Online Appendix of "Does the Potential to Merge Reduce Competition?"

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D Inverting prices to obtain the information vector

In this appendix we detail the linear transformation from the underlying state processes X_t^1 and X_t^2 to the price processes. The X_t processes are

$$dX_t^1 \;=\; A_t^1\, dt \,+\, \sigma_1\, dZ_t^1\,, \tag{D.30}$$

and

$$dX_t^2 \;=\; A_t^2\, dt \,+\, \sigma_2\, dZ_t^2\,, \tag{D.31}$$

We asserted that

$$dX_t^1 \;=\; \frac{(A_t^1 + A_t^2)}{2\,(\Pi_2 \,-\, \delta_2 A_t^1 \,-\, \beta_2 A_t^2)} dP_t^2 \,-\, \frac{(A_t^1 - A_t^2)}{2\,(\Pi_1 \,-\, \beta_1 A_t^1 \,-\, \delta_1 A_t^2)} dP_t^1 \tag{D.32}$$

and

$$dX_t^2 \;=\; \frac{(A_t^1 + A_t^2)}{2\,(\Pi_2 \,-\, \delta_2 A_t^1 \,-\, \beta_2 A_t^2)}\, dP_t^1 \,-\, \frac{(A_t^2 - A_t^1)}{2\,(\Pi_1 \,-\, \beta_1 A_t^1 \,-\, \delta_1 A_t^2)}\, dP_t^2 \tag{D.33}$$

Write this as the matrix formulation

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -(A_t^1 - A_t^2) & A_t^1 + A_t^2 \\ A_t^1 + A_t^2. & -(A_t^2 - A_t^1) \end{pmatrix} \begin{pmatrix} \frac{dP_t^1}{(\Pi_1 - \beta_1 A_t^1 - \delta_1 A_t^2)} \\ \frac{dP_t^2}{(\Pi_2 - \delta_2 A_t^1 - \beta_2 A_t^2)} \end{pmatrix} \tag{D.34}$$

Now invert the coefficient matrix to obtain

$$\begin{pmatrix} \frac{dP_t^1}{(\Pi_1 - \beta_1 A_t^1 - \delta_1 A_t^2)} \\ \frac{(dP_t^2}{(\Pi_2 - \delta_2 A_t^1 - \beta_2 A_t^2)} \end{pmatrix} = \frac{2}{(A_t^1 - A_t^2)(A_t^2 - A_t^1) - (A_t^1 + A_t^2)^2} \begin{pmatrix} (A_t^1 - A_t^2) & A_t^1 + A_t^2 \\ A_t^1 + A_t^2 & (A_t^2 - A_t^1) \end{pmatrix} \begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} \tag{D.35}$$

Simplification yields

$$\begin{pmatrix} \frac{dP_t^1}{(\Pi_1 - \beta_1 A_t^1 - \delta_1 A_t^2)} \\ \frac{(dP_t^2}{(\Pi_2 - \delta_2 A_t^1 - \beta_2 A_t^2)} \end{pmatrix} = -\frac{1}{{A_t^1}^2 + {A_t^2}^2} \begin{pmatrix} (A_t^1 - A_t^2) & A_t^1 + A_t^2 \\ A_t^1 + A_t^2 & (A_t^2 - A_t^1) \end{pmatrix} \begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} \tag{D.36}$$

Multiplying out on the right hand side yields

$$-\frac{1}{{A_t^1}^2 + {A_t^2}^2} \begin{pmatrix} (A_t^1 - A_t^2) dX_t^1 + (A_t^1 + A_t^2) dX_t^2 \\ (A_t^1 + A_t^2) dX_t^1 + (A_t^2 - A_t^1) dX_t^2 \end{pmatrix} \tag{D.37}$$

Thus, the noise terms can be fleshed out, yielding

$$-\frac{1}{{A_t^1}^2+{A_t^2}^2}\begin{pmatrix}((A_t^1-A_t^2)A_t^1+(A_t^1+A_t^2)A_t^2)dt\\((A_t^1+A_t^2)A_t^1+(A_t^2-A_t^1)A_t^2)dt\end{pmatrix}-\frac{1}{{A_t^1}^2+{A_t^2}^2}\begin{pmatrix}(A_t^1-A_t^2)\sigma_1 dZ_t^1+(A_t^1+A_t^2)\sigma_2 dZ_t^2\\(A_t^1+A_t^2)\sigma_1 dZ_t^1+(A_t^2-A_t^1)\sigma_2 dZ_t^2\end{pmatrix} \tag{D.38}$$

$$-\frac{1}{{A_t^1}^2+{A_t^2}^2}\begin{pmatrix}({A_t^1}^2+{A_t^2}^2)dt\\({A_t^1}^2+{A_t^2}^2)dt\end{pmatrix}-\frac{1}{{A_t^1}^2+{A_t^2}^2}\begin{pmatrix}(A_t^1-A_t^2)\sigma_1 dZ_t^1+(A_t^1+A_t^2)\sigma_2 dZ_t^2\\(A_t^1+A_t^2)\sigma_1 dZ_t^1+(A_t^2-A_t^1)\sigma_2 dZ_t^2\end{pmatrix} \tag{D.39}$$

which simplifies to

$$\begin{pmatrix}\frac{dP_t^1}{(\Pi_1-\beta_1 A_t^1-\delta_1 A_t^2)}\\\frac{(dP_t^2}{(\Pi_2-\delta_2 A_t^1-\beta_2 A_t^2)}\end{pmatrix}=-\begin{pmatrix}dt\\dt\end{pmatrix}-\frac{1}{{A_t^1}^2+{A_t^2}^2}\begin{pmatrix}(A_t^1-A_t^2)\sigma_1 dZ_t^1+(A_t^1+A_t^2)\sigma_2 dZ_t^2\\(A_t^1+A_t^2)\sigma_1 dZ_t^1+(A_t^2-A_t^1)\sigma_2 dZ_t^2\end{pmatrix} \tag{D.40}$$

Multiplying by the denominators on the left hand side then yields the price process equations (1) and (2) with $d\zeta_t^1$ and $d\zeta_t^2$ implicitly defined as linear combinations of the dZ_t^1 and dZ_t^2 processes. Thus, the noise processes in price are correlated across the two firms, but these noise processes can be inverted from the price signal.

E Converting to polar coordinates

The normal and tangent to the manifold are, in terms of the angle θ, respectively:

$$\mathbf{N}(\theta) = (\cos(\theta), \sin(\theta))\,, \qquad \text{and} \qquad \mathbf{T}(\theta) = (-\sin(\theta), \cos(\theta))\,. \tag{E.41}$$

We can then express the value process in polar coordinate form: the curvature $\kappa(W)$ of the equilibrium manifold, which is determined by the transformed form of the optimized Bellman equations of the firms, is the derivative of the polar coordinate with respect to movement along the equilibrium manifold. For example, κ would be zero if the manifold were locally a straight line. Thus, angle θ and arc length ℓ, we have along the manifold that:

$$\frac{d\theta}{d\ell} = \kappa\,, \qquad \text{and} \qquad \frac{d\ell}{d\theta} = \frac{1}{\kappa}\,, \tag{E.42}$$

The left-hand side derivative can be expressed in terms of the derivatives of the continuation values using polar coordinates, which leads us to the differential equation we solve numerically:

$$\begin{pmatrix} \frac{dW^1(\theta)}{d\theta} \\ \frac{dW^2(\theta)}{d\theta} \end{pmatrix} = \begin{pmatrix} -\frac{\sin(\theta)}{\kappa(\theta)} \\ \frac{\cos(\theta)}{\kappa(\theta)} \end{pmatrix} = \frac{\mathbf{T}(\theta)}{\kappa(\theta)}\,. \tag{E.43}$$

We can express the first-order cross-partial of W^1 in trigonometric form:

$$W^2_{W^1} = \frac{dW^2}{dW^1} = -\frac{\cos(\theta)}{\sin(\theta)}\,. \tag{E.44}$$

Hence the second-stage Bellman equation (A.13) becomes:

$$\begin{aligned} 0 = \max_{A^1} \Big\{ & r\left(g_1 - W^1\right) - r(g_2 - W^2)\left(-\frac{\sin(\theta)}{\cos(\theta)}\right) \\ & + \tfrac{1}{2}\left(-r\frac{\cos(\theta)}{\sin(\theta)} g_{1A^1}\right)^2 W^1_{W^2W^2} + \tfrac{1}{2}\left(r g_{2A^2}\right)^2 W^1_{W^2W^2} \Big\}\,, \end{aligned} \tag{E.45}$$

which then leads to:

$$W^1_{W^2W^2} = -\max_{A^1} \left\{ \frac{(g_1 - W^1) - (g_2 - W^2)\left(-\frac{\sin(\theta)}{\cos(\theta)}\right)}{r\left(\left(-\frac{\cos(\theta)}{\sin(\theta)} g_{1A^1}\right)^2 + (g_{2A^2})^2\right)} \right\}\,. \tag{E.46}$$

After some algebra, the equation can be restated as follows:

$$W^1_{W^2W^2} = -\max_{A^1}\left\{\frac{\frac{1}{\cos(\theta)}\left(\cos(\theta)(g_1 - W^1) - (g_2 - W^2)\left(-\sin(\theta)\right)\right)}{r\cos(\theta)^2\left(\left(\frac{g_{1A^1}}{\sin(\theta)}\right)^2 + \left(\frac{g_{2A^2}}{\cos(\theta)}\right)^2\right)}\right\}, \tag{E.47}$$

or

$$W^1_{W^2W^2} = -\max_{A^1}\left\{\frac{1}{\cos(\theta)^3}\frac{\cos(\theta)(g_1 - W^1) + \sin(\theta)(g_2 - W^2)}{r\left(\left(\frac{g_{1A^1}}{\sin(\theta)}\right)^2 + \left(\frac{g_{2A^2}}{\cos(\theta)}\right)^2\right)}\right\}. \tag{E.48}$$

The numerator term is $\mathbf{N}(g-W)$, and the denominator term is $r|\phi|^2$, just as in Sannikov's formula. Notice that this equation has a curvature on the left-hand side. The fact that it is a curvature will later be used in the numerical solution of the model. We repeat the exercise with firm 2 and obtain:

$$W^2_{W^1W^1} = -\max_{A^2}\left\{\frac{1}{\sin(\theta)^3}\frac{\cos(\theta)(g_1 - W^1) + \sin(\theta)(g_2 - W^2)}{r\left(\left(\frac{g_{1A^1}}{\sin(\theta)}\right)^2 + \left(\frac{g_{2A^2}}{\cos(\theta)}\right)^2\right)}\right\}. \tag{E.49}$$

Notice that the denominators in equations (E.48) and (E.49) are the same.

As shown in Appendix G, the second-order partial derivatives of the continuation values are weighted expressions of the curvature of the equilibrium manifold in the direction of the normal vector, $\kappa(W)$:

$$\cos(\theta)^3 W^1_{W^2W^2} \;=\; \sin(\theta)^3 W^2_{W^1W^1} \;\equiv\; \tfrac{1}{2}\,\kappa(W)\,. \tag{E.50}$$

We can add the two curvature values in equations (E.48) and (E.49) and denote $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ to obtain an expression for the curvature:

$$\kappa(W) \;=\; \max_{A\in\mathcal{A}\setminus\mathcal{A}^N}\frac{2\,\mathbf{N}\,(g-W)}{r|\phi|^2}\,, \tag{E.51}$$

which is Sannikov's (2007) optimality equation. Note that the maximization in equation (E.51) is over both A^1_t and A^2_t (excluding the set of pure strategy Nash equilibria, $\mathcal{A}^N$, as mentioned earlier). This is innocuous here because the numerator and denominator are each additively separable in A^1_t and A^2_t, so separate maximization for each firm taking its turn as the "principal" is satisfied. ■

Sannikov demonstrates that the differential equation system in (E.51) satisfies Lipschitz conditions and therefore has a solution:

> . . . the right-hand side of the optimality equation [(E.51)] is Lipschitz-continuous in w and T. This property guarantees the existence of solutions to the optimality equation from any initial conditions and helps us characterize the set $\mathcal{E}(r)$. [Sannikov p. 1320]

Sannikov then continues with his Lemma 3, culminating in his Proposition 4, establishing existence. (See also Hurewicz (1958), p. 28.)

F Why maximizing the Bellman equation is equivalent to maximizing a ratio in the curvature ODE

Having established that the "agency" optimization problem in equation (A.13) is equivalent to the ODE in equation (E.48) for the curvature of the equilibrium set boundary $\partial\mathcal{E}$, we need to show why the optimization of the Bellman equation is equivalent to the optimization of the ratio in the ODE.

Consider the abstract problem:

$$\max_x \; \{f(x) \, + \, A\, g(x)\} \, . \tag{F.52}$$

The first-order condition is:

$$f'(x) \, + \, A\, g'(x) \; = \; 0\,, \tag{F.53}$$

or:

$$A \; = \; -\frac{f'}{g'}\,. \tag{F.54}$$

Now consider the maximization problem:

$$\max_x \; \frac{f(x)}{g(x)}\,. \tag{F.55}$$

The first-order condition can be written as:

$$\frac{f}{g} \; = \; \frac{f'}{g'}\,, \tag{F.56}$$

and therefore, at the maximum, we have that:

$$\max_x \; \frac{f}{g} \; = \; \frac{f'}{g'}\,. \tag{F.57}$$

Therefore,

$$A \; = \; -\frac{f'}{g'} \; = \; -\max_x \; \frac{f}{g}\,. \tag{F.58}$$

Thus, the maximization of the ratio generates the same optimum (adjusted for the sign) as the Bellman equation. ■

G Curvature equality

We want to show that:

$$\cos(\theta)^3 \, W^1_{W^2 W^2} \;=\; \sin(\theta)^3 \, W^2_{W^1 W^1} \,. \tag{G.1}$$

To begin, note that:

$$\frac{d}{dW^1} W^1_{W^2} = W^1_{W^2 W^2} \frac{dW^2}{dW^1} \;=\; - \, W^1_{W^2 W^2} \frac{\cos(\theta)}{\sin(\theta)} \,. \tag{G.2}$$

This is equal to:

$$\frac{d}{dW^1} \frac{1}{W^2_{W^1}} \;=\; - \, \frac{1}{\left(W^2_{W^1}\right)^2} W^2_{W^1 W^1} \;=\; - \, W^2_{W^1 W^1} \left(\frac{\sin(\theta)}{\cos(\theta)} \right)^2 \,. \tag{G.3}$$

Equating the two terms and performing algebra yields the result. ■

H Optimality of the smooth-pasting condition

In this section we prove Lemma 2 and Proposition 4, which establish optimality of the smooth-pasting condition. Our strategy differs a bit from the more well-known approaches such as Dixit (1993); we will show that if the smooth-pasting condition is satisfied then the second-order condition associated with the Bellman equation is locally satisfied at the boundary point characterised by the the smooth-pasting condition. That is, we provide a verification of the sufficiency of local optimality at the merger point. More concretely, the steps in this demonstration are as follows:

(i) Calculate the second-order condition for the Bellman equation from the optimality condition;

(ii) Evaluate the "ratio" version of the Bellman equation at the value-matching point, using the value-matching condition and also the smooth-pasting condition, resulting in a reduced-form expression for the second partial derivative $W^1_{W^2W^2}$;

(iii) Substitute this reduced-form expression for $W^1_{W^2W^2}$ into the second-order condition, as well as the value-matching condition, establishing that the second-order condition is negative.

Proof: (Of Lemma 2) Commencing step (i), the reprise of the Bellman equation is

$$0 = \max_{A^1} \left\{ r(g_1 - W^1) - r(g_2 - W^2)W^1_{W^2} + \tfrac{1}{2}\left(-rW^2_{W^1}g_{1A^1}\right)^2 W^1_{W^2W^2} + \tfrac{1}{2}\left(rg_{2A^2}\right)^2 W^1_{W^2W^2} \right\}. \quad \text{(H.4)}$$

with optimality condition

$$rg_{1A^1} - rg_{2A^1}W^1_{W^2} + \left(\left(-r^2W^2_{W^1}g_{1A^1}\right)g_{1A^1A^1} + \left(r^2g_{2A^2}\right)g_{2A^2A^1}\right)W^1_{W^2W^2} = 0 \quad \text{(H.5)}$$

The second-order condition can make use of the quadratic structure of g_1 and g_2: the third derivatives are zero, so we have

$$rg_{1A^1A^1} - rg_{2A^1A^1}W^1_{W^2} + r^2\left(-W^2_{W^1}(g_{1A^1A^1})^2 + (g_{2A^2A^1})^2\right)W^1_{W^2W^2}$$

Now we can substitute the smooth-pasting condition: $W^1_{W^2} = -1$:

$$rg_{1A^1A^1} + rg_{2A^1A^1} + r^2\left(-W^2_{W^1}(g_{1A^1A^1})^2 + (g_{2A^2A^1})^2\right)W^1_{W^2W^2}$$

The same holds for the other player: $W^2_{W^1} = -1$:

$$rg_{1A^1A^1} + rg_{2A^1A^1} + r^2\left((g_{1A^1A^1})^2 \ + (g_{2A^2A^1})^2\right)W^1_{W^2W^2}$$

Now recall the structure of the stage game payoff function g_1:

$$g_1(a_1, a_2) = a_1(\Pi_1 \ - \ \beta_1 a_1 \ - \ \delta_1 a_2) \ ,$$

and similarly for g_2, so that

$$g_{1A^1A^1} = -2\beta_1 - \delta_1 \qquad g_{1A^2A^1} = -\delta_2 \, ,$$

which are both negative by assumption.

Carrying out step (ii), we next transform the system of Bellman equations to ratio form to isolate the second partials. Appendix F demonstrates by straightforward algebra that the maximization of the following ratio is equivalent to the original maximization in (A.13):

$$W^1_{W^2W^2} = -\max_{A^1}\left\{\frac{(g_1 - W^1) - r(g_2 - W^2)\frac{1}{r}W^1_{W^2}}{r\left(\left(W^2_{W^1}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)}\right\} . \tag{H.6}$$

Substituting $\frac{1}{W^1_{W^2}}$ for $W^2_{W^1}$ yields:

$$W^1_{W^2W^2} = -\max_{A^1}\left\{\frac{(g_1 - W^1) - r(g_2 - W^2)\frac{1}{r}W^1_{W^2}}{r\left(\left(\frac{1}{W^1_{W^2}}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)}\right\} , \tag{H.7}$$

which, along with the first-order condition in A^1, is an ordinary differential equation (ODE) in W^1. Thus, we have converted the Bellman equation from a partial to an ordinary differential equation.

Finally, carry out step (iii), substituting for $W^1_{W^2W^2}$ from the ratio reformulation of the Bellman equation in equation (H.7), evaluated at the value-matching and smooth-pasting point:

$$W^1_{W^2W^2} = -\frac{(g_1 - W^1) - r(g_2 - W^2)\frac{1}{r}W^1_{W^2}}{r\left(\left(\frac{1}{W^1_{W^2}}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)} = -\frac{(g_1 - W^1) + (g_2 - W^2)}{r\left(\left(\frac{1}{(-1)}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)}$$
$$= -\frac{(g_1 - W^1) + (g_2 - W^2)}{r\left((g_{1A^1})^2 + (g_{2A^2})^2\right)} \tag{H.8}$$

If we can demonstrate that the numerator of this expression is negative or zero then we will have demonstrated that the second-order condition holds at the smooth-pasting point.

Lemma 4 *At the smooth-pasting point, we have that:*

$$(g_1 - W^1) + (g_2 - W^2) = 0\,.$$

Proof: The expressions $(g_1 - W^1)$ and $(g_2 - W^2)$ are the drifts of the continuation value processes for W^1 and W^2, respectively (see equation (A.4)). At the smooth-pasting point, these drifts necessarily are equal and of the opposite sign in order to point along the merger line, which has a slope of -1, and they therefore sum to zero. (Equivalently, the curvature at the smooth-pasting point, $W^1_{W^2W^2}$, is zero.) □

This completes step (iii), establishing the result. □

We can use this result to prove global optimality.

Proof: (of Proposition 4) The optimality of the main action A^1_t as the solution of the HJB equation (A.13) follows directly from conventional optimal control considerations. Therefore the optimal value state path satisfies equation (E.48); further derivations lead to the differential equation (E.43), combined with equation(E), as demonstrated in E. Sannikov (Sannikov (2007) Theorem 2, p. 1309) establishes that any optimal path must satisfy this differential equation system.

The remaining issue is to provide two boundary conditions for the second-order differential equation, (E.48), that the optimal path necessarily satisfies; the value-matching condition (A.12) provides one boundary condition. By Lemma 2, the smooth-pasting condition is locally optimal. Suppose that an alternative optimal path exists that does not satisfy the smooth-pasting condition. In that case, there is a kink at the merger boundary, and thus local optimality cannot be satisfied. Any equilibrium path must satisfy optimality, and hence any equilibrium path satisfies smooth pasting.

The kink is "one-sided:" the local curvature at the kink point is *positive* infinity, but cannot be negative infinity because the differential equation can approach the merger line from above.

Therefore it is only necessary only to observe that the violation of the relevant inequality rests on the positive infinity property. □

We remark that our results have not established uniqueness of the equilibrium path. We know from numerical experiments that two manifolds that satisfy the smooth-pasting condition exist, with a smaller one fully contained within the larger one that we have analyzed here.

The "standard" proof of the optimality of the smooth-pasting point such as in Dixit (1993) uses a Taylor expansion of the solution of a second-order differential equation, exploiting the structure of the solution stemming from the assumption of fixed coefficients. Our model does not lead to an equation with fixed coefficients, so it is not possible to use Dixit's approach directly. Notice, however, that the smooth-pasting condition is *locally* optimal at the smooth-pasting point; by continuity of the solution that obeys the main differential equation this optimality argument must also hold in a neighborhood of the smooth-pasting point.

I Proof of the inclusion result, Proposition 1

Proof: The proof has three main parts.

(i) In the first part we observe that the boundary of the equilibrium set $\mathcal{E}$, $\partial\mathcal{E}$, is described by the same ordinary differential equation, equation (E.43), for any merger model with merger cost K and for the no-merger case. If the inclusion were violated there would be some point $(\tilde{W}^1, \tilde{W}^2)$ at which the differential equation would necessarily hold identically for both K and K' at $(\tilde{W}^1, \tilde{W}^2)$. However non-inclusion would necessarily imply that the local differential would be different at $(\tilde{W}^1, \tilde{W}^2)$, a contradiction. Thus, the merger manifold must lie either entirely inside the no-merger manifold or entirely outside it.

(ii) Now consider the case where the merger manifold is entirely inside the no-merger manifold, that is, $\partial\mathcal{E}_{\mathrm{M}}^K \subset \partial\mathcal{E}_{\mathrm{NM}}$. To begin, note that the equilibrium manifold is locally characterized by the angle θ, the slope of the manifold for each θ, and the curvature κ. The curvature κ is characterized by equation (E.51), which we reproduce here:

$$\kappa(\tilde{W}) \;=\; \max_{A\in\mathcal{A}\setminus\mathcal{A}^N} \frac{2\,\mathbf{N}\,(g-\tilde{W})}{r|\phi|^2} \tag{I.9}$$

The key part of this equation is the numerator term

$$-\cos(\theta)(g_1-\tilde{W}^1)-\sin(\theta)(g_2-\tilde{W}^2) \tag{I.10}$$

(See equations (E.48) and (E.49).)

We will use the standard convention for measuring the angle θ, that is, $\theta = 0$ along the x-axis.

- Suppose that we solve the differential equation for the manifold with initial point at the middle of the collusion point. In that case, $\theta = \frac{\pi}{4}$, so that $\cos(\theta) = \sin(\theta)$, resulting in equal weights on the numerator term in (I.10). Given that the smooth-pasting manifold must lie inside the no-merger manifold, if we reduce $\tilde{W}_1$ and $\tilde{W}_2$, that is, reduce the initial point of the differential equation, then the absolute value of the curvature must increase. Therefore the path of the differential equation will move away from the no-merger path toward the interior.
- Next, consider $\theta = -\frac{\pi}{4}$, corresponding to the "contestability" region. In this region $\sin(\theta) = -\cos(\theta)$, so that there is a negative weight, $\sin(\theta)$, on the $(g_2 - W^2)$ term, and a positive weight, $\cos(\theta)$, on the $(g_1 - W^1)$ term. Now suppose we increase $\tilde{W}_2$ and decrease $\tilde{W}_1$ along the negative-45 degree line, so that the sum $\tilde{W}^1 + \tilde{W}^2$ is constant. In that case, the negative

weight on $\tilde{W}_1$ and the positive weight on $\tilde{W}^2$ means that the net effect of the movement is to increase the curvature. Therefore as we move inward, the curvature increases.

This argument implicitly holds the actions A^i fixed; however, recall that the curvature is defined by the formula in (E.51), which has a maximization with respect to the vector A on the right hand side. Therefore, adjustments of A that are driven by our thought experiments with $\tilde{W}$ will not affect our argument, as the effects of A drop out due to the envelope condition.

It is clear that this logic holds for all other angles: moving toward the center increases the curvature. The result is that the manifold "curls up" near the merger line, so that a tangency to the merger line becomes possible; this is apparent in the numerical simulations presented in the figures.

The remaining possibility is that the inclusion is reversed. (This possibility is illustrated in Figure 10 of Appendix G; clearly it is possible for the ODE to have a solution lying outside the no-merger manifold.)

We can see that the tangency condition cannot be satisfied locally where the no-merger boundary $\partial\mathcal{E}_{\mathrm{NM}}$ crosses the merger line. By continuity of the *derivative* of the manifold satisfying the ODE at that point, it cannot flip over and become tangent.

As we move away from that point, the logic we developed above for the "inside" manifold, establishes that the slope continues to flatten, so it moves away from a potential tangency, and the smooth pasting condition therefore cannot be satisfied.

However, we need to rule out the possibility that the smooth pasting conditions are satisfied, but pointing the wrong way. However this would require that the curvature of the manifold boundary reverse itself, at some locus, which in turn would imply that the ODE (48) was locally linear at the locus where this reversal occurs, that is, the local second derivative would be zero. But the structure of the ODE precludes this: it would imply that the differential equation is "stuck" at the zero-curvature point; because κ would be negative the linear part of the manifold would be infinitely long (see equation (E.43)).

(iii) The third part of the proof is to establish that the first inclusion holds, namely that increasing the merger cost K increases the equilibrium set. But this case is similar to the previous one, in that it would require that the merger manifold would not be tangent as we decrease K. □

J A remark on social welfare

The continuation values in the model are discounted profits, so we can infer that when these profits increase, the firms are colluding to a greater extent, and consumer surplus is concomitantly reduced. Thus, social welfare is inversely related to the firms' joint profits.

The smooth-pasting equilibrium has lower firm profits when the firms are in the collusion phase than they do if they can never merge, that is, the equilibrium manifold moves in the southwesterly direction relative to the no-merger manifold.

However, the merger cost itself is part of the deadweight loss.[24] How do we incorporate this deadweight loss in the accounting? The answer is that we can ignore it, because the firms' discounted profits (their continuation values) in the collusion phase are still close to the monopoly line, despite the discounted merger cost. Thus, this deadweight cost is, from the firms' perspectives, just an alternative cost like the cost they would face if they entered into the price war in the no-merger model.

[24]We are grateful to Mikhail Panov for raising this question.

K Refusals to merge, punishment and jumps

In this appendix[25] we explore the impact of assumptions about the structure of the underlying game, especially the details of the moment of the merger, on outcomes. We focus on one issue: the consequences of the fact that the decision to merge is observable to the rival firm. We emphasize that our discussion is informal.

One of the key notions in the main text is that the firms find it optimal to merge when the value states attain a point on the merger line. Merging at that point is a binary decision, and each firm can observe whether or not the other firm has merged with it because in the event of the merger they subsequently share monopoly profits.

Because the decision by each firm to merge is observable, if the firms do not commit to the merger in advance, the possibility that a firm would refuse to merge can also be admitted as a strategy, and the continuation of the game must be specified in that case. Because the refusal to merge is observable, the structure of the game at that moment and the associated continuation are potentially very different from the noisy collusion game that has been played up to that point.

If one firm refuses to merge, then, given the noncooperative environment of the game, it is appropriate to consider how the other firm would subsequently punish it. If the punishment is effective then the firm contemplating refusal would be be deterred from refusing, and the merger would take place, but potentially without necessarily satisfying smooth pasting. One can broadly describe this punishment: it would be to revert to a continuation phase with the worst possible outcome for the firm that has refused to merge.

The punishment phase must must itself be an equilibrium. Given that the continuation game would not differ from the collusion game in the key respect of having actions that are obscured by noise, the punishment phase would be constituted from the same elements as the pre-merger (or pre-refusal) game. If such an equilibrium exists, we refer to it as a refusal-punishment (R-P) equilibrium.

What we will establish here is that credible punishment phases cannot in fact themselves be equilibria, and so alternative equilibria that are sustained by such punishments cannot exist.

The refusal-punishment construction

For a refusal-punishment equilibrium to function, there must be agreement between the firms about the punishments in the punishment game. These punishments, and in essence the structure of the continuation game, must be agreed to in a pre-play phase of the game the firms play in advance of

[25]The ideas and contributions of Mikhail Panov, Yuliy Sannikov, and Andrzej Skrzypacz led to the discussion in this appendix.

the noisy collusion game. Thus, there are really three phases of the overall game: (i) the pre-play phase in which the punishments that will be coordinated upon in the event of a refusal to merge; (ii) the pre-merger phase of the noisy duopoly game; (iii) the punishment phase in the event that a firm refuses to merge, which itself must be an equilibrium of a pre-merger phase of the game, including the potential for repeated refusal-punishment phases. Thus, in this final stage, any boundary conditions entailed by the potential for future mergers, or the lack thereof, must be delineated.

In the sequel we examine the simplest case: that if a merger has been refused and punishment invoked, there is no further potential to merge: the punishment is permanent.

If there is no potential to merge, then the game reverts to the collusion game described by Sannikov (2007): the value states of the firms stochastically and continuously transit around an egg-shaped manifold with payoffs that are strictly bounded by the monopoly line. The initial point on the manifold is determined by the extreme punishment: if firm 1 (with value state on the horizontal axis) has refused to merge, the worst continuation on the no-merger manifold is the leftmost point on it (point B in the figure); similarly if firm 2 (with value state on the vertical axis) has refused to merge, the continuation commences on the lowest point on the manifold (point B' in the figure).

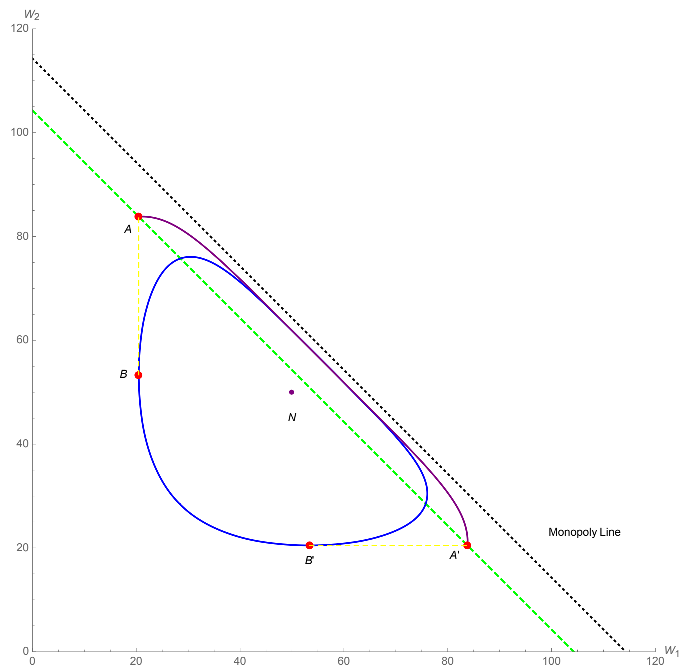
Punishments alter the boundary conditions of the pre-merger game. At the merger point, the punishment must be weakly worse than the worst state at the moment of the merger. Because the no-merger manifold is fixed in size and location, the leftmost and bottom-most punishment points on the manifold are fixed. This in turn dictates the corresponding upper and lower merger points on the merger line (points A and A' in Figure 6). Finally, because the equilibrium pre-merger manifold must be continuous and obey the underlying main differential equation, the pre-merger manifold must land on these two points, and is thus fully and uniquely determined.[26]

Because the landing points are determined by potential punishments, the smooth-pasting condition no longer determines the boundary conditions in this situation. The resulting manifold is semicircular and is not tangent at the landing points. We will refer to this manifold as the refusal-punishment (R-P) manifold. In addition to not being tangent at the merger line, the R-P manifold lies entirely between the no-merger manifold and the monopoly line. This is in contrast to the smooth-pasting manifold, which lies entirely inside the no-merger manifold.

This construction has *two* ingredients. The first is that the refusal and punishment are *observable*, unlike the pre-merger play, and also unlike the no-merger manifold that the firms jump to if the punishment is carried out. The second is that the punishment entails a *jump*. This is key: in

[26]Of course in equilibrium the punishments are not carried out. If they were carried out, the equilibrium manifold would then have to terminate on the punishment point, and this would in turn eliminate the original manifold from consideration. Thus, our example is already vitiated in the sense that it cannot be an equilibrium, but we continue with it to underscore an additional point.

Figure 6. The refusal-punishment equilibrium



In this figure one of the firms can refuse to merge; thereafter it is assumed that no merger is possible. If a firm refuses to merge, say at point A, then a punishment is initiated via a jump to point B, with symmetric possibilities at points A' and B'. The punishment manifold is simply the original no-merger manifold, with play evolving along this manifold as in the original Sannikov no-merger model.

the other stages of the game, jumps are not possible because actions, and the payoffs they generate, which are a flow, cannot generate discrete jumps in the continuation values, which are the integrals of these expected future payoffs and as such are continuous. But the *observability* of the refusal means that the game is no longer in its noisy mode at that instant, and a more conventional, static, full-information game can be played. Because there is a jump if the punishment is invoked, the punishment is far more severe than the incremental punishments of the pre-merger game. As the cost shrinks, the size of the jump interval actually increases, making the jump punishments even stronger. This is why the R-P manifold has the potential to achieve better cooperation than the no-merger game alone can achieve, if it is an equilibrium.

The punishments are not equilibria

The above reasoning breaks down for two reasons.

First, in equilibrium the punishments cannot be carried out with the proposed structure. If they were carried out, the equilibrium manifold would then have to terminate on the punishment point, because the continuation values are the integrals of the expected future payoffs, including the point at which the punishment commences, and as such are continuous and cannot accommodate jumps. This would in turn eliminate the original manifold from consideration.

Second, the punishment phase of the equilibrium would entail the firms traversing the punishment manifold that lies *below* the merger line. Every point on the punishment manifold has the property that it is worse for both firms relative to some point on the merger line. One of the firms could propose to move to such a point from the punishment manifold by merging, and the other firm could accept this proposal and improve its payoff.[27] This logic works because the proposal to merge, unlike the production action, is publicly observable. Therefore the punishment manifold is not immune to renegotiation in this sense, so in turn the punishment cannot be an equilibrium.[28]

This leaves only the manifolds that satisfy the smooth-pasting property as potential equilibria: because the smooth-pasting manifolds lie *above* the merger line, it is not better for both firms to immediately jump to the merger line. Thus, the R-P manifolds are ruled out. [29]

[27]The target locus would need to be decided in the pre-play phase of the game via a separate bargaining solution that we do not model. However any standard bargaining solution such as Nash bargaining or Kalai-Smorodinsky would have Pareto optimality as axiomatic.

[28]We further elaborate on such Pareto-improving jumps as non-cooperative coordination games, demonstrating that undertaking the merger in these situations is a dominant strategy under subgame perfection.

[29] It also bears emphasis that a Pareto-improving jump is not feasible in the pre-merger phase of the game because the continuation values are the integrals of these expected future payoffs and as such are continuous. The locus of the continuation values before the jump must be coterminous with the target point of the jump, simply because the continuation value prior to the jump is the expected discounted value of the continuation value at the target point.

L Ruling out never merging

In this appendix we again examine the potential for the firms to never merge, but focus on how these equilibria are ruled out by Pareto-improving mergers. Unlike the discussion in Appendix K, in this appendix we examine the opposite situation: instead of being punished by refusing to merge via a discrete and negative jump that reduces the continuation values of one or both firms, we examine the situation in which the firms have the potential to achieve a discrete positive improvement in their continuation values by merging.

Never merging might be viewed as a kind of coordination-failure continuation path where firms do not merge. Because the action space is such that both firms need to say they are ready to merge at the same time, there is a nominal potential for both firms never say they are ready to merge because each firm perceives that the rival will never agree to merge. In that case either firm refusing to merge seems to be a best response, since no matter what happens there is never going to be a merger.

If the firms never merging is an equilibrium then there is only one maximal equilibrium set and its associated boundary associated with this equilibrium, namely the manifold described by Sannikov, which we have replicated as the outer manifold in Figures 1 and 2 of the main text; we will refer to this manifold as the no-merger manifold. A key property of this manifold is that it dips below the merger line (except in the case where the merger cost is so high that the merger line is everywhere below the no-merger manifold, which we have not depicted).

We can focus on an arbitrary point on the no-merger manifold that lies below the merger line. For purposes of discussion denote this point by P. It is evident that there is a continuum of Pareto-improving mergers attained by jumping to the merger line, and the improvement in the continuation values is discrete. We can focus on one of these Pareto-improving jumps, namely the one such that the starting point P on the no-merger line is at right angles to the merger point.

Because there is a discrete improvement in the continuation payoffs, and because the improvement is the same for both firms, we can label this improvement as $\gamma > 0$ for both firms. Of course the magnitude of γ depends on the locus of the continuation values on the no-merger manifold, but the key property is that γ is positive.

We can now pose this situation faced by the firms at point P as an auxiliary game. We present the payoff matrix of the game in Table 1. The payoffs from agreeing to merge are the improvements in the continuation values, whilst the zero payoffs reflect the lack of improvement by refusing to merge or by both firms refusing. Another important detail of the auxiliary game is that it is a full-information game, unlike the tacit-collusion game that is our main object of study.

Table 1. Merger coordination game between firms

	Merge	Refuse
Merge	γ, γ	0, 0
Refuse	0, 0	0, 0

A key property of the payoff matrix is that there is no net punishment for refusing, and therefore the game is properly called a pure coordination game, also known as an assurance game: if the players play different strategies they both get zero. If they play the same strategy then there is a positive reward, however the reward is higher if they coordinate on the better option. This is a variant of the so-called stag hunt game that was originally described by Jean Jacques Rousseau: if the two players coordinate on hunting stags their reward is large, and if they take the safe option and hunt hares they get a smaller reward (see, e.g., Chapter 1 of Fudenberg and Tirole, 1991). The key feature of the game is that is a pure coordination game, so no punishments are available for players who choose to hunt hares. Therefore hunting stags is an equilibrium but so is hunting hares. Our game is an extreme version of this game in that there is a zero reward or punishment for hunting hares.

This multiplicity of equilibria of the auxiliary game can be broken by extending the Nash equilibrium concept to subgame perfection. The subgame perfect approach focuses on the extensive form and examines dominant moves if play is sequential.[30] The extensive form of the merger assurance game is depicted in Figure 7. Working from the end of the game with a move by firm 2, it is a dominant strategy to play Merge if player 1 has played Merge; in the penultimate stage where player 1 moves, it is then a dominant strategy to play Merge. Thus, {Merge, Merge} is the only equilibrium of the auxiliary game.[31]

This reasoning demonstrates that if the firms traverse the no-merger manifold to our arbitrary point P the firms then jump to the merger manifold. If we considered this as equilibrium play

[30] We emphasize that the sequential play structure is imposed at each point of any candidate equilibrium manifold, and does not entail the passage of time:

> "Often it is natural to identify the "stages" of the game with time periods, but this is not always the case. A counterexample is the Rubinstein-Stahl model of bargaining ... where each "time period" has two stages. In the first stage of each period, one payer proposes an agreement; in the second stage, the other player either accepts or rejects the proposal. The distinction is that time periods refer to some physical measure of the passage of time, such as the accumulation of delay costs in the bargaining model, whereas the stages need not have a direct temporal interpretation." [Fudenberg and Tirole (1991), p. 70]

[31] For further discussion see Fudenberg and Tirole (1991) chapter 3.

then the equilibrium manifold would terminate at our jump point, however it would then fail as an equilibrium in two senses: (i) it would fail to satisfy the main differential equation locally because at the jump point the differential equation is violated, and (ii) because the continuation values are defined as the integrals of future payoffs the jump point would incorrectly represent the discounted value of jumping to the merger.

In addition to these observations we can also note that our reasoning applies to any point on the no-merger manifold attained prior to our arbitrary jump point P, and so P will never be attained, as the jump will occur at an earlier point. Of course this reasoning applies at every point below the merger line, so no point below the merger line can survive as an equilibrium jumping off point.

By formally appending the assurance game as an auxiliary game at each point on the no-merger manifold and adopting subgame perfection as the equilibrium concept for this auxiliary game we then rule out the no-merger equilibrium. There is one additional detail however, which is that, given that no point on the no-merger manifold can survive a Pareto-improving jump, it is still possible for the upper part of the no-merger manifold to locally satisfy the main differential equation, and to terminate in a merger where it intersects the merger line. However it is clear that such a terminal point fails to satisfy the smooth pasting condition and as such is suboptimal for both firms. We can therefore rule out such candidate manifolds as equilibria.

The game structure outlined in Definition 1 is in fact the enhanced game; by replacing

(v) merging does not Pareto-dominate continuation prior to the merger.

in the definition of equilibrium in Definition 2 by

(v) the subgame-perfect equilibrium of the coordination game from every point on the manifold does not result in a jump.

We then formally rule out the no-merger manifold. For the sake of reducing distraction from our main focus, we have omitted this discussion from the main text.

We note that the subgame perfection mechanism that we have described here is the same underlying mechanism that we applied to rule out punishment manifolds in Appendix K, eliminating any candidate manifold in which the continuation values fall below the merger line.

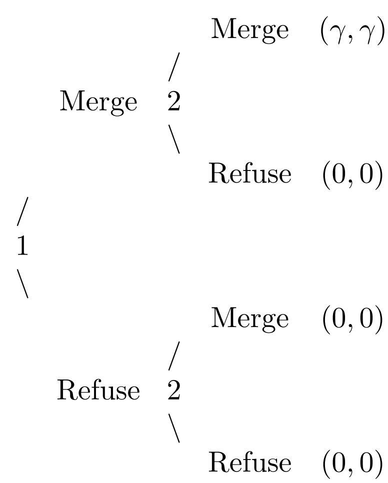


Figure 7. Extensive form of assurance game