Online Appendix to
"Taylor Approximation of Inventory Policies for One-Warehouse, Multi-Retailer Systems with Demand Feature Information"

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A. Proofs for Main Results

Proof of Proposition 1. For the second-stage problem (6), we define the Lagrange function $L(z_1, \cdots, z_J, \lambda, \mu_1, \cdots, \mu_J)$ as follows.

$$L(z_1, \cdots, z_J, \lambda, \mu_1, \cdots, \mu_J|\boldsymbol{x}) \triangleq \sum_{j=1}^{J} C_j(z_j|\boldsymbol{x}_j) + \lambda \cdot (\sum_{j=1}^{J} z_j - z_0) - \sum_{j=1}^{J} \mu_j \cdot z_j,$$

where λ and $\{\mu_j\}_{j=1}^J$ are the nonnegative Lagrange multipliers associated with constraints $\sum_{j=1}^J z_j \leq z_0$ and $z_j \geq 0$. Then, the optimal solutions z_j^*, μ_j^* and λ^* should satisfy the following conditions (we abbreviate $\lambda^*(\boldsymbol{x})$ as λ^* for notation simplicity):

$$\begin{aligned} \left.\frac{\partial L}{\partial z_j}\right|_{z_j=z_j^*} &= C_j'(z_j^*|\boldsymbol{x}_j) + \lambda^* - \mu_j^* = 0; \\ \lambda^* \cdot (\sum_{j=1}^J z_j^* - z_0) &= 0; \\ \mu_j^* z_j^* &= 0, \text{ for any } j \in [J]^+; \\ \sum_{j=1}^J z_j^* - z_0 &\leq 0; \\ \lambda^* \geq 0, \mu_j^* \geq 0, z_j^* &\geq 0, \text{ for any } j \in [J]^+. \end{aligned} \tag{A1}$$

The first equation represents the first-order conditions for decision variables z_j and the second and third equations represent the complementary slackness conditions. Conditions (A1) indicate that the optimal allocation quantities among retailers should balance the marginal costs at retailers when the warehouse's available inventory z_0 is limited.

We first consider the case with $z_j^* > 0$ which immediately implies $\mu_j^* = 0$ for all $j \in [J]^+$. This case corresponds to the result in (11). In this case, we further consider two subcases.

Subcase 1: $\lambda^* > 0$. By the first and second equations in (A1), the optimal allocation decisions should satisfy $z_j^* = F_j^{-1}\left(\frac{b_j + h_0 - \lambda^*}{b_j + h_j'}\middle|\boldsymbol{x}_j\right)$ and $\sum_{j=1}^J z_j^* = \sum_{j=1}^J F_j^{-1}\left(\frac{b_j + h_0 - \lambda^*}{b_j + h_j'}\middle|\boldsymbol{x}_j\right) = z_0$. To ensure $z_j^* > 0$, we need the condition that $\min_j(b_j + h_0 - \lambda^*) > 0$. Under the assumption that $b_1 \leq b_2 \leq \cdots \leq b_J$, this condition is equivalent to $\lambda^* < b_1 + h_0$. It directly implies that $z_0 > \sum_{j=1}^J F_j^{-1}\left(\frac{b_j - b_1}{b_j + h_j'}\middle|\boldsymbol{x}_j\right)$ due to the monotone property of $F_j^{-1}(\cdot|\boldsymbol{x}_j)$.

Subcase 2: $\lambda^*=0$. The first equation implies $C_j'(z_j^*|\boldsymbol{x}_j)=0$. Therefore, the optimal allocation decision is $z_j^*=F_j^{-1}\left(\dfrac{b_j+h_0}{b_j+h_j'}\middle|\boldsymbol{x}_j\right)$ for all $j\in[J]^+$. Obviously, the warehouse's inventory should be larger than the summation of retailers' order quantities, i.e., $z_0\geq\sum_{j=1}^J F_j^{-1}\left(\dfrac{b_j+h_0}{b_j+h_j'}\middle|\boldsymbol{x}_j\right)$.

We now turn to the case with $z_j^*=0$ which implies that some retailers may not be allocated with any inventory. There exists some $k\in\{1,2,\cdots,J-1\}$ such that $b_k+h_0\leq\lambda^*<b_{k+1}+h_0$. Then, the KKT conditions in (A1) imply that for retailer $j<k$, $C_j'(z_j^*|\boldsymbol{x}_j)+\lambda^*\geq C_j'(0|\boldsymbol{x}_j)+\lambda^*>-b_j+b_k\geq 0$. It implies $\mu_j^*>0$ for $j<k$ and thus, the optimal allocation decision for retailer j is $z_j^*=0$. For retailer $j>k$, we can similarly show that $z_j^*>0$ should satisfy $z_j^*=F_j^{-1}\left(\dfrac{b_j+h_0-\lambda^*}{b_j+h_j'}\middle|\boldsymbol{x}_j\right)$, where λ^* satisfies $\sum_{j=k}^J F_j^{-1}\left(\dfrac{b_j+h_0-\lambda^*}{b_j+h_j'}\middle|\boldsymbol{x}_j\right)=z_0$. When $j=k$, if $\lambda^*>b_k+h_0$, the analysis is similar with the case of $j<k$; otherwise, $\mu_j^*=0$ and $z_j^*=0$. This case corresponds to the result in (12). □

Proof of Proposition 2. Define the Lagrange function for problem (14) as follows.

$$\widetilde{L}(z_1,\cdots,z_J,\lambda,\mu_1,\cdots,\mu_J|\boldsymbol{x})\triangleq\sum_{j=1}^J\left[-h_0(z_j-s_j^l)+\frac{\left(z_j-s_j^l\right)^2}{2r_j}\right]+\lambda\cdot(\sum_{j=1}^J z_j-z_0)-\sum_{j=1}^J\mu_j\cdot z_j,$$

where we omit the constant term in (14) and λ and $\{\mu_j\}_{j=1}^J$ are the non-negative Lagrange multipliers associated with constraints $\sum_{j=1}^J z_j\leq z_0$ and $z_j\geq 0$. Then, the optimal solutions $\widetilde{z}_j$, μ_j^* and λ^* should satisfy the following conditions.

$$\begin{aligned}
\left.\frac{\partial\widetilde{L}}{\partial z_j}\right|_{z_j=\widetilde{z}_j} &= -h_0+\frac{\widetilde{z}_j-s_j^l}{r_j}+\lambda^*-\mu_j^*=0;\\
\lambda^*\cdot(\sum_{j=1}^J\widetilde{z}_j-z_0)=0;&\\
\mu_j^*\widetilde{z}_j=0,&\ \text{for any } j\in[J]^+;\\
\sum_{j=1}^J\widetilde{z}_j-z_0\leq 0;&\\
\lambda^*\geq 0,\mu_j^*\geq 0,\widetilde{z}_j\geq 0,&\ \text{for any } j\in[J]^+.
\end{aligned}$$

Then, the subsequent analysis is similar to that of Proposition 1 and thus omitted here. □

Proof of Proposition 3. Applying Taylor expansion to $C_j(z_j|\boldsymbol{x}_j)$, we can obtain

$$\begin{aligned}
C_{total}(z_0,\boldsymbol{z}|\boldsymbol{x}) &\triangleq\sum_{j=1}^J C_j(z_j|\boldsymbol{x}_j)\\
&=\sum_{j=1}^J\left[C_j(s_j^l|\boldsymbol{x}_j)+\left(z_j-s_j^l\right)C_j'(s_j^l|\boldsymbol{x}_j)+\frac{\left(z_j-s_j^l\right)^2}{2}C_j''(\xi_j|\boldsymbol{x}_j)\right],
\end{aligned}\tag{A2}$$

where C_j' and C_j'' represent the first and second order derivatives of $C_j(z_j|\boldsymbol{x}_j)$ over z_j and ξ_j is a constant between z_j and s_j^l. As $s_j^l = F_j^{-1}\left(\frac{b_j}{b_j+h_j'}\middle|\boldsymbol{x}_j\right)$, then $C_j'(s_j^l|\boldsymbol{x}_j) = -h_0$ and (A2) becomes

$$C_{total}(z_0, \boldsymbol{z}|\boldsymbol{x}) = \sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0(z_j - s_j^l) + \frac{\left(z_j - s_j^l\right)^2}{2}(b_j + h_j')f_j(\xi_j|\boldsymbol{x}_j)\right].$$

Immediately, the total cost under the Taylor approximation policy $\widetilde{\boldsymbol{z}} = (\widetilde{z}_1, \widetilde{z}_2, \cdots, \widetilde{z}_J)$ can be bounded by

$$\begin{aligned} C_{total}(z_0, \widetilde{\boldsymbol{z}}|\boldsymbol{x}) &= \sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0(\widetilde{z}_j - s_j^l) + \frac{\left(\widetilde{z}_j - s_j^l\right)^2}{2}(b_j + h_j')f_j(\widetilde{\xi}_j|\boldsymbol{x}_j)\right] \\ &\le \sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0(\widetilde{z}_j - s_j^l) + \frac{\left(\widetilde{z}_j - s_j^l\right)^2}{2}(b_j + h_j')u_j\right], \end{aligned} \tag{A3}$$

where the inequality holds as u_j is an upper bound of $f_j(\widetilde{\xi}_j|\boldsymbol{x}_j)$ (see Assumption 2).

Similarly, the total cost under the optimal allocation policy $\boldsymbol{z}^* = (z_1^*, z_2^*, \cdots, z_J^*)$ is bounded below by

$$\begin{aligned} C_{total}(z_0, \boldsymbol{z}^*|\boldsymbol{x}) &= \sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0(z_j^* - s_j^l) + \frac{\left(z_j^* - s_j^l\right)^2}{2}(b_j + h_j')f_j(\xi_j^*|\boldsymbol{x}_j)\right] \\ &\ge \min_{\sum_j z_j \le z_0} \sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0(z_j - s_j^l) + \frac{\left(z_j - s_j^l\right)^2}{2}(b_j + h_j')\ell_j\right] \\ &= \sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0(\bar{z}_j - s_j^l) + \frac{\left(\bar{z}_j - s_j^l\right)^2}{2}(b_j + h_j')\ell_j\right], \end{aligned} \tag{A4}$$

where the inequality holds as ℓ_j is a lower bound of $f_j(\xi_j^*|\boldsymbol{x}_j)$ (see Assumption 2) and $\bar{z}_j$ is the optimal solution of the optimization problem in the right-hand side of the inequality.

Define $r_j \triangleq \frac{1}{(b_j+h_j')f_j(s_j^l|\boldsymbol{x}_j)}$, $t_j \triangleq \frac{1}{(b_j+h_j')\ell_j}$, $p_j \triangleq \frac{1}{(b_j+h_j')u_j}$, $\alpha_j \triangleq \frac{u_j}{\ell_j}$ and $\alpha \triangleq \max_j \alpha_j$. It follows from the definition that $p_j \le r_j \le t_j$. Moreover, the solutions $\widetilde{z}_j$ and $\bar{z}_j$ in Eq. (A3) and (A4) are given by

$$\widetilde{z}_j = \begin{cases} \widetilde{s}_j^e - \frac{r_j}{\sum_j r_j}\cdot\left(\sum_j \widetilde{s}_j^e - z_0\right), & \text{if } \sum_{j=1}^{J}(\widetilde{s}_j^e - \underline{m}\cdot r_j) < z_0 < \sum_{j=1}^{J}\widetilde{s}_j^e; \\ \widetilde{s}_j^e, & \text{if } z_0 \ge \sum_{j=1}^{J}\widetilde{s}_j^e; \end{cases} \tag{A5}$$

and

$$\bar{z}_j = \begin{cases} \bar{s}_j^e - \frac{t_j}{\sum_j t_j}\cdot\left(\sum_j \bar{s}_j^e - z_0\right), & \text{if } z_0 < \sum_{j=1}^{J}\bar{s}_j^e; \\ \bar{s}_j^e, & \text{if } z_0 \ge \sum_{j=1}^{J}\bar{s}_j^e; \end{cases} \tag{A6}$$

where $\widetilde{s}_j^e = s_j^l + h_0\cdot r_j$ and $\bar{s}_j^e = s_j^l + h_0\cdot t_j$.

With (A3) and (A4), the performance bound of Taylor approximation can be written as

$$
\begin{aligned}
\tau(z_0,\boldsymbol{x}) &\triangleq \sum_{j=1}^{J}\left[C_j(\widetilde{z}_j(z_0,\boldsymbol{x})|\boldsymbol{x}_j) - C_j(z_j^*(z_0,\boldsymbol{x})|\boldsymbol{x}_j)\right] = C_{total}(z_0,\widetilde{\boldsymbol{z}}|\boldsymbol{x}) - C_{total}(z_0,\boldsymbol{z}^*|\boldsymbol{x}) \\
&\le \sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0(\widetilde{z}_j - s_j^l) + \frac{\left(\widetilde{z}_j - s_j^l\right)^2}{2\cdot p_j} - C_j(s_j^l|\boldsymbol{x}_j) + h_0(\bar{z}_j - s_j^l) - \frac{\left(\bar{z}_j - s_j^l\right)^2}{2\cdot t_j}\right] \\
&= \sum_{j=1}^{J}\left[h_0\cdot(\bar{z}_j - \widetilde{z}_j) + \frac{\left(\widetilde{z}_j - s_j^l\right)^2}{2\cdot p_j} - \frac{\left(\bar{z}_j - s_j^l\right)^2}{2\cdot t_j}\right], \qquad \text{(A7)}
\end{aligned}
$$

where $\widetilde{z}_j$ and $\bar{z}_j$ are given by (A5) and (A6).

As $t_j \ge r_j$, $\bar{s}_j^e \ge \widetilde{s}_j^e$. We now prove the result by considering the following three cases: (1) $z_0 \in (\sum_{j=1}^J(\widetilde{s}_j^e - \underline{m}\cdot r_j), \sum_{j=1}^J \widetilde{s}_j^e)$; (2) $z_0 \in [\sum_{j=1}^J \widetilde{s}_j^e, \sum_{j=1}^J \bar{s}_j^e)$; (3) $z_0 \in [\sum_{j=1}^J \bar{s}_j^e, \infty)$.

Case 1: $\sum_{j=1}^J(\widetilde{s}_j^e - \underline{m}\cdot r_j) < z_0 < \sum_{j=1}^J \widetilde{s}_j^e$. In this case, (A5) and (A6) imply that $\sum_j \widetilde{z}_j = \sum_j \bar{z}_j = z_0$. By (A7), the performance of the TA heuristic can be bounded by

$$
\begin{aligned}
\tau(z_0,\boldsymbol{x}) &= \sum_{j=1}^{J}\left[\frac{\left(\widetilde{z}_j - s_j^l\right)^2}{2\cdot p_j} - \frac{\left(\bar{z}_j - s_j^l\right)^2}{2\cdot t_j}\right] \\
&= \sum_{j=1}^{J}\left[\frac{(\frac{r_j}{\sum_j r_j})^2\cdot(\sum_j s_j^l - z_0)^2}{2\cdot p_j} - \frac{(\frac{t_j}{\sum_j t_j})^2\cdot(\sum_j s_j^l - z_0)^2}{2\cdot t_j}\right] \\
&\le \sum_{j=1}^{J}\left[\frac{t_j\cdot\alpha_j}{2\cdot(\sum_j p_j)^2}\cdot(\sum_j s_j^l - z_0)^2 - \frac{t_j}{2\cdot(\sum_j t_j)^2}\cdot(\sum_j s_j^l - z_0)^2\right] \\
&\le \sum_{j=1}^{J}\left[\frac{t_j\cdot\alpha^3}{2\cdot(\sum_j t_j)^2}\cdot(\sum_j s_j^l - z_0)^2 - \frac{t_j}{2\cdot(\sum_j t_j)^2}\cdot(\sum_j s_j^l - z_0)^2\right] \\
&= (\alpha^3 - 1)\cdot\frac{(\sum_j s_j^l - z_0)^2}{2\sum_j t_j}, \qquad \text{(A8)}
\end{aligned}
$$

where the first inequality holds due to $p_j \le r_j \le t_j$ and $p_j = \dfrac{t_j}{\alpha_j}$ and the second inequality holds due to $\sum_j p_j = \sum_j \dfrac{t_j}{\alpha_j} \ge \sum_j \dfrac{t_j}{\max_j \alpha_j} = \dfrac{\sum_j t_j}{\alpha}$.

Case 2: $\sum_{j=1}^J \widetilde{s}_j^e \le z_0 < \sum_{j=1}^J \bar{s}_j^e$. In this case, Eq. (A7) can be bounded by

$$
\begin{aligned}
\tau(z_0,\boldsymbol{x}) &< \sum_{j=1}^{J}\left[h_0\cdot(t_j-r_j)+\frac{\left(\widetilde{z}_j-s_j^l\right)^2}{2\cdot p_j}-\frac{\left(\bar{z}_j-s_j^l\right)^2}{2\cdot t_j}\right] \\
&= \sum_{j=1}^{J}\left[h_0\cdot(t_j-r_j)+\frac{(h_0)^2\cdot r_j^2}{2\cdot p_j}-\frac{t_j}{2\cdot(\sum_j t_j)^2}\cdot(\sum_j s_j^l-z_0)^2\right] \\
&\leq \sum_{j=1}^{J}\left[h_0\cdot(t_j-r_j)+\frac{(h_0)^2\cdot r_j^2}{2\cdot p_j}\right]-\frac{1}{2\cdot\sum_j t_j}\cdot(h_0)^2(\sum_j r_j)^2 \\
&\leq \frac{\alpha-1}{\alpha}h_0(\sum_j t_j)+\frac{1}{2}(h_0)^2\alpha\cdot(\sum_j t_j)-\frac{1}{2\cdot\alpha^2}(h_0)^2\cdot(\sum_j t_j) \\
&= \frac{\alpha-1}{\alpha}h_0(\sum_j t_j)+\frac{\alpha^3-1}{2\alpha^2}(h_0)^2(\sum_j t_j),
\end{aligned}
\tag{A9}
$$

where the first inequality holds according to $\sum_j \widetilde{z}_j=\sum_{j=1}^J \widetilde{s}_j^e \leq \sum_j \bar{z}_j = z_0 < \sum_{j=1}^J \bar{s}_j^e$, the second inequality follows from the fact that $-(\sum_j s_j^l - z_0)^2$ is decreasing in z_0 when $z_0 \geq \sum_{j=1}^J \widetilde{s}_j^e$ and the third inequality directly follows from the definitions of r_j, p_j, t_j and α.

Case 3: $z_0 \geq \sum_{j=1}^J \bar{s}_j^e$. In this case, Eq. (A7) can be bounded by

$$
\begin{aligned}
\tau(z_0,\boldsymbol{x}) &= \sum_{j=1}^{J}\left[h_0\cdot(t_j-r_j)+\frac{\left(\widetilde{z}_j-s_j^l\right)^2}{2\cdot p_j}-\frac{\left(\bar{z}_j-s_j^l\right)^2}{2\cdot t_j}\right] \\
&= \sum_{j=1}^{J}\left[h_0\cdot(t_j-r_j)+\frac{(h_0)^2\cdot r_j^2}{2\cdot p_j}-\frac{(h_0)^2\cdot t_j^2}{2\cdot t_j}\right] \\
&\leq \sum_{j=1}^{J}\left[h_0 t_j\cdot(1-\frac{1}{\alpha})+\frac{(h_0)^2}{2}t_j\cdot(\alpha-1)\right] \\
&= \frac{\alpha-1}{\alpha}h_0(\sum_j t_j)+\frac{\alpha-1}{2}(h_0)^2(\sum_j t_j).
\end{aligned}
\tag{A10}
$$

The results of (A8), (A9) and (A10) constitute Proposition 3. $\square$

Proof of Theorem 1. The proof of Theorem 1 is mainly divided into the following two parts, 1. We construct a lower bound of OPT policy; 2. We calculate the expected cost difference between the lower bound and our TA policy and to be specific, as the allocation policy under TA is piecewise, we discuss the following cases and calculate the upper bound of expected cost when: 2.1. $\widetilde{z}_0 \leq \sum_{j=1}^J(\widetilde{s}_j^e - \underline{m}\cdot r_j)$; 2.2. $\widetilde{z}_0 \geq \sum_{j=1}^J \widetilde{s}_j^e$; 2.3. $\sum_{j=1}^J(\widetilde{s}_j^e - \underline{m}\cdot r_j) < \widetilde{z}_0 < \sum_{j=1}^J \widetilde{s}_j^e$. Combine the performance bound in 2.1-2.3 and we get the asymptotic optimality result of the TA policy.

(i) We first construct an auxiliary system in which the decision-maker can observe the future realized feature in advance before making the replenishment decision. We call it *advanced feature* system (denoted by AF). We also define the replenishment decision as z_0^{AF} and the corresponding allocation decision as z_j^{AF} in the AF system.

For any given feature $\boldsymbol{x}$, the objective function of the AF system can be expressed as follows.

$$\min_{\substack{z_0>0,\\ z_j\geq 0,\forall j\in[J]^+}} \quad h_0\cdot z_0+\sum_{j=1}^{J} C_j(z_j|\boldsymbol{x}_j).$$

$$\text{s.t.} \quad \sum_{j=1}^{J} z_j \leq z_0.$$

We can use KKT conditions to solve the above problem and the optimal solutions are given by $z_j^{AF}=s_j^l$ and $z_0^{AF}=\sum_j s_j^l$ with $s_j^l \triangleq F_j^{-1}\left(\frac{b_j}{b_j+h_j'}\middle|\boldsymbol{x}_j\right)$. The inventory cost under AF system given feature $\boldsymbol{x}$ is $\mathbf{C_{AF}}(\boldsymbol{x})=h_0\cdot z_0^{AF}+\sum_{j=1}^{J} C_j(z_j^{AF}|\boldsymbol{x}_j)$ and the corresponding expected cost is $\mathbf{C_{AF}}=\mathbb{E}_{\boldsymbol{x}}\left[\mathbf{C_{AF}}(\boldsymbol{x})\right]$. Since $\mathbf{C_{OPT}}(\boldsymbol{x})=h_0\cdot z_0^*+\sum_{j=1}^{J} C_j(z_j^*(z_0^*,\boldsymbol{x})|\boldsymbol{x}_j)$ where z_0^* and z_j^* are the optimal solutions for Eq. (5) and (6), which is suboptimal for the objective function $h_0\cdot z_0+\sum_{j=1}^{J} C_j(z_j|\boldsymbol{x}_j)$. Thus we have that for any features $\boldsymbol{x}$, the result $\mathbf{C_{AF}}(\boldsymbol{x})\leq\mathbf{C_{OPT}}(\boldsymbol{x})$ holds and consequently we have $\mathbf{C_{AF}}\leq\mathbf{C_{OPT}}$, which indicates that this auxiliary system generates a lower bound on the optimal cost $\mathbf{C_{OPT}}$. According to Assumption 1, we have:

$$z_0^{AF}=\sum_{j=1}^{J}\left[\beta_{j0}+\sum_{p=1}^{P}\beta_{jp}\cdot x_{jp}\right], \tag{A11}$$

and

$$\widetilde{z}_0=\sum_{j=1}^{J}\left[\beta_{j0}+\sum_{p=1}^{P}\mathbb{E}\left[\beta_{jp}\cdot x_{jp}\right]\right]. \tag{A12}$$

Without loss of generality, we assume that $\mathbb{E}\left[x_{jp}\right]=0$. Otherwise, we can define new variables $x_{jp}'\triangleq x_{jp}-\mathbb{E}\left[x_{jp}\right]$ and $\beta_{j0}'\triangleq\beta_{j0}+\sum_{p=1}^{P}\mathbb{E}\left[x_{jp}\right]$. This transformation does not affect our subsequent analysis and the new feature variables have a zero mean.

Let $\mathscr{A}$ denote the domain of features $\boldsymbol{x}$, $\mathscr{A}_1\triangleq\left\{\boldsymbol{x}\in\mathscr{A}\ \middle|\ \sum_{j=1}^{J}(\widetilde{s}_j^e-\underline{m}\cdot r_j)\geq\widetilde{z}_0\right\}$, $\mathscr{A}_2\triangleq\left\{\boldsymbol{x}\in\mathscr{A}\ \middle|\ \sum_{j=1}^{J}(\widetilde{s}_j^e-\underline{m}\cdot r_j)<\widetilde{z}_0<\sum_{j=1}^{J}\widetilde{s}_j^e\right\}$, and $\mathscr{A}_3\triangleq\left\{\boldsymbol{x}\in\mathscr{A}\ \middle|\ \sum_{j=1}^{J}\widetilde{s}_j^e\leq\widetilde{z}_0\right\}$. Clearly, $\mathscr{A}=\mathscr{A}_1\cup\mathscr{A}_2\cup\mathscr{A}_3$. To construct the performance bound, we calculate the performance difference $\mathbf{C_{TA}}-\mathbf{C_{AF}}$ which is an upper bound of the real performance difference $\mathbf{C_{TA}}-\mathbf{C_{OPT}}$.

$$\begin{aligned}
&\mathbf{C_{TA}}-\mathbf{C_{OPT}}\leq\mathbf{C_{TA}}-\mathbf{C_{AF}}=\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left\{h_0\cdot\left(\widetilde{z}_0-z_0^{AF}\right)+\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j)-C_j(z_j^{AF}|\boldsymbol{x}_j)\right]\right\}\\
&=\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[h_0\cdot\left(\widetilde{z}_0-z_0^{AF}\right)\right]+\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j)-C_j(s_j^l|\boldsymbol{x}_j)\right]\right\}\\
&=\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j)-C_j(s_j^l|\boldsymbol{x}_j)\right]\right\}+\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j)-C_j(s_j^l|\boldsymbol{x}_j)\right]\right\}+\\
&\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_3}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j)-C_j(s_j^l|\boldsymbol{x}_j)\right]\right\},
\end{aligned} \tag{A13}$$

where the third equation holds due to $\mathbb{E}_{\boldsymbol{x}}\left[h_0 \cdot (\widetilde{z}_0 - z_0^{AF})\right] = 0$ by the definition of $\widetilde{z}_0$ and z_0^{AF} in (A11) and (A12).

We divide the expected cost difference into three different parts, as our Taylor approximation policy would have different forms of solutions in the three regions. We next bound each term in (A13).

- **The bound on the first term in (A13)**

The first term can be bounded as follows.

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j) - C_j(s_j^l|\boldsymbol{x}_j)\right]\right\} \le \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1}\left\{\sum_{j=1}^{J}\left[C_j(0|\boldsymbol{x}_j) - C_j(s_j^e|\boldsymbol{x}_j)\right]\right\} \\
&\le \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1}\left[\sum_{j=1}^{J}(b_j + h_0)\cdot s_j^e\right] \le \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1}\left[(\bar{b} + h_0)\bar{D}\cdot J\right] = (\bar{b}+h_0)\bar{D}J\cdot\mathbb{P}\left(\boldsymbol{x}\in\mathscr{A}_1\right),
\end{aligned} \tag{A14}$$

where the first inequality holds as $C_j(\cdot|\boldsymbol{x}_j)$ is a decreasing function in $[0, s_j^e]$, the second follows from Lagrange Mean Value Theorem and the last inequality follows from Assumption 2. It suffices to bound $\mathbb{P}\left(\boldsymbol{x}\in\mathscr{A}_1\right)$ which can be expressed as follows.

$$\begin{aligned}
\mathbb{P}\left(\boldsymbol{x}\in\mathscr{A}_1\right) &= \mathbb{P}\left(\widetilde{z}_0 \le \sum_{j=1}^{J}\left(\widetilde{s}_j^e - \underline{m}\cdot r_j\right)\right) \\
&= \mathbb{P}\left(\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp} \ge \sum_{j=1}^{J}(\underline{m} - h_0)\cdot r_j\right) \\
&\le \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \ge \sum_{j=1}^{J}(\underline{m} - h_0)\cdot r_j\right).
\end{aligned} \tag{A15}$$

We next bound $\sum_{j=1}^{J}(\underline{m} - h_0)\cdot r_j$. Before proceeding, we first construct the lower bound on s_j^l as follows.

$$\begin{aligned}
s_j^l &\triangleq F_j^{-1}\left(\frac{b_j}{b_j + h_j'}\middle|\boldsymbol{x}_j\right) = F_j^{-1}\left(0|\boldsymbol{x}_j\right) + \frac{b_j}{b_j + h_j'}\cdot\left(F_j^{-1}\left(\xi|\boldsymbol{x}_j\right)\right)' \\
&= 0 + \frac{b_j}{b_j + h_j'}\cdot\frac{1}{f_j\left(\eta|\boldsymbol{x}_j\right)} \ge \frac{b_j}{(b_j + h_j')\cdot u_j} = b_j\cdot p_j,
\end{aligned}$$

where $\xi \in (0, \frac{b_j}{b_j + h_j'})$ and $\eta \in (0, s_j^l)$. The first equation directly follows from Taylor expansion, the second from the property of the derivative of the inverse function, and the inequality holds according to Assumption 2 that u_j is the upper bound of $f_j(\cdot|\boldsymbol{x}_j)$ for all $\boldsymbol{x}_j$. Thus the term $\sum_{j=1}^{J}(\underline{m} - h_0)\cdot r_j$ has the following lower bound.

$$\begin{aligned}
\sum_{j=1}^{J}(\underline{m} - h_0)\cdot r_j &= \sum_{j=1}^{J} r_j\cdot\min\frac{s_j^l}{r_j} \ge \sum_{j=1}^{J} r_j\cdot\min\frac{b_j\cdot p_j}{r_j} \ge \sum_{j=1}^{J} p_j\cdot\min\frac{\frac{t_j}{\alpha_j}\cdot b_j}{t_j} \\
&\ge \frac{1}{\alpha}\sum_{j=1}^{J} p_j\cdot\min b_j = \frac{1}{\alpha}\underline{p}\underline{h}J,
\end{aligned} \tag{A16}$$

where $\underline{p} \triangleq \min_j p_j$ is a constant by Assumption 2. Putting (A16) into (A15), we can obtain

$$
\begin{aligned}
\mathbb{P}(\boldsymbol{x} \in \mathscr{A}_1) &\leq \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \geq \sum_{j=1}^{J}(\underline{m}-h_0)\cdot r_j\right) \\
&\leq \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \geq \frac{1}{\alpha}\underline{p}\underline{h}J\right).
\end{aligned} \tag{A17}
$$

According to the Hoeffding's inequality (see Theorem 2.6.3 by Vershynin 2018), for every $\delta \geq 0$, we have

$$
\mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \geq \delta\right) \leq 2\exp\left(-\frac{\delta^2}{2\bar{M}^2||\boldsymbol{\beta}||_2^2}\right) \leq 2\exp\left(-\frac{\delta^2}{2JP\bar{M}^2\bar{B}^2}\right), \tag{A18}
$$

where $\boldsymbol{\beta} = (\beta_{11}, \beta_{12}, \cdots, \beta_{1P}, \beta_{21}, \cdots, \beta_{JP})$, and $||\boldsymbol{\beta}||_2 = \sqrt{\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}^2}$ is the ℓ_2 norm.

Let $\delta = \frac{1}{\alpha}\underline{p}\underline{h}J$. Equations (A17) and (A18) imply the following result.

$$
\begin{aligned}
\mathbb{P}(\boldsymbol{x} \in \mathscr{A}_1) &\leq \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \geq \frac{1}{\alpha}\underline{p}\underline{h}J\right) \\
&\leq 2\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right).
\end{aligned} \tag{A19}
$$

Putting (A19) into (A14), we can have the bound on the first term as follows.

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j) - C_j(s_j^l|\boldsymbol{x}_j)\right]\right\} &\leq (\bar{b}+h_0)\bar{D}J\cdot\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_1) \\
&\leq 2(\bar{b}+h_0)\bar{D}J\cdot\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right),
\end{aligned} \tag{A20}
$$

which converges to 0 as J tends to infinity.

- **The bound on the third term in (A13)**

We now bound the third term in (A13).

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_3}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j) - C_j(s_j^l|\boldsymbol{x}_j)\right]\right\} = \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_3}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{s}_j^e|\boldsymbol{x}_j) - C_j(s_j^l|\boldsymbol{x}_j)\right]\right\} \\
&\leq \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_3}\left\{\sum_{j=1}^{J}\left[h_0 r_j\cdot C_j'(\widetilde{s}_j^e|\boldsymbol{x}_j)\right]\right\} \leq \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_3}\left\{\sum_{j=1}^{J}\left[h_0 r_j\cdot(h_j'-h_0)\right]\right\} \\
&\leq h_0\bar{t}\bar{h}J\cdot\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_3),
\end{aligned} \tag{A21}
$$

where $\bar{t} \triangleq \max_j t_j$ is a constant, the first inequality holds due to the convex property of $C_j(\cdot|\boldsymbol{x}_j)$ and the second inequality holds as $C_j'(\widetilde{s}_j^e|\boldsymbol{x}_j) \in (-h_0, h_j'-h_0]$. Next it suffices to bound $\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_3)$.

$$
\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_3) = \mathbb{P}\left(\widetilde{z}_0 \geq \sum_{j=1}^{J}\widetilde{s}_j^e\right) = \mathbb{P}\left(\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp} \leq -\sum_{j=1}^{J}h_0\cdot r_j\right)
$$

$$
\begin{aligned}
&\leq \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \geq \sum_{j=1}^{J} h_0 \cdot r_j\right) \\
&\leq \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \geq \underline{p}h_0 \cdot J\right) \\
&\leq 2\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right),
\end{aligned}
\tag{A22}
$$

where the last inequality is similar to the bound on $\mathbb{P}(\boldsymbol{x} \in \mathscr{A}_1)$ in (A17). Putting (A22) into (A21), we can obtain

$$
\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_3}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j) - C_j(s_j^l|\boldsymbol{x}_j)\right]\right\} \leq h_0\bar{t}\bar{h}J \cdot \mathbb{P}(\boldsymbol{x} \in \mathscr{A}_3) \leq 2h_0\bar{t}\bar{h}J \cdot \exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right), \tag{A23}
$$

which converges to 0 as J grows.

- **The bound on the second term in (A13)**

The second term can be bounded by

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left\{\sum_{j=1}^{J}\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j) - C_j(s_j^l|\boldsymbol{x}_j)\right]\right\} \\
&\leq \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left\{\sum_{j=1}^{J}\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0 \cdot (\widetilde{z}_j - s_j^l) + \frac{(\widetilde{z}_j - s_j^l)^2}{2 \cdot p_j} - C_j(s_j^l|\boldsymbol{x}_j)\right]\right\} \\
&= \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J}\frac{(\widetilde{z}_j - s_j^l)^2}{2 \cdot p_j}\right] + h_0 \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J} s_j^l - \widetilde{z}_0\right],
\end{aligned}
\tag{A24}
$$

where the first inequality follows from the upper bound of Taylor approximation (see (A3)). We can bound the first part in (A24) as follows.

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J}\frac{(\widetilde{z}_j - s_j^l)^2}{2 \cdot p_j}\right] = \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J}\frac{\left(\frac{r_j}{\sum_j r_j}\right)^2 \cdot (\sum_j s_j^l - \widetilde{z}_0)^2}{2 \cdot p_j}\right] \\
&\leq \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\frac{\alpha^3 \cdot (\sum_j s_j^l - \widetilde{z}_0)^2}{2\sum_{j=1}^{J} t_j}\right] \leq \alpha^3\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\frac{\left(\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right)^2}{2\sum_{j=1}^{J} t_j}\right] \\
&\leq \frac{\alpha^3}{2\underline{t} \cdot J} \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right]^2 \leq \frac{\alpha^3}{2\underline{t} \cdot J} \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right]^2 \\
&= \frac{\alpha^3}{2\underline{t} \cdot J} \cdot \sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}^2 \mathrm{Var}(x_{jp}) \leq \frac{\alpha^3 P\bar{B}^2\bar{\sigma}^2}{2\underline{t}} = O(1),
\end{aligned}
\tag{A25}
$$

where $t_j = \dfrac{1}{(b_j + h_j')\ell_j}$, $\alpha = \max_j \dfrac{u_j}{\ell_j}$, $\underline{t} \triangleq \min_j t_j$ is a constant, and $\bar{B}$ is the uniform bound for β_{jp} according to our Assumption 1. The first inequality holds due to $t_j \leq \alpha p_j$ and the last inequality holds due to the finite variance assumption that $\mathrm{Var}(x_{jp}) \leq \bar{\sigma}^2$ for all j and p.

The second part in (A24) can be bounded by

$$
\begin{aligned}
& h_0 \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J} s_j^l - \widetilde{z}_0\right] = h_0 \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right] \\
& = h_0 \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right] - h_0 \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1\cup\mathscr{A}_3}\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right] \\
& \le h_0 \cdot \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1\cup\mathscr{A}_3}\left[\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right|\right] \le h_0 J P \bar{B}\bar{M}\cdot\left(\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_1)+\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_3)\right) \\
& \le 4h_0 J P \bar{B}\bar{M}\cdot\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right),
\end{aligned}
\tag{A26}
$$

where the second inequality holds due to the bounded assumption on feature x_{jp} and the last inequality from (A19) and (A22).

Putting (A20), (A21), (A25) and (A26) into (A13), we can conclude the following result

$$
\begin{aligned}
\mathbf{C_{TA}} - \mathbf{C_{AF}} & \le 2\left[(\bar{b}+h_0)\bar{D} + h_0\bar{t}\bar{h} + 2h_0 P\bar{B}\bar{M}\right] J\cdot\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right) + \frac{\alpha^3 P\bar{B}^2\bar{\sigma}^2}{2\underline{t}} \\
& = J\cdot O(\exp(-J)) + O(1) = O(1)
\end{aligned}
\tag{A27}
$$

which shows that $\mathbf{C_{TA}} - \mathbf{C_{AF}}$ can be bounded by a constant.

Besides, we can easily verify that for any $\boldsymbol{x}$ in its domain,

$$
\begin{aligned}
\mathbf{C_{AF}}(\boldsymbol{x}) & = \sum_{j=1}^{J}\mathbb{E}\left[h_j'(q_j^* - \epsilon_j)^+ + b_j(\epsilon_j - q_j^*)^+\right] \\
& = \sum_{j=1}^{J}\int_{-\bar{\epsilon}}^{q_j^*} h_j'(q_j^* - x)\mathrm{d}F_{\epsilon_j}(x) + \int_{q_j^*}^{\bar{\epsilon}} b_j(x - q_j^*)\mathrm{d}F_{\epsilon_j}(x) \ge \underline{h}\cdot\underline{\ell}\bar{\epsilon}^2 J,
\end{aligned}
$$

where $\underline{\ell} \triangleq \min_j \ell_j$, $F_{\epsilon_j}(\cdot)$ is the CDF for random noise ϵ_j and $-\bar{\epsilon}$, $\bar{\epsilon}$ are its lower and upper bound in Assumption 2 for any $j\in[J]^+$. The first equality follows the fact that for any given $\boldsymbol{x}$, the cost under AF model is just the summation of J newsvendor costs w.r.t. random noise ϵ_j and $q_j^* \triangleq F_{\epsilon_j}^{-1}(\frac{b_j}{b_j+h_j'})$ is the optimal solution. The last inequality follows from Assumption 2 which assumes that $\mathrm{d}F_{\epsilon_j}(\cdot) \ge \ell_j$. Therefore, we have

$$
\frac{\mathbf{C_{TA}} - \mathbf{C_{OPT}}}{\mathbf{C_{OPT}}} \le \frac{\mathbf{C_{TA}} - \mathbf{C_{AF}}}{\mathbf{C_{OPT}}} \le \frac{\mathbf{C_{TA}} - \mathbf{C_{AF}}}{\mathbf{C_{AF}}} = O(J^{-1}).
$$

By definition, we have $\mathbb{E}_{\boldsymbol{x}}\left[\widetilde{z}_0 - \sum_{j=1}^{J} s_j^l\right] = \mathbb{E}_{\boldsymbol{x}}\left[\sum_{p=1}^{P}\beta_{jp}x_{jp}\right] = 0$. For $\mathbb{E}_{\boldsymbol{x}}\left[|\widetilde{z}_j - s_j^l|\right]$, we have:

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{x}}\left[|\widetilde{z}_j - s_j^l|\right] & = \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1\cup\mathscr{A}_3}\left[|\widetilde{z}_j - s_j^l|\right] + \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[|\widetilde{z}_j - s_j^l|\right] \\
& \le \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_1\cup\mathscr{A}_3}\left[\bar{D}\right] + \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\frac{r_j}{\sum_j r_j}\cdot\left|\sum_j s_j^l - \widetilde{z}_0\right|\right] \\
& \le 4\bar{D}\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right) + \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\frac{r_j}{\sum_j r_j}\cdot\left|\sum_j s_j^l - \widetilde{z}_0\right|\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq 4\bar{D}\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right) + \mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\frac{t_j}{\sum_{j=1}^J p_j}\cdot\left|\sum_{j=1}^J s_j^l - \widetilde{z}_0\right|\right] \\
&\leq 4\bar{D}\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right) + \frac{\bar{t}}{\underline{p}\cdot J}\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[\left|\sum_{j=1}^J\sum_{p=1}^P \beta_{jp}x_{jp}\right|\right] \\
&= 4\bar{D}\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right) + \frac{\bar{t}}{\underline{p}\cdot J}\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[\sqrt{(\sum_{j=1}^J\sum_{p=1}^P \beta_{jp}x_{jp})^2}\right] \\
&\leq 4\bar{D}\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right) + \frac{\bar{t}}{\underline{p}\cdot J}\sqrt{\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[(\sum_{j=1}^J\sum_{p=1}^P \beta_{jp}x_{jp})^2\right]} \\
&= 4\bar{D}\exp\left(-\frac{\underline{p}^2\underline{h}^2 J}{2\alpha^2 P\bar{M}^2\bar{B}^2}\right) + \frac{\bar{t}}{\underline{p}\cdot J}\sum_{j=1}^J\sum_{p=1}^P |\beta_{jp}|\sqrt{\mathrm{Var}(x_{jp})} \\
&= O(\exp(-J)) + O(1) = O(1),
\end{aligned}
$$

where the second inequality follows from (A19) and (A22) and that $\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_1\cup\mathscr{A}_3) = \mathbb{P}(\boldsymbol{x}\in\mathscr{A}_1) + \mathbb{P}(\boldsymbol{x}\in\mathscr{A}_3)$ and the last inequality holds according to the Jensen's inequality.

(ii) We first bound $(\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{OPT}}(\boldsymbol{x}))^+$. It directly follows the definition that $(\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{OPT}}(\boldsymbol{x}))^+ \leq (\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x}))^+ \leq |\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x})|$. Thus we instead bound the term $|\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x})|$ which can be bounded as follows.

$$
|\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x})| \leq |h_0\cdot\sum_{j=1}^J\sum_{p=1}^P \beta_{jp}x_{jp}| + |\sum_{j=1}^J\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j) - C_j(z_j^{AF}|\boldsymbol{x}_j)\right]|. \tag{A28}
$$

Recall that in (A18), we have for every $\delta \geq 0$,

$$
\mathbb{P}\left(\left|\sum_{j=1}^J\sum_{p=1}^P \beta_{jp}x_{jp}\right| \geq \delta\right) \leq 2\exp\left(-\frac{\delta^2}{2JP\bar{M}^2\bar{B}^2}\right),
$$

It directly follows that

$$
\mathbb{P}\left(\left|\sum_{j=1}^J\sum_{p=1}^P \beta_{jp}x_{jp}\right| < \delta\right) \geq 1 - 2\exp\left(-\frac{\delta^2}{2JP\bar{M}^2\bar{B}^2}\right). \tag{A29}
$$

We now turn to the second term on the right-hand side of (A28). By (A24), we can claim that when $\sum_{j=1}^J\left(\widetilde{s}_j^e - \underline{m}\cdot r_j\right) < \widetilde{z}_0 < \sum_{j=1}^J \widetilde{s}_j^e$, (i.e., $\boldsymbol{x}\in\mathscr{A}_2$), the second term on the right-hand side of (A28) can be bounded as follows.

$$
\begin{aligned}
|\sum_{j=1}^J\left[C_j(\widetilde{z}_j|\boldsymbol{x}_j) - C_j(z_j^{AF}|\boldsymbol{x}_j)\right]| &\leq \sum_{j=1}^J\left[C_j(s_j^l|\boldsymbol{x}_j) - h_0\cdot(\widetilde{z}_j - s_j^l) + \frac{(\sum_j s_j^l - \widetilde{z}_0)^2}{2\cdot p_j} - C_j(s_j^l|\boldsymbol{x}_j)\right] \\
&= h_0\cdot(\sum_j s_j^l - \widetilde{z}_0) + \sum_{j=1}^J\left[\frac{(\sum_j s_j^l - \widetilde{z}_0)^2}{2\cdot p_j}\right]
\end{aligned}
$$

$$\leq h_0 \cdot |\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}| + \frac{\alpha^3}{2\underline{t}J}|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}|^2,$$

where the first inequality follows from (A7) and the last inequality from (A25). Putting this result into (A28), we can bound $|\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x})|$ by

$$|\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x})| \leq 2h_0 \cdot |\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}| + \frac{\alpha^3}{2\underline{t}J}|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}|^2. \tag{A30}$$

We define the event

$$\mathscr{B} \triangleq \left\{\boldsymbol{x} \in \mathscr{A} \,\middle|\, |\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x})| < 2h_0\delta + \frac{\alpha^3}{2\underline{t}J}\delta^2\right\}.$$

We next bound $\mathbb{P}(\boldsymbol{x}\in\mathscr{B})$ by the fact that $\mathbb{P}(\boldsymbol{x}\in\mathscr{B}) \geq \mathbb{P}(\boldsymbol{x}\in\mathscr{B},\boldsymbol{x}\in\mathscr{A}_2) = \mathbb{P}(\boldsymbol{x}\in\mathscr{B}|\boldsymbol{x}\in\mathscr{A}_2)\cdot\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_2)$. To this end, we separately bound $\mathbb{P}(\boldsymbol{x}\in\mathscr{B}|\boldsymbol{x}\in\mathscr{A}_2)$ and $\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_2)$. The probability $\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_2)$ can be bounded by

$$\begin{aligned}
\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_2) &= 1 - \mathbb{P}(\boldsymbol{x}\in\mathscr{A}_1) - \mathbb{P}(\boldsymbol{x}\in\mathscr{A}_3)\\
&= 1 - \mathbb{P}\left(\widetilde{z}_0 \leq \sum_{j=1}^{J}(\widetilde{s}_j^e - \underline{m}\cdot r_j)\right) - \mathbb{P}\left(\widetilde{z}_0 \geq \sum_{j=1}^{J}\widetilde{s}_j^e\right)\\
&= 1 - \mathbb{P}\left(\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp} \geq \sum_{j=1}^{J}(\underline{m} - h_0)\cdot r_j\right) - \mathbb{P}\left(\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp} \leq -h_0\cdot\sum_{j=1}^{J} r_j\right)\\
&\geq 1 - \mathbb{P}\left(\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp} \geq \frac{1}{\alpha}\underline{p}h_0\cdot J\right) - \mathbb{P}\left(\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp} \leq -\frac{1}{\alpha}\underline{p}h_0\cdot J\right)\\
&= 1 - \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \geq \frac{1}{\alpha}\underline{p}h_0\cdot J\right)\\
&\geq 1 - 2\exp\left(-\frac{\underline{p}^2h_0^2J}{2\alpha^2P\bar{M}^2\bar{B}^2}\right),
\end{aligned} \tag{A31}$$

where the first inequality follows from (A16) and (A22). The probability $\mathbb{P}(\boldsymbol{x}\in\mathscr{B}|\boldsymbol{x}\in\mathscr{A}_2)$ can be bounded by

$$\begin{aligned}
\mathbb{P}(\boldsymbol{x}\in\mathscr{B}|\boldsymbol{x}\in\mathscr{A}_2) &\geq \mathbb{P}\left(2h_0\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| + \frac{\alpha^3}{2\underline{t}J}\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right|^2 < 2h_0\delta + \frac{\alpha^3}{2\underline{t}J}\delta^2\right)\\
&\geq 1 - 2\exp\left(-\frac{\underline{p}^2h_0^2\delta^2}{2\alpha^2JP\bar{M}^2\bar{B}^2}\right),
\end{aligned} \tag{A32}$$

where the last inequality follows from (A29). It immediately follows from (A31) and (A32) that

$$\mathbb{P}(\boldsymbol{x}\in\mathscr{B}) \geq \mathbb{P}(\boldsymbol{x}\in\mathscr{B}|\boldsymbol{x}\in\mathscr{A}_2)\cdot\mathbb{P}(\boldsymbol{x}\in\mathscr{A}_2) \geq 1 - 2\exp\left(-\frac{\underline{p}^2h_0^2J}{2\alpha^2P\bar{M}^2\bar{B}^2}\right) - 2\exp\left(-\frac{\underline{p}^2h_0^2\delta^2}{2\alpha^2JP\bar{M}^2\bar{B}^2}\right). \tag{A33}$$

Let $c_1 \triangleq \frac{\underline{p}^2 h_0^2}{2\alpha^2 P \bar{M}^2 \bar{B}^2}$, (A33) implies that for any $\boldsymbol{x} \in \mathscr{A}$, with probability $1 - 2\exp(-c_1 J) - 2\exp(-c_1\delta^2/J)$,

$$|\mathbf{C_{TA}}(\boldsymbol{x}) - \mathbf{C_{AF}}(\boldsymbol{x})| < 2h_0\delta + \frac{\alpha^3}{2\underline{t}J}\delta^2.$$

Follows from the results of (A11), (A12) and (A29):

$$\mathbb{P}\left(\left|\widetilde{z}_0 - \sum_{j=1}^{J} s_j^l\right| < \delta\right) = \mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| < \delta\right) \geq 1 - 2\exp\left(-\frac{\delta^2}{2JP\bar{M}^2\bar{B}^2}\right).$$

Notice that when $\boldsymbol{x} \in \mathscr{A}_2$, $\widetilde{z}_j = \widetilde{s}_j^e - \frac{r_j}{\sum_{j=1}^{J} r_j}(\sum_{j=1}^{J}\widetilde{s}_j^e - \widetilde{z}_0)$. Then, the bound on $|\widetilde{z}_j - s_j^l|$ is given by

$$|\widetilde{z}_j - s_j^l| = \left|\frac{r_j}{\sum_{j=1}^{J} r_j}\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| = \frac{r_j}{\sum_{j=1}^{J} r_j}\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \leq \frac{\bar{t}}{J\cdot\underline{p}}\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right|$$

According to the analysis for (A33), we can conclude that for any $\boldsymbol{x} \in \mathscr{A}$, with probability $1 - 2\exp(-c_1 J) - 2\exp(-c_1\delta^2/J)$, $|\widetilde{z}_j - s_j^l| < \bar{t}\delta/\underline{p}J$. $\square$

Proof of Proposition 4. To prove the result, we first show that the following results hold:

Result (a): $|\widehat{s}_j^l - s_j^l| = O_p(PN^{-1/2})$;

Result (b): $|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(s_j^l|\boldsymbol{x}_j)| = o_p(PN^{-1/3})$;

Result (c): $|\widehat{r}_j - r_j| = o_p(PN^{-1/3})$.

To derive result (a), we need the following lemma.

Lemma A1. *Under Assumptions 1,2 and 4, the coefficient estimations* $\widehat{\boldsymbol{\beta}}_j$ *in (20) are consistent with the real coefficient* $\boldsymbol{\beta}_j$*, and the rate of convergence* $|\widehat{\beta}_{jp} - \beta_{jp}| = O_p(N^{-1/2})$ *holds for any* $p \in [P]$ *and* $j \in [J]^+$*.*

Proof of Lemma A1. The consistency result directly follows from Chapter 4.1 in Koenker (2005). The convergence rate follows from Theorem 4.1 in Koenker (2005) which states that $\sqrt{N}\left(\widehat{\beta}_{jp} - \beta_{jp}\right)$ converges to a normal random variable with zero mean and a finite variance. Referring to Eq. (4.4) in this book, the Bahadur representation of estimators can be written as

$$\sqrt{N}\left(\widehat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j\right) = V_j^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{x}_j^i\psi_{\tau_j}\left(d_j^i - \xi_j^i(\tau_j)\right) + o_p(1), \tag{A34}$$

where $\boldsymbol{x}_j^i \triangleq (x_{j0}^i, x_{j1}^i, \cdots, x_{jP}^i)$ with $x_{j0}^i = 1$ for all i and j, $\psi_{\tau_j}(u) \triangleq \tau_j - \mathbf{1}(u < 0)$, $\xi_j^i(\tau_j)$ stands for the conditional τ_j-th quantile of D_j^i given $\boldsymbol{x}_j^i$ with $\tau_j \triangleq \frac{b_j}{b_j + h_j'}$, $d_j^i - \xi_j^i(\tau)$ can be viewed as the realization of random noise ϵ_j^i, and V_j stands for the covariance matrix for retailer j around the conditional quantile defined in Koenker's Condition A2(ii). As components in feature vector $\boldsymbol{x}_j$ are independent, V_j is a diagonal matrix and v_{jp}^{-1} is the p-th diagonal element of the inverse matrix.

For each component of $\widehat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j$, we have

$$\sqrt{N}\left(\widehat{\beta}_{jp} - \beta_{jp}\right) = v_{jp}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_{jp}^i \psi_{\tau_j}\left(\epsilon_j^i\right) + o_p(1), \tag{A35}$$

Notice that the random variable $x_{jp}^i \psi_{\tau_j}\left(\epsilon_j^i\right)$ has zero mean as x_{jp}^i and ϵ_j^i are independent and $\mathbb{E}[\psi_{\tau_j}\left(\epsilon_j^i\right)] = 0$ by its definition, and has finite variance due to its bounded property. Then by applying the Weak Law of Large Numbers (WLLN) in Theorem 2.2.6 of Durrett (2019), we have that for any positive $\eta_1 > 0$,

$$\frac{1}{N^{1/2+\eta_1}} \sum_{i=1}^{N} x_{jp}^i \psi_{\tau_j}\left(\epsilon_j^i\right) = o_p(1). \tag{A36}$$

Putting (A36) into (A35), we immediately have $|\widehat{\beta}_{jp} - \beta_{jp}| = o_p(N^{-1/2+\eta_1})$ for any $p \in [P]^+$ and $j \in [J]^+$ for any positive $\eta_1 > 0$, which is equivalent to say that $|\widehat{\beta}_{jp} - \beta_{jp}| = O_p(N^{-1/2})$. Compared with Koenker's assumptions, we assume a stronger condition in Assumption 4(ii) to replace the Condition A2(ii) so as to ensure the rate of convergence for $\widehat{s}_j^l$. □

According to Lemma A1, we can directly conclude $|\widehat{s}_j^l - s_j^l| = |\widehat{\beta}_{j0} - \beta_{j0} + \sum_{p=1}^{P}(\widehat{\beta}_{jp} - \beta_{jp}) \cdot x_{jp}| \leq |\widehat{\beta}_{j0} - \beta_{j0}| + \sum_{p=1}^{P} |\widehat{\beta}_{jp} - \beta_{jp}| \cdot \bar{M} = O_p(PN^{-1/2})$ holds.

To derive result (b), we first prove the following lemma.

Lemma A2. *Taking bandwidth in (21) as $w_N = c_1 N^{-1/3}$ for some constant $c_1 > 0$, the kernel density estimator $\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j)$ is consistent with the true conditional density $f_j(\widehat{s}_j^l|\boldsymbol{x}_j)$, and $|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(\widehat{s}_j^l|\boldsymbol{x}_j)| = o_p(N^{-1/3})$ holds for any $\boldsymbol{x}_j$ and $j \in [J]^+$ (ignoring log factors).*

Proof of Lemma A2. It is well-known that the kernel density estimator is a consistent estimator (see Wasserman 2006). The uniform rate of convergence result directly follows from Theorem 2 in Jiang (2017) and the reference therein, which state that for Lipschitz continuous probability density function (i.e., Hölder-continuous with the coefficient $\alpha = 1$), the following result holds. For any $\boldsymbol{x}_j$ and $j \in [J]^+$,

$$|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(\widehat{s}_j^l|\boldsymbol{x}_j)| = o_p\left(w_N + \sqrt{\frac{\ln N}{N \cdot w_N}}\right). \tag{A37}$$

Taking $w_N = c_1 N^{-1/3}$, the convergence rate can be written as $o_p(N^{-1/3})$ in (A37) (ignoring log factors).
□

Combining result (a) and Lemma A2, we can obtain that

$$|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(s_j^l|\boldsymbol{x}_j)| \leq |\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(\widehat{s}_j^l|\boldsymbol{x}_j)| + |f_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(s_j^l|\boldsymbol{x}_j)| = o_p(PN^{-1/3}), \tag{A38}$$

where the equality holds due to the facts that $|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(\widehat{s}_j^l|\boldsymbol{x}_j)| = o_p(N^{-1/3})$ by Lemma A2, $|f_j(\widehat{s}_j^l|\boldsymbol{x}_j) - f_j(s_j^l|\boldsymbol{x}_j)| \leq L \cdot |\widehat{s}_j^l - s_j^l| = o_p(PN^{-1/2+\eta_1})$ by Lemma A1, the assumption that the conditional probability density function f_j is Lipschitz continuous, and $o_p(N^{-1/3}) + o_p(PN^{-1/2+\eta_1}) = o_p(PN^{-1/3})$.

Now we prove result (c). Note that as f_j is always positive, it directly follows from (A38) that

$$|\widehat{r}_j - r_j| = \left| \frac{1}{(b_j + h_j') \cdot \widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j)} - \frac{1}{(b_j + h_j') \cdot f_j(s_j^l|\boldsymbol{x}_j)} \right| = o_p(PN^{-1/3}).$$

By the property of consistency that *if the estimator $\widehat{\theta}$ is consistent with respect to parameter θ, then for a continuous function f, $f(\widehat{\theta})$ is also a consistent estimator of $f(\theta)$*. We can conclude that $\widehat{z}_j(z_0, \boldsymbol{x})$ is consistent with $\widetilde{z}_j(z_0, \boldsymbol{x})$.

Now we are ready to prove the convergence rate of our data-driven allocation decisions $\widehat{z}_j$. To facilitate the proof, we assume that m_j are non-identical among all retailers $j \in [J]^+$; the proof can easily be extended to the case in which some retailers share the same value of m_j. Without loss of generality, we assume $m_1 < m_2 < \cdots < m_J$.

We will first show that when N tends to infinity, the result $\widehat{m}_1 < \widehat{m}_2 < \cdots < \widehat{m}_J$ holds with probability 1. Define $\Delta_j \triangleq m_j - m_{j-1} > 0$ for $j \in \{2, 3, \cdots, J\}$. We have $|m_j - \widehat{m}_j| = \left| \frac{s_j^l}{r_j} - \frac{\widehat{s}_j^l}{\widehat{r}_j} \right| \leq \frac{s_j^l}{r_j \widehat{r}_j} \cdot |\widehat{r}_j - r_j| + \frac{1}{\widehat{r}_j} \cdot |s_j^l - \widehat{s}_j^l| = o_p(PN^{-1/3})$. Thus, when N tends to infinity, $|m_{j-1} - \widehat{m}_{j-1}| + |m_j - \widehat{m}_j| < \Delta_j$ and $|m_j - \widehat{m}_j| + |m_{j+1} - \widehat{m}_{j+1}| < \Delta_{j+1}$ hold with probability 1. Consequently, we can conclude that $\widehat{m}_1 < \widehat{m}_2 < \cdots < \widehat{m}_J$ holds with probability 1 when N tends to infinity.

We consider the following two cases: $z_0 \geq \sum_{j=1}^J \widetilde{s}_j^e$ and $\sum_{j=1}^J (\widetilde{s}_j^e - m_1 \cdot r_j) < z_0 < \sum_{j=1}^J \widetilde{s}_j^e$.

Case 1: $z_0 \geq \sum_{j=1}^J \widetilde{s}_j^e$. When N tends to infinity, $z_0 > \sum_{j=1}^J (\widehat{s}_j^e - \underline{m} \cdot \widehat{r}_j) = \sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j)$ with probability 1, as $|\widehat{s}_j^e - \widetilde{s}_j^e| = o_p(PN^{-1/3})$. In this case, we further consider two subcases: $z_0 \geq \sum_{j=1}^J \widehat{s}_j^e$ and $\sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j) < z_0 < \sum_{j=1}^J \widehat{s}_j^e$.

Subcase 1(a): $z_0 \geq \sum_{j=1}^J \widehat{s}_j^e$. In this subcase, by results (a) and (c), we immediately have

$$|\widehat{z}_j(z_0, \boldsymbol{x}) - \widetilde{z}_j(z_0, \boldsymbol{x})| = |\widehat{s}_j^l - s_j^l| + h_0 \cdot |\widehat{r}_j - r_j| = o_p(PN^{-1/3}).$$

Subcase 1(b): $\sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j) < z_0 < \sum_{j=1}^J \widehat{s}_j^e$. Define $\epsilon_1 \geq 0$ and $\epsilon_2 > 0$ such that $z_0 = \epsilon_1 + \sum_{j=1}^J \widetilde{s}_j^e$ and $z_0 = -\epsilon_2 + \sum_{j=1}^J \widehat{s}_j^e$. Thus $\epsilon_1 + \epsilon_2 + \sum_{j=1}^J (s_j^l - \widehat{s}_j^l) + h_0 \cdot \sum_{j=1}^J (r_j - \widehat{r}_j) = 0$, and $\epsilon_2 \leq \epsilon_1 + \epsilon_2 = o_p(JPN^{-1/3})$ by results (a) and (c). Consequently, we have the following result.

$$\begin{aligned} |\widehat{z}_j(z_0, \boldsymbol{x}) - \widetilde{z}_j(z_0, \boldsymbol{x})| &= \left| \widehat{s}_j^e - \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} \cdot (\sum_{j=1}^J \widehat{s}_j^e - z_0) - \widetilde{s}_j^e \right| \\ &\leq |\widehat{s}_j^l - s_j^l| + \left| \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} \cdot (\sum_{j=1}^J \widehat{s}_j^l - z_0) - h_0 \cdot r_j \right| \end{aligned}$$

$$= |\widehat{s}_j^l - s_j^l| + \left| \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} \cdot (\epsilon_2 + \sum_{j=1}^J h_0 \cdot \widehat{r}_j) - h_0 \cdot r_j \right|$$
$$\leq |\widehat{s}_j^l - s_j^l| + \left| \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} \cdot \epsilon_2 \right| + h_0 \cdot |\widehat{r}_j - r_j| = o_p(PN^{-1/3}).$$

Case 2: $\sum_{j=1}^J (\widetilde{s}_j^e - m_1 \cdot r_j) < z_0 < \sum_{j=1}^J \widetilde{s}_j^e$. Similar to Case 1, one can show that $\sum_{j=2}^J (\widehat{s}_j^e - \widehat{m}_2 \cdot \widehat{r}_j) < z_0 \leq \sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j)$ holds with probability 1 when N tends to infinity. Therefore, in this case, we further consider three subcases: subcase 2(a): $z_0 \geq \sum_{j=1}^J \widehat{s}_j^e$, subcase 2(b): $\sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j) < z_0 < \sum_{j=1}^J \widehat{s}_j^e$, and subcase 2(c): $\sum_{j=2}^J (\widehat{s}_j^e - \widehat{m}_2 \cdot \widehat{r}_j) < z_0 \leq \sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j)$. The analysis of subcase 2(a) is similar with that of subcase 1(a) and thus omitted here. We next analyze two other subcases.

Subcase 2(b): $\sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j) < z_0 < \sum_{j=1}^J \widehat{s}_j^e$. In this subcase, we have

$$\begin{aligned} |\widehat{z}_j(z_0, \boldsymbol{x}) - \widetilde{z}_j(z_0, \boldsymbol{x})| &= \left| \widehat{s}_j^e - \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} \cdot (\sum_{j=1}^J \widehat{s}_j^e - z_0) - \widetilde{s}_j^e + \frac{r_j}{\sum_{j=1}^J r_j} \cdot (\sum_{j=1}^J \widetilde{s}_j^e - z_0) \right| \\ &\leq |\widehat{s}_j^l - s_j^l| + z_0 \cdot \left| \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} - \frac{r_j}{\sum_{j=1}^J r_j} \right| + \left| \frac{\widehat{r}_j \cdot \sum_{j=1}^J \widehat{s}_j^l}{\sum_{j=1}^J \widehat{r}_j} - \frac{r_j \cdot \sum_{j=1}^J s_j^l}{\sum_{j=1}^J r_j} \right|. \end{aligned} \tag{A39}$$

We next bound each term in (A39). Result (a) gives the bound on the first term. The second term can be bounded by

$$\begin{aligned} z_0 \cdot \left| \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} - \frac{r_j}{\sum_{j=1}^J r_j} \right| &\leq z_0 \cdot \left(\left| \frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j} - \frac{r_j}{\sum_{j=1}^J \widehat{r}_j} \right| + \left| \frac{r_j}{\sum_{j=1}^J \widehat{r}_j} - \frac{r_j}{\sum_{j=1}^J r_j} \right| \right) \\ &= z_0 \cdot \left(\frac{|\widehat{r}_j - r_j|}{\sum_{j=1}^J \widehat{r}_j} + \frac{r_j \cdot \sum_{j=1}^J |r_j - \widehat{r}_j|}{\sum_{j=1}^J \widehat{r}_j \cdot \sum_{j=1}^J r_j} \right) \\ &= O(J) \cdot o_p(J^{-1} PN^{-1/3}) = o_p(PN^{-1/3}), \end{aligned}$$

where the second equality holds due to result (c) and the third equality holds according to the property $o_p(\cdot)$ and that our proposed z_0 in the corresponding region would be bounded by $O(J)$.

For the third term in (A39), we can conduct a similar analysis and obtain the convergence rate of $o_p(PN^{-1/3})$. Therefore, we have $|\widehat{z}_j(z_0, \boldsymbol{x}) - \widetilde{z}_j(z_0, \boldsymbol{x})| = o_p(PN^{-1/3})$.

Subcase 2(c): $\sum_{j=2}^J (\widehat{s}_j^e - \widehat{m}_2 \cdot \widehat{r}_j) < z_0 \leq \sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j)$. We first consider retailer $j = 1$. Similar with Subcase 1(b), we define $\epsilon_1 \geq 0$ and $\epsilon_2 > 0$ such that $z_0 = -\epsilon_1 + \sum_{j=1}^J (\widehat{s}_j^e - \widehat{m}_1 \cdot \widehat{r}_j)$ and $z_0 = \epsilon_2 + \sum_{j=1}^J (\widetilde{s}_j^e - m_1 \cdot r_j)$. Thus $\epsilon_1 + \epsilon_2 + \sum_{j=1}^J (\widetilde{s}_j^e - \widehat{s}_j^e + \widehat{m}_1 \widehat{r}_j - m_1 r_j) = 0$ and $\epsilon_2 \leq \epsilon_1 + \epsilon_2 = o_p(JPN^{-1/3})$. With this result, we can immediately obtain the following result.

$$\begin{aligned} |\widehat{z}_1(z_0, \boldsymbol{x}) - \widetilde{z}_1(z_0, \boldsymbol{x})| &= \left| 0 - \widetilde{s}_1^e + \frac{r_1}{\sum_{j=1}^J r_j} \cdot (\sum_{j=1}^J \widetilde{s}_j^e - z_0) \right| \\ &= \left| -\widetilde{s}_1^e + \frac{r_1}{\sum_{j=1}^J r_j} \cdot (\sum_{j=1}^J \widetilde{s}_j^e - \epsilon_2 - \sum_{j=1}^J (\widetilde{s}_j^e - m_1 \cdot r_j)) \right| \\ &\leq \left| \frac{r_1}{\sum_{j=1}^J r_j} \cdot \epsilon_2 \right| + |m_1 \cdot r_1 - \widetilde{s}_1^e| = \left| \frac{r_1}{\sum_{j=1}^J r_j} \cdot \epsilon_2 \right| = o_p(PN^{-1/3}). \end{aligned}$$

The analysis for retailer $j \in \{2, 3, \cdots, J\}$ is similar and thus omitted here.

We are now ready to prove the consistent property of $\widehat{z}_0$ and the convergence rate. Referring to (17) and (24), $\widehat{z}_0$ and $\widetilde{z}_0$ are given by

$$\widehat{z}_0 = \frac{1}{N} \cdot \sum_{i=1}^{N} \sum_{j=1}^{J} \left[\widehat{\beta}_{j0} + \sum_{p=1}^{P} \widehat{\beta}_{jp} x_{jp}^i \right] \text{ and } \widetilde{z}_0 = \sum_{j=1}^{J} \left[\beta_{j0} + \sum_{p=1}^{P} \mathbb{E}\left[\beta_{jp} \cdot x_{jp}\right] \right].$$

Therefore, we immediately obtain that,

$$\begin{aligned} |\widehat{z}_0 - \widetilde{z}_0| &\le \sum_{j=1}^{J} |\widehat{\beta}_{j0} - \beta_{j0}| + \left| \sum_{j=1}^{J} \sum_{p=1}^{P} \cdot \left[\frac{1}{N} \sum_{i=1}^{N} \widehat{\beta}_{jp} x_{jp}^i - \mathbb{E}\left[\beta_{jp} x_{jp}\right] \right] \right| \\ &\le \sum_{j=1}^{J} |\widehat{\beta}_{j0} - \beta_{j0}| + \frac{1}{N} \cdot \left| \sum_{j=1}^{J} \sum_{p=1}^{P} \sum_{i=1}^{N} (\widehat{\beta}_{jp} - \beta_{jp}) x_{jp}^i \right| + \left| \sum_{j=1}^{J} \sum_{p=1}^{P} \beta_{jp} \left[\frac{1}{N} \sum_{i=1}^{N} x_{jp}^i - \mathbb{E}\left[x_{jp}\right] \right] \right| \\ &= O_p(JN^{-1/2}) + O_p(JPN^{-1/2}) + O_p(JPN^{-1/2}) = O_p(JPN^{-1/2}), \end{aligned} \tag{A40}$$

where the last inequality comes from the result in Lemma A1 and the convergence rate for WLLN that $\frac{1}{N} \sum_{i=1}^{N} x_{jp}^i - \mathbb{E}\left[x_{jp}\right] = O_p(N^{-1/2})$ (see Theorem 2.2.6 in Durrett 2019). The consistency result directly follows from the consistency of $\widehat{\beta}_{jp}$. □

Proof of Theorem 2. It follows from Proposition 4 that $|\widehat{z}_j(z_0, \boldsymbol{x}) - \widetilde{z}_j(z_0, \boldsymbol{x})| = o_p(PN^{-1/3})$. Notice that the derivative of function $C_j(\cdot|\boldsymbol{x}_j)$ is bounded by $|C_j'(\cdot|\boldsymbol{x}_j)| = |-b_j - h_0 + (b_j + h_j')F_j(\cdot|\boldsymbol{x}_j)| \le b_j + h_0$. By the convexity of $C_j(\cdot|\boldsymbol{x}_j)$, we have the following result.

$$|C_j(\widehat{z}_j(z_0, \boldsymbol{x})|\boldsymbol{x}_j) - C_j(\widetilde{z}_j(z_0, \boldsymbol{x})|\boldsymbol{x}_j)| \le (b_j + h_0) \cdot |\widehat{z}_j(z_0, \boldsymbol{x}) - \widetilde{z}_j(z_0, \boldsymbol{x})| = o_p(PN^{-1/3}). \tag{A41}$$

Therefore, the cost difference can be bounded as follows.

$$\begin{aligned} &|\mathbf{C_{DDTA}}(\boldsymbol{x}) - \mathbf{C_{TA}}(\boldsymbol{x})| \le h_0 \cdot |\widehat{z}_0 - \widetilde{z}_0| + \sum_{j=1}^{J} |C_j(\widehat{z}_j(\widehat{z}_0, \boldsymbol{x})|\boldsymbol{x}_j) - C_j(\widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})|\boldsymbol{x}_j)| \\ &\le h_0 \cdot |\widehat{z}_0 - \widetilde{z}_0| + \sum_{j=1}^{J} |C_j(\widehat{z}_j(\widehat{z}_0, \boldsymbol{x})|\boldsymbol{x}_j) - C_j(\widehat{z}_j(\widetilde{z}_0, \boldsymbol{x})|\boldsymbol{x}_j)| + \sum_{j=1}^{J} |C_j(\widehat{z}_j(\widetilde{z}_0, \boldsymbol{x})|\boldsymbol{x}_j) - C_j(\widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})|\boldsymbol{x}_j)|. \end{aligned} \tag{A42}$$

According to Proposition 4, the first term on the right-hand side of (A42) converges to 0 with the rate $O_p(JPN^{-1/2})$. Equation (A41) implies that the last term in (A42) converges to 0 with the rate $o_p(JPN^{-1/3})$. We now bound the second term. As $\widehat{z}_j(z_0, \boldsymbol{x})$ is a continuous function over z_0 and its derivative is upper bounded by 1, we can claim that $|\widehat{z}_j(\widehat{z}_0, \boldsymbol{x}) - \widehat{z}_j(\widetilde{z}_0, \boldsymbol{x})| \le |\widehat{z}_0 - \widetilde{z}_0|$. As a result, $|C_j(\widehat{z}_j(\widehat{z}_0, \boldsymbol{x})|\boldsymbol{x}_j) - C_j(\widehat{z}_j(\widetilde{z}_0, \boldsymbol{x})|\boldsymbol{x}_j)| \le (b_j + h_0) \cdot |\widehat{z}_j(\widehat{z}_0, \boldsymbol{x}) - \widehat{z}_j(\widetilde{z}_0, \boldsymbol{x})| \le (b_j + h_0) \cdot |\widehat{z}_0 - \widetilde{z}_0| = O_p(JPN^{-1/2})$. Combing the above results, we can obtain that

$$|\mathbf{C_{DDTA}}(\boldsymbol{x}) - \mathbf{C_{TA}}(\boldsymbol{x})| = O_p(JPN^{-1/2}) + O_p(J^2PN^{-1/2}) + o_p(JPN^{-1/3}) = o_p(J^2PN^{-1/3}). \tag{A43}$$

As Eq. (A43) uniformly holds for any $\boldsymbol{x}$ in its domain, we have $|\mathbf{C_{DDTA}} - \mathbf{C_{TA}}| = o_p(J^2PN^{-1/3})$ holds as well. □

Proof of Theorem 3. To prove Theorem 3, we first introduce the definition of *Uniform Stability* by Bousquet and Elisseeff (2002).

Definition A1 (Uniform Stability). An algorithm A has uniform stability parameter α with respect to the loss function ℓ if the following holds:

$$\forall S_N \in \mathcal{Z}^N, \forall i \in \{1, \ldots, N\}, ||\ell(A_{S_N}, .) - \ell(A_{S_{N\setminus i}}, .)||_\infty \leq \alpha,$$

where $S_N \triangleq \left\{ z^1 \triangleq (x^1, y^1), \ldots, z^N \right\}$ stands for training data set and $S_{N\setminus i} = \{z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^N\}$ stands for training data set by removing i-th instance.

Intuitively, an algorithm is uniformly stable if it is robust to changes in the training data set. We next provide the *Uniform Stability* parameter of our DDTA policy through the following proposition.

Proposition A1 (Uniform Stability of the DDTA Policy). *With the loss function for a single new instance* $(\boldsymbol{x}, \boldsymbol{d})$, $\ell_{\mathbf{DDTA}}(S_N; \boldsymbol{x}, \boldsymbol{d})$, *defined as follows*

$$\ell_{\mathbf{DDTA}}(S_N; \boldsymbol{x}, \boldsymbol{d}) \triangleq h_0 \cdot \widehat{z}_0 + \sum_{j=1}^{J} \left[-h_0 \cdot d_j + h_j \cdot (\widehat{z}_j(\widehat{z}_0, \boldsymbol{x}) - d_j) + (b_j + h_j') \cdot (d_j - \widehat{z}_j(\widehat{z}_0, \boldsymbol{x}))^+ \right],$$

the DDTA policy has uniform stability parameter $\alpha_{\mathbf{D}} = O(JPN^{-1}w_N^{-2})$, *where* w_N *is the bandwidth of kernel density estimator.*

Proof of Proposition A1. According to Ban and Rudin (2019)'s proof in their Proposition EC.1, we have that for any $j \in [J]^+$ and $\boldsymbol{x}_j$,

$$|\widehat{s}_j^l(\boldsymbol{x}_j) - \widehat{s}_j^{l\setminus i}(\boldsymbol{x}_j)| = \left| \sum_{p=0}^{P} \widehat{\beta}_{jp} x_{jp} - \sum_{p=0}^{P} \widehat{\beta}_{jp}^{\setminus i} x_{jp} \right| \leq \frac{b_j \bar{D}(P+1)}{h_j' N} = O(PN^{-1}), \tag{A44}$$

where $x_{j0} = 1$, $\widehat{\beta}_{jp}$ and $\widehat{\beta}_{jp}^{\setminus i}$ stand for the linear quantile regression parameters trained in data set S_N and $S_{N\setminus i}$, respectively, and $\widehat{s}_j^l(\boldsymbol{x}_j)$ and $\widehat{s}_j^{l\setminus i}(\boldsymbol{x}_j)$ are the corresponding estimated base-stock levels.

According to (A44), we have that for $\widehat{z}_0$ derived from S_N and $\widehat{z}_0^{\setminus i}$ derived from $S_{N\setminus i}$,

$$\begin{aligned}
|\widehat{z}_0 - \widehat{z}_0^{\setminus i}| &= \left| \frac{1}{N} \cdot \sum_{t=1}^{N} \sum_{j=1}^{J} \widehat{s}_j^l(\boldsymbol{x}_j^t) - \frac{1}{N-1} \cdot \sum_{t \neq i} \sum_{j=1}^{J} \widehat{s}_j^{l\setminus i}(\boldsymbol{x}_j^t) \right| \\
&\leq \left| \frac{1}{N} \cdot \sum_{t \neq i} \sum_{j=1}^{J} \widehat{s}_j^l(\boldsymbol{x}_j^t) - \frac{1}{N-1} \cdot \sum_{t \neq i} \sum_{j=1}^{J} \widehat{s}_j^{l\setminus i}(\boldsymbol{x}_j^t) \right| + \left| \frac{1}{N} \sum_{j=1}^{J} \widehat{s}_j^l(\boldsymbol{x}_j^i) \right| \\
&\leq \frac{1}{N-1} \left| \sum_{t \neq i} \sum_{j=1}^{J} (\widehat{s}_j^l(\boldsymbol{x}_j^t) - \widehat{s}_j^{l\setminus i}(\boldsymbol{x}_j^t)) \right| + \frac{1}{N(N-1)} \left| \sum_{t \neq i} \sum_{j=1}^{J} \widehat{s}_j^l(\boldsymbol{x}_j^i) \right| + \frac{1}{N} \left| \sum_{j=1}^{J} \widehat{s}_j^l(\boldsymbol{x}_j^i) \right| \\
&\leq \frac{\bar{b}\bar{D}J(P+1)}{\underline{h}N} + \frac{\bar{D}J}{N} + \frac{\bar{D}J}{N} = \frac{((P+1)\bar{b} + 2\underline{h})\bar{D}J}{\underline{h}N} = O(JPN^{-1}),
\end{aligned} \tag{A45}$$

where the last inequality follows from $\widehat{s}_j^l(\boldsymbol{x}_j) \le \bar{D}$. (As $\boldsymbol{x}_j$ has a bounded domain and $\widehat{s}_j^l(\boldsymbol{x}_j)$ does not scale with N and J, we could set $\bar{D}$ a very large constant to ensure this.)

We now study the stability of kernel density estimation. Notice that for any $\boldsymbol{x}_j$, $j \in [J]^+$, the gap between $\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j)$ and $\widehat{f}_j^{\setminus i}(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)$ can be bounded by

$$|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - \widehat{f}_j^{\setminus i}(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)| \le |\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - \widehat{f}_j(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)| + |\widehat{f}_j(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j) - \widehat{f}_j^{\setminus i}(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)|, \tag{A46}$$

where we define $\widehat{f}_j^{\setminus i}(\cdot|\boldsymbol{x}_j)$ as the kernel density estimator trained by the data set $S_{N\setminus i}$ and the bandwidth the same as that in $\widehat{f}_j(\cdot|\boldsymbol{x}_j)$, i.e., a given fixed bandwidth as in Ban and Rudin (2019). Note that for a varying bandwidth case, the following results still hold under a large-sample regime.

For the first term on the right-hand side of (A46),

$$|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - \widehat{f}_j(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)| = \frac{1}{N}\left|\sum_{t=1}^{N}\frac{1}{w_N}K\left(\frac{\widehat{s}_j^l(\boldsymbol{x}_j)-d_j^t}{w_N}\right) - \sum_{t=1}^{N}\frac{1}{w_N}K\left(\frac{\widehat{s}_j^{l\setminus i}(\boldsymbol{x}_j)-d_j^t}{w_N}\right)\right| \le \frac{b_j\bar{D}L(P+1)}{h_j' N\cdot w_N^2}, \tag{A47}$$

where L is the Lipschitz constant for Gaussian kernel $K(\cdot)$.

For the second term on the right-hand side of (A46), we abbreviate $\widehat{s}_j^{l\setminus i}(\boldsymbol{x}_j) - d_j^t$ as u_j^t and have

$$\begin{aligned} |\widehat{f}_j(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j) - \widehat{f}_j^{\setminus i}(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)| &= \left|\frac{1}{N}\sum_{t=1}^{N}\frac{1}{w_N}K\left(\frac{u_j^t}{w_N}\right) - \frac{1}{N-1}\sum_{t\neq i}\frac{1}{w_N}K\left(\frac{u_j^t}{w_N}\right)\right| \\ \le \left|\frac{1}{w_N}\cdot\frac{1}{N}\sum_{t\neq i}K\left(\frac{u_j^t}{w_N}\right) - \frac{1}{w_N}\cdot\frac{1}{N-1}\sum_{t\neq i}K\left(\frac{u_j^t}{w_N}\right)\right| &+ \left|\frac{1}{Nw_N}K\left(\frac{u_j^i}{w_N}\right)\right| \le \frac{2\bar{K}}{Nw_N}, \end{aligned} \tag{A48}$$

where $\bar{K}$ is the uniform upper bound of kernel function $K(\cdot)$. Combine results in (A46)-(A48), we can obtain

$$|\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j) - \widehat{f}_j^{\setminus i}(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)| = O(PN^{-1}w_N^{-2}). \tag{A49}$$

It directly follows from (A49) that the gap between $\widehat{r}_j(\boldsymbol{x}_j) \triangleq \dfrac{1}{(b_j+h_j')\widehat{f}_j(\widehat{s}_j^l|\boldsymbol{x}_j)}$ and $\widehat{r}_j^{\setminus i}(\boldsymbol{x}_j) \triangleq \dfrac{1}{(b_j+h_j')\widehat{f}_j^{\setminus i}(\widehat{s}_j^{l\setminus i}|\boldsymbol{x}_j)}$ can be expressed as

$$|\widehat{r}_j(\boldsymbol{x}_j) - \widehat{r}_j^{\setminus i}(\boldsymbol{x}_j)| = O(PN^{-1}w_N^{-2}). \tag{A50}$$

Then we conduct a similar analysis as in the proof of Proposition 4 and Theorem 2 to bound $|\widehat{z}_j(\widehat{z}_0,\boldsymbol{x}) - \widehat{z}_j^{\setminus i}(\widehat{z}_0^{\setminus i},\boldsymbol{x})|$ for any retailer $j \in [J]^+$ and feature $\boldsymbol{x}$, with the s_j^l and r_j therein replaced by $\widehat{s}_j^{l\setminus i}$ and $\widehat{r}_j^{\setminus i}$. We omit the details for brevity and conclude that for any $j \in [J]^+$ and $\boldsymbol{x}$:

$$|\widehat{z}_j(\widehat{z}_0,\boldsymbol{x}) - \widehat{z}_j^{\setminus i}(\widehat{z}_0^{\setminus i},\boldsymbol{x})| \le |\widehat{z}_j(\widehat{z}_0,\boldsymbol{x}) - \widehat{z}_j^{\setminus i}(\widehat{z}_0,\boldsymbol{x})| + |\widehat{z}_j^{\setminus i}(\widehat{z}_0,\boldsymbol{x}) - \widehat{z}_j^{\setminus i}(\widehat{z}_0^{\setminus i},\boldsymbol{x})| = O(PN^{-1}w_N^{-2}). \tag{A51}$$

Now we are ready to calculate the upper bound of the DDTA policy's stability parameter $\alpha_{\mathbf{D}}$. For any new instance $(\boldsymbol{x},\boldsymbol{d})$,

$$\begin{aligned}|\ell_{\mathbf{DDTA}}(S_N;\boldsymbol{x},\boldsymbol{d})-\ell_{\mathbf{DDTA}}(S_{N\setminus i};\boldsymbol{x},\boldsymbol{d})| &\leq h_0\cdot|\widehat{z}_0-\widehat{z}_0^{\setminus i}|+\left|\sum_{j=1}^{J}h_j\cdot\left(\widehat{z}_j(\widehat{z}_0,\boldsymbol{x})-\widehat{z}_j^{\setminus i}(\widehat{z}_0^{\setminus i},\boldsymbol{x})\right)\right| \\ &+\left|\sum_{j=1}^{J}(b_j+h_j')\cdot\left((d_j-\widehat{z}_j(\widehat{z}_0,\boldsymbol{x}))^+-(d_j-\widehat{z}_j^{\setminus i}(\widehat{z}_0^{\setminus i},\boldsymbol{x}))^+\right)\right|,\end{aligned} \tag{A52}$$

where according to (A45), the first term on the right-hand side of (A52) is bounded by $O(JPN^{-1})$. According to (A51), the second term on the right-hand side of (A52) is bounded by $O(JPN^{-1}w_N^{-2})$. Similarly, we have the third term on the right-hand side of (A52) also bounded by $O(JPN^{-1}w_N^{-2})$. As a result, we can conclude that $\alpha_{\mathbf{D}}\triangleq\sup_{i,(\boldsymbol{x},\boldsymbol{d})}|\ell_{\mathbf{DDTA}}(S_N;\boldsymbol{x},\boldsymbol{d})-\ell_{\mathbf{DDTA}}(S_{N\setminus i};\boldsymbol{x},\boldsymbol{d})|=O(JPN^{-1}w_N^{-2})$. □

With the uniform stability property of the DDTA policy, we can build its generalization error according to Corollary 8 in Bousquet et al. (2020). There exists a constant c_0 such that for any $0<\delta<1$, with probability at least $1-\delta$,

$$\begin{aligned}|\mathbf{R}_{\mathbf{DDTA}}(S_N)-\mathbf{C}_{\mathbf{DDTA}}(S_N)| &\lesssim \alpha_{\mathbf{D}}\ln N\ln(1/\delta)+(\bar{b}+h_0)J\bar{D}\sqrt{N^{-1}\ln(1/\delta)} \\ &\leq c_0\left(JPN^{-1}w_N^{-2}\ln N\ln(1/\delta)+J\sqrt{N^{-1}\ln(1/\delta)}\right).\end{aligned} \tag{A53}$$

Here $(\bar{b}+h_0)J\bar{D}$ is the uniform upper bound for $\mathbf{R}_{\mathbf{DDTA}}(\cdot)$. By taking w_N scaling as $N^{-1/3}$, we can obtain the result in Theorem 3. □

Proof of Corollary 1. Under the additional Assumption 6, we have access to the transition function $k_{jp}^t(\cdot)$, which can help us reduce the feature uncertainty by modifying the replenishment decision as $\widetilde{z}_0(\boldsymbol{x}^{[1,t]})\triangleq\sum_{j=1}^{J}\beta_{j0}+\sum_{p=1}^{P}\beta_{jp}\cdot k_{jp}^t(\boldsymbol{x}_{jp}^{[1,t]})$ (i.e., the replenishment decision under TA^{TC} policy). As a result, we have for any $m>0$:

$$\begin{aligned}\mathbb{P}\left\{|z_0^{AF}(\boldsymbol{x}^{t+1})-\widetilde{z}_0(\boldsymbol{x}^{[1,t]})|\geq m\right\} &=\mathbb{P}\left\{\left|\sum_{j=1}^{J}\left(\beta_{j0}+\sum_{p=1}^{P}\beta_{jp}x_{jp}^{t+1}\right)-\sum_{j=1}^{J}\left(\beta_{j0}+\sum_{p=1}^{P}\beta_{jp}\cdot k_{jp}^t(\boldsymbol{x}_{jp}^{[1,t]})\right)\right|\geq m\right\} \\ &=\mathbb{P}\left\{\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}\delta_{jp}^{t+1}\right|\geq m\right\}\leq 2\exp\left(-\frac{m^2}{2JP\bar{M}'\bar{B}^2}\right),\end{aligned} \tag{A54}$$

with $\bar{M}'$ the uniform upper bound for random noise δ_{jp}^{t+1} such that $|\delta_{jp}^{t+1}|\leq\bar{M}'$ holds almost surely for any $j\in[J]^+$, $p\in[P]^+$ and $t\in[T]^+$. As $\bar{M}'\leq\bar{M}$ by its definition, compared with (A18), knowing the feature transition function can help us lower the upper bound by ratio $\bar{M}'/\bar{M}$. The convergence rate can also be derived following Theorem 1 and details are omitted. □

Proof of Proposition 5. The proof logic is similar to that in Proposition 4. We again need to prove the following results when historical data is censored and Assumption 7 is satisfied.

Result (a): $|\widehat{s}_j^{lC} - s_j^l| = O_p(PN^{-1/2})$;

Result (b): $|\widehat{f}_j^C(\widehat{s}_j^{lC}|\boldsymbol{x}_j) - f_j(s_j^l|\boldsymbol{x}_j)| = O_p(PN^{-1/3})$;

Result (c): $|\widehat{r}_j^C - r_j| = O_p(PN^{-1/3})$, where $\widehat{r}_j^C \triangleq \dfrac{1}{(b_j + h_j') \cdot \widehat{f}_j^C(\widehat{s}_j^{lC}|\boldsymbol{x}_j)}$.

Result (a) is proved through the proof of Theorem 2 in Powell (1986) which implies

$$\sqrt{N}\left(\widehat{\beta}_{jp} - \beta_{jp}\right) = v_{jp}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{1}(\beta_{j0} + \sum_{p=1}^{P} \beta_{jp} x_{jp}^i < y_j^i) \cdot x_{jp}^i \psi_{\tau_j}(\epsilon_j^i) + o_p(1), \tag{A55}$$

where v_{jp}^{-1} is defined as the p-th diagonal element of the inverse of covariance matrix around the conditional quantile defined by Eq. (3.5) in Powell (1986). One can easily verify that Assumptions P.2 and R.2 in Powell (1986) are satisfied by our Assumptions 1 and 7(ii). By applying the weak law of large numbers, we can obtain the result (a).

Part (b) follows from Theorem 2.1 in Karunamuni and Yang (1991) by taking $w_N = a_1 N^{-1/6}$ for constant $a_1 > 0$:

$$|\widehat{f}_j^C(\cdot|\boldsymbol{x}_j) - f_j(\cdot|\boldsymbol{x}_j)| = O_p(N^{-1/3}). \tag{A56}$$

The remaining analysis just follows that of Proposition 4 and thus is omitted. □

B. Supplementary Analysis

In Online Appendix B, we provide the supplementary analysis in the paper.

B.1. Discussion on Theorem 1

We consider another simple heuristic policy as follows. First, the warehouse orders $z_0' = \mathbb{E}[\sum_{j=1}^J s_j^l(\boldsymbol{x}_j)] + c_0 \cdot g(J)$ for a constant $c_0 > 0$ and any $g(J) > 0$. After observing feature vector $\boldsymbol{x}_j$, the warehouse's allocation decision z_j' for each retailer j is $z_j' = s_j^l(\boldsymbol{x}_j)$ when $z_0' \geq \sum_{j=1}^J s_j^l(\boldsymbol{x}_j)$ and an arbitrary allocation with $\sum_j z_j' = z_0'$ when $z_0' < \sum_{j=1}^J s_j^l(\boldsymbol{x}_j)$. The main idea is to order extra inventory to ensure that each retailer j can be allocated with $s_j^l(\boldsymbol{x}_j)$ inventory with high probability and thus it is unnecessary to make the complex allocation decision. We refer to the policy as Extra Ordering (EO) policy. The following corollary provides its optimality gap.

Corollary B1 (Performance of the EO policy). *Denote by $\mathbf{C_{EO}}(\boldsymbol{x})$ the total inventory cost under the EO policy given feature $\boldsymbol{x}$ and $\mathbf{C_{EO}} \triangleq \mathbb{E}[\mathbf{C_{EO}}(\boldsymbol{x})]$. Then, the following result holds.*

$$h_0 c_0 \cdot g(J) \leq \mathbf{C_{EO}} - \mathbf{C_{AF}} \leq h_0 c_0 \cdot g(J) + c_1 \cdot J \exp(-g(J)^2/J) \tag{B1}$$

for a constant $c_1 > 0$.

Compared with the TA policy which has a constant optimality gap, the optimality gap for the EO policy is at least $g(J)$ due to the overordering in the warehouse. To be specific, the upper and lower bound match (up to a constant) when $g(J) = \sqrt{J \ln J}$.

Proof of Corollary B1. The left-hand side of inequality in (B1) is straightforward. The costs associated with retailers under EO policy are no less than that under AF policy according to its definition, but it charges an extra inventory holding cost of $\mathbb{E}_{\boldsymbol{x}} h_0(z_0' - z_0^{AF}) = h_0 c_0 \cdot g(J)$ in the warehouse according to (A13).

For the right-hand side of inequality in (B1), we first bound the total cost of the worst case. Obviously, it is bounded by $O(J)$ as the maximum inventory cost in each retailer is bounded by a constant.

Next we determine the upper bound of $\mathbb{P}\left(z_0' < \sum_{j=1}^J s_j^l(\boldsymbol{x}_j)\right)$. According to Hoeffding's inequality,

$$\mathbb{P}\left(z_0' < \sum_{j=1}^J s_j^l(\boldsymbol{x}_j)\right) \leq \mathbb{P}\left(\left|\sum_{j=1}^J \sum_{p=1}^P \beta_{jp} x_{jp}\right| \geq c_0 g(J)\right) \leq 2\exp\left(-\frac{c_0^2 g(J)^2}{2JP\bar{M}^2\bar{B}^2}\right). \tag{B2}$$

Thus the upper bound for the total inventory costs in retailers is $O(J\exp(-g(J)^2/J))$. $\square$

B.2. Time-Correlated Demand Features

In this subsection, we first discuss the performance of our proposed TA and DDTA policy under feature correlation. Then we further justify conditions under which DDTA^{TC} policy is consistent and provide its rate of convergence and generalization bound.

We have that under the stationary feature process in Assumption 5, TA policy can still work well with the rate of convergence equal to $1/J$, though the constant ratio is larger than that for TA^{TC} policy.

Proposition B2 (Performance of TA Policy Under Time-Correlated Feature). *Under Assumptions 1, 2, 5 and given feature vectors* $\boldsymbol{x}^{[1,t]}$, $\dfrac{\mathbf{C}^{\mathbf{t+1}}_{\mathbf{TA}}(\boldsymbol{x}^{[1,t]}) - \mathbf{C}^{\mathbf{t+1}}_{\mathbf{OPT}}(\boldsymbol{x}^{[1,t]})}{\mathbf{C}^{\mathbf{t+1}}_{\mathbf{OPT}}(\boldsymbol{x}^{[1,t]})} = O(J^{-1})$ *holds for any period* $t \in [T]^+$.

Proof of Proposition B2. As in time-correlated feature setting, for each period $t \in [T]^+$, obviously the AF policy defined before (observe feature $\boldsymbol{x}^{t+1}$ in advance and replenish according to $\boldsymbol{x}^{t+1}$) still serves as the lower bound of the optimal system cost. Due to the stationary assumption on feature which ensures the invariance of its mean[1], we have that $\mathbb{E}_{\boldsymbol{x}^{t+1}|\boldsymbol{x}^{[1,t]}} z_0^{AF}(\boldsymbol{x}^{t+1}) = \mathbb{E}_{\boldsymbol{x}^{t+1}|\boldsymbol{x}^{[1,t]}} \widetilde{z}_0(\boldsymbol{x}^{t+1})$. The remaining analysis is the same as in Theorem 1, and we can derive the same convergence rate. □

To derive the data-driven results parallel to Theorems 2 and 3 under correlation, we introduce the (strictly) stationary ϕ-mixing process:

Definition B2 (Stationary ϕ-Mixing). Let $\{X_t\}_{t=-\infty}^{\infty}$ be a strictly stationary sequence of random variables. For any $i, j \in \mathbb{Z} \cap \{-\infty, +\infty\}$, let σ_i^j denote the σ-algebra generated by the random variables $X_k, i < k < j$. Then, for any positive integer k, the ϕ-mixing coefficients of the stochastic process $\{X_t\}_{t=-\infty}^{\infty}$ are defined as

$$\phi(k) = \sup_{\substack{n \\ A \in \sigma_{n+k}^{\infty} \\ B \in \sigma_{-\infty}^{n}}} |\mathbb{P}(A|B) - \mathbb{P}(A)|$$

and $\{X_t\}_{t=-\infty}^{\infty}$ is said to be stationary ϕ-mixing if its ϕ-mixing coefficients $\phi(k) \to 0$ as $k \to \infty$.

In ϕ-mixing process, the dependence between X_t and X_{t+k} gradually vanishes as k grows. There are various examples in practice that follow a ϕ-mixing process, including Markov modulated process, bounded autoregressive moving-average (ARMA) process, etc., see Athreya and Pantula (1986), Qi et al. (2020) for details. The assumption below is a generalization of Assumption 4 for the stationary ϕ-mixing process.

[1] The weakly stationary assumption is sufficient to guarantee an invariant mean. We assume a strictly stationary process for the ease of following data-driven analysis.

Assumption B1 (Sample From Mixing Process). *For* $S_N \triangleq \{(d_1^i, \boldsymbol{x}_1^i), (d_2^i, \boldsymbol{x}_2^i), \cdots, (d_J^i, \boldsymbol{x}_J^i)\}_{i=1}^N$*, there exist positive definite matrices* M_j^0 *and a positive constant* $\bar{M} > 0$ *such that almost surely for any* $j \in [J]^+$*,*

(i) $\lim_{N\to\infty} \frac{1}{N} \cdot \sum_{i=1}^N \boldsymbol{x}_j^{i\top} \boldsymbol{x}_j^i = M_j^0$*;*

(ii) $\max_{i=1,\ldots,N} ||\boldsymbol{x}_j^i||_\infty \le \bar{M}$*;*

(iii) feature sequence $\{x_{jp}^i\}_{i=1}^N$ *follows a stationary* ϕ*-mixing process with geometrically bounded mixing coefficients* $\phi(k) \le \phi_0 \rho^k$ *for given constant* $\phi_0 > 0$ *and* $0 \le \rho < 1$ *for all* $p \in [P]^+$*;*

(iv) the sequence of random noise $\{\epsilon_j^i \triangleq d_j^i - \gamma_j - g_j(\boldsymbol{x}_j^i)\}_{i=1}^N$ *is i.i.d., where* $\gamma_j + g_j(\cdot)$ *is the demand function for retailer* j *in (1).*

Assumption B1 (i),(ii), and (iv) is the same as in Assumption 4; (iii) models the ϕ-mixing property of feature data. Specifically, when $\rho = 0$ it just degenerates to the i.i.d. setting in Assumption 4.

We study the performance of our proposed DDTA policy in (22)-(24) by verifying its estimation and generalization error bounds. Note that S_N can be regarded as sampling following time sequence, thus N and T are used interchangeably. The expected total inventory cost under the DDTA policy given S_N is defined according to (27):

$$\mathbf{C}_{\mathbf{DDTA}}^{\mathrm{N+1}}(\boldsymbol{x}^{[1,N]}; S_N) \triangleq \mathbb{E}_{\boldsymbol{x}^{N+1}|\boldsymbol{x}^{[1,N]}} \left[h_0 \cdot \widehat{z}_0(\boldsymbol{x}^{[1,N]}) + \sum_{j=1}^J C_j(\widehat{z}_j(\widehat{z}_0(\boldsymbol{x}^{[1,N]}), \boldsymbol{x}^{N+1}) | \boldsymbol{x}_j^{N+1}) \right], \tag{B3}$$

where we use S_N to emphasis its connection with data set, and that $\boldsymbol{x}^{[1,N]}$ are the feature vectors in S_N, $\boldsymbol{x}^{N+1}$ is any possible future feature vectors conditioned on $\boldsymbol{x}^{[1,N]}$ (or S_N, equivalently).

Proposition B3 (Convergence Rate for Estimation Error Under Mixing Process). *Under Assumptions 1,2 and B1,* $|\mathbf{C}_{\mathbf{DDTA}}^{\mathrm{N+1}}(\boldsymbol{x}^{[1,N]}; S_N) - \mathbf{C}_{\mathbf{TA}}^{\mathrm{N+1}}(\boldsymbol{x}^{[1,N]})| = o_p(J^2 P N^{-1/3})$.

Proof of Proposition B3. The proof logic is similar to that of Proposition 4. Firstly, We need to prove that the following results still hold when the sampling process follows Assumption B1.

Result (a): $|\widehat{s}_j^l - s_j^l| = O_p(PN^{-1/2})$;

Result (b): $|\widehat{f}_j(\widehat{s}_j^l | \boldsymbol{x}_j) - f_j(s_j^l | \boldsymbol{x}_j)| = o_p(PN^{-1/3})$;

Result (c): $|\widehat{r}_j - r_j| = o_p(PN^{-1/3})$.

Note that results (a)-(c) are all with respect to feature $\boldsymbol{x}_j^{N+1}$ for all $j \in [J]^+$, and we omit the superscript $N+1$ as we can observe $\boldsymbol{x}_j^{N+1}$ before estimating the base-stock level and the probability density. To derive result (a), we first need to prove a similar result as Lemma A1. The proof of Theorem 4.1 in Koenker (2005) is also valid for our problem as we can show (i) the Lindeberg-Feller condition is satisfied under Assumption B1; (ii) $\mathrm{Var}(Z_{2N}(\delta))$ in their proof still converges to 0 since

$\sum_{i=1}^{N-1}\sum_{j=i}^{N}\mathrm{Cov}(Z_{2Ni}(\delta),Z_{2Nj}(\delta))\leq\dfrac{c_0\phi_0\rho}{(1-\rho)N}$ for a constant $c_0>0$ which converges to 0 when N tends to infinity. Thus we can derive the same Bahadur representation as in (A35):

$$\sqrt{N}\left(\widehat{\beta}_{jp}-\beta_{jp}\right)=v_{jp}^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}x_{jp}^{i}\psi_{\tau_j}\left(\epsilon_j^i\right)+o_p(1). \tag{B4}$$

To apply the weak law of large numbers in Theorem 2.2.6 of Durrett (2019), we first need to calculate the variance of $\sum_{i=1}^{N}x_{jp}^{i}\psi_{\tau_j}\left(\epsilon_j^i\right)$. Define stationary random variables $\theta_{jp}^i\triangleq x_{jp}^{i}\psi_{\tau_j}\left(\epsilon_j^i\right)$ and $\sigma_{jp}^2\triangleq\mathrm{Var}(\theta_{jp}^i)$. Then, we have

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{N}x_{jp}^{i}\psi_{\tau}\left(d_j^i-\xi_j^i(\tau)\right)\right)&=N\sigma_{jp}^2+\sum_{i=1}^{N-1}\sum_{t=i}^{N}\mathrm{Cov}(\theta_{jp}^i,\theta_{jp}^t)\\&\leq N\sigma_{jp}^2+2\left(\frac{b_j\bar{M}}{b_j+h_j'}\right)^2\sum_{i=1}^{N-1}\sum_{t=i}^{N}\phi(t-i)\\&\leq N\left(\sigma_{jp}^2+2\frac{\phi_0\rho}{1-\rho}\left(\frac{b_j\bar{M}}{b_j+h_j'}\right)^2\right),\end{aligned} \tag{B5}$$

where the first inequality follows from the covariance estimation for mixing sequences in 10.1.a of Lin and Bai (2011) and the property (1.11) in Bradley (2005) that for any $1\leq i<j\leq N$,

$$\mathrm{Cov}(x_{ip},x_{jp})\leq 2\bar{M}^2\phi(j-i)\leq 2\bar{M}^2\phi_0\rho^{j-i}. \tag{B6}$$

Thus, for any $\eta_1>0$, the following result still holds.

$$v_{jp}^{-1}\frac{1}{N^{1/2+\eta_1}}\sum_{i=1}^{N}x_{jp}^{i}\psi_{\tau_j}\left(\epsilon_j^i\right)=o_p(1), \tag{B7}$$

which directly results in Result (a).

Result (b) can be derived directly as we assume that the random noise ϵ_j^i is i.i.d. for any $i\in[N]^+$, so does result (c). According we have $|\widehat{z}_j(z_0,\boldsymbol{x})-\widetilde{z}_j(z_0,\boldsymbol{x})|=o_p(PN^{-1/3})$ following the proof in Proposition 4.

For $|\widehat{z}_0-\widetilde{z}_0|$, we can still follow (A40) and WLLN that $\dfrac{1}{N}\sum_{i=1}^{N}x_{jp}^i-\mathbb{E}\left[x_{jp}\right]=O_p(N^{-1/2})$ by (B6) and obtain $|\widehat{z}_0-\widetilde{z}_0|=o_p(JPN^{-1/2+\eta_1})$ for any $\eta_1>0$. Then by the analysis in Theorem 2, we conclude $|\mathbf{C}_{\mathbf{DDTA}}^{\mathrm{N+1}}(\boldsymbol{x}^{[1,N]};S_N)-\mathbf{C}_{\mathbf{TA}}^{\mathrm{N+1}}(\boldsymbol{x}^{[1,N]})|=o_p(J^2PN^{-1/3})$. □

The result in Proposition B3 is parallel to Theorem 2. We can see that the convergence rate is the same as that in Theorem 2 only with a larger constant (see the proof). The result is derived through the weak law of large numbers (WLLN) and the fact that the variance of the sum of N random variables with geometrically bounded mixing coefficients is bounded by $O(N)$. For the probability bound of the DDTA policy on the generalization error, its in-sample cost $\mathbf{R}_{\mathbf{DDTA}}(S_N)$ is the same as in (25), and we have that:

Proposition B4 (Generalization Bound for DDTA Policy Under Mixing Process). *Under Assumptions 1,2 and B1, let the bandwidth for kernel density estimation* $w_N = O(N^{-\gamma})$ *for* $0<\gamma<1$. *Then for any* $N\geq 1$ *and* $0<\delta<1$, *there exists a constant* $c_0'>0$ *that with probability* $1-\delta$,

$$|\mathbf{R}_{\mathbf{DDTA}}(S_N) - \mathbf{C}^{\mathrm{N+1}}_{\mathbf{DDTA}}(\boldsymbol{x}^{[1,N]};S_N)| \leq \frac{c_0'}{|\phi_0 \ln\rho|}\left(\frac{(|JPN^{-1+2\gamma}\ln PN^{-1+2\gamma}| + JPN^{-1+2\gamma} + JN^{-1})}{\sqrt{1+2\phi_0\rho/(1-\rho)}}\cdot \sqrt{N\ln(2/\delta)} + |JPN^{-1+2\gamma}\ln PN^{-1+2\gamma}| + JPN^{-1+2\gamma}\right), \tag{B8}$$

where ϕ_0 *and* ρ *are defined in Assumption B1 (iii).*

Proof of Proposition B4. It follows from Proposition A1 that the DDTA policy is a uniform stable algorithm with the stability parameter $\alpha_{\mathbf{D}} = O(JPN^{-1}w_N^{-2})$. Following the discussion in Qi et al. (2020) and Theorem 11 in Mohri and Rostamizadeh (2010), we can conclude that for any $k\in\{0,1,\cdots,N\}$ and $\epsilon>0$,

$$\begin{aligned}
&\mathbb{P}\left(|\mathbf{R}_{\mathbf{DDTA}}(S_N) - \mathbf{C}^{\mathrm{N+1}}_{\mathbf{DDTA}}(\boldsymbol{x}^{[1,N]};S_N)| > \epsilon + (6k+2)\alpha_{\mathbf{D}} + 6(\bar{b}+h_0)J\bar{D}\phi(k)\right)\\
&\leq 2\exp\left(\frac{-2\epsilon^2(1+2\sum_{i=1}^N\phi(i))^{-2}}{N((k+2)2\alpha_{\mathbf{D}} + 2((\bar{b}+h_0)J\bar{D})\phi(k) + ((\bar{b}+h_0)J\bar{D})/N)^2}\right)\\
&\leq 2\exp\left(\frac{-2\epsilon^2(1+2\phi_0\rho/(1-\rho))}{N((k+2)2\alpha_{\mathbf{D}} + 2((\bar{b}+h_0)J\bar{D})\phi(k) + ((\bar{b}+h_0)J\bar{D})/N)^2}\right),
\end{aligned} \tag{B9}$$

which implies that for any $0<\delta<1$, there exists a constant $c_0'>0$ such that with probability $1-\delta$,

$$|\mathbf{R}_{\mathbf{DDTA}}(S_N) - \mathbf{C}^{\mathrm{N+1}}_{\mathbf{DDTA}}(\boldsymbol{x}^{[1,N]};S_N)| \leq c_0'\left(\frac{((k+2)\alpha_{\mathbf{D}} + J\phi(k) + J/N)\sqrt{N\ln(2/\delta)}}{\sqrt{1+2\phi_0\rho/(1-\rho)}} + (6k+2)\alpha_{\mathbf{D}} + J\phi(k)\right). \tag{B10}$$

Taking $\alpha_{\mathbf{D}} = JPN^{-1}w_N^{-2}$ and $k = c_1'\cdot\left|\frac{\ln -\alpha_{\mathbf{D}}/(J\phi_0\ln\rho)}{\ln\rho}\right|$ for some constant $c_1'>0$ (which minimizes the bound), we immediately have that for some constant $c_0'>0$, with probability $1-\delta$,

$$|\mathbf{R}_{\mathbf{DDTA}}(S_N) - \mathbf{C}^{\mathrm{N+1}}_{\mathbf{DDTA}}(\boldsymbol{x}^{[1,N]};S_N)| \leq \frac{c_0'}{|\phi_0\ln\rho|}\left(\frac{(|JPN^{-1}w_N^{-2}\ln PN^{-1}w_N^{-2}| + JPN^{-1}w_N^{-2} + JN^{-1})}{\sqrt{1+2\phi_0\rho/(1-\rho)}}\cdot\sqrt{N\ln(2/\delta)} + |JPN^{-1}w_N^{-2}\ln PN^{-1}w_N^{-2}| + JPN^{-1}w_N^{-2}\right). \tag{B11}$$

If we further set the bandwidth $w_N = O(N^{-\gamma})$ for $0<\gamma<1$, then (B11) can be simplified as

$$|\mathbf{R}_{\mathbf{DDTA}}(S_N) - \mathbf{C}^{\mathrm{N+1}}_{\mathbf{DDTA}}(\boldsymbol{x}^{[1,N]};S_N)| \leq \frac{c_0'}{|\phi_0 \ln\rho|}\left(\frac{(|JPN^{-1+2\gamma}\ln PN^{-1+2\gamma}| + JPN^{-1+2\gamma} + JN^{-1})}{\sqrt{1+2\phi_0\rho/(1-\rho)}}\cdot \sqrt{N\ln(2/\delta)} + |JPN^{-1+2\gamma}\ln PN^{-1+2\gamma}| + JPN^{-1+2\gamma}\right), \tag{B12}$$

which concludes the proof. □

For the i.i.d. data case with $\rho = 0$, the result degenerates to that in Bousquet and Elisseeff (2002). Notably, in Theorem 2, the optimal bandwidth γ is 1/3, but in the case of mixing, the literature suggests the bandwidth γ be less than 1/4 (see, e.g., Mohri and Rostamizadeh 2010). Proposition B4 suggests a bandwidth $\gamma \in (0, 1/4)$ to guarantee the convergence of generalization bound when N tends to infinity.

Next, we discuss how DDTA$^{\text{TC}}$ policy performs compared with its full-information counterpart TA$^{\text{TC}}$ policy. We follow the same sampling assumption in Assumption B1 as the allocation decisions under DDTA$^{\text{TC}}$ policy are the same as those in the DDTA policy (i.e., the same estimation for $\widehat{s}_j^l$, $\widehat{r}_j$). The major difference is the analysis of $|\widetilde{z}_0^{TC}(\boldsymbol{x}^{[1,t]}) - \widehat{z}_0^{TC}(\boldsymbol{x}^{[1,t]})|$, which involves the property of estimator $\widehat{k}_{jp}(\cdot)$. Therefore, we develop the following assumption which bounds the deviation of $\widehat{k}_{jp}(\cdot)$ and the generalization property of the algorithm used in deriving $\widehat{k}_{jp}(\cdot)$.

Assumption B2 (Properties of $\widehat{k}_{jp}^t(\cdot)$). *For $\forall j \in [J]^+, p \in [P]^+$ and $t \in [T]^+$:*

(i) Point-wise error rate: $|\widehat{k}_{jp}^t(\boldsymbol{x}^{[1,t]}) - \widehat{k}_{jp}^t(\boldsymbol{x}^{[1,t]})| = O_p(N^{-1/2})$ for any feasible $\boldsymbol{x}^{[1,t]}$;

(ii) Uniform stability: the algorithm used in learning $\widehat{k}_{jp}^t(\cdot)$ is uniform stable (following the definition in Definition A1) with stability parameter $\alpha = O(N^{-1})$.

If some algorithm satisfies Assumption B2 (e.g., the AR(1) process with autoregressive coefficient and estimated from OLS regression), we could then easily derive the following Corollary on the performance of DDTA$^{\text{TC}}$ policy and the proof is omitted.

Corollary B2 (Rate of Convergence and Generalization Bound of DDTA$^{\text{TC}}$ Policy). *Under Assumptions 1, 2, 5 and 6,*

(i) If Assumption B2 (i) holds, then $|\mathbf{C}^{\text{N+1}}_{\mathbf{DDTA^{TC}}}(\boldsymbol{x}^{[1,N]}; S_N) - \mathbf{C}^{\text{N+1}}_{\mathbf{TA^{TC}}}(\boldsymbol{x}^{[1,N]})|$ follows the convergence rate in Proposition B3.

(ii) If Assumption B2 (ii) holds, then $|\mathbf{R}_{\mathbf{DDTA^{TC}}}(S_N) - \mathbf{C}^{\text{N+1}}_{\mathbf{DDTA^{TC}}}(\boldsymbol{x}^{[1,N]}; S_N)|$ follows the probability bound in Proposition B4.

B.3. Retailer-Correlated Demand Features

First, we relax the assumption that demand features are independent across retailers by modeling the correlation according to the following assumption.

Assumption B3 (Correlated Feature Between Retailers). *Feature sequence $\{x_{jp}\}_{j=1}^J$ follows a (not necessarily stationary) ϕ-mixing process $\{X_{jp}\}_{j=-\infty}^{\infty}$ for any $p \in [P]^+$. Further, there exist constants $\phi_0 > 0$ and $0 \le \rho < 1$ such that the mixing coefficients $\phi(k) \le \phi_0 \rho^k$ for all $k \in [J]^+$.*

Assumption B3 implies that for any two retailers $j_1, j_2 \in [J]^+$, when the two locations are remote enough (i.e., $|j_1 - j_2| \to \infty$), their features can be approximately viewed as independent ones. With this weak dependence assumption, we can apply concentration inequality (see Samson 2000) to derive a similar optimality gap as in Theorem 1.

Proposition B5 (Performance of TA Policy Under Retailer-Correlated Feature). *Under Assumptions 1, 2 and B3, we have* $\dfrac{\mathbf{C_{TA}} - \mathbf{C_{OPT}}}{\mathbf{C_{OPT}}} = O(J^{-1})$.

Proof of Proposition B5. Before proving Proposition B5, we first give an intuitive illustration of how the correlation between retailers would affect the concentration results in Theorem 1. Recall that in (A25), we have

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J}\frac{(\widetilde{z}_j - s_j^l)^2}{2\cdot p_j}\right] &\le \frac{\alpha^3}{2\underline{t}J}\cdot\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right]^2 \\
&= \frac{\alpha^3}{2\underline{t}J}\cdot\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}^2\mathrm{Var}(x_{jp}) + 2\sum_{i=1}^{J-1}\sum_{j>i}^{J}\sum_{p=1}^{P}\beta_{ip}\beta_{jp}\mathrm{Cov}(x_{ip}, x_{jp})\right] \le \frac{c_0\alpha^3}{2\underline{t}}\cdot\bar{\sigma}^2\bar{B}^2 J,
\end{aligned} \tag{B13}$$

for some constant $c_0 > 0$. Further, (B13) would take the equality if $\sum_{i=1}^{J-1}\sum_{j>i}^{J}\mathrm{Cov}(x_{ip}, x_{jp}) = \Theta(J^2)$. That is, the asymptotic optimality property for TA policy would no longer hold unless $\lim_{|i-j|\to\infty}\mathrm{Cov}(x_{ip}, x_{jp}) = 0$ for any $p \in [P]^+$. One can readily show that the covariance indeed converges to 0 as J grows under the ϕ-mixing assumption in Assumption B3.

We next turn to the proof for Proposition B5. Similar to the proof of Theorem 1, it is critical to provide a lower bound on $\mathbb{P}(\boldsymbol{x} \in \mathscr{A}_2)$ which is equivalent to find a lower bound on $\mathbb{P}\left(|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}| \ge \delta\right)$. We need the following lemma which directly follows from Corollary 4 in Samson (2000).

Lemma A3. *Under Assumption B3, for any $\delta > 0$ we have the following probability inequality:*

$$\mathbb{P}\left(\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right| \ge \delta\right) \le 2\exp\left(-\frac{\delta^2}{8JP\bar{M}^2\bar{B}^2||\Gamma||^2}\right) \le 2\exp\left(-\frac{\delta^2(1-\sqrt{\rho})^2}{16JP\bar{M}^2\bar{B}^2\phi_0}\right), \tag{B14}$$

where ϕ_0 and ρ are constants defined in Assumption B3 and the operator norm of the matrix Γ is bounded by $||\Gamma|| \le \dfrac{\sqrt{2\phi_0}}{1-\sqrt{\rho}}$.

With Lemma A3, we can derive a similar lower bound for $\mathbb{P}(\boldsymbol{x} \in \mathscr{A}_2)$ as in Theorem 1 by replacing (A18) with (B14).

Another issue different from Theorem 1 is that we need to bound the covariance in (B13) which is no longer zero under the case of correlated features. By (B6), we immediately have that for any $1 \le i < j \le J$,

$$\mathrm{Cov}(x_{ip}, x_{jp}) \le 2\bar{M}^2\phi(j-i) \le 2\bar{M}^2\phi_0\rho^{j-i}, \tag{B15}$$

which implies

$$\sum_{i=1}^{J-1}\sum_{j>i}^{J}\sum_{p=1}^{P}\beta_{ip}\beta_{jp}\mathrm{Cov}(x_{ip},x_{jp}) \le 2\bar{M}^2\bar{B}^2\phi_0 P\sum_{i=1}^{J-1}\sum_{j>i}^{J}\rho^{j-i} \le 2\bar{M}^2\bar{B}^2\phi_0 JP\cdot\frac{\rho}{1-\rho}.$$

As a result, we can rewrite (A25) as follows.

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}_2}\left[\sum_{j=1}^{J}\frac{(\widetilde{z}_j-s_j^l)^2}{2\cdot p_j}\right] \le \frac{\alpha^3}{2\underline{t}J}\cdot\mathbb{E}_{\boldsymbol{x}\in\mathscr{A}}\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}x_{jp}\right]^2\\
&=\frac{\alpha^3}{2\underline{t}J}\cdot\left[\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}^2\mathrm{Var}(x_{jp})+2\sum_{i=1}^{J-1}\sum_{j>i}^{J}\sum_{p=1}^{P}\beta_{ip}\beta_{jp}\mathrm{Cov}(x_{ip},x_{jp})\right]\\
&\le\frac{\alpha^3}{2\underline{t}J}\cdot\left(\bar{\sigma}^2\bar{B}^2J+4\bar{M}^2\bar{B}^2\phi_0 JP\cdot\frac{\rho}{1-\rho}\right)=O(1).
\end{aligned}\tag{B16}$$

With a similar analysis to Theorem 1, we can show that the optimality gap can be bounded by a constant that is correlated with mixing coefficients and larger than that in Theorem 1 (i.e., independent between retailers) when $\rho \neq 0$. □

With retailer-correlated features, the estimation and generalization error bounds in Theorems 2 and 3 still hold as the base-stock levels are estimated separately for each retailer and we do not require the historical data to be independent among retailers in Assumption 4. In fact, the retailer correlation mainly affects the replenishment decisions and would not affect the allocation decisions directly. We conduct a numerical study on out-of-sample costs for different data-driven methods following the same setting as that in deriving Table 3, except that we let the covariance $\mathrm{Cov}(x_{ip},x_{jp}) = 16\cdot 0.5^{j-i}$ for any $i, j \in [J]^+$ and $j > i$. Note that here we apply $\widehat{z}_0$ as the replenishment quantity for all the data-driven methods. Table B1 provides the performance of different data-driven methods under retailer-correlated demand features: the DDTA policy performs better than SAA and LL and slightly worse than PTSO; the computational time of DDTA is significantly less than LL and PTSO.

Table B1 Out-of-sample costs for different data-driven methods under retailer-correlated demand features. Values in parentheses denote the computational time (second) for each instance.

	SAA	LL	PTSO	DDTA	TA	AF
J=4	80.21(-)	67.12(0.63)	56.33(0.50)	59.16(0.02)	54.23	28.84
J=8	158.52(-)	124.15(3.02)	88.59(2.72)	91.48(0.02)	88.78	57.06
J=16	333.53(-)	272.15(10.24)	159.72(8.85)	163.41(0.04)	161.20	114.53
J=32	656.07(-)	524.58(44.55)	267.78(30.37)	275.03(0.08)	254.80	225.52
J=64	1271.57(-)	921.76(208.17)	495.32(122.07)	508.50(0.21)	484.44	449.14

In summary, our TA/DDTA policies still perform well under weakly correlated demand features between retailers. However, it remains to be an open question on developing joint replenishment and allocation policy when features/random shocks are under general correlation between retailers, e.g., the retailers share the same features.

B.4. Fixed-design Demand Features

In the paper, we assume the demand features follow a random-design setting, i.e., the features are random variables. According to the case study of Hema, the assumption is reasonable as the features we finally choose in our demand model are indeed random variables. However, the demand for some products may be significantly influenced by deterministic features like the day of the week, the price of the product, etc., which are recognized as fixed-design features. We refer to Qi et al. (2022) for a more detailed discussion on the data-driven inventory problems with the deterministic features. We next discuss how to incorporate the fixed-design features into our model.

For each retailer $j \in [J]^+$, we denote the fixed-design feature vector as $\boldsymbol{x}_j^f \in \mathbb{R}^{P_1}$ and the random-design feature vector as $\boldsymbol{x}_j^r \in \mathbb{R}^{P_2}$. We model the demand model as follows.

$$D_j = \gamma_j + \boldsymbol{\beta}_j^f \boldsymbol{x}_j^f + \boldsymbol{\beta}_j^r \boldsymbol{x}_j^r + \epsilon_j = \gamma_j + \sum_{p=1}^{P_1} \beta_{jp}^f x_{jp}^f + \sum_{p=1}^{P_2} \beta_{jp}^r x_{jp}^r + \epsilon_j. \tag{B17}$$

The local base-stock level can be expressed as $s_j^l = \beta'_{j0} + \sum_{p=1}^{P_2} \beta_{jp}^r x_{jp}^r$, where $\beta'_{j0} \triangleq \gamma_j + \sum_{p=1}^{P_1} \beta_{jp}^f x_{jp}^f + F_{\epsilon_j}^{-1}(\frac{b_j}{b_j + h'_j})$. The TA policy can be applied directly following the same routine with the replenishment quantity $\widetilde{z}_0 = \sum_{j=1}^{J} (\beta'_{j0} + \sum_{p=1}^{P_2} \mathbb{E}[x_{jp}^r])$ since according to our fixed-design setting, x_{jp}^f is known to the decision-maker before the replenishment decision. In other words, we can treat the fixed-design features as the advanced feature introduced in our paper.

For the implementation of the DDTA policy, it becomes a statistics problem of quantile estimation and density estimation under the fixed-design setting. We refer to Qi et al. (2022) and Linke et al. (2022) for the parametric and non-parametric approaches to deal with the quantile estimation under the fixed-design setting respectively, and Efromovich (2007) for the conditional density estimation with the fixed-design data. As a result, we can incorporate the fixed-design features into our model through (B17), and derive some performance guarantees for both the TA and the DDTA policies.

B.5. General Lead Time under Time-Correlated Demand Features

Here we will discuss how the feature correlation may benefit our decision-making when considering general lead time. To see the point, we model the feature correlation according to Assumption 6. We further simplify the transition function in (28) by an AR(1) process $x_{jp}^{t+1} = \eta_{jp} x_{jp}^t + \delta_{jp}^{t+1}$ with autoregressive coefficient $|\eta_{jp}| < 1$ for any $j \in [J]^+$, $p \in [P]^+$ for notation simplicity reason and our analysis can still be generalized to the general transition $k_{jp}^t(\cdot)$ case.

Under previous known feature transition settings, we define inventory policy $\pi = \mathbf{TA^{GL}}$ (Taylor Approximation under General Lead Time) as follows: for given historical feature $\boldsymbol{x}^t$, the replenishment decision at period t is $\widetilde{z}_0^{GL}(\boldsymbol{x}^t) \triangleq \sum_{j=1}^{J} \theta_j^t + \sum_{p=1}^{P} \beta_{jp}(\eta_{jp})^{L_0+L} x_{jp}^t$, where θ_j^t denotes the $\dfrac{b_j}{b_j+h_j'}$-th quantile of random variable $\epsilon_j + \sum_{p=1}^{P}\sum_{\ell=0}^{L-1}(\eta_{jp})^{L-1-\ell}\cdot\delta_{jp}^{t+L_0+\ell}$; and the allocation decisions decisions at period $t+L_0$ after the order placed at period t arrives and the observation of feature $\boldsymbol{x}^{t+L_0}$ follow (15)-(16) as well, with regard to the predicted feature $\widehat{x}_{jp}^{t+L_0+L} \triangleq (\eta_{jp})^L \cdot x_{jp}^{t+L_0}$. We have

Corollary B3 (Performance of $\mathrm{TA}^{\mathrm{GL}}$ Policy For General Lead Time). *Under Assumptions 1, 2, 5, 6, with transition functions follow AR(1) process with bounded random noise, warehouse's lead time $L_0 \geq 1$ and retailers' lead time $L \geq 0$, and given feature vectors $\boldsymbol{x}^{[1,t]}$, $\dfrac{\mathbf{C}^{\mathrm{t+1}}_{\mathbf{TA^{GL}}}(\boldsymbol{x}^{[1,t]}) - \mathbf{C}^{\mathrm{t+1}}_{\mathbf{OPT}}(\boldsymbol{x}^{[1,t]})}{\mathbf{C}^{\mathrm{t+1}}_{\mathbf{OPT}}(\boldsymbol{x}^{[1,t]})} = O(J^{-1})$ holds for any period $t \in [T]^+$.*

Proof of Corollary B3. We follow a similar logic as in the proof of Theorem 1 and Proposition B2. The difference is that now our allocation decision at period t is based on the prediction of the feature at period $t+L$, which will lead to additional prediction error when $L > 0$. Thus, we need to redefine the Advance Feature (AF) model such that the decision-maker can observe future L_0 periods' features in advance at period t (i.e., acquiring feature information L_0 period ahead). Then, the optimization problem can be rewritten as:

$$\begin{aligned} \min_{\substack{z_0>0,\\ z_j\geq 0,\forall j\in[J]^+}} \quad & h_0\cdot z_0 + \sum_{j=1}^{J} C_j(z_j|\boldsymbol{x}_j^{[1,t+L_0]}). \\ \text{s.t.} \quad & \sum_{j=1}^{J} z_j \leq z_0. \end{aligned}$$

By applying the KKT conditions, we can easily derive the optimal decisions as follows.

$$\begin{aligned} z_j^{AF}(\boldsymbol{x}_j^{[1,t+L_0]}) &= s_j^l(\boldsymbol{x}_j^{[1,t+L_0]}); \\ z_0^{AF}(\boldsymbol{x}^{[1,t+L_0]}) &= \sum_{j=1}^{J} s_j^l(\boldsymbol{x}_j^{[1,t+L_0]}), \end{aligned}$$

where $s_j^l(\boldsymbol{x}_j^{[1,t+L_0]}) \triangleq \theta_j^t + \sum_{p=1}^{P}\beta_{jp}(\eta_{jp})^L \cdot x_{jp}^{t+L_0}$ and θ_j^t denotes the $\dfrac{b_j}{b_j+h_j'}$-th quantile of random variable $\epsilon_j + \sum_{p=1}^{P}\sum_{\ell=1}^{L}(\eta_{jp})^{L-\ell}\cdot\delta_{jp}^{t+L_0+\ell}$.

It is obvious the total cost under the AF policy defined above is the lower bound of that under the optimal policy (with no access to advanced features) as the proof in Theorem 1. Thus $\mathbf{C}^{\mathrm{t+1}}_{\mathbf{TA^{GL}}}(\boldsymbol{x}^{[1,t]}) - \mathbf{C}^{\mathrm{t+1}}_{\mathbf{OPT}}(\boldsymbol{x}^{[1,t]}) \leq \mathbf{C}^{\mathrm{t+1}}_{\mathbf{TA^{GL}}}(\boldsymbol{x}^{[1,t]}) - \mathbf{C}^{\mathrm{t+1}}_{\mathbf{AF}}(\boldsymbol{x}^{[1,t]})$. Similar to the proof of Theorem 1, the major issue is providing an upper bound on the probability $\mathbb{P}\left(|z_0^{AF}(\boldsymbol{x}^{[1,t+L_0]}) - \widetilde{z}_0(\boldsymbol{x}^{[1,t]})| \geq m\right)$ for any $m > 0$. We can immediately have the following result holds following previous analysis for some constant $c_0 > 0$:

$$\begin{aligned}\mathbb{P}\left\{|z_0^{AF}(\boldsymbol{x}^{[1,t+L_0]}) - \widetilde{z}_0(\boldsymbol{x}^{[1,t]})| \geq m\right\} &= \mathbb{P}\left\{\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}\left((\eta_{jp})^L \cdot x_{jp}^{t+L_0} - (\eta_{jp})^{L_0+L}\cdot x_{jp}^t\right)\right| \geq m\right\} \\ &= \mathbb{P}\left\{\left|\sum_{j=1}^{J}\sum_{p=1}^{P}\beta_{jp}(\eta_{jp})^L \cdot (\sum_{\ell=0}^{L_0-1}(\eta_{jp})^{\ell}\cdot\delta_{jp}^{t+L_0-\ell})\right| \geq m\right\} \\ &\leq 2\exp\left(-c_0\frac{m^2}{2JP\bar{M}_{L_0}^2\bar{B}^2\bar{\eta}^{2L}}\right),\end{aligned} \tag{B18}$$

where $\bar{\eta} < 1$ stands for the uniform upper bound of η_{jp}, $\bar{M}_{L_0}$ stands for the uniform upper bound of random variable $\sum_{\ell=0}^{L_0-1}(\eta_{jp})^{\ell}\cdot\delta_{jp}^{t+L_0-\ell}$ such that $|\sum_{\ell=0}^{L_0-1}(\eta_{jp})^{\ell}\cdot\delta_{jp}^{t+L_0-\ell}| \leq \bar{M}_{L_0}$ holds almost surely for all $j \in [J]^+$, $p \in [P]^+$ and $t \in [T]^+$. □

Through the proof of Corollary B3 and under the i.i.d. assumption on δ_{jp}^t, we can derive intuition on how lead time affects the performance of TA$^{\text{GL}}$ policy: as $\bar{\eta}^{2L}$ is decreasing in retailers' lead time L; $\bar{M}_{L_0}$ is increasing in warehouse's lead time L_0, according to (B18) we can conclude that: under AR(1) feature transition process, a longer warehouse's lead time L_0 or a shorter retailers' lead time L would lead to a better performance of our TA$^{\text{GL}}$ policy (i.e., a smaller constant ratio). Following the analysis, we can derive DDTA$^{\text{GL}}$ policy as well. We omit the detailed analysis.

B.6. Details for Numerical Study

Value of Feature Information. We assume that each retailer j's demand D_j follows a normal distribution with mean $\mu_{D_j} = \sum_{p=1}^{P}(49+p)\cdot\beta_{jp}$ and variance $\sigma_{D_j}^2 = \sum_{p=1}^{P}\beta_{jp}^2\sigma_{x_{jp}}^2 + \sigma_{\epsilon_j}^2$. We fix the dimension of feature vector $P = 3$, holding cost $h'_j = 2$ and varies lost-sales cost $b_j = 3, 8, 18, 38$ (i.e., with corresponding critical ratio $0.6, 0.8, 0.9, 0.95$) for all retailer $j \in [J]^+$. For the variances, we test different combinations of $\sigma_{x_{jp}}^2 \in \{1, 2, 4, 8, 16\}$ and $\sigma_{\epsilon_j}^2 \in \{1, 2, 4, 8\}$. Finally, we set $J \in \{8, 16, 32\}$. Following Tables B2-B5 summarize the numerical results.

Asymptotic Optimality of the TA Policy. We fix $P = 3$, $\sigma_{x_{jp}}^2 = 16$ for all $j \in [J]^+$ and $p \in [P]^+$. The variance of noise $\sigma_{\epsilon_j}^2 = 4$, $h'_j = 2$ and $b_j = 18$ for all retailers $j \in [J]^+$. Figure B1 provides additional numerical results which verify Theorem 1. Figure B1(a) calculates the percentage of loss for $J \in \{8, 16, 32, 64, 128, 256, 512\}$ and $P \in \{3, 5, 10\}$. It can be seen that the percentage of loss would be smaller if the dimension of feature P is small, which is consistent with our bound in (A27). Figure B1(b) examines the result in Theorem 1(ii) by illustrating the pdf of the performance loss ϕ_{AF}^{TA} with $P = 3$. As the number of retailers increases, the TA policy performs better and is also more robust. Specifically, when $J = 8$, the percentage loss is almost uniformly distributed between 0 and 0.2. Besides, the extreme event may occur and cause a relatively large error rate when J is small. However, when $J = 128$, the percentage loss rate is very close to 0.

Table B2 **Value of feature information for $h'_j = 2$ and $b_j = 3$**

	ϕ_{TA}^{NF}	$\sigma^2_{x_{jp}} = 1$	$\sigma^2_{x_{jp}} = 2$	$\sigma^2_{x_{jp}} = 4$	$\sigma^2_{x_{jp}} = 8$	$\sigma^2_{x_{jp}} = 16$
J=8	$\sigma^2_{\epsilon_j} = 1$	30.36%	51.15%	79.43%	112.55%	145.78%
	$\sigma^2_{\epsilon_j} = 2$	16.88%	30.36%	51.15%	79.43%	112.55%
	$\sigma^2_{\epsilon_j} = 4$	8.97%	16.88%	30.36%	51.15%	79.43%
	$\sigma^2_{\epsilon_j} = 8$	4.63%	8.97%	16.88%	30.36%	51.15%
J=16	$\sigma^2_{\epsilon_j} = 1$	32.59%	56.41%	91.12%	136.48%	188.57%
	$\sigma^2_{\epsilon_j} = 2$	17.75%	32.59%	56.41%	91.12%	136.48%
	$\sigma^2_{\epsilon_j} = 4$	9.23%	17.75%	32.59%	56.41%	91.12%
	$\sigma^2_{\epsilon_j} = 8$	4.65%	9.23%	17.75%	32.59%	56.41%
J=32	$\sigma^2_{\epsilon_j} = 1$	37.37%	64.77%	105.57%	160.93%	228.52%
	$\sigma^2_{\epsilon_j} = 2$	20.45%	37.37%	64.77%	105.57%	160.93%
	$\sigma^2_{\epsilon_j} = 4$	10.74%	20.45%	37.37%	64.77%	105.57%
	$\sigma^2_{\epsilon_j} = 8$	5.49%	10.74%	20.45%	37.37%	64.77%

Table B3 **Value of feature information for $h'_j = 2$ and $b_j = 8$**

	ϕ_{TA}^{NF}	$\sigma^2_{x_{jp}} = 1$	$\sigma^2_{x_{jp}} = 2$	$\sigma^2_{x_{jp}} = 4$	$\sigma^2_{x_{jp}} = 8$	$\sigma^2_{x_{jp}} = 16$
J=8	$\sigma^2_{\epsilon_j} = 1$	28.29%	48.33%	75.31%	104.86%	129.51%
	$\sigma^2_{\epsilon_j} = 2$	15.48%	28.29%	48.33%	75.31%	104.86%
	$\sigma^2_{\epsilon_j} = 4$	8.09%	15.48%	28.29%	48.33%	75.31%
	$\sigma^2_{\epsilon_j} = 8$	4.13%	8.09%	15.48%	28.29%	48.33%
J=16	$\sigma^2_{\epsilon_j} = 1$	30.66%	53.45%	87.17%	130.98%	178.51%
	$\sigma^2_{\epsilon_j} = 2$	16.69%	30.66%	53.45%	87.17%	130.98%
	$\sigma^2_{\epsilon_j} = 4$	8.74%	16.69%	30.66%	53.45%	87.17%
	$\sigma^2_{\epsilon_j} = 8$	4.47%	8.74%	16.69%	30.66%	53.45%
J=32	$\sigma^2_{\epsilon_j} = 1$	35.10%	61.61%	101.57%	156.20%	221.96%
	$\sigma^2_{\epsilon_j} = 2$	18.95%	35.10%	61.61%	101.57%	156.20%
	$\sigma^2_{\epsilon_j} = 4$	9.81%	18.95%	35.10%	61.61%	101.57%
	$\sigma^2_{\epsilon_j} = 8$	4.96%	9.81%	18.95%	35.10%	61.61%

Convergence of the DDTA policy. We fix $P = 3$, $\sigma^2_{x_{jp}} = 16$ for all j and p and the variance of noise $\sigma^2_{\epsilon_j} = 4$, $h'_j = 2$ and $b_j = 18$ for all retailers $j \in [J]^+$. The training feature and demand data are generated according to (32).

Figures B2(a) and (b) respectively illustrate the consistency of the warehouse's replenishment decision $\widehat{z}_0$ and the allocation decision $\widehat{z}_j(z_0, \boldsymbol{x})$ which aim to verify the result of Proposition 4. We measure the absolute error $|\widehat{z}_0 - \widetilde{z}_0|$ for replenishment decisions and percentage error $|\widehat{z}_j(\widetilde{z}_0, \boldsymbol{x}) -$

Table B4 Value of feature information for $h'_j = 2$ and $b_j = 18$

	ϕ_{TA}^{NF}	$\sigma^2_{x_{jp}} = 1$	$\sigma^2_{x_{jp}} = 2$	$\sigma^2_{x_{jp}} = 4$	$\sigma^2_{x_{jp}} = 8$	$\sigma^2_{x_{jp}} = 16$
J=8	$\sigma^2_{\epsilon_j} = 1$	27.27%	46.19%	70.72%	94.44%	107.18%
	$\sigma^2_{\epsilon_j} = 2$	15.15%	27.27%	46.19%	70.72%	94.44%
	$\sigma^2_{\epsilon_j} = 4$	8.19%	15.15%	27.27%	46.19%	70.72%
	$\sigma^2_{\epsilon_j} = 8$	4.34%	8.19%	15.15%	27.27%	46.19%
J=16	$\sigma^2_{\epsilon_j} = 1$	28.88%	50.71%	82.88%	123.62%	163.73%
	$\sigma^2_{\epsilon_j} = 2$	15.68%	28.88%	50.71%	82.88%	123.62%
	$\sigma^2_{\epsilon_j} = 4$	8.28%	13.55%	28.88%	50.71%	82.88%
	$\sigma^2_{\epsilon_j} = 8$	4.29%	7.65%	13.55%	28.88%	50.71%
J=32	$\sigma^2_{\epsilon_j} = 1$	34.27%	60.09%	99.01%	151.75%	212.91%
	$\sigma^2_{\epsilon_j} = 2$	18.59%	34.27%	60.09%	99.01%	151.75%
	$\sigma^2_{\epsilon_j} = 4$	9.72%	18.59%	34.27%	60.09%	99.01%
	$\sigma^2_{\epsilon_j} = 8$	4.94%	9.72%	18.59%	34.27%	60.09%

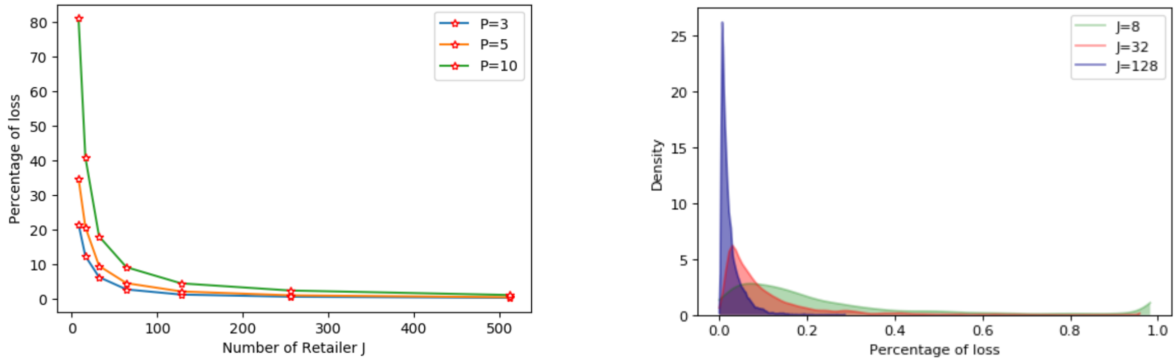
Table B5 Value of feature information for $h'_j = 2$ and $b_j = 38$

	ϕ_{TA}^{NF}	$\sigma^2_{x_{jp}} = 1$	$\sigma^2_{x_{jp}} = 2$	$\sigma^2_{x_{jp}} = 4$	$\sigma^2_{x_{jp}} = 8$	$\sigma^2_{x_{jp}} = 16$
J=8	$\sigma^2_{\epsilon_j} = 1$	26.05%	44.09%	65.93%	82.75%	82.62%
	$\sigma^2_{\epsilon_j} = 2$	14.01%	26.05%	44.09%	65.93%	82.75%
	$\sigma^2_{\epsilon_j} = 4$	7.08%	14.01%	26.05%	44.09%	65.93%
	$\sigma^2_{\epsilon_j} = 8$	3.44%	7.08%	14.01%	26.05%	44.09%
J=16	$\sigma^2_{\epsilon_j} = 1$	26.40%	47.67%	78.74%	116.26%	148.03%
	$\sigma^2_{\epsilon_j} = 2$	13.60%	26.40%	47.67%	78.74%	116.26%
	$\sigma^2_{\epsilon_j} = 4$	6.62%	13.60%	26.40%	47.67%	78.74%
	$\sigma^2_{\epsilon_j} = 8$	3.09%	6.62%	13.60%	26.40%	47.67%
J=32	$\sigma^2_{\epsilon_j} = 1$	33.91%	59.39%	97.58%	148.27%	203.64%
	$\sigma^2_{\epsilon_j} = 2$	18.43%	33.91%	59.39%	97.58%	148.27%
	$\sigma^2_{\epsilon_j} = 4$	9.70%	18.43%	33.91%	59.39%	97.58%
	$\sigma^2_{\epsilon_j} = 8$	5.02%	9.70%	18.43%	33.91%	59.39%

$\widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})|/\widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})$ for allocation decisions under the same $\widetilde{z}_0$ with $J = 4$.[2] We calculate the average distance in 20 different out-of-sample data sets with the size of 100 each. We observe that as the size of training data set becomes larger, both $|\widehat{z}_0 - \widetilde{z}_0|$ and $|\widehat{z}_j(\widetilde{z}_0, \boldsymbol{x}) - \widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})|/\widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})$ converge to 0. Moreover, the convergence rate decreases in the number of retailers.

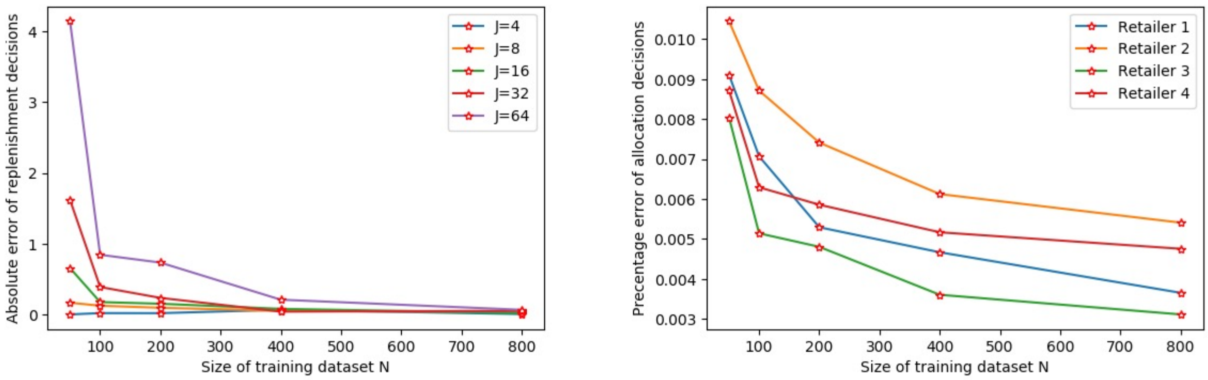
[2] We use absolute error for replenishment decisions in order to see the effect of J in the convergence result. For allocation decisions, J does not affect the rate of convergence and we show $J = 4$ case in the figure for simplicity.

Figure B1 Asymptotic optimality of TA policy



(a) Asymptotic optimality over J and P (b) Density of Performance Loss

Figure B2 Consistency of replenishment and allocation decision.



(a) Average of $|\widehat{z}_0 - \widetilde{z}_0|$ (b) Average of $|\widehat{z}_j(\widetilde{z}_0, \boldsymbol{x}) - \widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})| / \widetilde{z}_j(\widetilde{z}_0, \boldsymbol{x})$

Comparison of Other Data-Driven Policies. We consider the following three policies generated from data-driven methods.

• **Sample Average Approximation (SAA)**: The SAA method ignores the demand feature information and equally weights the samples in the training set to approximate the objective function. Let $\boldsymbol{z}^{SAA}(z_0)$ be the optimal allocation decision for a given z_0, where

$$\begin{aligned} \boldsymbol{z}^{SAA}(z_0) = \ & \underset{z_j \geq 0}{\operatorname{argmin}} \ \frac{1}{N} \sum_{i=1}^{N} \left\{ h_0 \cdot (z_0 - \sum_{j=1}^{J} z_j) + \sum_{j=1}^{J} \left[h_j' \cdot (z_j - d_j^i)^+ + b_j \cdot (d_j^i - z_j)^+ \right] \right\} \\ & \text{s.t.} \quad \sum_{j=1}^{J} z_j \leq z_0. \end{aligned} \tag{B19}$$

We use a grid search to find the best z_0 that minimizes the total cost in an independently sampled cross-validation data set of size 100 and use this z_0 as given to test the out-of-sample performance.

• **Local Learning (LL) Method**: Bertsimas and Kallus (2020) use this method to generate data-driven solutions. The LL method uses non-parametric approaches, such as kNN, Kernel Regression, CART to estimate the weight function from in-sample training data. Then, for a given $\boldsymbol{x}^{N+1}$ (i.e., the new feature observation), the method solves the best replenishment and allocation decision. Note that the computational time is extensive during training to find the best non-parametric model parameters, as the approach needs to solve an allocation problem for each in-sample observation. We test all these approaches and find that they have a close performance in our problem. Thus, we use kNN to present the numerical results. The kNN determines the weights $w(\boldsymbol{x}^{N+1}, \boldsymbol{x}^i) = \dfrac{1}{k}$ if $\boldsymbol{x}^{N+1}$ belongs to the k nearest neighbors of sample i and 0 otherwise. The corresponding optimization problem can be written as follows.

$$\begin{aligned}
\boldsymbol{z}^{LL}(z_0, \boldsymbol{x}^{N+1}) = \ & \underset{z_j \geq 0}{\operatorname{argmin}} \ \sum_{i=1}^{N} w(\boldsymbol{x}^{N+1}, \boldsymbol{x}^i) \left\{ h_0 \cdot (z_0 - \sum_{j=1}^{J} z_j) + \sum_{j=1}^{J} \left[h_j' \cdot (z_j - d_j^i)^+ + b_j \cdot (d_j^i - z_j)^+ \right] \right\} \\
& \text{s.t.} \quad \sum_{j=1}^{J} z_j \leq z_0,
\end{aligned}$$

where the hyper-parameter k is chosen according to the cross-validation technique.[3]

• **Predict Then Smart Optimize (PTSO)**: The PTSO method adopts a two-step process by first predicting the demand and then optimizing the allocation decision for a given z_0 and feature $\boldsymbol{x}^{N+1}$. First, we train a linear regression model $\widehat{d}_j = \widehat{\gamma}_0 + \sum_{p=1}^{P} \widehat{\beta}_{jp} x_{jp}$ using OLS and calculate the corresponding residuals $\widehat{\epsilon}_j^i \triangleq d_j^i - \widehat{\gamma}_0 - \sum_{p=1}^{P} \widehat{\beta}_{jp} x_{jp}^i$ for each $i \in [N]^+$ for each retailer $j \in [J]^+$.

In the second step, for new feature observation $\boldsymbol{x}^{N+1}$, we predict its mean demand as $\widehat{\gamma}_0 + \sum_{p=1}^{P} \widehat{\beta}_{jp} x_{jp}^{N+1}$, and its residual sampled from $\{\widehat{\epsilon}_j^i\}_{i=1}^{N}$ with equal probability $1/N$. We then optimize the allocation decisions $\boldsymbol{z}^{PTSO}(z_0, \boldsymbol{x}^{N+1})$ by solving the following problem.

$$\begin{aligned}
\boldsymbol{z}^{PTSO}(z_0, \boldsymbol{x}^{N+1}) = \ & \underset{z_j \geq 0}{\operatorname{argmin}} \ h_0 \cdot (z_0 - \sum_{j=1}^{J} z_j) + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} \left[h_j'(z_j - \widehat{\gamma}_0 - \sum_{p=1}^{P} \widehat{\beta}_{jp} x_{jp}^{N+1} - \widehat{\epsilon}_j^i)^+ \right. \\
& \left. + b_j(\widehat{\gamma}_0 + \sum_{p=1}^{P} \widehat{\beta}_{jp} x_{jp}^{N+1} + \widehat{\epsilon}_j^i - z_j)^+ \right] \\
& \text{s.t.} \quad \sum_{j=1}^{J} z_j \leq z_0.
\end{aligned}$$

For the LL policy and the PTSO policy, we use the same z_0 as that of the $\widehat{z}_0$ calculated through the DDTA policy since the computational effort to conduct a grid search for the best in-sample z_0 is significantly larger than that for the SAA policy and hard to implement. Note that the PTSO approach is also recognized as "Data-driven SAA with Covariate" according to Kannan et al. (2022), in which they derive its convergence property. In fact, we can regard it as a semi-nonparametric approach, which first utilizes a parametric model to train the demand estimator and then conducts optimization based on a nonparametric approach (i.e., SAA).

[3] When the sample size N and the number of retailer J is relatively large, we use the conventional choice of $k = \lfloor \sqrt{N} \rfloor$.

For Table 3, we fix the training set size $N = 400$, the dimension of feature $P = 3$, the variance of feature $\sigma^2_{x_{jp}} = 16$ for all j and p, the variance of noise $\sigma^2_{\epsilon_j} = 4$, $h'_j = 2$ and $b_j = 18$ for all retailer $j \in [J]^+$. In addition, Table B6 also provides the inventory costs and computation time with the size of training data $N \in \{50, 100, 200, 400, 800\}$. We can see that the performance of each policy becomes better as the training data size increases and are quite robust to the data size.

Table B6 Out-of-sample costs for different Data-driven policies over training set size N

	Cost	N=50	N=100	N=200	N=400	N=800
SAA	J=4	68.57(-)	67.01(-)	68.72(-)	67.31(-)	67.14(-)
	J=8	150.64(-)	151.92(-)	156.22(-)	147.52(-)	149.79(-)
	J=16	337.15(-)	340.45(-)	326.23(-)	321.67(-)	326.90(-)
	J=32	681.34(-)	658.66(-)	650.10(-)	647.67(-)	643.36(-)
	J=64	1305.27(-)	1279.63(-)	1269.54(-)	1252.98(-)	1249.16(-)
LL	J=4	58.06(0.09)	53.16(0.13)	53.09(0.28)	47.21(0.53)	47.09(1.02)
	J=8	138.96(0.33)	127.73(0.54)	123.69(1.03)	114.35(2.42)	109.36(4.16)
	J=16	312.29(1.31)	305.81(2.35)	279.08(4.31)	264.14(9.44)	261.02(16.84)
	J=32	664.47(5.59)	625.92(11.01)	594.13(18.15)	571.20(39.53)	545.90(74.42)
	J=64	1334.09(22.57)	1303.36(53.72)	1221.86(92.78)	1184.76(198.68)	1129.06(354.50)
PTSO	J=4	37.53(0.10)	36.87(0.16)	37.55(0.40)	36.78(0.48)	36.67(1.01)
	J=8	73.99(0.33)	73.48(0.62)	70.87(1.66)	68.32(2.12)	68.86(4.25)
	J=16	143.06(1.48)	137.58(2.39)	133.50(4.09)	132.87(7.85)	132.09(15.45)
	J=32	261.89(5.70)	253.72(8.95)	247.42(15.91)	246.07(28.37)	245.33(55.30)
	J=64	498.73(22.57)	493.94(42.63)	476.07(74.46)	471.14(119.79)	469.47(221.78)
DDTA	J=4	40.46(0.01)	38.47(0.01)	37.31(0.01)	37.01(0.02)	36.51(0.01)
	J=8	77.08(0.02)	72.70(0.02)	69.94(0.03)	68.90(0.03)	68.26(0.03)
	J=16	150.70(0.04)	141.81(0.05)	135.76(0.05)	133.40(0.05)	131.66(0.05)
	J=32	279.53(0.08)	260.28(0.09)	251.47(0.10)	246.73(0.09)	243.53(0.09)
	J=64	532.54(0.16)	496.61(0.18)	479.72(0.18)	471.07(0.21)	466.84(0.21)

While the performance of the PTSO policy is slightly better than the DDTA policy according to the numerical results, the computational time is more relevant for Hema because its goal is to leverage the most recent feature information to achieve same-day delivery for more than 400 SKUs of fresh goods. We discuss this point from theoretical and implementation perspectives below.

- From the theoretical perspective, (for a given replenishment decision z_0) the computational complexity of the DDTA policy is $O(JN)$.[4] For the PTSO policy, its objective function can be rewritten as a linear program with $2JN + J$ variables and $4JN + 1$ constraints. To the best of our knowledge, the computational complexity of the current fastest algorithm is $O((JN)^\omega \log(JN/\delta))$ according to

[4] The true complexity is $O(P + JN)$, and by the assumption that $P \ll JN$ we can simplify it to $O(JN)$.

Cohen et al. (2020), where $\omega \geq 2$ is the matrix multiplication exponent and $\delta > 0$ is the relative accuracy level.[5] Different from the DDTA policy which only involves estimation, the PTSO policy has to conduct iterations to find the solutions within the relative accuracy level: for each iteration, the computational complexity is $O((JN)^{\omega})$ and it has $O(\log(JN/\delta))$ iterations in total.

Consider the rate of convergence for the DDTA policy: if the decision-maker has some prior knowledge of the distribution family of the random noise, i.e., a normal distribution, then she can use the Maximum Likelihood Estimation (MLE) to estimate corresponding distribution parameters and the corresponding probability density instead of KDE, with a rate of convergence $N^{-1/2}$, which is the same as that of the PTSO policy.

- From the implementation perspective, although it seems tolerable to spend several seconds (7.85s) for a moderate number of retailers ($J = 16$) and sample size ($N = 400$), the computational time of DDTA policy is much less than that of the PTSO policy, which is within 0.05 seconds. Note that this is the computational time for a single SKU. Fresh Hema has more than 20,000 SKUs in total, which includes more than 400 "Daily Fresh" products. Thus, fast computation is an important criterion for Hema to achieve same-day delivery. Specifically, the computational time for the PTSO policy to generate the solutions for, say, 450 SKUs requires about one hour (= 450 SKUs $\times$ 8 seconds/SKU = one hour). This will make Hema's same-day delivery hard to implement as the time window between receiving fresh goods at the warehouse and shelving them at the retail store is less than 6 hours.

Besides, the DDTA policy is a general policy that can be applied to any one-warehouse-multi-retailer system which may contain more fresh SKUs than Hema. In other words, fast computation is always an attractive feature for data-driven solutions.

Meanwhile, the DDTA policy has a closed-form solution, which is more transparent and thus enhances interpretability. For example, the decision-maker can learn how the system parameters affect the inventory allocation (see (15)-(16), (22)-(23)); she can also learn the maximum allocation quantity (i.e., $\widehat{s}^e_j$) level for each retailer j, the marginal inventory cost for each retailer j when the inventory is insufficient (i.e, $\frac{\widehat{r}_j}{\sum_{j=1}^J \widehat{r}_j}$), etc., which are all of the operational importance and interests for the decision-maker. In contrast, the PTSO is more like a black-box optimization system that only returns a set of solutions without further explanations.

To summarize, although the proposed PTSO policy has some advantages over the DDTA policy, i.e., its consistency property to the optimal allocation decisions, it is really hard to implement in Hema's same-day delivery according to our previous analysis. In contrast, the DDTA policy is more practical with still relatively good performance, which makes it more attractive to the decision-maker in Hema.

[5] So far, the best matrix multiplication exponent ω (i.e., the smallest real number for which any two $n \times n$ matrices can be multiplied together using $n^{\omega+o(1)}$ operations) is proven to be 2.37188 by Duan et al. (2022). We hide $(JN)^{o(1)}$ and $\log^{O(1)}(1/\delta)$ terms in the computational complexity.

B.7. Feature Selection for Case Study

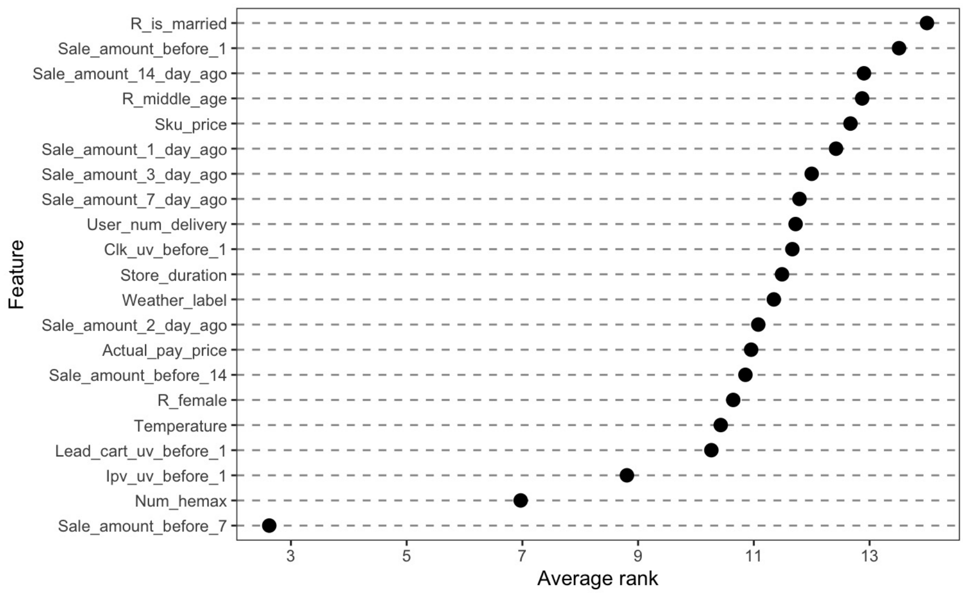
In the original data set, Hema has listed more than 50 demand features, divided into four categories:

- Item-specific features, e.g., past sales, selling price, page views.
- Environment-related features, e.g., weather conditions, temperature.
- Date-related features, e.g., day of the week, holiday.
- Location-specific features, e.g., number of members, the ratio of genders.

Data sparsity (i.e., high-dimensional features) is also a problem studied in data-driven operation management (see Feng and Shanthikumar 2022). To deal with the issue, we first remove features that have incomplete data or share strong collinearity with the other features. With that, we have 21 candidate features left.

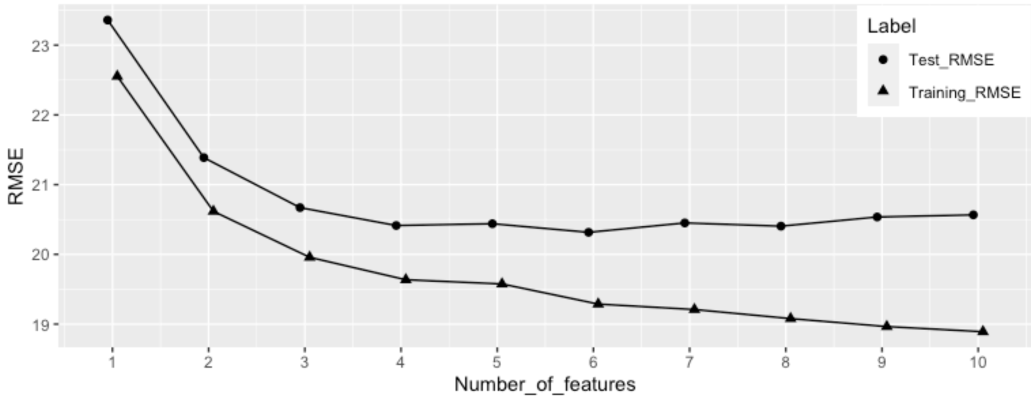
We then select the demand features for our model by using step-wise regression, which is a popular data-mining tool that selects the explanatory variables used in the multiple-regression model by statistical significance criterion. We randomly select 85% instances from the original data set and treat the selected data set as the training set and the remaining instances as the test set. We repeatedly conduct step-wise regression by the forward-selection rule in the 16 retailers 40 times and record the order when a specific attribute is selected to enter the regression model. Figure B3 shows the average rank of features among all retailers and the features with smaller rank is more important in predicting the demand. The final step is to choose the number of features that we shall include in the demand

Figure B3 Importance rank for demand features



model. We increase the number of features from 1 to 10 according to the importance rank of demand features in Figure B3 and calculate the average rooted mean square error (RMSE) of corresponding linear models in both training and test sets.

Figure B4 RMSE under different number of features



From Figure B4, we choose $P = 4$ features for the linear demand model as it has the best bias-variance trade-off. In addition, we use variance inflation factor (VIF) to detect multicollinearity in the regression model, and all these four features: historical sales; the number of Hema members; item page views; the number of items in the shopping cart, have VIFs less than 10, indicating that no strong multicollinearity exists in the model for each retailer.

After feature selection, we model the correlation of features between time periods by fitting them with the best ARIMA(p,d,q) model (with respect to model parameters p, d, q) according to Bayesian Information Criterion (BIC) and test the significance of estimated parameters, as well as the independent property of residuals through QQ-plot and Ljung–Box test. The results show that nearly all the features in each retailer can be approximated well by the ARIMA(p,d,q) model and the following table provides corresponding model parameters p, d, q for each feature. (F1 for historical sales; F2 for number of Hema members; F3 for item page views; F4 for number of items in shopping cart)

Table B7 Best Fitted ARIMA Paramaters

	F1	F2	F3	F4
Retailer1	ARIMA(4,2,2)	ARIMA(2,0,2)	ARIMA(1,0,1)	ARIMA(5,0,2)
Retailer2	ARIMA(5,1,2)	ARIMA(2,0,1)	ARIMA(5,0,5)	ARIMA(5,0,2)
Retailer3	ARIMA(5,1,2)	ARIMA(2,0,1)	ARIMA(2,0,1)	ARIMA(2,0,1)
Retailer4	ARIMA(5,1,5)	ARIMA(5,0,3)	ARIMA(1,0,1)	ARIMA(3,0,5)
Retailer5	ARIMA(3,1,3)	ARIMA(4,0,3)	ARIMA(1,0,1)	ARIMA(1,0,1)
Retailer6	ARIMA(2,1,4)	ARIMA(2,0,4)	ARIMA(1,1,1)	ARIMA(0,1,1)
Retailer7	ARIMA(2,2,4)	ARIMA(5,0,5)	ARIMA(2,0,1)	ARIMA(0,1,1)
Retailer8	ARIMA(5,1,5)	ARIMA(5,0,5)	ARIMA(1,0,1)	ARIMA(0,0,1)
Retailer9	ARIMA(2,1,5)	ARIMA(2,0,2)	ARIMA(2,0,2)	ARIMA(0,0,1)
Retailer10	ARIMA(5,1,4)	ARIMA(2,0,4)	ARIMA(4,0,1)	ARIMA(0,0,1)
Retailer11	ARIMA(2,1,3)	ARIMA(5,0,5)	ARIMA(1,0,1)	ARIMA(0,0,1)
Retailer12	ARIMA(5,1,5)	ARIMA(5,0,4)	ARIMA(2,0,1)	ARIMA(0,0,1)
Retailer13	ARIMA(5,1,5)	ARIMA(3,0,1)	ARIMA(0,1,1)	ARIMA(0,1,1)
Retailer14	ARIMA(5,1,2)	ARIMA(5,0,5)	ARIMA(1,0,1)	ARIMA(5,0,5)
Retailer15	ARIMA(1,0,1)	ARIMA(2,0,1)	ARIMA(1,1,1)	ARIMA(0,1,1)
Retailer16	ARIMA(5,1,2)	ARIMA(5,0,5)	ARIMA(1,0,1)	ARIMA(2,1,3)

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