Appendix

PROPOSITION 14. *Consider a trajectory of size $x(t)$ and balance $\gamma(t)$ that are continuously differentiable. Then it can be uniquely determined that the stock of supply is*

$$s(t) = \beta_1^s \gamma(t) x(t) \tag{29}$$

the stock of demand is

$$d(t) = \beta_1^d (1 - \gamma(t)) x(t) \tag{30}$$

the price paid to the seller is

$$p_s(t) = c + \frac{\gamma'(t)x(t) + \gamma(t)x'(t) + \beta_0^s \gamma(t) x(t)}{g(\gamma(t), x(t))} \tag{31}$$

and the price charged to the buyer is

$$p_d(t) = v - \frac{-\gamma'(t)x(t) + (1-\gamma(t))x'(t) + \beta_0^d (1-\gamma(t))x(t)}{g(\gamma(t), x(t))} \tag{32}$$

Proof. First, note that (30) and (29) are directly given by the definition of $\gamma(t)$. Next, by (6),

$$p_s = c + \frac{s' + \beta_0^s s}{\beta_1^s g(s,d)}$$

Since $s(t) = \beta_1^s \gamma(t) x(t)$ and $\gamma(t), x(t)$ are both differentiable, by the chain rule we have

$$s' = \beta_1^s (\gamma'(t)x(t) + \gamma(t)x'(t)) \tag{33}$$

Substituting s' in the expression of p_s with (33) gives $p_s(t)$. The expression of $p_d(t)$ can be obtained using similar steps by (7) and (30). $\square$

PROPOSITION 15. *Suppose there are finite number of jumps in the trajectory of $\gamma(t), x(t)$. Then for t around such a discontinuous time point t_0, the price paid to the seller is*

$$p_s(t) = \frac{\gamma(t_0^+)x(t_0^+) - \gamma(t_0^-)x(t_0^-)}{g(\gamma(t_0^-), x(t_0^-))} \delta(t_0 - t)$$

and the price charged to the buyer is

$$p_d(t) = \frac{(1-\gamma(t_0^-))x(t_0^-) - (1-\gamma(t_0^+))x(t_0^+)}{g(\gamma(t_0^-), x(t_0^-))} \delta(t_0 - t)$$

Proof. By proposition 14, around time t_0,

$$s(t_0^-) = \beta_1^s \gamma(t_0^-) x(t_0^-), s(t_0^+) = \beta_1^s \gamma(t_0^+) x(t_0^+)$$

We have shown in the proof of Proposition 1 that a price shock on the supply side $p_s(t) = \frac{s_1 - s_0}{\beta_1^s g(s_0, d_0)} \delta(\tau - t)$ can instantly shift the stock of supply from s_0 to s_1 at time τ. As a result, a price shock given below will shift the stock of supply from $s(t_0^-)$ to $s(t_0^+)$:

$$\begin{aligned} p_s(t) &= \frac{\beta_1^s \gamma(t_0^+) x(t_0^+) - \beta_1^s \gamma(t_0^-) x(t_0^-)}{\beta_1^s g(\gamma(t_0^-), x(t_0^-))} \delta(t_0 - t) \\ &= \frac{\gamma(t_0^+) x(t_0^+) - \gamma(t_0^-) x(t_0^-)}{g(\gamma(t_0^-), x(t_0^-))} \delta(t_0 - t) \end{aligned}$$

The price shock for the demand side can be shown using identical analysis. □

Proof of Lemma 1

Proof. By the definition of h,

$$h(\gamma) = (\theta \gamma^m (\beta_1^s)^m + (1-\theta)(1-\gamma)^m (\beta_1^d)^m)^{\frac{\alpha}{m}}$$

The smoothness can be checked by taking the derivative of $h(\gamma)$. Moreover, given that $m \geq 0$, $h(\gamma)$ is defined on $\gamma = 0$ and $\gamma = 1$. Hence, Assumption 1 is confirmed.

For Assumption 4, Denote $l(\gamma) = \theta \gamma^m (\beta_1^s)^m + (1-\theta)(1-\gamma)^m (\beta_1^d)^m$. $l(\gamma) > 0$. Then

$$h'(\gamma) = \frac{\alpha}{m} l(\gamma)^{\frac{\alpha}{m}-1} l'(\gamma) \tag{34}$$

$$h''(\gamma) = \frac{\alpha}{m}\left(\frac{\alpha}{m} - 1\right) l(\gamma)^{\frac{\alpha}{m}-2} l'(\gamma)^2 + \frac{\alpha}{m} l(\gamma)^{\frac{\alpha}{m}-1} l''(\gamma) \tag{35}$$

Then

$$\begin{aligned} &(1-\alpha) h'(\gamma)^2 + \alpha h(\gamma) h''(\gamma) \\ &= \frac{\alpha^2}{m^2} l(\gamma)^{\frac{2\alpha}{m}-2} l'(\gamma)^2 - \frac{\alpha^3}{m^2} l(\gamma)^{\frac{2\alpha}{m}-2} l'(\gamma)^2 + l(\gamma)^{\frac{2\alpha}{m}-2}\left(\frac{\alpha^3}{m^2} - \frac{\alpha^2}{m}\right) l'(\gamma)^2 + \frac{\alpha^2}{m} l(\gamma)^{\frac{2\alpha}{m}-1} l''(\gamma) \\ &= \frac{\alpha^2}{m^2} l(\gamma)^{\frac{2\alpha}{m}-2} l'(\gamma)^2 + l(\gamma)^{\frac{2\alpha}{m}-2}\left(-\frac{\alpha^2}{m}\right) l'(\gamma)^2 + \frac{\alpha^2}{m} l(\gamma)^{\frac{2\alpha}{m}-1} l''(\gamma) \\ &= \frac{\alpha^2}{m} l(\gamma)^{\frac{2\alpha}{m}-2}\left\{\left(\frac{1}{m} - 1\right) l'(\gamma)^2 + l(\gamma) l''(\gamma)\right\} \end{aligned}$$

To check the sign of $\left(\frac{1}{m} - 1\right) l'(\gamma)^2 + l(\gamma) l''(\gamma)$,

$$\begin{aligned} l'(\gamma) &= \theta (\beta_1^s)^m m \gamma^{m-1} - (1-\theta)(\beta_1^d)^m m (1-\gamma)^{m-1} \\ l''(\gamma) &= \theta (\beta_1^s)^m m(m-1) \gamma^{m-2} + (1-\theta)(\beta_1^d)^m m(m-1)(1-\gamma)^{m-2} \end{aligned}$$

By some algebraic manipulation,

$$\left(\frac{1}{m} - 1\right) l'(\gamma)^2 + l(\gamma) l''(\gamma) = (m-1) m \theta (1-\theta)(\beta_1^s)^m (\beta_1^d)^m \gamma^{m-2} (1-\gamma)^{m-2}$$

Then

$$(1-\alpha) h'(\gamma)^2 + \alpha h(\gamma) h''(\gamma) = \alpha^2 l(\gamma)^{\frac{2\alpha}{m}-2} (m-1) \theta (1-\theta)(\beta_1^s)^m (\beta_1^d)^m \gamma^{m-2} (1-\gamma)^{m-2}$$

Since $m<1$, $(1-\alpha)h'(\gamma)^2+\alpha h(\gamma)h''(\gamma)<0$. Hence Assumption 4 is confirmed.

For Assumption 3, setting $h'(\gamma)=0$ gives

$$\gamma=\frac{1}{1+\left(\frac{\theta}{1-\theta}(\frac{\beta_1^s}{\beta_1^d})^m\right)^{\frac{1}{m-1}}}$$

Since $\left(\frac{\theta}{1-\theta}(\frac{\beta_1^s}{\beta_1^d})^m\right)^{\frac{1}{m-1}}>0$, the solution of γ is always in the range of $(0,1)$.

For Assumption 2, we have shown in (35) that

$$h''(\gamma)=\frac{\alpha}{m}l(\gamma)^{\frac{\alpha}{m}-2}\left((\frac{\alpha}{m}-1)l'(\gamma)^2+l(\gamma)l''(\gamma)\right)$$

Since $l(\gamma)$ does not contain any term related to α, for $m>0$, one can show that $(\frac{\alpha}{m}-1)l'(\gamma)^2+l(\gamma)l''(\gamma)$ is increasing and continuous in α. If $\alpha<1$,

$$(\frac{\alpha}{m}-1)l'(\gamma)^2+l(\gamma)l''(\gamma)<(\frac{1}{m}-1)l'(\gamma)^2+l(\gamma)l''(\gamma)<0$$

If $\alpha>1$, by continuity, there always exists $\epsilon>0$ that is sufficiently small, such that

$$(\frac{1}{m}-1)l'(\gamma)^2+l(\gamma)l''(\gamma)<(\frac{1+\epsilon}{m}-1)l'(\gamma)^2+l(\gamma)l''(\gamma)<0$$

Therefore, if $1<\alpha\le 1+\epsilon$, $h''(\gamma)<0$ still holds.

For $m=0$, $h(\gamma)=\gamma^{\alpha\theta}(1-\gamma)^{\alpha(1-\theta)}$, which is the Cobb-Douglas function. The proof for Assumption 1 to 4 still hold by taking the limit of $m\to 0$. In particular, for Assumption 2, it can be shown that $h''(\gamma)<0$ for any $\alpha<\min\{1/\theta,1/(1-\theta)\}$, given $m=0$:

$$\begin{aligned}&\lim_{m\to 0}\frac{\alpha}{m}l'(\gamma)=\lim_{m\to 0}(\frac{\alpha}{m}-1)l'(\gamma)=\alpha(\theta\gamma^{-1}-(1-\theta)(1-\gamma)^{-1})\\&\lim_{m\to 0}\frac{\alpha}{m}l''(\gamma)=\alpha\{-\theta\gamma^{-1}-(1-\theta)(1-\gamma)^2\}\\&\lim_{m\to 0}l(\gamma)^{\frac{\alpha}{m}-2}=\gamma^{\alpha\theta}(1-\gamma)^{\alpha(1-\theta)}\end{aligned}$$

Hence,

$$\lim_{m\to 0}h''(\gamma)=\gamma^{\alpha\theta}(1-\gamma)^{\alpha(1-\theta)}\left((\frac{\alpha\theta}{\gamma}-\frac{\alpha(1-\theta)}{1-\gamma})^2-\frac{\alpha\theta}{\gamma^2}-\frac{\alpha(1-\theta)}{(1-\gamma)^2}\right) \tag{36}$$

$$=\gamma^{\alpha\theta}(1-\gamma)^{\alpha(1-\theta)}\left(\frac{\alpha\theta(\alpha\theta-1)}{\gamma^2}+\frac{\alpha(1-\theta)(\alpha(1-\theta)-1)}{(1-\gamma)^2}-\frac{2\theta(1-\theta)\alpha^2}{\gamma(1-\gamma)}\right) \tag{37}$$

Since $\alpha\theta<1$, $\alpha(1-\theta)<1$, (36) is negative. Hence, $h''(\gamma)<0$. $\square$

LEMMA 5. *For any finite trajectory $x(t)$, the following equation holds:*

$$\int_0^T e^{-\rho t}\pi g(\gamma,x)dt=\int_0^T e^{-\rho t}G(\gamma,x)dt-e^{-\rho T}x(T)+x(0)$$

Proof. By Definition 4,

$$G(\gamma,x) = (v-c)g(\gamma,x) - (\rho + \beta_0^s\gamma + \beta_0^d(1-\gamma))x$$

By the formulation of $\dot{x}$,

$$\pi g(\gamma,x) = (v-c)g(\gamma,x) - (\beta_0^s\gamma + \beta_0^d(1-\gamma))x - \dot{x}$$

Then

$$\int_0^T e^{-\rho t}\pi g(\gamma,x)dt = \int_0^T ((v-c)g(\gamma,x) - (\beta_0^s\gamma + \beta_0^d(1-\gamma))x)e^{-\rho t}dt - \int_0^T \dot{x}e^{-\rho t}dt$$

Integration by parts gives

$$\int_0^T \dot{x}e^{-\rho t}dt = \int_0^T x(t)\rho e^{-\rho t}dt + e^{-\rho T}x(T) - x(0)$$

Therefore,

$$\int_0^T e^{-\rho t}\pi g(\gamma,x)dt = \int_0^T ((v-c)g(\gamma,x) - (\rho + \beta_0^s\gamma + \beta_0^d(1-\gamma))x)e^{-\rho t}dt - e^{-\rho T}x(T) + x(0)$$

LEMMA 6. *Under Assumption 2 and 4, $G(\gamma,x)$ is strictly concave in γ. Moreover, if $\alpha < 1$, then $\max_\gamma G(\gamma,x)$ is strictly concave in x and reaches maximum at $x = x^*$; if $\alpha > 1$, then $\max_\gamma G(\gamma,x)$ is strictly convex in x and reaches global minimum at $x = x^*$.*

Proof. By the concavity of $h(\gamma)$ (Assumption 2),

$$\frac{\partial^2 G(\gamma,x)}{\partial\gamma^2} = (v-c)h''(\gamma)x^\alpha < 0$$

Then $\arg\max_\gamma G(\gamma,x)$ can be obtained by setting

$$\frac{\partial G(\gamma,x)}{\partial\gamma} = -(\beta_0^s - \beta_0^d)x + (v-c)h'(\gamma)x^\alpha = 0$$

Note that $h'(\gamma)$ is bounded by $h'(0)$ and $h'(1)$, due to $\gamma \in [0,1]$. Thus, the optimal γ can be solved by

$$\gamma^*(x) = \max\left\{\min\{(h')^{-1}(\frac{(\beta_0^s - \beta_0^d)x^{1-\alpha}}{v-c}), 1\}, 0\right\} \tag{38}$$

Then for those x that $\gamma^*(x)$ has an interior solution,

$$\frac{d\gamma^*(x)}{dx} = \frac{(\beta_0^s - \beta_0^d)(1-\alpha)}{(v-c)h''(\gamma)x^\alpha}$$

Since $h''(\gamma) < 0$,

$$\text{sgn}(\frac{d\gamma^*(x)}{dx}) = \text{sgn}((\alpha-1)(\beta_0^s - \beta_0^d)) \tag{39}$$

By the envelope theorem,

$$\frac{\partial G(\gamma^*(x),x)}{\partial x} = \frac{\partial G(\gamma,x)}{\partial x}\Big|_{\gamma=\gamma^*(x)} = -(\rho+\beta_0^s\gamma^*(x)+\beta_0^d(1-\gamma^*(x)))+(v-c)\alpha h(\gamma^*(x))x^{\alpha-1}$$

When (38) has an interior solution, plugging in (38)gives

$$\frac{\partial G(\gamma^*(x),x)}{\partial x} = \alpha(\beta_0^s-\beta_0^d)\frac{h(\gamma^*(x))}{h'(\gamma^*(x))} - (\beta_0^s\gamma^*(x)+\beta_0^d(1-\gamma^*(x))+\rho)$$

Then

$$\frac{\partial G^2(\gamma^*(x),x)}{\partial x^2} = (\beta_0^s-\beta_0^d)\left(\frac{\alpha h'(\gamma)^2-\alpha h(\gamma)h''(\gamma)}{h'(\gamma)^2}-1\right)\frac{d\gamma^*(x)}{dx}$$

By (39), $\mathrm{sgn}(\frac{\partial G^2(\gamma^*(x),x)}{\partial x^2}) = \mathrm{sgn}((\alpha-1)((\alpha-1)h'(\gamma)^2-\alpha h(\gamma)h''(\gamma)))$. When (38) does not have an interior solution, $\mathrm{sgn}(\frac{\partial G^2(\gamma^*(x),x)}{\partial x^2}) = \mathrm{sgn}(\alpha-1)$.

Therefore, if $\alpha > 1$, $G(\gamma^*(x),x)$ is strictly convex; if $\alpha < 1$, $G(\gamma^*(x),x)$ is strictly concave under Assumption 4.

Setting $\frac{\partial G(\gamma^*(x),x)}{\partial x} = 0$ gives

$$x = \left(\frac{\rho+\beta_0^s\gamma^*(x)+\beta_0^d(1-\gamma^*(x))}{(v-c)\alpha h(\gamma^*(x))}\right)^{\frac{1}{\alpha-1}}$$

The solution to the above equation is γ^*, and the corresponding market size x is x^*. To check the existence of x^*, it can be verified that $\lim_{x\to 0} G_x(\gamma^*(x),x))$ and $\lim_{x\to+\infty} G_x(\gamma^*(x),x)$ have opposite signs. Since $G_x(\gamma^*(x),x)$ is monotone and continuous, $G_x(\gamma^*(x),x)=0$ must have a unique solution.

Therefore, x^* is the global minimum (maximum) for $\max_\gamma G(\gamma,x)$ under $\alpha > 1$ $(\alpha < 1)$.

Proof of Theorem 1

Proof. By Lemma 5, the infinite-horizon problem (13) can be written as

$$\int_0^\infty e^{-\rho t}G(\gamma,x)dt + x(0) \tag{40}$$

Since the selection of γ only affects the term $G(\gamma,x)$, by Lemma 6, it is optimal to set $\gamma = \arg\max G(\gamma,x)$. Steps for obtaining $\gamma^*(x)$ and its monotonicity can be found in the proof of Lemma 6.

To see the limit of γ^*, consider $\beta_0^s < \beta_0^d$ and $\alpha < 1$. $\lim_{x\to 0} h'^{-1}(\frac{(\beta_0^s-\beta_0^d)x^{1-\alpha}}{v-c}) = h'^{-1}(0) = \gamma^*$. By Assumption 3, $\min(1,\gamma^*) = \gamma^*$. For $x\to+\infty$, $\lim_{x\to+\infty} h'^{-1}(\frac{(\beta_0^s-\beta_0^d)x^{1-\alpha}}{v-c}) = h'^{-1}(-\infty)$. if $h'(1)$ is bounded, then $h'(1) > -\infty$, $h'^{-1}(h'(1)) < h'^{-1}(-\infty)$, and $\min(1, h'^{-1}(-\infty)) = 1$. If $h'(1)$ is unbounded, by Assumption 2 and 3, $h'(1)\to-\infty$. Then $h'^{-1}(-\infty)$ just equals to 1. The other three cases can be checked similarly. □

Proof of Theorem 2

Proof. The current value Hamiltonian to infinite-horizon problem (13) is

$$H(x,\pi,\gamma,\psi) = \pi g(\gamma,x) + \psi\left(-(\beta_0^s\gamma + \beta_0^d(1-\gamma))x + (v-c-\pi)g(\gamma,x)\right)$$

By the maximum principle, a stationary solution must satisfy

$$H_\pi = 0, H(x^*,\pi^*,\gamma^*,\psi) > H(x^*,\pi^*,\gamma,\psi), \psi' = \rho\psi - H_x = 0, x' = 0$$

By $H_\pi = 0$ and $g(\gamma,x) = h(\gamma)x^\alpha$,

$$(1-\psi)\pi h(\gamma)x^\alpha = 0$$

Since $x > 0$, $0 < \gamma < 1$, $h(\gamma) \neq 0$,

$$\psi = 1$$

which satisfies the limiting transversality condition

$$\lim_{T\to\infty} e^{-\rho T}\psi(T) = 0$$

To maximize $H(x^*,\pi^*,\gamma,\psi)$, set $H_\gamma = 0$ assuming it has a solution on $[0,1]$:

$$(\pi - \psi\pi + v - c)h'(\gamma)x^\alpha - \psi(\beta_0^s - \beta_0^d)x = 0$$

Substitute by $\psi = 1$ and rearrange the terms,

$$(v-c)h'(\gamma) = (\beta_0^s - \beta_0^d)x^{1-\alpha} \tag{41}$$

When (41) does not have a solution on $[0,1]$, the optimal γ is on the boundary; whether $\gamma = 1$ or 0 depends on its gradient. One can check that the balance that maximizes H is given by

$$\gamma^*(x) = \max\left\{\min\left\{h'^{-1}\left(\frac{(\beta_0^s - \beta_0^d)x^{1-\alpha}}{v-c}\right), 1\right\}, 0\right\}$$

By $\psi' = \rho\psi - H_x$ and $\psi = 1$,

$$0 = \rho - H_x$$

which gives

$$\rho = -(\beta_0^s\gamma^*(x) + \beta_0^d(1-\gamma^*(x))) + (v-c)\alpha h(\gamma^*(x))x^{\alpha-1} \tag{42}$$

Both x and φ are constants, so the last two equations are satisfied. Combining (41) and (42) gives the stationary solution.

We also need to check whether H is jointly concave in (γ, x) for sufficiency.

$$H(x,\pi,\gamma^*(x),\psi) = \{(1-\psi)\pi + \psi(v-c)\}h(\gamma^*(x))x^\alpha - \psi(\beta_0^s\gamma^*(x) + \beta_0^d(1-\gamma^*(x)))x$$

$$H_x(x,\pi,\gamma^*(x),\psi) = \{(1-\psi)\pi + \psi(v-c)\}h(\gamma^*(x))\alpha x^{\alpha-1} - \psi(\beta_0^s\gamma^*(x) + \beta_0^d(1-\gamma^*(x)))$$

$$H_{xx}(x,\pi,\gamma^*(x),\psi) = ((1-\psi)\pi + \psi(v-c))\alpha\left(h'(\gamma)\frac{d\gamma^*(x)}{dx} + h(\gamma)(\alpha-1)x^{\alpha-2}\right) - \psi(\beta_0^s - \beta_0^d)\frac{d\gamma^*(x)}{dx}$$

When $\gamma^*(x)$ is a boundary solution, i.e. $\gamma^*(x)=1$ or 0, H_{xx} can be simplified as

$$H_{xx}(x,\pi,\gamma^*(x),\psi)=((1-\psi)\pi+\psi(v-c))\alpha h(\gamma^*(x))(\alpha-1)x^{\alpha-2}$$

When $\gamma^*(x)$ is an interior solution, by $H_\gamma=0$,

$$((1-\psi)\pi+\psi(v-c))h'(\gamma)x^{\alpha-1}=\psi(\beta_0^s-\beta_0^d)$$

Moreover,

$$\frac{d\gamma^*(x)}{dx}=\frac{\psi(\beta_0^s-\beta_0^d)(1-\alpha)}{((1-\psi)\pi+\psi(v-c))h''(\gamma)x^\alpha}=\frac{h'(\gamma)(1-\alpha)}{h''(\gamma)x}$$

Hence,

$$H_{xx}(x,\pi,\gamma^*(x),\psi)=\frac{(\alpha-1)\psi^2(\beta_0^s-\beta_0^d)^2}{((1-\psi)\pi+\psi(v-c))h''(\gamma)x^\alpha}\left\{\frac{\alpha h(\gamma)h''(\gamma)}{h'(\gamma)^2}+1-\alpha\right\}$$

By the maximum principle, $H_\pi=0$, which implies that $\psi(t)=1$. Therefore, when $\gamma^*(x)$ is an interior solution,

$$H_{xx}(x,\pi,\gamma^*(x),\psi)=\frac{(\alpha-1)(\beta_0^s-\beta_0^d)^2}{(v-c)h''(\gamma)x^\alpha}\left\{\frac{\alpha h(\gamma)h''(\gamma)}{h'(\gamma)^2}+1-\alpha\right\}$$

When $\gamma^*(x)$ is a boundary solution,

$$H_{xx}(x,\pi,\gamma^*(x),\psi)=(v-c)\alpha h(\gamma^*(x))(\alpha-1)x^{\alpha-2}$$

If $\alpha<1$, under Assumption 2 and 4, $h''(\gamma)<0$, $\frac{\alpha h(\gamma)h''(\gamma)}{h'(\gamma)^2}+1-\alpha<0$, which implies that $H_{xx}<0$. Therefore, Hamiltonian is jointly concave in (γ,x). If $\alpha>1$, $\frac{\alpha h(\gamma)h''(\gamma)}{h'(\gamma)^2}<0$, $1-\alpha<0$, and again by Assumption 2, $h''(\gamma)<0$. Therefore, $H_{xx}>0$. This implies that the Hamiltonian $H(x,\pi,\gamma^*(x),\psi)$ attains its minimum at $x=x^*$. Thus, the stationary solution characterizes a saddle point. □

THEOREM 6 **(Fast vs. Slow)**. *Consider a set of increasing growth paths from $\underline{x}$ to $\bar{x}$ and the fixed endpoint problem*

$$\max\int_0^{t_0}e^{-\rho t}\pi g(\gamma^*(x),x)dt$$
$$s.t.\quad x(0)=\underline{x},\ x(t_0)=\bar{x}$$

Then in a decreasing returns to scale market, if $0<\underline{x}<\bar{x}\le x^*$, faster growth dominates slower growth; and if $x^*\le\underline{x}<\bar{x}$, slower growth dominates faster growth. Conversely, in an increasing returns to scale market, if $0<\underline{x}<\bar{x}\le x^*$, then slower growth dominates faster growth; if $x^*\le\underline{x}<\bar{x}$, then faster growth dominates slower growth. To summarize,

	$\alpha>1$	$\alpha<1$
$\underline{x}>x^*$	*faster is better*	*slower is better*
$\bar{x}<x^*$	*slower is better*	*faster is better*

This result shows that the stationary point in Theorem 2 defines a threshold between growth strategies over an interval – determining when it is optimal to grow fast and when it is optimal to grow slow. We next apply this result to analyze the optimal growth strategies overall.

Proof. By Lemma 5 in the Appendix,

$$\int_0^{t_0} e^{-\rho t}\pi g(\gamma^*(x),x)dt = \int_0^{t_0} G(\gamma^*(x),x)e^{-\rho t}dt - e^{-\rho t_0}\bar{x} + \underline{x}$$

Let $x_1(t)$ be a faster growth path from $\underline{x}$ to $\bar{x}$ over $[0,t_0]$ than $x_2(t)$. Let the corresponding pricing policies be $\pi_1(t)$ and $\pi_2(t)$. Then

$$\begin{aligned}\int_0^{t_0} e^{-\rho t}\pi_1 g(\gamma^*(x_1),x_1)dt &- \int_0^{t_0} e^{-\rho t}\pi_2 g(\gamma^*(x_2),x_2)dt \\ &= \int_0^{t_0}(G(\gamma^*(x_1),x_1) - G(\gamma^*(x_2),x_2))e^{-\rho t}dt \end{aligned} \tag{43}$$

Setting $\frac{dG(\gamma^*(x),x)}{dx} = 0$ gives

$$-(\rho + \beta_0^s\gamma^*(x) + \beta_0^d(1-\gamma^*(x))) + (v-c)\alpha h(\gamma^*(x))x^{\alpha-1} = 0$$

This is the same equation as (42). Therefore, $x = x^*$ is the solution to $\frac{dG(\gamma^*(x),x)}{dx} = 0$.

By Lemma 6 in the Appendix, when $\alpha > 1$, $G(\gamma^*(x),x)$ is convex in x. $x = x^*$ is the global minimum. $G(\gamma^*(x),x)$ is strictly decreasing for $x < x^*$ and strictly increasing for $x > x^*$.

If $\bar{x} < x^*$, then $x_2(t) < x_1(t) < x^*$, and

$$G(\gamma^*(x_1),x_1) < G(\gamma^*(x_2),x_2)$$

for all $0 \leqslant t \leqslant t_0$. As a result, (43) is negative. This implies that slower growth paths dominate faster growth paths.

Similarly, if $x^* < \underline{x}$, then $x^* < x_2(t) < x_1(t)$, and

$$G(\gamma^*(x_1),x_1) > G(\gamma^*(x_2),x_2)$$

for all $0 \leqslant t \leqslant t_0$. As a result, (43) is positive. This implies that faster growth paths dominate slower growth paths.

For $\alpha < 1$, under Assumption 4, $G(\gamma^*(x),x)$ is concave in x. So the monotonicity flips for $x > x^*$ and $x < x^*$. Using a similar analysis as for $\alpha > 1$ completes the proof. □

LEMMA 7. *Any increasing growth path from x_0 to $\bar{x}$ is weakly dominated by*

$$x(t;t_i) = \begin{cases} x_0, & t \le t_i \\ F(t-t_i), & t_i < t \le t_i + F^{-1}(\bar{x}) \\ \bar{x}, & t_i + F^{-1}(\bar{x}) < t \le T \end{cases} \tag{44}$$

where $0 \le t_i \le T - F^{-1}(\bar{x})$.

Proof. If $x_0 < \bar{x} < x^*$, by Theorem 6, it is optimal to grow the market as slow as possible. In this case, the slowest growth path is given by

$$x_s(t) = x(t; T - F^{-1}(\bar{x})) \tag{45}$$

To see why, suppose there is another increasing growth path $y(t)$ from x_0 to $\bar{x}$ over $[0,T]$ that is admissible and not faster than $x_s(t)$ given here. By definition, there exists a time point $t' \in [0,T]$ such that $y(t') < x_s(t')$. if $t' \leqslant T - F^{-1}(\bar{x})$, then $y(t') < x_s(t') = x_0$. This can't be true because $y(t)$ is an increasing growth path, and thus $y(t) \geqslant x_0$. If $t' > T - F^{-1}(\bar{x})$, then $y(t') < F(t' - T + F^{-1}(\bar{x}))$. Then $F^{-1}(y(t')) < t' - T + F^{-1}(\bar{x})$. The shortest time it takes to grow from $y(t')$ to $\bar{x}$ follows

$$F^{-1}(\bar{x}) - F^{-1}(y(t')) > T - t'$$

Therefore, $y(t)$ cannot reach $\bar{x}$ before or at $t = T$. Contradiction.

Similarly, if $x_0 > x^*$, again by Theorem 6, it is optimal to grow the market as fast as possible. The fastest growth path is given by

$$x_f(t) = x(t; 0) \tag{46}$$

The proof is similar to that for the slowest growth path (45).

For $x_0 < x^* < \bar{x}$, again consider an increasing growth path $y(t)$ from x_0 to $\bar{x}$ over $[0,T]$ that is admissible but doesn't satisfy (44). Since the growth rate is bounded, $y(t)$ is continuous, and thus must cross x^*. Denote the time $y(t) = x^*$ as $t_{y=x^*}$. We construct a the following growth path:

$$x_{s-f}(t) = x(t; t_{y=x^*}) \tag{47}$$

One can check that this growth path is the slowest from x_0 to x^* over $[0, t_{y=x^*}]$ and the fastest from x^* to $\bar{x}$ over $[t_{y=x^*}, T]$ using similar arguments for proving (45). □

Proof of Theorem 3

Proof. By Lemma 5, the infinite-horizon problem (13) can be written as

$$\int_0^\infty e^{-\rho t} G(\gamma, x) dt + x(0) \tag{48}$$

Since the selection of γ only affects the term $G(\gamma, x)$, by Lemma 6 and Theorem 1, it is optimal to set $\gamma = \gamma^*(x(t))$. Then (48) is just a function of $x(t)$. We show that the solution has the property of a *most rapid approach path* (see Spence and Starrett (1975)). The next part is also similar to the steps taken in Spence and Starrett (1975):

(1) Any path from x_0 to $\bar{x} > x_0$ is feasible. This is true by having $f(x) > 0$ for all x.

(2) The optimal growth path either (a) stays at x_0 forever, or (b) goes to $\bar{x}$. This can be shown by contradiction. Suppose (a) and (b) are both not optimal; Then the optimal increasing growth path must grow to a market size y such that $x_0 < y < \bar{x}$. If $G(\gamma^*(y), y) \le G(\gamma^*(x_0), x_0)$, by convexity, $G(\gamma^*(x), x) < G(\gamma^*(x_0), x_0)$ for any x that $x_0 < x < y$. Then any path from x_0 to y is clearly dominated by (a); If $G(\gamma^*(y), y) > G(\gamma^*(x_0), x_0)$, by Lemma 6, it must be true that $G(\gamma^*(x), x) > G(\gamma^*(y), y)$ for any x that $y < x \le \bar{x}$. But this implies that a growth path from x_0 to y is strictly dominated by (b). Contradiction.

(3) By Lemma 7, a candidate for an optimal growth path from x_0 to $\bar{x}$ must have the following property:

$$x(t; t_i) = \begin{cases} x_0, & 0 \le t \le t_i \\ F(t - t_i), & t_i < t \le t_i + F^{-1}(\bar{x}) \\ \bar{x}, & t_i + F^{-1}(\bar{x}) < t \end{cases} \tag{49}$$

We show that the optimal t_i is either 0 or $+\infty$. (48) can be further expanded as

$$\begin{aligned} J = & \int_{t_i}^{t_i + F^{-1}(\bar{x})} e^{-\rho t} G(\gamma^*(F(t - t_i)), F(t - t_i)) dt + G(\gamma^*(x_0), x_0) \frac{1 - e^{-\rho t_i}}{\rho} + G(\gamma^*(\bar{x}), \bar{x}) \frac{e^{-\rho(t_i + F^{-1}(\bar{x}))}}{\rho} \\ = & e^{-\rho t_i} \int_0^{F^{-1}(\bar{x})} e^{-\rho t} G(\gamma^*(F(t)), F(t)) dt + G(\gamma^*(x_0), x_0) \frac{1 - e^{-\rho t_i}}{\rho} + G(\gamma^*(\bar{x}), \bar{x}) \frac{e^{-\rho(t_i + F^{-1}(\bar{x}))}}{\rho} \end{aligned}$$

The second equality is by the change of variable. Taking derivative of J over t_i gives:

$$\begin{aligned} \frac{\partial J}{\partial t_i} &= -e^{-\rho t_i} \left\{ \rho \int_0^{F^{-1}(\bar{x})} e^{-\rho t} G(\gamma^*(F(t)), F(t)) dt - G(\gamma^*(x_0), x_0) + e^{-\rho F^{-1}(\bar{x})} G(\gamma^*(\bar{x}), \bar{x}) \right\} \\ &= -e^{-\rho t_i} \int_0^{F^{-1}(\bar{x})} e^{-\rho t} G_x(\gamma^*(F(t)), F(t)) F'(t) dt \end{aligned}$$

The second equality is obtained by integration by parts. Hence, the sign of $\frac{\partial J}{\partial t_i}$ does not change in t_i. In particular, if $\int_0^{F^{-1}(\bar{x})} e^{-\rho t} G_x(\gamma^*(F(t)), F(t)) F'(t) dt > 0$, $\frac{\partial J}{\partial t_i} < 0$, the growth path $x(t; 0)$ is optimal; otherwise, the growth path $x(t; +\infty)$ is optimal.

(4) It can be checked that $x(t; +\infty)$ generates the same profit (48) as $x(t) = x_0, \forall t$. So it is sufficient to compare $x(t; 0)$ and $x(t; +\infty)$. Define

$$S(\bar{x}) = \int_0^{F^{-1}(\bar{x})} e^{-\rho t} G_x(\gamma^*(F(t)), F(t)) F'(t) dt \tag{50}$$

We show that for any $x_0 < x^*$, there exists an $\tilde{x}$ such that $S(\bar{x}) > 0$ for any $\bar{x} > \tilde{x}$, and $S(\bar{x}) < 0$ for any $\bar{x} < \tilde{x}$:

$$S'(\bar{x}) = \frac{dF^{-1}(\bar{x})}{d\bar{x}} e^{-\rho F^{-1}(\bar{x})} G_x(\gamma^*(\bar{x}), \bar{x}) F'(F^{-1}(\bar{x})) = e^{-\rho F^{-1}(\bar{x})} G_x(\gamma^*(\bar{x}), \bar{x})$$

The derivative is obtained following Leibniz integral rule. Then by Lemma 6, for $\bar{x} > x^*(\bar{x} < x^*)$, $S'(\bar{x}) > 0 (S'(\bar{x}) < 0)$. Moreover, $S(x_0) = 0$. Therefore, if $x_0 < x^*$, then $S(\bar{x}) = 0$ must have a unique positive root. Denote it as $\tilde{x}$. Then $S(\bar{x}) < 0$ for $\bar{x} < \tilde{x}$ and $S(\bar{x}) > 0$ for $\bar{x} > \tilde{x}$.

If $x_0 > x^*$, then $G_x(\gamma^*(F(t)), F(t)) > 0$ on $[x_0, \bar{x}]$. Hence, $S(\bar{x}) > 0$ as well.

Hence, given that $x_0 < x^*$, for $\bar{x} > \tilde{x}$, $S(\bar{x}) > 0$, $\frac{\partial J}{\partial t_i} < 0$, the optimal policy is to grow as fast as possible, and the optimal growth path is $x(t;0)$; for $\bar{x} < \tilde{x}$, $S(\bar{x}) < 0$, $\frac{\partial J}{\partial t_i} > 0$, the optimal policy is to grow as slow as possible, and the optimal growth path is $x(t) = x_0, \forall t$. Given that $x_0 > x^*$, it is optimal to grow as fast as possible, and the optimal growth path is $x(t;0)$. $\square$

Proof of Theorem 4

Proof. By Theorem 2, in a decreasing returns to scale market, the long-run optimal size is the saturation size x^*. By Theorem 6, faster growth dominates slower growth below x^*. Hence, the optimal growth policy is to grow to the saturation size as fast as possible. $\square$

Proof of Theorem 5

Proof. This is a combination of Theorem 3 and Theorem 4. $\square$

Proof of Proposition 2

Proof. We will construct a feasible policy with unbounded value. Denote the initial market size as x_0. Fix $\gamma = 0.5$ and $\pi = 0.5(v-c)$. Apply an impulse to instantly increase the market size from $x(0)$ to $\tilde{x}$, where $\tilde{x} > (\frac{\beta_0^s + \beta_0^d}{(v-c)h(0.5)})^{\frac{1}{\alpha-1}}$. Then we can directly obtain the expression of $x(t)$ by integrating the differential equation:

$$\dot{x} = -0.5(\beta_0^s + \beta_0^d)x + 0.5(v-c)h(0.5)x^{\alpha}$$

which gives

$$x(t) = \left(\frac{0.5(\beta_0^s + \beta_0^d)}{0.5(v-c)h(0.5) - C_0 e^{0.5(\beta_0^s + \beta_0^d)(\alpha-1)t}} \right)^{\frac{1}{\alpha-1}}$$

$C_0 = 0.5(v-c)h(0.5) - 0.5(\beta_0^s + \beta_0^d)\tilde{x}^{1-\alpha}$. Then for this policy, x goes to infinity as t approaches $\frac{1}{0.5(\alpha-1)(\beta_0^s + \beta_0^d)} \ln(0.5(v-c)h(0.5)/C_0)$. Since the integrand becomes unbounded in a finite amount of time, the discounted objective value is unbounded. $\square$

Proof of Proposition 3

Proof. For $\alpha > 1$, by Proposition 2, there exists a feasible growth path that leads to unbounded profit, while keeping $x(t) = x_0$ generates finite profit. Thus, an increasing returns to scale market is viable.

For $\alpha < 1$, if $x_0 < x^*$, then by Lemma 6, $G(\gamma^*(x), x) > G(\gamma^*(x_0), x_0)$ for all $x_0 < x \le x^*$. Hence, by Lemma 5, any increasing growth path from x_0 to x^* generates a higher profit than $x(t) = x_0$. $\square$

Proof of Proposition 4

Proof. Prove by contradiction. Suppose there exists some $\tilde{x} < x_c$ such that $\dot{x} \ge 0$ and $\pi \ge 0$. When the market size is $\tilde{x}$ and $\dot{x} \ge 0$, by (12),

$$-(\beta_0^s\gamma + \beta_0^d(1-\gamma))\tilde{x} + (v-c-\pi)h(\gamma)\tilde{x}^{\alpha} \ge 0$$

Since $h(\gamma) > 0$,

$$\pi \le v - c - \frac{\beta_0^s \gamma + \beta_0^d (1-\gamma)}{h(\gamma)\tilde{x}^{\alpha-1}}$$

Maximize the right-hand side over γ on $[0,1]$:

$$\left(v - c - \frac{\beta_0^s \gamma + \beta_0^d (1-\gamma)}{h(\gamma)\tilde{x}^{\alpha-1}}\right)_\gamma = \frac{-(\beta_0^s - \beta_0^d) h(\gamma) + (\beta_0^s \gamma + \beta_0^d (1-\gamma)) h'(\gamma)}{h(\gamma)^2 \tilde{x}^{\alpha-1}} \tag{51}$$

Setting the numerator of (51) to 0 gives

$$\frac{h'(\gamma)}{h(\gamma)} (\beta_0^s \gamma + \beta_0^d (1-\gamma)) = \beta_0^s - \beta_0^d \tag{52}$$

When (17) has a solution on $[0,1]$, it is the expression for γ_c. Moreover, (51) is positive for $\gamma < \gamma_c$ and negative for $\gamma > \gamma_c$. To see why, the numerator in (51) is decreasing in γ. This can be seen by checking its first-order derivative:

$$-(\beta_0^s - \beta_0^d) h'(\gamma) + (\beta_0^s - \beta_0^d) h'(\gamma) + (\beta_0^s \gamma + \beta_0^d (1-\gamma)) h''(\gamma) = (\beta_0^s \gamma + \beta_0^d (1-\gamma)) h''(\gamma) < 0$$

The last step is by Assumption 2. Therefore, $\gamma = \gamma_c$ is the global maximizer. When (17) does not have a solution on $[0,1]$, it means that the maximizer is a boundary solution. Extending the definition of γ_c to include the boundary solutions gives by

$$\gamma_c^\dagger = \begin{cases} 0, & \frac{h'(0)}{h(0)} < \frac{\beta_0^s - \beta_0^d}{\beta_0^d} \\ \gamma_c, & \beta_0^s \frac{h'(1)}{h(1)} < \beta_0^s - \beta_0^d < \beta_0^d \frac{h'(0)}{h(0)} \\ 1, & \frac{h'(1)}{h(1)} > \frac{\beta_0^s - \beta_0^d}{\beta_0^s} \end{cases}$$

This means

$$v - c - \frac{\beta_0^s \gamma + \beta_0^d (1-\gamma)}{h(\gamma)\tilde{x}^{\alpha-1}} \le v - c - \frac{\beta_0^s \gamma_c^\dagger + \beta_0^d (1-\gamma_c^\dagger)}{h(\gamma_c^\dagger)\tilde{x}^{\alpha-1}}$$

Since $\alpha > 1$ and $\tilde{x} < x_c$,

$$v - c - \frac{\beta_0^s \gamma_c^\dagger + \beta_0^d (1-\gamma_c^\dagger)}{h(\gamma_c^\dagger)\tilde{x}^{\alpha-1}} < v - c - \frac{\beta_0^s \gamma_c^\dagger + \beta_0^d (1-\gamma_c^\dagger)}{h(\gamma_c^\dagger){x_c}^{\alpha-1}}$$

By (16), the right-hand side equals to 0. Then

$$\pi \le v - c - \frac{\beta_0^s \gamma + \beta_0^d (1-\gamma)}{h(\gamma)\tilde{x}^{\alpha-1}} \le v - c - \frac{\beta_0^s \gamma_c^\dagger + \beta_0^d (1-\gamma_c^\dagger)}{h(\gamma_c^\dagger)\tilde{x}^{\alpha-1}} < v - c - \frac{\beta_0^s \gamma_c^\dagger + \beta_0^d (1-\gamma_c^\dagger)}{h(\gamma_c^\dagger){x_c}^{\alpha-1}} = 0$$

Therefore, such an $\tilde{x}$ does not exist. □

Proof of Proposition 5

Proof. By the proof of proposition 4, for any $x < x_c$, $\dot{x} \ge 0$ implies that $\pi < 0$. □

Proof of Proposition 6

Proof. By Lemma 5, for the infinite horizon problem, the objective function is equivalent to

$$\int_0^\infty e^{-\rho t} G(\gamma, x) dt + x(0)$$

By Theorem 1, $\gamma(t) = \gamma^*(x(t))$. By the convexity of $G(\gamma^*(x), x)$ by Lemma 6, $\arg\max_{x \leqslant \bar{x}} G(\gamma^*(x), x)$ is either x_0 or $\bar{x}$. If $G(\gamma^*(\bar{x}), \bar{x}) > G(\gamma^*(x_0), x_0)$, then a jump from x_0 to $\bar{x}$ is optimal; if not, $x(t) = x_0$ is optimal.

By Lemma 6, $G(\gamma^*(x), x)$ is continuous. $\lim_{x\to 0} G(\gamma^*(x), x) \to 0$. $\lim_{x\to\infty} G(\gamma^*(x), x) \to \infty$. Then for $x_0 < x^*$, $\tilde{x}$ defined in Proposition 6 must always exist. Moreover, if $\bar{x} > \tilde{x}$, $G(\gamma^*(\bar{x}), \bar{x}) > G(\gamma^*(x_0), x_0)$; if $x_0 < \bar{x} < \tilde{x}$, $G(\gamma^*(\bar{x}), \bar{x}) < G(\gamma^*(x_0), x_0)$. □

Proof of Lemma 2

Proof. We first show that $G(\gamma^*(x_c), x_c) < 0$. By Theorem 1, $\gamma^*(x_c) = \gamma_c$, and

$$-(\beta_0^s \gamma_c + \beta_0^d (1 - \gamma_c)) x_c + (v - c) h(\gamma_c) x_c^\alpha = 0$$

Therefore, by Lemma 5,

$$G(\gamma^*(x_c), x_c) = -\rho x_c < 0$$

Since $\alpha > 1$,

$$\lim_{x\to 0} G(\gamma^*(x), x) = 0$$

By Lemma 6, $\gamma^*(x)$ is continuous. By Assumption 1, $h(\gamma)$ is continuous. Hence, $G(\gamma^*(x), x)$ is continuous too. Then there always exists an $x_0 > 0$ sufficiently small such that

$$G(\gamma^*(x_c), x_c) < G(\gamma^*(x_0), x_0) = G(\tilde{x})$$

Then for such an x_0, it must be true that $x_c < \tilde{x})$, since for any $x > \tilde{x}$, $G(\gamma^*(x), x))$ is increasing in x by the definition of $\tilde{x}$.

Proof of Proposition 7

Proof. This is a direct result of Theorem 4. □

Proof of Proposition 8, 9, 10, 11

Proof. Proposition 8 and 10 are direct results of Theorem 3; Proposition 9 and 11 are direct results of Theorem 4.

Proof of Lemma 3

Proof. Integrating by parts and (18) gives

$$\begin{aligned}
&\int_0^{F^{-1}(\tilde{x})} e^{-\rho t} G_x(\gamma^*(F(t)), F(t)) F'(t) dt \\
&= \rho \int_0^{F^{-1}(\tilde{x})} e^{-\rho t} G(\gamma^*(F(t)), F(t)) - G(\gamma^*(x_0), x_0) + e^{-\rho F^{-1}(\tilde{x})} G(\gamma^*(\tilde{x}), \tilde{x}) \\
&= \int_0^{F^{-1}(\tilde{x})} \rho e^{-\rho t} \left(G(\gamma^*(F(t)), F(t)) - G(\gamma^*(x_0), x_0) \right) dt
\end{aligned}$$

Since $G(\gamma^*(x_0),x_0)=G(\gamma^*(\tilde{x}),\tilde{x})$, by convexity of G, $G(\gamma^*(x),x)<\gamma^*(x_0),x_0)$ for all $x\in[x_0,\tilde{x}]$. Hence,

$$\int_0^{F^{-1}(\tilde{x})} e^{-\rho t}G_x(\gamma^*(F(t)),F(t))F'(t)dt<0$$

Then it is implied that

$$\int_0^{F^{-1}(\tilde{x})} e^{-\rho t}G(\gamma^*(F(t)),F(t))+\frac{e^{-\rho F^{-1}(\tilde{x})}}{\rho}G(\gamma^*(\tilde{x}),\tilde{x})<\frac{1}{\rho}G(\gamma^*(x_0),x_0)$$

Hence, $\tilde{x}<\tilde{x}_b$.

Proof of Proposition 12 and Proposition 13

Proof. First, Proposition 1 still holds because it only requires no constraint on p_s,p_d. Lemma 6 holds too because it only requires g to be integrable. Hence,

$$\begin{aligned}G(\gamma,x)&=(v-c)A\min\{\beta_1^s\gamma,\beta_1^d(1-\gamma)\}x-(\rho+\beta_0^s\gamma+\beta_0^d(1-\gamma))x\\&=\begin{cases}(((v-c)A\beta_1^s-(\beta_0^s-\beta_0^d))\gamma-(\rho+\beta_0^d))x, & \gamma<\frac{\beta_1^d}{\beta_1^d+\beta_1^s}\\((-(v-c)A\beta_1^d-(\beta_0^s-\beta_0^d))\gamma+(v-c)A\beta_1^d-(\rho+\beta_0^d))x, & \gamma\geqslant\frac{\beta_1^d}{\beta_1^d+\beta_1^s}\end{cases}\end{aligned}$$

There are three cases:

(i) $\beta_0^s-\beta_0^d\geqslant(v-c)A\beta_1^s$. For a given $x>0$, $G(\gamma,x)$ is decreasing in γ. $G(\gamma,x)\leqslant G(0,x)=-(\rho+\beta_0^d)x$. The optimal balance is then $\gamma^*(x)=0$, and the optimal policy is to keep $x(t)=x_0$. The market is not viable.

(ii) $\beta_0^s-\beta_0^d\leqslant-(v-c)A\beta_1^d$. For a given $x>0$, $G(\gamma,x)$ is increasing in γ. $G(\gamma,x)\leqslant G(1,x)=-(\rho+\beta_0^s)x$. The optimal balance is then $\gamma^*(x)=1$, and the optimal policy is again to keep $x(t)=x_0$. The market is not viable.

(iii) $-(v-c)A\beta_1^d<\beta_0^s-\beta_0^d<(v-c)A\beta_1^s$ For a given $x>0$, $G(\gamma,x)$is increasing in γ in the first piece and decreasing in γ in the second piece. Hence, $\gamma^*(x)=\frac{\beta_1^d}{\beta_1^d+\beta_1^s}$. $G\left(\frac{\beta_1^d}{\beta_1^d+\beta_1^s},x\right)=\frac{(v-c)A\beta_1^s\beta_1^d-\beta_1^s(\rho+\beta_0^d)-\beta_1^d(\rho+\beta_0^s)}{\beta_1^s+\beta_1^d}x$. $h(\gamma^*(x))=\frac{(v-c)A\beta_1^s\beta_1^d-\beta_1^s(\rho+\beta_0^d)-\beta_1^d(\rho+\beta_0^s)}{\beta_1^s+\beta_1^d}$.

If (28) holds, $h(\gamma^*(x))>0$, $G^*(\gamma^*(x),x)$ is increasing in x. Hence, faster growth is better than slower growth. Otherwise, $h(\gamma^*(x))\leq 0$, $G^*(\gamma^*(x),x)$ is decreasing in x. Hence, it is optimal to keep $x(t)=x_0$. The market is not viable ☐

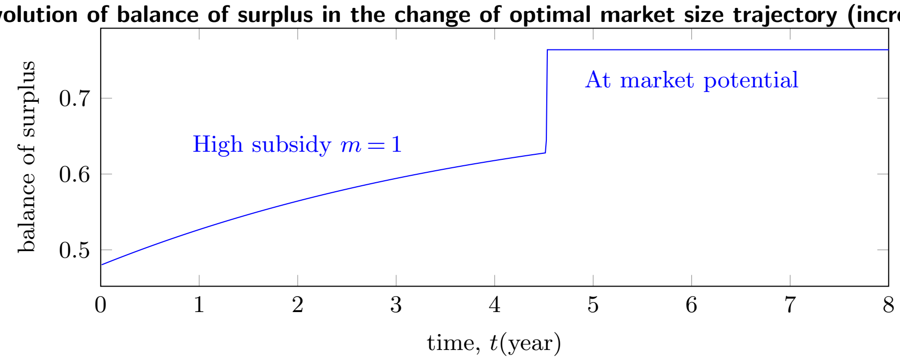
Proof of Lemma 4

Proof. Since $\rho>0$, (28) implies that $(v-c)A\beta_1^s\beta_1^d>\beta_0^s\beta_1^d+\beta_0^d\beta_1^s$. Then

$$(v-c)A\beta_1^s\beta_1^d>\beta_0^s\beta_1^d+\beta_0^d\beta_1^s>\max\{\beta_0^s\beta_1^d,\beta_0^d\beta_1^s\}>\max\{(\beta_0^s-\beta_0^d)\beta_1^d,(\beta_0^d-\beta_0^s)\beta_1^s\}$$

Dividing each term by $\beta_1^s\beta_1^d$ and rearranging the terms give (25).

Figure 7 Evolution of balance of surplus in the change of optimal market size trajectory (increasing returns)



Note. The market size trajectory $x(t)$ is the optimal policy under high subsidy $m = 1$

An illustrative example of supply and demand surpluses In the analytical model, we look at optimal balance as a measure for the market value of the supply relative to the total market value. Here we take a similar approach and calculate the ratio of the supply side surplus relative to the total surplus, which we call the *balance of surplus*. The formula is given by

$$\frac{(p_s - c)g(s,d)}{(p_s - c)g(s,d) + (v - p_d)g(s,d)} = \frac{\Delta(\gamma x)/\Delta t + \beta_0^s \gamma x}{\Delta x/\Delta t + \beta_0^s \gamma x + \beta_0^d (1-\gamma)x}$$

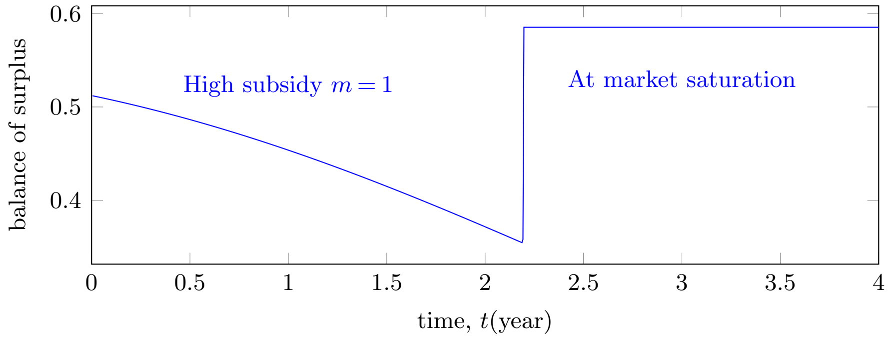
The above expression can be obtained by Proposition 14. Hence, if the trajectory of $x(t)$ is fixed, γ is also fixed at $\gamma^*(x(t))$. The trajectory of the balance of surplus can then be computed as a function of $x(t)$. Here we provide two examples.

For the increasing returns to scale market, we use the optimal market size trajectory under the high-subsidy policy $m = 1$ (shown in Figure 1). The trajectory of the balance of surplus is shown in Figure 7. The balance of surplus is increasing before the market reaches its potential at $t = 4.5$ years. Compared with Figure 3, it shows that the balance of surplus is increasing as the market grows, similar to the optimal balance.

For the decreasing returns to scale market, we use the optimal market size trajectory under the high-subsidy policy $m = 1$ (shown in Figure 2). The balance of surplus is decreasing before the market reaches the saturation size at $t = 2.2$ years. It also shows a similar trend as as the optimal balance in Figure 4.

Hence, although it is intractable to analytically show the connection between the optimal balance and the balance of surplus, numerical examples suggest that a larger market balance implies that the total surplus of supply and demand is more concentrated on the supply side, and a smaller market balance implies that the total surplus of supply and demand is more concentrated on the demand side.

Figure 8 **Evolution of balance of surplus in the change of optimal market size trajectory (decreasing returns)**



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