$, $r = \frac{\beta s_j(s_i+s_j)}{s_j-\beta s_i}\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right)^{-1}$, and $m_{ir} = 1 - \frac{s_i}{s_j}$. Therefore, Store i's distribution has a mass point if and only if $s_i < s_j$. Profits become

$$\pi_i = \beta s_j\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right)^{-1} \text{and } \pi_j = \beta s_i\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right)^{-1}.$$

We have $\frac{\partial\gamma_i}{\partial s_i} = -\frac{(1+\beta)s_j}{(s_i+s_j)^2}$ and $\frac{\partial\gamma_j}{\partial s_j} = -\frac{(1+\beta)s_i}{(s_i+s_j)^2}$. Moreover, $\frac{\partial p_0}{\partial s_i} = -\frac{\beta(y^2-y+Log[1+y])}{(1+\beta)(y-Log[1+y])^2} < 0$, where $y \equiv \frac{\beta(s_i+s_j)}{s_j-\beta s_i}$. Hence, $\frac{\partial\pi_i}{\partial s_i} < 0$ as both p_0 and γ_i decrease with s_i. Furthermore, $\frac{\partial\pi_j}{\partial s_j} = \frac{(\beta-y)^2 y}{(1+\beta)(1+y)(y-Log[1+y])^2} > 0$. Therefore, Store j's profit increases with its own search cost, whereas Store i's profit decreases with its own search cost.

The region of Case 5 consists of $r < v, \gamma_i > 0, \gamma_j > 0, m_{iv} > 0, \pi_i > \alpha_i v$, and $\pi_j > \alpha_j v$. These imply $s_j < \frac{s_j-\beta s_i}{\beta(s_i+s_j)} v\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right)$, $\frac{\alpha_j+\beta}{1-\alpha_j} < \frac{s_i}{s_j} < \frac{1-\alpha_i}{\alpha_i+\beta}$, $s_i < s_j$, $\beta s_j\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right)^{-1} > \alpha_i v$, and $\beta s_i\left(\frac{(1+\beta)s_j}{s_j-\beta s_i} - Log\frac{(1+\beta)s_j}{s_j-\beta s_i} - 1\right)^{-1} > \alpha_j v$.

Case 6

In this case, all costly-searchers visit and buy from Store i ($\gamma_0 = 0$, $\gamma_i = \gamma$, and $\gamma_j = 0$). Also, the price distributions are of type B (bounded from above by r_j). In terms of reservation pries, we have $r_i < r_j = r < v$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma + \beta)p_0 = (\alpha_i + \gamma)r_j, \pi_j = (\alpha_j + \beta)p_0 = \alpha_j r_j + \beta r_j m_{ir}.$$

Search equations are $s_i = \frac{r_i p_0(1-m_{ir})}{r_i-p_0}\left(\frac{r_i}{p_0} - Log\frac{r_i}{p_0} - 1\right)$, $s_j = \frac{r_j p_0}{r_j-p_0}\left(\frac{r_j}{p_0} - Log\frac{r_j}{p_0} - 1\right)$. Using the above equations, we get $p_0 = \frac{\beta s_j}{\alpha_i+\gamma+\beta}\left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right)^{-1}$, and $r_j = \frac{\beta s_j}{\alpha_i+\gamma}\left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right)^{-1}$, $m_{ir} = 1 - F_{ir} = \frac{\alpha_i-\alpha_j+\gamma}{\alpha_i+\beta+\gamma}$. Profits become

$$\pi_i = \beta s_j\left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right)^{-1} \text{and } \pi_j = \frac{\beta(\alpha_j+\beta)s_j}{\alpha_i+\gamma+\beta}\left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right)^{-1}.$$

Hence, profits do not change with s_i but linearly increase with s_j.

The region of Case 6 consists of $r_i < r_j < v$, $m_{iv} > 0$, $\pi_i > \alpha_i v$, and $\pi_j > \alpha_j v$. It follows that $s_i < \frac{\alpha_j+\beta}{\alpha_i+\beta+\gamma} s_j$ and $\max\left\{\frac{\alpha_i+\gamma+\beta}{\alpha_j+\beta}\frac{\alpha_j v}{\beta}, \frac{\alpha_i v}{\beta}\right\} < s_j \left(\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - Log\,\frac{\alpha_i+\gamma+\beta}{\alpha_i+\gamma} - 1\right)^{-1} < \frac{\alpha_i+\gamma}{\beta} v$.

Type C Price Distributions (Cases 7-8)

Case 7

Search costs are low enough, so that all costly-searchers visit and purchase from one of the stores ($\gamma_0 = 0$, $\gamma_i > 0$, and $\gamma_j > 0$). Hence, we have $r_i = r_j < v$. Store j has the top price of v with positive probability. Indifference equations are

$$\pi_i = (\alpha_i + \gamma_i + \beta)p_0 + \gamma_j p_0 m_{jv} = (\alpha_i + \gamma_i)r + (\beta + \gamma_j)r m_{jv}$$

$$\pi_j = (\alpha_j + \gamma_j + \beta)p_0 = (\alpha_j + \gamma_j)r + \beta r m_{ir} = \alpha_j v.$$

Search equations are $s_i = \frac{r p_0(1-m_{ir})}{r-p_0}\left(\frac{r}{p_0} - Log\,\frac{r}{p_0} - 1\right)$, $s_j = \frac{r p_0(1-m_{jv})}{r-p_0}\left(\frac{r}{p_0} - Log\,\frac{r}{p_0} - 1\right)$.

From π_j indifference equations, we have $(\alpha_j + \gamma_j + \beta)(r - p_0) = r\beta(1 - m_{ir})$. Thus, $\frac{r p_0(1-m_{ir})}{r-p_0} = \frac{(\alpha_j+\gamma_j+\beta)p_0}{\beta} = \frac{\alpha_j v}{\beta}$. Plugging into the search equation, we get $\frac{\beta s_i}{\alpha_j v} = x - Log(x) - 1$, where $x = \frac{r}{p_0}$. This equation always has a unique positive root for x. Then, combining the Store j search equation with the Store j indifference equation, we get $\beta s_i x(\alpha_i + \beta + \gamma)p_0^2 - \alpha_j v s_j(\alpha_j - x\alpha_j + \beta)p_0 - \alpha_j^2 v^2 s_j(x - 1) = 0$, which can be solved for p_0. Because $x > 1$, this quadratic equation always has one positive root. Having x and p_0, we have $r = x p_0$, $\gamma_i = \alpha_j + \beta + \gamma - \frac{\alpha_j v}{p_0}$, and $\gamma_j = \gamma - \gamma_i = \frac{\alpha_j v}{p_0} - \alpha_j - \beta$. Finally, $m_{ir} = \frac{\alpha_j v - (\alpha_j+\gamma_j)r}{\beta r} = 1 - \frac{\alpha_j v(x-1)}{\beta x p_0}$ and $m_{jv} = 1 - \frac{(1-m_{ir})s_j}{s_i} = 1 - \frac{\alpha_j v(x-1)}{\beta x p_0}\frac{s_j}{s_i}$. Profits are $\pi_i = \frac{s_j}{s_i}\alpha_j v$ and $\pi_j = \alpha_j v$. Thus, Store i's profit decreases with s_i, whereas $\pi_j = \alpha_j v$ does not depend on the search costs.

Case 8

In this case, all costly-searchers visit and purchase from Store i ($\gamma_0 = 0$, $\gamma_i = \gamma$, and $\gamma_j = 0$). Hence, $r_i < r_j = r < v$. Moreover, Store j charges the top price with positive probability. Indifference equations are

$$\pi_i = (\alpha_i + \gamma + \beta)p_0 = (\alpha_i + \gamma)r_j + \beta r_j m_{jv}\ ,\ \pi_j = (\alpha_j + \beta)p_0 = \alpha_j r_j + \beta r_j m_{ir} = \alpha_j v.$$

Search equations are $s_i = \frac{r_i p_0(1-m_{ir})}{r_i-p_0}\left(\frac{r_i}{p_0} - Log\,\frac{r_i}{p_0} - 1\right)$, $s_j = \frac{r_j p_0(1-m_{jv})}{r_j-p_0}\left(\frac{r_j}{p_0} - Log\,\frac{r_j}{p_0} - 1\right)$.

From the second equation, $p_0 = \frac{\alpha_j v}{\alpha_j+\beta}$. Then, $r_j = x p_0$, where x solves $x - Log(x) - 1 = \frac{\beta(\alpha_j+\beta)s_i}{\alpha_j(\alpha_i+\gamma+\beta)v}$.

Profits are $\pi_i = \frac{\alpha_i+\gamma+\beta}{\alpha_j+\beta}\alpha_j v$ and $\pi_j = \alpha_j v$, which do not depend on search costs.

Type D Price Distributions (Cases 9-12)

Case 9

Costly-searchers are divided between the two stores: $\gamma_0 = 0$, $\gamma_i > 0$, and $\gamma_j > 0$. In terms of reservation prices, we have $r = r_i = r_j < v$. Also, $F_{iv} = 1$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma_i + \beta)p_0 + \gamma_j p_0 m_{jr} = (\alpha_i + \gamma_i)r + (\beta + \gamma_j)r m_{jr} = \alpha_i p_m + (\beta + \gamma)p_m m_{jr} = \alpha_i v + (\beta + \gamma)v m_{jv}$$

$$\pi_j = (\alpha_j + \gamma_j + \beta)p_0 + \gamma_i p_0 m_{irb} = (\alpha_j + \gamma_j)r + \beta r m_{ira} + \gamma_i r m_{irb} = \alpha_j p_m + (\beta + \gamma)p_m(1 - F_{irb}) = \alpha_j v$$

Search equations are $s_i = \frac{r p_0(1-m_{ira})}{r-p_0}\left(\frac{r}{p_0} - Log\frac{r}{p_0} - 1\right)$, $s_j = \frac{r p_0(1-m_{jr})}{r-p_0}\left(\frac{r}{p_0} - Log\frac{r}{p_0} - 1\right)$.

These equations are solved for nine unknowns: $\{p_0, r, p_m, \gamma_i, \gamma_j, m_{ira}, m_{irb}, m_{jr}, m_{jv}\}$ as follows. First, we find $x = r/p_0$ from $x - Log(x) - 1 = \frac{\beta s_i}{\alpha_j v}$. Second, we find p_m by solving the quadratic equation, $(1-\alpha_j)s_i\left((1-\alpha_i)(x(\beta+\gamma) - 2\beta - \gamma)s_j - (1-\alpha_j)x\beta s_i\right)p_m^2 + \alpha_j \beta s_j v\left((1-\alpha_i)s_j + (1-\alpha_j)(1+2x)s_i\right)p_m + \alpha_j^2 s_j^2 v^2\left(\gamma - x(2\beta+\gamma)\right) = 0$. Having p_m, we can find all other variables from indifference equations. Profits are $\pi_i = \frac{s_j}{s_i}\alpha_j v$ and $\pi_j = \alpha_j v$. Thus, Store i's profit decreases with s_i, whereas $\pi_j = \alpha_j v$ does not depend on the search costs.

Case 10

In this case, $\gamma_0 = 0$, $\gamma_i = \gamma$, and $\gamma_j = 0$ ($r_i < r_j = r$). costly-searchers strictly prefer Store i to Store j and hence visit Store i. However, (with positive probability) they may make a second visit to Store j, but only if they find Store i's price too high. Moreover, $F_{iv} = 1$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma + \beta)p_0 = (\alpha_i + \gamma)r_j + \beta r_j m_{jr} = \alpha_i p_m + (\beta + \gamma)p_m m_{jr} = \alpha_i v + (\beta + \gamma)v m_{jv}$$

$$\pi_j = (\alpha_j + \beta)p_0 + \gamma p_0 m_{irb} = \alpha_j r_j + \beta r_j m_{ira} + \gamma r_j m_{irb} = \alpha_j p_m + (\beta + \gamma)p_m(1 - F_{irb}) = \alpha_j v.$$

Search equations are $s_i = \frac{r_i p_0(1-m_{ira})}{r_i-p_0}\left(\frac{r_i}{p_0} - Log\frac{r_i}{p_0} - 1\right)$, $s_j = \frac{r_j p_0(1-m_{jr})}{r_j-p_0}\left(\frac{r_j}{p_0} - Log\frac{r_j}{p_0} - 1\right)$.

These equations are solved for eight unknowns: $\{p_0, r_i, r_j, p_m, m_{ira}, m_{irb}, m_{jr}, m_{jv}\}$. First, we find $x = r_j/p_0$ from $x - Log(x) - 1 = \frac{\beta^3(\alpha_j+\beta+\gamma)s_j}{\alpha_j(\alpha_i+\gamma+\beta)v}\frac{x}{(\beta^2+\gamma^2+\beta\gamma)x-\beta\gamma-\gamma^2}$. Then, $p_0 = \frac{\alpha_j v\left((\beta^2+\gamma^2+\beta\gamma)x-\beta\gamma-\gamma^2\right)}{\beta^2(\alpha_j+\gamma+\beta)x}$. Having solved p_0, all other variables are obtained from indifference equations. Profits are $\pi_i = (\alpha_i + \gamma + \beta)p_0$ and $\pi_j = \alpha_j v$. Because x does not vary with s_i, p_0 and π_i also do not change with s_i. Store j's profit $\pi_j = \alpha_j v$ does not depend on the search costs.

Case 11

In this case, costly-searchers are divided between the two stores; hence, $\gamma_0 = 0$, $\gamma_i > 0$, and $\gamma_j > 0$. In terms of reservation prices, we have $r = r_i = r_j < v$. Also, we have $F_{jv} = 1$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma_i + \beta)p_0 + \gamma_j p_0 m_{jr} = (\alpha_i + \gamma_i)r + (\beta + \gamma_j)r m_{jr} = \alpha_i p_m + (\beta + \gamma)p_m m_{jr} = \alpha_i v$$

$$\pi_j = (\alpha_j + \beta + \gamma_j + \gamma_i m_{irb})p_0 = (\alpha_j + \gamma_j + \beta m_{ira} + \gamma_i m_{irb})r = (\alpha_j + \beta m_{irb} + \gamma m_{irb})p_m = (\alpha_j + \beta m_{iv} + \gamma m_{iv})v.$$

Search equations are $s_i = \frac{r p_0(1-m_{ira})}{r-p_0}\left(\frac{r}{p_0} - Log\frac{r}{p_0} - 1\right)$, $s_j = \frac{r p_0(1-m_{jr})}{r-p_0}\left(\frac{r}{p_0} - Log\frac{r}{p_0} - 1\right)$.

These equations are solved for nine unknowns: $\{p_0, r, p_m, \gamma_i, \gamma_j, m_{ira}, m_{irb}, m_{iv}, m_{jr}\}$. Profits are $\pi_i = \alpha_i v$ and $\pi_j = \frac{s_i}{s_j}\alpha_i v$. Thus, Store j's profit decreases with s_j, whereas $\pi_i = \alpha_i v$ is does not depend on the search costs.

Case 12

In this case, $\gamma_0 = 0$, $\gamma_i = \gamma$, and $\gamma_j = 0$ ($r_i < r_j = r$). costly-searchers strictly prefer Store i to Store j and hence visit Store i. However, (with positive probability) they might make a second visit to Store j, only if they find Store i's price too high. Also, $F_{jv} = 1$. Indifference equations are

$$\pi_i = (\alpha_i + \gamma + \beta)p_0 = (\alpha_i + \gamma)r_j + \beta r_j m_{jr} = \alpha_i p_m + (\beta + \gamma)p_m m_{jr} = \alpha_i v$$

$$\pi_j = (\alpha_j + \beta)p_0 + \gamma p_0 m_{irb} = \alpha_j r_j + \beta r_j m_{ira} + \gamma r_j m_{irb} = \alpha_j p_m + (\beta + \gamma)p_m(1 - F_{irb}) = \alpha_j v + (\beta + \gamma)v m_{iv}.$$

Search equations are $s_i = \frac{r_i p_0(1-m_{ira})}{r_i-p_0}\left(\frac{r_i}{p_0} - Log\frac{r_i}{p_0} - 1\right)$, $s_j = \frac{r_j p_0(1-m_{jr})}{r_j-p_0}\left(\frac{r_j}{p_0} - Log\frac{r_j}{p_0} - 1\right)$.

These eight equations are solved for eight unknowns: $\{p_0, r_i, r_j, p_m, m_{ira}, m_{irb}, m_{iv}, m_{jr}\}$. From indifference equations, we get $p_0 = \frac{\alpha_i v}{\alpha_i+\gamma+\beta}$. We find $x = r_j/p_0$ from $x - Log(x) - 1 = \frac{\beta s_j}{\alpha_i v}$. Then, $r_j = p_0 x = \frac{\alpha_i v}{\alpha_i+\gamma+\beta}x$. All other variables are obtained respectively. Profits are $\pi_i = \alpha_i v$ and $\pi_j = \frac{\alpha_i\beta^2(\alpha_j+\gamma+\beta)r_j v}{(\alpha_i+\gamma+\beta)(\beta^2+\gamma^2+\beta\gamma)r_j - \alpha_i\gamma(\gamma+\beta)v}$. We have $\frac{\partial x}{\partial s_j} > 0$. Therefore, $\frac{\partial r_j}{\partial s_j} > 0$. Thus, $\frac{\partial \pi_j}{\partial s_j} = \frac{\partial \pi_j}{\partial r_j}\frac{\partial r_j}{\partial s_j} < 0$. Hence, Store j's profit decreases with s_j, whereas $\pi_i = \alpha_i v$ does not depend on the search costs.

Derivation of Store Best-Response Functions

Denote stores' best-response functions by $s_1^*(s_2)$ and $s_2^*(s_1)$. Figure A2 shows these functions.

Store 1's Best-Response Function

When s_2 is small, Store 1's profit is increasing in s_1 in the region of Case $6b$, and is constant in other cases. We assumed facilitation of search has an infinitesimal cost, implying a store prefers a higher s to a lower one when both levels generate the same revenue. Therefore, Store 1's best response is to set $s_1^*(s_2) = s_0$ when s_2 is sufficiently small. Store 1's profit is then $\pi_1 = \frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{\alpha_2+\gamma+\beta}$ (the profit in Case $4b$). When s_2 increases, Store 1's incentive to undercut s_2 also increases. By undercutting s_2 (Case $6a$), Store 1's profit becomes $\pi_1 = \beta s_2 \left(\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - 1\right)^{-1}$. Denote the level of s_2 that makes Store 1 indifferent between undercutting s_1 and staying at $s_1 = s_0$ by s_{2f}. Then, we calculate s_{2f} from

$$\frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{\alpha_2+\gamma+\beta} = \beta s_{2f}\left(\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - 1\right)^{-1} \Rightarrow s_{2f} = \frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{\beta(\alpha_2+\beta+\gamma)}\left(\frac{\beta}{\alpha_1+\gamma} - Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma}\right).$$

Therefore, for $s_2 > s_{2f}$, Store 1 facilitates and attracts all costly-searchers. The facilitation should be just enough to attract all of the costly-searchers. This implies the best response is on the boundary of Case $5a$ and Case $6a$. Recall that in Case $5a$, we had $\frac{s_1}{s_2} = \frac{\alpha_2+\gamma_2+\beta}{\alpha_1+\gamma_1+\beta}$. Plugging in $\gamma_1 = \gamma$ and $\gamma_2 = 0$, we get $s_1^*(s_2) = \frac{\alpha_2+\beta}{\alpha_1+\gamma+\beta}s_2$. When s_2 increases, the type of price distribution changes and stores extend their upper bound of distributions to v. We equate $r_j = v$ in Case $6a$ to obtain the boundary of Case $6a$ and Case $4a$:

$$v = \frac{\beta s_{2b}}{\alpha_1+\gamma}\left(\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - 1\right)^{-1} \Rightarrow\ s_{2v} = \frac{(\alpha_1+\gamma)v}{\beta}\left(\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} - 1\right).$$

For $s_2 > s_{2v}$, Store 1's best response does not change with s_2 and stays constant at $s_1^*(s_{2v}) = \underline{s_1} = \frac{\alpha_2+\beta}{\alpha_1+\gamma+\beta}s_{2v}$. To summarize, Store 1's best-response function is

$$s_1^*(s_2) = \begin{cases} s_0 & s_2 \leq s_{2f} \\ \frac{\alpha_2+\beta}{\alpha_1+\gamma+\beta}s_2 & s_{2f} < s_2 \leq s_{2v}. \\ \underline{s_1} & s_2 > s_{2v} \end{cases} \quad (4)$$

Store 2's Best-Response Function

We denote Store 2's best-response function by $s_2^*(s_1)$. When s_1 is small enough, Store 2 does not facilitate: $s_2^*(s_1) = s_0$. Store 2's profit is (in Case $4a$) $\pi_2 = \frac{(\alpha_1+\gamma)(\alpha_2+\beta)v}{\alpha_1+\gamma+\beta}$. When s_1 increases, Store 2 may want to facilitate search—not to steal the costly-searchers from Store 1, but rather to take the portion of costly-searchers who stay out of the market (γ_0). Therefore, Store 2 sets the search cost at the boundary of Case $5a$ and Case $4a$, where γ_0 becomes zero. Decreasing s_2 further is not beneficial for Store 2, because doing so increases the competition and derives the profits down. At the boundary of Case $5a$ and Case $4a$, we have $\gamma_0 = 0$ and $r_1 = r_2 = v$. Therefore, we plug $r = v$ into Case $5a$ to get the boundary of Case $5a$ and Case $4a$:

$$v = \frac{\beta s_2(s_1+s_2)}{s_2-\beta s_1}\left(\frac{\beta(s_1+s_2)}{s_2-\beta s_1} - Log\frac{(1+\beta)s_2}{s_2-\beta s_1}\right)^{-1} \Rightarrow s_{2m}^*(s_1). \quad (5)$$

Store 2's profit at this boundary is $\pi_2 = \beta s_1\left(\frac{\beta(s_1+s_2)}{s_2-\beta s_1} - Log\frac{(1+\beta)s_2}{s_2-\beta s_1}\right)^{-1}$. Store 2's profit is increasing in Case 5a and decreasing in Case 4a. Hence, the indifference level $s_{1f} = \underline{s_1}$.

When s_1 increases further, Store 2's incentive to undercut s_1 and get all costly-searchers increases. When $s_1 > s_{1m}$, Store 2 undercuts Store 1's search cost and receives all costly-searchers. To calculate the indifference level s_{1m}, we know s_{1m} should be on the boundary of Case $5a$ and Case $4a$. Moreover, Store 2 should be indifferent between sharing costly-searchers with Store 1 (boundary of Case $5a$ and Case $4a$) and getting all the costly-searchers (boundary of Case $5b$ and Case $6b$). These two conditions together yield

$$s_{1m} = \frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{\beta(\alpha_2+\gamma+\beta)}\left(\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma} - Log\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma} - 1\right) = \underline{s}_2.$$

For $s_1 > s_{1m}$, Store 2's best response is at the boundary of Case $5b$ and Case $6b$: $s_2^*(s_1) = \frac{\alpha_1+\beta}{\alpha_2+\gamma+\beta}s_1$. Finally, for $s_1 > s_{1v}$, stores charge the top price. Store 2's best response is constant at $s_2^*(s_1) = \underline{s}_2 = \frac{\alpha_1+\beta}{\alpha_2+\gamma+\beta}s_{1v} = s_{1m}$ where, $s_{1v} = \frac{(\alpha_2+\gamma)v}{\beta}\left(\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma} - Log\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma} - 1\right)$. We summarize Store 2's best response as

$$s_2^*(s_1) = \begin{cases} s_0 & s_1 \le s_{1f} \\ s_{2m}^*(s_1) & s_{1f} < s_1 \le s_{1m} \\ \frac{\alpha_1+\beta}{\alpha_2+\gamma+\beta}s_1 & s_{1m} < s_1 \le s_{1v} \\ \underline{s}_2 & s_1 > s_{1v} \end{cases}. \tag{6}$$

Proof of Proposition 1

To show that $(s_1^*, s_2^*) = (\underline{s}_1, s_0)$ is NE (as shown in Figure A2), we need to prove that stores are at the best response. Because s_0 was assumed to be high enough, Store 1's best-response function in (4) shows that $s_1^*(s_0) = \underline{s}_1$. To show that Store 2 is at its best response, we show that Store 2 does not want to facilitate when $s_1 = s_{1f} = \underline{s}_1$; that is, we should prove $\pi_2(\underline{s}_1, s_0) > \pi_2(\underline{s}_1, 0)$. Store 2's profit when it facilitates falls in the Case $8b$ region, so $\pi_2(\underline{s}_1, 0) = \frac{(\alpha_2+\gamma+\beta)\alpha_1 v}{\alpha_1+\beta}$. We have

$$\alpha_1 < \beta^2 + \alpha_2\beta + \alpha_1\alpha_2 \Leftrightarrow 0 < (1-\beta-\alpha_1-\alpha_2)(\beta^2+\alpha_2\beta+\alpha_1\alpha_2-\alpha_1) \Leftrightarrow$$

$$\frac{(1-\alpha_1)\alpha_1}{(\alpha_1+\beta)} < \frac{(1-\alpha_2-\beta)(\alpha_2+\beta)}{(1-\alpha_2)} \Leftrightarrow \frac{(\alpha_2+\gamma+\beta)\alpha_1 v}{\alpha_1+\beta} < \frac{(\alpha_1+\gamma)(\alpha_2+\beta)v}{\alpha_1+\gamma+\beta} \Leftrightarrow \pi_2(\underline{s}_1, 0) < \pi_2(\underline{s}_1, s_0).$$

Therefore, $s_2^*(\underline{s}_1) = s_0$. To show the equilibrium is unique, we should prove the other possible candidate $(s_1^*, s_2^*) = (s_0, \underline{s}_2)$ is not a NE. To do so, we prove that Store 1 in fact does facilitate when $s_2 = \underline{s}_2$. In other words, $\pi_1(0, \underline{s}_2) > \pi_1(s_0, \underline{s}_2)$:

$$\pi_1(0, \underline{s}_2) > \pi_1(s_0, \underline{s}_2) \Leftrightarrow (\alpha_1+\gamma)v > \frac{(\alpha_2+\gamma)(\alpha_1+\beta)v}{\alpha_2+\gamma+\beta} \Leftrightarrow (\alpha_1-\alpha_2)\beta + \alpha_2\gamma + \gamma^2 > 0,$$

which is always correct because $\alpha_1 > \alpha_2$. Therefore, $(s_1^*, s_2^*) = (s_0, \underline{s}_2)$ in not NE. Therefore, the only equilibrium is when Store 1 facilitates search and Store 2 does not. We note that the existence of a sufficient mass of non-loyals in the market ($\gamma + \beta > 1 - \beta^2$ condition) implies

$$\gamma + \beta > 1 - \beta^2 \Leftrightarrow \alpha_1 + \alpha_2 < \beta^2 \Rightarrow \alpha_1 + \alpha_2 < \beta^2 + \alpha_2 + \alpha_2\beta + \alpha_1\alpha_2 \Leftrightarrow \alpha_1 < \beta^2 + \alpha_2\beta + \alpha_1\alpha_2,$$

which is the condition in the proposition.

Illustration of the Best-Response Functions for Small shopper Size β

Best-response functions for β that do not satisfy the condition in Proposition 1 are illustrated in Figure A3. Because the best-response functions do not intersect, no Nash equilibrium in pure strategies for search costs exists. To understand the incentives in this case, consider how the best responses react to each other: if Store 1 sets its search cost high, Store 2 chooses a lower search cost just sufficient to obtain the demand from all costly-searchers, but at a level that Store 1 strictly prefers to undercut to win over all the costly-searchers. The best response of Store 2 is then to further fight for the costly-searchers and decrease its search cost to a lower level. However, Store 1 then wants to give up and increase its search cost to high level, and we are back to the high search cost at Store 1 and the cycle continues. The arrows in Figure A3 illustrate the cycle of best responses. It is easy to see that starting from any pair of search costs, best-response reactions quickly settle into this cycle.

Note that when the condition in the statement of Proposition 1 is satisfied, in order to attract all the costly-searchers, Store 1 has to set a search cost so low that Store 2 does not want to decrease its search cost to successfully attract the costly-searchers, and the cycle stops at the point where Store 2's search cost is high and Store 1's search cost is low (but just sufficiently to entice all the costly-searchers to its store).

Proof of Proposition 2

As proven in Proposition 1, $(s_1^*, s_2^*) = (\underline{s}_1, s_0)$ is the unique equilibrium with $\pi_1^* = (\alpha_1 + \gamma)v$ and $\pi_2^* = \frac{\alpha_2+\beta}{\alpha_1+\gamma+\beta}(\alpha_1 + \gamma)v$. Further, $\Delta\pi^* \equiv \pi_1^* - \pi_2^* = \left(1 - \frac{\alpha_2+\beta}{\alpha_1+\beta+\gamma}\right)(\alpha_1 + \gamma)v = \frac{(\Delta\alpha+\gamma)(\Delta\alpha+1-\beta+\gamma)v}{\Delta\alpha+1+\beta+\gamma}$, where $\Delta\alpha \equiv \alpha_1 - \alpha_2$.

Since $\frac{d\Delta\pi^*}{d\Delta\alpha} = \frac{1-\beta^2+\Delta\alpha^2+2\gamma+2\beta\gamma+\gamma^2+\Delta\alpha(2+2\beta+2\gamma)}{(\Delta\alpha+1+\beta+\gamma)^2}v > 0$, we have that $\Delta\pi^*$ is increasing in $\Delta\alpha$. Thus, $\Delta\pi^* > \Delta\pi^*|_{\Delta\alpha=0}$. Therefore, $\pi_1^* - \pi_2^* > \frac{(1-\beta+\gamma)\gamma v}{1+\beta+\gamma}$. The last inequality of the proposition now follows if one notes that $1 - \beta + \gamma > \gamma$ (since $\beta < 1$) and $1 + \beta + \gamma < 2$ (since $\beta + \gamma < 1$), and the proof of the proposition is complete.

Proof of Proposition 3

When $\alpha_1 < \beta^2 + \alpha_2\beta + \alpha_1\alpha_2$, the equilibrium characterized in Proposition 1 occurs with $(s_1^*, s_2^*) = (\underline{s}_1, s_0)$, which was at the border of Case $3a$ and Case $2a$ in which all costly-searchers visit and buy from Store 1 ($\gamma_1 = \gamma, \gamma_2 = 0$), and we have $F_i(p) = \frac{\alpha_j+\gamma_j+\beta}{\beta}\left(1 - \frac{p_0}{p}\right)$ with $p_0 = \frac{(\alpha_1+\gamma)v}{\alpha_1+\gamma+\beta}$. To calculate the number of shoppers who buy from Store (β_1), we calculate the probability that Store 1's price is lower:

$\beta_1 = \beta \Pr(p_1 < p_2) = \int_{p_0}^{v} F_1(p)dF_2(p) \ = \frac{(\alpha_1+\gamma+\beta)(\alpha_2+\beta)}{\beta}\int_{p_0}^{v}\left(1 - \frac{p_0}{p}\right)d\left(1 - \frac{p_0}{p}\right) =$

$\frac{(\alpha_1+\gamma+\beta)(\alpha_2+\beta)}{2\beta}\left(1-\frac{p_0}{v}\right)^2 = \frac{\beta(\alpha_2+\beta)}{2(\alpha_1+\gamma+\beta)}$.

Thus, we have $\beta_2 = \beta - \beta_1 = \frac{\beta}{2}\left(2 - \frac{(\alpha_2+\beta)}{(\alpha_1+\gamma+\beta)}\right) > \beta_1$ because $\frac{(\alpha_2+\beta)}{(\alpha_1+\gamma+\beta)} < 1$. Store 1's market share is $\alpha_1 + \beta_1 + \gamma = \alpha_1 + \frac{\beta(\alpha_2+\beta)}{2(\alpha_1+\gamma+\beta)} + \gamma = \frac{2\alpha_1^2+\beta^2+2\gamma^2+2\alpha_1\beta+\alpha_2\beta+4\alpha_1\gamma+2\beta\gamma}{2(\alpha_1+\gamma+\beta)}$, whereas Store 2's market share is $\alpha_2 + \beta_2 = \alpha_2 + \frac{\beta}{2}\left(\frac{(2\alpha_1-\alpha_2+\beta+2\gamma)}{(\alpha_1+\gamma+\beta)}\right) = \frac{2\alpha_1\alpha_2+2\alpha_2\gamma+\alpha_2\beta+2\alpha_1\beta+2\beta\gamma+\beta^2}{2(\alpha_1+\gamma+\beta)}$. So, the difference in market shares is

$$MS_1 - MS_2 = \frac{2\gamma^2+2\gamma(2\alpha_1-\alpha_2)+2\alpha_1(\alpha_1-\alpha_2)}{2(\alpha_1+\gamma+\beta)} > 0.$$

The difference in the market share of non-loyals is $(\beta_1+\gamma)-(\beta_2)$. We plug in β_1 and β_2 and simplify to obtain $(\beta_1+\gamma)-\beta_2 = \frac{\alpha_2(2\beta-\gamma)+\beta^2-\beta+\gamma}{1-\alpha_2}$, which is positive if and only if $\alpha_2(2\beta-\gamma)+\beta^2-\beta+\gamma$, resulting in Store 1's market share among non-loyals being higher than Store 2's.

Proof of Proposition 4

In Case $3a$—to which the equilibrium store search costs s_1^*, s_2^* belong—we have $F_i(p) = \frac{\alpha_j+\gamma_j+\beta}{\beta}\left(1-\frac{p_0}{p}\right)$ with $\gamma_1 = \gamma, \gamma_2 = 0$, and $p_0 = \frac{(\alpha_1+\gamma)v}{\alpha_1+\gamma+\beta}$. Only Store 1 has a mass point at v with size $m_{1v} = \frac{\alpha_1-\alpha_2+\gamma}{\alpha_1+\beta+\gamma}$. We have $\frac{\partial m_{1v}}{\partial \alpha_1} = \frac{\partial m_{1v}}{\partial \gamma} = \frac{\beta+\alpha_2}{(\alpha_1+\beta+\gamma)^2} > 0$ and $\frac{\partial p_0}{\partial \alpha_1} = \frac{\beta v}{(\alpha_1+\beta+\gamma)^2}$. The average prices are

$$E_i(p) = \int_{p_0}^{v} p\,dF_i(p) = \frac{\alpha_j+\gamma_j+\beta}{\beta}\int_{p_0}^{v^-} p\,d\left(1-\frac{p_0}{p}\right) + v m_{iv} = \frac{\alpha_j+\gamma_j+\beta}{\beta} p_0 Log\frac{v}{p_0} + v m_{iv}.$$

So, $E_1(p) = \frac{(\alpha_1+\gamma)(\alpha_2+\beta)v}{(\alpha_1+\gamma+\beta)\beta} Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma} + \frac{\alpha_1-\alpha_2+\gamma}{\alpha_1+\beta+\gamma}v$ and $E_2(p) = \frac{(\alpha_1+\gamma)v}{\beta} Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma}$. Therefore, $E_1(p) - E_2(p) = \left(\frac{\alpha_1-\alpha_2+\gamma}{\alpha_1+\beta+\gamma}\right)\frac{\alpha_1+\gamma}{\beta}v\left(\frac{\beta}{\alpha_1+\gamma} - Log\left(1+\frac{\beta}{\alpha_1+\gamma}\right)\right)$, which is positive because $x > Log(1+x)$. Finally,

$$E_1(p|p<v) = \frac{\int_{p_0}^{v^-} p\,dF_1(p)}{F_1(v)} = \frac{\alpha_2+\beta}{\beta\left(1-\frac{\alpha_1-\alpha_2+\gamma}{\alpha_1+\beta+\gamma}\right)} p_0 Log\frac{v}{p_0} = \frac{\alpha_1+\beta+\gamma}{\beta} p_0 Log\frac{v}{p_0} = \frac{(\alpha_1+\gamma)v}{\beta} Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma},$$

$$E_2(p|p<v) = \frac{\int_{p_0}^{v^-} p\,dF_2(p)}{F_2(v)} = \frac{\alpha_1+\beta+\gamma}{\beta} p_0 Log\frac{v}{p_0} = \frac{(\alpha_1+\gamma)v}{\beta} Log\frac{\alpha_1+\gamma+\beta}{\alpha_1+\gamma}.$$

Thus, we have $E_1(p|p<v) = E_2(p|p<v)$.

Proof of Proposition 5

Store 1's profit should be higher with facilitation at $s_2 = \underline{s_2}$, i.e., $\pi_1\left(s_1^*\left(\underline{s_2}\right), \underline{s_2}\right) > \pi_1\left(s_0, \underline{s_2}\right)$. We have

$$\pi_1\left(s_1^*\left(\underline{s_2}\right),\underline{s_2}\right)=\beta\underline{s_2}\left(\frac{\alpha_1+\beta+\gamma}{\alpha_1+\gamma}-Log\frac{\alpha_1+\beta+\gamma}{\alpha_1+\gamma}-1\right)^{-1}-c\left(s_0-s_1^*\left(\underline{s_2}\right)\right),$$

where $\underline{s_2}=\frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{\beta(\alpha_2+\gamma+\beta)}\left(\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-Log\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-1\right)$. Moreover, $\pi_1(s_0,\underline{s_2})=\frac{(\alpha_2+\gamma)(\alpha_1+\beta)v}{\alpha_2+\gamma+\beta}$. Thus, $\pi_1\left(s_1^*\left(\underline{s_2}\right),\underline{s_2}\right)>\pi_1(s_0,\underline{s_2})\Leftrightarrow$

$$\frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{(\alpha_2+\gamma+\beta)}\left(\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-Log\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-1\right)\left(\frac{\alpha_1+\beta+\gamma}{\alpha_1+\gamma}-Log\frac{\alpha_1+\beta+\gamma}{\alpha_1+\gamma}-1\right)^{-1}-c\left(s_0-s_1^*\left(\underline{s_2}\right)\right)>\frac{(\alpha_2+\gamma)(\alpha_1+\beta)v}{\alpha_2+\gamma+\beta}$$

Hence, $c<c^*\equiv\frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{(\alpha_2+\beta+\gamma)\left(s_0-s_1^*\left(\underline{s_2}\right)\right)}\left(\left(\frac{\beta}{\alpha_2+\gamma}-Log\frac{\alpha_2+\beta+\gamma}{\alpha_2+\gamma}\right)\left(\frac{\beta}{\alpha_1+\gamma}-Log\frac{\alpha_1+\beta+\gamma}{\alpha_1+\gamma}\right)^{-1}-1\right)$.

Proof of Proposition 6

When $s_M<\frac{\alpha_1+\beta}{\alpha_2+\gamma+\beta}\underline{s_2}$, we have $\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}<\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}$. Because, $x-Log(x)-1$ is increasing, we have

$$\left(\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-Log\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-1\right)<\left(\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-Log\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-1\right)\Leftrightarrow$$

$$1<\left(\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-Log\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-1\right)\left(\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-Log\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-1\right)^{-1}\Leftrightarrow$$

$$\frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{(\alpha_2+\gamma+\beta)}<\beta\frac{(\alpha_1+\beta)(\alpha_2+\gamma)v}{\beta(\alpha_2+\gamma+\beta)}\left(\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-Log\frac{\alpha_2+\gamma+\beta}{\alpha_2+\gamma}-1\right)\left(\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-Log\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-1\right)^{-1}\Leftrightarrow$$

$$\pi_1(s_0,\underline{s_2})=\frac{(\alpha_2+\gamma)(\alpha_1+\beta)v}{\alpha_2+\gamma+\beta}<\beta\underline{s_2}\left(\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-Log\frac{(1+\beta)\underline{s_2}}{\underline{s_2}-\beta s_M}-1\right)^{-1}=\pi_1(s_M,\underline{s_2})$$

Thus, Store 1's best response to $s_2=\underline{s_2}$ is to facilitate. Hence, $s_{2f}<\underline{s_2}$ as illustrated in Figure 7. Thus, $s_1^*(s_2)=s_0$ for sufficiently small s_2, then $s_1^*(s_2)=s_M$ for $s_2<s_{2f}$. We note that because Store 1's profit does not change with s_1 in Case $4a$, the Store 1 does not want to facilitate fully and the best response will lie on the boundary of Case $4a$ and Case $5a$ ($s_1^*(s_2)>s_M$) until s_2 is sufficiently high such that Store 2 does not receive any costly-searcher in which case the best response is again $s_1^*(s_2)=s_M$. This will create the multiplicity of equilibria illustrated in Figure 7. Denote the intersection of s_M with the boundary of Case $4a$-Case $5a$ and the boundary of Case $2a$-Case $4a$ by s_{2c} and s_{2d}, respectively. Thus, when $s_{2c}<s_2<s_{2d}$, then $s_1^*(s_2)$ as well as $s_2^*(s_1)$ are on the boundary of Case $4a$-Case $5a$:

$$\text{NE: } s_2^*=v\left(1-\frac{s_2^*-\beta s_1^*}{\beta(s_1^*+s_2^*)}Log\frac{(1+\beta)s_2^*}{s_2^*-\beta s_1^*}\right)\quad\text{for } s_{2c}<s_2<s_{2d},$$

where s_{2c} solves $s_{2c}=v\left(1-\frac{s_{2c}-\beta s_M}{\beta(s_M+s_{2c})}Log\frac{(1+\beta)s_{2c}}{s_{2c}-\beta s_M}\right)$ and s_{2d} solves $s_M=\frac{(\alpha_2+\beta)s_{d2}-\beta s_M}{\beta s_{2d}}v\left(\frac{\beta s_M}{(\alpha_2+\beta)s_{2d}-\beta s_M}-Log\frac{(\alpha_2+\beta)s_{2d}}{(\alpha_2+\beta)s_{2d}-\beta s_M}\right)$. We have $s_M<s_1^*<s_2^*<s_0$ because in Case 5 $s_1<s_2<s_0$. To show that $\gamma_2^*<\gamma_1^*$, we note that in Case $5a$ we have $\frac{s_1}{s_2}=\frac{\alpha_2+\gamma_2+\beta}{\alpha_1+\gamma_1+\beta}$. If $\gamma_1=\gamma_2=\gamma/2$, then at the boundary of Case $4a$-Case $5a$, we have $s_{1h} = \frac{\alpha_2+\beta+\gamma/2}{\alpha_1+\beta+\gamma/2}\frac{\alpha_1+\gamma/2}{\beta} v\left(\frac{\beta}{\alpha_1+\gamma/2} - Log\frac{\alpha_1+\beta+\gamma/2}{\alpha_1+\gamma/2}\right)$ and $s_{2h} = \frac{\alpha_1+\gamma/2}{\beta} v\left(\frac{\beta}{\alpha_1+\gamma/2} - Log\frac{\alpha_1+\beta+\gamma/2}{\alpha_1+\gamma/2}\right)$, where s_{1h} and s_{2h} indicate levels at which $\gamma_1 = \gamma_2$ at the boundary of Case $4a$-Case $5a$. If $s_{2h} = s_{2d}$, then $s_M = \frac{(\alpha_1+\gamma/2)(\alpha_2+\beta)v}{\beta(\alpha_1+\beta+\gamma/2)}\left(\frac{\beta}{\alpha_1+\gamma/2} - Log\frac{\alpha_1+\beta+\gamma/2}{\alpha_1+\gamma/2}\right)$. Thus, if condition $s_M < \tilde{s} = \frac{(\alpha_1+\gamma/2)(\alpha_2+\beta)v}{\beta(\alpha_1+\beta+\gamma/2)}\left(\frac{\beta}{\alpha_1+\gamma/2} - Log\frac{\alpha_1+\beta+\gamma/2}{\alpha_1+\gamma/2}\right)$ holds, then $s_{2h} > s_{2d}$ and thus $\gamma_2^* < \gamma_1^*$.

Proof of Proposition 7

When K is sufficiently small, in equilibrium stores invest to split the costly-searchers. Thus, we have

$$\frac{C_2^*}{C_2^*+K} + \frac{C_1^*}{C_1^*+K} = 1 \implies C_2^* = \frac{K^2}{C_1^*} = \frac{K^2}{-K+\sqrt{K\gamma v}}.$$

Moreover, Store 1 gets most of the costly-searchers and it does not have incentive to invest more. We have $\pi_1 = (\alpha_1+\gamma_1)v - C_1 = \left(\alpha_1 + \frac{C_1}{C_1+K}\gamma\right)v - C_1$. Thus, $\frac{d\pi_1}{dC_1} = 0 \;\; \frac{K\gamma v}{(C_1+K)^2} - 1 = 0 \implies C_1^* = -K + \sqrt{K\gamma v}$. From these two equations we get $\gamma_1^* = \left(1 - \frac{K}{\sqrt{K\gamma v}}\right)\gamma$ and $\gamma_2^* = \frac{K}{\sqrt{K\gamma v}}\gamma$. We see that as $K \to 0$, $\gamma_1^* \to \gamma$. Moreover, $\gamma_1^* > \gamma_2^* \Leftrightarrow K < \frac{\gamma v}{4}$.

Moreover, we have $K < \frac{(\alpha_1-\alpha_2)^2 v}{\gamma} \Leftrightarrow \frac{K\gamma}{\sqrt{K\gamma v}} < \alpha_1 - \alpha_2 \Leftrightarrow \alpha_2 + \gamma < \alpha_1 + \frac{-K+\sqrt{K\gamma v}}{\sqrt{K\gamma v}}\gamma \Leftrightarrow \alpha_2 + \gamma < \alpha_1 + \frac{C_1^*}{C_1^*+K}\gamma$. Thus, the sufficient condition $K < \frac{(\alpha_1-\alpha_2)^2 v}{\gamma}$ guarantees that, given Store 1's optimal investment C_1^*, Store 2 cannot become the store with loyalty advantage even if it invests infinitely to reach to all costly-searchers.

Proof of Proposition 8

We have $\alpha_H = \frac{\alpha t_H}{1+t_H+t_L}$ and $\alpha_L = \frac{\alpha t_L}{1+t_H+t_L}$. The condition $\alpha^2 + \alpha(\beta+\gamma) < \beta^2$ (a part of "high enough β" assumption) guarantees the pure-strategy NE in search-cost decisions (second stage of the game) exist for any investment level t_L and t_H. Then, the H-type and L-type store profits are (see Proposition 1 and 2)

$$\pi_H(t_H, t_L) = (\alpha_H + \gamma)v - t_H = \left(\frac{\alpha t_H}{t_H+t_L+c} + \gamma\right)v - t_H$$

$$\pi_L(t_H, t_L) = \frac{\alpha_L+\beta}{\alpha_H+\beta+\gamma}(\alpha_H+\gamma)v - t_L = \frac{(\alpha+\beta)t_L+\beta t_H+\beta}{(\beta+\gamma)t_L+t_H+(\beta+\gamma)}\left(\frac{\alpha t_H}{t_H+t_L+1} + \gamma\right)v - t_L.$$

We take the first derivatives of $\pi_H(t_H, t_L)$, $\frac{\partial \pi_H}{\partial t_H} = \frac{\alpha v(t_L+1)}{(t_H+t_L+1)^2} - 1$. The first FOC yields $t_H = \sqrt{\alpha v(t_L+1)} - t_L - 1$. We show that $(t_L^*, t_H^*) = (0, \sqrt{\alpha v} - 1)$ is NE. We need to assume $\alpha v > 1$ ("high enough v") so that $t_H^* = \sqrt{\alpha v} - 1 > 0$. Given equilibrium investment, we get $(\alpha_L^*, \alpha_H^*) = \left(0, \alpha - \frac{\sqrt{\alpha v c}}{v}\right)$. These investment strategies result in equilibrium profits of

$$\pi_L^* = \beta v\left(1 - \frac{\beta v}{(\alpha+\beta+\gamma)v - \sqrt{\alpha v}}\right), \pi_H^* = (\alpha+\gamma)v + 1 - 2\sqrt{\alpha v}.$$

From FOC, we know the H-Type store is at its best response. The L-type store should not have an incentive to invest more than H-type in order to get the costly-searchers. Thus, the L type's equilibrium profit π_L^* should be higher than the profit it would obtain if it deviated to investment level $t_{dev} = \sqrt{\alpha v(t_H^* + 1)} - t_H^* - 1 = \sqrt{\alpha v\sqrt{\alpha v}} - \sqrt{\alpha v}$. Therefore, we need to have $\pi_L^* = \beta v\left(1 - \frac{\beta v}{(\alpha+\beta+\gamma)v - \sqrt{\alpha v}}\right) > \pi_H(t_{dev}, t_H^*) = \left(\frac{\alpha t_{dev}}{t_{dev}+t_H^*+c} + \gamma\right)v - t_{dev} = (\alpha+\gamma)v - 2\sqrt{\alpha v\sqrt{\alpha v}} + \sqrt{\alpha v} \Longrightarrow \beta > \frac{\left((\alpha+\gamma)v - \sqrt{\alpha v}\right)\left((\alpha+\gamma)v + \sqrt{\alpha v} - 2\sqrt{\alpha v\sqrt{\alpha v}}\right)}{2v\left(\sqrt{\alpha v\sqrt{\alpha v}} - \sqrt{\alpha v}\right)}$. Thus, we obtain the claimed Nash equilibrium when β is high enough.

Appendix Figures and Tables

Figure A1. Regions of Pricing Cases For $\alpha_1 = 0.2, \alpha_2 = 0.1, \beta = 0.5, \gamma = 0.2$

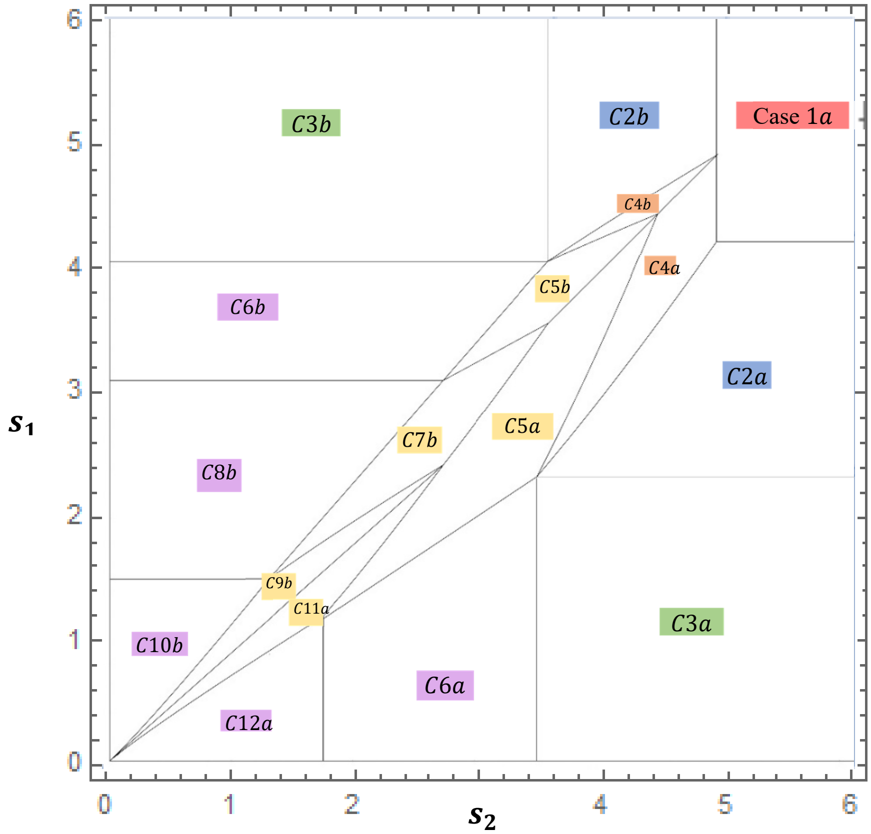
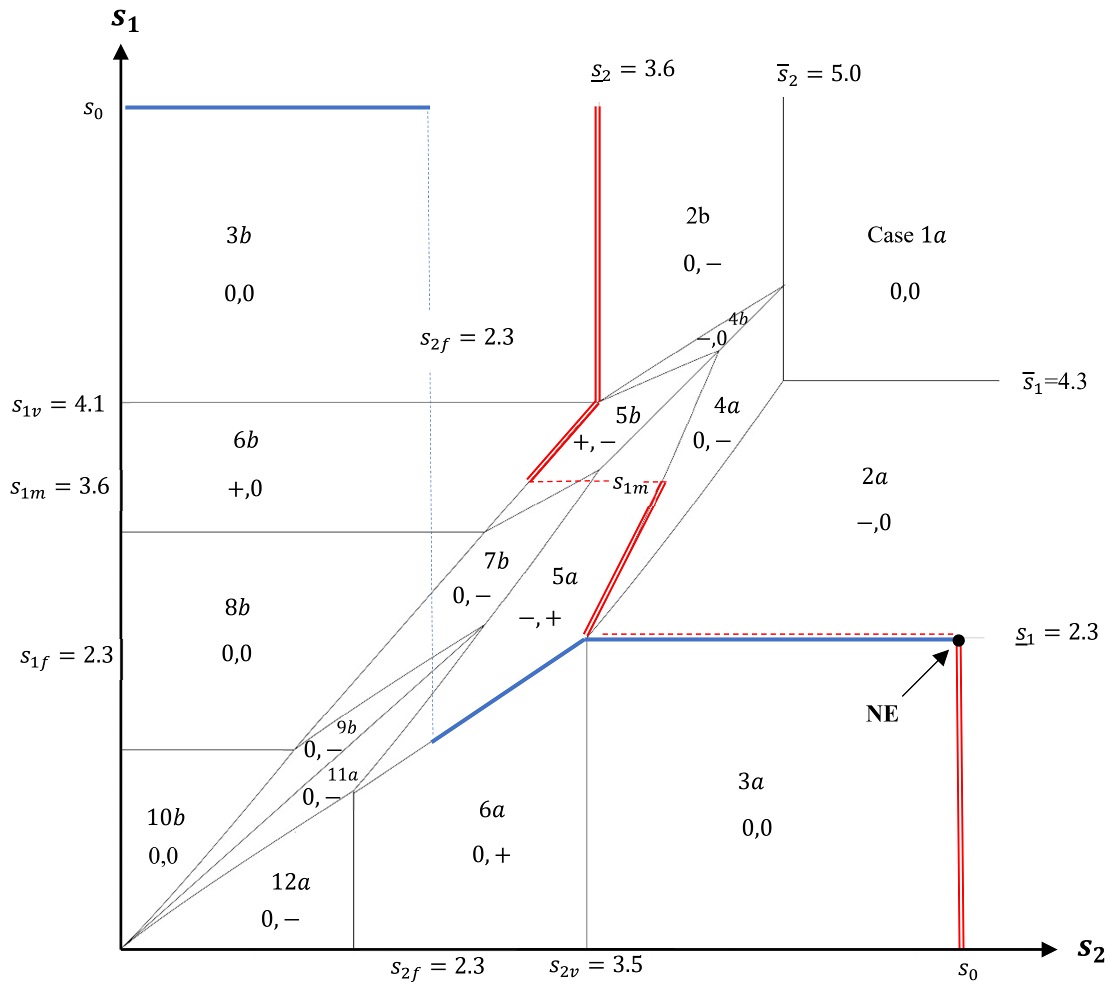


Table A1. Reservation Prices and Costly-Searchers Behavior in Pricing Cases

Case	Reservation Prices	Costly-Searcher First Search			Price Distribution
Case 1	$\boldsymbol{v < r_i, v < r_j}$	$\gamma_0 = \gamma$	$\gamma_i = 0$	$\gamma_j = 0$	Type A
Case 2	$\boldsymbol{r_i = v < r_j}$	$\gamma_0 > 0$	$\gamma_i > 0$	$\gamma_j = 0$	Type A
Case 3	$\boldsymbol{r_i < v < r_j}$	$\gamma_0 = 0$	$\gamma_i = \gamma$	$\gamma_j = 0$	Type A
Case 4	$\boldsymbol{r_i = r_j = v}$	$\gamma_0 > 0$	$\gamma_i > 0$	$\gamma_j > 0$	Type A
Case 5	$\boldsymbol{r_i = r_j < v}$	$\gamma_0 = 0$	$\gamma_i > 0$	$\gamma_j > 0$	Type B
Case 7	$\boldsymbol{r_i = r_j < v}$	$\gamma_0 = 0$	$\gamma_i > 0$	$\gamma_j > 0$	Type C
Case 9	$\boldsymbol{r_i = r_j < v}$	$\gamma_0 = 0$	$\gamma_i > 0$	$\gamma_j > 0$	Type D with $m_{iv} = 0$
Case 11	$\boldsymbol{r_i = r_j < v}$	$\gamma_0 = 0$	$\gamma_i > 0$	$\gamma_j > 0$	Type D with $m_{jv} = 0$
Case 6	$\boldsymbol{r_i < r_j < v}$	$\gamma_0 = 0$	$\gamma_i = \gamma$	$\gamma_j = 0$	Type B
Case 8	$\boldsymbol{r_i < r_j < v}$	$\gamma_0 = 0$	$\gamma_i = \gamma$	$\gamma_j = 0$	Type C
Case 10	$\boldsymbol{r_i < r_j < v}$	$\gamma_0 = 0$	$\gamma_i = \gamma$	$\gamma_j = 0$	Type D with $m_{iv} = 0$
Case 12	$\boldsymbol{r_i < r_j < v}$	$\gamma_0 = 0$	$\gamma_i = \gamma$	$\gamma_j = 0$	Type D with $m_{jv} = 0$

Figure A2. Regions of Pricing Cases and Best Response Functions
For $\alpha_1 = 0.2, \alpha_2 = 0.1, \beta = 0.5, \gamma = 0.2$



Notes:

I. The numbers followed by a or b are Case numbers.

II. The two of $0, +$ or $-$ signs separated by comma are the signs of $\frac{\partial \pi_1}{\partial s_1}$ and $\frac{\partial \pi_2}{\partial s_2}$ in each region.

III. The thick-single-line plot in blue is the best response function of Store 1. The thick-double-line plot in red is the best response function of Store 2.

Figure A3. Store 1 (Blue/Single-Line) and Store 2 (Red/Double-Line) Best Response Functions For $\alpha_1 = 0.2, \alpha_2 = 0.1, \beta = 0.3,\ \gamma = 0.4$