Online Supplement Appendix to Manuscript "Product Selling Versus Pay-per-use Service: A Strategic Analysis of Competing Business Models,"

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Appendix A: Supplement Online Appendix

A.1. Base model:Monopoly

A.1.1. Monopolist's prices and profits When the monopolist sets a price p_s for the product, a customer of consumption profile q enjoys utility $U(q;p_s) = aq - p_s$. The customer of profile q buys the product if $U \geqslant 0$. Hence, customers with frequency $q \geqslant p_s/a$ will purchase, and customers with $q < p_s/a$ do not buy. The maximization of the monopolist's profit Π_s is can be written as follows: $\max_{p_s>0} \Pi_s(p_s)$ where $\Pi_s = \int_{\frac{p_s}{a}}^{q_m} (p_s - c) f(q) dq = \frac{1}{q_m}(p_s - c)(q_m - \frac{p_s}{a})$. The value p_s that maximizes the profit $\Pi_s(p_s)$ satisfies the first order condition $\partial \Pi_s / \partial p_s = 0$. It is straightforward to show that the price and the maximum profits of the product provider will respectively be $p_s^* = (aq_m + c)/2$ and $\Pi_s^*(p_s) = \frac{1}{4aq_m}(aq_m - c)^2$. The market share for the product provider is $M_s = 1 - p_s^*/(aq_m)$. Note that when $c = 0$, customers with use profile $q \geqslant q_m/2$ buy the product and the market share is $q_m/2$. As the cost increases $(c \to aq_m)$, the marginal customer shifts to higher use moving towards q_m and the monopolist's market share diminishes gradually to zero.

Instead of selling, a monopolist can choose to deliver a service where she charges p_u for a single use of the product. Then a customer with profile q will enjoy utility $U(q;p_u) = q(a - p_u)$. Under such a selling mechanism, any customers enjoys a positive utility for every $p_u \leqslant a$. If $p_u > a$ no consumer purchases the service and the monopolist gets zero profit. The profit maximization problem can be written as follows: $\max_{p_u \leqslant a} \Pi_u(p_u)$ where $\Pi_u(p_u) = \int_0^{q_m} (p_u - c_u) q f(q) dq = \frac{1}{2} q_m (p_u - c_u)$ It is straightforward to establish that $\partial \Pi_u / \partial p_u = q_m/2 > 0$, in other words, $\Pi_u(p_u)$ is an increasing function of p_u and hence the maximum profit is attained at $p_u = a$. Then the optimal price and the maximum profit for the pay-per-use scheme are respectively: $p_u^* = a$ and $\Pi_u^* = \frac{1}{2}\big(q_m(a - \delta) - c\big)$ and the market is covered (i.e, $M_u = 1$). Note that because the service provider bears a cost per use δ is limited to offer the pay-per-use solution up to costs $c \leqslant q_m(a - \delta)$. The following table summarizes the optimum prices, market share, and profits for the seller and the PPU provider

	Selling	Pay-per-use
Price	$p_s^* = \frac{1}{2}(aq_m + c)$	$p_u^* = a$
Market share	$M_s = \frac{1}{2} - \frac{c}{2aq_m}$	$M_u = 1$
Profit	$\Pi_s^* = \frac{(aq_m - c)^2}{4aq_m}$	$\Pi_u^* = \frac{1}{2}\big(q_m(a-\delta) - c\big)$

A.1.2. Proof of Proposition 1 There exists a value of c that makes the maximum profits of the pay per use and the selling mechanism equal, i.e., $\Pi_s^* = \Pi_u^*$. Using the expressions for the profits presented in section 3.1 it is straightforward to show that the value of c that makes the profits equal is given by $c = q_m\sqrt{a(a-2\delta)}$ and that $\Pi_u > \Pi_s$ (i.e., the pay-per-use mechanism) outperforms the selling mechanism) for $c \leqslant q_m\sqrt{a(a-2\delta)}$.

A.2. Base model: Duopoly

A.2.1. Duopoly prices and profits In a duopoly we assume that one firm acts as a product provider who uses the selling mechanism setting a price at p_s offering unrestricted usage of the product, and the other firm acts as a service provider and employs the pay-per-use mechanism asking p_u per one use of the product. The indifferent customer gets the same utility both from buying the product and from buying uses of the product. In other words, the indifferent customer is at frequency $\bar{q}$ for which $U_s(\bar{q}; p_s) = U_u(\bar{q}; p_u)$. Solving this equation for $\bar{q}$ it can be easily seen that $\bar{q} = p_s/p_u$. Customers with frequency $q \in [\bar{q}, q_m]$ prefer to buy the product and customers with frequency $q \in [0, \bar{q}]$ prefer to use the service and to pay according to their use. The profits each firm expects Π_s and Π_u are given by:

$$\Pi_s(p_s, p_u) = \int_{\bar{q}}^{q_m} (p_s - c) f(q) dq = \frac{1}{q_m}(p_s - c)(q_m - \frac{p_s}{p_u}) \tag{1}$$

$$\Pi_u(p_s, p_u) = \int_0^{\bar{q}} (p_u - c_u) q f(q) dq = \frac{1}{2q_m}\frac{p_s^2}{p_u^2}(p_u - c_u) \tag{2}$$

The product provider will set the price p_s as to maximize her profit and the condition $\partial\Pi_s/\partial p_s = 0$ gives $p_s = \frac{1}{2}(c + q_m p_u)$ as the seller's best response. The service provider will choose its price p_u as to maximize her profits Π_u subject to the constraint $p_u \leq a$. The Lagrangian for the service provider's constrained optimization problem is $\Lambda(p_u, \mu) = \Pi_u(p_u) + \mu(a - p_u)$. The solution for p_u and μ must satisfy the KKT conditions

$$\frac{\partial\Pi_u}{\partial p_u} - \mu = 0 \quad , \quad \mu(a - p_u) = 0 \quad , \quad p_u \leqslant a \quad , \quad \mu \geqslant 0$$

When $\mu = 0$ we have from the KKT conditions that $\partial\Pi_u/\partial p_u = 0$. The derivative of Π_u with respect to p_u is

$$\frac{\partial\Pi_u}{\partial p_u} = \frac{p_s^2(2c_u - p_u)}{2q_m p_u^3}$$

Setting the above expression to zero gives the solution $p_{u,1} = 2c_u$ which satisfies all the KKT conditions when $c_u < a/2$. When $\mu \neq 0$ we have $p_{u,2} = a$ which gives $\mu = p_s^2(2c_u - a)/(2q_m a^3)$ which is positive provided $c_u > a/2$. Therefore, when the cost $c \leqslant \frac{1}{2}q_m(a-2\delta)$, the service provider will ask a price per use at $p_u = 2(\frac{c}{q_m} + \delta)$ (a price double than the cost of producing one use of the product or service), he will capture market form zero frequency up to $\frac{1}{2}q_m + \frac{1}{4}q_m c/(c + q_m\delta)$, and will end up with profits

$$\Pi_{u,d} = \frac{(3c + 2q_m\delta)^2}{32(c + q_m\delta)}$$

In response, the product provider will set a price at $p_s = \frac{1}{2}(3c + 2\delta q_m)$ and will capture profits

$$\Pi_{s,d} = \frac{(c + 2q_m\delta)^2}{8(c + q_m\delta)}$$

When $c > \frac{1}{2}q_m(a - 2\delta)$ the service provider cannot continue this policy (pricing at double the cost) as this price will exceed the customer's willingness to pay a. He will then price at $p_u = a$ and the product provider in response will price at $\frac{1}{2}(c + aq_m)$. The profits for the product provider (seller) and the service provider they will be

$$\Pi_s = \frac{1}{4q_m a}(aq_m - c)^2 \quad \text{and} \quad \Pi_u = \frac{1}{8q_m^2 a^2}(q_m(a-\delta) - c)(aq_m + c)^2$$

The market shares for the product provider M_s and the service provider M_u are respectively, $M_s = 1 - \bar{q}/q_m$ and $M_u = \bar{q}/q_m$. The overall solution for the duopoly problem for both firms is given in the following table.

	$c \leqslant q_m(\frac{a}{2} - \delta)$	$c > q_m(\frac{a}{2} - \delta)$
p_s	$\frac{1}{2}(3c + 2\delta q_m)$	$\frac{1}{2}(c + aq_m)$
p_u	$2(\frac{c}{q_m} + \delta)$	a
M_s^*	$\frac{1}{2} - \frac{c}{4(c+\delta q_m)}$	$\frac{1}{2} - \frac{c}{2aq_m}$
M_u^*	$\frac{1}{2} + \frac{c}{4(c+\delta q_m)}$	$\frac{1}{2} + \frac{c}{2aq_m}$
Π_s	$\frac{(c+2\delta q_m)^2}{8(c+\delta q_m)}$	$\frac{(aq_m - c)^2}{4q_m a}$
Π_u	$\frac{(3c+2\delta q_m)^2}{32(c+\delta q_m)}$	$\frac{(q_m(a-\delta)-c)(aq_m+c)^2}{8q_m^2 a^2}$

It can be easily verified the solutions for the profits of the service provider and for the product provider are continuous in c.

A.2.2. Equilibrium analysis of the base model (Proof of Proposition 2) If both firms use the same business model they compete themselves down to zero profits under Bertrand competition. Each firm can lower its selling price p_s or pay-per-use fee p_u to undercut the other. Therefore, the (s,s) and (u,u) sub-games will have zero payoffs for both firms. On the other hand, if each firm is committing to a different business model, they will assume non zero profits $\Pi_s > 0$ for the selling firm and $\Pi_u > 0$ for the firm using PPU. The expressions for profits Π_s and Π_u are derived in appendix (A.2.1).

We consider the "hybrid" case where one firm is offering both business models (sell and PPU) at prices (p_{s_1}, p_{u_1}) and the other is offering just PPU at p_{u_2}. The firm offering only pay-per-use will compete down to c_u with the firm that is offering both models. and hence the pay-per-use fee will be $p_{u_1} = p_{u_2} = c_u$. The firm that is offering only PPU will realize zero profits and the firm that is offering both models will realize profits only form selling. Given that the PPU price is at $p_u = c_u$, the indifferent customer is at $\bar{q} = p_s/c_u$ and the profits for the firm that is offering both models Π'_s will $\Pi'_s = \frac{1}{q_m}(p_s - c)(q_m - p_s/c_u)$. The firm will set p_s as to maximize the profits. The optimum price will be $p_s = \frac{1}{2}(c + q_m c_u)$ and the maximum profits will be $\Pi'_s = \frac{1}{4q_m c_u}(q_m c_u - c)^2$. It is straight forward to show that $\Pi'_s < \Pi_s$.

The symmetric problem is when one firm offers both models at prices (p_{s1}, p_{u1}) and the other is offering just selling at p_{s2}. The firm offering only the selling model will compete down to $p_{s_1} = p_{s_2} = c$ with the firm that is offering both models. So the firm offering only selling will realize zero profits and the firm that offers both models will realize profits Π'_u only from customers attracted by the pay-per-use model. Given that the selling price is at $p_s = c$, the indifferent customer is at $\bar{q} = c/p_u$ and the profits will be $\Pi'_u = \frac{c^2}{2q_m}\frac{p_u - c_u}{p_u^2}$. The firm will try to set p_u as to maximize the profits. The optimum price will be $p_u = 2c_u$ for $c_u \leqslant a/2$ and $p_u = a$ for $c_u > a/2$ and the maximum profits will be $\Pi'_u = \frac{c^2}{8q_m c_u}$ for $c_u \leqslant a/2$ and $\Pi'_u = \frac{(a-c_u)c^2}{2q_m a^2}$ for $c_u > a/2$. It is straightforward to show that $\Pi'_u < \Pi_u$.

In case where both firms offer both business models (selling and PPU) they will compete down to cost for both the selling part and the PPU part. The prices will be $p_u = c_u$ and $p_s = c$ and both firms will end up with zero profits.

The normal form representation of the game is:

	Sell	PPU	Both
Sell	0,0	Π_s,Π_u	0, Π'_u
PPU	Π_u,Π_s	0,0	0,Π'_s
Both	Π'_u,0	Π'_s,0	0,0

There are two asymmetric pure-strategy Nash equilibria in the game where the two firms offer identical products, with one firm selling and the other using PPU business model.

For the mixed-strategy equilibrium for the normal game in which one firm is using selling and the competitor uses PPU we have the following. Each firm can adopt a mixed-strategy in which the probability λ_s of selling makes the other firm indifferent between selling and PPU. This probability then satisfies $(1-\lambda_s)\Pi_s = \lambda_s \Pi_u$ which gives $\lambda_s = \Pi_s/(\Pi_s + \Pi_u)$.

A.2.3. Imperfect Market As in the base model, suppose customers want q uses of the product, with q uniformly distributed between zero and q_m. The customers have a reservation price a per use. To model market imperfection, we assume that customers come in two types, informed and uninformed. Uninformed customers select a firm at random; a customer of type q buys the product from a seller that asks a price p_s if $aq \leqslant p_s$ or buys q uses of the product from a PPU provider if the price $p_u \leqslant a$. Informed customers, in contrast, know the business models and prices and select that firm that maximizes their utility. Let λ be the percentage of uninformed customers where $0 \leqslant \lambda \leqslant 1$. Firms attempt to price discriminate between informed and uninformed customers. In what follows we use the notation Π_{im}, Π_{id} for the monopoly and duopoly profits for $i = \{s, u\}$ and p_{im}, p_{id} for the corresponding optimal prices as presented in the base model. We repeat here the functional forms for convenience:

$$\begin{aligned} \Pi_{sm}(p_s) &= \tfrac{1}{q_m}(p_s - c) * (q_m - p_s/a) \\ \Pi_{sd}(p_s, p_u) &= \tfrac{1}{q_m}(p_s - c) * (q_m - p_s/p_u) \\ \Pi_{um}(p_u) &= \tfrac{1}{2} q_m (p_u - c_u) \\ \Pi_{ud}(p_s, p_u) &= \tfrac{1}{2q_m}(p_u - c_u) p_s^2/p_u^2 \end{aligned}$$

The (s,s) and (u,u) subgame: Assume that two firms sell identical goods at prices p_{s1} and p_{s2}. If firm 1 post a price $p_{s1} < p_{s2}$, she will get $\frac{\lambda}{2}$ customers that have a usage profile $q \leqslant p_{s1}/a$ and all the $1-\lambda$ customers with usage profile $q \leqslant p_{s1}/a$. If firm1 posts a price $p_{s1} > p_{s2}$ she will get only $\frac{\lambda}{2}$ of the uninformed customers whose q profile is $q \leqslant p_{s1}/a$. If firms post the same price they split the market. In other words, firm 1 will get profits

$$\Pi_{s1}(p_{s1}) = \begin{cases} \frac{\lambda}{2}\Pi_{sm}(p_{s1}) & p_{s1} > p_{s2} \\ \frac{1}{2}\Pi_{sm}(p_{s1}) & p_{s1} = p_{s2} \\ (1 - \frac{\lambda}{2})\Pi_{sm}(p_{s1}) & p_{s1} < p_{s2} \end{cases}$$

where $\Pi_{sm}(p_{s1})$ are the monopoly profits for price p_{s1}. In this duopoly game firms enter the market if they get at least $\frac{\lambda}{2}\Pi_{sm}(p_{sm})$ profits. Varian (1980, p. 654) shows this game has no symmetric price equilibrium in pure strategies. He assumes that firms choose prices randomly out of a distribution density $f(p_s)$ which has a cumulative distribution function $F(p_s)$. Varian (1980) shows that there exists a cumulative distribution function $F(p_s)$ that maximizes the expected profits for each firm. Moreover, the expected profits for each firms are the optimal monopolist profits for the uninformed customer segment. Therefore, the expected profits of both firms will be

$$\mathbb{E}(\Pi_{si}) = \tfrac{\lambda}{2}\Pi_{sm}(p_{sm}) = \frac{\lambda}{8aq_m}(aq_m - c)^2$$

for $i = 1, 2$ where p_{sm} is the monopoly price $p_{sm} = \frac{1}{2}(c + aq_m)$. When there is no market friction $(\lambda = 0)$, the expected profits are zero inline with Bertrand's competition for identical products result.

The (u, u) sub-game can be treated in exactly the same way. The expected profits of both firms will be the optimal monopolist profits for the uninformed customer segment

$$\mathbb{E}(\Pi_{ui}) = \tfrac{\lambda}{2}\Pi_{um}(p_{um}) = \frac{\lambda}{4}q_m(a - c_u)$$

for $i = 1, 2$, where p_{um} is the monopolist price $p_{um} = a$. When there is no market friction, $(\lambda = 0)$ the expected profits drop to zero inline with Bertrand's competition result for identical products.

The (s,u) subgame: We assume that one firm sells the product at price p_s and the other firms offers uses of the product at price p_u per use. Each provider gets monopoly profits from $\frac{\lambda}{2}$ part of the uninformed customers and engages in competition for the $1 - \lambda$ uninformed customers from which she gets duopoly profits according to the posted prices p_s and p_u. The profits for the seller $\Pi_{s\lambda}$ and the PPU provider $\Pi_{u\lambda}$ are therefore,

$$\begin{aligned}\Pi_{s\lambda}(p_s, p_u) &= \tfrac{\lambda}{2}\Pi_{sm}(p_s) + (1-\lambda)\Pi_{sd}(p_s, p_u)\\ \Pi_{u\lambda}(p_s, p_u) &= \tfrac{\lambda}{2}\Pi_{um}(p_s) + (1-\lambda)\Pi_{ud}(p_s, p_u)\end{aligned}$$

The partial derivatives of seller's and PPU's profits with respect to the seller's and PPU's price are:

$$\begin{aligned}\frac{\partial \Pi_{s\lambda}}{\partial p_{s\lambda}} &= \left(1 - \tfrac{\lambda}{2}\right) q_m - \left(\tfrac{\lambda}{2a} + \tfrac{1-\lambda}{p_u}\right)(2p_s - c)\\ \frac{\partial \Pi_{u\lambda}}{\partial p_{u\lambda}} &= \tfrac{\lambda}{2} q_m^2 p_u^3 - (1-\lambda)p_s^2(p_u - 2c_u) = 0\end{aligned}$$

The equilibrium prices $p_{s\lambda}$ and $p_{u\lambda}$ for λ market imperfection have to satisfy the following equations

$$\left(1 - \tfrac{\lambda}{2}\right) q_m - \left(\tfrac{\lambda}{2a} + \tfrac{1-\lambda}{p_{u\lambda}}\right)(2p_{s\lambda} - c) = 0 \tag{3}$$

$$\tfrac{\lambda}{2} q_m^2 p_{u\lambda}^3 - (1-\lambda)p_{s\lambda}^2(p_{u\lambda} - 2c_u) = 0 \tag{4}$$

$$p_{u\lambda} \leqslant a \tag{5}$$

The equilibrium profits are $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda})$ and $\Pi_{u\lambda}(p_{s\lambda}, p_{u\lambda})$. Note that equations (3) and (4) form a system of nonlinear equations of fifth degree in p_u and one cannot expect to track it analytically.

Equation (4) implies that $p_u > 2c_u$, otherwise the second term is negative and subtracted from the first term which is positive gives always a positive result. But if $c_u \geqslant a/2$ then $p_u \geqslant a$ and therefore the solution for p_u is limited to $p_{u\lambda} = p_{sm} = a$. Note that when $p_{u\lambda}p_{um} = a$ then equation (3) gives $p_{s\lambda} = p_{sm} = \frac{1}{2}(c + aq_m)$. Therefore, when $c_u \geqslant a/2$ the equilibrium prices for the λ problem are the monopoly prices allowing equilibrium profits to be expressed in closed form as shown below

$$\Pi_{s\lambda} = (1 - \tfrac{\lambda}{2})\Pi_{sm}(p_{sm}) = (1 - \tfrac{\lambda}{2})\frac{(aq_m - c)^2}{4aq_m}$$
$$\Pi_{u\lambda} = \tfrac{\lambda}{2}\Pi_{um}(a) + (1 - \lambda)\Pi_{ud}(p_{sm}, a) = \frac{a - cu}{2q_m}\left(\tfrac{\lambda}{2}q_m^2 + (1 - \lambda)\frac{(aq_m + c)^2}{4a^2}\right)$$

As is evident from the above equations, equilibrium profits are functions of c and $c_u = c/q_m + \delta$.

For $c_u < a/2$ there are no closed form solutions. But we can compare the equilibrium profits. Let $p_{s\lambda}$ and $p_{u\lambda}$ denote the equilibrium prices. Because $p_{u\lambda}$ is the Nash price, any other price different from this will give lower profits. Therefore, $\Pi_{u\lambda}(p_{s\lambda}, p_{u\lambda}) > \Pi_{u\lambda}(p_{s\lambda}, a)$. But $\Pi_{u\lambda}(p_{s\lambda}, a) = \frac{\lambda}{2}\Pi_{um}(a) + (1 - \lambda)\Pi_{ud}(p_{s\lambda}, a) > \frac{\lambda}{2}\Pi_{um}(a) > \frac{\lambda}{2}\Pi_{sm}(p_{sm})$. The last inequality comes from the fact that when $c_u < a/2$, the monopoly profits of PPU are greater than the monopoly profits of the selling scheme as we have shown in the base section. Therefore, for all λ and for $c_u < a/2$, we have that $\Pi_{u\lambda}(p_{s\lambda}, p_{u\lambda}) > \frac{\lambda}{2}\Pi_{sm}(p_{sm})$. This implies that for $c_u < a/2$ the equilibrium can be either (u, s), or (s, u), or (u, u). Said differently, the equilibrium cannot be (s, s) for $c_u < a/2$.

For $c_u < a/2$, the equilibrium prices $p_{s\lambda}$ and $p_{u\lambda}$ are smaller than the monopoly prices so we have that $p_{s\lambda} < p_{sm}$ and $p_{u\lambda} < a$. We can show that $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda}) < \Pi_{s\lambda}(p_{sm}, p_{um})$. Indeed, $\Pi_{s\lambda}(p_{sm}, p_{um}) = \frac{\lambda}{2}\Pi_{sm}(p_{sm}) + (1 - \lambda)\Pi_{sd}(p_{sm}, p_{um}) = \frac{\lambda}{2}\Pi_{sm}(p_{sm}) + (1 - \lambda)\Pi_{sm}(p_{sm})$. On the other hand, $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda}) = \frac{\lambda}{2}\Pi_{sm}(p_{s\lambda}) + (1 - \lambda)\Pi_{sd}(p_{s\lambda}, p_{u\lambda})$. But $\Pi_{sm}(p_{s\lambda}) < \Pi_{sm}(p_{sm})$ because p_{sm} is the optimum monopoly price. Also, $\Pi_{sd}(p_{s\lambda}, p_{u\lambda}) < \Pi_{sd}(p_{s\lambda}, p_{um})$ because $\Pi_{sd}(p_s, p_u)$ is an increasing function in p_u[12]. Therefore, we have that $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda}) < \Pi_{s\lambda}(p_{sm}, p_{um})$. But $\Pi_{s\lambda}(p_{sm}, p_{um}) = (1 - \frac{\lambda}{2})\Pi_{sm}(p_{sm})$ and hence we have that $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda}) < (1 - \frac{\lambda}{2})\Pi_{sm}(p_{sm})$. Because $\Pi_{sm}(p_{sm}) < \Pi_{um}(p_{um})$ for $c_u < a/2$, it is straightforward to show[13] that there is $\lambda_0 < 1$ such that for $\lambda > \lambda_0$, $(1 - \frac{\lambda}{2})\Pi_{sm}(p_{sm}) < \frac{\lambda}{2}\Pi_{um}(p_{um})$. We have therefore shown that for $c_u < a/2$ and for $\lambda > \lambda_0$, $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda}) < \frac{\lambda}{2}\Pi_{um}(p_{um})$ which implies that the equilibrium is (u, u).

We can also show that there exists $\lambda_1 < 1$ such that for $\lambda < \lambda_1$ $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda}) > \frac{\lambda}{2}\Pi_{um}(p_{um})$. Indeed, because $p_{s\lambda}$ is the equilibrium price, $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda}) > \Pi_{s\lambda}(p_{sd}, p_{u\lambda}) > (1 - \lambda)\Pi_{s\lambda}(p_{sd}, p_{u\lambda}) > (1 - \lambda)\Pi_{s\lambda}(p_{sd}, p_{ud})$. The last equality comes from the fact that Π_{sd} is an increasing function in

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[12] It is straightforward to show that $\partial\Pi_{sd}/\partial p_u > 0$.

[13] Indeed, $\lambda_0 = 2\Pi_{sm}(p_{sm})/(\Pi_{sm}(p_{sm}) + \Pi_{um}(p_{um})$. Because $\Pi_{sm}(p_{sm}) < \Pi_{um}(p_{um})$, then $\lambda_0 < 1$

p_u and that $p_{u\lambda} > p_{ud} = 2c_u$. It is straightforward to show[14] that there exists $\lambda_1 < 1$ such that for $\lambda < \lambda_1$, $(1-\lambda)\Pi_{sd}(p_{sd}, p_{ud}) > \frac{\lambda}{2}\Pi_{um}(p_{um})$. Therefore, there exists a $\lambda_1 < 1$ such that for $\lambda < \lambda_1$ we have asymmetric equilibrium is (u,s) or (s,u).

The overall game Each firm can choose one of the two strategies: either to play (s) or to play (u). The normal form representation of the game is given in the following table

	Selling	PPU
Selling	$\frac{\lambda}{2}\Pi_{sm}(p_{sm})$, $\frac{\lambda}{2}\Pi_{sm}(p_{sm})$	$\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda})$, $\Pi_{u\lambda}(p_{s\lambda}, p_{u\lambda})$
PPU	$\Pi_{u\lambda}(p_{s\lambda}, p_{u\lambda})$, $\Pi_{s\lambda}(p_{s\lambda}, p_{u\lambda})$	$\frac{\lambda}{2}\Pi_{um}(p_{um})$, $\frac{\lambda}{2}\Pi_{um}(p_{um})$

The following figures depict the outcomes of a family of games for all feasible values of c and δ for specific values of the market friction parameter λ. We present two scenarios: a small market friction $\lambda = 10\%$, where just 10% of the customers are uninformed, and significant market friction, where 50% of the customers are uninformed.

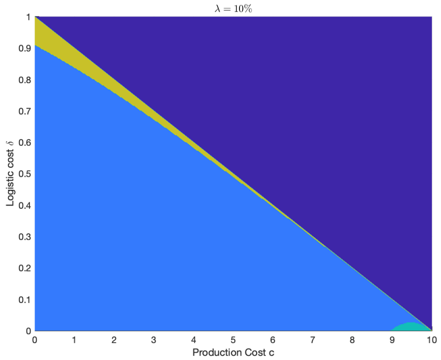

Figure 14 Equilibria for small friction $\lambda = 10\%$ case in the (c,δ) space. Light Blue area: Equilibrium at (u,s) or (s,u), Yellow area: Equilibrium at (s,s), Green area: Equilibrium at (u,u), Dark Blue area: PPU not feasible.

Figures 14 and 15 suggest that for small values of c and large values of δ, the equilibrium is (s,s) (selling dominates even under head-to-head competition). For large values of c and small values of δ the imperfect market game equilibrium becomes (u,u) (PPU is more attractive even under head-to-head competition). For all other regions in the (c,δ) space, we get an asymmetric equilibrium (u,s) or (s,u) as in the base model. It is striking that even in the scenario of significant market friction (50% of customers are uninformed), the asymmetric equilibria arise across the largest portion of the feasible parameter space.

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[14] Indeed, $\lambda_1 = 2\Pi_{sd}(p_{sd}, p_{ud})/(2\Pi_{sd}(p_{sd}, p_{ud}) + \Pi_{um}(p_{um}))$ which is always positive and smaller than one.

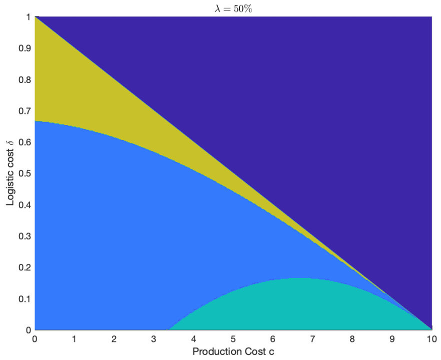

Figure 15 Equilibria for significant friction $\lambda = 50\%$ case. Light Blue area: Equilibrium at (u,s) or (s,u), Yellow area: Equilibrium at (s,s), Green area: Equilibrium at (u,u), Dark Blue area: PPU not feasible

A.2.4. Equilibrium profits comparison (Proof of Proposition 3) The iso-profit line consists of points in (c,δ) space where the profits for the two mechanisms are equal. For $c \leqslant q_m(\frac{a}{2} - \delta)$ the condition $\Pi_s = \Pi_u$, after substituting the relevant expressions for the profits from Appendix (A.2.1) leads to the equation $(5c + 6\delta q_m)(c - 2\delta q_m) = 0$, which gives $c = 2\delta q_m$ or $\delta = c/(2q_m)$. The second solution of the quadratic equation gives a negative δ. Because this is valid for $c \leqslant q_m(\frac{a}{2} - \delta)$ this leads to an upper bound for δ i.e., $2\delta q_m \leqslant q_m(\frac{a}{2} - \delta)$, which implies that $\delta \leqslant a/6$. The parameter δ achieves the value $a/6$ at $c = \frac{1}{3} a q_m$.

For $c > q_m(\frac{a}{2} - \delta)$ the condition $\Pi_s = \Pi_u$, after substituting the relevant expressions from Appendix (A.2.1), leads to the equation $2aq_m(aq_m - c)^2 = (q_m(a - \delta) - c)(aq_m + c)^2$. Solving for δ we find

$$\delta = a - \frac{c}{q_m} - 2a\frac{(aq_m - c)^2}{(aq_m + c)^2}$$

The derivative with respect to c is

$$\partial\delta/\partial c = \frac{c^3 + 3ac^2 + 11a^2 q_m^2 c - 7a^3 q_m^3}{q_m(aq_m + c)^3}$$

The condition $\partial\delta/\partial c = 0$ gives the value of c that maximizes δ which is $c \approx 0.542 a q_m$ and the maximum value of δ is $\delta_{max} \approx 0.286a$.

A.3. Heterogeneity in valuation

A.3.1. Monopoly The product provider will set a price p_s. Customers with $qa \geqslant p_s$ will buy the product and customers with $qa < p_s$ will abstain from the market. In this case the iso-utility curves are hyperbolas in the (q,a) plane. The expected profit for the seller is

$$\Pi_s = \int_{p_s}^{q_m} f(q)dq \int_{p_s/q}^{1} (p_s - c)da = (p_s - c)\left(1 - \frac{p_s}{q_m} + \frac{p_s}{q_m}\ln(p_s/q_m)\right) \tag{6}$$

The price p_s that maximizes seller's profits must satisfy condition $\partial\Pi_s/\partial p_s = 0$, which gives,

$$\frac{1}{q_m}(2p_s - c)\ln(p_s/q_m) + 1 - \frac{p_s}{q_m} = 0 \tag{7}$$

Equation 7 is a transcendental equation that can be solved with the help of the following lemma.

LEMMA 1. *Equations of the form* $(Ax+C)\ln x - Bx = a$ *for* A, B, a *real numbers and* $A \neq 0$ *and* $B \neq 0$ *have solution given by the following expression*

$$x = \exp\left[\frac{B}{A} + \mathcal{W}\left(-\frac{C}{A}e^{-B/A}; 0, \frac{aA-BC}{AC}\right)\right]$$

where $\mathcal{W}$ *is the generalized Lambert function*

We note that equation 7 is in the form of the equation in Lemma 1 with $A=2$, $B=1$, $C=-c/q_m$, and $a=-1$. Applying the results of Lemma 1 we arrive at

$$p_s = q_m \exp\left[\frac{1}{2} + \mathcal{W}\left(\frac{c}{2q_m}e^{-1/2}; 0, \frac{2q_m - c}{2c}\right)\right] \tag{8}$$

The corresponding profits can be expressed in terms of the price p_s. We can solve equation 7 for $\ln(p_s/q_m)$ and substitute into equation 6 to express the seller's profits in terms of the price p_s as

$$\Pi_s = \frac{(p_s - c)^2(q_m - p_s)}{q_m(2p_s - c)} \tag{9}$$

The service provider, in a monopoly setting, will set a price p_u as to maximize his profits. Customers with $a \geqslant p_u$ i.e, customers that are willing to pay more than p_u will participate in the market irrespective of the frequency of use. The service provider's profits are given by

$$\Pi_u = \int_0^{q_m} qf(q)dq \int_{p_u}^1 (p_u - c_u)g(a)da = \frac{q_m}{2}(p_u - c_u)(1 - p_u)$$

The condition $\partial\Pi_u/\partial p_u = 0$ gives as $p_u = \frac{1}{2}(1 + c_u)$. Substituting back into the equations for the profits, we find that the profits the service provider expects as a monopolist are $\Pi_u = \frac{q_m}{8}(1 - c_u)^2$.

A.3.2. Duopoly In duopoly one firm decides to offer a fixed-based pricing scheme at p_s and the other to offer a pay-per use scheme at p_u. The two firms compete by truncating the others segment along the line $q = p_s/p_u$ which is the locus of the indifference between fixed and usage based pricing. The profits each firm expect are then given by

$$\Pi_u = \int_0^{\frac{p_s}{p_u}} \int_{p_u}^1 (p_u - c_u)qf(q)dadq \quad \text{and} \quad \Pi_s = \int_{\frac{p_s}{p_u}}^{q_m} \int_{\frac{p_s}{q}}^1 (p_s - c)f(q)dadq$$

Carrying out the integration for the profits of the service provider, it can be readily found that $\Pi_u = \frac{1}{2q_m}(p_u - c_u)(1 - p_u)(p_s/p_u)^2$ and that $\partial\Pi_u/\partial p_u = -p_s^2(c_u(p_u - 2) + p_u)/2q_m p_u^3$. The condition $\partial\Pi_u/\partial p_u = 0$ leads to $p_u = 2c_u/(1 + c_u)$. It can be readily seen that $p_u \leqslant 1$ as long as $c_u \leqslant 1$ and there is no need for a break point in prices as we has in the base model. Substituting the solution for the service provider's price to the equation for the service provider's profits we arrive at $\Pi_u = \frac{(1-c_u)^2 p_s^2}{8q_m c_u}$. Carrying out the integration for the product provider's profits we can find that

$$\Pi_s = (p_s - c)\left(1 - \frac{p_s}{q_m p_u} + \frac{p_s}{q_m}\ln\frac{p_s}{q_m p_u}\right) \tag{10}$$

The condition $\partial\Pi_s/\partial p_s = 0$ gives

$$q_m - \frac{2p_s - c}{p_u} + (2p_s - c)\ln(p_s/q_m p_u) + (p_s - c) = 0 \tag{11}$$

Rearranging terms we can write the above equation as

$$\frac{2p_s - c}{q_m p_u}\ln\frac{p_s}{q_m pu} - \left(\frac{2}{pu} - 1\right)\frac{p_s}{q_m pu} = \frac{p_u(c - q_m) - c}{p_u^2 q_m}$$

We can use Lemma 1 with $A = 2$, $B = (2 - p_u)/pu$, $C = -c/(q_m p_u)$, and $a = (p_u(c - q_m) - c)/(p_u^2 q_m)$. The solution for the price of the seller then becomes

$$p_s = p_u q_m \exp\left[\frac{2 - p_u}{2p_u} + \mathcal{W}\left(\frac{c}{2q_m p_u}e^{(p_u - 2)/2p_u}; 0, \frac{2q_m - c}{c}\right)\right] \tag{12}$$

Substituting the expression for p_u in the above formula we arrive at

$$p_s = \frac{2c_u q_m}{1 + c_u}\exp\left[\frac{c_u}{2} + \mathcal{W}\left(\frac{c(1 + c_u)}{4q_m c_u}e^{c_u/2}; 0, \frac{2q_m - c}{c}\right)\right] \tag{13}$$

Solving equation 11 for $\ln\frac{p_s}{q_m p_u}$ and substituting into equation 10, we can express the profits of the seller in terms of p_s as

$$\Pi_s = \frac{(p_s - c)^2(q_m - p_s)}{q_m(2p_s - c)}$$

A.3.3. Comparison Between the Base Model and the Extension with Heterogeneous Valuation Customers The base model and the heterogeneous valuation model cannot be directly compared as they have a different total willingness to pay and a different cost support. The base model can support costs up to aq_m, while the heterogeneous valuation model (with willingness to pay up to a_m) can support costs up to $a_m q_m$. For both models having the same cost support it is required that $a = a_m$. But then the models have a different total willingness to pay. In the base model, where the customers place a constant value of a for the willingness to pay for a single use, the total willingness of the market to pay is

$$\int_0^{q_m} aqf(q)dq = \frac{1}{2}aq_m.$$

In the heterogeneous valuation model, in which customers are willing to pay up to a_m per use, the total willingness of the market to pay is

$$\int_0^{a_m}\int_0^{q_m} aqf(q)g(a)dadq = \frac{1}{4}a_m q_m.$$

This means that the two models (uniform a versus heterogeneous willingness to pay) can have the same total willingness to pay but they support different cost ranges, or they can have the same cost support but with different total willingness to pay.

A.4. Endogenous q model

In this section we examine a model the usage q is endogenous. Customers observe the price under the given business model and decide on usage by maximizing their utility. Customers are heterogeneous in two parameters: the type a which is roughly equivalent to the willingness to pay for a unit of usage and in usage q. Customers with large a derive more utility from the same usage than customers with small a. We assume that both a and q are uniformly distributed in $[0,1]$. The net utility a customer of type a derives from q units of usage under the selling model is $U_s(a,q) = qa - \frac{1}{2}q^2 - p_s$ and $U_u(a,q) = qa - \frac{1}{2}q^2 - qp_u$ under the PPU model. Note that $U(a,q)$ is increasing and concave in q.

A.4.1. Monopoly for the selling or the PPU model Given a price p_s, a customer of type a will choose $q_{s,m}$ that will maximize his utility. The first order condition $\partial U_s/\partial q = 0$ gives $q_{s,m} = a$ and the utility at the maximum usage is $U_{s,m}(a) = \frac{1}{2}a^2 - p_s$. Customers with type $a \geqslant \sqrt{2p_s}$ have a positive utility and participate in the market and customers with $a < \sqrt{2p_s}$ will abstain. The number of customers that will buy is $1 - \sqrt{2p_s}$ and the profit the seller will realize will be $\Pi_s(p_s) = (p-c)(1-\sqrt{2p_s})$. The first order condition $\partial \Pi_s/\partial p_s = 0$ gives $p_s^* = (1 + 3c + \sqrt{6c+1})/9$ and maximum profit $\Pi_s^* = (1 - 18c + \sqrt{(6c+1)^3})/27$.

Given a price per use p_u, a customer of type q will choose $q_{u,m}$ that will maximize his utility. The first order condition $\partial U_u/\partial q = 0$ gives $q_{u,m} = a - p_u$ and the utility at the maximum usage is $U_{u,m}(a) = \frac{1}{2}(a-p_u)^2$. As the usage has to be positive, customers with type $a \geqslant p_u$ participate in the market, while customers with $a < p_u$ abstain. The number of uses M_u required by the customers is $M_u = \int_{p_u}^{1}(a-p_u)\,da = \frac{1}{2}(1-p_u)^2$. The number of customers that participate in the market are $N_u = 1 - p_u$. The profits for the monopolist are $\Pi_u(p_u) = M_u(p_u - \delta) - \lambda N_u c = \frac{1}{2}(p_u - \delta)(1-p_u)^2 - \lambda c(1-p_u)$ where $\lambda < 1$ is the pooling factor that determines the level of service of the PPU provider. The first order condition $\partial \Pi_u/\partial p_u = 0$ gives $p_u^* = (2 + \delta + \sqrt{1 + \delta^2 - 3\lambda c - 2\delta})/3$. For a level of service equivalent to the one in the base model (strong pooling), we assume that the maximum uses the product can deliver is one and the cost per use is $c_u = c + \delta$. Therefore the profit for the PPU provider becomes $\Pi_u(p_u) = \frac{1}{2}(p_u - c_u)(1-p_u)^2$. The first order condition $\partial \Pi_u/\partial p_u = 0$ gives the optimal price at $p_u^* = \frac{1}{3}(1+2c_u)$ and substituting into the profit equation, the maximum profit is at $\Pi_u^* = \frac{2}{27}(1-c_u)^3$.

A.4.2. Competition between selling and PPU model Given prices p_s and p_u customers find the quantities $q_{s,m}$ and $q_{u,m}$ that maximize their utility. These optimum quantities are as in monopoly $q_{s,m} = a$ and $q_{u,m} = a - p_u$. Customers' utility at the optimum quantities are $U_{s,m} = \frac{a^2}{2} - p_s$ and $U_{u,m} = \frac{1}{2}(a-p_u)^2$. Customers choose among three options: not participating (remain inactive), buying uses from the PPU provider or buying the product from the seller. The indifferent customer

a_1 between remain inactive and buying is at $U_{u,m}(a_1)=0$ which gives $a_1=p_u$. The indifferent customer a_2 between buying uses and buying product is at $U_{u,m}(a_2)=U_{s,m}(a_2)$, which gives $a_2=\frac{1}{2}p_u+\frac{p_s}{p_u}$. So the number of customers in the inactive, PPU, selling segments are respectively $p_u, \frac{p_s}{p_u}-\frac{1}{2}p_u, 1-\frac{p_s}{p_u}-\frac{1}{2}p_u$. The number of uses required by customers that adopt PPU model are $M_u=\int_{a_1}^{a_2}(a-p_u)\,da=\frac{1}{2}(\frac{p_s}{p_u}-\frac{p_u}{2})^2$. The profits for the seller and the PPU provider (under pooling factor λ) are

$$\Pi_s(p_s,p_u)=(p_s-c)\left(1-\frac{p_s}{p_u}-\frac{p_u}{2}\right) \quad \text{and} \quad \Pi_u(p_s,p_u)=\tfrac{1}{2}(p_u-\delta)\left(\frac{p_s}{p_u}-\frac{p_u}{2}\right)^2-\lambda c\left(\frac{p_s}{p_u}-\frac{p_u}{2}\right)$$

The first order conditions $\partial\Pi_s/\partial p_s=0$ and $\partial\Pi_u/\partial p_u=0$ give

$$p_u-\tfrac{1}{2}p_u^2+c-2p_s=0$$

$$\frac{1}{8p_u^3}\left(8\lambda c p_s p_u+4\lambda c p_u^3+8\delta p_s^2-2\delta p_u^4-4p_s^2p_u-4p_sp_u^3+3p_u^5\right)=0$$

The last two equations are a system of nonlinear equations for the prices of p_s and p_u that has to be treated numerically for the optimum prices p_s^* and p_u^*. Under strong pooling the profits of the PPU provider are

$$\Pi_u(p_s,p_u)=\tfrac{1}{2}(p_u-c_u)\left(\frac{p_s}{p_u}-\frac{p_u}{2}\right)^2$$

where $c_u=c+\delta$. The first order conditions $\partial\Pi_s/\partial p_s=0$ and $\partial\Pi_u/\partial p_u=0$ gives the following system of equations for the optimum prices

$$p_u-\tfrac{1}{2}p_u^2+c-2p_s=0$$

$$3p_u^3+2p_sp_u-2c_u(2p_s+p_u^2)=0$$

Solving the first equation for p_s and substituting into the second equation we arrive at a third degree equation for the price of the PPU provider $\frac{5}{2}p_u^3+(1-c_u)p_u^2-(2c_u-c)p_u-2cc_u=0$. The cubic discriminant is $\Delta=\frac{q^2}{4}+\frac{p^3}{27}$, where $p=\beta-\frac{1}{3}\alpha^2$ and $q=\frac{2\alpha^3}{27}-\frac{\alpha\beta}{3}+\gamma$ and where $\alpha=\frac{2}{5}(1-c_u)$, $\beta=-\frac{2}{5}(2c_u-c)$, and $\gamma=-\frac{4}{5}cc_u$. The cubic discriminant Δ depends on c and c_u. It is straightforward to show numerically that there are regions in (c,c_u) space in which $\Delta>0$ and therefore there is only one real solution $(0<p_u<1)$, and regions where $\Delta\leqslant 0$ and the cubic equation has three real solutions (one positive $0<p_u<1$ and two negatives). Therefore, for all values of c and c_u there is one solution $0<p_u<1$, which implies also that $0<p_s<1$ because p_s has to satisfy $p_u-\frac{1}{2}p_u^2+c-2p_s=0$.

A.5. Uncertainty in the usage profile (Proof of proposition 4)

We assume that the maximum use frequency q_m is a random variable $\tilde{q}_m$ that takes values q_m-v with probability $\frac{1}{2}$ and q_m+v with probability $\frac{1}{2}$ where $v\geqslant 0$ is a measure of the uncertainty in the value of the maximum use frequency.

A.5.1. Monopoly In the monopoly setting the seller will set a price p_s and the profits she will expect will become

$$\begin{aligned}\Pi_s &= \frac{(p_s-c)(q_m+v-p_s/a)}{2(q_m+v)} + \frac{(p_s-c)(q_m-v-p_s/a)}{2(q_m-v)}\\ &= (p_s-c)\left(1-\frac{q_m p_s}{a(q_m^2-v^2)}\right)\end{aligned}$$

The price p_s that maximizes the seller's profits satisfies the equation $\partial\Pi_s/\partial p_s = 0$. Comparing the seller's profits with the ones of the base model (A.1.1), we find the the seller behaves as if he is faced with a smaller maximum frequency $q'_m = (q_m^2 - v^2)/q_m \leqslant q_m$. Therefore, the optimum price and the expected profits under uncertainty will be smaller than the ones of the base model for the monopolist seller. It can readily shown that the optimum price p_s^* and the maximum profits Π_s^* for the seller in a monopoly situation are

$$p_s^* = \frac{1}{2}(c + aq'_m) \quad , \quad \Pi_s^* = \frac{1}{4aq'_m}(aq'_m - c)^2 \quad \text{where} \quad q'_m = (q_m^2 - v^2)/q_m$$

If the monopolist chooses to sell uses of the product and offer a pay-per-use service at a price p_u, he will face per use cost $c/(q_m+v)+\delta$ with probability $\frac{1}{2}$ and $c/(q_m-v)+\delta$ again with probability $\frac{1}{2}$. She will set the price p_u that will maximize the expected profits

$$\begin{aligned}\Pi_u &= \frac{1}{4}(p_u-\delta-c/(q_m+v))(q_m+v) + \frac{1}{4}(p_u-\delta-c/(q_m-v))(q_m-v)\\ &= \frac{1}{2}(q_m(p_u-\delta)-c)\end{aligned}$$

Note that for the service provider, the effect of the uncertainty cancels out and the expression for the profits is the same as in the base model in A.1.1. Therefore, the optimum price and the profits for the service provider are exactly the same as in the base case. The seller on the other hand, faced with uncertainty, she will choose a lower price than the price chosen in the base model. She will price as if the maximum use frequency is $q'_m = (q_m^2 - v^2)/q_m$ and of course, $q'_m \leqslant q_m$. This choice brings more market share than the base model but less profits. Therefore, in monopoly the effect of uncertainty in the maximum use frequency q_m make the service provider more attractive than the seller.

A.5.2. Duopoly In duopoly, the indifferent customer is not affected by uncertainty in the maximum use frequency q_m and is, just like the base model, at $\bar{q} = p_s/p_u$. The expected profits as functions of the prices p_s and p_u for the seller and the service provider are respectively

$$\begin{aligned}\Pi_s(p_s,p_u) &= \tfrac{1}{2}(p_s-c)\left[\frac{1}{q_m-v}\left(q_m-v-\frac{p_s}{p_u}\right) + \frac{1}{q_m+v}\left(q_m+v-\frac{p_s}{p_u}\right)\right]\\ \Pi_s(p_s,p_u) &= (p_s-c)\left[1-\frac{q_m p_s}{p_u(q_m^2-v^2)}\right] \qquad (14)\end{aligned}$$

$$\Pi_u(p_s,p_u)=\tfrac{1}{2}\frac{p_s^2}{p_u^2}\left[\frac{1}{2(q_m-v)}\left(p_u-\delta-\frac{c}{q_m-v}\right)+\frac{1}{2(q_m+v)}\left(p_u-\delta-\frac{c}{q_m+v}\right)\right]$$

$$\Pi_u(p_s,p_u)=\tfrac{1}{2}\frac{p_s^2}{p_u^2}\left[(p_u-\delta)\frac{q_m}{q_m^2-v^2}-\frac{c(q_m^2+v^2)}{(q_m+v)^2(q_m-v)^2}\right] \tag{15}$$

Equation 14 can be written in the form $(p_s-c)(1-p_s/(q_m'p_u))$ with $q_m'=q_m-v^2/q_m$. In other words, the seller behaves as if the maximum use frequency is $q_m-v^2/q_m<q_m$. For example if $v^2=0.1q_m$ then the seller behaves as if the maximum frequency is $0.9q_m$. Similarly, equation 15 can be put in the form $\frac{1}{2q_m'}(p_u-\delta)p_s^2/p_u^2-\frac{c'}{2q_m'^2}p_s^2/p_u^2$, where $q_m'=q_m-v^2/q_m$ and $c'=c(1+v^2/q_m^2)$. In other words, the service provider behaves as if she is facing a smaller maximum use frequency and a larger cost. Following the previous example, if there is a 10% uncertainty in the maximum use frequency (i.e, $v^2=0.1q_m$), the service provider will behave as if the maximum use frequency is $0.9q_m$ and the cost is $1.1c$.

The equilibrium (p_s^*,p_u^*) satisfies the equations $\partial\Pi_s/\partial p_s=0$ and $\partial\Pi_u/\partial p_u=0$. The equation $\partial\Pi_s/\partial p_s=0$ gives the seller's best response

$$p_s=\frac{1}{2}\left(c+\frac{p_u}{q_m}(q_m^2-v^2)\right)$$

The service provider is constrained, as in the base case, by $p_u\leqslant a$. The solution to the maximization problem for the service provider gives

$$p_u^*=\begin{cases}2\delta+\frac{2c(q_m^2+v^2)}{q_m(q_m^2-v^2)} & \text{for}\quad c\leqslant\frac{1}{2}q_m(a-2\delta)(q_m^2-v^2)/(q_m^2+v^2)\\ a & \text{for}\quad c>\frac{1}{2}q_m(a-2\delta)(q_m^2-v^2)/(q_m^2+v^2)\end{cases} \tag{16}$$

Finally, substituting the optimum price of the service provider to the seller's best response we find the optimum price for the seller

$$p_s^*=\begin{cases}\frac{1}{2q_m^2}(3cq_m^2+2cv^2+2\delta q_m(q_m^2-v^2)) & \text{for}\quad c\leqslant\frac{1}{2}q_m(a-2\delta)(q_m^2-v^2)/(q_m^2+v^2)\\ \frac{1}{2q_m}(cq_m+a(q_m^2-v^2)) & \text{for}\quad c>\frac{1}{2}q_m(a-2\delta)(q_m^2-v^2)/(q_m^2+v^2)\end{cases} \tag{17}$$

The optimum profits can be computed by substituting the optimum prices from equations 16 and 17 to the expressions for the profits 14 and 15.

A.6. Overcapacity and queuing effects

We assume that customer's utility under the selling model is $U_s=qa-p_s$ where a is the willingness to pay for a single use and q is the desired usage. We also assume that customer's utility under the PPU model is $U_u=q(A(\rho)-p_u)$ where $A(\rho)=ae^{-k\rho/(1-\rho)}$ for $\rho\in[0,1]$. The parameter ρ is a proxy for the service level and is a decision variable for the PPU model. $A(\rho)$ is the willingness to pay for a single use when the service level is ρ. When $\rho\to0$, $A(\rho)\to a$ and when $\rho\to1$, $A(\rho)\to0$. Clearly $A(\rho)\leqslant a$. On the cost side we assume also that the cost per use depends on the service level and is given by $c_u(\rho)=c/(\rho q_m)+\delta$. Note that because $0\leqslant\rho\leqslant1$ we have that $c_u(\rho)\geqslant c/q_m+\delta$ i.e., the cost per use is greater than the strong pooling cost. The PPU provider chooses first the service level ρ and subsequently the price p_u.

The indifferent customer Q is at

$$Q = Q(p_s, p_u; \rho) = \frac{p_s}{a - A(\rho) + p_u}$$

and the profits for the seller Π_s and the PPU provider Π_u are respectively

$$\begin{aligned}
\Pi_s &= \Pi_s(p_s, p_u; \rho) = \frac{1}{q_m}(p_c - c)(q_m - Q) \\
\Pi_u &= \Pi_u(p_s, p_u; \rho) = \frac{1}{2q_m} Q^2 (p_u - c_u(\rho))
\end{aligned}$$

The seller's and the PPU provider's best response functions satisfy $\partial \Pi_s / \partial p_s = 0$ and $\partial \Pi_u / \partial p_u = 0$ which gives respectively

$$\begin{aligned}
p_s &= \tfrac{1}{2}\left(c + q_m(a - A(\rho) + p_u)\right) \\
p_u &= a - A(\rho) + 2c_u(\rho)
\end{aligned}$$

The PPU provider is constrained to have $p_u \leqslant A(\rho)$ because if he chooses a price more than customer's willingness to pay he will get no customers and no profits. Therefore there are two cases to consider: case A where $p_u = A(\rho)$ which applies when $c_u(\rho) > A(\rho) - a/2$ and case B where $p_u = a - A(\rho) + 2c_u(\rho)$ which applies when $c_u(\rho) \leqslant A(\rho) - a/2$.

Case A: When $p_u = A(\rho)$ then $p_s = \frac{1}{2}(c + aq_m)$ and the indifferent customer is at $Q = \frac{1}{2a}(c + aq_m)$. Note that seller's price and the indifferent customer do not depend on p_u or ρ; the seller behaves as in the monopoly case. The profits of the PPU provider are then given by

$$\Pi_u = \frac{1}{8q_m a^2}(c + aq_m)^2 \left(A(\rho) - c_u(\rho)\right)$$

The condition $\partial \Pi_u / \partial \rho = 0$ implies that $\partial A / \partial \rho - \partial c_u / \partial \rho = 0$, which in turns gives

$$\frac{k^2 \rho^2}{(1-\rho)^2} e^{-k\rho/(1-\rho)} = \frac{kc}{aq_m}$$

Set $z = -k\rho/(1-\rho)$. Then the above equation becomes $z^2 e^z = c/(aq_m)$ which has three possible solutions:

$$z_1 = 2\mathbb{W}_0\left(\tfrac{1}{2}\sqrt{kc/(aq_m)}\right) \text{ or } z_2 = 2\mathbb{W}_0\left(-\tfrac{1}{2}\sqrt{kc/(aq_m)}\right) \text{ or } z_3 = 2\mathbb{W}_{-1}\left(-\tfrac{1}{2}\sqrt{kc/(aq_m)}\right)$$

where $\mathbb{W}_j$ is jth branch of the Lambert function. Note that for the solutions to be a real number we must have $-1/e \leqslant -\frac{1}{2}\sqrt{kc/(aq_m)} \leqslant 0$ which in turn limits the parameter kc to $0 \leqslant kc \leqslant 4aq_m/e^2 \approx 0.541 aq_m$.

Case B: When $p_u = a - A(\rho) + 2c_u(\rho)$ then set $B(\rho) = a - A(\rho) + c_u(\rho)$. The seller's price is $p_s = \frac{1}{2}(c + 2q_m B(\rho))$ and the indifferent customer is at $Q = \frac{1}{4}(c + 2q_m B(\rho))/B(\rho)$. The profits of the PPU provider are then

$$\Pi_u = \frac{1}{32q_m}\frac{(c + 2q_m B(\rho))^2}{B(\rho)}$$

The derivative of PPU profits with respect to ρ is then

$$\frac{\partial \Pi_u}{\partial \rho} = \frac{1}{32q_m}\frac{B'(\rho)(2q_m B(\rho) + c)(2q_m B(\rho) - c)}{B^2(\rho)}$$

Where $B'(\rho) = \partial B/\partial \rho$. Note that $B(\rho) > 0$ because $a > A(\rho)$ and $c_u(\rho) > 0$. Therefore, $2q_m B(\rho) + c > 0$. Similarly, $2q_m B(\rho) - c = 2q_m(a - A(\rho)) + \frac{2c}{\rho} - c > 0$ because $\rho \in [0, 1]$. Hence for $\partial \Pi_u/\partial \rho = 0$, we must have $B'(\rho) = 0$ which gives us the same equation for ρ as in case A above.

A.7. Service and price discrimination

A.7.1. Service Discrimination The PPU provider can decide to service the patient customers only, leaving the impatient customers to the seller (Case A) or service the impatient customers only leaving the patient segment to the seller.

Case A When customers are patient the PPU provider can schedule service in a way that demand in non overlapping. In this case the cost per use is given by the strong pooling approximation as $c_u = c/q_m + \delta$. The PPU provider and the seller have to find prices p_u and p_s to maximize their profits. The profits of the seller and the PPU provider are given by

$$\Pi_u = \tfrac{1}{2q_m} w(p_u - c_u)(p_s/p_u)^2$$
$$\Pi_s = \tfrac{1}{q_m}(w(p_s - c)(q_m - p_s/p_u) + (1 - w)(p_s - c)(q_m - p_s/a))$$

The condition $\partial \Pi_u/\partial p_u = 0$ gives $p_u = 2c_u$ for $c_u \leqslant a/2$ and $p_u = a$ for $c_u > a$. The condition $\partial \Pi_s/\partial p_s = 0$ gives

$$p_s = \frac{1}{2}\left(c + \frac{a q_m p_u}{wa + (1 - w)p_u}\right)$$

Case B The PPU provider has to find a price p_u and a service level ρ and the seller has to find a price p_s that will maximize their profits

$$\Pi_u = \tfrac{1}{2q_m}(1 - w)(p_u - c_u(\rho))\frac{p_s^2}{(a - A(\rho) + p_u)^2}$$
$$\Pi_s = \tfrac{1}{q_m}\left(w(p_s - c)(q_m - p_s/a) + (1 - w)(p_s - c)\left(q_m - \frac{p_s}{a - A(\rho) + p_u}\right)\right)$$

The problem for the PPU provider solves as in appendix (A.6). The optimum service level is $\rho = z/(z - k)$ where $z = 2\mathbb{W}_0\left(-\frac{1}{2}\sqrt{kc/(aq_m)}\right)$. The optimum price is $p_u = a - A(\rho) + 2c_u(\rho)$ for $c_u(\rho) \leqslant A(\rho) - a/2$ and $p_u = A(\rho)$ for $c_u(\rho) > A(\rho) - a/2$ where $A(\rho) = ae^{k\rho/(1-\rho)}$ and $c_u(\rho) = c/(\rho q_m) + \delta$. The condition $\partial \Pi_s/\partial p_s = 0$ gives

$$p_s = \frac{1}{2}\left(c + \frac{a q_m (a - A(\rho) + p_u)}{a - wA(\rho) + wp_u}\right)$$

A.7.2. Price Discrimination Assume that the PPU provider knows in which segment each customer belongs and decides to offer two prices p_{u1} for the patient customers and p_{u2} for the impatient ones. Note that for patient customers the PPU provider can schedule their demand and hence the cost per use is given by the strong pooling approximation $c_u = c/q_m + \delta$. For the impatient segment, the PPU provider has to build overcapacity $c_u(\rho)$ and decide on the service level ρ. The profits for the PPU provider and the seller are

$$\Pi_u = \tfrac{1}{2q_m}\left(w(p_{u1} - c_u)\frac{p_s^2}{p_{u1}^2} + (1-w)(p_{u2} - c_u(\rho))\frac{p_s^2}{(a - A(\rho) + p_{u2})^2}\right)$$
$$\Pi_s = \tfrac{1}{q_m}\left(w(p_s - c)(q_m - p_s/p_{u1}) + (1-w)(p_s - c)\left(q_m - \frac{p_s}{a - A(\rho) + p_{u2}}\right)\right)$$

It is straight forward to show that $p_{u1} = 2c_u$ for $c_u \leqslant a/2$ and $p_{u1} = a$ for $c_u > a/2$. The optimum ρ is $\rho = z/(z-k)$ where $z = 2\mathbb{W}_0\left(-\tfrac{1}{2}\sqrt{kc/(aq_m)}\right)$. The optimum price for the impatient segment is $p_{u2} = a - A(\rho) + 2c_u(\rho)$ for $c_u(\rho) \leqslant A(\rho) - a/2$ and $p_{u2} = A(\rho)$ for $c_u(\rho) > A(\rho) - a/2$, where $A(\rho) = ae^{k\rho/(1-\rho)}$. Finally, the condition $\partial\Pi_s/\partial p_s = 0$ gives seller's price

$$p_s = \frac{1}{2}\left(c + \frac{q_m}{\frac{w}{p_{u1}} + \frac{1-w}{a - A(\rho) + p_{u2}}}\right)$$

A.8. Proof of Lemma 1: The Generalized Lambert Function

Equations in the form $xe^x = \alpha$ are solved in terms of the Lambert W function, and the solution is given as $x = W(\alpha)$. For more details on the properties of the Lambert W function see Corless et al. (1996). Equations of the form

$$(x - t)e^x = \beta(x - s) \tag{18}$$

are solved through the *generalized Lambert W function.* The solution of $(x - t)e^x = \beta(x - s)$ is $x = \mathcal{W}(\beta; t, s)$. For details and properties of the generalized Lambert W function see Siewert and Burniston (1974), Mező (2017), and Wright (1961). For solving the equation $(Ax + C)\ln x - Bx = a$ we let $y = \ln x$, or equivalently $x = e^y$. In terms of y the previous equation becomes $(Ay - B)e^y + Cy = a$. We let $z = Ay - B$, or equivalently, $y = (z + B)/A$. In terms of z the previous equation then becomes $ze^{z/A}e^{B/A} + \frac{C}{A}(z + B) = a$. Rearranging terms and multiplying both sides by $1/A$ we arrive at

$$\frac{z}{A}e^{z/A} + \frac{Ce^{-B/A}}{A}\frac{z}{A} = \frac{aA - BC}{A^2}e^{-B/A}$$

It can be readily shown that the above equation for z/A is of the form of equation 18 with $t = 0$, $\beta = -Ce^{-B/A}/A$, and $s = (aA - BC)/(AC)$. The solution therefore for z is

$$z = A\mathcal{W}\left(-\frac{C}{A}e^{-B/A}; 0; \frac{aA - BC}{AC}\right)$$

Carrying out the inverse transformations, the solution for x is then

$$x = \exp\left[\frac{B}{A} + \mathcal{W}\left(-\frac{C}{A}e^{-B/A}; 0, \frac{aA - BC}{AC}\right)\right] \tag{19}$$