Stars in their Constellations: Great Person or Great Team?

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Appendix 1: Sensitivity analysis on the magnitude of value capture intervals and dominance patterns

The calculation of the value capture intervals reported in the paper (Step 2) is based on the point estimates of the matching production function (Step 1). However, while the point estimates are the best possible estimates, the population of true parameters characterizing the matching production function lies in a confidence interval. In this appendix we describe the sensitivity checks conducted to assess how the uncertainty regarding the estimates obtained in Step 1 might influence the results in Step 2, in particular, the dominance patterns. To do so, we use a form of bootstrapping as follows:

Step 1) We extract 2500 draws, each draw consisting of a subsample of 28 (out of 33) markets (subsamples are extracted with replacement).

Step 2) We re-estimate the matching production function in each of these 2500 draws.

Step 3) For each draw, we recompute the minimum and maximum bounds of value capture for each PI-star and their constellations.[1]

We reassess the dominance relationship of PI-star vs. constellation in the 2500 draws. Reporting the results presents inherent difficulties as we are interested in specific pairs of PI-constellations across multiple realizations. We chose to focus on the dominance patterns.

Table 1 shows, for each of the dominating Star PI (respectively, constellation), the percentage of draws in which these agents continue to dominate.[2] We find that the average of dominance confirmation for stars is 89% (min 66.50 % and max 99.96%) and for teams it is 80% (min 60.00 % and max 97.20%).

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[1] When estimated coefficients are not jointly in the confidence intervals, the value of a matched pair in the subsample produced by the draw can vary significantly from that in the baseline model, which in turn might impact values of minima and maxima. Likewise, matching errors impact the total value created in a market after agents are removed from the market and the remaining ones are rematched optimally. As a general rule, it is thus important to ensure that Step 2 is applied to a matching model that predicts a high number of inequalities and is robust to specification errors. To conclude, min and max can be sensitive to the explanatory power of the model.

[2] The percentage increases in draws in which all matching coefficients are simultaneously in their confidence intervals, with six stars and six constellations confirmed 100%.

In addition, we also calculated the number of stars (respectively, constellations) that are dominating across all runs. On average, we find that 13.3 stars dominate (with a range of 6 to 15) and 8 constellations dominate (with a range of 2 to 10). It is reassuring to note that the minimum of two dominating constellation occurs in only one draw, and all dominance cases under 6 occur in less than 2% of the draws. Therefore, the point estimates produced the highest count of dominating stars and constellations. However, when considering all draws, on average, the number of dominating agents-whether stars or constellations- is very close to that generated by the point estimates, differing by only 2 units.

Table 1

Stars	Dominance confirmed	Constellations	Dominance confirmed
1	99.28%	1	65.68%
2	99.96%	2	60.00%
3	96.36%	3	96.88%
4	68.20%	4	97.20%
5	72.44%	5	85.00%
6	66.84%	6	67.44%
7	95.96%	7	62.56%
8	66.50%	8	84.28%
9	89.12%	9	89.76%
10	98.48%	10	92.68%
11	95.72%		
12	94.08%		
13	91.64%		
14	97.28%		
15	99.84%		

Appendix 2

Below we establish a simple analytic solution for the computation of bounds for the core of a two-sided matching game with transferable utility. Consider the general two-sided matching problem, where the two sides consist of I and respectively J agents which can combine to generate output. In our empirical set-up the first side consists of the set of principal investigators, whereas the second side is the collection of "constellations". Characterize the set of agents on each side of market by their respective attributes (endowments)

$$\{x_i\}_{i \in I} \text{ and } \{y_j\}_{j \in J}$$

where the vectors $x_i = \left(x_i^1, x_i^2, ..., x_i^m\right)$ and $y_j = \left(y_j^1, y_j^2, ..., y_j^k\right)$ respectively describe the value of the each of attribute for each player. Throughout the proof, we will identify players on each side of the market, by a slight abuse of notation, through their respective endowments.

The output generated by a matched pair in this market will be described by

$$\theta(x_i, y_j),$$

where $\theta : R^{m+k} \to R_+$ is a general production function. We will assume that production can occur only in matched pairs. Alternatively, this means that we normalize the reservation utility from participation to 0.

The overall assignment problem in the transferable utility (TU) formulation is to find the allocation $\varphi : I \to J \cup \{\Phi\}$ amongst all injective (one-to-one) functions, which maximizes

$$\sum_i \theta(x_i, y_{\varphi(i)}),$$

where we understand $\varphi(i) = \{\Phi\}$ to indicate that player i is unmatched. This solution is generically unique and we will define the value of this problem by

$$V = \arg\max_{\varphi} \sum_i \theta(x_i, y_{\varphi(i)}).$$

For the purposes of this paper we will adopt the following simplifying notation. Relabel the set J so that the assignment function for the game is $\varphi(i) = \begin{cases} i \text{ for all } i \leq |J| \\ \Phi \text{ otherwise} \end{cases}$ so that when $|I| = |J|$ we have

$$V = \sum_{i \in I} \theta(x_i, y_i).$$

A coalition S is a subset of players $S \subseteq I \cup J$ and thus we can define

$$V_S = \arg\max_{\varphi:\{S \cap I\} \to \{S \cap J\} \cup \{\Phi\}} \sum_i \theta(x_i, y_{\varphi(i)}),$$

the value of the market when restricted only to the players in coalition S.

It is convenient to write $V_{-\{x_i\}}, V_{-\{y_j\}}$ and $V_{-\{x_i,y_j\}}$ as descriptors for the value function when the coalition is $I\setminus\{i\}\cup J,\ I\cup J\setminus\{j\}$ and $I\setminus\{i\}\cup J\setminus\{j\}$ respectively.

The core of the matching game is as usual

$$C\equiv\left\{\left(p_1,..,p_{|I|},q_1,..q_{|J|}\right)\in R_+^{|I|+|J|}|\sum_{i,j}p_i+q_j=V \text{ and} \sum_{i,j\in S}p_i+q_j\geq V_S \text{ for all } S\subset I\cup J\right\}.$$

Whenever the core is non-empty, then according to Shapley and Shubik it is convex and closed, as it consists of an intersection of hyperplanes. It is straightforward to observe that finding the core is a rather tedious procedure, but in particular, in matching games it is equivalent to finding the vectors $\left((p_i)_{i\in I},(q_j)_{j\in J}\right)\in R_+^{|I|+|J|}$ such that

$$p_i+q_i=\theta(x_i,y_i) \text{ and } p_i+q_j\geq\theta(x_i,y_j) \text{ for all } i\neq j.$$

To show this, note that the following needs to hold

$$\theta(x_i,y_i)+\theta(x_j,y_j)\geq\theta(x_i,y_j)+\theta(x_j,y_i)$$

as a direct consequence of the optimal matching. Also note that

$$V_{-\{x_i,y_i\}}=\sum_{k\neq i}\theta(x_k,y_k)$$

as $V_{-\{x_i,y_i\}}\geq\sum_{k\neq i}\theta(x_k,y_k)$ and the fact that if this inequality would be strict then the allocation over the coalition $I\setminus\{i\}\cup J\setminus\{i\}$ coupled with matching players indexed by i on each side of the market would yield an allocation with a value in excess of $\sum_{k\neq i}\theta(x_k,y_k)+\theta(x_i,y_i)=V$, a contradiction. The defining inequalities of the core imply

$$\sum_{i,j}p_i+q_j=V \text{ and } \sum_{i\neq i_0,j\neq i_0}p_i+q_j\geq V_{-\left\{x_{i_0},y_{i_0}\right\}} \text{ and } p_{i_0}+q_{i_0}\geq\theta(x_{_{i_0}},y_{_{i_0}})$$

and so subtracting the first two inequalities

$$p_{i_0}+q_{i_0}\leq V-V_{-\left\{x_{i_0},y_{i_0}\right\}}=\theta(x_{_{i_0}},y_{_{i_0}})$$

which implies

$$p_{i_0}+q_{i_0}=\theta(x_{_{i_0}},y_{_{i_0}})$$

for all i_0. Thus, the inequalities defining the core imply our restricted set of inequalities. To prove the equivalence between the two sets start from our restricted inequalities and observe that for any strict sub-coalition S we will have

$$\sum_{i,j\in S}p_i+q_j\geq V_S$$

as the value of the coalition consists of individual pairings via its optimal assignment function, whereas for the total coalition

$$\sum_i p_i + q_i = \sum_i \theta(x_i, y_i) = V,$$

which concludes our proof.

Finding a closed-form expression or at least a workable characterization for the upper lower bound of the projection of the core is key from the perspective of bi-form games as Brandenburger and Stuart (1996) and MacDonald and Ryall (2004), require it in the order to characterize the impact of the investment and entry decisions in matching games.

As the core is convex, closed and non-empty, define

$$\underline{p}_i = \min\left\{p_i | \left((p_s)_{s\in I}, (q_t)_{t\in J}\right) \in C\right\} \text{ and } \bar{p}_i = \max\left\{p_i | \left((p_s)_{s\in I}, (q_t)_{t\in J}\right) \in C\right\}$$

as well as

$$\underline{q}_j = \min\left\{q_j | \left((p_s)_{s\in I}, (q_t)_{t\in J}\right) \in C\right\} \text{ and } \bar{q}_j = \max\left\{q_j | \left((p_s)_{s\in I}, (q_t)_{t\in J}\right) \in C\right\}$$

as the lowest and highest values corresponding to player i and respectively j in the core.

Shapley and Shubik (1971) show that

$$\left((\underline{p}_i), (\bar{q}_j)\right) \in C \text{ and } \left((\bar{p}_i), \left(\underline{q}_j\right)\right) \in C,$$

but stop short of characterizing the quantities $\bar{p}_i$ and $\bar{q}_j$.

The object of this note is to offer the following characterization

Lemma 1 *The upper and lower bounds of the projections of the core in a matching game can be found as follows*

$$\begin{aligned}
\bar{p}_i &= V - V_{-\{x_i\}}, &(1)\\
\underline{p}_i &= V_{-\{y_i\}} - V_{-\{x_i, y_i\}}, &(2)\\
\bar{p}_i + \underline{q}_i &= \underline{p}_i + \bar{q}_i = \theta(x_i, y_i). &(3)
\end{aligned}$$

Proof. The proof will proceed in steps. Observe first that since $\left((\underline{p}_i), (\bar{q}_j)\right) \in C$ and $\left((\bar{p}_i), \left(\underline{q}_j\right)\right) \in C$ we have that the restricted inequalities need to hold and as such we have that (3) holds by definition. Further note that, in general, $p_i \leq V - V_{-\{x_i\}}$ for every $\left((p_s)_{s\in I}, (q_t)_{t\in J}\right) \in C$. To see this revert to the general definition of the core and consider that

$$\sum_{s,t} p_s + q_t = V \text{ and } \sum_{s\neq i,t} p_s + q_t \geq V_{-\{x_i\}},$$

which after subtraction yields the "marginal contribution" bound.[1] In a similar fashion, note that in a matching game we also have $p_i \geq V_{-\{y_i\}} - V_{-\{x_i, y_i\}}$

[1] McDonald and Ryall (2004) coin the term of marginal contribution principle.

as one can revert again to the general definition of the core and consider the coalitions $I \cup J \setminus \{i\}$ and $I \setminus \{i\} \cup J \setminus \{i\}$ and observe that ■

$$p_i + \sum_{s \neq i, t \neq i} p_s + q_t \geq V_{-\{y_i\}} \text{ and } \sum_{s \neq i, t \neq i} p_s + q_t = V_{-\{x_i, y_i\}}.$$

After subtraction we conclude that

$$V_{-\{y_i\}} - V_{-\{x_i, y_i\}} \leq \underline{p}_i \leq \bar{p}_i \leq V - V_{-\{x_i\}}.$$

Since the choice of the side was arbitrary, a similar set of inequalities holds for the bounds on q_j.[2] Our proof boils down to showing that these bounds are indeed tight $(1), (2)$. Consider the case $\underline{p}_i > 0$, then by our equivalent conditions on the core it follows that

$$\underline{p}_i + \bar{q}_j \geq \theta(x_i, y_j)$$

for all $j \neq i$. Observe that all of these inequalities cannot be strict. If they were strict then an increase of $\bar{q}_i$ by $\varepsilon > 0$ coupled with a corresponding decrease of $\underline{p}_i$ would still satisfy the set of inequalities in the core. This would be a contradiction to the definition of $\underline{p}_i$ as the lowest core imputation of player i. Thus, for any i_0 such that $\underline{p}_{i_0} > 0$ there exists $i_1 \neq i_0$ such that

$$\underline{p}_{i_0} + \bar{q}_{i_1} = \theta(x_{i_0}, y_{i_1}).$$

Assume by way of contradiction that $\underline{p}_{i_1} + \bar{q}_{i_0} = \theta(x_{i_1}, y_{i_0})$ then we would have

$$\theta(x_{i_0}, y_{i_0}) + \theta(x_{i_1}, y_{i_1}) = V_S = \underline{p}_{i_0} + \underline{p}_{i_1} + \bar{q}_{i_0} + \bar{q}_{i_1} = \theta(x_{i_0}, y_{i_1}) + \theta(x_{i_1}, y_{i_0}),$$

a contradiction to the supermodularity, when we compare the extreme terms. It follows that $\underline{p}_{i_1} + \bar{q}_{i_0} > \theta(x_{i_1}, y_{i_0})$ and thus either $\underline{p}_{i_1} = 0$ or there exists $i_2 \neq i_1$ such that

$$\underline{p}_{i_2} + \bar{q}_{i_1} = \theta(x_{i_2}, y_{i_1}).$$

Repeating said argument, it follows that $\underline{p}_{i_2} + \bar{q}_{i_0} > \theta(x_{i_1}, y_{i_0})$ or else we have a cycle where

$$\sum_{j=0}^{2} \theta(x_{i_j}, y_{i_j}) = V_S = \theta(x_{i_0}, y_{i_1}) + \theta(x_{i_1}, y_{i_2}) + \theta(x_{i_2}, y_{i_0})$$

contradicting the essential uniqueness of the allocation. The same argument will apply for cycles of any length and since the set of players is finite we conclude that each such chain $T_{i_0} = \{i_0, i_1, ..., i_k\}$ must end up with an element such that

[2] The interested reader will observe in the inequalities above the more general finding that the two sides are indeed complementary i.e.

$$V_{-\{y_j\}} - V_{-\{x_i, y_j\}} \leq V - V_{-\{x_i\}}$$

for all i and j.

$\underline{p}_{i_k} = 0$. Thus for any player i_0 we can start our construction iteratively and identify such a chain T_{i_0} and thus

$$\begin{aligned} & \sum_{j=0}^{k-1} \theta(x_{i_j}, y_{i_{j+1}}) + \sum_{t \notin T_{i_0}} \theta(x_t, y_t) \\ = \; & \underline{p}_{i_0} + \sum_{t \neq i_0} \underline{p}_t + \bar{q}_t = \underline{p}_{i_0} + \bar{q}_{i_1} + ... + \underline{p}_{i_{k-1}} + \bar{q}_{i_k} + \underline{p}_{i_k} + \sum_{t \notin T_{i_0}} \underline{p}_t + \bar{q}_t \geq V_{-\{y_{i_0}\}}, \end{aligned}$$

which means that the inequality cannot be strict as otherwise the constructed chain would exceed the value generated by the optimal allocation inside of the coalition $I \cup J \setminus \{y_{i_0}\}$. We thus conclude that

$$\underline{p}_{i_0} + \sum_{t \neq i_0} \underline{p}_t + \bar{q}_t = V_{-\{y_{i_0}\}}$$

and since

$$\sum_{t \neq i_0} \underline{p}_t + \bar{q}_t = V_{-\{x_{i_0}, y_{i_0}\}},$$

it follows that

$$\underline{p}_{i_0} = V_{-\{y_{i_0}\}} - V_{-\{x_{i_0}, y_{i_0}\}}.$$

Since in the previous argument the side was arbitrarily chosen, it follows that a similar inequality holds for $\underline{q}_{i_0}$. Combining this with (3) yields (1).

In order to understand the implication of the main theorem consider the simplest case which considers a two-sided market where the attributes are unidimensional - i.e. a situation where each side of the market consists of N agents characterized as before by the size of their attributes $x_1 \leq x_2 \leq .. \leq x_N$ and $y_1 \leq y_2 \leq .. \leq y_N$ respectively. We will assume a production function which exhibits complementarities across the sides, i.e. $\frac{\partial^2}{\partial x \partial y} \theta(x, y) \geq 0$. These assumptions imply that all optimal assignments will be assortative - i.e. the assignment games in every coalition lead to top-down pairings. We can easily characterize the objects $V, V_{-\{x_i\}}$ and $V_{-\{x_i, y_i\}}$ as a function of θ and the original attributes.

Corollary 2 *In the one-dimensional matching problem when production θ is supermodular then*

$$\begin{aligned} \bar{p}_i &= \sum_{s=1}^{i} [\theta(x_s, y_s) - \theta(x_{s-1}, y_s)] \\ \underline{p}_i &= \sum_{s=1}^{i-1} [\theta(x_{s+1}, y_s) - \theta(x_s, y_s)]. \end{aligned}$$

Since each term in the sum is positive, by the monotonicity assumption on θ, it follows that both $\bar{p}_i$ and $\underline{p}_i$ are monotone in i. However $\bar{p}_i$ and $\underline{p}_i$ are not necessarily convex in i as

$$\bar{p}_i - \bar{p}_{i-1} = \theta(x_i, y_i) - \theta(x_{i-1}, y_i)$$

clearly depends on the gap between x_i and x_{i-1}.