Online Appendix

Optimizing Offline Product Design and Online Assortment Policy: Measuring the Relative Impact of Each Decision

Mengxin Wang, Heng Zhang, Paat Rusmevichientong, Max Shen

Appendix A: Differences Between Our Model and Existing Models on Reusable Products

Our problem formulation involving product returns differs from the existing models on reusable products because in the reusable product setting, each selected product generates revenue proportional to its usage time. However, in our setting, the selected product generates **zero** revenue during the trial period, no matter how long it is being tried by a customer. We only receive the revenue if the customer decides to purchase it at the end of the trial duration.

Because in our setting outstanding units that are being tried by customers do not generate any revenue, the system dynamics differ from the traditional reusable product setting. For example, Gong et al. (2021) show that a myopic policy is $\frac{1}{2}$-competitive for the reusable product setting under adversarial arrivals, if the usage time distribution is independent of customers. However, no online deterministic algorithm can achieve any meaningful competitive ratio result for our problem even if the trial duration is deterministic, as shown in the following example.

Suppose the retailer has one product and one unit of inventory. The unit profit of this product is \$1. Every customer tries the product for exactly two periods before deciding whether or not to buy. There are two types of customers: Type 1 and Type 2. We assume that after the trial duration of exactly two periods, Type-1 customer will choose to buy the product with probability ϵ, and Type-2 customer will choose to buy the product with probability 1.

Consider a time horizon T such that $T \gg 2$ and fix $i \in \{1, 2, \dots, T\}$. Consider two arrival sequences with T customers:

- Sequence A: The i^{th} arrival is a Type-2 customer, and the rest are Type-1 customers.
- Sequence B: The $(i+1)^{\text{th}}$ arrival is a Type-2 customer, and the rest are Type-1 customers.

In order to be competitive at sequence A, the retailer should at least offer the product to the i^{th} customer. However, such a policy under sequence B causes the competitive ratio to be at most $\frac{\epsilon T}{1} = \epsilon T$, which can be arbitrarily small. This happens because, under sequence B, the Type-1 customer who is the i^{th} arrival holds on to the product for two periods and that unit is thus not available to be shown to the Type-2 customer who arrives in the $(i+1)^{\text{th}}$ period.

Appendix B: Supporting Arguments for Section 3

B.1. Proof of Theorem 3.1

Proof: Fix an arbitrary $\boldsymbol{x} \in \mathcal{X}$ at the beginning of the selling season. Under any policy $\boldsymbol{\pi}$, for each $t \in \mathcal{T}$ and $A \in \mathcal{F}$, let $Z^t(S) = 1$ if we offer assortment S at period t; otherwise, we have $Z^t(S) = 0$.

We let $\Phi_i^t = 1$ if the customer arriving at time period t selects product i from the assortment offered in period t; otherwise $\Phi_i^t = 0$. Note that $Z^t(A)$ and Φ_i^t are random variables. Also, once a customer selects product i, its expected revenue is $r_i(\boldsymbol{x})$.

Let $\bar{z}^t(S) = \mathbb{E}[Z^t(S)] = \Pr\{Z^t(S) = 1\}$. To complete the proof, we will show that $(\bar{z}^t(S) : S \in \mathcal{F})$ is a feasible solution to $\mathsf{UB}(\boldsymbol{x})$ and its objective value associated is equal to $\mathbb{E}\left[\sum_{t\in\mathcal{T}} R_t^{\boldsymbol{\pi}}(\boldsymbol{x})\right]$. By definition,

$$\mathbb{E}\left[\sum_{t\in\mathcal{T}} R_t^{\boldsymbol{\pi}}(\boldsymbol{x})\right] = \mathbb{E}\left[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} r_i(\boldsymbol{x})\Phi_i^t\right] = \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} r_i(\boldsymbol{x}) \sum_{S\in\mathcal{F}} \bar{z}^t(A)\phi_i^t(S\,|\,\boldsymbol{x}) \ ,$$

where second equality follows because $\mathbb{E}[\Phi_i^t] = \sum_{S\in\mathcal{F}} \Pr\{Z^t(S) = 1\} \Pr\{\Phi_i^t = 1 \,|\, Z^t(S) = 1\} = \sum_{S\in\mathcal{F}} \bar{z}^t(S)\phi_i^t(S\,|\,\boldsymbol{x})$ and conditional on $\Phi_i^t = 1$, the expected revenue from item i is

$$\sum_{d\in\mathcal{D}_i} \left((1-\gamma_{id})p_{id} + \gamma_{id}(p_{id} - c_{id})\right) x_{id} = \sum_{d\in\mathcal{D}_i} r_{id}x_{id} = r_i(\boldsymbol{x}).$$

We will now establish the feasibility of $(\bar{z}^t(A) : A \in \mathcal{F})$. Clearly, $\sum_{S\in\mathcal{F}} \bar{z}^t(S) \leq 1$ because at $\sum_{S\in\mathcal{F}} Z^t(S) = 1$ by definition. For each $\tau \leq t$ and $i \in \mathcal{N}$, let $\mathsf{Hold}_i^\tau(t) = 1$ if product i that was selected in period τ by a customer still remains with the customer in period t; otherwise, $\mathsf{Hold}_i^\tau(t) = 0$. Note that, by definition in Eq. (2.1), we have that $\mathbb{E}[\mathsf{Hold}_i^\tau(t)\,|\,\Phi_i^\tau = 1] = \alpha_i(t-\tau\,|\,\boldsymbol{x})$. Consider product i and period t. The, number of units of product i that remains with a customer in period t is given $\sum_{\tau=1}^t \Phi_i^t \mathsf{Hold}_i^\tau(t)$. Because of the inventory constraint, we have that $C_i(\boldsymbol{x}) \geq \sum_{\tau=1}^t \Phi_i^t \mathsf{Hold}_i^\tau(t)$ almost surely. Taking expectation on both sides, we obtain

$$C_i(\boldsymbol{x}) \geq \sum_{\tau=1}^t \mathbb{E}[\Phi_i^\tau \mathsf{Hold}_i^\tau(t)] = \sum_{\tau=1}^t \mathbb{E}[\mathsf{Hold}_i^\tau(t)\,|\,\Phi_i^\tau = 1]\Pr\{\Phi_i^\tau = 1\} = \sum_{S\in\mathcal{F}}\sum_{\tau=1}^t \bar{z}^t(S)\phi_i^\tau(S\,|\,\boldsymbol{x})\alpha_i(t-\tau\,|\,\boldsymbol{x}) \ ,$$

which completes the proof. ■

B.2. Consistency of Random Utility Maximization (RUM) Choice Models

In fact, any RUM-based model satisfies the consistency requirement. Consequently, well-known choice models under the RUM framework, such as mixture of logit and nested logit, are all consistent. To show this, let us suppose the utility of design $d \in \mathcal{D}_i$ of each product $i \in \mathcal{N}$ can be expressed as $u_{id} = v_{id} + \epsilon_{id}$, where v_{id} is the mean utility and ϵ_{id} is a random noise. Let the utility of product i under design $\boldsymbol{x}$ be $u_i(\boldsymbol{x}) = v_i(\boldsymbol{x}) + \epsilon_i(\boldsymbol{x})$, where $v_i(\boldsymbol{x}) = \sum_{d\in\mathcal{D}_i} v_{id}x_{id}$ and $\epsilon_i(\boldsymbol{x}) = \sum_{d\in\mathcal{D}_i} \epsilon_{id}x_{id}$. For a RUM-based choice function ϕ, we have

$$\phi_i(S\,|\,\boldsymbol{x}) = \mathbb{P}\left\{v_i(\boldsymbol{x}) + \epsilon_i(\boldsymbol{x}) > v_j(\boldsymbol{x}) + \epsilon_j(\boldsymbol{x}), \quad \forall j \neq i,\, j \in S\right\}.$$

To show that ϕ is consistent, consider a RUM model ψ on the ground set $\mathcal{M}$ where the utility for $i^d \in \mathcal{M}$ is $u_{i,d} = v_{id} + \epsilon_{id}$. For any $S \subseteq \mathcal{N}$, $i \in \mathcal{S}$, and $d \in \mathcal{D}_i$ such that $x_{id} = 1$,

$$\begin{aligned}
\phi_i(S\,|\,\boldsymbol{x}) &= \mathbb{P}\left\{v_i(\boldsymbol{x}) + \epsilon_i(\boldsymbol{x}) > v_j(\boldsymbol{x}) + \epsilon_j(\boldsymbol{x}),\ \forall j \neq i,\, j \in S\right\} \\
&\overset{(a)}{=} \mathbb{P}\left\{v_{id} + \epsilon_{id} > \epsilon_{jh} + v_{jh},\ \forall j \neq i\,,\, j \in S\,,\, h \in \mathcal{D}_j\,,\, x_{jh} = 1\right\} \\
&\overset{(b)}{=} \mathbb{P}\left\{u_{i,d} > u_{j,h},\ \forall j^h \neq i^d\,,\, j^h \in \boldsymbol{A}(S,\boldsymbol{x})\right\} = \psi_{i^d}(\boldsymbol{A}(S,\boldsymbol{x})),
\end{aligned}$$

where (a) follows from the definition of $v_i(\boldsymbol{x})$ and $\epsilon_i(\boldsymbol{x})$ for all $i \in S$ and (b) follows from the definition of $u_{i,d}$ and the definition of $\boldsymbol{A}(S,\boldsymbol{x})$. Therefore, $\phi(\cdot)$ is consistent.

B.3. Specifying the Structure of Feasible Assortments on the Enlarged Ground Set

In this section, we discussion how to specify $\mathcal{G}$ under different $\mathcal{F}$ such that (i) the three desired properties of $\mathcal{G}$ are satisfied, and (ii) the static assortment optimization problem (F.3) can be solved. Since the choice of $\mathcal{G}$ depends on $\mathcal{F}$, which varies from applications to applications, our discussion cannot be exhaustive, but we cover a range of applications that are of practical interests.

Example B.1 (Unconstrained Assortment Problem) Suppose that the assortment optimization is unconstrained so $\mathcal{F}$ contains all possible subsets of $\mathcal{N}$. In this case, we may let $\mathcal{G}$ be all possible subsets of $\mathcal{M}$; the first and second properties are automatically met. Let us fix $\boldsymbol{x} \in \mathcal{X}$. Given any $\boldsymbol{A} \in \mathcal{G} \setminus \mathcal{G}(\boldsymbol{x})$, let $S = \{i \in \mathcal{N} : \exists\, d \in D_i \text{ such that } x_{id} = 1 \text{ and } i^d \in \boldsymbol{A}\}$. Then

$$\boldsymbol{A}' = \left\{i^d \in \boldsymbol{A} : x_{id} = 1\right\} = \boldsymbol{A}(S,\boldsymbol{x}) \in \mathcal{G}(\boldsymbol{x}),$$

so the third property is also satisfied. We observe that in the case, the static assortment optimization problem (F.3) is unconstrained. There exist polynomial-time algorithms to compute the exact solution for a wide range of choice models, such as the MNL model, the nested logit model and the MCCM (Davis et al. 2014, Feldman and Topaloglu 2017).

Example B.2 (Capacity Constrained Problem) The capacity constraint on the set of feasible assortments is also common in the literature. We may assume that for each $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$, there is an associated weight $w_{id} \geq 0$. We also define $w_i(\boldsymbol{x}) = \sum_{d \in \mathcal{D}_i} w_{id} x_{id}$. In this case, $\mathcal{F} = \left\{S \subset \mathcal{N} : \sum_{i \in S} w_i(\boldsymbol{x}) \leq \widehat{C}\right\}$, where $\widehat{C} > 0$.[2] To constrain $\mathcal{G}$, we may set $\tilde{w}_{id} = w_{id}$ for each $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$ and let $\mathcal{G} = \left\{\boldsymbol{A} \subset \mathcal{M} : \sum_{i^d \in \boldsymbol{A}} \tilde{w}_{id} \leq \widehat{C}\right\}$. Then because $w_{id} \geq 0$, the downward inclusiveness is satisfied. Given $\boldsymbol{x}$, $S \in \mathcal{F}$ implies

$$\sum_{i \in S} w_i(\boldsymbol{x}) \;=\; \sum_{i \in S} \sum_{d \in \mathcal{D}_i} w_{id} x_{id} \;\overset{(a)}{=}\; \sum_{i^d \in \boldsymbol{A}(S,\boldsymbol{x})} w_{id} \leq \hat{C},$$

where (a) follows from the definition of $\boldsymbol{A}(S,\boldsymbol{x})$. To verify the third property, we again define $S = \{i \in \mathcal{N} : \exists\, d \in \mathcal{D}_i \text{ such that } x_{id} = 1 \text{ and } i^d \in \boldsymbol{A}\}$ for a fixed $\boldsymbol{x}$ and $\boldsymbol{A} \in \mathcal{G} \setminus \mathcal{G}(\boldsymbol{x})$. Note that

$$\sum_{i \in S} w_i(\boldsymbol{x}) \;=\; \sum_{i^d \in \boldsymbol{A} : x_{id} = 1} w_{id} \;\leq\; \sum_{i^d \in \boldsymbol{A}} w_{id} = \sum_{i^d \in \boldsymbol{A}} \tilde{w}_{id} \;\leq\; \widehat{C},$$

[2] Technically, since in this case the set of feasible assortments depends on $\boldsymbol{x}$, we should write $\mathcal{F}(\boldsymbol{x})$ instead. We note that even when the set of feasible assortments $\mathcal{F}$ depends on $\boldsymbol{x}$, all of our results in this paper are still valid following exactly the same arguments. However, to reduce notation overload, and because most of the applications in e-retailing that we are interested in can be modelled with a fixed $\mathcal{F}$, we focus on the fixed $\mathcal{F}$ throughout the entire paper and here in this example we still keep $\mathcal{F}$.

so $S \in \mathcal{F}$. Then using exactly the same argument as in the previous example, we obtain that $\boldsymbol{A}' = \left\{i^d \in \boldsymbol{A} : x_{id} = 1\right\} \in \mathcal{G}(\boldsymbol{x})$.

Examples of static assortment optimization problem (F.3) with a capacity constraint are abundant in the literature. To name a few, we note that when $w_{id} = 1$ for all $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$, the capacity constraint reduces to a single cardinality constraint. Then, the static assortment optimization problem under the MNL model can be solved exactly in polynomial time (Rusmevichientong et al. 2010). Under a general capacity constraint, one can obtain an FPTAS for the static problem under the MNL model (Désir and Goyal 2014).

Example B.3 (Totally Unimodular Constrained Problem) Let us use binary variables $\left(y_i \in \{0,1\}^n : i \in \mathcal{N}\right)$ to model the assortment decisions. That is, $i \in S$ if and only if $y_i = 1$. Then, we can impose totally unimodular constraints $\mathcal{F} = \left\{\boldsymbol{y} : \mathbf{B}\boldsymbol{y} \leq \bar{C}\right\}$ where $\mathbf{B} \in \mathbb{R}^{m \times n}$ is a totally unimodular matrix and $\bar{C}$ is an integer. We impose the technical assumption that all components of $\mathbf{B}$ are non-negative. In this case, we may define $\mathcal{G}$ by letting $\widehat{\mathbf{B}} \in \mathbb{R}^{m \times L}$ be defined such that the (i,d)th column of $\widehat{\mathbf{B}}$ is precisely the ith column of matrix $\mathbf{B}$, and setting

$$\mathcal{G} = \left\{\hat{\boldsymbol{y}} : \widehat{\mathbf{B}}\hat{\boldsymbol{y}} \leq \bar{C}\right\}.$$

Here we again use $\hat{y}_{id} = 1$ to indicate if i^d is included in an assortment. By assumption, the downward inclusiveness and the requirement $\mathcal{G}(\boldsymbol{x}) \in \mathcal{G}$ are clearly satisfied. Using the same argument as in Example B.2 for each constraint separately, we can verify the third desired property of $\mathcal{G}$.

The key observation is that $\widehat{\mathbf{B}}$ remains totally unimodular. To see this, consider any submatrix $\widehat{\mathbf{B}}_1$ of $\widehat{\mathbf{B}}$. If no two columns of $\widehat{\mathbf{B}}_1$ correspond to two different designs $d_1, d_2 \in \mathcal{D}_i$ for some product $i \in \mathcal{N}$, $\widehat{\mathbf{B}}_1$ is also a submatrix of $\mathbf{B}$. In this case, its determinant must be 0, +1 or −1. If there exists $i \in \mathcal{N}$ such that there are $d_1, d_2 \in \mathcal{D}_i$ with their corresponding columns included in $\widehat{\mathbf{B}}_1$, the determinant of $\widehat{\mathbf{B}}_1$ must be 0. This proves the totally unimodularity of $\widehat{\mathbf{B}}$, a property that can helps solving the static assortment optimization problem (F.3). Indeed, we refer to the readers to Sumida et al. (2021) for exact algorithms and FPTAS for the static assortment optimization under totally unimodular constraints.

B.4. Proof of Lemma 3.4

Proof: Fix $\boldsymbol{x} \in \mathcal{X}$. Consider an arbitrary optimal solution $(y^t(\boldsymbol{A}) : \boldsymbol{A} \in \mathcal{G}, t \in \mathcal{T})$ to the linear program associated with $\mathsf{VB}(\boldsymbol{x})$. Assume that there exist $\boldsymbol{A}_1 \in \mathcal{G}$, $\boldsymbol{A}_1 \notin \mathcal{G}(\boldsymbol{x})$, and $y^t(\boldsymbol{A}_1) > 0$ for some t. Then for each $d \in \mathcal{D}_i$, let us define $\boldsymbol{A}_1' = \{i^d \in \boldsymbol{A}_1 : x_{id} = 1\}$. By the third property of $\mathcal{G}$, it holds that $\boldsymbol{A}_1' \in \mathcal{G}(\boldsymbol{x}) \subset \mathcal{G}$.

In the next, we claim that for any $i^d \in \boldsymbol{A}_1 \backslash \boldsymbol{A}_1'$, it must be that $\psi^t_{i^d}(\boldsymbol{A}_1) = 0$. Indeed, by definition of $\boldsymbol{A}_1$, $x_{id} = 0$. Then by the feasibility of $(y^t(\boldsymbol{A}_1) : \boldsymbol{A}_1 \in \mathcal{G}, t \in \mathcal{T})$ to the linear program associated with $\mathsf{VB}(\boldsymbol{x})$, we have

$$0 \ \geq \ \sum_{\tau=1}^{t} \sum_{\boldsymbol{A}_1 \in \mathcal{G}} y^\tau(\boldsymbol{A}_1) \psi^t_{i^d}(\boldsymbol{A}_1) \alpha_{id}(t-\tau) \ \geq \ (1-\gamma_{id}) \sum_{\tau=1}^{t} \sum_{\boldsymbol{A}_1 \in \mathcal{G}} y^\tau(\boldsymbol{A}_1) \psi^t_{i^d}(\boldsymbol{A}_1) \ ,$$

where the last inequality follows because $\alpha_{id} \geq 1 - \gamma_{id}$ for all $i \in \mathcal{N}$ and $d \in \mathcal{D}$. Because $y^t(\boldsymbol{A}_1) > 0$, we obtain that $\psi^t_{i^d}(\boldsymbol{A}_1) = 0$.

We then claim that $\psi^t_{i^d}(\boldsymbol{A}_1') = \psi^t_{i^d}(\boldsymbol{A}_1)$ for all $i^d \in \mathcal{M}$. Indeed, we have

$$1 = \psi^t_0(\boldsymbol{A}_1) + \sum_{i^d \in \boldsymbol{A}_1} \psi^t_{i^d}(\boldsymbol{A}_1) \stackrel{(a)}{=} \psi^t_0(\boldsymbol{A}_1) + \sum_{i^d \in \boldsymbol{A}_1'} \psi^t_{i^d}(\boldsymbol{A}_1) \stackrel{(b)}{\leq} \psi^t_0(\boldsymbol{A}_1') + \sum_{i^d \in \boldsymbol{A}_1'} \psi^t_{i^d}(\boldsymbol{A}_1') = 1$$

where (a) follows because $\psi^t_{i^d}(\boldsymbol{A}_1) = 0$ for all $i^d \in \boldsymbol{A}_1 \backslash \boldsymbol{A}_1'$, and (b) follows because by the substitutability condition it must that $\psi^t_0(\boldsymbol{A}_1) \leq \psi^t_0(\boldsymbol{A}_1')$ and $\psi^t_{i^d}(\boldsymbol{A}_1) \leq \psi^t_{i^d}(\boldsymbol{A}_1')$ for all $i^d \in \boldsymbol{A}_1'$. As a result, it must be that $\psi^t_{i^d}(\boldsymbol{A}_1) = \psi^t_{i^d}(\boldsymbol{A}_1')$ for all $i^d \in \boldsymbol{A}_1$. If $i^d \in \boldsymbol{A}_1 \backslash \boldsymbol{A}_1'$, as we have argued, $\psi^t_{i^d}(\boldsymbol{A}_1) = 0 = \psi^t_{i^d}(\boldsymbol{A}_1')$. If $i^d \notin \boldsymbol{A}_1$, then clearly $\psi^t_{i^d}(\boldsymbol{A}_1) = \psi^t_{i^d}(\boldsymbol{A}_1') = 0$. This proves the claim.

Let us define a new solution $(\tilde{y}^t(\boldsymbol{A}) : \boldsymbol{A} \in \mathcal{G})$ such that

$$\tilde{y}^t(\boldsymbol{A}) \ = \ \begin{cases} y^t(\boldsymbol{A}_1) & \text{if } \boldsymbol{A} = \boldsymbol{A}_1', \\ 0 & \text{if } \boldsymbol{A} = \boldsymbol{A}_1, \\ y^t(\boldsymbol{A}) & \text{if otherwise.} \end{cases}$$

Since $\psi^t_{i^d}(\boldsymbol{A}_1') = \psi^t_{i^d}(\boldsymbol{A}_1)$ for all $i^d \in \mathcal{M}$, it must be that $\tilde{y}^t(\cdot)$ is feasible to the linear program associated with $\mathsf{VB}(\boldsymbol{x})$ with an objective function value exactly the same with that of $y^t(\cdot)$. By repeating the same procedure for each t and each $\boldsymbol{A} \notin \mathcal{G}(\boldsymbol{x})$ such that $y^t(\boldsymbol{A}) > 0$, we establish the desired result. ■

B.5. Proof of Theorem 3.5

Proof: Fix an arbitrary $\boldsymbol{x} \in \mathcal{X}$. Consider an arbitrary feasible solution $\boldsymbol{z} = \{z^t(S) : t \in \mathcal{T}, S \in \mathcal{F}\}$ to the upper bound $\mathsf{UB}(\boldsymbol{x})$ in Eq. (3.1). Recall that $\boldsymbol{A}(S, \boldsymbol{x}) = \{i^d : i \in S, \, d \in \mathcal{D}_i, \, x_{id} = 1\}$. For each $\boldsymbol{A} \in \mathcal{G}$, let

$$y^t(\boldsymbol{A}) = \begin{cases} z^t(S) & \text{if there exists } S \in \mathcal{F} \text{ such that } \boldsymbol{A} = \boldsymbol{A}(S, \boldsymbol{x}) \\ 0 & \text{otherwise.} \end{cases}$$

We note that the mapping $\boldsymbol{A}(\cdot, \boldsymbol{x})$ from $\mathcal{F}$ to $\mathcal{G}(\boldsymbol{x})$ is a bijection. Therefore, it is clear that $y^t(\boldsymbol{A}) \geq 0$ and $\sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \ \leq \ 1$ for all t, so the second sets of constraints of the linear program associated with $\mathsf{VB}(\boldsymbol{x})$ in Eq. (3.2) are satisfied. Consider an arbitrary $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$. To show that the first sets of constraints are also satisfied, note that for each $i \in \mathcal{N}$, $d \in \mathcal{D}_i$ and $t \in \mathcal{T}$, we have the following relationships.

$$\sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \psi^t_{i^d}(\boldsymbol{A}) \ \stackrel{(a)}{=} \ \begin{cases} \sum_{S \in \mathcal{F}} z^t(S) \phi^t_i(S \,|\, \boldsymbol{x}) & \text{if } \ x_{id} = 1, \\ 0 & \text{if } \ x_{id} = 0. \end{cases}$$

We will prove the equality (a) as follow. By our construction,

$$\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A}) = \sum_{S\in\mathcal{F}} z^t(S)\psi^t_{id}(\boldsymbol{A}(S,\boldsymbol{x}))\ .$$

If $x_{id}=1$, then $\psi^t_{id}(A(S,\boldsymbol{x})) = \phi^t_i(S\,|\,\boldsymbol{x})$ for all S by Assumption 3.2, so $\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A}) = \sum_{S\in\mathcal{F}} z^t(S)\phi^t_i(S\,|\,\boldsymbol{x})$. If $x_{id}=0$, then $i^d\notin \boldsymbol{A}(S,\boldsymbol{x})$, so $\psi^t_{id}(\boldsymbol{A}(S,\boldsymbol{x}))=0$ for all S, giving us the desired result. This proves equality (a) above and immediately imply that $y^t(\boldsymbol{A})$ satisfy the first sets of constraints. It also shows that for all $i\in\mathcal{N}$,

$$\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\sum_{d\in\mathcal{D}_i}\psi^t_{id}(\boldsymbol{A})r_{id} \ = \ \sum_{S\in\mathcal{F}} z^t(S)\phi^t_i(S\,|\,\boldsymbol{x})r_i(\boldsymbol{x})\ ,$$

so both $\boldsymbol{z}$ and $\boldsymbol{y}$ have the same objective value. Since $\boldsymbol{z}$ is arbitrary, it follows that $\mathsf{UB}(\boldsymbol{x})\le\mathsf{VB}(\boldsymbol{x})$.

We will now prove the reverse inequality. Consider an optimal solution $\{\bar{y}^t(\boldsymbol{A}):\boldsymbol{A}\in\mathcal{G}\}$ to the linear program associated with $\mathsf{VB}(\boldsymbol{x})$ in Eq. (3.2) such that $\bar{y}^t(\boldsymbol{A})=0$ for all $t\in\mathcal{T}$ and $\boldsymbol{A}\notin\mathcal{G}(\boldsymbol{x})$, where $\mathcal{G}(\boldsymbol{x})=\{\boldsymbol{A}(S,\boldsymbol{x}) : S\in\mathcal{F}\}$. By Lemma 3.4, we know that such solution exists. For each $S\in\mathcal{F}$, let $\bar{z}^t(S)=\bar{y}^t(\boldsymbol{A}(S,\boldsymbol{x}))$. We will now show that $\bar{z}^t(S)$ is feasible for $\mathsf{UB}(\boldsymbol{x})$ and has the same objective value as $\mathsf{VB}(\boldsymbol{x})$. Clearly, $\bar{z}^t(S)\ge 0$ and $\sum_{S\in\mathcal{F}}\bar{z}^t(S)\le 1$ for all $S\in\mathcal{F}$. For an arbitrary $i\in\mathcal{N}$ and $t\in\mathcal{T}$, it follows from our construction of $\bar{z}^t(S)$ and Assumption 3.2, there exists an $d\in\mathcal{D}_i$ such that $x_{id}=1$,

$$\sum_{S\in\mathcal{F}}\bar{z}^t(S)\phi^t_i(S\,|\,\boldsymbol{x}) \ = \ \sum_{S\in\mathcal{F}}\bar{y}^t(\boldsymbol{A}(S,\boldsymbol{x}))\psi^t_{id}(\boldsymbol{A}(S,\boldsymbol{x})) \ \overset{(c)}{=} \ \sum_{\boldsymbol{A}\in\mathcal{G}(\boldsymbol{x})}\bar{y}^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A}) \ \overset{(d)}{=} \ \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A}) \tag{B.1}$$

where the equality (c) follows because $\mathcal{G}(\boldsymbol{x})=\{\boldsymbol{A}(S,\boldsymbol{x}) : S\in\mathcal{F}\}$, and the equality (d) follows because $\bar{y}^t(\boldsymbol{A})=0$ for all $\boldsymbol{A}\notin\mathcal{G}(\boldsymbol{x})$. Then $\sum_{\tau=1}^t\sum_{S\in\mathcal{F}}\bar{z}^\tau(S)\phi^\tau_i(S\,|\,\boldsymbol{x})\alpha_{id}(t-\tau) \ \le\ C_{id}x_{id}$ for all t by the feasibility of $\bar{y}^t(\boldsymbol{A})$. Therefore, we have that $\sum_{\tau=1}^t\sum_{S\in\mathcal{F}}\bar{z}^\tau(S)\phi^\tau_i(S\,|\,\boldsymbol{x})\alpha_i(t-\tau\,|\,\boldsymbol{x}) \ \le\ C_i(\boldsymbol{x})$ for all $i\in\mathcal{N}$ and $t\in\mathcal{T}$, establishing the feasibility of $\bar{z}^t(S)$.

We will now show that the objective value under $\bar{z}^t(S)$ is the same as $\mathsf{VB}(\boldsymbol{x})$. For an arbitrary $i\in\mathcal{N}$, if $x_{id}=1$, then because $\alpha_{ih}\ge 1-\gamma_{ih}>0$ for all ℓ, it follows from the first set of constraints of the linear program associated with $\mathsf{VB}(\boldsymbol{x})$ that $\sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\psi^t_{ih}(\boldsymbol{A}) \ = \ 0$ for all $h\ne d, h\in\mathcal{D}_i$. Therefore, the objective function under $\mathsf{VB}(\boldsymbol{x})$ can be written as $\sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\Big(\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\psi^t_{id}(\boldsymbol{A})r_{id}x_{id}\Big)$. Then, to complete the proof, note that the objective function of $\mathsf{UB}(\boldsymbol{x})$ is

$$\begin{aligned}\sum_{t\in\mathcal{T}}\sum_{S\in\mathcal{F}}\bar{z}^t(S)\sum_{i\in\mathcal{N}}\phi^t_i(S\,|\,\boldsymbol{x})r_i(\boldsymbol{x}) \ &= \ \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}r_i(\boldsymbol{x})\sum_{S\in\mathcal{F}}\bar{z}^t(S)\phi^t_i(S\,|\,\boldsymbol{x}) \ \overset{(e)}{=} \ \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}r_i(\boldsymbol{x})\sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A})\\ &= \ \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in D_i}r_{id}x_{id}\sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A}) \ = \ \mathsf{VB}(\boldsymbol{x})\,,\end{aligned}$$

where (e) follows because of (B.1). This shows that $\mathsf{VB}(\boldsymbol{x})\le\mathsf{UB}(\boldsymbol{x})$, which is the desired result. ■

Appendix C: Proofs of Theorem 4.1 and Complexity of Two Customer Types

We give the proof of Theorem 4.1 and show that even when there are only two designs and two customer types, optimizing the upper bound remains NP-hard.

C.1. Proof of Theorem 4.1

We are interested in the special case where there are ample capacities and two designs ($\mathcal{D}_i = \{1,2\}$), with $C_{i1} = C_{i2} = T$ for all i, so the capacity constraint can be dropped from the optimization problem. Finally, we assume that $r_{1i} = r_{i2} = 1$ and $\gamma_{i1} = \gamma_{i2} = 0$ for all i. Thus, the optimization problem that we want to solve is $\max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{UB}(\boldsymbol{x})$, which reduces to

$$Z^* = \max_{\boldsymbol{x}\in\mathcal{X}} \sum_{t=1}^{T} \max_{S^t\subseteq\mathcal{N}} \sum_{i\in\mathcal{N}} \phi_i^t(S^t \mid \boldsymbol{x}) = \max_{\boldsymbol{x}\in\mathcal{X}} \sum_{t=1}^{T} \frac{\sum_{i\in\mathcal{N}} v_{i1}^t x_i^1 + v_{i2}^t x_i^2}{1+\sum_{i\in\mathcal{N}} v_{i1}^t x_i^1 + v_{i2}^t x_i^2}$$
$$= \max_{\boldsymbol{y}\in\{0,1\}^n} \sum_{t=1}^{T} \frac{\sum_{i\in\mathcal{N}} v_{i1}^t (1-y_i) + v_{i2}^t y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^t (1-y_i) + v_{i2}^t y_i} \tag{C.1}$$

where the second equality follows because we are maximizing the probability of purchase for each customer, so the optimal assortment is to offer all products.

We will use a reduction from the MAXCUT problem, which is a well known APX-complete problem. Consider an arbitrary instance of the MAXCUT problem given by:

Input: An undirected graph $\mathsf{G} = (\mathsf{V},\mathsf{E})$, where V is the set of vertices and E is the set of edges.
Problem: Find a partition of the vertices resulting in the maximum cut, corresponding to the following optimization problem:

$$Y^* = \max_{A\subseteq\mathsf{V}} \sum_{\{u,v\}\in\mathsf{E}} \mathbb{1}_{\{\{u,v\}\cap A\neq\varnothing \ \text{ and } \ \{u,v\}\cap(\mathsf{V}\setminus A)\neq\varnothing\}}$$

For the MAXCUT problem, it is well known that it is NP-hard to obtain an approximation guarantee that is strictly better than $\frac{16}{17}$; see, for example, Håstad (2001). Also, it is well known that the maximum cut-value is at least half of the total number of edges, so $Y^* \geq |\mathsf{E}|/2$; see, for example, Bollobás and Scott (2002).

Fix an arbitrary $0<\epsilon<\frac{1}{51}$. We will show that a $(1-\epsilon)$-approximation algorithm for computing Z^* yields an approximation algorithm for the MAXCUT problem with a performance guarantee that is strictly better than $\frac{16}{17}$. This is, of course, impossible unless P = NP, which will give us the desired result

Choose $\Gamma>0$ such that $\Gamma > \frac{2}{3}\left(\frac{1}{51\epsilon}-1\right)^{-1}$, which implies that $\frac{1}{\Gamma} < \frac{3}{2}\left(\frac{1}{51\epsilon}-1\right)$. Given an arbitrary instance of the MAXCUT problem, we will construct the following instance of our upper bound optimization problem. The products will correspond to the vertices of the graph, so $\mathcal{N} = \mathsf{V}$. Associated with each edge $\{u,v\} \in \mathsf{E}$, we construct **two** customer types: $(\{u,v\},1)$ and $(\{u,v\},2)$, and the preference weight parameters for the two customer types are given by: for each $i \in \mathsf{V}$,

$$v_{i1}^{(\{u,v\},1)} = \begin{cases} \Gamma & \text{if } i \in \{u,v\}, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{and} \qquad v_{i2}^{(\{u,v\},1)} = 0$$

$$v_{i1}^{(\{u,v\},2)} = 0 \qquad \text{and} \qquad v_{i2}^{(\{u,v\},2)} = \begin{cases} \Gamma & \text{if } i \in \{u,v\}, \\ 0 & \text{otherwise}, \end{cases}$$

Therefore, the corresponding upper bound optimization problem is given by:

$$Z^* = \max_{\boldsymbol{y} \in \{0,1\}^{|\mathsf{V}|}} f(\boldsymbol{y}) \qquad \text{where} \qquad f(\boldsymbol{y}) \equiv \sum_{\{u,v\} \in \mathsf{E}} \frac{\Gamma y_u + \Gamma y_v}{1 + \Gamma y_u + \Gamma y_v} + \frac{\Gamma(1-y_u) + \Gamma(1-y_v)}{1 + \Gamma(1-y_u) + \Gamma(1-y_v)} .$$

Consider an optimal solution $A^* \subseteq \mathsf{V}$ to the MAXCUT problem. Let $\tilde{y}_i = \mathbb{1}_{\{i \in A^*\}}$ for all i. Then,

$$\begin{aligned} Z^* \geq f(\tilde{\boldsymbol{y}}) &= \sum_{\{u,v\} \in \mathsf{E}} \frac{\Gamma \tilde{y}_u + \Gamma \tilde{y}_v}{1 + \Gamma \tilde{y}_u + \Gamma \tilde{y}_v} + \frac{\Gamma(1-\tilde{y}_u) + \Gamma(1-\tilde{y}_v)}{1 + \Gamma(1-\tilde{y}_u) + \Gamma(1-\tilde{y}_v)} \\ &\overset{(a)}{=} \sum_{\{u,v\} \in \mathsf{E}} \mathbb{1}_{\{\{u,v\} \cap A^* \neq \varnothing \text{ and } \{u,v\} \cap (\mathsf{V} \setminus A^*) \neq \varnothing\}} \frac{2\Gamma}{1+\Gamma} + \mathbb{1}_{\{\{u,v\} \subseteq A^* \text{ or } \{u,v\} \subseteq (\mathsf{V} \setminus A^*)\}} \frac{2\Gamma}{1+2\Gamma} \\ &\overset{(b)}{=} \frac{2\Gamma}{1+\Gamma} Y^* + (|\mathsf{E}| - Y^*) \frac{2\Gamma}{1+2\Gamma} \overset{(c)}{=} \frac{2\Gamma}{1+2\Gamma} \left(|\mathsf{E}| + \frac{\Gamma Y^*}{1+\Gamma} \right) \end{aligned}$$

where (a) follows because if $\tilde{y}_u \neq \tilde{y}_v$, so the edge $\{u,v\}$ is part of a cut, then the two terms associated with $\{u,v\}$ in the objective function add up to $2\Gamma/(1+\Gamma)$. But if $\tilde{y}_u = \tilde{y}_v$, then the edge $\{u,v\}$ is not part of a cut, the only one term is nonzero and it contributes $2\Gamma/(1+2\Gamma)$ toward the objective function. The equality (b) follows from the definition of Y^*, and (c) follows from simplification.

Suppose that we have a $(1-\epsilon)$-approximation algorithm for the optimization problem associated with Z^*. Thus, the algorithm generates a vector $\hat{\boldsymbol{y}} \in \{0,1\}^{|\mathsf{V}|}$ such that $f(\hat{\boldsymbol{x}}) \geq (1-\epsilon) Z^*$. Let $\hat{A} = \{i \in \mathsf{V} : \hat{y}_i = 1\}$ denote the subset of vertices associated with $\hat{\boldsymbol{y}}$, and let $\hat{Y}$ denote the *cut-value* associated with the set $\hat{A}$; that is,

$$\hat{Y} = \sum_{\{u,v\} \in \mathsf{E}} \mathbb{1}_{\left\{\{u,v\} \cap \hat{A} \neq \varnothing \text{ and } \{u,v\} \cap (\mathsf{V} \setminus \hat{A}) \neq \varnothing\right\}} .$$

Using exactly the same argument as above, note that

$$\begin{aligned} f(\hat{\boldsymbol{y}}) &= \sum_{\{u,v\} \in \mathsf{E}} \frac{\Gamma \hat{y}_u + \Gamma \hat{y}_v}{1 + \Gamma \hat{y}_u + \Gamma \hat{y}_v} + \frac{\Gamma(1-\hat{y}_u) + \Gamma(1-\hat{y}_v)}{1 + \Gamma(1-\hat{y}_u) + \Gamma(1-\hat{y}_v)} \\ &= \frac{2\Gamma}{1+\Gamma} \hat{Y} + \left(|\mathsf{E}| - \hat{Y} \right) \frac{2\Gamma}{1+2\Gamma} = \frac{2\Gamma}{1+2\Gamma} \left(|\mathsf{E}| + \frac{\Gamma \hat{Y}}{1+\Gamma} \right) \end{aligned}$$

Because $f(\hat{\boldsymbol{x}}) \geq (1-\epsilon) Z^*$, it follows that

$$\frac{2\Gamma}{1+2\Gamma} \left(|\mathsf{E}| + \frac{\Gamma \hat{Y}}{1+\Gamma} \right) \geq (1-\epsilon) Z^* \geq (1-\epsilon) \frac{2\Gamma}{1+2\Gamma} \left(|\mathsf{E}| + \frac{\Gamma Y^*}{1+\Gamma} \right) ,$$

which implies that

$$\frac{\Gamma \hat{Y}}{1+\Gamma} \ \geq\ -\epsilon\,|\mathsf{E}| + \frac{(1-\epsilon)\Gamma\, Y^*}{1+\Gamma} \qquad \Leftrightarrow \qquad \hat{Y} \ \geq\ -\epsilon\,|\mathsf{E}|\,\frac{1+\Gamma}{\Gamma} \ +\ (1-\epsilon)Y^*$$

Since $|\mathsf{E}| \leq 2Y^*$, it follows that

$$\hat{Y} \ \geq\ -\epsilon\left(1+\frac{1}{\Gamma}\right)2Y^* \ +\ (1-\epsilon)Y^* \ =\ \left(1-3\epsilon-\frac{2\epsilon}{\Gamma}\right)Y^* \ >\ \frac{16}{17}Y^*\ ,$$

where the last (strict) inequality follows because our choice of Γ implies that

$$\frac{1}{\Gamma} < \frac{3}{2}\left(\frac{1}{51\epsilon}-1\right) \quad \Leftrightarrow \quad \frac{2\epsilon}{\Gamma} < 3\left(\frac{1}{51}-\epsilon\right) \quad \Leftrightarrow \quad 3\epsilon+\frac{2\epsilon}{\Gamma} < \frac{1}{17}$$

Thus, if we have an approximation algorithm for optimizing the MNL upper bound whose performance guarantee is better than $\frac{50}{51}$, we can use the algorithm to obtain an approximation method for the MAXCUT problem whose guarantee is better than $\frac{16}{17}$. Therefore, it is NP-hard to approximate the upper bound with a performance guarantee that is better than $\frac{50}{51}$.

C.2. Complexity of the Problem with Two Customer Types

We are interested in the special case with two designs ($\mathcal{D}_i = \{1,2\}$) and just two customers ($T=2$), with the corresponding choice models $\phi_i^1(S\,|\,\boldsymbol{x})$ and $\phi_i^2(S\,|\,\boldsymbol{x})$. We also assume there are ample capacities, with $C_{i1} = C_{i2} = 2$ for all i, so the capacity constraint can be dropped from the optimization problem. Finally, $r_{1i} = r_{i2} = 1$ and $\gamma_{i1} = \gamma_{i2} = 0$ for all i, and selected products cannot be returned. Thus, the optimization problem $\max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{UB}(\boldsymbol{x})$ reduces to

$$\begin{aligned}
Z^* \ &=\ \max_{\boldsymbol{x}\in\mathcal{X}} \left\{ \max_{S^1\subseteq\mathcal{N}} \sum_{i\in\mathcal{N}} \phi_i^1(S^1\,|\,\boldsymbol{x}) \ +\ \max_{S^2\subseteq\mathcal{N}} \sum_{i\in\mathcal{N}} \phi_i^2(S^2\,|\,\boldsymbol{x}) \right\}, \\
&=\ \max_{\boldsymbol{x}\in\mathcal{X}} \ \frac{\sum_{i\in\mathcal{N}} v_{i1}^1 x_i^1 + v_{i2}^1 x_i^2}{1+\sum_{i\in\mathcal{N}} v_{i1}^1 x_i^1 + v_{i2}^1 x_i^2} \ +\ \frac{\sum_{i\in\mathcal{N}} v_{i1}^2 x_i^1 + v_{i2}^2 x_i^2}{1+\sum_{i\in\mathcal{N}} v_{i1}^2 x_i^1 + v_{i2}^2 x_i^2} \\
&=\ \max_{\boldsymbol{y}\in\{0,1\}^n} \ \frac{\sum_{i\in\mathcal{N}} v_{i1}^1 (1-y_i) + v_{i2}^1 y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^1 (1-y_i) + v_{i2}^1 y_i} \ +\ \frac{\sum_{i\in\mathcal{N}} v_{i1}^2 (1-y_i) + v_{i2}^2 y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^2 (1-y_i) + v_{i2}^2 y_i}
\end{aligned} \tag{C.2}$$

where the second equality follows because we are maximizing the probability of purchase for each customer, so the optimal assortment is to offer all products. Here is the main result.

Proposition C.1 (Complexity of Two Customers) *The decision-theoretic version of the optimization problem with two designs and two customer types in Eq.* (C.2) *is NP-complete.*

Proof: We will use a reduction from the PARTITION problem, which is a well known NP-complete problem. Consider an arbitrary instance of the PARTITION problem given by:

Input: A set $\mathcal{N} = \{1,2,\ldots,n\}$ and each item $i\in\mathcal{N}$ has a weight $w_i \in \mathbb{Z}_+$. Let $T = \frac{1}{2}\sum_{i\in\mathcal{N}} w_i$.
Question: Is there a subset $A \subseteq \mathcal{N}$ such that $\sum_{i\in A} w_i = T$?

Without loss of generality, we can assume that $T \in \mathbb{Z}_+$. Given the above instance of the PARTITION problem, we construct the following instance of our upper bound optimization problem. Define the preference weights for the two customers as follow:

$$v_{i1}^1 = \left(4 + \frac{3}{2T}\right) w_i \quad \text{and} \quad v_{i2}^1 = \frac{w_i}{2T} \quad \forall i \in \mathcal{N},$$
$$v_{i1}^2 = \frac{w_i}{2T} \quad \text{and} \quad v_{i2}^2 = \left(4 + \frac{3}{2T}\right) w_i \quad \forall i \in \mathcal{N}\ .$$

Define a function $F : [0, 2T] \to \mathbb{R}_+$ as follows: for each $z \in [0, 2T]$,

$$F(z) = \frac{3 + 8T - (4 + \frac{1}{T})z}{4 + 8T - (4 + \frac{1}{T})z} \ + \ \frac{1 + (4 + \frac{1}{T})z}{2 + (4 + \frac{1}{T})z}\ .$$

For all $0 \le z \le 2T$, $F(z)$ is well defined and strictly positive. Define our target value as follow:

$$K \ = \ F(T) \ = \ \frac{2 + 4T}{3 + 4T} \ + \ \frac{2 + 4T}{3 + 4T} \ = \ \frac{4 + 8T}{3 + 4T}\ .$$

To complete the proof, we will show that there is a subset $A \subseteq \mathcal{N}$ such that $\sum_{i \in A} w_i = T$ if and only if there is a vector $\boldsymbol{x} \in \{0,1\}^n$ such that $\frac{\sum_{i\in\mathcal{N}} v_{i1}^1(1-y_i)+v_{i2}^1 y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^1(1-y_i)+v_{i2}^1 y_i} \ + \ \frac{\sum_{i\in\mathcal{N}} v_{i1}^2(1-y_i)+v_{i2}^2 y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^2(1-y_i)+v_{i2}^2 y_i} \ge K$.

For any arbitrary $\boldsymbol{x} \in \{0,1\}^n$, we have that

$$\begin{aligned}
&\frac{\sum_{i\in\mathcal{N}} v_{i1}^1(1-y_i)+v_{i2}^1 y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^1(1-y_i)+v_{i2}^1 y_i} \ + \ \frac{\sum_{i\in\mathcal{N}} v_{i1}^2(1-y_i)+v_{i2}^2 y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^2(1-y_i)+v_{i2}^2 y_i} \\
&\quad = \ \frac{\sum_{i\in\mathcal{N}} v_{i1}^1 + \sum_{i\in\mathcal{N}} \left(v_{i2}^1 - v_{i1}^1\right) y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^1 + \sum_{i\in\mathcal{N}} \left(v_{i2}^1 - v_{i1}^1\right) y_i} \ + \ \frac{\sum_{i\in\mathcal{N}} v_{i1}^2 + \sum_{i\in\mathcal{N}} \left(v_{i2}^2 - v_{i1}^2\right) y_i}{1+\sum_{i\in\mathcal{N}} v_{i1}^2 + \sum_{i\in\mathcal{N}} \left(v_{i2}^2 - v_{i1}^2\right) y_i} \\
&\quad \overset{(a)}{=} \ \frac{3 + 8T - (4+\frac{1}{T})\sum_{i\in\mathcal{N}} w_i y_i}{4 + 8T - (4+\frac{1}{T})\sum_{i\in\mathcal{N}} w_i y_i} \ + \ \frac{1 + (4+\frac{1}{T})\sum_{i\in\mathcal{N}} w_i y_i}{2 + (4+\frac{1}{T})\sum_{i\in\mathcal{N}} w_i y_i} \ \overset{(b)}{=} \ F\Big(\sum_{i\in\mathcal{N}} w_i y_i\Big)
\end{aligned}$$

where (a) follows because $v_{i2}^1 - v_{i1}^1 = -(4 + \frac{1}{T})w_i$, $v_{i2}^2 - v_{i1}^2 = (4 + \frac{1}{T})w_i$, and $\sum_{i\in\mathcal{N}} w_i = 2T$, so $\sum_{i\in\mathcal{N}} v_{i1}^1 = 3 + 8T$ and $\sum_{i\in\mathcal{N}} v_{i1}^2 = 1$. The last equality (b) follows from the definition of F.

Note that $\sum_{i\in\mathcal{N}} w_i y_i$ is a number between 0 and $2T$. We will show that, on the interval $[0, 2T]$, the function F achieves the unique maximum at T. To see this, note that

$$\begin{aligned}
F'(z) \ &= \ -\frac{4+\frac{1}{T}}{(4+8T-(4+\frac{1}{T})z)^2} + \frac{4+\frac{1}{T}}{(2+(4+\frac{1}{T})z)^2} \\
&= \ \frac{4+\frac{1}{T}}{(4+8T-(4+\frac{1}{T})z)^2}\left\{\left(\frac{4+8T-(4+\frac{1}{T})z}{2+(4+\frac{1}{T})z}\right)^2 \ - \ 1\right\} \\
&= \ \frac{4+\frac{1}{T}}{(4+8T-(4+\frac{1}{T})z)^2} \times \left\{\frac{4+8T-(4+\frac{1}{T})z}{2+(4+\frac{1}{T})z} \ + \ 1\right\} \times \left\{\frac{4+8T-(4+\frac{1}{T})z}{2+(4+\frac{1}{T})z} \ - \ 1\right\}\ ,
\end{aligned}$$

and the first two terms are strictly positive over the interval $[0, 2T]$, and it is easy to check that

$$\frac{4+8T-(4+\frac{1}{T})z}{2+(4+\frac{1}{T})z} \ - \ 1 \ = \ \begin{cases} > 0 & \text{if} \quad z \in [0,\, T), \\ = 0 & \text{if} \quad z = T, \\ < 0 & \text{if} \quad z \in (T,\, 2T], \end{cases}$$

This means that the function $F(z)$ is strictly increasing over the interval $[0,T]$ and strictly decreasing over the interval $[T,2T]$, and thus, it has a unique maximum at $z=T$. Therefore, there is a vector $\boldsymbol{y}\in\{0,1\}^n$ such that $F(\sum_{i\in\mathcal{N}} w_i y_i)\geq K=F(T)$ if and only if there is a vector $\boldsymbol{y}\in\{0,1\}^n$ such that $\sum_{i\in\mathcal{N}} w_i y_i = T$, which is the desired result. ■

Appendix D: Supporting Arguments for Section 4

In the next section, we describe auxiliary lemmas needed for the proofs. The proofs of Theorems 4.2 and 4.4 are given in Appendices D.2 and D.4, respectively.

D.1. Auxiliary Lemmas

To prove Theorem 4.2, we need the following auxiliary lemmas. We start with some preliminary definitions. Let $\mathcal{P}=\{\boldsymbol{\mathcal{B}}=(\mathcal{B}_1,\ldots,\mathcal{B}_n):\mathcal{B}_i\subseteq\mathcal{D}_i,\ \forall i\in\mathcal{N}\}$. Let $\bar{\mathcal{X}}=[0,1]^L$. Note that $\mathcal{X}^{\mathsf{LP}}\subset\bar{\mathcal{X}}$. For each $\boldsymbol{\mathcal{B}}\in\mathcal{P}$ and $\boldsymbol{u}\in\bar{\mathcal{X}}$, define the linear programs $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ and $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ be defined as follow:

$$\begin{aligned}
Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \;=\; \max_{\boldsymbol{y}\geq\boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\psi^t_{id}(\boldsymbol{A})r_{id} && \\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A})\alpha_{id}(t-\tau) \;\leq\; C_{id}u_{id} && \forall\, i\in\mathcal{N},\ d\in\mathcal{B}_i,\ t\in\mathcal{T},\\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \;\leq\; 1 && \forall\, t\in\mathcal{T},
\end{aligned}$$

and

$$\begin{aligned}
Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \;=\; \max_{\boldsymbol{y}\geq\boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}}\psi^t_{id}(\boldsymbol{A})r_{id} && \\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A})\alpha_{id}(t-\tau) \;\leq\; C_{id}u_{id} && \forall\, i\in\mathcal{N},\ d\in\mathcal{B}_i,\ t\in\mathcal{T},\\
& \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A})\alpha_{id}(t-\tau) \;\leq\; 0 && \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i\setminus\mathcal{B}_i,\ t\in\mathcal{T},\\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \;\leq\; 1 && \forall\, t\in\mathcal{T}.
\end{aligned}$$

The following lemma show that the above two linear programs are equivalent.

Lemma D.1 *For each $\boldsymbol{\mathcal{B}}\subseteq\mathcal{P}$ and $\boldsymbol{u}\in\bar{\mathcal{X}}$, $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})=Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$.*

Proof: Fix arbitrary $\boldsymbol{\mathcal{B}}\in\mathcal{P}$ and $\boldsymbol{u}\in\bar{\mathcal{X}}$. From the constraints associated with $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ and the fact that $0<1-\gamma_{id}\leq\alpha_{id}(t-\tau)$ for all $i\in\mathcal{N}$ and $d\in\mathcal{D}_i$, it follows that

$$\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\psi^t_{id}(\boldsymbol{A}) \;=\; 0 \qquad \forall\, i\in\mathcal{N},\ d\in\mathcal{D}_i\setminus\mathcal{B}_i,\ t\in\mathcal{T}.$$

Therefore, the objective function for the linear program for $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ is equal to

$$\sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\psi^t_{id}(\boldsymbol{A})r_{id}\ ,$$

which is the same as the objective function for $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$. Since the optimization problem for $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ have more constraints and we are maximizing, it follows that $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \leq Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$.

To establish the reverse inequality, we use strong duality of linear program, which shows that

$$\begin{aligned} Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) = \min_{(\boldsymbol{\lambda},\boldsymbol{\theta})\geq \mathbf{0}} \quad & \sum_{t=1}^T \lambda^t + \sum_{t=1}^T \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} C_{id} u_{id} \theta_{id}^t \\ \text{s.t.} \quad & \lambda^t \geq \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) \quad \forall\, t\in\mathcal{T}, \boldsymbol{A}\in\mathcal{G}, \\ = \min_{(\boldsymbol{\lambda},\boldsymbol{\theta})\geq \mathbf{0}} \quad & \sum_{t=1}^T \lambda^t + \sum_{t=1}^T \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} C_{id} u_{id} \theta_{id}^t \\ \text{s.t.} \quad & \lambda^t \geq \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) \quad \forall\, t\in\mathcal{T}, \end{aligned}$$

and similarly,

$$\begin{aligned} Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u}) = \min_{(\boldsymbol{\lambda},\boldsymbol{\theta})\geq \mathbf{0}} \quad & \sum_{t=1}^T \lambda^t + \sum_{t=1}^T \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} C_{id} u_{id} \theta_{id}^t \\ \text{s.t.} \quad & \lambda^t \geq \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) \quad \forall\, t\in\mathcal{T}, \boldsymbol{A}\in\mathcal{G}, \\ = \min_{(\boldsymbol{\lambda},\boldsymbol{\theta})\geq \mathbf{0}} \quad & \sum_{t=1}^T \lambda^t + \sum_{t=1}^T \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} C_{id} u_{id} \theta_{id}^t \\ \text{s.t.} \quad & \lambda^t \geq \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) \quad \forall\, t\in\mathcal{T}. \end{aligned}$$

To prove that $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \geq Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$, we will first establish the following claim.

Claim: For each $\boldsymbol{\theta}\geq\mathbf{0}$ and $t\in\mathcal{T}$,

$$\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) \geq \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right)$$

The above claim shows that the feasible region associated with $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ is a <u>*subset*</u> of the feasible region for $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$. Because we are minimizing and the objective function is the same in $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ and $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$, it follows that $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \geq Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$, which is the desired result.

To prove the claim, let $\boldsymbol{Q} = \{i^d \in \mathcal{M} : i\in\mathcal{N}, d\in\mathcal{B}_i\}$. Then, we have

$$\begin{aligned} \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) &\overset{(a)}{=} \max_{\boldsymbol{A}\in\mathcal{G}\,:\,\boldsymbol{A}\subseteq\boldsymbol{Q}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) \\ &\overset{(b)}{=} \max_{\boldsymbol{A}\in\mathcal{G}\,:\,\boldsymbol{A}\subseteq\boldsymbol{Q}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right) \\ &\leq \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta_{id}^t \right), \end{aligned}$$

where (a) follows because we only consider products in $\boldsymbol{Q}$ in our objective function, so by the substitutability property of the choice function ψ^t, it is not optimal to include products outside $\boldsymbol{Q}$. Indeed, let us assume

$$\widetilde{\boldsymbol{A}} \;:=\; \arg\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \psi^t_{id}(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=t}^{T} \alpha_{id}(\tau-t)\theta^t_{id} \right).$$

We can remove products i^d from $\widetilde{\boldsymbol{A}}$ with $i^d \in \widetilde{\boldsymbol{A}} \cap Q$, $r_{id} - \sum_{\tau=t}^{T} \alpha_{id}(\tau-t)\theta^t_{id} \le 0$ and $i^d \in \widetilde{\boldsymbol{A}} \setminus Q$. Substitutability implies the choice probability of any $i^d \in \widetilde{\boldsymbol{A}} \cap Q$ with $r_{id} - \sum_{\tau=t}^{T} \alpha_{id}(\tau-t)\theta^t_{id} > 0$ will increase and we obtain a (weakly) better assortments. Note that this argument requires the downward inclusiveness property of $\mathcal{G}$. Equality (b) follows because $\boldsymbol{A} \subseteq \boldsymbol{Q}$, so $\psi^t_{id}(\boldsymbol{A}) = 0$ for all $i^d \notin \boldsymbol{Q}$. This proves the desired claim. ■

The next lemma relates the upper bound $\mathsf{UB}(\boldsymbol{x})$ with the linear program $Z^1(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$.

Lemma D.2 *For each $\boldsymbol{x} \in \{0,1\}^L$ and $\boldsymbol{u} \in \bar{\mathcal{X}}$, $\mathsf{UB}(\boldsymbol{x}) \ge Z^1\big(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u}\big)$, where $\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = (\mathcal{B}_1(\boldsymbol{x}), \ldots, \mathcal{B}_n(\boldsymbol{x}))$, where $\mathcal{B}_i(\boldsymbol{x}) = \{d \in \mathcal{D}_i : x_i^d = 1\}$.*

Proof: Fix arbitrary $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{u} \in \bar{\mathcal{X}}$. By Theorem 3.5, $\mathsf{UB}(\boldsymbol{x}) = \mathsf{VB}(\boldsymbol{x})$ and by definition of $\mathsf{VB}(\boldsymbol{x})$ in Lemma 3.2,

$$\begin{aligned}
\mathsf{VB}(\boldsymbol{x}) \;=\; \max_{\boldsymbol{y}\ge \boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}} \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) r_{id} \\
\text{s.t.} \quad & \sum_{\tau=1}^{t} \sum_{\boldsymbol{A}\in\mathcal{G}} y^\tau(\boldsymbol{A}) \psi^\tau_{id}(\boldsymbol{A}) \alpha_{id}(t-\tau) \;\le\; C_{id} && \forall\, i\in\mathcal{N}, d\in\mathcal{B}_i(\boldsymbol{x}),\ t\in\mathcal{T}, \\
& \sum_{\tau=1}^{t} \sum_{\boldsymbol{A}\in\mathcal{G}} y^\tau(\boldsymbol{A}) \psi^\tau_{id}(\boldsymbol{A}) \alpha_{id}(t-\tau) \;\le\; 0 && \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i \setminus \mathcal{B}_i(\boldsymbol{x}),\ t\in\mathcal{T}, \\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \;\le\; 1 && \forall\, t\in\mathcal{T}.
\end{aligned}$$

Note that both $\mathsf{VB}(\boldsymbol{x})$ and $Z^2(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u})$ have the same objective function. Moreover, the left-hand side of each constraint in $\mathsf{VB}(\boldsymbol{x})$ and $Z^2(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u})$ are the same, but the constants on the right-hand side of the constraints in $Z^2(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u})$ is bounded above by the corresponding constants in $\mathsf{UB}(\boldsymbol{x})$. Therefore, the feasible region of $Z^2(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u})$ is a subset of the feasible for $\mathsf{UB}(\boldsymbol{x})$. Because we are maximizing, it follows that $\mathsf{UB}(\boldsymbol{x}) \ge Z^2(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u}) = Z^1(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u})$, where the last equality follows from Lemma D.1. ■

The next lemma establishes a relation between the revenue of each product and the linear program $Z^1(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$. Recall that for each $\boldsymbol{y} = (y^t(\boldsymbol{A}) : t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G})$, $\mathsf{Rev}_{id}(\boldsymbol{y}) = r_{id} \sum_{t\in\mathcal{T}} \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \psi^t_{id}(\boldsymbol{A})$.

Lemma D.3 *For each $\boldsymbol{\mathcal{B}} \in \mathcal{P}$ and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ that is a feasible solution to the linear program Z^{LP},*

$$Z^1(\boldsymbol{\mathcal{B}}, \bar{\boldsymbol{x}}) \;\geq\; \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \mathsf{Rev}_{id}(\bar{\boldsymbol{y}}) \;.$$

Proof: Fix $\boldsymbol{\mathcal{B}} \in \mathcal{P}$ and a feasible solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ to the linear program Z^{LP}. By definition of Z^{LP}, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ satisfies the following constraints:

$$\sum_{\tau=1}^{t} \sum_{\boldsymbol{A}\in\mathcal{G}} \bar{y}^{\tau}(\boldsymbol{A}) \psi_{id}^{\tau}(\boldsymbol{A}) \alpha_{id}(t-\tau) \;\leq\; C_{id}\bar{x}_i^d \qquad \forall\, i\in\mathcal{N},\; d\in\mathcal{D},\; t\in\mathcal{T},$$

$$\sum_{\boldsymbol{A}\in\mathcal{G}} \bar{y}^t(\boldsymbol{A}) \;\leq\; 1 \qquad \forall\, t\in\mathcal{T}\;,$$

and it thus follows immediately from the definition that $\bar{\boldsymbol{y}}$ is also a feasible solution to the linear program $Z^1(\boldsymbol{\mathcal{B}}, \bar{\boldsymbol{x}})$. This implies that

$$Z^1(\boldsymbol{\mathcal{B}}, \bar{\boldsymbol{x}}) \;\geq\; \sum_{t\in\mathcal{T}} \sum_{\boldsymbol{A}\in\mathcal{G}} \bar{y}^t(\boldsymbol{A}) \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \psi_{id}^t(\boldsymbol{A}) r_{id} \;\;=\;\; \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \mathsf{Rev}_{id}(\bar{\boldsymbol{y}}) \;,$$

which is the desired result. ■

D.2. Proof of Theorem 4.2

For each $i \in \mathcal{N}$, let $\mathcal{B}_i = \{d \in \mathcal{D}_i : x_{id}^H = 1\}$. Note that $\boldsymbol{\mathcal{B}} \in \mathcal{P}$. By Lemma D.3,

$$\begin{aligned} Z^1(\boldsymbol{\mathcal{B}}, \boldsymbol{x}^{\mathsf{LP}}) \;\geq\; \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{B}_i} \mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right) \;&\overset{(a)}{=}\; \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} x_{id}^H \mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right) \;\overset{(b)}{=}\; \sum_{i\in\mathcal{N}} \max_{d\in\mathcal{D}_i} \mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right) \\ \overset{(c)}{\geq}\; \sum_{i\in\mathcal{N}} \frac{1}{D_i} \sum_{d\in\mathcal{D}_i} \mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right) \;&\overset{(d)}{\geq}\; \frac{1}{D} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right) \;\overset{(e)}{=}\; \frac{1}{D} Z^{\mathsf{LP}} \;\overset{(f)}{\geq}\; \frac{1}{D} \max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{UB}(\boldsymbol{x}) \;, \end{aligned}$$

where (a) follows the definition of $\boldsymbol{\mathcal{B}}$ and (b) follows from our construction of x_{id}^H where $x_{id}^H = 1$ if and only if $\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right) \geq \max_{h\neq d} \mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right)$. The inequality (c) follows because the maximum is at least as large as the average. The inequality d follows becuase D is defined as the maximum value of all D_i. The equality (e) is from the fact that $(\boldsymbol{x}^{\mathsf{LP}}, \boldsymbol{y}^{\mathsf{LP}})$ is an optimal solution to Z^{LP}, so $Z^{\mathsf{LP}} = \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{LP}}\right)$. The final inequality (f) follows because

$$Z^{\mathsf{LP}} \;=\; \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{LP}}} \mathsf{VB}(\boldsymbol{x}) \;\geq\; \max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{VB}(\boldsymbol{x}) \;=\; \max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{UB}(\boldsymbol{x}) \;,$$

where the last equality follows from Theorem 3.5. Because $\mathsf{UB}(\boldsymbol{x}^H) \geq Z^1(\boldsymbol{\mathcal{B}}, \boldsymbol{x}^{\mathsf{LP}})$ by Lemma D.2, it follows that $\mathsf{UB}(\boldsymbol{x}^H) \geq \frac{1}{D} \max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{UB}(\boldsymbol{x})$, which is the desired result.

D.3. Proof of Theorem 4.3

In Appendix F.1, we show that the CDLP has an sale-based formulation when the choice model is the MNL model and $\mathcal{F} = 2^{\mathcal{N}}$. If we further assume that $\alpha_{id}(\ell) = 1$ for all $i \in \mathcal{N}$, $d \in \mathcal{D}_i$ and $\ell \in \mathcal{T}$, and that the customers are homogeneous, one can simplify $W^{\mathsf{LP}}(\boldsymbol{x})$ and recover essentially the classical sales-based LP form in Gallego et al. (2015). In this case, since the customers will never return the purchased items before the ending of the selling season, we might just treat the problem as a classical dynamic assortment problem. The next result formalizes this observation.

Lemma D.4 (Sales-based LP for MNL II) *If $f_{id}(\ell)=0$ for all $\ell \le T$, $v_{id}^t = v_{id}$ for some $v_{id} > 0$ for all $i \in \mathcal{N}$, $d \in \mathcal{D}_i$ and $t \in \mathcal{T}$, and $v_0^t = 1$ for all $t \in \mathcal{T}$, then $\mathsf{VB}(\boldsymbol{x}) = W^{\mathsf{LP}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}^{\mathsf{LP}}$, where*

$$
\begin{aligned}
W^{\mathsf{LP}}(\boldsymbol{x}) = \max_{\boldsymbol{w}} \quad & T \cdot \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} r_{id} w_{id} \\
\text{s.t.} \quad & w_{id} \le \frac{C_{id}}{T} \cdot x_{id} \quad \textit{and} \quad w_{id} \le v_{id} w_0 \quad \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, \\
& w_0 + \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} w_{id} \le 1 \\
& w_0 \ge 0, \quad w_{id} \ge 0 \qquad \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i.
\end{aligned}
$$

<u>*Constructing an ϵ-Instance*</u>: Consider an arbitrary fixed small number $\epsilon > 0$. We assume without loss of generality that $1/\epsilon$ is an integer. In the next, we show that we can construct an ϵ-instance such that its integrality gap, defined by

$$
\max_{\boldsymbol{x}\in\mathcal{X}} W^{\mathsf{LP}}(\boldsymbol{x}) \Big/ \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{LP}}} W^{\mathsf{LP}}(\boldsymbol{x})
$$

is at most $\frac{1}{D} \cdot \frac{1+D\epsilon}{1+\epsilon}$. The details are as follows. First, we let all products have the same number of designs, i.e., $\mathcal{D}_i = \mathcal{D} = \{1, 2, ..., D\}$ for all $i \in \mathcal{N}$. We let the number of time periods be $T = 1/\epsilon + D$. We assume that $\mathcal{N} = \{1\}$, so from now on we suppress the index of the product i. We assume that $v_d^t = \epsilon$ for all $d \in \mathcal{D}$ and $t \in \mathcal{T}$, and that $v_0^t = 1$ for all $t \in \mathcal{T}$. Therefore, Lemma D.4 applies. We also let the capacity $C_d = D$ and revenue $r_{id} = 1$ for all $d \in \mathcal{D}$. Consequently, we have

$$
\begin{aligned}
W^{\mathsf{LP}}(\boldsymbol{x}) = \max_{\boldsymbol{w}} \quad & \left(\frac{1}{\epsilon} + D\right) \cdot \sum_{d\in\mathcal{D}} w_d \\
\text{s.t.} \quad & w_d \le \frac{D\epsilon}{1+D\epsilon} \cdot x_d \quad \text{and} \quad w_d \le \epsilon w_0 \qquad \forall\, d \in \mathcal{D}, t \in \mathcal{T}, \\
& w_0 + \sum_{d\in\mathcal{D}} w_d \le 1 \\
& w_0 \ge 0, \; w_d \ge 0 \qquad \forall\, d \in \mathcal{D}.
\end{aligned}
$$

We prove Theorem 4.3 in the following. We consider a fractional design vector $\widetilde{\boldsymbol{x}}$ such that $\widetilde{x}_d = 1/D$ for all $i \in \mathcal{N}$ and $d \in \mathcal{D}$ and construct a feasible solution to $W^{\mathsf{LP}}(\widetilde{\boldsymbol{x}})$ by setting

$$
\widetilde{w}_0 = \frac{1}{1+D\epsilon} \quad \text{and} \quad \widetilde{w}^d = \frac{\epsilon}{1+D\epsilon} \;\; \forall d \in \mathcal{D}.
$$

One can verify the feasibility of $\widetilde{\boldsymbol{w}}$ and obtain that

$$
\max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{LP}}} W^{\mathsf{LP}}(\boldsymbol{x}) \ge W^{\mathsf{LP}}(\widetilde{\boldsymbol{x}}) \ge \left(\frac{1}{\epsilon} + D\right) \cdot \sum_{d\in\mathcal{D}} \widetilde{w}^d = \left(\frac{1}{\epsilon} + D\right) \cdot \frac{D\epsilon}{1+D\epsilon}. \tag{D.1}
$$

Now consider $\max_{\boldsymbol{x}\in\mathcal{X}} W^{\mathsf{LP}}(\boldsymbol{x})$. By symmetry, we can just let $\hat{x}_1=1$ and $\hat{x}_d=0$ for all $d\neq 1$ and $d\in\mathcal{D}$ to construct an optimal design $\hat{\boldsymbol{x}}$. Therefore, the problem is equivalent to

$$\begin{aligned} W^{\mathsf{LP}}(\hat{\boldsymbol{x}}) = \max_{\boldsymbol{w}} \quad & \left(\frac{1}{\epsilon}+D\right)\cdot w_1 \\ \text{s.t.} \quad & w_1 \le \frac{D\epsilon}{1+D\epsilon} \quad \text{and} \quad w_1 \le \epsilon w_0, \\ & w_0+w_1 \le 1 \quad \text{and} \quad w_0, w_1 \ge 0. \end{aligned}$$

Note that the constraints $w_1\le \epsilon w_0$ and $w_0+w_1\le 1$ implies $(1+1/\epsilon)\,w_1\le w_0+w_1\le 1$, or equivalently $w_1\le \epsilon/(1+\epsilon)$. So,

$$W^{\mathsf{LP}}(\hat{\boldsymbol{x}}) \le \left(\frac{1}{\epsilon}+D\right)\cdot\frac{\epsilon}{1+\epsilon}.$$

By exactly setting $\hat{w}_1=\epsilon/(1+\epsilon)$ and $\hat{w}_0=1/(1+\epsilon)$, we can verify that this upper bound is obtained and all constraints are met. Therefore,

$$\max_{\boldsymbol{x}\in\mathcal{X}} W^{\mathsf{LP}}(\boldsymbol{x}) = \left(\frac{1}{\epsilon}+D\right)\cdot\frac{\epsilon}{1+\epsilon} \quad \text{and} \quad \max_{\boldsymbol{x}\in\mathcal{X}} W^{\mathsf{LP}}(\boldsymbol{x}) \Big/ \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{LP}}} W^{\mathsf{LP}}(\boldsymbol{x}) \le \frac{1}{D}\cdot\frac{1+D\epsilon}{1+\epsilon},$$

where the last inequality follows from Eq.(D.1). So the ϵ-instance indeed has the claimed integrality gap. Also, we notice that

$$\lim_{\epsilon\to 0}\ \frac{1}{D}\cdot\frac{1+D\epsilon}{1+\epsilon} = \frac{1}{D}.$$

Therefore, the performance guarantee based on rounding the continuous relaxation is tight.

D.4. Proof of Theorem 4.4

For each $i\in\mathcal{N}$, let $\mathcal{B}_i=\{d\in\mathcal{D}_i : \hat{x}_{id}=1\}$. Note that $\boldsymbol{\mathcal{B}}\in\mathcal{P}$. Because $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ is feasible to the linear program Z^{LP}, it follows from Lemma D.3 and the same argument as in the proof of Theorem 4.2 that

$$\begin{aligned} Z^1(\boldsymbol{\mathcal{B}},\bar{\boldsymbol{x}}) \;&\ge\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\mathsf{Rev}_{id}\left(\bar{\boldsymbol{y}}\right) \;=\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} x^H_{id}\mathsf{Rev}_{id}\left(\bar{\boldsymbol{y}}\right) \;=\; \sum_{i\in\mathcal{N}}\max_{d\in\mathcal{D}_i}\ \mathsf{Rev}_{id}\left(\bar{\boldsymbol{y}}\right) \\ &\ge\; \frac{1}{D}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\mathsf{Rev}_{id}\left(\bar{\boldsymbol{y}}\right) \;\overset{(a)}{\ge}\; \frac{1}{D}(1-\delta)Z^{\mathsf{LP}} \;\ge\; \frac{1}{D}(1-\delta)\max_{\boldsymbol{x}\in\mathcal{X}}\mathsf{UB}(\boldsymbol{x})\ , \end{aligned}$$

where (a) follows from the theorem's hypothesis that $(\bar{\boldsymbol{x}}\,,\,\bar{\boldsymbol{y}})$ is a feasible solution whose objective value is at least $(1-\delta)Z^{\mathsf{LP}}$, so $\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\mathsf{Rev}_{id}\left(\bar{\boldsymbol{y}}\right) \;\ge\; (1-\delta)Z^{\mathsf{LP}}$. The final inequality is because Z^{LP} is an upper bound on the maximum value of the upper bound. The desired result follows because Because $\mathsf{UB}(\,\hat{\boldsymbol{x}}\,)\ge Z^1(\boldsymbol{\mathcal{B}},\bar{\boldsymbol{x}})$ by Theorem D.2.

Appendix E: Details of Partial Enumeration

In this part, we discuss the partial enumeration algorithm discussed after Theorem 4.3. Let $\mathcal{X}_{k^-}$ denote the set of all vectors of the form $\boldsymbol{x}_{k^-} = (x_{id} : i = 1, \ldots, k-1, d \in \mathcal{D}_i)$ and let $\mathcal{X}_{k^+}$ be all vectors $\boldsymbol{x}_{k^+} = (x_{id} : i = k, \ldots, m, d \in \mathcal{D}_i)$ for each $k = 2, \ldots, n$. Then, for each fixed $\boldsymbol{x}_{k^-} \in \mathcal{X}_{k^-}$, we use revenue-based rounding to find an approximate solution $\boldsymbol{x}_{k^+}(\boldsymbol{x}_{k^-})$ to the problem $\max_{\boldsymbol{x}_{k^+} \in \mathcal{X}_{k^+}} \mathsf{VB}\big((\boldsymbol{x}_{k^-}, \boldsymbol{x}_{k^+})\big)$, by treating the number of designs as <u>*one*</u> for each product i with $i = 1, \ldots, k$. Finally, return $\boldsymbol{x}^{H,k} = (\boldsymbol{x}^H_{k^-}, \boldsymbol{x}_{k^+}(\boldsymbol{x}^H_{k^-}))$, where

$$\boldsymbol{x}^H_{k^-} = \operatorname*{arg\,max}_{\boldsymbol{x}_{k^-} \in \mathcal{X}_{k^-}} \mathsf{VB}\big(\,(\boldsymbol{x}_{k^-}, \boldsymbol{x}_{k^+}(\boldsymbol{x}_{k^-}))\,\big).$$

The performance guarantee of this algorithm is summarized in the next result. It shows that if $D_k \ll D$, we can achieve a significantly improved performance bound. This technique can be advantageous if $\prod_{\ell=1}^{k-1} D_\ell$ is of a reasonable scale, because linear programs are usually easy to solve. We remark on the performance guarantee of this algorithm in the next result.

Corollary E.1 (Partial Enumeration) *For each $k = 2, \ldots, n$, we can find an offline design $\boldsymbol{x}^{H,k}$ such that $\mathsf{Eff}_{\text{off}}(\boldsymbol{x}^{H,k}) \geq \frac{1}{D_k}$ with $\prod_{\ell=1}^{k-1} D_\ell$ iterations of the revenue-based rounding algorithm.*

Proof: For any $k = 2, \ldots, n$, clearly we have in total $\prod_{\ell=1}^{k-1} D_\ell$ runs of the revenue-based rounding procedure in the algorithm described above. By the same arguments leading to Theorem 4.2,

$$\mathsf{VB}\big(\,(\boldsymbol{x}_{k^-}, \boldsymbol{x}_{k^+}(\boldsymbol{x}_{k^-}))\,\big) \;\geq\; \frac{1}{D_k} \max_{\boldsymbol{x}_{k^+} \in \mathcal{X}_{k^+}} \mathsf{VB}\big((\boldsymbol{x}_{k^-}, \boldsymbol{x}_{k^+})\big),$$

for each $\boldsymbol{x}_{k^-} \in \mathcal{X}_{k^-}$, because the number of designs is bounded by D_k in the revenue-based rounding. Then, it holds that

$$\mathsf{VB}(\boldsymbol{x}^{H,k}) = \max_{\boldsymbol{x}_{k^-} \in \mathcal{X}_{k^-}} \mathsf{VB}\big(\,(\boldsymbol{x}_{k^-}, \boldsymbol{x}_{k^+}(\boldsymbol{x}_{k^-}))\,\big) \geq \frac{1}{D_k} \max_{\boldsymbol{x}_{k^-} \in \mathcal{X}_{k^-}} \max_{\boldsymbol{x}_{k^+} \in \mathcal{X}_{k^+}} \mathsf{VB}\big((\boldsymbol{x}_{k^-}, \boldsymbol{x}_{k^+})\big) = \frac{1}{D_k} \max_{\boldsymbol{x} \in \mathcal{X}} \mathsf{VB}(\boldsymbol{x}),$$

because $\mathcal{X} = \mathcal{X}_{k^-} \times \mathcal{X}_{k^+}$. Then, by Theorem 3.5, we have $\mathsf{UB}(\boldsymbol{x}^{H,k}) \geq \frac{1}{D_k} \max_{\boldsymbol{x} \in \mathcal{X}} \mathsf{UB}(\boldsymbol{x})$, which implies $\mathsf{Eff}_{\text{off}}(\boldsymbol{x}^{H,k}) \geq \frac{1}{D_k}$. ■

Appendix F: Solving the Continuous Relaxation

In this section, we focus on solving

$$\begin{aligned}
Z^{\mathsf{LP}} = \max_{\boldsymbol{x}, \boldsymbol{y}} \quad & \sum_{t \in \mathcal{T}} \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) r_{id} && \text{(F.1)} \\
\text{s.t.} \quad & \sum_{\tau=1}^{t} \sum_{\boldsymbol{A} \in \mathcal{G}} y^\tau(\boldsymbol{A}) \psi^\tau_{id}(\boldsymbol{A}) \alpha_{id}(t-\tau) \;\leq\; C_{id} x_{id} && \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T} \\
& \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \;\leq\; 1 && \forall\, t \in \mathcal{T}, \\
& \sum_{d \in \mathcal{D}_i} x_{id} = 1 && \forall i \in \mathcal{N}, \\
& x_{id} \geq 0, \quad y^t(\boldsymbol{A}) \geq 0, && \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G}\;.
\end{aligned}$$

In Appendix F.1, we show how to utilize the ellipsoid method to solve (F.1), if the corresponding static assortment admits an exact solution In Appendix F.2, we extend this discussion to more complex choice models, provided that there is an FPTAS available. In Appendix F.3, we show that Z^{LP} admits a more compact formulation that facilitates the computation when the underlying choice model is given by the MNL model, the generalized attraction model (GAM), and the Markov chain choice model (MCCM).

F.1. Efficient Methods for Solving the Continuous Relaxation

To solve the (F.1), we associate the dual variables $\boldsymbol{\theta}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\eta}$ with the first, second, and third sets of constraints of the program, respectively. By strong duality, the dual is given by

$$\begin{aligned} Z^{\mathsf{LP}} \;=\; \min_{\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\eta}} \quad & \sum_{i\in\mathcal{N}} \eta_i \;+\; \sum_{t\in\mathcal{T}} \lambda^t & & \text{(F.2)} \\ \text{s.t.} \quad & \eta_i \;\geq\; \sum_{t\in\mathcal{T}} C_i^d \theta_{id}^t & & \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, \\ & \lambda^t \;\geq\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} \psi_{i^d}^t(\boldsymbol{A})\Big(r_{id} - \sum_{\tau=1}^{t} \alpha_{id}(\tau-t)\theta_{id}^{\tau}\Big) & & \forall\, t\in\mathcal{T}, \boldsymbol{A}\in\mathcal{G}, \\ & \theta_{id}^t \geq 0, \quad \lambda^t \geq 0 & & \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}. \end{aligned}$$

The dual has $TL+T+n$ decision variables, but exponentially many constraints. It is well-known that a linear program with exponentially many constraints can still be solved in polynomial time, provided that there is a "separation oracle" that can determine in polynomial time whether an arbitrary vector $(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\eta})\in\mathbb{R}^{TL+T+n}$ is feasible and, if not, find a violated constraint.

Definition F.1 (Complexity of Static Assortment Optimization) *A choice function ψ defined on the ground set $\mathcal{M}$ admits an efficient static assortment optimization if for each $\boldsymbol{\mu}\in\mathbb{R}^L$, we can solve exactly the following optimization problem*

$$\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i^d\in\mathcal{M}} \psi_{i^d}(\boldsymbol{A})\mu_{id} \tag{F.3}$$

in time that is polynomial in $L=O(nD)$ and the input size.

The choice functions that admit an efficient static assortment optimization abound. Indeed, a fundamental building block of our joint optimization method is how to solve such static assortment optimization (also see Section 5.2). Our discussion assumes that there are practical algorithms for such problems, given the substantial literature on this topic. To wit, since the publication of seminal work of Talluri and van Ryzin (2004) and Gallego et al. (2004), researchers have developed methods for static assortment optimization under various choice models, such as the nested logit (Davis et al. 2014, Feldman and Topaloglu 2015), Markov chain choice model (Désir et al. 2020), locational choice model (Gaur and Honhon 2006), and mixture of MNL model (Désir and Goyal 2014). Other authors expand on this topic by incorporating practical considerations, including production cost (Kunnumkal and de Albéniz 2019), operating with omnichannel (Dzyabura and Jagabathula 2018, Lo and Topaloglu 2021), or customer utility tradeoffs (Sumida et al. 2021).

If the static assortment optimization problem can be solved exactly in polynomial time, we can also compute Z^{LP} efficiently. The proof of the next result follows from classical results in linear program on the connection between separation and optimization.

Theorem F.2 (Exact Solution of Continuous Relaxation) *Under Assumption 3.2, if the choice function ψ^t admits an efficient static assortment optimization for all $t \in \mathcal{T}$, then Z^{LP} can be computed exactly in polynomial time.*

Proof: Under the hypothesis, for each $(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\eta}) \in \mathbb{R}^{TL+T+n}$, checking whether $(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\eta})$ is a feasible solution of the dual linear program in Eq. (F.2) can be done in polynomial time. Indeed, checking for the non-negativity constraint and the first set of constraints ($\eta_i \geq \sum_{t \in \mathcal{T}} C_{id} \theta^t_{id} \ \ \forall\, i \in \mathcal{N}, \forall d \in \mathcal{D}_i$) is easy. To check the second set of constraints, for each $t \in \mathcal{T}$, we solve the following static assortment optimization problem:

$$\max_{\boldsymbol{A} \in \mathcal{G}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) \Big(r_{id} - \sum_{\tau=1}^{t} \alpha_{id}(\tau - t) \theta^\tau_{id} \Big),$$

which can be done in polynomial time by the hypothesis. The second set of constraints is satisfied if and only if λ^t is at least as large as the above maximum value for all $t \in \mathcal{T}$. Moreover, clearly, if $(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\eta})$ is not feasible, we can also identify the violated constraint.

Because we can check for feasibility and identify a violated constraint in polynomial time, the desired result then follows immediately from the classical result in linear program that shows the connection between separation and linear optimization (Grötschel et al. 1981). ■

Lastly, we note that when $\phi^t(S \mid \boldsymbol{x})$ admits certain choices models such as the MNL model, GAM, and the MCCM, and $\mathcal{F}$ consists of all subsets of $\mathcal{N}$, we can prove stronger results. In these cases, the linear program in Eq. (F.1) admits a compact representation with $O(nDT)$ decision variables and constraints. Then, we can use the compact representation to derive constant approximate solution (ones that do not depend on the maximum number of designs) for $\max_{\boldsymbol{x} \in \mathcal{X}} \mathsf{UB}(\boldsymbol{x})$ that holds in specific settings. See relevant discussions in Appendix F.3.

F.2. Extension to more complex choice models

We note that Theorem F.2 enables us to solve the linear programming relaxation Z^{LP} for a broad class of choice models. However, there are situations in which it is NP-hard to solve the static assortment optimization problem exactly. Examples include a knapsack constraint on the assortments, such as $\mathcal{F} = \{S \subseteq \mathcal{N} : \sum_{i \in S} c_i \leq B\}$, under the MNL model, or the mixture of MNLs. For instance, under the mixture of MNLs, it is well-kown that if the number of customer types, say K, is fixed, the static assortment problem admits an FPTAS, even though generally there does not exist any polynomial-time algorithm to obtain an approximation for mixture of MNLs better than $O(1/K^{1-\delta})$, where $\delta > 0$ is arbitrary (Désir and Goyal 2014). For these examples, we only have a fully polynomial-time approximation scheme (FPTAS) for the static assortment problem; that is, for each $\epsilon \in (0,1]$, we can compute an assortment whose objective value is at least $(1-\epsilon)$ of the optimal value and the running time is polynomial in n, D, $1/\epsilon$, and the input size (Désir and Goyal 2014). When a choice model admits an FPTAS for the static assortment optimization problem, we can show that the linear program Z^{LP} also admits an FPTAS and construct in polynomial time a feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ whose objective value is at least $(1-\epsilon)Z^{\mathsf{LP}}$. It then follows from Theorem 4.4 that we can determine the designs whose revenue is at least $\frac{1}{D}(1-\epsilon)$ of the maximum upper bound for these choice models. To flesh out the analysis, we introduce the following definition.

Definition F.3 A choice function ψ defined on the ground set $\mathcal{M}$ admits a *FPTAS for static assortment optimization* if for each $0 < \epsilon < 1$ and $(\mu_{id}, : i \in \mathcal{N}, d \in \mathcal{D}_i) \in \mathbb{R}^L$, we can construct an assortment $\boldsymbol{A}(\epsilon) \in \mathcal{G}$ in time that is polynomial in n, D, $1/\epsilon$, and the input size such that

$$\sum_{i^d \in \mathcal{M}} \psi_{i^d}\big(\boldsymbol{A}(\epsilon)\big)\mu_{id} \;\geq\; (1-\epsilon) \max_{\boldsymbol{A} \in \mathcal{G}} \sum_{i^d \in \mathcal{M}} \psi_{i^d}(\boldsymbol{A})\mu_{id}.$$

In this section, we make the following assumption.

Assumption F.4 *The functions $\psi^1, \ldots, \psi^T$ admit a FPTAS for static assortment optimization.*

It is known that often one can approximately solve a linear program with an exponential number of decision variables using an FPTAS as a separation oracle in the dual space. For example, one can refer to Karmarkar and Karp (1982) for such an algorithm for the bin-packing problem. However, the construction details differ, depending on the problems at hand. To be self-contained, we show in this section that this can be done for our problem as well. The main result of this section is stated in the following theorem. Recall that $D = \max_{i \in N} D_i$.

Theorem F.5 *Suppose that $C_{id} \geq D$ for all $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$. Under Assumptions 3.2 and F.4, for each $0 < \epsilon < 1/2$, there is an algorithm that computes a feasible solution to Z^{LP} with an objective function value at least $(1-2\epsilon)Z^{\mathsf{LP}}$, and the algorithm's running time is polynomial in n, D, $1/\epsilon$, and the input size.*

We devote the rest of this section to the proof of Theorem F.5, which makes use of the following lemmas. The first lemma establishes upper and lower bounds on the optimal objective value. By

Assumption F.4, for each $t \in \mathcal{T}$ and $0 < \epsilon < 1$, we can construct in polynomial time $\boldsymbol{A}^t(\epsilon) \in \mathcal{G}$ such that

$$\sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}\big(\boldsymbol{A}^t(\epsilon)\big)\, r_{id} \;\geq\; (1-\epsilon)\; \max_{\boldsymbol{A} \in \mathcal{G}} \; \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}\big(\boldsymbol{A}\big)\, r_{id} \; .$$

Define

$$M_\epsilon \;=\; \frac{1}{1-\epsilon} \times \max_{t \in \mathcal{T}} \; \Big\{ \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}\big(\boldsymbol{A}^t(\epsilon)\big)\, r_{id} \Big\}.$$

Lemma F.6 *For each $0 < \epsilon < 1$, $(1-\epsilon) M_\epsilon \;\leq\; Z^{\mathsf{LP}} \;\leq\; T\, M_\epsilon$ and for each $\boldsymbol{\theta} \geq \boldsymbol{0}$ and $t \in \mathcal{T}$,*

$$\max_{\boldsymbol{A} \in \mathcal{G}} \; \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) \left(\frac{r_{id}}{M_\epsilon} - \sum_{\tau=t}^{T} \theta^\tau_{id} \alpha_{id}(\tau - t) \right) \;\leq\; 1 \, .$$

Proof: Note that for each $\boldsymbol{\theta} \geq \boldsymbol{0}$ and $t \in \mathcal{T}$,

$$\max_{\boldsymbol{A} \in \mathcal{G}} \; \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) \left(\frac{r_{id}}{M_\epsilon} - \sum_{\tau=t}^{T} \theta^\tau_{id} \alpha_{id}(\tau - t) \right) \;\leq\; \max_{\boldsymbol{A} \in \mathcal{G}} \; \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) \frac{r_{id}}{M_\epsilon} \;\leq\; 1 \; ,$$

where the last inequality follows from the definition of M_ϵ.

To establish a lower bound on Z^{LP}, suppose that the maximum in the definition of M_ϵ occurs at $\tau \in \mathcal{T}$. Setting $x^d_i = 1/D_i$ for all i and $y^\tau(\boldsymbol{A}^\tau(\epsilon)) = 1$ and all other $y^t(\boldsymbol{A}) = 0$. Because $C_{id} \geq D \geq D_i$ for all i and d, this is a feasible solution to Z^{LP} and its objective value is $(1-\epsilon) M_\epsilon$, giving us the lower bound. To establish an upper bound on Z^{LP}, it follows from Eq. (F.1) that

$$Z^{\mathsf{LP}} \;\leq\; \sum_{t \in \mathcal{T}} \max_{\boldsymbol{A} \in \mathcal{G}} \; \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}\big(\boldsymbol{A}\big)\, r_{id} \;\leq\; T M_\epsilon \; ,$$

where the last inequality follows from the definition of M_ϵ. ■

We will consider a "scaled" problem where the revenue of every product design is multiplied by $1/M_\epsilon$; that is,

$$\begin{aligned}
\frac{Z^{\mathsf{LP}}}{M_\epsilon} \;=\; & \max_{\boldsymbol{x} \in \mathcal{X}, \boldsymbol{y} \geq \boldsymbol{0}} && \sum_{t \in \mathcal{T}} \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \left(\psi^t_{id}(\boldsymbol{A}) \frac{r_{id}}{M_\epsilon} \right) && \text{(P-scaled)} \\
& \text{s.t.} && \sum_{\tau=1}^{t} \sum_{\boldsymbol{A} \in \mathcal{G}} y^\tau(\boldsymbol{A}) \psi^\tau_{id}(\boldsymbol{A}) \alpha_{id}(t-\tau) \;\leq\; C_{id} x^d_i && \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \\
& && \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \;\leq\; 1 && \forall\, t \in \mathcal{T}, \\
=\; & \min_{(\boldsymbol{\theta}, \boldsymbol{\lambda}) \geq \boldsymbol{0}, \boldsymbol{\eta}} && \sum_{t \in \mathcal{T}} \lambda^t \;+\; \sum_{i \in \mathcal{N}} \eta_i && \text{(D-Scaled)} \\
& \text{s.t.} && \eta_i \;\geq\; \sum_{t \in \mathcal{T}} C_{id} \theta^t_{id} && \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, \\
& && \lambda^t \;\geq\; \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) \left(\frac{r_{id}}{M_\epsilon} - \sum_{\tau=1}^{t} \alpha_{id}(\tau - t) \theta^\tau_{id} \right) && \forall\, t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G}.
\end{aligned}$$

Note that for the D-Scaled optimization problem, it follows from Lemma F.6 that we can add the following redundant constraints as crude upper bounds:

$$\lambda^t \le 2 \quad \text{and} \quad \theta_{id}^t \le 2T \quad \text{and} \quad \eta_i \le 2T \qquad \forall\, t \in \mathcal{T}, i \in \mathcal{N}, d \in \mathcal{D}_i \ .$$

This ensures that dual variables are always bounded. Also, by Lemma F.6, these inequalities are never tight in any optimal solution. The next lemma shows that for the D-Scaled optimization problem, we can construct a weak separation oracle.

Lemma F.7 (Weak Separation) *The* D-Scaled *optimization problem admits a weak separation oracle; that is, for each* $(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\eta}) \in \mathbb{R}^{(Dn+1)T+n}$ *and* $\epsilon > 0$, *we can find in polynomial-time either*
(i) a vector $\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\eta}}\right)$ *that is feasible to the problem in Eq.* (D-Scaled) *with*

$$\sum_{i \in \mathcal{N}} \hat{\eta}_i \ + \ \sum_{t=1}^{T} \hat{\lambda}^t \ \le \ \sum_{i \in N} \eta_i \ + \ \sum_{t \in \mathcal{T}} \lambda^t \ + \ \epsilon \ ,$$

(ii) or a constraint that is violated by $(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\eta})$.

Proof: Given any $(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\eta})$, we first check if the constraints

$$\eta_i \ \ge \ \sum_{t \in \mathcal{T}} C_{id} \theta_{id}^t \quad \forall i \in \mathcal{N}, d \in \mathcal{D}_i \ \text{ and } \ \lambda^t \ \le \ 2, \ \theta_{id}^t \ \le \ 2T, \ \eta_i \ \le \ 2T \quad \forall\, t \in \mathcal{T}, \ i \in \mathcal{N}, \ d \in \mathcal{D}_i$$

are violated. If any of the constraints is violated, we are done. If none of the above constraints are violated, we focus on the remaining constraints. For ease of exposition, define a function over assortments as follow:

$$h^t(\boldsymbol{A} \,|\, \boldsymbol{\theta}) \ = \ \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \left(\frac{r_{id}}{M_\epsilon} - \sum_{\tau=1}^{t} \alpha_{id}(\tau - t) \theta_{id}^\tau \right)$$

so the second set of constraints in D-Scaled corresponds to $\lambda^t \ge \max_{\boldsymbol{A} \in \mathcal{G}} h^t(\boldsymbol{A} \,|\, \boldsymbol{\theta}) \ \ \forall\, t \in \mathcal{T}$.

Under Assumption F.4, for each $t \in \mathcal{T}$, we construct an assortment $\hat{\boldsymbol{A}}^t \in \mathcal{G}$ such that

$$\max_{\boldsymbol{A} \in \mathcal{G}} h^t(\boldsymbol{A} \,|\, \boldsymbol{\theta}) \ \ge \ h^t(\hat{\boldsymbol{A}}^t \,|\, \boldsymbol{\theta}) \ \ge \ \left(1 - \frac{\epsilon}{3T}\right) \max_{\boldsymbol{A} \in \mathcal{G}} h^t(\boldsymbol{A} \,|\, \boldsymbol{\theta}) \ .$$

If $\lambda^q < h^q(\hat{\boldsymbol{A}}^q \,|\, \boldsymbol{\theta})$ for some q, then we have found a violated constraint and we are done.

Otherwise, it must be the case that $\lambda^t \ge h^t(\hat{\boldsymbol{A}}^t \,|\, \boldsymbol{\theta})$ for all t. Let $\hat{\lambda}^t = \lambda^t / \left(1 - \frac{\epsilon}{3T}\right)$. Then, for all $t \in \mathcal{T}$

$$\hat{\lambda}^t \ = \ \frac{\lambda^t}{1 - \frac{\epsilon}{3T}} \ge \frac{1}{1 - \frac{\epsilon}{3T}} \, h^t(\hat{\boldsymbol{A}}^t \,|\, \boldsymbol{\theta}) \ \ge \ \max_{\boldsymbol{A} \in \mathcal{G}} h^t(\boldsymbol{A} \,|\, \boldsymbol{\theta}),$$

where the last inequality follows from the definition of $\hat{\boldsymbol{A}}^t$. Then we know that $(\boldsymbol{\theta}, \hat{\boldsymbol{\lambda}}, \boldsymbol{\eta})$ is feasible to the D-Scaled optimization problem. Furthermore, it also holds that

$$\sum_{t \in \mathcal{T}} \hat{\lambda}^t \ = \ \sum_{t \in \mathcal{T}} \frac{\lambda^t}{1 - \frac{\epsilon}{3T}} \ = \ \sum_{t \in \mathcal{T}} \left(1 \ + \ \frac{\epsilon/(3T)}{1 - \frac{\epsilon}{3T}}\right) \lambda^t \ \le \ \left(\sum_{t \in \mathcal{T}} \lambda^t\right) + \frac{2\epsilon/3}{1 - \frac{\epsilon}{3}} \ \le \ \left(\sum_{t \in \mathcal{T}} \lambda^t\right) \ + \ \epsilon \ ,$$

where the second-to-last inequality follows from the fact that $\lambda^t \le 2$ for all t and $T \ge 1$, and the final inequality follows because $\frac{2\epsilon/3}{1-(\epsilon/3)} \le \epsilon$. This completes the proof. ■

We are now ready to prove Theorem F.5.

Proof of Theorem F.5: With the result of Lemma F.7, it follows from Theorem 4.19 in Korte and Vygen (2011) that we can use the ellipsoid method to solve the scaled optimization problem in D-Scaled up to an *additive error* ϵ, in time that is polynomial in the input size and $1/\epsilon$. In other words, the algorithm terminates with $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\eta}})$ that is feasible to D-Scaled and satisfies

$$\sum_{i\in\mathcal{N}} \tilde{\eta}_i + \sum_{t=1}^{T} \tilde{\lambda}^t \ \le\ \frac{Z^{\mathsf{LP}}}{M_\epsilon} + \epsilon \ .$$

The algorithm makes a polynomial number of calls of the separation oracle and therefore identifies at most a polynomial number of violated constraints. We denote this set of constraints by Π.

Then, we can solve a relaxed problem, in which we replace the second set of constraints in D-Scaled with a "smaller" number of constraints of the form:

$$\lambda^t \ \ge\ \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \left(\frac{r_{id}}{M_\epsilon} - \sum_{\tau=1}^{t} \alpha_{id}(\tau - t)\theta_{id}^{\tau} \right) \quad \forall\ (t, \boldsymbol{A}) \in \Pi \ .$$

That is, we consider the following optimization problem:

$$\begin{aligned}
\widetilde{W} \ =\ \min_{(\boldsymbol{\theta},\boldsymbol{\lambda})\ge \mathbf{0},\boldsymbol{\eta}} \quad & \sum_{i\in\mathcal{N}} \eta_i \ +\ \sum_{t\in\mathcal{T}} \lambda^t && \text{(D-Scaled-Small)} \\
\text{s.t.} \quad & \eta_i \ \ge\ \sum_{t\in\mathcal{T}} C_{id}\theta_{id}^t && \forall\ i\in\mathcal{N}, d\in\mathcal{D}_i \\
& \lambda^t \ \ge\ \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \left(\frac{r_{id}}{M_\epsilon} - \sum_{\tau=1}^{t} \alpha_{id}(\tau - t)\theta_{id}^{\tau} \right) && \forall\ (t, \boldsymbol{A}) \in \Pi.
\end{aligned}$$

An important observation at this point is that, if one solves both D-Scaled and D-Scaled-Small problems using the ellipsoid method, it is impossible to distinguish which problem is being solved.[3] Therefore, the **same** solution $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\eta}})$ is returned for D-Scaled-Small, and thus,

$$\sum_{i\in\mathcal{N}} \tilde{\eta}_i + \sum_{t\in\mathcal{T}} \tilde{\lambda}^t - \epsilon \ \overset{(a)}{\le}\ \widetilde{W} \ \overset{(b)}{\le}\ \frac{Z^{\mathsf{LP}}}{M_\epsilon} \ \le\ \sum_{i\in\mathcal{N}} \tilde{\eta}_i + \sum_{t\in\mathcal{T}} \tilde{\lambda}^t,$$

where (a) follows since $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\eta}})$ also weakly optimizes D-Scaled-Small. The inequality (b) is because D-Scaled-Small is a relaxation of D-Scaled. The last inequality follows because $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\eta}})$ is feasible for D-Scaled.

This implies that $\widetilde{W} \le \frac{Z^{\mathsf{LP}}}{M_\epsilon} \le \widetilde{W} + \epsilon$. The dual of D-Scaled-Small is a restricted version of P-scaled in the sense that we set $y^t(\boldsymbol{A}) = 0$ for $(t, \boldsymbol{A}) \notin \Pi$. Also, we note that in any optimal solution to

[3] Technically, this requires us to start with the same initial ellipsoid when solving the D-Scaled and the D-Scaled-Small problems. One can construct this ellipsoid using the redundant box constraints we add to D-Scaled.

this restricted linear program, the variables corresponding to the constraints $\lambda_t \le 2$, $\theta_{id}^t \le 2T$, and $\eta_i \le 2T$ must all be zero because of complementary slackness and the fact that these constraints are never tight.

Therefore, if we solve the restricted linear program (aka the dual of D-Scaled-Small) optimally, which clearly can be done in a polynomial running time, we obtain a feasible solution $\left\{\tilde{x}_i^d,\, \tilde{y}^t(\boldsymbol{A}) : i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G}\right\}$ to P-scaled whose objective function value is equal to $\widetilde{W}$ by the strong duality of linear program.

Next, we notice that we have scaled the coefficients in the objective function by $1/M_\epsilon$. This implies for the original problem in Eq. (F.1), the objective value of $\left\{\tilde{x}_i^d,\, \tilde{y}^t(\boldsymbol{A}) : i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G}\right\}$ is equal to

$$M_\epsilon \widetilde{W} \overset{(a)}{\ge} M_\epsilon \left(\frac{Z^{\mathsf{LP}}}{M_\epsilon} - \epsilon\right) = Z^{\mathsf{LP}} - \epsilon M_\epsilon \overset{(b)}{\ge} Z^{\mathsf{LP}} - \epsilon \frac{Z^{\mathsf{LP}}}{1-\epsilon} = \frac{1-2\epsilon}{1-\epsilon} \times Z^{\mathsf{LP}} \overset{(c)}{\ge} (1-2\epsilon) Z^{\mathsf{LP}} ,$$

where (a) follows from the fact that $\widetilde{W} \le \frac{Z^{\mathsf{LP}}}{M_\epsilon} \le \widetilde{W} + \epsilon$, (b) is from Lemma F.6, and (c) is from the fact that $0 < \epsilon < 1/2$. This completes the proof. ■

F.3. Sales-based Formulation of CDLP

In this section, we discuss how to reformulate the CDLP into a compact form under the MNL model, GAM, and MCCM. In particular, this reformulation helps us to design an alternative approximation algorithm to compute a heuristic offline decision in Appendix H.1 and facilitates the computational experiments in Section 6.

Multinomial logit (MNL) model: When $\phi^t(S \,|\, \boldsymbol{x})$ is a MNL model with preference weights given by $\{v_{id}^t : i \in \mathcal{N}, d \in \mathcal{D}_i\}$, as in Example 3.3, and $\mathcal{F}$ consists of all subsets of $\mathcal{N}$, we can prove a stronger result. In this case, the linear program in Eq. (F.1) admits a compact representation with $O(nDT)$ decision variables and constraints, as shown in the next. For each $\boldsymbol{x} \in [0,1]^L$, let $W^{\mathsf{LP}}(\boldsymbol{x})$ be defined as follows:

$$\begin{aligned}
W^{\mathsf{LP}}(\boldsymbol{x}) = \max_{\boldsymbol{w}} \quad & \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} r_{id} w_{id}^t \\
\text{s.t.} \quad & \sum_{\tau=1}^{t} \alpha_{id}(t-\tau) w_{id}^\tau \le C_{id} x_{id} \qquad \text{and} \qquad w_{id}^t \le v_{id}^t w_0^t \qquad \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \\
& w_0^t + \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} w_{id}^t \le 1 \qquad \forall\, t \in \mathcal{T}, \\
& w_0^t \ge 0,\ w_{id}^t \ge 0, \qquad \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}.
\end{aligned}$$

The following result is a generalization of the well-known sales-based LP for the MNL choice model (Gallego et al. 2015), featuring additional decision variables that control each product's design decision. In a similar vein, we can also represent the CDLP as a sales-based LP under the generalized attraction model (Gallego et al. 2015) and the Markov chain choice model (Blanchet et al. 2016, Feldman and Topaloglu 2017, Désir et al. 2020), as will be discussed later. As a result, off-the-shelf solvers can be used to implement our revenue-based rounding algorithm under these choice models. We take advantage of this in our numerical experiments in Section 6.

Theorem F.8 (Sales-based LP for MNL) *If the customer in period t follows a MNL choice model with parameters $\left(v_{id}^t : i \in \mathcal{N}, d \in \mathcal{D}_i\right)$ with $v_0^t = 1$ and there is no constraint on the assortment, then $\mathsf{VB}(\boldsymbol{x}) = W^{\mathsf{LP}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in [0,1]^L$.*

Proof: Fix an arbitrary $\boldsymbol{x} \in [0,1]^L$. We will first show that $\mathsf{VB}(\boldsymbol{x}) \leq W^{\mathsf{LP}}(\boldsymbol{x})$. Let $\{\bar{y}^t(\boldsymbol{A}) : t \in \mathcal{T}, i \in \mathcal{N}, \boldsymbol{A} \in \mathcal{G}\}$ denote an optimal solution for $\mathsf{VB}(\boldsymbol{x})$. For each $t \in \mathcal{T}$, $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$, let $\bar{w}_{id}^t = \sum_{\boldsymbol{A} \in \mathcal{G}} \bar{y}^t(\boldsymbol{A}) \psi_{id}^t(\boldsymbol{A})$, and let $\bar{w}_0^t = \sum_{\boldsymbol{A} \in \mathcal{G}} \bar{y}^t(\boldsymbol{A}) \psi_0^t(\boldsymbol{A})$, where under the MNL model, $\psi_0^t(\boldsymbol{A}) = 1/\left(1 + \sum_{i^d \in \boldsymbol{A}} v_{id}^t\right)$. Note that by our construction

$$\sum_{t \in \mathcal{T}} \sum_{\boldsymbol{A} \in \mathcal{G}} \bar{y}^t(\boldsymbol{A}) \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) r_{id} \;=\; \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} r_{id} \bar{w}_{id}^t.$$

To prove that $\mathsf{VB}(\boldsymbol{x}) \leq W^{\mathsf{LP}}(\boldsymbol{x})$, it suffices to show that $\bar{\boldsymbol{w}}$ is a feasible solution to the linear program associated with $W^{\mathsf{LP}}(\boldsymbol{x})$. Clearly, $\bar{\boldsymbol{w}} \geq 0$. Furthermore, the first constraint of $W^{\mathsf{LP}}(\boldsymbol{x})$ follow immediately from the first set of constraints of $\mathsf{VB}(\boldsymbol{x})$ in (3.2). For all $i \in \mathcal{N}, d \in \mathcal{D}_i$ and $t \in \mathcal{T}$, by the definition of the MNL,

$$\frac{\bar{w}_{id}^t}{v_{id}^t} \;=\; \frac{1}{v_{id}^t} \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \frac{v_{id}^t \cdot \mathbb{1}_{\{i^d \in \boldsymbol{A}\}}}{1 + \sum_{j^h \in \boldsymbol{A}} v_{jh}^t} \;=\; \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \frac{\mathbb{1}_{\{i^d \in \boldsymbol{A}\}}}{1 + \sum_{j^h \in \boldsymbol{A}} v_{jh}^t} \;\leq\; \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \psi_0^t(\boldsymbol{A}) \;=\; \bar{w}_0^t\,.$$

This proves the second constraint. The last constraint of W^{LP} follows because $\bar{w}_0^t + \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \bar{w}_{id}^t = \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \leq 1$, by the last constraint for $\mathsf{VB}(\boldsymbol{x})$. Thus, $\mathsf{VB}(\boldsymbol{x}) \leq W^{\mathsf{LP}}(\boldsymbol{x})$.

To establish the reversed inequality, we will consider the dual linear programs of $\mathsf{VB}(\boldsymbol{x})$ and $W^{\mathsf{LP}}(\boldsymbol{x})$. By strong duality of linear program,

$$\begin{aligned}
\mathsf{VB}(\boldsymbol{x}) \;=\; & \min_{(\boldsymbol{\theta},\boldsymbol{\lambda}) \geq \mathbf{0}} && \sum_{t \in \mathcal{T}} \lambda^t \;+\; \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} C_{id} x_{id} \theta_{id}^t \\
& \text{s.t.} && \lambda^t \;\geq\; \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \left(r_{id} - \sum_{\tau=1}^{t} \alpha_{id}(\tau - t) \theta_{id}^\tau \right) && \forall\, t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G}. \\
W^{\mathsf{LP}}(\boldsymbol{x}) \;=\; & \min_{(\boldsymbol{\gamma},\boldsymbol{\sigma},\boldsymbol{u}) \geq \mathbf{0}} && \sum_{t \in \mathcal{T}} u^t \;+\; \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} C_{id} x_{id} \gamma_{id}^t \\
& \text{s.t.} && u^t - \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} v_{id}^t \sigma_{id}^t \;\geq\; 0 && \forall\, t \in \mathcal{T}, \\
& && u^t + \sigma_{id}^t + \sum_{\tau=t}^{T} \alpha_{id}(\tau - t) \gamma_{id}^t \geq\; r_{id} && \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}.
\end{aligned}$$

Let $(\bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\lambda}})$ denote an optimal solution to the dual linear program associated with $\mathsf{VB}(\boldsymbol{x})$. For each $t \in \mathcal{T}$, $i \in \mathcal{N}$, and $d \in \mathcal{D}_i$, define $\bar{u}^t = \bar{\lambda}^t$, $\bar{\gamma}_{id}^t = \bar{\theta}_{id}^t$, $\bar{\sigma}_{id}^t = \left[r_{id} - \bar{\lambda}^t - \sum_{\tau=t}^T \alpha_{id}(\tau - t)\bar{\theta}_{id}^t \right]^+$. By our construction, $\mathsf{VB}(\boldsymbol{x}) = \sum_{t\in\mathcal{T}} \bar{u}^t + \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} C_{id} x_{id} \bar{\gamma}_{id}^t$. To show that $\mathsf{VB}(\boldsymbol{x}) \geq W^{\mathsf{LP}}(\boldsymbol{x})$, it suffices to show that $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{u}})$ is a feasible solution for the dual linear program of W^{LP}. The last set of constraints of W^{LP} is clearly satisfied by our construction of $\bar{\sigma}_{id}^t$. Thus, it suffices to check the first constraint. Let $P^t = \left\{ i^d : r_{id} - \bar{\lambda}^t - \sum_{\tau=t}^T \alpha_{id}(\tau - t)\bar{\theta}_{id}^t \geq 0 \right\}$. Then, by definition of $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{u}})$ and collecting terms,

$$
\begin{aligned}
\bar{u}^t \geq \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} v_{id}^t \bar{\sigma}_{id}^t \;\Leftrightarrow\;& \bar{\lambda}^t \;\geq\; \sum_{i^d \in P^t} v_{id}^t \Big(r_{id} - \bar{\lambda}^t - \sum_{\tau=t}^T \alpha_{id}(\tau - t)\bar{\gamma}_{id}^t \Big) \\
\Leftrightarrow\;& \Big(1 + \sum_{i^d\in P^t} v_{id}^t \Big)\bar{\lambda}^t \;\geq\; \sum_{i^d\in P^t} v_{id}^t \Big(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau - t)\bar{\gamma}_{id}^t \Big) \\
\overset{(a)}{\Leftrightarrow}\;& \bar{\lambda}^t \;\geq\; \sum_{i^d\in P^t} \psi_{id}^t(P^t) \Big(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau - t)\bar{\gamma}_{id}^t \Big) \\
\overset{(b)}{\Leftrightarrow}\;& \bar{\lambda}^t \;\geq\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} \psi_{id}^t(P^t) \Big(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau - t)\bar{\gamma}_{id}^t \Big) \;,
\end{aligned}
$$

where (a) follows from the definition of the MNL choice model and (b) follows because $\psi_{ih}^t(P^t) = 0$ for all $i^h \notin P^t$. Because $\bar{\lambda}^t$ is a feasible solution to the dual of $\mathsf{VB}(\boldsymbol{x})$, it follows that $\bar{\lambda}^t \geq \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) \Big(r_{id} - \sum_{\tau=1}^t \alpha_{id}(\tau - t)\bar{\theta}_{id}^\tau \Big)$. Since $P^t \in \mathcal{G}$, which consists of all sets of products, it follows that $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{u}})$ satisfies the first set of constraint of W^{LP}. Therefore, $\mathsf{VB}(\boldsymbol{x}) \geq W^{\mathsf{LP}}(\boldsymbol{x})$. ■

<u>*Generalized attraction model* (GAM)</u>: Let us fix $t \in \mathcal{T}$. Under the GAM, for each product $i \in \mathcal{N}$, the parameters v_{id}^t denote the preference weights of design d of product i if these designs are offered to a customer. Under GAM, we also use ξ_{id}^t to denote the "shadow" preference weights of the D_i designs of product i if they are **<u>not</u>** offered to a customer. The shadow preference weights capture the possibility that, even if the firm does not offer a particular design of a product in its store, a customer may search it somewhere else. We require that $0 \leq \xi_{id}^t \leq v_{id}^t$ for all $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$. Then, for each assortment $S \subseteq \mathcal{N}$ and $i \in S$,

$$
\phi_i^t(S \,|\, \boldsymbol{x}) \;=\; \frac{\sum_{d\in\mathcal{D}_i} v_{id}^t x_{id}}{1 \;+\; \sum_{j\in\mathcal{N}\setminus S}\sum_{h\in\mathcal{D}_j} \xi_{jh}^t \;+\; \sum_{j\in S}\sum_{h\in\mathcal{D}_j} \xi_{jh}^t(1 - x_{jh}) \;+\; \sum_{j\in S}\sum_{h\in\mathcal{D}_j} v_{jh}^t x_{jh}} \;,
$$

where in the denominator, the expression $\sum_{j\in\mathcal{N}\setminus S}\sum_{h\in\mathcal{D}_j} \xi_{jh}^t$ capture the shadow weights of all designs of all the products not in S. The second term in the denominator, $\sum_{j\in S}\sum_{h\in\mathcal{D}_j} \xi_{jh}^t(1 - x_{jh})$ capture the shadow weight of the designs of the products in S that are not offered. The final term $\sum_{j\in S}\sum_{h\in\mathcal{D}_j} v_{jh}^t x_{jh}$ reflects the preference weight of all the designs shown to the customer in the assortment S. Note that $\phi_0^t(S\,|\,\boldsymbol{x}) = 1 - \sum_{i\in S}\phi_i^t(S\,|\,\boldsymbol{x})$. Note that here we will allow for heterogeneous customers and the customers' choice behavior is characterized by the set of parameters $\{v_{id}^t, \xi_{id}^t : i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}\}$.

Then, for all $\boldsymbol{A}\subset\mathcal{M}$ and $i^d\in\mathcal{M}$

$$\psi_{i^d}^t(\boldsymbol{A}) \;=\; \frac{v_{id}^t\mathbb{1}_{\{i^d\in\boldsymbol{A}\}}}{1+\sum_{j^h\in\mathcal{M}\setminus\boldsymbol{A}}\xi_{jh}^t + \sum_{j^h\in\boldsymbol{A}} v_{jh}^t}. \tag{F.4}$$

It is easy to check that

$$\phi_i^t(S\,|\,\boldsymbol{x}) \;=\; \sum_{d\in\mathcal{D}_i}\psi_{i^d}^t(\boldsymbol{A}(S,\boldsymbol{x}))x_{id}.$$

We will show that under GAM, the upper bound admits a compact sales-based representation. To facilitate our exposition, let $\tilde{v}_0^t = 1+\sum_{j\in\mathcal{N}}\sum_{d\in\mathcal{D}_j}\xi_{jh}^t$ and for all $i\in\mathcal{N}$, $d\in\mathcal{D}_i$, $\tilde{v}_{id}^t = v_{id}^t - \xi_{id}^t$. Then,

$$1+\sum_{j^h\in\mathcal{M}\setminus\boldsymbol{A}}\xi_{jh}^t + \sum_{j^h\in\boldsymbol{A}}v_{jh}^t \;=\; \tilde{v}_0^t + \sum_{j^h\in\boldsymbol{A}}\tilde{v}_{jh}^t \quad\text{and}\quad \psi_{i^d}^t(\boldsymbol{A}) \;=\; \frac{v_{id}^t\mathbb{1}_{\{i^d\in\boldsymbol{A}\}}}{\tilde{v}_0^t+\sum_{j^h\in\boldsymbol{A}}\tilde{v}_{jh}^t}\;\forall i\in\mathcal{N},\; d\in\mathcal{D}_i\;. \tag{F.5}$$

For each $\boldsymbol{x}\in[0,1]^L$, let $W^{\mathsf{LP}}(\boldsymbol{x})$ be defined as

$$\begin{aligned}
W^{\mathsf{LP,GAM}}(\boldsymbol{x}) = \max_{\boldsymbol{w}} \quad & \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} r_{id}w_{id}^t \\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\alpha_{id}(t-\tau)w_{id}^\tau \;\le\; C_{id}x_{id} \qquad \text{and} \qquad w_{id}^t \;\le\; v_{id}^t w_0^t \quad \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}, \\
& \tilde{v}_0^t w_0^t + \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\left(\frac{\tilde{v}_{id}^t}{v_{id}^t}w_{id}^t\right) \;\le\; 1 \qquad \forall\, t\in\mathcal{T}, \\
& w_0^t\ge 0,\; w_{id}^t\ge 0, \qquad \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}.
\end{aligned}$$

The next result follows from a proof similar to that of Theorem F.8.

Theorem F.9 (Sales-based LP for GAM) *If the customer in period t follows the GAM with parameters $\{v_{id}^t, \xi_{id}^t : i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}\}$ and there is no constraint on the assortment, then* $\mathsf{VB}(\boldsymbol{x}) = W^{\mathsf{LP,GAM}}(\boldsymbol{x})$ *for all $\boldsymbol{x}\in[0,1]^L$.*

Proof: Fix $\boldsymbol{x}\in[0,1]^L$. We will first show that $\mathsf{VB}(\boldsymbol{x}) \le W^{\mathsf{LP,GAM}}(\boldsymbol{x})$. Let $\{\bar{y}^t(\boldsymbol{A}) : t\in\mathcal{T}, i\in\mathcal{N}, \boldsymbol{A}\in\mathcal{G}\}$ denote an optimal solution for $\mathsf{VB}(\boldsymbol{x})$. For each $t\in\mathcal{T}$, $i\in\mathcal{N}$ and $d\in\mathcal{D}_i$, let $\bar{w}_{id}^t \;=\; \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\psi_{i^d}^t(\boldsymbol{A})$ and let

$$\bar{w}_0^t = \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\frac{1}{1+\sum_{i^d\in\mathcal{M}\setminus\boldsymbol{A}}\xi_{id}^t+\sum_{i^d\in\boldsymbol{A}}v_{id}^t} = \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\frac{1}{\tilde{v}_0^t+\sum_{i^d\in\boldsymbol{A}}\tilde{v}_{id}^t}.$$

Note that $\bar{w}_0^t \ne \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\psi_0^t(\boldsymbol{A})$ here unlike under the MNL model. Further, under GAM, it follows from (F.5) that

$$\psi_0^t(\boldsymbol{A}) = 1-\sum_{i^d\in\boldsymbol{A}}\psi_{i^d}^t(\boldsymbol{A}) \;=\; \frac{1+\sum_{i^d\in\mathcal{M}\setminus\boldsymbol{A}}\xi_{id}^t}{1+\sum_{i^d\in\mathcal{M}\setminus\boldsymbol{A}}\xi_{id}^t+\sum_{i^d\in\boldsymbol{A}}v_{id}^t} \;=\; \frac{\tilde{v}_0^t}{\tilde{v}_0^t+\sum_{i^d\in\boldsymbol{A}}\tilde{v}_{id}^t}\;.$$

Thus,

$$\sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\left(\psi^t_{i^d}(\boldsymbol{A})r_{id}\right) \;=\; \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} r_{id}\bar{w}^t_{id}\ .$$

Thus, to prove that $\mathsf{VB}(\boldsymbol{x}) \le W^{\mathsf{LP}}(\boldsymbol{x})$, it suffices to show that $\{\bar{w}^t_0,\ \bar{w}^t_{id},: t\in\mathcal{T}, i\in\mathcal{N}, d\in\mathcal{D}_i\}$ defined above is a feasible solution to the linear program associated with $W^{\mathsf{LP}}(\boldsymbol{x})$. The capacity constraints of $W^{\mathsf{LP}}(\boldsymbol{x})$ follow immediately from the capacity constraints of $\mathsf{VB}(\boldsymbol{x})$ in (3.2). For all i and t, by the definition of $\bar{w}^t_{id}$ and (F.5),

$$\frac{\bar{w}^t_{id}}{v^t_{id}} \;=\; \frac{1}{v^t_{id}}\sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\psi^t_{i^d}(\boldsymbol{A}) \;=\; \frac{1}{v^t_{id}}\sum_{\boldsymbol{A}\in\mathcal{G}}y^t(\boldsymbol{A})\frac{v^t_{id}\mathbb{1}_{\{i^d\in\boldsymbol{A}\}}}{\tilde{v}^t_0+\sum_{j^h\in\boldsymbol{A}}\tilde{v}^t_{jh}} \;\le\; \sum_{\boldsymbol{A}\in\mathcal{G}}y^t(\boldsymbol{A})\frac{1}{\tilde{v}^t_0+\sum_{j^h\in\boldsymbol{A}}\tilde{v}^t_{jh}} \;=\; \bar{w}^t_0\,.$$

To establish the last constraint of $W^{\mathsf{LP}}(\boldsymbol{x})$, it follows from our construction that

$$\begin{aligned}
\tilde{v}^t_0\bar{w}^t_0+\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\left(\frac{\tilde{v}^t_{id}}{v^t_{id}}\bar{w}^t_{id}\right) &\overset{(a)}{=} \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\times\left\{\frac{\tilde{v}^t_0}{\tilde{v}^t_0+\sum_{j^h\in\boldsymbol{A}}\tilde{v}^t_{jh}} \;+\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\frac{\tilde{v}^t_{id}}{v^t_{id}}\cdot\frac{v^t_{id}\mathbb{1}_{\{i^d\in\boldsymbol{A}\}}}{\tilde{v}^t_0+\sum_{j^h\in\boldsymbol{A}}\tilde{v}^t_{jh}}\right\}\\
&= \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\times\frac{\tilde{v}^t_0+\sum_{i^d\in\boldsymbol{A}}\tilde{v}^t_{id}}{\tilde{v}^t_0+\sum_{j^h\in\boldsymbol{A}}\tilde{v}^t_{jh}}\\
&= \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A}) \;\overset{(b)}{\le}\; 1\ ,
\end{aligned}$$

where (a) follows from the definition of $\bar{w}^t_0$ and $\bar{w}^t_{id}$ and (F.5). Inequality (b) follows from the last inequality of $\mathsf{VB}(\boldsymbol{x})$, and thus, $\mathsf{VB}(\boldsymbol{x}) \le W^{\mathsf{LP,GAM}}(\boldsymbol{x})$.

To establish the reverse inequality, we will consider the dual linear programs of $\mathsf{VB}(\boldsymbol{x})$ and $W^{\mathsf{LP,GAM}}(\boldsymbol{x})$. By strong duality,

$$\begin{aligned}
\mathsf{VB}(\boldsymbol{x}) \;=\; &\min_{(\boldsymbol{\theta},\boldsymbol{\lambda})\ge\boldsymbol{0}} \quad \sum_{t\in\mathcal{T}}\lambda^t \;+\; \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}C_{id}x_{id}\theta^t_{id}\\
&\quad\text{s.t.}\quad \lambda^t \;\ge\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\psi^t_{i^d}(\boldsymbol{A})\left(r_{id}-\sum_{\tau=1}^{t}\alpha_{id}(\tau-t)\theta^\tau_{id}\right) \qquad \forall\ t\in\mathcal{T}, \boldsymbol{A}\in\mathcal{G},
\end{aligned}$$

and

$$\begin{aligned}
W^{\mathsf{LP}}(\boldsymbol{x}) \;=\; &\min_{(\boldsymbol{\gamma},\boldsymbol{\sigma},\boldsymbol{u})\ge\boldsymbol{0}} \quad \sum_{t\in\mathcal{T}}u^t \;+\; \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}C_{id}x_{id}\gamma^t_{id}\\
&\quad\text{s.t.}\quad \tilde{v}^t_0u^t-\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}v^t_{id}\sigma^t_{id} \;\ge\; 0 \qquad\qquad t\in\mathcal{T},\\
&\qquad\quad\ \frac{\tilde{v}^t_{id}}{v^t_{id}}u^t+\sigma^t_{id}+\sum_{\tau=t}^{T}\alpha_{id}(\tau-t)\gamma^t_{id} \;\ge\; r_{id} \qquad\qquad i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}.
\end{aligned}$$

Let $\left(\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\lambda}}\right)$ denote an optimal solution to the dual linear program associated with $\mathsf{VB}(\boldsymbol{x})$. For each $t\in\mathcal{T}$ and $i\in\mathcal{N}$, define

$$\bar{u}^t=\bar{\lambda}^t,\ \bar{\gamma}^t_{id}=\bar{\theta}^t_{id},\ \bar{\sigma}^t_{id}=\left[r_{id}-\frac{\tilde{v}^t_{id}}{v^t_{id}}\bar{\lambda}^t-\sum_{\tau=t}^{T}\alpha_{id}(\tau-t)\bar{\theta}^t_{id}\right]^+.$$

By our construction, $\mathsf{VB}(\boldsymbol{x}) = \sum_{t\in\mathcal{T}} \bar{u}^t + \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} C_{id}x_{id}\bar{\gamma}^t_{id}$. To show that $\mathsf{VB}(\boldsymbol{x}) \geq W^{\mathsf{LP}}(\boldsymbol{x})$, it suffices to show that $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{u}})$ is a feasible solution for the dual linear program of $W^{\mathsf{LP}}(\boldsymbol{x})$. The last constraint of W^{LP} are clearly satisfied by our construction of $\bar{\sigma}^t_{id}$. Thus, it suffices to check the first constraint. Let $P^t = \left\{ i^d \in \mathcal{M} : r_{id} - \frac{\tilde{v}^t_{id}}{v^t_{id}}\bar{\lambda}^t - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\bar{\theta}^t_{id} \geq 0 \right\}$. Then, by definition of $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{u}})$ and collecting terms,

$$
\begin{aligned}
\tilde{v}^t_0\bar{u}^t \geq \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}(v^t_{id}\bar{\sigma}^t_{id}) \quad &\Leftrightarrow \quad \tilde{v}^t_0\bar{\lambda}^t \;\geq\; \sum_{i^d\in P^t} v_{id}\left(r_{id} - \frac{\tilde{v}^t_{id}}{v^t_{id}}\bar{\lambda}^t - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\bar{\gamma}^t_{id}\right) \\
&\Leftrightarrow \quad \Big(\tilde{v}^t_0 + \sum_{i^d\in P^t}\tilde{v}^t_{id}\Big)\bar{\lambda}^t \;\geq\; \sum_{i^d\in P^t} v_{id}\Big(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\bar{\gamma}^t_{id}\Big) \\
&\overset{(a)}{\Leftrightarrow} \quad \bar{\lambda}^t \;\geq\; \sum_{i^d\in P^t}\psi^t_{id}(P^t)\Big(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\bar{\gamma}^t_{id}\Big) \\
&\overset{(b)}{\Leftrightarrow} \quad \bar{\lambda}^t \;\geq\; \sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{N}}\psi^t_{id}(P^t)\Big(r_{id} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\bar{\gamma}^t_{id}\Big)\ ,
\end{aligned}
$$

where (a) follows from the definition of the choice probabilities under GAM given in Equation (F.5) and (b) follows because $\psi^t_{id}(P^t) = 0$ for all $i^d \notin P^{d,t}$. Because $\bar{\lambda}^t$ is a feasible solution to the (dual) of $\mathsf{VB}(\boldsymbol{x})$, it follows that

$$
\bar{\lambda}^t \;\geq\; \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\psi^t_{id}(\boldsymbol{A})\left(r_{id} - \sum_{\tau=1}^t \alpha_{id}(\tau-t)\bar{\theta}^\tau_{id}\right).
$$

Since $P^t \in \mathcal{G}$, it follows that $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{u}})$ satisfies the first constraint of $W^{\mathsf{LP}}(x)$. Therefore, $\mathsf{VB}(\boldsymbol{x}) \geq W^{\mathsf{LP,GAM}}(\boldsymbol{x})$, which is the desired result. ■

Markov chain choice model (MCCM): We next extend the sales-based LP to the MCCM. Our discussion on the MCCM closely follows that in Feldman and Topaloglu (2017). Let us fixed $t \in \mathcal{T}$. Under the MCCM, we have parameters $\{\lambda^t_{id} : i \in \mathcal{N}, d \in \mathcal{D}_i\}$ where $\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\lambda^t_{id} \leq 1$. Particularly, in period t, the customer arrives to purchase version d of product i with probability λ^t_{id}. In addition, we have a transition matrix $\left(\rho^t_{idjh} : j^d, i^h \in \mathcal{M}\right)$ where ρ^t_{idjh} denotes the probability that the t^{th} customer transition to design h of item j, given that design d of item i is not available. As shown in Feldman and Topaloglu (2017), for each assortment $\boldsymbol{A} \subseteq \mathcal{M}$, the probability $\psi^t_{id}(\boldsymbol{A})$ that a customer chooses item $i^d \in \boldsymbol{A}$ is given implicitly through the following system of linear equations:

$$
\psi^t_{id}(\boldsymbol{A}) + E^t_{id}(\boldsymbol{A}) = \lambda^t_{id} + \sum_{j^h\in\mathcal{M}} E^t_{jh}(\boldsymbol{A})\rho^t_{jhid} \quad \forall\ i^d \in \mathcal{M},
$$

and

$$
\psi^t_{jd}(\boldsymbol{A}) = 0 \quad \forall\ j^d \notin \boldsymbol{A}, \quad E^t_{jd}(\boldsymbol{A}) = 0 \quad \forall\ j^d \in \boldsymbol{A},
$$

where $\psi^t_{j^d}(\boldsymbol{A})$ can be interpreted as the expected number of times that a customer visits item j^d that is available for purchase and $E^t_{j^d}(\boldsymbol{A})$ is the expected number of times that a customer visits item j^d that is not available for purchase.

For each $\boldsymbol{x}\in[0,1]^L$, let $W^{\mathsf{LP,MCCM}}(\boldsymbol{x})$ be defined as follows:

$$\begin{aligned} W^{\mathsf{LP,MCCM}}(\boldsymbol{x}) = \max_{(\boldsymbol{w},\boldsymbol{z})\geq \boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} r_{id} w^t_{id} \\ \text{s.t.} \quad & \sum_{\tau=1}^{t} \alpha_{id}(t-\tau) w^{d,\tau}_j \ \leq \ C_{id}x_{id} && \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}, \\ & w^t_{id} + z^t_{id} \ = \ \lambda^t_{id} + \sum_{j\in\mathcal{N}}\sum_{h\in\mathcal{D}_j} z^t_{jh}\rho^t_{jhid} && \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}. \end{aligned}$$

The main result of this discussion is stated in the following theorem.

Theorem F.10 (Sales-based LP for MCCM) *If the customer in period t follows the MCCM and there is no constraint on the assortment, then $\mathsf{VB}(\boldsymbol{x}) = W^{\mathsf{LP,MCCM}}(\boldsymbol{x})$ for all $\boldsymbol{x}\in[0,1]^L$.*

Proof: Fix $\boldsymbol{x}\in[0,1]^L$. We will first show that $W^{\mathsf{LP,MCCM}}(\boldsymbol{x})\leq \mathsf{VB}(\boldsymbol{x})$. Let $(\hat{\boldsymbol{w}},\hat{\boldsymbol{z}})$ denote an optimal solution associated with $W^{\mathsf{LP,MCCM}}(\boldsymbol{x})$. For each $t\in\mathcal{T}$, let $\mathcal{H}^t$ be defined by:

$$\mathcal{H}^t = \left\{ (\boldsymbol{w},\boldsymbol{z})\in\mathbb{R}^{4n}_+ : w^t_{id} + z^t_{id} \ = \ \lambda^t_{id} + \sum_{j\in\mathcal{N}}\sum_{h\in\mathcal{D}_j} z^t_{jh}\rho^t_{jhid} \quad \forall\, t\in\mathcal{T}, j\in\mathcal{N}, d\in\mathcal{D}_j \right\} .$$

It follows from Lemma 10 in Feldman and Topaloglu (2017) that $\mathcal{H}^t$ is a bounded polytope, so there are extreme points $(\boldsymbol{w}^{t,1},\boldsymbol{z}^{t,1})$, $(\boldsymbol{w}^{t,2},\boldsymbol{z}^{t,2})$, ... $(\boldsymbol{w}^{t,K_t},\boldsymbol{z}^{t,K_t})$ of $\mathcal{H}^t$ such that for each $t\in\mathcal{T}$,

$$\hat{\boldsymbol{w}}^t = \sum_{k=1}^{K_t}\gamma^{t,k}\boldsymbol{w}^{t,k}, \qquad \hat{\boldsymbol{z}}^t = \sum_{k=1}^{K_t}\gamma^{t,k}\boldsymbol{z}^{t,k} \quad \text{and} \quad \sum_{k=1}^{K_t}\gamma^{t,k} = 1, \quad \gamma^{t,k}\geq 0 \ \ \forall \ \ t,k \ .$$

Define $\boldsymbol{A}^{t,k} = \{i\in\mathcal{N}, d\in\mathcal{D}_i : w^{t,k}_{id} > 0\}$. By Lemma 1 in Feldman and Topaloglu (2017), we know that

$$\psi^t_{i^d}(\boldsymbol{A}^{t,k}) = w^{t,k}_{id} \quad \text{and} \quad E^t_{j^d}(\boldsymbol{A}^{t,k}) = z^{t,k}_{id} \ , \tag{F.6}$$

and therefore for all $i^d\in\mathcal{M}$

$$\hat{w}^t_{id} = \sum_{k=1}^{K_t}\gamma^{t,k}w^{t,k}_{id} = \sum_{k=1}^{K_t}\gamma^{t,k}\psi^t_{i^d}(\boldsymbol{A}^{t,k}) \quad \text{and} \quad \hat{z}^t_{id} = \sum_{k=1}^{K_t}\gamma^{t,k}z^{t,k}_{id} = \sum_{k=1}^{K_t}\gamma^{t,k}E^t_{i^d}(\boldsymbol{A}^{t,k}) \ . \tag{F.7}$$

For each $t\in\mathcal{T}$ and $\boldsymbol{A}\subseteq\mathcal{M}$, define $\hat{y}^t(\boldsymbol{A})$ as follows:

$$\hat{y}^t(\boldsymbol{A}) = \begin{cases} \gamma^{t,k} & \text{if} \quad S = \boldsymbol{A}^{t,k} \\ 0 & \text{otherwise} \end{cases} .$$

We will show that $\hat{y}^t(\boldsymbol{A})$ is a feasible solution to $\mathsf{VB}(\boldsymbol{x})$. By definition of $\hat{\boldsymbol{y}}$, for any $i\in\mathcal{N}$ and $d\in\mathcal{D}_i$,

$$\sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^\tau(\boldsymbol{A})\psi^\tau_{jd}(\boldsymbol{A})\alpha_{id}(t-\tau) \;=\; \sum_{\tau=1}^{t}\sum_{k=1}^{K_\tau}\gamma^{\tau,k}\psi^\tau_{id}(\boldsymbol{A}^{k,\tau})\alpha_{jd}(t-\tau) \;\overset{(a)}{=}\; \sum_{\tau=1}^{t}\sum_{k=1}^{K_\tau}\gamma^{\tau,k}w^{\tau,k}_{jd}\alpha_{jd}(t-\tau)$$
$$\overset{(b)}{=}\; \sum_{\tau=1}^{t}\hat{w}^\tau_{id}\alpha_{id}(t-\tau) \;\overset{(b)}{\le}\; C_{jd}x_{id}\ ,$$

where the (a) and (b) follow from (F.6) and (F.7), respectively. The inequality (c) follows because $\hat{\boldsymbol{w}}$ is feasible for $W^{\mathsf{LP}}(\boldsymbol{x})$. This argument establishes the feasibility of $\hat{y}^t(S)$. Finally, note that

$$W^{\mathsf{LP}}(\boldsymbol{x}) = \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} r_{id}\hat{w}^t_{id} \;=\; \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} r_{id}\sum_{k=1}^{K_t}\gamma^{t,k}\psi^t_{i^d}(\boldsymbol{A}^{t,k}) \;=\; \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} r_{id}\sum_{\boldsymbol{A}\in\mathcal{G}}\hat{y}^t(\boldsymbol{A})\psi^t_{i^d}(\boldsymbol{A})\ ,$$

which shows that $W^{\mathsf{LP,MCCM}}(\boldsymbol{x})\le \mathsf{VB}(\boldsymbol{x})$.

We will now establish the reverse inequality. Let $\bar{\boldsymbol{y}}$ denote an optimal solution to $\mathsf{VB}(\boldsymbol{x})$. Without loss of generality, we can assume that for all $t\in\mathcal{T}$, $\sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})=1$ because we can always put weights on the empty set. Define $(\bar{\boldsymbol{w}},\bar{\boldsymbol{z}})$ as follow: for all $t\in\mathcal{T}$ and $i^d\in\mathcal{M}$,

$$\bar{w}^t_{id} = \sum_{\boldsymbol{A}\in\mathcal{G}}\psi^t_{i^d}(\boldsymbol{A})\bar{y}^t(\boldsymbol{A}) \quad\text{and}\quad \bar{z}^t_{id} = \sum_{\boldsymbol{A}\in\mathcal{G}}E^t_{i^d}(\boldsymbol{A})\bar{y}^t(\boldsymbol{A})\ .$$

We will now show that $(\bar{\boldsymbol{w}},\bar{\boldsymbol{z}})$ is a feasible solution to $W^{\mathsf{LP}}(\boldsymbol{x})$. By definition,

$$\sum_{\tau=1}^{t}\alpha_{id}(t-\tau)\bar{w}^\tau_{id} \;=\; \sum_{\tau=1}^{t}\alpha_{jd}(t-\tau)\sum_{\boldsymbol{A}\in\mathcal{G}}\psi^\tau_{i^d}(\boldsymbol{A})\bar{y}^\tau(\boldsymbol{A}) \;\le\; C_{id}x_{id}\ ,$$

where the inequality is from the feasibility of $\bar{\boldsymbol{y}}$. Moreover, by the property of the Markov chain choice model, we have that for all $\boldsymbol{A}\in\mathcal{G}$, $t\in\mathcal{T}$, and $i^d\in\mathcal{M}$, $\psi^t_{i^d}(\boldsymbol{A})+E^t_{i^d}(\boldsymbol{A})=\lambda^t_{id}+\sum_{j^h\in\mathcal{M}}R^t_{j^h}(\boldsymbol{A})\rho^t_{jhid}$. Multiplying both sides by $\bar{y}^t(\boldsymbol{A})$, summing over $\boldsymbol{A}$, and using the fact that $\sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})=1$, we get

$$\bar{w}^t_{id}+\bar{z}^t_{id} \;=\; \sum_{\boldsymbol{A}\in\mathcal{G}}\phi^t_{i^d}(\boldsymbol{A})\bar{y}^t(\boldsymbol{A})+\sum_{\boldsymbol{A}\in\mathcal{G}}E^t_{j^d}(\boldsymbol{A})\bar{y}^t(\boldsymbol{A}) \;=\; \lambda^t_{id}+\sum_{\boldsymbol{A}\in\mathcal{G}}\sum_{j^h\in\mathcal{M}}\bar{y}^t(\boldsymbol{A})E^t_{j^h}(\boldsymbol{A})\rho^t_{jhid}$$
$$=\; \lambda^t_{id}+\sum_{j^h\in\mathcal{M}}\bar{z}^t_{jh}\rho^t_{jhid}\ ,$$

which establishes the feasibility of $(\bar{\boldsymbol{w}},\bar{\boldsymbol{z}})$. Finally, note that

$$\mathsf{VB}(\boldsymbol{x}) = \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\psi^t_{i^d}(\boldsymbol{A})r_{id} = \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\bar{w}^t_{id}r_{id}\ ,$$

where the last equality follows from the definition of $\bar{w}^t_{id}$. This shows that $\mathsf{VB}(\boldsymbol{x})\le W^{\mathsf{LP,MCCM}}(\boldsymbol{x})$, which completes the proof. ■

Appendix G: Supporting Arguments for Section 5

In this section, we present supporting arguments for results in Section 5.

G.1. Derivation of the Dynamic Programming Formulation

In this part, we justify the dynamic programming formulation (5.1). Suppose that the assortment $S \subset \mathcal{N}$ is offered at period t. Provided that the customer does not make a purchase, which happens with probability $\left(1-\sum_{i\in\mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x})\right)$, clearly the next system state will be $\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})$. If the customer purchases item $i \in \mathcal{N}$, there are several cases to consider. First, the customer may decide to keep the item for at least one period, which happens with probability $\big(1-\gamma_{i0}(\boldsymbol{x})\big)$, the state will be $\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i1}$. Second, the customer may return the item immediately with probability $\gamma_{i0}(\boldsymbol{x})$. There are several scenarios afterward. With probability $h_i(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})$, the item will become immediately available, so the next system state is still $\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})$. With probability $h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)$, the item needs to be reprocessed for at least one period. Then the system state changes to $\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i,-1}$. Lastly, with probability $1-h_i(\boldsymbol{x})$, the item fails the inspection and the system states will be $\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0}$. Summarize all cases together, we have:

$$
\begin{aligned}
J^t(\boldsymbol{q}\,|\,\boldsymbol{x}) = \max_{S\in\mathcal{F}:\, q_{i0}\geq 1\ \forall i\in S} \Bigg\{ & \Big(1-\sum_{i\in\mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x})\Big)\,\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big] \\
& + \sum_{i\in\mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x}) \Big[r_i(\boldsymbol{x}) + \big(1-\gamma_{i0}(\boldsymbol{x})\big)\,\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i1} \;\big|\; \boldsymbol{x}\big)\Big] \\
& + \gamma_{i0}(\boldsymbol{x}) h_i(\boldsymbol{x}) \zeta_{i0}(\boldsymbol{x}) \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big] \\
& + \gamma_{i0}(\boldsymbol{x}) h_i(\boldsymbol{x}) \big(1-\zeta_{i0}(\boldsymbol{x})\big) \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i,-1} \;\big|\; \boldsymbol{x}\big)\Big] \\
& + \gamma_{i0}(\boldsymbol{x}) \big(1-h_i(\boldsymbol{x})\big) \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} \;\big|\; \boldsymbol{x}\big)\Big]\Big]\Bigg\}\,.
\end{aligned}
$$

Extracting the term $\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\,|\,\boldsymbol{x}\big)\Big]$ out, we obtain the formulation given by (5.1).

G.2. Derivation of Equation (5.6)

In this part, we justify (5.6). Suppose we are in state $\boldsymbol{q} \in \mathcal{Q}(\boldsymbol{x})$ and period $t \in \mathcal{T}$. As explained in Section 5.2, policy $\boldsymbol{\pi}^H$ chooses an assortment $G^t(\boldsymbol{q}\,|\,\boldsymbol{x})$ that maximizes the right-hand side of the dynamic programming equation (5.1), where we replace the optimal value function with our approximation. Therefore, by definition,

$$
\begin{aligned}
G^t(\boldsymbol{q}\,|\,\boldsymbol{x}) = \arg\max_{\substack{S\in\mathcal{F}:\\ q_{i0}\geq 1\\ \forall i\in S}} \Bigg\{ & \sum_{i\in\mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x}) \Big[r_i(\boldsymbol{x}) \\
& + \big(1-\gamma_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i1} \;\big|\; \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big]\Big) \\
& + \gamma_{i0}(\boldsymbol{x}) h_i(\boldsymbol{x}) \big(1-\zeta_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i,-1} | \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) | \boldsymbol{x}\big)\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \gamma_{i0}(\boldsymbol{x})\big(1-h_i(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})-\boldsymbol{e}_{i0}\ \big|\ \boldsymbol{x}\big)\Big]-\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\,\big|\,\boldsymbol{x}\big)\Big]\Big)\Big]\Big\} \\
\overset{(a)}{=}\ & \arg\max_{\substack{S\in\mathcal{F}:\\ q_{i0}\geq 1\\ \forall i\in S}} \Bigg\{\sum_{i\in\mathcal{N}}\phi_i^t(S\,|\,\boldsymbol{x})\Big[r_i(\boldsymbol{x})+\big(1-\gamma_{i0}(\boldsymbol{x})\big)\big(\beta_{i1}^{t+1}(\boldsymbol{x})-\beta_{i0}^{t+1}(\boldsymbol{x})\big) \\
&+\gamma_{i0}(\boldsymbol{x})h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\big(\beta_{i,-1}^{t+1}(\boldsymbol{x})-\beta_{i0}^{t+1}(\boldsymbol{x})\big)-\gamma_{i0}(\boldsymbol{x})\big(1-h_i(\boldsymbol{x})\big)\beta_{i0}^{t+1}(\boldsymbol{x}) \\
\overset{(b)}{=}\ & \arg\max_{\substack{S\in\mathcal{F}:\\ q_{i0}\geq 1\\ \forall i\in S}} \Bigg\{\sum_{i\in\mathcal{N}}\phi_i^t(S\,|\,\boldsymbol{x})\xi_i^t(\boldsymbol{x})\Bigg\}\,,
\end{aligned}
\tag{G.1}
$$

where (a) follows from the linearity of the approximate value function in (5.2). For example,

$$
\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})-\boldsymbol{e}_{i0}+\boldsymbol{e}_{i1}\ \big|\ \boldsymbol{x}\big)\Big]-\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\ \big|\ \boldsymbol{x}\big)\Big]\ =\ \beta_{i1}^{t+1}(\boldsymbol{x})-\beta_{i0}^{t+1}(\boldsymbol{x})
$$

and other terms can be derived similarly. Also, (b) follows from the definition of $\xi_i^t(\boldsymbol{x})$ in (5.3). This justifies (5.6), as desired.

G.3. Proof of Lemma 5.1

Let us fix a design $\boldsymbol{x}$ and a policy $\boldsymbol{\pi}$. Under model Υ, we let $X_{it}=1$ if a unit of item i is selected by the customer at time t and $X_{it}=0$ otherwise. We also let $Y_{it}=1$ if the unit of item i purchased by customer t is returned. When item i is not selected by the customer, Y_{it} can be arbitrarily defined. We define $Z_{it}=1$ if the unit of item i purchased and returned by customer t passes the inspection; similar to above, when a unit of item i is not purchased or returned by customer t, Z_{it} can be arbitrarily defined. Then, we can write the expected revenue from the dynamic assortment optimization as

$$
\begin{aligned}
\mathbb{E}_{\Upsilon,\boldsymbol{\pi}}\Big[\sum_{t\in\mathcal{T}}R_t^{\boldsymbol{\pi}}(\boldsymbol{x})\Big] &= \mathbb{E}_{\Upsilon,\boldsymbol{\pi}}\Bigg[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}X_{it}\Big(p_i(\boldsymbol{x})-Y_{it}\big(c_i(\boldsymbol{x})+(1-Z_{it})p_i^s(\boldsymbol{x})-Z_{it}c_i^r(\boldsymbol{x})\big)\Big)\Bigg] \\
&\overset{(a)}{=} \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\mathbb{E}_{\Upsilon,\boldsymbol{\pi}}\Big[X_{it}\Big(p_i(\boldsymbol{x})-Y_{it}\big(c_i(\boldsymbol{x})+(1-Z_{it})p_i^s(\boldsymbol{x})-Z_{it}c_i^r(\boldsymbol{x})\big)\Big)\Big] \\
&\overset{(b)}{=} \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\mathbb{E}_{\Upsilon,\boldsymbol{\pi}}\Big[X_{it}p_i(\boldsymbol{x})-X_{it}\mathbb{E}_{\Upsilon,\boldsymbol{\pi}}[Y_{it}c_i(\boldsymbol{x})\,|\,X_{it}] \\
&\qquad\qquad -X_{it}\mathbb{E}_{\Upsilon,\boldsymbol{\pi}}\Big[Y_{it}\mathbb{E}_{\Upsilon,\boldsymbol{\pi}}[(1-Z_{it})p_i^s(\boldsymbol{x})-Z_{it}c_i^r(\boldsymbol{x})\,|\,X_{it},\,Y_{it}]\,\Big|\,X_{it}\Big]\Big] \\
&\overset{(c)}{=} \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\mathbb{E}_{\Upsilon,\boldsymbol{\pi}}\Big[X_{it}\Big(p_i(\boldsymbol{x})-\gamma_{it}(\boldsymbol{x})\big(c_i(\boldsymbol{x})+(1-h_i(\boldsymbol{x}))p_i^s(\boldsymbol{x})-h_i(\boldsymbol{x})c_i^r(\boldsymbol{x})\big)\Big)\Big] \\
&\overset{(d)}{=} \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\mathbb{E}_{\widetilde{\Upsilon},\boldsymbol{\pi}}\Big[X_{it}\Big(p_i(\boldsymbol{x})-\gamma_{it}(\boldsymbol{x})\big(c_i(\boldsymbol{x})+(1-h_i(\boldsymbol{x}))p_i^s(\boldsymbol{x})-h_i(\boldsymbol{x})c_i^r(\boldsymbol{x})\big)\Big)\Big] \\
&\overset{(e)}{=} \mathbb{E}_{\widetilde{\Upsilon},\boldsymbol{\pi}}\Bigg[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}X_{it}r_i(\boldsymbol{x})\Bigg]
\end{aligned}
$$

where (a) follows by the linearity of expectations, (b) follows by the iterated law of expectation, (c) follows from the definitions of X_{it} and Y_{it}, (d) follows by the definition of $\widetilde{\Upsilon}$, and (e) follows from linearity of expectations and the definition of $r_i(\boldsymbol{x})$ for all $i \in \mathcal{N}$. Clearly, $\mathbb{E}_{\widetilde{\Upsilon},\boldsymbol{\pi}}\left[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} X_{it} r_i(\boldsymbol{x})\right] = \mathbb{E}_{\widetilde{\Upsilon},\boldsymbol{\pi}}\left[\sum_{t\in\mathcal{T}} R_t^{\boldsymbol{\pi}}(\boldsymbol{x})\right]$. This concludes the proof.

G.4. Proof of Theorem 5.2

The proof primarily consists of the following two lemmas. The first lemma establishes a relationship between the approximate value function $\hat{J}^t$ and the upper bound.

Lemma G.1 (Approximate Value Function and the Upper Bound) *For each $\boldsymbol{x} \in \mathcal{X}$, $\hat{J}^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0} \,\big|\, \boldsymbol{x}\Big) \geq \frac{1}{2}\mathsf{UB}(\boldsymbol{x})$.*

Let $V^t(\boldsymbol{q}\,|\,\boldsymbol{x})$ denote the expected revenue under the greedy policy $\boldsymbol{\pi}^H$ in periods $t, t+1, \ldots T$, given that the system state is $\boldsymbol{q}$ in period t. Then, for all $\boldsymbol{q} \in \mathcal{Q}(\boldsymbol{x})$, $V^{T+1}(\boldsymbol{q}\,|\,\boldsymbol{x}) = 0$. To facilitate our exposition, we will write G^t to denote $G^t(\boldsymbol{q}\,|\,\boldsymbol{x})$. For each $\boldsymbol{q} \in \mathcal{Q}(\boldsymbol{x})$ and $t \leq T$, using the same argument as in (5.1), we have the following recursion for the function $V^t(\boldsymbol{q}\,|\,\boldsymbol{x})$:

$$\begin{aligned}
V^t(\boldsymbol{q}\,|\,\boldsymbol{x}) =& \mathbb{E}\Big[V^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big] + \sum_{i\in\mathcal{N}} \phi_i^t(G^t\,|\,\boldsymbol{x}) \Bigg[r_i(\boldsymbol{x}) \\
&+ \big(1-\gamma_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[V^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i1} \;\big|\; \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[V^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big]\Big) \\
&+ \gamma_{i0}(\boldsymbol{x}) h_i(\boldsymbol{x}) \big(1-\zeta_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[V^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i,-1} | \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[V^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) | \boldsymbol{x}\big)\Big]\Big) \\
&+ \gamma_{i0}(\boldsymbol{x})\big(1-h_i(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[V^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} \;\big|\; \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[V^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\,\big|\,\boldsymbol{x}\big)\Big]\Big)\Bigg]\Bigg\},
\end{aligned} \tag{G.2}$$

The following lemma then gives a lower bound on V^t.

Lemma G.2 (Lower Bound on Greedy Policy Performance) *For each $t \in \mathcal{T}$, $\boldsymbol{x} \in \mathcal{X}$, and $\boldsymbol{q} \in \mathcal{Q}(\boldsymbol{x})$, $V^t(\boldsymbol{q}\,|\,\boldsymbol{x}) \geq \hat{J}^t(\boldsymbol{q}\,|\,\boldsymbol{x})$.*

Here we present the proof of Theorem 5.2.

Proof: It follows from Lemmas G.1 and G.2 that the expected revenue under the greedy policy $\boldsymbol{\pi}^H$ satisfies

$$V^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0}\,\big|\,\boldsymbol{x}\Big) \;\geq\; \hat{J}^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0}\,\big|\,\boldsymbol{x}\Big) \;\geq\; \frac{1}{2}\mathsf{UB}(\boldsymbol{x}) \;,$$

which is the desired result. ■

G.5. Proof of Lemma G.1

We first present an auxiliary lemma as follows. We define $\prod_{h=0}^{-1} = 1$ and $\beta_{i0}^t = 0$ for all $t \geq T+1$ for mathematical convenience.

Lemma G.3 *For each $\boldsymbol{x} \in \mathcal{X}$, $t \in \mathcal{T}$, and $i \in \mathcal{N}$, it holds that* *(i)* $\beta_{i0}^t(\boldsymbol{x}) \geq \beta_{i0}^{t+1}(\boldsymbol{x})$;

(ii)

$$\beta_{i\ell}^t(\boldsymbol{x}) = \begin{cases} h_i(\boldsymbol{x}) \sum_{s=0}^{\infty} \left[\gamma_{i,\ell+s}(\boldsymbol{x}) \left(\prod_{h=0}^{s-1} \left(1-\gamma_{i,\ell+h}(\boldsymbol{x})\right)\right) \left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x}) \beta_{i0}^{t+s+p+1}\right)\right] & \text{if } \ell \geq 1 \\ \sum_{s=0}^{\infty} \left[\zeta_{i,-\ell+s}(\boldsymbol{x}) \left(\prod_{h=0}^{s-1} \left(1-\zeta_{i,-\ell+h}(\boldsymbol{x})\right)\right) \beta_{i0}^{t+s+1}(\boldsymbol{x})\right] & \text{if } \ell \leq -1 \end{cases}; \tag{G.3}$$

(iii) and for all $\ell \geq 0$

$$f_i(\ell\,|\,\boldsymbol{x}) = \gamma_{i\ell}(\boldsymbol{x}) \prod_{s=0}^{\ell-1} \left(1-\gamma_{is}(\boldsymbol{x})\right) \quad \textit{and} \quad g_i(\ell\,|\,\boldsymbol{x}) = \zeta_{i\ell}(\boldsymbol{x}) \prod_{s=0}^{\ell-1} \left(1-\zeta_{is}(\boldsymbol{x})\right). \tag{G.4}$$

The first part shows the monotonicity of $\beta_{i0}^t(\boldsymbol{x})$. The second part gives an expansion of $\beta_{i\ell}^t(\boldsymbol{x})$ and $\beta_{i,-\ell}^t(\boldsymbol{x})$ for all $\ell \geq 1$ in terms of the basis $\beta_{i,0}^t(\boldsymbol{x})$. The last part presents a useful representation of the pmf in terms of the hazard rates. Equipped with this lemma, we prove Lemma G.1 as follows. One can equivalently write $\mathsf{UB}(\boldsymbol{x})$ in the dual form of Eq.(3.1) given by

$$\begin{aligned} \mathsf{UB}(\boldsymbol{x}) \; = \; \min_{(\boldsymbol{\lambda},\boldsymbol{\theta}) \geq \mathbf{0}} \quad & \sum_{t=1}^{T} \lambda^t + \sum_{t=1}^{T} \sum_{i \in \mathcal{N}} C_i(\boldsymbol{x}) \theta_i^t \\ \text{s.t.} \quad & \lambda^t \geq \sum_{i \in \mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x}) \left[r_i(\boldsymbol{x}) \; - \; \sum_{\tau=t}^{T} \alpha_i(\tau - t\,|\,\boldsymbol{x}) \theta_i^\tau \right] \quad \forall\, S \in \mathcal{F}, t \in \mathcal{T}, \end{aligned}$$

We next verify that $\left(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\theta}}\right)$, defined by $\hat{\lambda}^t = \sum_{i \in \mathcal{N}} C_i(\boldsymbol{x})(\beta_{i0}^t(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}))$ and $\hat{\theta}_i^t = \beta_{i0}^{t+1}(\boldsymbol{x}) - \beta_{i0}^{t+2}(\boldsymbol{x})$ for all $t \in \mathcal{T}$ is a feasible solution to the dual linear program above. Fix an arbitrary $S \in \mathcal{F}$. By definition of $\hat{\theta}_i^t$,

$$\begin{aligned} & \sum_{i \in \mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x}) \left[r_i(\boldsymbol{x}) \; - \; \sum_{\tau=t}^{T} \alpha_i(\tau - t\,|\,\boldsymbol{x}) \hat{\theta}_i^\tau \right] \\ = \; & \sum_{i \in \mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x}) \left[r_i(\boldsymbol{x}) \; - \; \sum_{\tau=t}^{T} \alpha_i(\tau - t\,|\,\boldsymbol{x}) \left(\beta_{i0}^{\tau+1}(\boldsymbol{x}) - \beta_{i0}^{\tau+2}(\boldsymbol{x})\right) \right] \\ \overset{(a)}{=} \; & \sum_{i \in \mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x}) \left[r_i(\boldsymbol{x}) \; - \; \alpha_i(0\,|\,\boldsymbol{x}) \beta_{i0}^{t+1}(\boldsymbol{x}) \; + \; \sum_{\ell=0}^{\infty} \beta_{i0}^{t+2+\ell}(\boldsymbol{x}) \big(\alpha_i(\ell\,|\,\boldsymbol{x}) - \alpha_i(\ell+1\,|\,\boldsymbol{x})\big) \right] \\ \overset{(b)}{=} \; & \sum_{i \in \mathcal{N}} \phi_i^t(S\,|\,\boldsymbol{x}) \left[r_i(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x}) \sum_{\ell=0}^{\infty} \beta_{i0}^{t+\ell+1}(\boldsymbol{x}) \left(\sum_{s=0}^{\ell} f_i(s\,|\,\boldsymbol{x}) g_i(\ell - x\,|\,\boldsymbol{x}) \right) \right] \end{aligned} \tag{G.5}$$

where the equality (a) follows from the fact that $\beta_{i0}^t = 0$ for all $t > T$, and one can verify (b) as follows. We focus on the term inside the bracket for each $i \in \mathcal{N}$

$$- \, \alpha_i(0\,|\,\boldsymbol{x}) \beta_{i0}^{t+1}(\boldsymbol{x}) + \sum_{\ell=0}^{\infty} \beta_{i0}^{t+2+\ell}(\boldsymbol{x}) \big(\alpha_i(\ell\,|\,\boldsymbol{x}) - \alpha_i(\ell+1\,|\,\boldsymbol{x})\big)$$

$$\stackrel{(c)}{=} -\Big(1-h_i(\boldsymbol{x})f_i(0\,|\,\boldsymbol{x})g_i(0\,|\,\boldsymbol{x})\Big)\beta_{i0}^{t+1}(\boldsymbol{x}) + \sum_{\ell=0}^{\infty}\beta_{i0}^{t+2+\ell}(\boldsymbol{x})\left[\left(1-h_i(\boldsymbol{x})\sum_{s=0}^{\ell}f_i(s\,|\,\boldsymbol{x})\sum_{\tau=0}^{\ell-s}g_i(\tau\,|\,\boldsymbol{x})\right)-\left(1-h_i(\boldsymbol{x})\sum_{s=0}^{\ell+1}f_i(s\,|\,\boldsymbol{x})\sum_{\tau=0}^{\ell-s}g_i(\tau\,|\,\boldsymbol{x})\right)\right]$$

$$\stackrel{(d)}{=} -\Big(1-h_i(\boldsymbol{x})f_i(0\,|\,\boldsymbol{x})g_i(0\,|\,\boldsymbol{x})\Big)\beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{\ell=0}^{\infty}\beta_{i0}^{t+2+\ell}(\boldsymbol{x})\left(\sum_{s=0}^{\ell+1}f_i(s\,|\,\boldsymbol{x})g_i(\ell+1-s\,|\,\boldsymbol{x})\right)$$

$$\stackrel{(e)}{=} -\Big(1-h_i(\boldsymbol{x})f_i(0\,|\,\boldsymbol{x})g_i(0\,|\,\boldsymbol{x})\Big)\beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{\ell'=1}^{\infty}\beta_{i0}^{t+\ell'+1}(\boldsymbol{x})\left(\sum_{s=0}^{\ell'}f_i(s\,|\,\boldsymbol{x})g_i(\ell'-s\,|\,\boldsymbol{x})\right)$$

$$= -\beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{\ell'=0}^{\infty}\beta_{i0}^{t+\ell'+1}(\boldsymbol{x})\left(\sum_{s=0}^{\ell'}f_i(s\,|\,\boldsymbol{x})g_i(\ell'-s\,|\,\boldsymbol{x})\right),$$

where (c) follows from plugging the definition of $\alpha_i(\cdot\,|\,\boldsymbol{x})$, (d) follows from algebra and (e) follows from a change of index, $\ell'=\ell+1$. This verifies (b) in (G.5).

Next, we claim that

$$\xi_i^t(\boldsymbol{x}) \;=\; r_i(\boldsymbol{x})-\beta_{i0}^{t+1}(\boldsymbol{x})+h_i(\boldsymbol{x})\sum_{\ell=0}^{\infty}\beta_{i0}^{t+\ell+1}(\boldsymbol{x})\left(\sum_{s=0}^{\ell}f_i(s\,|\,\boldsymbol{x})g_i(\ell-x\,|\,\boldsymbol{x})\right). \tag{G.6}$$

Indeed, we note that (5.3) implies $\xi_i^t(\boldsymbol{x})$ can be expanded as

$$r_i(\boldsymbol{x})-\big(1-h_i(\boldsymbol{x})\gamma_{i0}(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})\big)\beta_{i0}^{t+1}(\boldsymbol{x}) \;+\; \underbrace{\big(1-\gamma_{i0}(\boldsymbol{x})\big)\beta_{i1}^{t+1}(\boldsymbol{x})}_{(**)} \;+\; \gamma_{i0}(\boldsymbol{x})h_i(\boldsymbol{x})\underbrace{\big(1-\zeta_{i0}(\boldsymbol{x})\big)\beta_{i,-1}^{t+1}(\boldsymbol{x})}_{(*)}.$$

Applying the definition of $\beta_{i,-1}^{t+1}$ in (G.3), we can simplify $(*)$ as

$$\begin{aligned}
\big(1-\zeta_{i0}(\boldsymbol{x})\big)\beta_{i,-1}^{t+1}(\boldsymbol{x}) \;&=\; \big(1-\zeta_{i0}(\boldsymbol{x})\big)\sum_{s=0}^{\infty}\left[\zeta_{i,1+s}(\boldsymbol{x})\left(\prod_{h=0}^{s-1}\big(1-\zeta_{i,1+h}(\boldsymbol{x})\big)\right)\beta_{i0}^{t+s+2}(\boldsymbol{x})\right]\\
&\stackrel{(f)}{=}\; \big(1-\zeta_{i0}(\boldsymbol{x})\big)\sum_{s=0}^{\infty}\left[\zeta_{i,1+s}(\boldsymbol{x})\left(\prod_{h'=1}^{s}\big(1-\zeta_{i,h'}(\boldsymbol{x})\big)\right)\beta_{i0}^{t+s+2}(\boldsymbol{x})\right]\\
&\stackrel{(g)}{=}\; \big(1-\zeta_{i0}(\boldsymbol{x})\big)\sum_{s'=1}^{\infty}\left[\zeta_{i,s'}(\boldsymbol{x})\left(\prod_{h'=1}^{s'-1}\big(1-\zeta_{i,h'}(\boldsymbol{x})\big)\right)\beta_{i0}^{t+s'+1}(\boldsymbol{x})\right]\\
&=\; \sum_{s'=1}^{\infty}\left[\zeta_{i,s'}(\boldsymbol{x})\left(\prod_{h'=0}^{s'-1}\big(1-\zeta_{i,h'}(\boldsymbol{x})\big)\right)\beta_{i0}^{t+s'+1}(\boldsymbol{x})\right]\\
&\stackrel{(h)}{=}\; \sum_{s=0}^{\infty}g_i(s\,|\,\boldsymbol{x})\beta_{i0}^{t+s+1}(\boldsymbol{x}) \;-\; g_i(0\,|\,\ell)\beta_{i0}^{t+1}(\boldsymbol{x}),
\end{aligned}$$

where (f) and (g) follows by setting $h'=h+1$ and $s'=s+1$, respectively, and (h) follows from part (iii) of Lemma G.3. Similarly, it follows from (G.3) that $(**)$ can be simplified as

$$\big(1-\gamma_{i0}(\boldsymbol{x})\big)\beta_{i1}^{t+1}(\boldsymbol{x}) = \big(1-\gamma_{i0}(\boldsymbol{x})\big)h_i(\boldsymbol{x})\sum_{s=0}^{\infty}\left[\gamma_{i,1+s}(\boldsymbol{x})\prod_{h=0}^{s-1}\big(1-\gamma_{i,1+h}(\boldsymbol{x})\big)\left(\sum_{p=0}^{\infty}g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+2}\right)\right]$$

$$
\begin{aligned}
&\overset{(i)}{=} \big(1-\gamma_{i0}(\boldsymbol{x})\big)h_i(\boldsymbol{x})\sum_{s'=1}^{\infty}\left[\gamma_{i,s'}(\boldsymbol{x})\prod_{h'=1}^{s'-1}\big(1-\gamma_{i,h'}(\boldsymbol{x})\big)\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s'+p+1}\right)\right]\\
&= h_i(\boldsymbol{x})\sum_{s'=1}^{\infty}\left[\gamma_{i,s'}(\boldsymbol{x})\prod_{h'=0}^{s'-1}\big(1-\gamma_{i,h'}(\boldsymbol{x})\big)\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s'+p+1}\right)\right]\\
&\overset{(j)}{=} h_i(\boldsymbol{x})\sum_{s'=1}^{\infty}\left[f_i(s'\,|\,\boldsymbol{x})\cdot\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s'+p+1}\right)\right],
\end{aligned}
$$

where (i) follows from changing the indexes and (j) follows from Lemma G.3. Therefore, noticing that $\gamma_{i0}=f_i(0\,|\,\boldsymbol{x})$ and $\zeta_{i0}=g_i(0\,|\,\boldsymbol{x})$, it follows that

$$
\begin{aligned}
\xi_i^t(\boldsymbol{x}) \;&=\; r_i(\boldsymbol{x}) - \big(1-h_i(\boldsymbol{x})f_i(0\,|\,\boldsymbol{x})g_i(0\,|\,\boldsymbol{x})\big)\beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{s=1}^{\infty}\left[f_i(s\,|\,\boldsymbol{x})\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+1}\right)\right]\\
&\qquad + f_i(0\,|\,\boldsymbol{x})h_i(\boldsymbol{x})\left(\sum_{s=0}^{\infty} g_i(s\,|\,\boldsymbol{x})\beta_{i0}^{t+s+1}(\boldsymbol{x}) \;-\; g_i(0\,|\,\ell)\beta_{i0}^{t+1}(\boldsymbol{x})\right)\\
&\overset{(k)}{=}\; r_i(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{s=0}^{\infty}\left[f_i(s\,|\,\boldsymbol{x})\cdot\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+1}\right)\right]\\
&\overset{(\ell)}{=}\; r_i(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{\ell=0}^{\infty}\beta_{i0}^{t+\ell+1}(\boldsymbol{x})\cdot\left(\sum_{s=0}^{\ell} f_i(s\,|\,\boldsymbol{x})g_i(\ell-s\,|\,\boldsymbol{x})\right),
\end{aligned}
$$

where both (k) and (ℓ) follow from algebra and this verifies (G.6).

Combining (G.5) and (G.6), we have that

$$
\begin{aligned}
\sum_{i\in\mathcal{N}}\phi_i^t(S\,|\,\boldsymbol{x})\Big[r_i(\boldsymbol{x})-\sum_{\tau=t}^{T}\alpha_i(\tau-t\,|\,\boldsymbol{x})\hat{\theta}_i^{\tau}\Big] \;&=\; \sum_{i\in\mathcal{N}}\phi_i^t(S\,|\,\boldsymbol{x})\,\xi_i^t(\boldsymbol{x}) \;\overset{(m)}{\le}\; \sum_{i\in\mathcal{N}}\phi_i^t(A^t(\boldsymbol{x})\,|\,\boldsymbol{x})\,\xi_i^t(\boldsymbol{x})\\
&\overset{(n)}{=}\; \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})(\beta_{i0}^t(\boldsymbol{x})-\beta_{i0}^{t+1}(\boldsymbol{x})) \;=\; \hat{\lambda}^t, \qquad\qquad \text{(G.7)}
\end{aligned}
$$

where (m) follows by the optimality of $A^t(\boldsymbol{x})$ and (n) follows from Eq.(5.5). Next, we notice that $\beta_{i0}^t(\boldsymbol{x})\ge\beta_{i0}^{t+1}(\boldsymbol{x})$ for all $i\in\mathcal{N}$, $t\ge 0$ and $\boldsymbol{x}$ from the monotonicity property in Lemma G.3. As a result, $(\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\theta}})$ is feasible to the dual program.

Then, plug the values $(\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\theta}})$ into the objective function of the dual program, we obtain

$$
\begin{aligned}
\mathsf{UB}(\boldsymbol{x}) \;&\le\; \sum_{t=1}^{T}\hat{\lambda}^t \;+\; \sum_{t=1}^{T}\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\hat{\theta}_i^t\\
&=\; \sum_{t=1}^{T}\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\Big(\beta_{i0}^t(\boldsymbol{x})-\beta_{i0}^{t+1}(\boldsymbol{x})\Big) \;+\; \sum_{t=1}^{T}\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\Big(\beta_{i0}^{t+1}(\boldsymbol{x})-\beta_{i0}^{t+2}(\boldsymbol{x})\Big)\\
&=\; \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\sum_{t=1}^{T}\Big(\beta_{i0}^t(\boldsymbol{x})-\beta_{i0}^{t+1}(\boldsymbol{x})\Big) \;+\; \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\sum_{t=1}^{T}\Big(\beta_{i0}^{t+1}(\boldsymbol{x})-\beta_{i0}^{t+2}(\boldsymbol{x})\Big)\\
&=\; \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\beta_{i0}^1(\boldsymbol{x}) + \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\beta_{i0}^2(\boldsymbol{x}) \;\overset{(o)}{\le}\; \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\beta_{i0}^1(\boldsymbol{x}) + \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\beta_{i0}^1(\boldsymbol{x})
\end{aligned}
$$

where (o) follows from the monotonicity property in Lemma G.3. By the definition of $\hat{J}^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0}\,\Big|\,\boldsymbol{x}\Big)$, the right-hand side of this chain of inequalities is bounded by $2\hat{J}^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0}\,\Big|\,\boldsymbol{x}\Big)$, and we conclude the proof.

G.6. Proof of Lemma G.2

In this part, we finish the proof of Theorem 5.2 by lower bounding the performance of the greedy policy π^H by $\hat{J}^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0}\,\Big|\,\boldsymbol{x}\Big)$. We use induction to show that $V^t(\boldsymbol{q}\,|\,\boldsymbol{x}) \geq \hat{J}^t(\boldsymbol{q}\,|\,\boldsymbol{x})$ for all $\boldsymbol{q}\in\mathcal{Q}(\boldsymbol{x})$ and $t\in\mathcal{T}$. This clearly holds when $t=T+1$. Then, assume that $V^{t+1}(\boldsymbol{q}\,|\,\boldsymbol{x}) \geq \hat{J}^{t+1}(\boldsymbol{q}\,|\,\boldsymbol{x})$. It holds by definition that for all $i\in\mathcal{N}$

$$\mathbb{E}[\mathsf{A}_{i\ell}(\boldsymbol{x})] = \gamma_{i\ell}(\boldsymbol{x}), \quad \text{and} \quad \mathbb{E}[\mathsf{D}_{i,-\ell}(\boldsymbol{x})] \;=\; q_{i,-\ell}\zeta_i(\ell\,|\,\boldsymbol{x}) \;\forall\,\ell\geq 1,$$

and

$$\mathbb{E}\left[\mathsf{B}_i\left(\boldsymbol{x}\right)\right] \;=\; h_i(\boldsymbol{x})\sum_{\ell=1}^{\infty} q_{i\ell}\gamma_{i\ell}(\boldsymbol{x}), \quad \text{and} \quad \mathbb{E}\left[\mathsf{C}_i(\boldsymbol{x})\right] \;=\; h_i(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})\sum_{\ell=1}^{\infty} q_{i\ell}\gamma_{i\ell}(\boldsymbol{x})$$

Therefore, by the definition of $W_{i\ell}(\boldsymbol{q},\boldsymbol{x})$, it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big] \qquad\qquad \text{(G.8)}\\
&= \sum_{i\in\mathcal{N}}\Bigg[\sum_{\ell=1}^{\infty} q_{i\ell}(1-\gamma_{i\ell}(\boldsymbol{x}))\beta^{t+1}_{i,\ell+1}(\boldsymbol{x}) \;+\; \sum_{\ell=-1}^{-\infty} q_{i,\ell}(1-\zeta_{i,-\ell}(\boldsymbol{x}))\beta^{t+1}_{i,\ell-1}(\boldsymbol{x})\\
&\quad + \; h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\sum_{\ell=1}^{\infty} q_{i\ell}\gamma_{i\ell}(\boldsymbol{x})\beta^{t+1}_{i,-1}(\boldsymbol{x}) + \left(q_{i0} + h_i(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})\sum_{\ell=1}^{\infty} q_{i\ell}\gamma_{i\ell}(\boldsymbol{x}) + \sum_{\ell=-1}^{-\infty} q_{i\ell}\zeta_{i\ell}\right)\beta^{t+1}_{i0}(\boldsymbol{x})\Bigg]\\
&= \sum_{i\in\mathcal{N}}\Bigg[\sum_{\ell=1}^{\infty} q_{i\ell}\Big(\big(1-\gamma_{i\ell}(\boldsymbol{x})\big)\beta^{t+1}_{i,\ell+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\gamma_{i\ell}(\boldsymbol{x})\beta^{t+1}_{i,-1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})\gamma_{i\ell}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x})\Big)\\
&\qquad + \; q_{i0}\beta^{t+1}_{i0}(\boldsymbol{x}) + \sum_{\ell=-1}^{-\infty} q_{i\ell}\Big(\zeta_{i\ell}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) \;+\; (1-\zeta_{i\ell}(\boldsymbol{x}))\beta^{t+1}_{i,\ell-1}(\boldsymbol{x})\Big)\Bigg]\\
&\overset{(a)}{=} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\in\mathbb{Z}:\, i\neq 0} q_{i\ell}\beta^t_{i\ell}(\boldsymbol{x}) \,+\, q_{i0}\beta^{t+1}_{i0}(\boldsymbol{x})\right) \qquad\qquad \text{(G.9)}
\end{aligned}$$

where (a) follows by definition of $\beta^t_{i\ell}(\boldsymbol{x})$ for all $\ell\in\mathbb{Z}$ and $\ell\neq 0$. Using the induction hypothesis and noticing that the coefficient of each $\mathbb{E}\big[V^{t+1}\big(\,\cdot\,\big|\,\boldsymbol{x}\big)\big]$ term on the right-hand side of (G.2) is non-negative, we obtain that

$$\begin{aligned}
&V^t(\boldsymbol{q}\,|\,\boldsymbol{x})\\
&\geq \; \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big] + \sum_{i\in\mathcal{N}} \phi^t_i(G^t\,|\,\boldsymbol{x})\Big[r_i(\boldsymbol{x})\\
&\quad + \big(1-\gamma_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i1} \;\big|\; \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big]\Big)\\
&\quad + \gamma_{i0}(\boldsymbol{x})h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i,-1}\big|\boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\big|\boldsymbol{x}\big)\Big]\Big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \gamma_{i0}(\boldsymbol{x})\big(1-h_i(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})-\boldsymbol{e}_{i0} \mid \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \,|\, \boldsymbol{x}\big)\Big]\Big)\Big]\Big\} \\
&\overset{(b)}{=} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\in\mathbb{Z}:\, i\neq 0}^{\infty} q_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + q_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) + \sum_{i\in\mathcal{N}}\phi_i^t(G^t\,|\,\boldsymbol{x})\Big[r_i(\boldsymbol{x}) - \gamma_{i0}(\boldsymbol{x})\big(1-h_i(\boldsymbol{x})\big)\beta_{i0}^{t+1}(\boldsymbol{x}) \\
&\quad + \big(1-\gamma_{i0}(\boldsymbol{x})\big)\big(\beta_{i1}^{t+1}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x})\big) + \gamma_{i0}(\boldsymbol{x})h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\big(\beta_{i,-1}^{t+1}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x})\big) \\
&\overset{(c)}{=} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\in\mathbb{Z}:\, i\neq 0}^{\infty} q_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + q_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) + \sum_{i\in\mathcal{N}}\phi_i^t(G^t\,|\,\boldsymbol{x})\,\xi_i^t(\boldsymbol{x}) \\
&\overset{(d)}{\geq} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\in\mathbb{Z}:\, i\neq 0}^{\infty} q_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + q_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) + \sum_{i\in\mathcal{N}}\phi_i^t\Big(A^t\cap\{i:\ q_{i0}\geq 1\}\,|\,\boldsymbol{x}\Big)\,\xi_i^t(\boldsymbol{x}) \\
&\overset{(e)}{\geq} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\in\mathbb{Z}:\, i\neq 0}^{\infty} q_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + q_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) + \sum_{i\in\mathcal{N}}\mathbb{1}_{\{q_{i0}\geq 1\}}\phi_i^t(A^t\,|\,\boldsymbol{x})\,\xi_i^t(\boldsymbol{x}) \\
&\overset{(f)}{\geq} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\in\mathbb{Z}:\, i\neq 0}^{\infty} q_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + q_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) + \sum_{i\in\mathcal{N}}\frac{q_{i0}}{C_i(\boldsymbol{x})}\phi_i^t(A^t\,|\,\boldsymbol{x})\,\xi_i^t(\boldsymbol{x}) \\
&= \sum_{i\in\mathcal{N}}\left[q_{i0}\cdot\left(\beta_{i0}^{t+1}(\boldsymbol{x}) + \frac{1}{C_i(\boldsymbol{x})}\phi_i^t(A^t\,|\,\boldsymbol{x})\,\xi_i^t(\boldsymbol{x})\right) + \sum_{\ell\in\mathbb{Z}:\, i\neq 0}^{\infty} q_{i\ell}\beta_{i\ell}^t(\boldsymbol{x})\right] \\
&\overset{(g)}{=} \sum_{i\in\mathcal{N}}\sum_{\ell=-\infty}^{\infty} q_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) = \hat{J}^t(\boldsymbol{q}\,|\,\boldsymbol{x}).
\end{aligned}
$$

The inequality (b) follows from (G.8) and the linearity of expectation. The inequality (c) follows from the definition of $\xi_i^t(\boldsymbol{x})$ for all $i\in\mathcal{N}$ and $t\in\mathcal{T}$. The inequality (d) holds due to the optimality of G^t. Also, notice that $\phi_i^t\Big(A^t\cap\{i:\ q_{i0}\geq 1\}\,|\,\boldsymbol{x}\Big)\leq \phi_i^t(A^t\,|\,\boldsymbol{x})$ by our assumption on the choice model $\phi_i^t(\cdot\,|\,\boldsymbol{x})$. Also, by the definition of A^t, it must be that $\xi_i^t(\boldsymbol{x})\geq 0$ if $\phi_i^t(A^t\,|\,\boldsymbol{x})>0$. Therefore, inequality (e) follows. (f) follows because $q_{i0}\leq C_i(\boldsymbol{x})$. The inequality ($g$) follows by the definition of $\beta_{i0}^t(\boldsymbol{x})$. This finishes our proof.

G.7. Proof of Lemma G.3

The first part of the Lemma follows from the definition of $\beta_{i0}^t(\boldsymbol{x})$ in (5.5). Indeed, for any $i\in\mathcal{N}$ such that $\phi_i^t(A^t(\boldsymbol{x})\,|\,\boldsymbol{x})>0$, it must be that $i\in A^t(\boldsymbol{x})$ and $\xi_i^t(\boldsymbol{x})\geq 0$. Otherwise, if $\xi_i^t(\boldsymbol{x})<0$, we can remove product i from $A^t(\boldsymbol{x})$ and all other products from $A^t(\boldsymbol{x})$ to obtain an assortment A' with strictly larger value of $\sum_{i\in\mathcal{N}}\phi_i^t(S\,|\,\boldsymbol{x})\xi_i^t(\boldsymbol{x})$. This is a contradiction. Therefore $\phi_i^t(A^t(\boldsymbol{x})\,|\,\boldsymbol{x})\xi_i^t(\boldsymbol{x})\geq 0$ so the first part follows. We next prove the third part. We first show the recursive equation $\sum_{s=0}^{\ell} f_i(s\,|\,\boldsymbol{x}) = 1-\prod_{s=0}^{\ell}(1-\gamma_{is}(\boldsymbol{x}))$. The base case ($\ell=0$) is true since both sides of the equation is zero. Suppose the result is true at $\ell-1$, then we have

$$
\begin{aligned}
\sum_{s=0}^{\ell} f_i(\ell\,|\,\boldsymbol{x}) &= \sum_{s=0}^{\ell-1} f_i(s\,|\,\boldsymbol{x}) + f_i(\ell\,|\,\boldsymbol{x}) = \sum_{s=0}^{\ell-1} f_i(s\,|\,\boldsymbol{x}) + \gamma_{i\ell}(\boldsymbol{x})\left(1-\sum_{s=0}^{\ell-1} f_i(s\,|\,\boldsymbol{x})\right) \\
&= (1-\gamma_{i,\ell}(\boldsymbol{x}))\sum_{s=0}^{\ell-1} f_i(s\,|\,\boldsymbol{x}) + \gamma_{i,\ell}(\boldsymbol{x}) = (1-\gamma_{i,\ell}(\boldsymbol{x}))\left(1-\prod_{s=0}^{\ell-1}\big(1-\gamma_{is}(\boldsymbol{x})\big)\right) + \gamma_{i\ell}(\boldsymbol{x})
\end{aligned}
$$

$$= 1-\prod_{s=0}^{\ell}(1-\gamma_{is}(\boldsymbol{x})).$$

Then we have

$$f_i(\ell\,|\,\boldsymbol{x}) = \sum_{s=0}^{\ell} f_i(\ell\,|\,\boldsymbol{x}) - \sum_{s=0}^{\ell-1} f_i(\ell\,|\,\boldsymbol{x}) = \gamma_{i\ell}(\boldsymbol{x})\prod_{s=0}^{\ell-1}(1-\gamma_{is}(\boldsymbol{x})).$$

We now show the second part. Fix $\boldsymbol{x}\in\mathcal{X}$ and $i\in\mathcal{N}$. We will verify (G.3) by induction on t. Due to the definition of $\beta^T_{i,\ell}$ and the assumption that $\beta^t_{i0}=0$ for all $t\geq T+1$, it must be that $\beta^T_{i,\ell}=0$ for all $\ell\leq -1$. This proves the base case for $\ell\leq -1$. Assume that these equations are true for $t+1$. Then by (5.5), for $\ell\leq -1$

$$\begin{aligned}
\beta^t_{i,\ell}(\boldsymbol{x}) &= \zeta_{i,-\ell}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + (1-\zeta_{i,-\ell}(\boldsymbol{x}))\beta^{t+1}_{i,\ell-1}(\boldsymbol{x})\\
&= \zeta_{i,-\ell}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \sum_{s=0}^{\infty}\left[\zeta_{i,-\ell+s+1}(\boldsymbol{x})\left(\prod_{h=0}^{s-1}\big(1-\zeta_{i,-\ell+h+1}(\boldsymbol{x})\big)\right)\beta^{t+s+2}_{i0}(\boldsymbol{x})\right]\\
&\overset{(a)}{=} \zeta_{i,-\ell}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \sum_{s'=1}^{\infty}\left[\zeta_{i,-\ell+s'}(\boldsymbol{x})\left(\prod_{h=0}^{s'-2}\big(1-\zeta_{i,-\ell+h+1}(\boldsymbol{x})\big)\right)\beta^{t+s'+1}_{i0}(\boldsymbol{x})\right]\\
&\overset{(b)}{=} \zeta_{i,-\ell}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \sum_{s'=1}^{\infty}\left[\zeta_{i,-\ell+s'}(\boldsymbol{x})\left(\prod_{h'=1}^{s'-1}\big(1-\zeta_{i,-\ell+h'}(\boldsymbol{x})\big)\right)\beta^{t+s'+1}_{i0}(\boldsymbol{x})\right]\\
&= \sum_{s'=0}^{\infty}\left[\zeta_{i,-\ell+s'}(\boldsymbol{x})\left(\prod_{h'=1}^{s'-1}\big(1-\zeta_{i,-\ell+h'}(\boldsymbol{x})\big)\right)\beta^{t+s'+1}_{i0}(\boldsymbol{x})\right],
\end{aligned}\tag{G.10}$$

where (a) and (b) follows by setting $s'=s+1$ and $h'=h+1$, respectively. This proves the expression for (G.3) for $t\in\mathcal{T}$ and $\ell\leq -1$. Furthermore,

$$\begin{aligned}
&\zeta_{i0}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \big(1-\zeta_{i0}(\boldsymbol{x})\big)\beta^{t+1}_{i,-1}(\boldsymbol{x})\\
&\overset{(c)}{=} \zeta_{i0}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \big(1-\zeta_{i0}(\boldsymbol{x})\big)\sum_{s=0}^{\infty}\left[\zeta_{i,1+s}(\boldsymbol{x})\left(\prod_{h=0}^{s-1}\big(1-\zeta_{i,1+h}(\boldsymbol{x})\big)\right)\beta^{t+s+2}_{i0}(\boldsymbol{x})\right]\\
&\overset{(d)}{=} \zeta_{i0}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \big(1-\zeta_{i0}(\boldsymbol{x})\big)\sum_{s'=1}^{\infty}\left[\zeta_{i,s'}(\boldsymbol{x})\left(\prod_{h=0}^{s'-2}\big(1-\zeta_{i,1+h}(\boldsymbol{x})\big)\right)\beta^{t+s'+1}_{i0}(\boldsymbol{x})\right]\\
&\overset{(e)}{=} \zeta_{i0}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \big(1-\zeta_{i0}(\boldsymbol{x})\big)\sum_{s'=1}^{\infty}\left[\zeta_{i,s'}(\boldsymbol{x})\left(\prod_{h'=1}^{s'-1}\big(1-\zeta_{i,h'}(\boldsymbol{x})\big)\right)\beta^{t+s'+1}_{i0}(\boldsymbol{x})\right]\\
&= \sum_{s'=0}^{\infty}\left[\zeta_{i,s'}(\boldsymbol{x})\left(\prod_{h'=0}^{s'-1}\big(1-\zeta_{i,h'}(\boldsymbol{x})\big)\right)\beta^{t+s'+1}_{i0}(\boldsymbol{x})\right],
\end{aligned}$$

where (c) follows from applying (G.3) for $\beta^{t+1}_{i,-1}(\boldsymbol{x})$, and (d) and (e) follows by setting $s'=s+1$ and $h'=h+1$, respectively. Therefore, for any $i\in\mathcal{N}$, if we denote by

$$\chi^t_{i0}(\boldsymbol{x}) = h_i(\boldsymbol{x})\cdot\sum_{s=0}^{\infty}\left[\zeta_{i,s}(\boldsymbol{x})\left(\prod_{h=0}^{s-1}\Big(1-\zeta_{i,h}(\boldsymbol{x})\Big)\right)\beta^{t+s}_{i0}(\boldsymbol{x})\right] \overset{(f)}{=} h_i(\boldsymbol{x})\cdot\sum_{s=0}^{\infty}g_i(s\,|\,\boldsymbol{x})\beta^{t+s}_{i0}(\boldsymbol{x}),$$

where (f) follows from part (iii) of the Lemma. For $t \in \mathcal{T}$ and set $\chi_{i0}^t(\boldsymbol{x}) = 0$ for all $t \geq T+1$. It then holds that

$$\beta_{i,\ell}^t(\boldsymbol{x}) = \gamma_{i\ell}(\boldsymbol{x})\chi_{i0}^{t+1}(\boldsymbol{x}) \; + \; \big(1 - \gamma_{i\ell}(\boldsymbol{x})\big)\beta_{i,\ell+1}^{t+1}(\boldsymbol{x}).$$

Applying the same argument that leads to (G.10), we have for all $t \in \mathcal{T}$ and $i \in \mathcal{N}$

$$\beta_{i,\ell}^t(\boldsymbol{x}) = \sum_{s=0}^{\infty} \left[\zeta_{i,\ell+s}(\boldsymbol{x}) \left(\prod_{h=1}^{s-1} \Big(1 - \zeta_{i,\ell+h}(\boldsymbol{x})\Big) \right) \chi_{i0}^{t+s+1}(\boldsymbol{x}) \right].$$

Plugging the definition of $\chi_{i0}^{t+s+1}(\boldsymbol{x})$, we conclude the proof of (G.3).

Appendix H: Extensions

In this section, we present two major extensions to complement our discussion in the main body. Utilizing the sales-based linear program under the MNL model discussed in Appendix F.3, we can obtain an alternative approach for maximizing offline effectiveness with a constant performance guarantee that is independent of the number of designs in Section H.1, provided that the return rate is moderately small. In Section H.2, we consider the joint assortment and discrete pricing problem in the online selling stage. In Section H.3, we extend our revenue-based rounding technique to a setting in which one can simultaneously offer multiple designs. In Section H.4, we discuss how to extend our model to case with stochastic customer arrivals.

H.1. Constant Performance bound under the MNL Model with Moderate Return Rate

We start by further approximating $W^{\mathsf{LP}}(\boldsymbol{x})$ with the following linear program by fixing the value of $w_0^t = \frac{1}{2}$ and tightening the capacity constraints:

$$\begin{aligned}
W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{x}) \;=\; \max_{\boldsymbol{w}} \quad & \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} r_{id} w_{id}^t \\
\text{s.t.} \quad & \sum_{t\in\mathcal{T}} w_{id}^t \;\leq\; C_{id} x_{id} \;\; \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i \quad \text{and} \quad w_{id}^t \;\leq\; \frac{1}{2} v_{id}^t \;\; \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \\
& \sum_{i\in\mathcal{N}} \sum_{d\in\mathcal{D}_i} w_{id}^t \;\leq\; \frac{1}{2} \;\; \forall\, t \in \mathcal{T} \qquad\qquad \text{and} \quad w_{id}^t \geq 0, \qquad \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}.
\end{aligned}$$

When the likelihood of return is moderate or small, we can show in the following Lemma that $W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{x})$ is a half-approximation to $W^{\mathsf{LP}}(\boldsymbol{x})$.

Lemma H.1 (Half-Relaxation of Sales-based LP) *If $\gamma_{id} \leq \frac{1}{2}$ for all $i \in \mathcal{N}, d \in \mathcal{D}_i$, then we have $\frac{1}{2} W^{\mathsf{LP}}(\boldsymbol{x}) \leq W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{x}) \leq W^{\mathsf{LP}}(\boldsymbol{x})$.*

Proof: We obtain $W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{x})$ from $W^{\mathsf{LP}}(\boldsymbol{x})$ by fixing $w_0^t = \frac{1}{2}$ for all $t \in \mathcal{T}$ and shrinking the feasible region in the first constraint of $W^{\mathsf{LP}}(\boldsymbol{x})$. Therefore, $W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{x}) \leq W^{\mathsf{LP}}(\boldsymbol{x})$.

Now let $\tilde{\boldsymbol{w}}$ be the optimal solution to $W^{\mathsf{LP}}(\boldsymbol{x})$. We claim that $\left(\frac{1}{2}\tilde{w}_{id}:\ i\in\mathcal{N},\, d\in\mathcal{D}_i\right)$ is a feasible solution to $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$. Since $\alpha_{id}\geq 1-\gamma_{id}\geq\frac{1}{2}$ for all $i\in\mathcal{N}$, $d\in\mathcal{D}_i$, it follows that

$$\sum_{\tau=1}^{T}\frac{1}{2}\tilde{w}_{id}^{\tau} \;\leq\; \sum_{\tau=1}^{T}\alpha_{id}(t-\tau)\tilde{w}_{id}^{\tau} \;\leq\; C_{id}x_{id}.$$

Thus, the first constraint of $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ is satisfied. The second constraint of $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ is satisfied because

$$\frac{1}{2}\tilde{w}_i^{d,t} \;\leq\; \frac{1}{2}v_{id}^t\tilde{w}_0^t\leq\frac{1}{2}v_{id}^t,$$

which follows because $\tilde{w}_0^t \;\leq\; \tilde{w}_0^t+\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}w_{id}^t \;\leq\; 1$ due to the third constraint in $W^{\mathsf{LP}}(\boldsymbol{x})$. The third constraint of $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ is satisfied because $\sum_{i\in\mathcal{N}}\tilde{w}_{id}^t \;\leq\; \tilde{w}_0^t+\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}w_{id}^t \;\leq\; 1$. Therefore, $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})\geq\frac{1}{2}\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}r_{id}\tilde{w}_{id}^t=\frac{1}{2}W^{\mathsf{LP}}(\boldsymbol{x})$. ■

In the next, we show that $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ is also submodular with respect to $\boldsymbol{x}$.

Theorem H.2 (Submodularity of Half-Relaxation) *$W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ is submodular with respect to $\boldsymbol{x}$. Equivalently, for any $\boldsymbol{x}\geq\boldsymbol{y}\in\{0,1\}^L$, $i\in\mathcal{N}$, and $d\in\mathcal{D}_i$, we have*

$$W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x}+\boldsymbol{e}_{id})-W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x}) \;\leq\; W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{y}+\boldsymbol{e}_{id})-W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{y}).$$

Before discussing the proof of this result, we remark on its significance. Note that $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ is non-decreasing in $\boldsymbol{x}$ and $\mathcal{X}$ is described by a matroid constraint. Therefore, there is a $1-\frac{1}{e}$ approximation algorithm to $\max_{\boldsymbol{x}\in\mathcal{X}}W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ (Calinescu et al. 2011) so we can obtain a $\frac{1}{2}\left(1-\frac{1}{e}\right)$ approximate solution to $\max_{\boldsymbol{x}\in\mathcal{X}}W^{\mathsf{LP}}(\boldsymbol{x})$ as long as the assumption in Lemma H.1 is met. In this case, combining with Theorem 2.1 and our discussion in Section 5.1, we can obtain a $\frac{1}{4}\left(1-\frac{1}{e}\right)$ approximation to the joint optimization problem. Clearly, this result is independent of the number of designs we have, which serves as a complement to the $\frac{1}{D}$-approximate algorithm that we have developed in Section 4.1.

Proof of Theorem H.2: Using dual variables $\boldsymbol{\mu}=(\mu_{id}:i\in\mathcal{N},d\in\mathcal{D}_i)$, $\boldsymbol{\sigma}=(\sigma_{id}^t:i\in\mathcal{N},d\in\mathcal{D}_i,t\in\mathcal{T})$, and $\boldsymbol{\theta}=(\theta^t:t\in\mathcal{T})$ for the first three constraints, the dual of $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ can be written as

$$\begin{aligned}
&W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})\\
&=\min_{(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\theta})\geq\boldsymbol{0}}\left\{\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}C_{id}x_{id}\mu_{id}+\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}}\sum_{d\in\mathcal{D}_i}\frac{v_{id}^t}{2}\sigma_{id}^t+\sum_{t\in\mathcal{T}}\frac{\theta^t}{2}\,:\,\mu_{id}+\sigma_{id}^t+\theta^t\geq r_{id},\ \forall i\in\mathcal{N},t\in\mathcal{T},d\in\mathcal{D}_i\right\}\\
&=\min_{\boldsymbol{\mu}\geq\boldsymbol{0}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}C_{id}x_{id}\mu_{id}+\frac{1}{2}\sum_{t\in\mathcal{T}}\min_{(\boldsymbol{\sigma}^t,\theta^t)\geq\boldsymbol{0}}\left\{\theta^t+\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}v_{id}^t\sigma_{id}^t:\sigma_{id}^t+\theta^t\geq r_{id}-\mu_{id}\ \forall i\in\mathcal{N},d\in\mathcal{D}_i\right\}.
\end{aligned}$$

Define $L(\boldsymbol{x};\boldsymbol{\mu})=\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}C_{id}x_{id}\mu_{id}$ and

$$G^t(\boldsymbol{\mu}) \;=\; \min_{(\boldsymbol{\sigma}^t,\theta^t)\geq\boldsymbol{0}}\left\{\theta^t+\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}v_{id}^t\sigma_{id}^t:\sigma_{id}^t+\theta^t\geq r_{id}-\mu_{id}\ \forall i\in\mathcal{N},d\in\mathcal{D}_i\right\}.$$

Then $G^t(\boldsymbol{\mu})$ can also be written in its dual form, which is a knapsack problem:

$$\begin{aligned} G^t(\boldsymbol{\mu}) \;=\; & \max_{\boldsymbol{z}^t \geq 0} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} (r_{id}-\mu_{id})\, z^t_{id} \\ & \text{s.t.} \;\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} z^t_{id} \leq 1 \\ & \qquad z^t_{id} \leq v^t_{id}, \qquad\qquad \forall i\in\mathcal{N}, d\in\mathcal{D}_i. \end{aligned}$$

Define $F(\boldsymbol{x},\boldsymbol{\mu}) = L(\boldsymbol{x},\boldsymbol{\mu}) + \frac{1}{2}\sum_{t\in\mathcal{T}} G^t(\boldsymbol{\mu})$. By definition, $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x}) = \min_{\boldsymbol{\mu}\geq 0} F(\boldsymbol{x},\boldsymbol{\mu})$. We show the submodularity of $W^{\mathsf{LP}}_{\mathsf{app}}(\boldsymbol{x})$ with the following property of $F(\boldsymbol{x},\boldsymbol{\mu})$.

Lemma H.3 *For any* $\boldsymbol{x}\geq \boldsymbol{y}$ *and* $\boldsymbol{\mu}\geq 0$, $t\in\mathcal{T}$, *we have the inequality*

$$F(\boldsymbol{x},\boldsymbol{\mu}) + F(\boldsymbol{y},\boldsymbol{\eta}) \;\geq\; F(\boldsymbol{x},\boldsymbol{\mu}\wedge\boldsymbol{\eta}) + F(\boldsymbol{y},\boldsymbol{\mu}\vee\boldsymbol{\eta}).$$

Proof: By the definition of F, it suffices to show the following two inequalities hold:

$$L(\boldsymbol{x},\boldsymbol{\mu}) + L(\boldsymbol{y},\boldsymbol{\eta}) \;\geq\; L(\boldsymbol{x},\boldsymbol{\mu}\wedge\boldsymbol{\eta}) + L(\boldsymbol{y},\boldsymbol{\mu}\vee\boldsymbol{\eta}), \tag{H.1}$$

$$G^t(\boldsymbol{\mu}) + G^t(\boldsymbol{\eta}) \;\geq\; G^t(\boldsymbol{\mu}\wedge\boldsymbol{\eta}) + G^t(\boldsymbol{\mu}\vee\boldsymbol{\eta}). \tag{H.2}$$

(H.2) follows from Lemma C.1 in Bai et al. (2022). (H.1) follows from the following arguments:

$$\begin{aligned} & L(\boldsymbol{x},\boldsymbol{\mu}) + L(\boldsymbol{y},\boldsymbol{\eta}) \\ =\; & \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} C_{id}\,(x_{id}\mu_{id} + y_{id}\eta_{id}) \\ =\; & \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} C_{id}\,\{x_{id}\,(\mu_{id}\wedge\eta_{id}) + x_{id}\,(\mu_{id}-\mu_{id}\wedge\eta_{id}) + y_{id}\,(\eta_{id}\vee\eta_{id}) + y_{id}\,(\eta_{id}-\mu_{id}\vee\eta_{id})\} \\ \geq\; & \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} C_{id}\,\{x_{id}\,(\mu_{id}\wedge\eta_{id}) + y_{id}\,(\mu_{id}-\mu_{id}\wedge\eta_{id}) + y_{id}\,(\eta_{id}\vee\eta_{id}) + y_{id}\,(\eta_{id}-\mu_{id}\vee\eta_{id})\} \\ =\; & L(\boldsymbol{x},\boldsymbol{\mu}\wedge\boldsymbol{\eta}) + L(\boldsymbol{y},\boldsymbol{\mu}\vee\boldsymbol{\eta}) + \sum_{i\in\mathcal{N},d\in\mathcal{D}_i} C_{id}y_{id}\,(\mu_{id}-\mu_{id}\wedge\eta_{id}+\eta_{id}-\mu_{id}\vee\eta_{id}) \\ =\; & L(\boldsymbol{x},\boldsymbol{\mu}\wedge\boldsymbol{\eta}) + L(\boldsymbol{y},\boldsymbol{\mu}\vee\boldsymbol{\eta}). \end{aligned}$$

This finishes the proof. ■

We next establish the proof of Theorem H.2. It is easy to verify from definition that $L(\boldsymbol{x}+\boldsymbol{e}_{id},\boldsymbol{x}) = L(\boldsymbol{x},\boldsymbol{\mu}) + C_{id}\mu_{id}$ for all $\boldsymbol{x}\in\{0,1\}^L$, $\boldsymbol{\mu}\geq 0$, $i\in\mathcal{N}$, and $d\in\mathcal{D}_i$. Since G does not depend on $\boldsymbol{x}$, it follows that for all $\boldsymbol{x}\in\{0,1\}^L$ and $\boldsymbol{\mu}\geq 0$,

$$F(\boldsymbol{x}+\boldsymbol{e}_{id},\boldsymbol{x}) = F(\boldsymbol{x},\boldsymbol{\mu}) + C_{id}\mu_{id},\; \forall i\in\mathcal{N}, d\in\mathcal{D}_i \tag{H.3}$$

Define $\boldsymbol{\mu}_{\boldsymbol{x}} = \arg\min_{\boldsymbol{\mu}\geq\boldsymbol{0}} F(\boldsymbol{x},\boldsymbol{\mu})$, $\boldsymbol{\mu}^+_{\boldsymbol{x}} = \arg\min_{\boldsymbol{\mu}\geq\boldsymbol{0}} F(\boldsymbol{x}+\boldsymbol{e}_{id},\boldsymbol{\mu})$. Define $\boldsymbol{\mu}_{\boldsymbol{y}} = \arg\min_{\boldsymbol{\mu}\geq\boldsymbol{0}} F(\boldsymbol{y},\boldsymbol{\mu})$,

$\boldsymbol{\mu}_{\boldsymbol{y}}^{+} = \arg\min_{\boldsymbol{\mu}\geq \boldsymbol{0}} F(\boldsymbol{y}+\boldsymbol{e}_{id},\boldsymbol{\mu})$. It then follows that

$$
\begin{aligned}
W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{x}) + W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{y}+\boldsymbol{e}_{id}) &= F(\boldsymbol{x},\boldsymbol{\mu}_{\boldsymbol{x}}) + F(\boldsymbol{y}+\boldsymbol{e}_{id},\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) \\
&\overset{(a)}{=} F(\boldsymbol{x},\boldsymbol{\mu}_{\boldsymbol{x}}) + F(\boldsymbol{y},\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) + C_{id}(\boldsymbol{\mu}_{\boldsymbol{y}}^{+})_{id} \qquad \text{(H.4)} \\
&\overset{(b)}{=} F(\boldsymbol{x},\boldsymbol{\mu}_{\boldsymbol{x}}\wedge\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) + F(\boldsymbol{y},\boldsymbol{\mu}_{\boldsymbol{x}}\vee\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) + C_{id}(\boldsymbol{\mu}_{\boldsymbol{y}}^{+})_{id} \qquad \text{(H.5)} \\
&\geq F(\boldsymbol{x},\boldsymbol{\mu}_{\boldsymbol{x}}\wedge\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) + F(\boldsymbol{y},\boldsymbol{\mu}_{\boldsymbol{x}}\vee\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) + C_{id}(\boldsymbol{\mu}_{\boldsymbol{x}}\wedge\boldsymbol{\mu}_{\boldsymbol{y}}^{+})_{id} \\
&\overset{(c)}{=} F(\boldsymbol{x}+\boldsymbol{e}_{id},\boldsymbol{\mu}_{\boldsymbol{x}}\wedge\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) + F(\boldsymbol{y},\boldsymbol{\mu}_{\boldsymbol{x}}\vee\boldsymbol{\mu}_{\boldsymbol{y}}^{+}) \qquad \text{(H.6)} \\
&\overset{(d)}{\geq} F(\boldsymbol{x}+\boldsymbol{e}_{id},\boldsymbol{\mu}_{\boldsymbol{x}}^{+}) + F(\boldsymbol{y},\boldsymbol{\mu}_{\boldsymbol{y}}) \qquad \text{(H.7)} \\
&= W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{x}+\boldsymbol{e}_{id}) + W_{\mathsf{app}}^{\mathsf{LP}}(\boldsymbol{y}).
\end{aligned}
$$

where (a) and (c) follow from (H.3), and (b) and (d) from Lemma H.3. This is the desired result. ■

H.2. Dynamic Assortment with Discrete Pricing

In this section, we introduce dynamic pricing into our model. Specifically, while continuous pricing might require a new framework with new techniques developed, we show that we can fully extend our approach and incorporate discrete pricing into the joint optimization problem. Our notations and assumptions largely follow our previous discussion, and we assume that we need to choose one design for each product as the offline design decision. For each product $i\in\mathcal{N}$ and design $d\in\mathcal{D}_i$, there is a price menu from which we can choose one price to offer to each arriving customer. The price menus across different products, or across different designs for the same product, can be different from each other. To simplify the presentation, we assume without loss of generality that there is a set of K price indexes $\mathcal{K}=\{1,\dots K\}$ such that p_{ikd} represents the kth price of product i with design d. If the number of prices are different across different products or different designs, we can guarantee this assumption by introducing duplicate prices.

With any design vector $\boldsymbol{x}\in\mathcal{X}$, where again we let

$$
\mathcal{X} = \Big\{\boldsymbol{x}\in\{0,1\}^L : \sum_{d\in\mathcal{D}_i} x_{id} = 1,\ \forall i\in\mathcal{N}\Big\},
$$

committed before the selling season, the retailer needs to decide an assortment on the *product-price* pairs offered to each arriving customer in the selling season, or equivalently choosing an assortment from $\mathcal{F}=\{\mathsf{S}\subset\mathcal{N}\times\mathcal{K}:\ |S\cap S_i|\leq 1\ \forall i\in\mathcal{N}\}$ where $S_i=\{(i,k):\ k\in\mathcal{K}\}$ for each $i\in\mathcal{N}$. We assume for each $i\in\mathcal{N}$ and $k\in\mathcal{K}$, different designs may impact different customer valuation and therefore choice probability, which we capture through $\phi_{ik}^{t}(S\,|\,\boldsymbol{x})$, which is heterogeneous across different customer $t\in\mathcal{T}$. Further, we also assume that the price might impact customer's willingness of return and let $f_{ikd}(\ell)$ denote the probability that the customer returns product i in acutely ℓ time periods with design d and price p_{ikd}. We also define the return probability $\gamma_{ikd}=\sum_{\ell=0}^{\infty} f_{ikd}(\ell)$ and the hazard rate for return $\gamma_{ikd\ell} < 1$ in a similar way to Section 2 and Section 5.1. We assume that other parameters, such as the probability of passing the inspection before reselling, h_{id}, the probability of different reprocessing time, $g_{id}(\ell)$, the return cost c_{id}, and the salve value p^s_{id}, among others, are homogeneous across different prices to simplify presentation. Our objective is very similar to the discussion in the main body; we seek a effective joint decision, including the offline design $\boldsymbol{x}$ and an online policy π, which (approximately) solves

$$\mathsf{OPT} \;=\; \max_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{\pi}\in\Pi} \; \mathbb{E}\Bigg[\sum_{t\in\mathcal{T}} R^{\boldsymbol{\pi}}_t(\boldsymbol{x})\Bigg], \tag{H.8}$$

where Π contains policies that map the history to a joint assortment and discrete pricing decision $S\in\mathcal{F}$. We follow the approximation framework discussed in Theorem 2.1 to solve the problem.

Offline decisions with revenue-based rounding: Defining

$$r_{ikd} \;=\; p_{ikd} - \gamma_{ikd}c_{id} + \gamma_{ikd}(1-h_{id})p^s_{id} - \gamma_{ikd}h_{id}c^p_{id},$$

which is assumed to be positive, and

$$\alpha_{ikd}(\ell) \;=\; 1 - h_{id}\sum_{s=0}^{\ell} f_{ikd}(s)\sum_{\tau=0}^{\ell-s} g_{id}(\tau) \;\geq\; 1-\gamma_{ikd} \;>\; 0,$$

we can write the CDLP for the dynamic joint assortment and discrete pricing problem as

$$\begin{aligned}
\mathsf{UB}(\boldsymbol{x}) \;=\; \max_{\boldsymbol{z}} \quad & \sum_{t\in\mathcal{T}}\sum_{S\in\mathcal{F}} z^t(S)\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}} \phi^t_{ik}(S\,|\,\boldsymbol{x})\, r_{ik}(\boldsymbol{x}) && \text{(H.9)}\\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{S\in\mathcal{F}} z^\tau(S)\sum_{k\in\mathcal{K}} \phi^\tau_{ik}(S\,|\,\boldsymbol{x})\alpha_{ik}\left(t-\tau\,|\,\boldsymbol{x}\right) \;\leq\; C_i(\boldsymbol{x}) && \forall t\in\mathcal{T}, i\in\mathcal{N},\\
& \sum_{S\in\mathcal{F}} z^t(S) \;\leq\; 1 && \forall t\in\mathcal{T},\\
& z^t(S) \;\geq\; 0 && \forall t\in\mathcal{T}, S\in\mathcal{F}\;.
\end{aligned}$$

Here the notations such as $r_{ik}(\boldsymbol{x})$ and $C_i(\boldsymbol{x})$ are defined in a similar fashion to our previous discussions. The readers may compare this formulation with the original upper bound (3.1) and we remark on two facts. First, compared with the setting discussed in the main body, we exclude the pricing decision from the offline product design decision. Instead, we can adjust the prices for each arriving customer. Both formulations root from industrial practices and a practitioner can choose the model that best fits his or her application. Second, from an assortment optimization perspective, it is helpful to think of each product-price pair as a "product." However, a naive approach of introducing binary variables x_{ikd} requires additional constraints to maintain that the same design applies all prices of the same product. This causes difficulties in optimization. Instead, in (H.9), we keep the decision variable x_{id} and adapt the capacity constraints by summing $\phi^\tau_{ik}(S\,|\,\boldsymbol{x})\alpha_{ik}$ over all $k \in \mathcal{K}$ in the left-hand side for each $i \in \mathcal{N}$ and $t \in \mathcal{T}$. It turns out that we can fully extend our previous approach under this formulation.

To start, it is easy to verify that this is indeed an upper bound to $\mathbb{E}\left[\sum_{t\in\mathcal{T}} R_t^{\boldsymbol{\pi}}(\boldsymbol{x})\right]$ for every $\boldsymbol{x} \in \mathcal{X}$ and $\pi \in \Pi$, following the same argument as in Theorem 3.1. Further, similar to the discussion in Section 3.1, we introduce the extended ground set $\mathcal{M} = \{i_k^d : i \in \mathcal{N}, k \in \mathcal{K}, d \in \mathcal{D}_i\}$. On the ground set $\mathcal{M}$, we are interested in assortments in $\mathcal{G} = \{\boldsymbol{A} \subset \mathcal{M} : |\boldsymbol{A} \cap \boldsymbol{A}_{id}| \le 1, \ \forall i \in \mathcal{N}, d \in \mathcal{D}_i\}$, where $\boldsymbol{A}_{id} = \{i_k^d : k \in \mathcal{K}\}$ for each $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$. Let $\boldsymbol{A}(S,\boldsymbol{x}) = \left\{i_k^d : (i,k) \in S, x_{id} = 1\right\}$ and $\mathcal{G}(\boldsymbol{x}) = \left\{\boldsymbol{A}(S,\boldsymbol{x}) : S \in \mathcal{F}\right\}$. We assume that the choice models $\phi^1, \phi^2, \ldots, \phi^T$ are consistent in the sense that there are choice models $\psi^1, \psi^2, \ldots, \psi^T$ such that for any $\boldsymbol{x} \in \mathcal{X}$ and $S \subset \mathcal{N}$, $\phi_{ik}(S \mid \boldsymbol{x}) = \psi_{i^d}(\boldsymbol{A}(S,\boldsymbol{x}))$ for all $d \in \mathcal{D}_i$ with $x_{id} = 1$; ψ^t satisfy the substitutability assumption for all $t \in \mathcal{T}$. We also note that both $\mathcal{F}$ and $\mathcal{G}$ are downward inclusive. We remark on a key property that will be useful later.

Lemma H.4 *Let us fix* $\boldsymbol{x} \in \mathcal{X}$*.*

(i) $\mathcal{G}(\boldsymbol{x}) \subset \mathcal{G}$.

(ii) Assume that $\boldsymbol{A} \in \mathcal{G} \backslash \mathcal{G}(\boldsymbol{x})$*. Define* $\boldsymbol{A}' = \left\{i_k^d \in \boldsymbol{A} : x_{id} = 1\right\}$*. Then* $\boldsymbol{A}' \in \mathcal{G}(\boldsymbol{x})$*.*

Proof: We first argue the first part. Consider $\boldsymbol{A}(S,\boldsymbol{x})$ where $S \in \mathcal{F}$. Fix $i \in \mathcal{N}$. If $|S \cap S_i| = 0$, then clearly $\boldsymbol{A}(S,\boldsymbol{x}) \cap \boldsymbol{A}_{id} = \varnothing$ for all $d \in \mathcal{D}_i$. Otherwise, assume that $S \cap S_i = (i,k)$. If $x_{id} = 0$, it must be that $\boldsymbol{A}(S,\boldsymbol{x}) \cap \boldsymbol{A}_{id} = 0$ by the definition of $\boldsymbol{A}(S,\boldsymbol{x})$. If $x_{id} = 1$, clearly $\boldsymbol{A}(S,\boldsymbol{x}) \cap \boldsymbol{A}_{id} = \{i_k^d\}$. This proves the first item.

For the second part, we fix any $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$. By the definition of $\mathcal{G}$, if $\boldsymbol{A} \cap \boldsymbol{A}_{id} \neq \varnothing$, it must be there is a unique $k(i,d) \in \boldsymbol{A}_{id}$ such that $i_{k(i,d)}^d \in \boldsymbol{A}$. Then clearly

$$\boldsymbol{A}' = \left\{i_{k(i,d)}^d \in \boldsymbol{A} : |\boldsymbol{A} \cap \boldsymbol{A}_{id}| = 1 \ \text{ and } \ x_{id} = 1\right\} = \boldsymbol{A}(S,\boldsymbol{x}),$$

where $S \in \mathcal{F}$ is the collection of (i,k) such that for $i \in \mathcal{N}$ there is a unique $d \in \mathcal{D}_i$ with $|\boldsymbol{A} \cap \boldsymbol{A}_{id}| = 1$, $x_{id} = 1$ and $k = k(i,d)$. ■

With the definitions so far, we consider an alternative CDLP formulation defined on $\mathcal{M}$ as

$$\begin{aligned}
\mathsf{VB}(\boldsymbol{x}) = \max_{\boldsymbol{y} \ge \boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}} \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{i\in\mathcal{N}} \sum_{k\in\mathcal{K}} \sum_{d\in\mathcal{D}_i} \psi_{i_k^d}^t(\boldsymbol{A}) r_{ikd} && \text{(H.10)} \\
\text{s.t.} \quad & \sum_{\tau=1}^{t} \sum_{\boldsymbol{A}\in\mathcal{G}} y^\tau(\boldsymbol{A}) \sum_{k\in\mathcal{K}} \psi_{i_k^d}^\tau(\boldsymbol{A}) \alpha_{ikd}(t-\tau) \le C_{id} x_{id} && \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \le 1 && \forall\, t \in \mathcal{T}\ ,
\end{aligned}$$

which is an analogue to (3.2). We next prove the following.

Theorem H.5 (Equivalent Representation with Pricing) *For each* $\boldsymbol{x}\in\mathcal{X}$*,* $\mathsf{UB}(\boldsymbol{x})=\mathsf{VB}(\boldsymbol{x})$*.*

Proof: We only sketch the proof here because the argument is very similar to that of Theorem 3.5. Let us fix $\boldsymbol{x}\in\mathcal{X}$. Consider an arbitrary feasible solution $\boldsymbol{z}=\{z^t(S):t\in\mathcal{T},S\in\mathcal{F}\}$ to the upper bound $\mathsf{UB}(\boldsymbol{x})$ in (H.9). Then for each $\boldsymbol{A}\in\mathcal{G}$, we define

$$y^t(\boldsymbol{A})=\begin{cases} z^t(S) & \text{if there exists } S\in\mathcal{F} \text{ such that } \boldsymbol{A}=\boldsymbol{A}(S,\boldsymbol{x}) \\ 0 & \text{otherwise.}\end{cases}$$

We notice that the mapping $\boldsymbol{A}(\cdot,\boldsymbol{x})$ from $\mathcal{F}$ to $\mathcal{G}(\boldsymbol{x})$ is a bijection so $y^t(\boldsymbol{A})\geq 0$ and $\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A})\leq 1$ for all t. Also, because of the consistency assumption, it holds that $\psi^t_{i^d_k}(A(S,\boldsymbol{x}))=\phi^t_{ik}(S\,|\,\boldsymbol{x})$ if $x_{id}=1$. If $x_{id}=0$, $\psi^t_{i^d_k}(\boldsymbol{A}(S,\boldsymbol{x}))=0$ by definition of $\boldsymbol{A}(S,\boldsymbol{x})$. Therefore, by the same argument as in Theorem 3.5, the capacity constraint is satisfied and the objective value of (H.10) with $\boldsymbol{y}$ is the same as $\mathsf{UB}(\boldsymbol{x})$. Therefore, $\mathsf{UB}(\boldsymbol{x})\leq\mathsf{VB}(\boldsymbol{x})$.

To prove the reversed equation $\mathsf{UB}(\boldsymbol{x})\geq\mathsf{VB}(\boldsymbol{x})$, the key is to observe that it still holds that there always exists an optimal solution $\bar{\boldsymbol{y}}$ to $\mathsf{VB}(\boldsymbol{x})$ such that $\bar{y}^t(\boldsymbol{A})=0$ if $\boldsymbol{A}\notin\mathcal{G}(\boldsymbol{x})$. Indeed, if $\bar{y}^t(\boldsymbol{A}_1)>0$ and $A_1\notin\mathcal{G}(\boldsymbol{x})$, we can define $\boldsymbol{A}_1'=\{i^d_k\in\boldsymbol{A}_1 : x_{id}=1\}$. By Lemma H.4, $\boldsymbol{A}_1'\in\mathcal{G}(\boldsymbol{x})$. For any $i^d_k\in\boldsymbol{A}_1\backslash\boldsymbol{A}_1'$, we argue that it must be that $\psi^t_{i^d_k}(\boldsymbol{A}_1)=0$. Indeed, feasibility implies

$$0\;\geq\;\sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^\tau(\boldsymbol{A})\sum_{k\in\mathcal{K}}\psi^\tau_{i^d_k}(\boldsymbol{A})\alpha_{ikd}(t-\tau)\;\geq\;\bar{y}^t(\boldsymbol{A}_1)\psi^t_{i^d_k}(\boldsymbol{A}_1)(1-\gamma_{ikd})$$

so $\psi^t_{i^d_k}(\boldsymbol{A}_1)=0$. Therefore, we can apply the same iterative procedure as in Lemma 3.4 to construct a solution $\tilde{\boldsymbol{y}}$ with exactly the same objective value such that $\tilde{\boldsymbol{y}}(\boldsymbol{A})>0$ only when $\boldsymbol{A}\in\mathcal{G}(\boldsymbol{x})$. Then, we let $\{\bar{y}^t(\boldsymbol{A}):\boldsymbol{A}\in\mathcal{G}\}$ be optimal to the linear program associated with $\mathsf{VB}(\boldsymbol{x})$ such that $\bar{y}^t(\boldsymbol{A})=0$ for all $t\in\mathcal{T}$ and $\boldsymbol{A}\notin\mathcal{G}(\boldsymbol{x})$. For each $S\in\mathcal{F}$, let $\bar{z}^t(S)=\bar{y}^t(\boldsymbol{A}(S,\boldsymbol{x}))$. Using the same argument leading to Theorem 3.5, we can show $\bar{z}^t(S)$ is feasible to $\mathsf{UB}(\boldsymbol{x})$ with the objective value equal to $\mathsf{VB}(\boldsymbol{x})$. For instance, we can verify the capacity constraint as follows. Fix any $i\in\mathcal{N}$ and $t\in\mathcal{T}$, there exists an $d\in\mathcal{D}_i$ such that $x_{id}=1$ with

$$\begin{aligned}\sum_{S\in\mathcal{F}}\bar{z}^t(S)\sum_{k\in\mathcal{K}}\phi^t_{ik}(S\,|\,\boldsymbol{x}) &\overset{(a)}{=} \sum_{S\in\mathcal{F}}\bar{y}^t(\boldsymbol{A}(S,\boldsymbol{x}))\sum_{k\in\mathcal{K}}\psi^t_{i^d_k}(\boldsymbol{A}(S,\boldsymbol{x})) \overset{(b)}{=} \sum_{\boldsymbol{A}\in\mathcal{G}(\boldsymbol{x})}\bar{y}^t(\boldsymbol{A})\sum_{k\in\mathcal{K}}\psi^t_{i^d_k}(\boldsymbol{A})\\ &\overset{(c)}{=} \sum_{\boldsymbol{A}\in\mathcal{G}}\bar{y}^t(\boldsymbol{A})\sum_{k\in\mathcal{K}}\psi^t_{i^d_k}(\boldsymbol{A})\end{aligned}$$

where (a) follows form the construction of $\boldsymbol{z}$ and the consistency assumption of the choice models, (b) follows because the mapping is a bijection, (c) follows because $\bar{y}^t(\boldsymbol{A})=0$ if $\boldsymbol{A}\notin\mathcal{G}(\boldsymbol{x})$ by construction. This shows that the capacity constraints are satisfied. In summary, it must be that $\mathsf{UB}(\boldsymbol{x})\geq\mathsf{VB}(\boldsymbol{x})$, so the desired result holds. ■

Therefore, we solve a linear programming relaxation problem $Z^{\mathsf{LP}} = \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{LP}}} \mathsf{VB}(\boldsymbol{x})$, where

$$\mathcal{X}^{\mathsf{LP}} = \left\{ \boldsymbol{x} \in [0,1]^L \;:\; \sum_{d\in\mathcal{D}_i} x_{id} = 1,\; x_{id} \geq 0 \;\; \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i \right\} .$$

We allow each x_{id} to be any fractional value in $[0,1]$ and round the fractional solution to obtain an integer solution. For each $\boldsymbol{y} = \big(y^t(\boldsymbol{A}) : \boldsymbol{A}\in\mathcal{G}, t\in\mathcal{T}\big)$, let

$$\mathsf{Rev}_{id}(\boldsymbol{y}) = \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{k\in\mathcal{K}} \psi^t_{id}(\boldsymbol{A}) r_{ikd},$$

which is the revenue contribution of design d of product i in the optimal solution of the linear relaxation. With this definition, the execution of the revenue-based rounding is exactly the same as the discussion in Section 4.1.

Recall that as defined in Section 2.3, $\mathsf{Eff}_{\text{off}}(\boldsymbol{x}) = \mathsf{UB}(\boldsymbol{x})/\mathsf{UB}^*$ for any $\boldsymbol{x}$, where $\mathsf{UB}^* = \max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{UB}(\boldsymbol{x})$. Then the next result follows. We only provide a sketch of the proof, because it follows the same line of analysis leading to Theorem 4.2.

Theorem H.6 (Offline Effectiveness with Pricing) $\mathsf{Eff}_{\text{off}}(\boldsymbol{x}^H) \geq \frac{1}{D}$.

Proof: We start with some preliminary definitions. Let $\mathcal{P} = \{\boldsymbol{\mathcal{B}} = (\mathcal{B}_1,\ldots,\mathcal{B}_n) : \mathcal{B}_i \subseteq \mathcal{D}_i, \forall i\in\mathcal{N}\}$. Let $\bar{\mathcal{X}} = [0,1]^L$. Note that $\mathcal{X}^{\mathsf{LP}} \subset \bar{\mathcal{X}}$. For each $\boldsymbol{\mathcal{B}}\in\mathcal{P}$ and $\boldsymbol{u}\in\bar{\mathcal{X}}$, define the linear programs $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ and $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ be defined as follow:

$$\begin{aligned}
Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \;=\; \max_{\boldsymbol{y}\geq\boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A}) r_{ikd} \\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A}) \alpha_{ikd}(t-\tau) \;\leq\; C_{id} u_{id} && \forall\, i\in\mathcal{N},\; d\in\mathcal{B}_i,\; t\in\mathcal{T}, \\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \;\leq\; 1 && \forall\, t\in\mathcal{T},
\end{aligned}$$

and

$$\begin{aligned}
Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \;=\; \max_{\boldsymbol{y}\geq\boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A}) r_{id} \\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A}) \alpha_{ikd}(t-\tau) \;\leq\; C_{id} u_{id} && \forall\, i\in\mathcal{N},\; d\in\mathcal{B}_i,\; t\in\mathcal{T}, \\
& \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A}) \alpha_{ikd}(t-\tau) \;\leq\; 0 && \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i\setminus\mathcal{B}_i,\; t\in\mathcal{T}, \\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \;\leq\; 1 && \forall\, t\in\mathcal{T}.
\end{aligned}$$

With these definitions, we can establish the following identities.

(*i*) For each $\boldsymbol{\mathcal{B}}\subseteq\mathcal{P}$ and $\boldsymbol{u}\in\bar{\mathcal{X}}$, $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \;=\; Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$.

(*ii*) For each $\boldsymbol{x} \in \{0,1\}^L$ and $\boldsymbol{u} \in \bar{\mathcal{X}}$, $\mathsf{UB}(\boldsymbol{x}) \ \geq \ Z^1\big(\boldsymbol{\mathcal{B}}(\boldsymbol{x}), \boldsymbol{u}\big)$, where $\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = (\mathcal{B}_1(\boldsymbol{x}), \ldots, \mathcal{B}_n(\boldsymbol{x}))$, where $\mathcal{B}_i(\boldsymbol{x}) = \{d \in \mathcal{D}_i : x_i^d = 1\}$.

(*iii*) For each $\boldsymbol{\mathcal{B}} \in \mathcal{P}$ and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ that is a feasible solution to the linear program Z^{LP},

$$Z^1(\boldsymbol{\mathcal{B}}, \bar{\boldsymbol{x}}) \ \geq \ \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} \mathsf{Rev}_{id}(\bar{\boldsymbol{y}}) \ .$$

Let us verify the item (*i*), which involves a slightly different dual formulation, compared with the proof of Lemma D.1. Fix arbitrary $\boldsymbol{\mathcal{B}} \in \mathcal{P}$ and $\boldsymbol{u} \in \bar{\mathcal{X}}$. Consider any feasible solution to $Z^2(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$. From the constraints associated with $Z^2(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$ and the fact that $0 < 1 - \gamma_{ikd} \leq \alpha_{ikd}(t - \tau)$ for all $i \in \mathcal{N}$ and $d \in \mathcal{D}_i$, it follows that

$$y^t(\boldsymbol{A}) \psi^t_{i_k^d}(\boldsymbol{A}) \ = \ 0 \qquad \forall \ (i,k) \in \mathcal{N} \times \mathcal{K}, \ d \in \mathcal{D}_i \setminus \mathcal{B}_i, \ t \in \mathcal{T}.$$

Therefore, the objective function for the linear program for $Z^2(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$ is equal to

$$\sum_{t \in \mathcal{T}} \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} \sum_{k \in \mathcal{K}} \psi^t_{i_k^d}(\boldsymbol{A}) r_{ikd} \ ,$$

which is the same as the objective function for $Z^1(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$. Since the optimization problem for $Z^2(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$ have more constraints and we are maximizing, it follows that $Z^2(\boldsymbol{\mathcal{B}}, \boldsymbol{u}) \leq Z^1(\boldsymbol{\mathcal{B}}, \boldsymbol{u})$.

To establish the reverse inequality, we use strong duality of linear program, which shows that

$$\begin{aligned}
Z^2(\boldsymbol{\mathcal{B}}, \boldsymbol{u}) \ = \ & \min_{(\boldsymbol{\lambda}, \boldsymbol{\theta}) \geq \boldsymbol{0}} \quad \sum_{t \in \mathcal{T}} \lambda^t \ + \ \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} C_{id} u_{id} \theta^t_{id} \\
& \quad \text{s.t.} \quad \lambda^t \ \geq \ \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \sum_{k \in \mathcal{K}} \psi^t_{i_k^d}(\boldsymbol{A}) \left(r_{ikd} - \sum_{\tau = t}^{T} \alpha_{ikd}(\tau - t) \theta^t_{id} \right) \quad \forall \ t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G}, \\
= \ & \min_{(\boldsymbol{\lambda}, \boldsymbol{\theta}) \geq \boldsymbol{0}} \quad \sum_{t \in \mathcal{T}} \lambda^t \ + \ \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} C_{id} u_{id} \theta^t_{id} \\
& \quad \text{s.t.} \quad \lambda^t \ \geq \ \max_{\boldsymbol{A} \in \mathcal{G}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \sum_{k \in \mathcal{K}} \psi^t_{i_k^d}(\boldsymbol{A}) \left(r_{ikd} - \sum_{\tau = t}^{T} \alpha_{ikd}(\tau - t) \theta^t_{id} \right) \quad \forall \ t \in \mathcal{T},
\end{aligned}$$

and similarly,

$$\begin{aligned}
Z^1(\boldsymbol{\mathcal{B}}, \boldsymbol{u}) \ = \ & \min_{(\boldsymbol{\lambda}, \boldsymbol{\theta}) \geq \boldsymbol{0}} \quad \sum_{t \in \mathcal{T}} \lambda^t \ + \ \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} C_{id} u_{id} \theta^t_{id} \\
& \quad \text{s.t.} \quad \lambda^t \ \geq \ \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} \sum_{k \in \mathcal{K}} \psi^t_{i_k^d}(\boldsymbol{A}) \left(r_{ikd} - \sum_{\tau = t}^{T} \alpha_{ikd}(\tau - t) \theta^t_{id} \right) \quad \forall \ t \in \mathcal{T}, \boldsymbol{A} \in \mathcal{G}, \\
= \ & \min_{(\boldsymbol{\lambda}, \boldsymbol{\theta}) \geq \boldsymbol{0}} \quad \sum_{t \in \mathcal{T}} \lambda^t \ + \ \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} C_{id} u_{id} \theta^t_{id} \\
& \quad \text{s.t.} \quad \lambda^t \ \geq \ \max_{\boldsymbol{A} \in \mathcal{G}} \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{B}_i} \sum_{k \in \mathcal{K}} \psi^t_{i_k^d}(\boldsymbol{A}) \left(r_{ikd} - \sum_{\tau = t}^{T} \alpha_{ikd}(\tau - t) \theta^t_{id} \right) \quad \forall \ t \in \mathcal{T}.
\end{aligned}$$

To prove that $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \geq Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$, we will first establish the following claim.

Claim: For each $\boldsymbol{\theta} \geq \mathbf{0}$ and $t \in \mathcal{T}$,

$$\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A})\left(r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id}\right) \geq \max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A})\left(r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id}\right)$$

The above claim shows that the feasible region associated with $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ is a <u>*subset*</u> of the feasible region for $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$. Because we are minimizing and the objective function is the same in $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ and $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$, it follows that $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u}) \geq Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$, which is the desired result.

To prove the claim, let $\boldsymbol{Q} = \{i^d_k \in \mathcal{M} : (i,k) \in \mathcal{N}\times\mathcal{K}, d \in \mathcal{B}_i\}$. Then, we have

$$\begin{aligned}
&\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A})\left(r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id}\right)\\
\overset{(a)}{=}\; &\max_{\boldsymbol{A}\in\mathcal{G}\,:\,\boldsymbol{A}\subseteq\boldsymbol{Q}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A})\left(r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id}\right)\\
\overset{(b)}{=}\; &\max_{\boldsymbol{A}\in\mathcal{G}\,:\,\boldsymbol{A}\subseteq\boldsymbol{Q}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A})\left(r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id}\right)\\
\leq\; &\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}}\sum_{k\in\mathcal{K}} \psi^t_{i^d_k}(\boldsymbol{A})\left(r_{ikd} - \sum_{\tau=t}^T \alpha_{id}(\tau-t)\theta^t_{ikd}\right),
\end{aligned}$$

where (a) follows because we only consider products in $\boldsymbol{Q}$ in our objective function, so by the substitutability property of the choice function ψ^t, it is not optimal to include products outside $\boldsymbol{Q}$. Indeed, let us assume

$$\widetilde{\boldsymbol{A}} := \arg\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\sum_{k\in\mathcal{K}} \psi^t_{id}(\boldsymbol{A})\left(r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id}\right).$$

We can remove products i^d_k from $\widetilde{\boldsymbol{A}}$ with $i^d_k \in \widetilde{\boldsymbol{A}} \cap Q$, $r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id} \leq 0$ and $i^d_k \in \widetilde{\boldsymbol{A}} \setminus Q$. Substitutability implies the choice probability of any $i^d_k \in \widetilde{\boldsymbol{A}} \cap Q$ with $r_{ikd} - \sum_{\tau=t}^T \alpha_{ikd}(\tau-t)\theta^t_{id} > 0$ will increase and we obtain a (weakly) better assortments. Equality (b) follows because $\boldsymbol{A} \subseteq \boldsymbol{Q}$, so $\psi^t_{i^d_k}(\boldsymbol{A}) = 0$ for all $i^d_k \notin \boldsymbol{Q}$. The proves the desired claim.

We note that items (ii) and (iii) can be established with exactly the same argument of Lemma D.2 and Lemma D.3. With these items established, we can verify the proof of Theorem 4.2 line by line to prove Theorem H.6. ■

<u>*Solving the continuous relaxation*</u>: Along the same line of discussions as in the Appendix F, to solve $Z^{\mathsf{LP}} = \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{LP}}} \mathsf{VB}(\boldsymbol{x})$, we can derive its dual as

$$Z^{\mathsf{LP}} = \min_{\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\eta}} \sum_{i\in\mathcal{N}} \eta_i + \sum_{t\in\mathcal{T}} \lambda^t \tag{H.11}$$

$$
\begin{aligned}
\text{s.t.} \quad & \eta_i \ \geq \ \sum_{t\in\mathcal{T}} C_i^d \theta_{id}^t && \forall\ i\in\mathcal{N}, d\in\mathcal{D}_i,\\
& \lambda^t \ \geq \ \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\sum_{k\in\mathcal{K}} \psi^t_{i_k^d}(\boldsymbol{A})\Big(r_{ikd} - \sum_{\tau=1}^{t}\alpha_{ikd}(\tau-t)\theta_{id}^{\tau}\Big) && \forall\ t\in\mathcal{T}, \boldsymbol{A}\in\mathcal{G},\\
& \theta_{id}^t \geq 0, \quad \lambda^t \geq 0 && \forall\ i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}.
\end{aligned}
$$

The dual has exponentially many constraints, but we can approach the problem with the ellipsoid method. In particular, this requires us to solve the assortment optimization problem

$$
\max_{\boldsymbol{A}\in\mathcal{G}} \sum_{i^d\in\mathcal{M}} \psi_{i_k^d}(\boldsymbol{A})\mu_{id} \tag{H.12}
$$

for each $\boldsymbol{\mu}\in\mathbb{R}^L$. Let us recall that $\mathcal{G} = \{\boldsymbol{A}\subset\mathcal{M} : |\boldsymbol{A}\cap\boldsymbol{A}_{id}|\leq 1,\ \forall i\in\mathcal{N},\, d\in\mathcal{D}_i\}$, where $\boldsymbol{A}_{id} = \{i_k^d : k\in\mathcal{K}\}$. Then (H.12) is referred to as the static assortment and discrete pricing problem in the literature. Under the MNL model, (Sumida et al. 2021) show that this problem can be solved exactly and efficiently as a parameter linear programming problem although no polynomial running time guarantee is not provided. Following this approach and using the same analysis leading to Theorem F.2, we can solve $Z^{\mathsf{LP}} = \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{LP}}}\mathsf{VB}(\boldsymbol{x})$ exactly by solving a polynomial number of parametric linear programs. Furthermore, Sumida et al. (2021) also provide an FPTAS for the problem under the MNL model If we use this approach as our separation oracle, we can obtain an FPTAS for Z^{LP} following the analysis of Theorem F.5. Then using the counterpart of Theorem 4.4, we can obtain a $\frac{1}{D}(1-\delta)$ approximation to the offline design problem.

Dynamic joint assortment and discrete pricing problem: Next, we consider optimizing the online allocation policy following the value function approximation discussed in Section 5.1. Let us fix an offline design $\boldsymbol{x}$. We also assume without loss of generality that $c_{id}^r = p_{id}^s = c_{id}^p = 0$ and therefore $r_{ikd} = p_{ikd}$; see Lemma 5.1. We need to define our state space differently to accommodate the pricing decision. In particular, we define the state vector $\boldsymbol{q} = (\boldsymbol{w}, \tilde{\boldsymbol{w}})$, such that for each $i\in\mathcal{N}$, w_{i0} corresponds to the on-hand inventory available for selling, $w_{i\ell}$ captures the number of units that are being reprocessed for exactly ℓ periods for every $\ell\geq 1$, $\tilde{w}_{ik\ell}$ is the number of units sold with price p_{ikd} that are with a customer for exactly ℓ periods, where $d\in\mathcal{D}_i$ is such that $x_{id}=1$. The set of feasible states are given by

$$
\mathcal{Q}(\boldsymbol{x}) = \left\{\boldsymbol{q} = (\boldsymbol{w}, \tilde{\boldsymbol{w}}) : \sum_{\ell=0}^{\infty} w_{i\ell} + \sum_{k\in\mathcal{K}}\sum_{\ell=1}^{\infty}\tilde{w}_{ik\ell} \leq C_i(\boldsymbol{x})\ \forall i\in\mathcal{N}\right\}.
$$

We further define

$$
\mathsf{A}_{ik\ell}(\boldsymbol{x}) \ = \ \mathsf{Ber}\big(\gamma_{ik\ell}(\boldsymbol{x})\big),\ \mathsf{B}_i(\boldsymbol{x}) \ = \ \mathsf{Bin}\left(\sum_{k\in\mathcal{K}}\sum_{\ell=1}^{\infty}\tilde{w}_{ik\ell}\mathsf{A}_{ik\ell}(\boldsymbol{x}),\, h_i(\boldsymbol{x})\right),
$$

$$
\mathsf{C}_i(\boldsymbol{x}) \ = \ \mathsf{Bin}\big(\mathsf{B}_i(\boldsymbol{x}), \zeta_{i0}(\boldsymbol{x})\big) \ \text{ and } \ \mathsf{D}_{i\ell}(\boldsymbol{x}) \ = \ \mathsf{Bin}\big(w_{i\ell}, \zeta_{i\ell}(\boldsymbol{x})\big).
$$

We let $\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) = (\boldsymbol{U}(\boldsymbol{q},\boldsymbol{x}), \widetilde{\boldsymbol{U}}(\boldsymbol{q},\boldsymbol{x}))$ describe the system dynamics when we do not offer any products. Particularly, for each $i \in \mathcal{N}$,

$$U_{i\ell}(\boldsymbol{q},\boldsymbol{x}) = \begin{cases} w_{i0} + \mathsf{C}_i(\boldsymbol{x}) + \sum_{s=1}^{\infty} \mathsf{D}_{is}(\boldsymbol{x}) & \text{if } \ell = 0 \\ \mathsf{B}_i(\boldsymbol{x}) - \mathsf{C}_i(\boldsymbol{x}) & \text{if } \ell = -1 \\ w_{i,\ell-1} - \mathsf{D}_i(\ell - 1 \,|\, \boldsymbol{x}) & \text{if } \ell \geq 2 \end{cases},$$

and for each $i \in \mathcal{N}$ and $k \in \mathcal{K}$,

$$\widetilde{U}_{ik\ell}(\boldsymbol{q},\boldsymbol{x}) = \begin{cases} \tilde{w}_{i,\ell-1}\big(1 - \mathsf{A}_{i,\ell-1}(\boldsymbol{x})\big) & \text{if } \ell \geq 2 \\ 0 & \text{if } \ell = 1 \end{cases}.$$

Using $\mathbf{0}$ to denote a vector of zeros of appropriate dimensions. We can describe the dynamic programming formulation as

$$\begin{aligned} J^t(\boldsymbol{q} \,|\, \boldsymbol{x}) =& \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big] + \max_{S \in \mathcal{F} \,:\, w_{i0} \geq 1 \;\forall (i,k) \in S} \Bigg\{ \sum_{i\in\mathcal{N}} \sum_{k \in \mathcal{K}} \phi^t_{ik}(S \,|\, \boldsymbol{x}) \Bigg[r_{ik}(\boldsymbol{x}) \\ &+ \big(1 - \gamma_{ik0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) + (-\boldsymbol{e}_{i0}, \boldsymbol{e}_{ik1}) \;\big|\; \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \;\big|\; \boldsymbol{x}\big)\Big]\Big) \\ &+ \gamma_{ik0}(\boldsymbol{x}) h_i(\boldsymbol{x})\big(1 - \zeta_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) + (-\boldsymbol{e}_{i0} + \boldsymbol{e}_{i1}, \mathbf{0}) \big| \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \big| \boldsymbol{x}\big)\Big]\Big) \\ &+ \gamma_{ik0}(\boldsymbol{x})\big(1 - h_i(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - (\boldsymbol{e}_{i0}, 0) \;\big|\; \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \,|\, \boldsymbol{x}\big)\Big]\Big)\Bigg]\Bigg\}, \end{aligned} \tag{H.13}$$

which is analogous to (5.1). Computing the value function $J^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0} \;\Big|\; \boldsymbol{x}\Big)$ exactly is again intractable and we propose a greedy assortment policy $\boldsymbol{\pi}^H$ based the following linear value function approximation: for each state $\boldsymbol{q} \in \mathcal{Q}(\boldsymbol{x})$, define

$$\hat{J}^t(\boldsymbol{q} \,|\, \boldsymbol{x}) \;=\; \sum_{i\in\mathcal{N}} \left(\sum_{\ell=0}^{\infty} \beta^t_{i\ell}(\boldsymbol{x}) w_{i\ell} + \sum_{\ell=1}^{\infty} \sum_{k\in\mathcal{K}} \tilde{\beta}^t_{ik\ell}(\boldsymbol{x})\, \tilde{w}_{ik\ell} \right), \tag{H.14}$$

where the coefficient $\beta^t_{i0}(\boldsymbol{x})$ denotes the marginal value of product i's on-hand inventory, $\beta^t_{i\ell}(\boldsymbol{x})$ represents the marginal value of each unit of product i that is being reprocessed for ℓ periods, for each $\ell \geq 1$, and $\tilde{\beta}^t_{ik\ell}(\boldsymbol{x})$ captures the the marginal value of each unit of product i that was sold with price p_{ikd}, where d is such that $x_{id} = 1$ and is being tried by a customer for ℓ periods, for each $\ell \geq 1$.

We define these coefficients recursively as follows. We initialize by setting $\beta^{T+1}_{i\ell}(\boldsymbol{x}) = 0$ for all $i \in \mathcal{N}$ and $\ell \geq 0$ and $\tilde{\beta}^{T+1}_{ik\ell}(\boldsymbol{x}) = 0$ for all $(i,k) \in \mathcal{N} \times \mathcal{K}$ and $\ell \geq 1$. We also recursively define for $t = T, T-1, \dots, 1$

$$\begin{aligned} \xi^t_{ik}(\boldsymbol{x}) \;=\; & r_{ik}(\boldsymbol{x}) + \big(1 - \gamma_{ik0}(\boldsymbol{x})\big)\big(\tilde{\beta}^{t+1}_{ik1}(\boldsymbol{x}) - \beta^{t+1}_{i0}(\boldsymbol{x})\big) + \gamma_{ik0}(\boldsymbol{x}) h_i(\boldsymbol{x})\big(1 - \zeta_{i0}(\boldsymbol{x})\big)\big(\beta^{t+1}_{i1}(\boldsymbol{x}) - \beta^{t+1}_{i0}(\boldsymbol{x})\big) \\ & - \gamma_{ik0}(\boldsymbol{x})\big(1 - h_i(\boldsymbol{x})\big)\beta^{t+1}_{i0}(\boldsymbol{x}) \end{aligned} \tag{H.15}$$

for all $i \in \mathcal{N}$. Furthermore,

$$A^t(\boldsymbol{x}) \;=\; \arg\max_{S\in\mathcal{F}} \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}} \phi^t_{ik}(S\,|\,\boldsymbol{x})\xi^t_{ik}(\boldsymbol{x}). \tag{H.16}$$

We also define for each $i \in \mathcal{N}$ and $t \in \mathcal{T}$,

$$\beta^t_{i\ell}(\boldsymbol{x}) = \begin{cases} \beta^{t+1}_{i0}(\boldsymbol{x}) \;+\; \frac{1}{C_i(\boldsymbol{x})}\sum_{k\in\mathcal{K}} \phi^t_i(A^t(\boldsymbol{x})\,|\,\boldsymbol{x})\,\xi^t_{ik}(\boldsymbol{x}) & \text{if } \ell = 0 \\ \zeta_{i\ell}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) \;+\; (1-\zeta_{i\ell}(\boldsymbol{x}))\beta^{t+1}_{i,\ell+1}(\boldsymbol{x}) & \text{if } \ell \geq 1 \end{cases}. \tag{H.17}$$

and

$$\tilde{\beta}^t_{ik\ell}(\boldsymbol{x}) = \gamma_{ik\ell}(\boldsymbol{x})h_i(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})\beta^{t+1}_{i0}(\boldsymbol{x}) + \gamma_{ik\ell}(\boldsymbol{x})h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\beta^{t+1}_{i1}(\boldsymbol{x}) \;+\; \big(1-\gamma_{ik\ell}(\boldsymbol{x})\big)\tilde{\beta}^{t+1}_{ik,\ell+1} \tag{H.18}$$

for all $\ell \geq 1$. These coefficients admit similar intuitive explanations of those defined in Section 5.1, but are adapted to fit the current model. For example, in (H.17), the marginal value of a unit of on-hand inventory of product i, $\beta^t_{i0}(\boldsymbol{x})$, takes into account of the possibilities of all possible price allocations. With these definitions, define $\boldsymbol{\pi}^H$ as the policy that offers the following assortment to customer $t \in \mathcal{T}$.

$$G^t(\boldsymbol{q}\,|\,\boldsymbol{x}) \;=\; \arg \max_{\substack{S\in\mathcal{F}:\\ w_{i0}\geq 1\\ \forall\,(i,k)\in S}} \left\{ \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}} \phi^t_i(S\,|\,\boldsymbol{x})\xi^t_{ik}(\boldsymbol{x}) \right\}. \tag{H.19}$$

We state the key result regarding the bound on the conditional online effectiveness of policy $\boldsymbol{\pi}^H$.

Theorem H.7 (Offline Effectiveness of Pricing) *For each $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{\pi}^H$,*

$$\mathsf{Eff}_{\text{on}}(\boldsymbol{\pi}^H\,|\,\boldsymbol{x}) \;=\; \frac{1}{\mathsf{UB}(\boldsymbol{x})}\cdot\mathbb{E}\Big[\sum_{t\in\mathcal{T}} R^{\boldsymbol{\pi}^H}_t(\boldsymbol{x})\Big] \;\geq\; \frac{1}{2}\;.$$

Note that this result assumes that we can solve (H.16) and (H.19) optimally, both of which takes the form of a static joint assortment and discrete pricing problem. As remarked, this can be done using parametric linear programming (Sumida et al. 2021). Otherwise, FPTAS for the two programs also possible, and in this case, we can obtain a policy with online effectiveness at least $\frac{1}{2(1+\epsilon)}$ with total running time polynomial in n, $1/\epsilon$ and T, following the discussion in Rusmevichientong et al. (2020). We next give the sketch of the proof of Theorem H.7.

Proof of Theorem H.7: We first remark on the following critical properties that hold for all $\boldsymbol{x} \in \mathcal{X}$, $t \in \mathcal{T}$ and $i \in \mathcal{N}$. They follow from the same argument that leading to Lemma G.3. We skip the details of these arguments.

(*i*) $\beta^t_{i0}(\boldsymbol{x}) \geq \beta^{t+1}_{i0}(\boldsymbol{x})$;

(ii) For each $\ell \geq 1$,

$$\beta_{i\ell}^t(\boldsymbol{x}) = \sum_{s=0}^{\infty}\left[\zeta_{i,\ell+s}(\boldsymbol{x})\left(\prod_{h=0}^{s-1}\left(1-\zeta_{i,\ell+h}(\boldsymbol{x})\right)\right)\beta_{i0}^{t+s+1}(\boldsymbol{x})\right] \tag{H.20}$$

and for all $k \in \mathcal{K}$

$$\tilde{\beta}_{ik\ell}^t(\boldsymbol{x}) = h_i(\boldsymbol{x})\sum_{s=0}^{\infty}\left[\gamma_{ik,\ell+s}(\boldsymbol{x})\left(\prod_{h=0}^{s-1}\left(1-\gamma_{ik,\ell+h}(\boldsymbol{x})\right)\right)\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+1}\right)\right] \tag{H.21}$$

(iii) For all $\ell \geq 0$ and all $k \in \mathcal{K}$

$$f_{ik}(\ell\,|\,\boldsymbol{x}) = \gamma_{ik\ell}(\boldsymbol{x})\prod_{s=0}^{\ell-1}\left(1-\gamma_{iks}(\boldsymbol{x})\right) \quad \text{and} \quad g_i(\ell\,|\,\boldsymbol{x}) = \zeta_{i\ell}(\boldsymbol{x})\prod_{s=0}^{\ell-1}\left(1-\zeta_{is}(\boldsymbol{x})\right). \tag{H.22}$$

Let $V^t(\boldsymbol{q}\,|\,\boldsymbol{x})$ denote the expected revenue under the greedy policy $\boldsymbol{\pi}^H$ in periods $t, t+1, \ldots T$, given that the system state is $\boldsymbol{q}$ in period t. With the three properties listed above, we can prove the following two critical results.

(iv) For each $\boldsymbol{x} \in \mathcal{X}$, $\hat{J}^1\left(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0}\,\middle|\,\boldsymbol{x}\right) \geq \frac{1}{2}\mathsf{UB}(\boldsymbol{x})$.

(v) For each $t \in \mathcal{T}$, $\boldsymbol{x} \in \mathcal{X}$, and $\boldsymbol{q} \in \mathcal{Q}(\boldsymbol{x})$, $V^t(\boldsymbol{q}\,|\,\boldsymbol{x}) \geq \hat{J}^t(\boldsymbol{q}\,|\,\boldsymbol{x})$.

One can verify these two results step by step following the analysis of Lemmas G.2 and G.2. Particularly, one can equivalently write $\mathsf{UB}(\boldsymbol{x})$ in the dual form of Eq.(H.9) given by

$$\begin{aligned}
\mathsf{UB}(\boldsymbol{x}) \;=\; \min_{(\boldsymbol{\lambda},\boldsymbol{\theta})\geq \boldsymbol{0}} \quad & \sum_{t=1}^{T}\lambda^t + \sum_{t=1}^{T}\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\theta_i^t \\
\text{s.t.} \quad & \lambda^t \geq \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(S\,|\,\boldsymbol{x})\left[r_{ik}(\boldsymbol{x}) \;-\; \sum_{\tau=t}^{T}\alpha_{ik}(\tau-t\,|\,\boldsymbol{x})\theta_i^\tau\right] \quad \forall\, S\in\mathcal{F}, t\in\mathcal{T},
\end{aligned}$$

We will verify that $\left(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\theta}}\right)$, defined by $\hat{\lambda}^t = \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})(\beta_{i0}^t(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}))$ and $\hat{\theta}_i^t = \beta_{i0}^{t+1}(\boldsymbol{x}) - \beta_{i0}^{t+2}(\boldsymbol{x})$ for all $t \in \mathcal{T}$ is a feasible solution to the dual linear program above. Fix an arbitrary $S \in \mathcal{F}$. By definition of $\hat{\theta}_i^t$,

$$\begin{aligned}
&\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(S\,|\,\boldsymbol{x})\left[r_{ik}(\boldsymbol{x}) \;-\; \sum_{\tau=t}^{T}\alpha_{ik}(\tau-t\,|\,\boldsymbol{x})\hat{\theta}_i^\tau\right] \\
=\; &\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(S\,|\,\boldsymbol{x})\left[r_{ik}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{\ell=0}^{\infty}\beta_{i0}^{t+\ell+1}(\boldsymbol{x})\left(\sum_{s=0}^{\ell} f_{ik}(s\,|\,\boldsymbol{x})g_i(\ell-x\,|\,\boldsymbol{x})\right)\right]
\end{aligned} \tag{H.23}$$

following exactly the same argument as (G.5).

Next, we claim that

$$\xi_{ik}^t(\boldsymbol{x}) \;=\; r_{ik}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{\ell=0}^{\infty}\beta_{i0}^{t+\ell+1}(\boldsymbol{x})\left(\sum_{s=0}^{\ell} f_{ik}(s\,|\,\boldsymbol{x})g_i(\ell-x\,|\,\boldsymbol{x})\right). \tag{H.24}$$

Indeed, we note that $\xi_{ik}^t(\boldsymbol{x})$ can be expanded as

$$r_{ik}(\boldsymbol{x}) - \big(1 - h_i(\boldsymbol{x})\gamma_{ik0}(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})\big)\beta_{i0}^{t+1}(\boldsymbol{x}) \; + \; \underbrace{\big(1-\gamma_{ik0}(\boldsymbol{x})\big)\tilde{\beta}_{i1}^{t+1}(\boldsymbol{x})}_{(**)} \; + \; \gamma_{ik0}(\boldsymbol{x})h_i(\boldsymbol{x})\underbrace{\big(1-\zeta_{i0}(\boldsymbol{x})\big)\beta_{i1}^{t+1}(\boldsymbol{x})}_{(*)}.$$

For $(*)$, using exactly the same steps as in the proof of of Lemma G.2, we obtain

$$\big(1-\zeta_{i0}(\boldsymbol{x})\big)\beta_{i,1}^{t+1}(\boldsymbol{x}) \; = \; \sum_{s=1}^{\infty}\left[f_{ik}(s\,|\,\boldsymbol{x})\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+1}\right)\right]$$

Also, it follows from (H.21) that $(**)$ can be simplified as

$$\begin{aligned}
\big(1-\gamma_{ik0}(\boldsymbol{x})\big)\tilde{\beta}_{i1}^{t+1}(\boldsymbol{x}) &= \big(1-\gamma_{ik0}(\boldsymbol{x})\big)h_i(\boldsymbol{x})\sum_{s=0}^{\infty}\left[\gamma_{i\ell,1+s}(\boldsymbol{x})\prod_{h=0}^{s-1}\big(1-\gamma_{i\ell,1+h}(\boldsymbol{x})\big)\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+2}\right)\right] \\
&\overset{(a)}{=} \big(1-\gamma_{ik0}(\boldsymbol{x})\big)h_i(\boldsymbol{x})\sum_{s'=1}^{\infty}\left[\gamma_{i\ell s'}(\boldsymbol{x})\prod_{h'=1}^{s'-1}\big(1-\gamma_{i\ell h'}(\boldsymbol{x})\big)\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s'+p+1}\right)\right] \\
&= h_i(\boldsymbol{x})\sum_{s'=1}^{\infty}\left[\gamma_{i\ell s'}(\boldsymbol{x})\prod_{h'=0}^{s'-1}\big(1-\gamma_{i\ell h'}(\boldsymbol{x})\big)\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s'+p+1}\right)\right] \\
&\overset{(b)}{=} h_i(\boldsymbol{x})\sum_{s'=1}^{\infty}\left[f_{ik}(s'\,|\,\boldsymbol{x})\cdot\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s'+p+1}\right)\right],
\end{aligned}$$

where (a) follows from changing the indexes and (b) follows from (H.22). Therefore, noticing that $\gamma_{ik0} = f_{ik}(0\,|\,\boldsymbol{x})$ and $\zeta_{i0} = g_i(0\,|\,\boldsymbol{x})$, it follows that

$$\begin{aligned}
\xi_{ik}^t(\boldsymbol{x}) \; &= \; r_{ik}(\boldsymbol{x}) - \big(1 - h_i(\boldsymbol{x})f_{ik}(0\,|\,\boldsymbol{x})g_i(0\,|\,\boldsymbol{x})\big)\beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{s=1}^{\infty}\left[f_{ik}(s\,|\,\boldsymbol{x})\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+1}\right)\right] \\
&\qquad + f_{ik}(0\,|\,\boldsymbol{x})h_i(\boldsymbol{x})\left(\sum_{s=0}^{\infty} g_i(s\,|\,\boldsymbol{x})\beta_{i0}^{t+s+1}(\boldsymbol{x}) \; - \; g_i(0\,|\,\ell)\beta_{i0}^{t+1}(\boldsymbol{x})\right) \\
&\overset{(c)}{=} \; r_{ik}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{s=0}^{\infty}\left[f_{ik}(s\,|\,\boldsymbol{x})\cdot\left(\sum_{p=0}^{\infty} g_i(p\,|\,\boldsymbol{x})\beta_{i0}^{t+s+p+1}\right)\right] \\
&\overset{(d)}{=} \; r_{ik}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x}) + h_i(\boldsymbol{x})\sum_{\ell=0}^{\infty}\beta_{i0}^{t+\ell+1}(\boldsymbol{x})\cdot\left(\sum_{s=0}^{\ell} f_{ik}(s\,|\,\boldsymbol{x})g_i(\ell-s\,|\,\boldsymbol{x})\right),
\end{aligned}$$

where both (c) and (d) follow from algebra and this verifies (G.6).

Combining (H.23) and (H.24), we have that

$$\begin{aligned}
\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(S\,|\,\boldsymbol{x})\Big[r_{ik}(\boldsymbol{x}) - \sum_{\tau=t}^{T}\alpha_{ik}(\tau-t\,|\,\boldsymbol{x})\hat{\theta}_i^{\tau}\Big] \; &= \; \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(S\,|\,\boldsymbol{x})\,\xi_{ik}^t(\boldsymbol{x}) \; \overset{(e)}{\leq} \; \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(A^t(\boldsymbol{x})\,|\,\boldsymbol{x})\,\xi_{ik}^t(\boldsymbol{x}) \\
&\overset{(f)}{=} \; \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})(\beta_{i0}^t(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x})) \; = \; \hat{\lambda}^t, \qquad \text{(H.25)}
\end{aligned}$$

where (e) follows by the optimality of $A^t(\boldsymbol{x})$ and (f) follows from (H.17). Next, we notice that $\beta_{i0}^t(\boldsymbol{x}) \geq \beta_{i0}^{t+1}(\boldsymbol{x})$ for all $i\in\mathcal{N}$, $t\geq 0$ and $\boldsymbol{x}$. As a result, $\big(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\theta}}\big)$ is feasible to the dual program.

Then, plugging the values $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\theta}})$ into the objective function of the dual program and using the same steps as in the proof of Lemma G.2, we obtain

$$\mathsf{UB}(\boldsymbol{x}) \;\le\; \sum_{t=1}^{T} \hat{\lambda}^t \;+\; \sum_{t=1}^{T}\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\hat{\theta}_i^t \le \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\beta_{i0}^1(\boldsymbol{x}) + \sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\beta_{i0}^1(\boldsymbol{x}) \le 2\hat{J}^1\Big(\sum_{i\in\mathcal{N}} C_i(\boldsymbol{x})\boldsymbol{e}_{i0}\,\big|\,\boldsymbol{x}\Big)$$

This verifies (iv). For (v), we use induction to show that $V^t(\boldsymbol{q}\,|\,\boldsymbol{x}) \ge \hat{J}^t(\boldsymbol{q}\,|\,\boldsymbol{x})$ for all $\boldsymbol{q}\in\mathcal{Q}(\boldsymbol{x})$ and $t\in\mathcal{T}$. This clearly holds when $t = T+1$. Then, assume that $V^{t+1}(\boldsymbol{q}\,|\,\boldsymbol{x}) \ge \hat{J}^{t+1}(\boldsymbol{q}\,|\,\boldsymbol{x})$. It holds by definition that for all $i\in\mathcal{N}$

$$\mathbb{E}[\mathsf{A}_{ik\ell}(\boldsymbol{x})] = \gamma_{ik\ell}(\boldsymbol{x}), \quad \text{and} \quad \mathbb{E}[\mathsf{D}_{i\ell}(\boldsymbol{x})] \;=\; w_{i\ell}\zeta_{i\ell}(\boldsymbol{x}) \;\forall\,\ell\ge 1,$$

and

$$\mathbb{E}\left[\mathsf{B}_i\left(\boldsymbol{x}\right)\right] \;=\; h_i(\boldsymbol{x})\sum_{k\in\mathcal{K}}\sum_{\ell=1}^{\infty}\tilde{w}_{ik\ell}\mathsf{A}_{ik\ell}(\boldsymbol{x}), \quad \text{and} \quad \mathbb{E}\left[\mathsf{C}_i(\boldsymbol{x})\right] \;=\; h_i(\boldsymbol{x})\zeta_{i0}(\boldsymbol{x})\sum_{k\in\mathcal{K}}\sum_{\ell=1}^{\infty}\tilde{w}_{ik\ell}\mathsf{A}_{ik\ell}(\boldsymbol{x}).$$

Therefore, using the same computation as (G.8), we can show that

$$\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\;\big|\;\boldsymbol{x}\big)\Big] \;=\; \sum_{i\in\mathcal{N}}\left(\sum_{\ell\ge 1}\left(w_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + \sum_{k\in\mathcal{K}}\tilde{w}_{i\ell}\tilde{\beta}_{i\ell}^t(\boldsymbol{x})\right) + w_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) \tag{H.26}$$

where (a) follows by definition of $\beta_{i\ell}^t(\boldsymbol{x})$ for all $\ell\in\mathbb{Z}$ and $\ell\ne 0$. Using the induction hypothesis and noticing that the coefficient of each $\mathbb{E}\big[V^{t+1}\big(\,\cdot\,\big|\,\boldsymbol{x}\big)\big]$ term on the right-hand side of (G.2) is non-negative, we obtain that

$$\begin{aligned}
&V^t(\boldsymbol{q}\,|\,\boldsymbol{x})\\
\overset{(e)}{\ge}\;& \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\;\big|\;\boldsymbol{x}\big)\Big] + \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(S\,|\,\boldsymbol{x})\Bigg[r_{ik}(\boldsymbol{x})\\
&+\big(1-\gamma_{ik0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) + (-\boldsymbol{e}_{i0},\boldsymbol{e}_{i\ell 1})\;\big|\;\boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\;\big|\;\boldsymbol{x}\big)\Big]\Big)\\
&+\gamma_{ik0}(\boldsymbol{x})h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) + (-\boldsymbol{e}_{i0}+\boldsymbol{e}_{i1},\boldsymbol{0})|\boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})|\boldsymbol{x}\big)\Big]\Big)\\
&+\gamma_{ik0}(\boldsymbol{x})\big(1-h_i(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - (\boldsymbol{e}_{i0},0)\;\big|\;\boldsymbol{x}\big)\Big] - \mathbb{E}\Big[\hat{J}^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x})\,\big|\,\boldsymbol{x}\big)\Big]\Big)\Bigg]\Bigg\},\\
=\;& \sum_{i\in\mathcal{N}}\left(\sum_{\ell\ge 1}\left(w_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + \sum_{k\in\mathcal{K}}\tilde{w}_{i\ell}\tilde{\beta}_{i\ell}^t(\boldsymbol{x})\right) + w_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) + \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(S\,|\,\boldsymbol{x})\Bigg[r_{ik}(\boldsymbol{x})\\
&-\gamma_{ik0}(\boldsymbol{x})\big(1-h_i(\boldsymbol{x})\big)\beta_{i0}^{t+1}(\boldsymbol{x}) + \big(1-\gamma_{i0}(\boldsymbol{x})\big)\big(\tilde{\beta}_{i1}^{t+1}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x})\big)\\
&+\gamma_{ik0}(\boldsymbol{x})h_i(\boldsymbol{x})\big(1-\zeta_{i0}(\boldsymbol{x})\big)\big(\beta_{i1}^{t+1}(\boldsymbol{x}) - \beta_{i0}^{t+1}(\boldsymbol{x})\big)\\
\overset{(f)}{=}\;& \sum_{i\in\mathcal{N}}\left(\sum_{\ell\ge 1}\left(w_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + \sum_{k\in\mathcal{K}}\tilde{w}_{i\ell}\tilde{\beta}_{i\ell}^t(\boldsymbol{x})\right) + w_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) \;+\; \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t(G^t\,|\,\boldsymbol{x})\,\xi_{ik}^t(\boldsymbol{x})\\
\overset{(g)}{\ge}\;& \sum_{i\in\mathcal{N}}\left(\sum_{\ell\ge 1}\left(w_{i\ell}\beta_{i\ell}^t(\boldsymbol{x}) + \sum_{k\in\mathcal{K}}\tilde{w}_{i\ell}\tilde{\beta}_{i\ell}^t(\boldsymbol{x})\right) + w_{i0}\beta_{i0}^{t+1}(\boldsymbol{x})\right) \;+\; \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\phi_{ik}^t\Big(A^t\cap\{(i,k):\; w_{i0}\ge 1\}\,|\,\boldsymbol{x}\Big)\,\xi_{ik}^t(\boldsymbol{x})
\end{aligned}$$

$$\begin{aligned}
&\overset{(h)}{\geq} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\geq 1}\left(w_{i\ell}\beta^t_{i\ell}(\boldsymbol{x})+\sum_{k\in\mathcal{K}}\tilde{w}_{i\ell}\tilde{\beta}^t_{i\ell}(\boldsymbol{x})\right)+w_{i0}\beta^{t+1}_{i0}(\boldsymbol{x})\right)+\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\mathbb{1}_{\{w_{i0}\geq 1\}}\phi^t_{ik}(A^t\,|\,\boldsymbol{x})\,\xi^t_{ik}(\boldsymbol{x})\\
&\overset{(i)}{\geq} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\geq 1}\left(w_{i\ell}\beta^t_{i\ell}(\boldsymbol{x})+\sum_{k\in\mathcal{K}}\tilde{w}_{i\ell}\tilde{\beta}^t_{i\ell}(\boldsymbol{x})\right)+w_{i0}\beta^{t+1}_{i0}(\boldsymbol{x})\right)+\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}}\frac{w_{i0}}{C_i(\boldsymbol{x})}\phi^t_{ik}(A^t\,|\,\boldsymbol{x})\,\xi^t_{ik}(\boldsymbol{x})\\
&= \sum_{i\in\mathcal{N}}\left[\sum_{\ell\geq 1}\left(w_{i\ell}\beta^t_{i\ell}(\boldsymbol{x})+\sum_{k\in\mathcal{K}}\tilde{w}_{i\ell}\tilde{\beta}^t_{i\ell}(\boldsymbol{x})\right)+w_{i0}\left(\beta^{t+1}_{i0}(\boldsymbol{x})+\sum_{k\in\mathcal{K}}\frac{1}{C_i(\boldsymbol{x})}\phi^t_{ik}(A^t\,|\,\boldsymbol{x})\,\xi^t_{ik}(\boldsymbol{x})\right)\right]\\
&\overset{(j)}{=} \sum_{i\in\mathcal{N}}\left(\sum_{\ell\geq 0}w_{i\ell}\beta^t_{i\ell}(\boldsymbol{x})+\sum_{\ell\geq 1}\sum_{k\in\mathcal{K}}\tilde{w}_{ik\ell}\tilde{\beta}^t_{ik\ell}(\boldsymbol{x})\right)=\hat{J}^t(\boldsymbol{q}\,|\,\boldsymbol{x}).
\end{aligned}$$

The inequality (e) follows from the induction hypothesis. The inequality (f) follows from the definition of $\xi^t_{ik}(\boldsymbol{x})$ for all $i\in\mathcal{N}$ and $t\in\mathcal{T}$. The inequality (g) holds due to the optimality of G^t. Also, notice that $\phi^t_{ik}\Big(A^t\cap\{(i,k):\ w_{i0}\geq 1\}\,|\,\boldsymbol{x}\Big)\leq\phi^t_{ik}\big(A^t\,|\,\boldsymbol{x}\big)$ by our assumption on the choice model $\phi^t_{ik}(\cdot\,|\,\boldsymbol{x})$. Also, by the definition of A^t, it must be that $\xi^t_{ik}(\boldsymbol{x})\geq 0$ if $\phi^t_{ik}(A^t\,|\,\boldsymbol{x})>0$. Therefore, inequality (h) follows. (i) follows because $w_{i0}\leq C_i(\boldsymbol{x})$. The inequality (j) follows by the definition of $\beta^t_{i0}(\boldsymbol{x})$. This establishes (v).

With claims (iv) and (v) established, one can show the desired result following the proof of Theorem 5.2 This finishes our proof. ■

H.3. Simultaneously Offering Multiple Designs

In this part, we consider the setting in which multiple designs of the same product can be offered at the same time. For each product $i\in\mathcal{N}$, we assume that we can offer at most K_i designs, so the set of feasible offline design decisions can be denoted as $\mathcal{X}^{\mathsf{mul}}=\Big\{\boldsymbol{x}\in\{0,1\}^L:\sum_{d\in\mathcal{D}_i}x_{id}\leq K_i\ \ \forall\, i\in\mathcal{N}\Big\}$. Other than offering multiple designs, we maintain the same setup as discussed in Section 2. Clearly, allowing offering multiple designs requires a different method for solving $\max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{mul}}}\mathsf{UB}(\boldsymbol{x})$, which is the focus of discussion in this section.

In this setting, it is more convenient to direct define choice probabilities $\psi^t_{id}(\boldsymbol{A})$ where $\boldsymbol{A}\in\mathcal{G}$ and $\mathcal{G}$ represents the set of feasible assortments defined on the ground set $\mathcal{M}=\{i^d:i\in\mathcal{N},\,d\in\mathcal{D}_i\}$. We assume that $\mathcal{G}$ is downward inclusive. Then, let us define

$$\mathcal{G}(\boldsymbol{x})=\big\{\boldsymbol{A}\in\mathcal{G}:\boldsymbol{A}\cap\{i^d\in\mathcal{M}:x_{id}=0\}=\varnothing\big\}$$

and we may adopt the following CDLP formulation for each design $\boldsymbol{x}\in\mathcal{X}^{\mathsf{mul}}$:

$$\begin{aligned}
\mathsf{UB}(\boldsymbol{x})=\max_{\boldsymbol{y}\geq\boldsymbol{0}}\quad&\sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}(\boldsymbol{x})}y^t(\boldsymbol{A})\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\psi^t_{id}(\boldsymbol{A})r_{id} && \text{(H.27)}\\
\text{s.t.}\quad&\sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}(\boldsymbol{x})}y^\tau(\boldsymbol{A})\psi^\tau_{id}(\boldsymbol{A})\alpha_{id}(t-\tau)\leq C_{id}x_{id} && \forall\, i\in\mathcal{N},d\in\mathcal{D}_i,t\in\mathcal{T},\\
&\sum_{\boldsymbol{A}\in\mathcal{G}(\boldsymbol{x})}y^t(\boldsymbol{A})\leq 1 && \forall\, t\in\mathcal{T}\,.
\end{aligned}$$

Note that $\mathcal{G}(\boldsymbol{x})$ is the set of feasible assortments that are aligned with the product decision $\boldsymbol{x}$. It is easy to verify that the CDLP is indeed an upper bound of the dynamic assortment optimization problem for each $\boldsymbol{x} \in \mathcal{X}^{\mathsf{mul}}$. Further, let

$$\begin{aligned} \mathsf{VB}(\boldsymbol{x}) = \max_{\boldsymbol{y} \geq \boldsymbol{0}} \quad & \sum_{t \in \mathcal{T}} \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \sum_{i \in \mathcal{N}} \sum_{d \in \mathcal{D}_i} \psi^t_{id}(\boldsymbol{A}) r_{id} \\ \text{s.t.} \quad & \sum_{\tau=1}^{t} \sum_{\boldsymbol{A} \in \mathcal{G}} y^\tau(\boldsymbol{A}) \psi^\tau_{id}(\boldsymbol{A}) \alpha_{id}(t-\tau) \ \leq \ C_{id} x_{id} \qquad \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i, t \in \mathcal{T}, \\ & \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \ \leq \ 1 \qquad \forall\, t \in \mathcal{T}\,. \end{aligned} \tag{H.28}$$

Following the same argument leading to Theorem 3.5, we can show that there is an optimal solution to $\mathsf{VB}(\boldsymbol{x})$, in which we only offer products in $\mathcal{G}(\boldsymbol{x})$, so $\mathsf{UB}(\boldsymbol{x}) = \mathsf{VB}(\boldsymbol{x})$. Let us also define $Z^{\mathsf{mul,LP}} = \max_{\boldsymbol{x} \in \mathcal{X}^{\mathsf{mul,LP}}} \mathsf{VB}(\boldsymbol{x})$, where

$$\mathcal{X}^{\mathsf{mul,LP}} = \left\{ \boldsymbol{x} \in [0,1]^L \ : \ \sum_{d \in \mathcal{D}_i} x_{id} \leq K_i,\ x_{id} \geq 0 \ \ \forall\, i \in \mathcal{N}, d \in \mathcal{D}_i \right\} .$$

Then, we can modify the revenue-based rounding for the multiple-design setting as follows. We name this as the revenue-based ranking algorithm. Similar to Z^{LP}, $Z^{\mathsf{mul,LP}}$ is also a linear program and we can solve it to obtain $\boldsymbol{x}^{\mathsf{mul,LP}} = \left(x^{\mathsf{mul,LP}}_{id} : i \in \mathcal{N}, d \in \mathcal{D}_i\right)$ and $\boldsymbol{y}^{\mathsf{mul,LP}} = (y^{\mathsf{mul,LP},t}(\boldsymbol{A}) : \boldsymbol{A} \in \mathcal{G}, t \in \mathcal{T})$ as the optimal solution to $Z^{\mathsf{LP,mul}}$. Recall that we have defined in Section 4.1 that, $\mathsf{Rev}_{id}(\boldsymbol{y}) = r_{id} \sum_{t \in \mathcal{T}} \sum_{\boldsymbol{A} \in \mathcal{G}} y^t(\boldsymbol{A}) \psi^t_{id}(\boldsymbol{A})$ for any given $\boldsymbol{y}$, which is the revenue contribution of design d of product i in the optimal solution of the linear relaxation. We may assume that $\mathsf{Rev}_{i1}(\boldsymbol{y}^{\mathsf{mul,LP}}) \ \geq \ \mathsf{Rev}_{i2}(\boldsymbol{y}^{\mathsf{mul,LP}}) \ \geq \ \ldots \ \geq \ \mathsf{Rev}_{iD_i}(\boldsymbol{y}^{\mathsf{mul,LP}})$ for each $i \in \mathcal{N}$ without loss of generality. We can then obtain the offline design decision $\boldsymbol{x}^{\mathsf{mul},H}$ by setting $x^{\mathsf{mul},H}_{id} = 1$ for $d \leq K_i$ and $x^{\mathsf{mul},H}_{id} = 0$.

The following theorem characterizes a guarantee for the design decision $\boldsymbol{x}^{\mathsf{mul},H}$, extending the performance bound of the revenue-based round in Theorem 4.2 to the multiple-design setting. Again we let $\mathsf{Eff}_{\text{off}}(\boldsymbol{x}) = \mathsf{UB}(\boldsymbol{x})/\mathsf{UB}^*$ for any $\boldsymbol{x}$, where we have $\mathsf{UB}^* = \max_{\boldsymbol{x} \in \mathcal{X}^{\mathsf{mul}}} \mathsf{UB}(\boldsymbol{x})$. Note that when K_i is close to D_i, the performance guarantee is close to one, which is desired.

Theorem H.8 (Offline Effectiveness with Multiple Designs) $\mathsf{Eff}_{\text{off}}(\boldsymbol{x}^H) \geq \min_{i \in \mathcal{N}} \frac{K_i}{D_i}$.

Before proceeding to the proof, we remark that exactly the same value function approximation as discussed in Section 5.1 can be used to compute an effective online policy in our new setting. With the same analysis, we can show the expected revenue of this online assortment policy is at least half of $\mathsf{UB}(\boldsymbol{x})$. Following Theorem 2.1, we obtain a performance guarantee of $\frac{1}{2} \min_{i \in \mathcal{N}} \frac{K_i}{D_i}$ for the joint optimization problem.

Proof of Theorem H.8: The proof is similar to that of Theorem 4.2. Let us recall $Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ and $Z^2(\boldsymbol{\mathcal{B}},\boldsymbol{u})$ defined in Appendix D.1 for any $\boldsymbol{\mathcal{B}}\in\mathcal{P}$, where $\mathcal{P}=\{\boldsymbol{\mathcal{B}}=(\mathcal{B}_1,\ldots,\mathcal{B}_n):\mathcal{B}_i\subset\mathcal{D}_i,,\forall i\in\mathcal{N}\}$, and $\boldsymbol{u}\in\bar{X}=[0,1]^L$. We note that using the same argument as in Lemma D.3, we can show that for each $\boldsymbol{\mathcal{B}}\in\mathcal{P}$ and $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ that is a feasible solution to the linear program $Z^{\mathsf{mul,LP}}$,

$$Z^1(\boldsymbol{\mathcal{B}},\bar{\boldsymbol{x}}) \;\geq\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\mathsf{Rev}_{id}(\bar{\boldsymbol{y}}) \;. \tag{H.29}$$

For each $i\in\mathcal{N}$, let $\mathcal{B}_i=\{d\in\mathcal{D}_i : x_{id}^{\mathsf{mul},H}=1\}$. Note that $\boldsymbol{\mathcal{B}}\in\mathcal{P}$. By (H.29),

$$\begin{aligned}
Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{x}^{\mathsf{mul,LP}}) \;&\geq\; \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{B}_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right) \overset{(a)}{=} \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}x_{id}^{\mathsf{mul},H}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right) \overset{(b)}{=} \sum_{i\in\mathcal{N}}\sum_{d\leq K_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right)\\
&\overset{(c)}{\geq} \sum_{i\in\mathcal{N}}\frac{K_i}{D_i}\sum_{d\in\mathcal{D}_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right) \;\geq\; \left(\min_{i\in\mathcal{N}}\frac{K_i}{D_i}\right)\cdot\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right)\\
&\overset{(d)}{=} \left(\min_{i\in\mathcal{N}}\frac{K_i}{D_i}\right)\cdot Z^{\mathsf{mul,LP}} \;\overset{(e)}{\geq}\; \left(\min_{i\in\mathcal{N}}\frac{K_i}{D_i}\right)\cdot\max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{mul}}}\mathsf{UB}(\boldsymbol{x}) \;,
\end{aligned}$$

where (a) follows the definition of $\boldsymbol{\mathcal{B}}$ and (b) follows from our construction of $x_{id}^{\mathsf{mul},H}$ where $x_{id}^H=1$ if and only if $i\leq K_i$. The inequality (c) follows because by assumption

$$\mathsf{Rev}_{i1}(\boldsymbol{y}^{\mathsf{mul,LP}}) \;\geq\; \mathsf{Rev}_{i2}(\boldsymbol{y}^{\mathsf{mul,LP}}) \;\geq\; \ldots \;\geq\; \mathsf{Rev}_{iD_i}(\boldsymbol{y}^{\mathsf{mul,LP}})$$

and therefore for each $i\in\mathcal{N}$

$$\begin{aligned}
\frac{1}{D_i}\sum_{d\in\mathcal{D}_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right) \;&=\; \frac{K_i}{D_i}\cdot\left(\frac{1}{K_i}\sum_{d\leq K_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right)\right)+\frac{D_i-K_i}{D_i}\cdot\left(\frac{1}{D_i-K_i}\sum_{d>K_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right)\right)\\
&\leq\; \frac{K_i}{D_i}\cdot\left(\frac{1}{K_i}\sum_{d\leq K_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right)\right)+\frac{D_i-K_i}{D_i}\cdot\left(\frac{1}{K_i}\sum_{d\leq K_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right)\right)\\
&\leq \frac{1}{K_i}\sum_{d\leq K_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right).
\end{aligned}$$

The inequality (d) is from the fact that $(\boldsymbol{x}^{\mathsf{mul,LP}},\boldsymbol{y}^{\mathsf{mul,LP}})$ is an optimal solution to Z^{LP}, so $Z^{\mathsf{mul,LP}}=\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i}\mathsf{Rev}_{id}\left(\boldsymbol{y}^{\mathsf{mul,LP}}\right)$. The final inequality (e) follows because

$$Z^{\mathsf{mul,LP}} \;=\; \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{mul,LP}}}\mathsf{VB}(\boldsymbol{x}) \;\geq\; \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{mul}}}\mathsf{VB}(\boldsymbol{x}) \;=\; \max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{mul}}}\mathsf{UB}(\boldsymbol{x}) \;,$$

where the last equality follows from Theorem 3.5. Because $\mathsf{UB}(\boldsymbol{x}^{\mathsf{mul},H})\geq Z^1(\boldsymbol{\mathcal{B}},\boldsymbol{x}^{\mathsf{mul,LP}})$ by Lemma D.2, it follows that $\mathsf{UB}(\boldsymbol{x}^{\mathsf{mul},H})\geq\left(\min_{i\in\mathcal{N}}\frac{K_i}{D_i}\right)\cdot\max_{\boldsymbol{x}\in\mathcal{X}^{\mathsf{mul}}}\mathsf{UB}(\boldsymbol{x})$, which is the desired result. ■

H.4. Probabilistic Customer Arrivals

In this part, we briefly remark on how to extend our model to a setting in which at most one customer arrival in each time period. In particular, we assume that with probability $\rho^t\in[0,1]$, a customer arrives on the platform at each period $t\in\mathcal{T}$. After the customer arrives, all the subsequent events follow the same manner as our original model. If no customer arrives, then no assortment is offered and no purchase would take place at period t. Our original model corresponds to the case where $\rho^t = 1$ for all $t \in \mathcal{T}$. Following our notation in Section 5, for $t = T, T-1, \ldots, 2, 1$, the value function satisfies the following dynamic program equation given the aforementioned system dynamics:

$$\begin{aligned} J^t(\boldsymbol{q} \mid \boldsymbol{x}) = & \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \mid \boldsymbol{x}\big)\Big] + \rho^t \cdot \max_{S \in \mathcal{F}:\, q_{i0} \geq 1\ \forall i \in S} \Bigg\{ \sum_{i \in \mathcal{N}} \phi_i^t(S \mid \boldsymbol{x}) \Big[r_i(\boldsymbol{x}) \\ & + \big(1 - \gamma_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i1} \mid \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \mid \boldsymbol{x}\big)\Big]\Big) \\ & + \gamma_{i0}(\boldsymbol{x}) h_i(\boldsymbol{x})\big(1 - \zeta_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i,-1} | \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) | \boldsymbol{x}\big)\Big]\Big) \\ & + \gamma_{i0}(\boldsymbol{x})\big(1 - h_i(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} \mid \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \mid \boldsymbol{x}\big)\Big]\Big)\Big]\Bigg\}. \end{aligned}$$

We can transform the above dynamic program equations into our original formulation in (5.1). In particular, we define a new choice function $\tilde{\phi}^t$ as follows: For all $t \in \mathcal{T}$, let

$$\tilde{\phi}_i^t(S|\boldsymbol{x}) = \rho^t \phi_i^t(S|\boldsymbol{x}) \quad \text{and} \quad \tilde{\phi}_0^t = 1 - \rho^t + \rho^t \phi_0^t(S|\boldsymbol{x})$$

for all $i \in \mathcal{N}$, $S \in \mathcal{N}$, and $\boldsymbol{x} \in \mathcal{X}$. We can verify that $\tilde{\phi}$ is a valid choice function because for all $S \in \mathcal{N}$ and $\boldsymbol{x} \in \mathcal{X}$:

- $\sum_{i \in \mathcal{N}} \tilde{\phi}_i^t(S|\boldsymbol{x}) + \tilde{\phi}_0(S|\boldsymbol{x}) = \rho^t(1 - \phi_0^t(S|\boldsymbol{x})) + 1 - \rho^t + \rho^t \phi_0^t(S|\boldsymbol{x}) = 1$,
- $0 \leq \tilde{\phi}_i^t(S|\boldsymbol{x}) = \rho^t \phi_i^t(S|\boldsymbol{x}) \leq 1$ for all $i \in \mathcal{N}$ and $0 \leq \tilde{\phi}_0^t(S|\boldsymbol{x}) = 1 - \rho^t + \rho^t \phi_0^t(S|\boldsymbol{x}) \leq 1$, and
- $\tilde{\phi}_i^t(S|\boldsymbol{x}) = \rho^t \phi_i^t(S|\boldsymbol{x}) = 0$ for all $i \in \mathcal{N} \setminus S$.

With the definition of $\tilde{\phi}^t$, we can transform the dynamic program equation above as

$$\begin{aligned} J^t(\boldsymbol{q} \mid \boldsymbol{x}) = & \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \mid \boldsymbol{x}\big)\Big] + \max_{S \in \mathcal{F}:\, q_{i0} \geq 1\ \forall i \in S} \Bigg\{ \sum_{i \in \mathcal{N}} \tilde{\phi}_i^t(S \mid \boldsymbol{x}) \Big[r_i(\boldsymbol{x}) \\ & + \big(1 - \gamma_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i1} \mid \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \mid \boldsymbol{x}\big)\Big]\Big) \\ & + \gamma_{i0}(\boldsymbol{x}) h_i(\boldsymbol{x})\big(1 - \zeta_{i0}(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} + \boldsymbol{e}_{i,-1} | \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) | \boldsymbol{x}\big)\Big]\Big) \\ & + \gamma_{i0}(\boldsymbol{x})\big(1 - h_i(\boldsymbol{x})\big)\Big(\mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) - \boldsymbol{e}_{i0} \mid \boldsymbol{x}\big)\Big] - \mathbb{E}\Big[J^{t+1}\big(\boldsymbol{W}(\boldsymbol{q},\boldsymbol{x}) \mid \boldsymbol{x}\big)\Big]\Big)\Big]\Bigg\}, \end{aligned}$$

which is equivalent to our original setting in (5.1). Therefore, the value function approximation as discussed in Section 5.1 can be applied in this setting.

For optimizing the offline decision, it remains to show the $\tilde{\phi}^t$ consistent as long as ϕ^t is consistent. Suppose that ψ^t is a discrete choice function defined on the extended ground set $\mathcal{M}$ that is substitutable and consistent with ϕ^t. We can define a discrete choice model $\tilde{\psi}^t$ on the extended ground set $\mathcal{M}$ in a similar vein as $\tilde{\phi}^t$. Then, we can verify that for each $\boldsymbol{x} \in \mathcal{X}$ and $S \subseteq \mathcal{N}$,

$$\tilde{\phi}_i^t(S \mid \boldsymbol{x}) = \rho^t \phi_i^t(S \mid \boldsymbol{x}) = \rho^t \psi_{id}^t\left(\boldsymbol{A}(S, \boldsymbol{x})\right) = \tilde{\psi}_{id}^t\left(\boldsymbol{A}(S, \boldsymbol{x})\right)$$

for $d \in \mathcal{D}_i$ such that $x_{id} = 1$. It is easy to verify that $\tilde{\psi}^t$ is also substitutable. Therefore, $\tilde{\phi}^t$ is consistent and all results on performance guarantees in our paper, such as Theorem 2.1, Theorem 4.2, Theorem 5.2, can be applied.

Last, we remark that the static assortment optimization problem

$$\max_{\boldsymbol{A} \in \mathcal{G}} \sum_{i^d \in \mathcal{M}} \tilde{\psi}^t_{i^d}(\boldsymbol{A}) \mu_{id} = (1 - \rho^t) + \rho^t \cdot \max_{\boldsymbol{A} \in \mathcal{G}} \psi^t_{i^d}(\boldsymbol{A}) \mu_{id}$$

can be efficiently solved, as long as the original assortment optimization problem $\max_{\boldsymbol{A} \in \mathcal{G}} \psi^t_{i^d}(\boldsymbol{A}) \mu_{id}$ is efficiently solvable. This allows us to solve the continuous relaxation Z^{LP} in (F.1), compute the coefficients for value function approximation using (5.4), and decide the assortments to offer in the greedy policy by solving (5.6). With this. we conclude our discussion.

Appendix I: Parameter estimation

In this part, we explain the data estimation procedure based on the ISMI durable good dataset that is used in Section 6 and will be used subsequently in later sections of the appendix. We estimate the experiment parameters using the data of the Cordless Phone subcategory from 1999 to 2003. We select the items with sales greater than or equal to 11, resulting in $n = 91$ items and 1589 transaction data. We estimate two critical components of the problem: the customer's choice model and the distribution of the trial duration.

Customer choice model: We assume that the customer choice model follows the MNL choice model and is non-stationary. We model the non-stationarity by partitioning total time span (5 years) into several seasons, denoted as set $\mathcal{S}$. Specifically, we first partition the data based on year since the retailer sells very different products across different years. Then, we partition the 12 months in each year into seasons, which we explain in detail later. We assume that the customers in the same season share the same choice model. In other words, the demand is stationary in the same season. We assume the preference weight of item j under price p_j for customers in season s is $v_j(p_j \mid s) = \exp\{u_{sj} - \beta_s p_j\}$, where u_{sj} denotes the nominal utility for each item j during season s, and β_s denotes the price sensitivity parameter in season s. We assume that the store offers the full assortment to every customer. Therefore, we can write the probability that item i is purchased in a transaction sample during season s given item prices $\{r_j\}_{j \in \mathcal{N}}$ as

$$\mathbb{P}(\text{purchase } i \mid s) \ = \ \frac{v_i(r_i \mid s)}{\sum_{j \in \mathcal{N}} v_j(r_j \mid s)} \ = \ \frac{\exp\{u_{si} - \beta_s r_i\}}{\sum_{j \in \mathcal{N}} \exp\{u_{sj} - \beta_s r_j\}}, \quad \forall i \in \mathcal{N}.$$

Let $\left\{t_k, (p_k^1, p_k^2, \cdots, p_k^n), y_k\right\}_{k=1}^m$ denote m transaction data points, where t_k is the month that the k-th transaction took place, p_k^i is the price for product i at the time of the k-th transaction, and y_k indicates the product that is purchased in the k-th transaction. For each data point, we only observe the price of the purchased item. Thus, we interpolate the missing price values using the average price of each item. Based on our assumption, the log-likelihood for a data point given seasons $\mathcal{S}$, $\boldsymbol{u}$, and $\boldsymbol{\beta}$ is

$$\begin{aligned}\mathcal{LL}\Big(\big\{t_k,(p_k^1,p_k^2,\cdots,p_k^n),y_k\big\}_{k=1}^m \,|\, \mathcal{S},\boldsymbol{u},\boldsymbol{\beta}\Big) &= \log\left(\prod_{k=1}^m\prod_{s\in\mathcal{S}}\mathbb{P}(\text{purchase } y_k \,|\, s)^{\mathbb{1}[t_k\in s]}\right)\\ &= \log\left(\prod_{k=1}^m\prod_{s\in\mathcal{S}}\left(\frac{\exp\{u_{s,y_k}-\beta_s p_k^{y_k}\}}{\sum_{j\in\mathcal{N}}\exp\{u_{sj}-\beta_s p_k^j\}}\right)^{\mathbb{1}[t_k\in s]}\right)\\ &= \sum_{k=1}^m\sum_{s\in\mathcal{S}}\mathbb{1}[t_k\in s]\left(u_{s,y_k}-\beta_s p_k^{y_k}-\log\sum_{j\in\mathcal{N}}\exp\{u_{sj}-\beta_s p_k^j\}\right).\end{aligned}$$

We carefully determine the selling seasons $\mathcal{S}$ using a clustering algorithm and the cross-validation (CV) procedure. We randomly pick 80% percent of the data as the training data and the rest 20% as the testing data. We conduct a top-down hierarchical clustering on the training data to find the best partition of months into seasons. We start with the trivial season set {{Jan, Feb, ..., December}} for each year and iteratively partition the months until our stop criterion is met. At each step, given a potential partition of seasons, we can estimate $\boldsymbol{u}$, $\boldsymbol{\beta}$ via the maximum likelihood E=estimation (MLE) method under the constraint that $\boldsymbol{u}\geq 0$ and $\boldsymbol{\beta}\geq 0$. Then we select the best partition among all partitions based on the log-likelihood of the five-fold CV. We terminate the clustering procedure when the gain in CV log-likelihood is less than 0.001. The resulting seasons are shown in Table EC.1. Our proposed estimation model gives an average of CV log-likelihood $=-1.902$ annually.

Note that if we fit a standard MNL model, i.e., $v_j(p_j\,|\,s)=\exp\{u_{sj}\}$ as our baseline, the average CV log-likelihood of the baseline is -4.278. Significant goodness-of-fit improvement is achieved. We re-train the final model on the entire training set with this partition. This demonstrates the advantage of our incorporating the effect of prices. The out-of-sample log-likelihood of the final model is -2.264 on the testing set.

Year	Seasons
1999	{{Jan, Feb, Mar, Apr, May, June, July, Aug}, {Sept, Oct}, {Nov, Dec}}
2000	{{Jan, Feb, Mar, Apr, May, June}, {July, Aug, Sept}, {Oct, Nov, Dec}}
2001	{{Jan, Feb, Mar}, {Apr, May}, {June, July, Aug, Sept, Oct}, {Nov, Dec}}
2002	{{Jan, Feb}, {Mar, Apr}, {May}, {June, July, Aug, Sept, Oct, Nov, Dec}}
2003	{{Jan, Feb, Mar}, {Apr, May, June, July}, {Aug, Sept, Oct, Nov}, {Dec}}

Table EC.1 Estimated Seasons From Hierarchical Clustering

Take year 2002 for example, the estimated price sensitivity parameter is 0.008 for $s_1=${Jan, Feb}, 0.0289 for $s_2=${Mar, Apr}, and 0.0537 for $s_3=${May}, and 0.0263 for $s_4=${June, July, Aug,

Item	s_1	s_2	s_3	s_4
0	1.00	4.01	1.00	1.00
1	3.15	6.74	10.22	6.50
2	1.95	3.89	1.95	5.88
3	2.28	4.11	2.28	2.28
4	2.95	2.95	2.95	2.95
5	3.48	7.24	10.49	6.91
6	2.67	3.60	2.67	3.35
7	2.95	2.95	2.95	2.95
8	2.50	3.00	5.43	2.50
9	0.73	3.34	3.99	3.29
10	1.04	4.65	4.36	3.30
11	3.12	8.30	3.12	3.12
12	0.51	4.30	0.51	4.94
13	2.87	2.87	2.87	6.06
14	2.70	6.00	2.70	2.70
15	0.00	6.35	0.00	6.21
16	0.00	0.00	0.00	5.08
17	0.00	0.00	8.40	6.36
18	0.00	0.00	0.00	7.39
19	0.00	0.00	0.00	5.15
20	0.00	0.00	0.00	6.12
21	0.00	0.00	0.00	5.07

Table EC.2 Estimated nominal utility parameters

Sept, Oct, Nov, Dec}. The estimated nominal utilities of the products sold in 2002 are summarized in Table EC.2. Clearly, there is a notable difference in utilities between seasons.

Distribution of the trial duration: As explained, the retailer offers different set of products over time. Furthermore, the customers choice behavior also seems to evolve over time, evidenced by differential utility parameters across different years even for the same product. Therefore, it is difficult to "pool" all the data together and estimate only one model for our numerical experiments; doing so leads to inferior fits in experiments. As a result, we build our experiments based on the estimation result and data for 2002 (the year with the largest amount of sales) to narrow down the set of products in a relatively stationary operational environment. In this part, we aim to estimate the distribution of the return time, or trial duration, in terms of its hazard rate from the data in 2002, which contains 306 transaction samples. The main challenge lies in the fact that we do no know the actual trial duration of the items that are not returned in the dataset, unless we have observation of an infinite number of time periods. To deal with this censored data, we assume that a time window of length T (days) such that no customer returns will be allowed after T. We treat T as a hyper-parameter and choose the best value through cross-validation.

According to our model, trial ends independently in each period. Let $\big((p_k, t, y_k^t)_{t=1}^{T_k}\big)_{k=1}^m$ denote the trial data, where for each data point k, p_k is the price for the purchased product, T_k is the length of the trial period during the observation time window, and y_k^t denotes whether the item is being returned at period $t = 1, \ldots, T_k$. Note that T_k can be equal to zero, meaning that the item is returned in the same day. If the item is returned with in the observation time window, then $y_k^t = 0$ for all $t = 0, \ldots, T_k - 1$ and $y_k^{T_k} = 1$. Otherwise, there is no valid observation of the trial duration, and we let $T_k = T$ for simplicity of analysis. In this case, $y_k^t = 0$ for all $t = 1, \ldots, T_k$. Note that, for each of the 306 transactions in 2002, we have T_k independent observations.

We assume the trial ends with probability h independently in each period. In other words, h is the hazard rate of the trial duration. We assume that the hazard rate of an item being tried for t periods satisfies $h(r,t) = 1/(1+\exp(\beta_1 p + \beta_2 t))$, where r is the price of the item and $\beta_1 \in \mathbb{R}$ is a coefficient associated with the price, and $\beta_2 \in \mathbb{R}$ is a coefficient associated with the time period. The log-likelihood of the dataset given β_1 and β_2 is

$$\begin{aligned}
\mathcal{LL}\left(\left\{\{p_k, t, y_k^t\}_{t=1}^{T_k}\right\}_{k=1}^{m} \,\Big|\, \beta_1, \beta_2\right) &= \log\left(\prod_{k=1}^{m}\prod_{t=1}^{T_k}\left(1 - h(p_k,t)\right)^{1-y_k^t} h(p_k,t)^{y_k^t}\right) \\
&= \sum_{k=1}^{m}\sum_{t=1}^{T_k}\Big[(1-y_k^t)(\beta_1 p_k + \beta_2 t) - \log\left(1 + \exp(\beta_1 p_k + \beta_2 t)\right)\Big].
\end{aligned}$$

We use the MLE method to estimate the value of β_1 and β_2. We split the data into 80% training set and 20% testing set. We conduct a 10-fold cross-validation on the training set to find the best observation time window over $T = \{50, 100, \ldots, 500\}$. Since we have multiple observations for each transaction data point, a training set with around 200 transaction samples would yield a dataset with around 9000 observations even with the smallest stopping time $T = 50$. Therefore, the dataset is sufficiently large for cross-validation. The best observation time window is $T^* = 200$, with CV log-likelihood $= -0.007$. We compare the baseline model that assumes a constant hazard rate and observation time window $T = 500$. In particular, the MLE estimator for the baseline model is $\hat{h} = \sum_{k=1}^{m} y_k / (\sum_{k=1}^{m} T_k + m)$. The CV log-likelihood of the baseline model is -0.021. We re-train the entire training set with $T^* = 200$ to obtain the final model. The out-of-sample log-likelihood of the final model is -0.008 on the testing set.

Appendix J: Numerical Experiments on Return Eligibility

In this section, we discuss the numerical experiments on the RE application, still on the ISMI data set. We adopt the same estimation procedure as discussed in Appendix I, and the same joint optimization schemes as described in Section 6. We will discuss the experiment setup, computational results, and main findings in the next. This discussion shows that our findings in Section 6 are robust with respect to different applications.

J.1. Experiment Setup

We let $\mathcal{D}=\{1,2\}$ and designate $d=1$ as the non-returnable version of a product and $d=2$ as the returnable version the product. For each product $i\in\mathcal{N}$, we let the maximum price in the dataset be the price of the designs, so $p_{i1}=p_{i2}=p_i^{\max}$. We randomly sample the initial inventory level C_i from the uniform distribution over $[10,20]\cap\mathbb{Z}$. We set $C_{i1}=C_{i2}=C_i$ for all $i\in\mathcal{N}$. We generate a sequence of $T=\sigma_{\mathrm{L}}\sum_{i\in N}C_i$ customers, where σ_{L} is the experimental parameter that controls the load of the customers relative to the inventory level. A larger σ_{L} indicates a more congested system. Furthere, we let v_{id}^t be the preference weight of the d-th design of item i at time t. We assume that the customers' choice decisions follow the MNL choice model $\{\phi^t\}_{t\in\mathcal{T}}$ such that the choice probability under assortment $A\subseteq\mathcal{N}$ is:

$$\phi_i^t(A\,|\,\boldsymbol{x}) \;=\; \frac{v_{i1}^1x_{i1} \;+\; v_{i2}^t x_{i2}}{v_0^t \;+\; \sum_{j\in A}v_{j1}^t x_{j1} \;+\; v_{j2}^t x_{j2}}\mathbb{1}[i\in A],\quad \forall i\in\mathcal{N}.$$

Our estimation procedure yields the nominal utility $\hat{u}_i^t\geq 0$ for each item and the price sensitivity $\hat{\beta}^t$ for each $t\in\mathcal{T}$. The net utility of item i given price p is assumed to be $\hat{u}_i^t-\hat{\beta}^tp$. We assume the data is collected under a fully returnable design. Therefore, the estimated utility parameters correspond to design $d=1$. We set $v_{i2}^t=\exp(\hat{u}_i^t-\hat{\beta}^tp_{i2})$ for all $t\in\mathcal{T}$. We assume that the nominal utility of the non-returnable version of each product is $\eta\hat{u}_i^t$, where η is generated uniformly at random over $[0,1]$. In particular, η represents the discount of the nominal utility when not being allowed to return the product. Thus, $v_{i1}^1=\exp(\eta\hat{u}_i^t-\hat{\beta}^tp_{i1})$ for all $t\in\mathcal{T}$. It remains to generate the preference weight for the no-purchase option. Let α_0^t denote the no-purchase probability when we offer full returnable products and full assortment for arrival t. We randomly sample α_0^t from $\{0.25,0.75\}$. The preference weight of the no-purchase option is therefore $v_0^t=\alpha_0^t/(1-\alpha_0^t)\cdot\sum_{i\in\mathcal{N}}v_{i2}^t$.

Similar to the experiments for the BOGO design, we set the hazard rate of the trial duration for each item as $(1-\sigma_{\mathrm{R}})\hat{\gamma}_i+\sigma_{\mathrm{R}}$, where σ_{R} is an experimental parameter that controls the likelihood of return. We consider different load parameters $\sigma_{\mathrm{L}}\in\{1.8,2.0,2.2\}$ and return likelihood parameters $\sigma_{\mathrm{R}}\in\{0.0,0.5,1.0\}\times 10^{-3}$. For each combination of experimental parameters, we randomly generate 50 problem instances, and for each problem instance, we generate 50 sample paths and calculate the sample average revenue.

J.2. Computational Results

We present our results in Tables EC.3 and EC.4, which correspond to Tables 2 and 3 for the BOGO application discussed in Section 6. Specifically, Table EC.3 highlights the effectiveness of our proposed joint optimization scheme. Among various joint optimization schemes tested, the combination of $(x_{\mathsf{REV}},\mathsf{VFA})$ consistently outperforms the others in most experimental setups. This result emphasizes the significance of simultaneously considering both product design and dynamic

Problem Param.		Upper Bound	Effectiveness of Offline Dec. and Policy Comb. (% of UB*)								
			$\boldsymbol{x}_{\mathsf{REV}}$			$\boldsymbol{x}_{\mathsf{NoR}}$			$\boldsymbol{x}_{\mathsf{NoC}}$		
σ_{L}	$\sigma_{\mathrm{R}}(\times 10^{-3})$	UB*	VFA	MY	IB	VFA	MY	IB	VFA	MY	IB
1.8	0.0	10195	96.19	93.55	93.78	95.95	93.50	93.62	90.89	88.21	88.98
2.0	0.0	10229	96.68	94.08	94.80	95.61	93.24	94.22	91.08	88.39	89.43
2.2	0.0	9966	95.84	95.17	94.85	95.22	94.38	94.25	92.50	90.35	90.77
1.8	0.5	9404	96.25	94.38	94.41	95.62	94.06	93.94	89.56	87.88	88.22
2.0	0.5	9388	96.01	94.53	94.51	95.11	93.51	93.56	89.33	88.81	88.71
2.2	0.5	9054	95.83	94.78	94.40	94.64	93.84	93.44	91.03	89.97	89.77
1.8	1.0	8742	96.51	95.00	94.75	95.96	94.42	94.30	90.53	89.82	89.47
2.0	1.0	8674	96.77	95.50	95.11	96.24	95.06	94.62	90.96	89.93	89.80
2.2	1.0	8310	97.03	96.16	95.57	95.93	95.25	94.79	91.88	91.14	91.03

Table EC.3 The upper bound UB^* and the effectiveness (%) of the joint optimization schemes for each problem parameter. The table shows that (x_{REV}, VFA) outperforms the other schemes by a margin of 1%-8%.

		$\mathsf{Eff}_{\mathrm{on}}(\cdot \mid \boldsymbol{x}_{\mathsf{REV}})$			$\mathsf{Eff}_{\mathrm{on}}(\cdot \mid \boldsymbol{x}_{\mathsf{NoR}})$			$\mathsf{Eff}_{\mathrm{on}}(\cdot \mid \boldsymbol{x}_{\mathsf{NoC}})$			$\mathsf{Eff}_{\mathrm{off}}(\cdot)$		
σ_{L}	$\sigma_{\mathrm{R}}(\times 10^{-3})$	VFA	MY	IB	VFA	MY	IB	VFA	MY	IB	$\boldsymbol{x}_{\mathsf{REV}}$	$\boldsymbol{x}_{\mathsf{NoR}}$	$\boldsymbol{x}_{\mathsf{NoC}}$
1.8	0.0	97.10	94.43	94.66	97.33	94.82	94.96	97.58	94.49	95.44	99.07	98.58	93.06
2.0	0.0	97.74	95.12	95.84	97.80	95.36	96.36	97.96	94.90	96.13	98.91	97.75	92.88
2.2	0.0	97.26	96.58	96.27	97.28	96.41	96.28	98.41	95.97	96.52	98.53	97.88	93.90
1.8	0.5	97.15	95.26	95.29	97.26	95.68	95.55	97.81	95.83	96.29	99.07	98.30	91.46
2.0	0.5	97.35	95.84	95.82	97.57	95.91	95.96	97.67	96.94	96.87	98.63	97.47	91.40
2.2	0.5	97.77	96.69	96.30	97.76	96.92	96.50	98.45	97.22	97.08	98.02	96.79	92.34
1.8	1.0	97.29	95.77	95.51	97.29	95.73	95.60	97.79	96.91	96.59	99.20	98.64	92.51
2.0	1.0	97.72	96.44	96.04	97.81	96.59	96.14	98.06	96.90	96.78	99.03	98.40	92.66
2.2	1.0	98.21	97.33	96.73	98.11	97.39	96.93	98.41	97.54	97.45	98.80	97.77	93.28

Table EC.4 Conditional online and offline effectiveness (%) averaged across the 50 problem instances.

assortment optimization, as demonstrated in Section 6.4. Moreover, this finding suggests that the benefits of joint optimization can be extended to other contexts. Similarly, in the RE application, it is crucial to account for both capacity and returns. Table EC.4 separates the performance of offline and online decisions. We observe a similar trend as in the BOGO application. The offline design decision plays a primary role in differentiating the performance of joint optimization schemes.

We carry out a two-way ANOVA on the data detailed in Table EC.3, similar to our application of the BOGO design. For further details, please refer to Appendix K. The analysis reveals that offline decisions account for 85.0% of the total explainable variance observed in the data, while online decisions explain 7.4% of the total variance. In line with our BOGO design application discussed in the appendix, offline decisions continue to exert a more significant influence. However, in this application, online assortment policies gain more importance. This underlines the fact that the relative significance of these two types of decisions is dependent on the specific application. Consequently, effective optimization strategies for the online decision, such as the value function approximation proposed in Section 5.1, can be pivotal in achieving high joint effectiveness.

Appendix K: Two-Way Analysis of Variance (ANOVA) on the Experiment Results

In this section, we present the details of the ANOVA on results from the numerical experiments on the BOGO design and RE design presented in Section 6 and Appendix J, respectively. We first discuss the ANOVA analysis for BOGO. Our methods follow the standard literature on the ANOVA (e.g., see Chapter 3 of Wu and Hamada (2011)). In particular, let $s \in \{\mathsf{REV}, \mathsf{NoR}, \mathsf{NoC}\}$ and $t \in \{\mathsf{VFA}, \mathsf{MY}, \mathsf{IB}\}$. Also, let each k correspond to a (σ_L, σ_R) combination. Then, we use Y_{stk} to denote the average percentage of the upper bound achieved by a (s,t,k) combination, and there are a total of 81 data points as in Table 2. We treat s and t as two treatment factors. To focus on the effects of offline and online designs, we consider the variation arising from different (σ_L, σ_R) configurations as noises. Therefore, we assume a two-way layout with interactions as

$$Y_{stk} = \eta + \theta_s + \xi_t + \omega_{st} + \epsilon_{stk}, \ \forall s,t,k,$$

and correspondingly decompose observed data points as

$$y_{stk} = \bar{y}_{\cdots} + (\bar{y}_{s\cdot\cdot} - \bar{y}_{\cdots}) + (\bar{y}_{\cdot t\cdot} - \bar{y}_{\cdots}) + (\bar{y}_{st\cdot} - \bar{y}_{s\cdot\cdot} - \bar{y}_{\cdot t\cdot} + \bar{y}_{\cdots}) + (y_{stk} - \bar{y}_{st\cdot}), \ \forall s,t,k.$$

As a result, we can generate the following two-way decomposition of the sum of squares in Table EC.5. The first column of Table EC.5 gives the source of variation. The second column exhibits the formulas for the decomposition and the third column shows the values for each sum of squares. The last column presents the ratio between the corresponding sum of squares value and the total sum of squares in percentages.

In Table EC.5, we first observe the dominating impact of offline decisions (i.e., 94.0% of total variations explained) and the minor role of online decisions (i.e., 2.5% of total variations explained). This contrast clearly shows that for the BOGO design application, the offline decision plays a vital role. We also observe that the impacts from interactions are very small (i.e., 0.03% of total variation as shown in the third row of Table EC.5). Therefore, the impacts of offline and online decisions are mostly additive.

Source	Sum of squares formulas	Sum of squares value	Percentages
Offline (s)	$27\sum_s (\bar{y}_{s\cdot\cdot} - \bar{y}_{\cdots})^2$	4730.5	94.0%
Online (t)	$27\sum_t (\bar{y}_{\cdot t\cdot} - \bar{y}_{\cdots})^2$	124.3	2.5%
Offline × Online ($s \times t$)	$9\sum_s \sum_t (\bar{y}_{st\cdot} - \bar{y}_{s\cdot\cdot} - \bar{y}_{\cdot t\cdot} + \bar{y}_{\cdots})^2$	1.5	0.0%
Residual	$\sum_s \sum_t \sum_k (y_{stk} - \bar{y}_{st\cdot})^2$	178.9	3.6%
Total	$\sum_s \sum_t \sum_k (y_{stk} - \bar{y}_{\cdots})^2$	5035.2	100.0%

Table EC.5 Two-way ANOVA table for the BOGO design experiment data

We conduct the same ANOVA for the RE design on the 81 data points presented in Table EC.3 and present the result in Table EC.6. It is interesting to observe from Table EC.6 that the online design is of much more significance for the RE design compared to the ANOVA results for the BOGO design in Table EC.5. We also consistently observe a virtually non-existent interaction effect in Table EC.6 for the RE design experiment.

Source	Sum of squares formulas	Sum of squares value	Percentages
Offline (s)	$27\sum_s(\bar{y}_{s\cdot\cdot}-\bar{y}_{\cdot\cdot\cdot})^2$	455.6	85.0%
Online (t)	$27\sum_t(\bar{y}_{\cdot t\cdot}-\bar{y}_{\cdot\cdot\cdot})^2$	39.8	7.4%
Offline × Online ($s\times t$)	$9\sum_s\sum_t(\bar{y}_{st\cdot}-\bar{y}_{s\cdot\cdot}-\bar{y}_{\cdot t\cdot}+\bar{y}_{\cdot\cdot\cdot})^2$	0.4	0.1%
Residual	$\sum_s\sum_t\sum_k(y_{stk}-\bar{y}_{st\cdot})^2$	40.5	7.6%
Total	$\sum_s\sum_t\sum_k(y_{stk}-\bar{y}_{\cdot\cdot\cdot})^2$	536.2	100.0%

Table EC.6 Two-way ANOVA table for the RE design experiment data

Appendix L: Computational Robustness of Revenue-based Rounding

In this section, we provide additional numerical experiments on our revenue-based rounding. In the first experiment, we focus on the computational time of reveune-based rounding, and in the second we gauge its performance under multiple designs. We use the same dataset, estimation procedure, and joint optimization schemes as described in Section 6 and in Appendix I.

L.1. Computation Time of Revenue-based Rounding

We examine the computation time under the same experimental setting in Section 6. We consider the number of items $n \in \{5, 10, 15, 20\}$ and the length of the planning horizon $T \in \{1.0, 1.2, 1.4, 1.6\} \times 50$. We generate 50 instances of the problem for each combination of (n, T) following the protocols described in Section 6. The computation times in seconds for each (n, T) combination are recorded in Table EC.7. The results of our study show that revenue-based rounding is computationally efficient; notably, the computation time appears to increase linearly in relation to both the number of items and the length of the planning horizon.

n	T (× 50)			
	1.0	1.2	1.4	1.6
5	1.27	1.82	2.54	3.23
10	2.78	3.94	4.98	6.10
15	4.15	5.39	6.73	7.74
20	4.91	5.86	6.91	7.92

Table EC.7 Computation times (in seconds) of the revenue-based rounding algorithm

L.2. Effectiveness of Revenue-based Rounding with Multiple Designs

To augment our numerical experiments thus far, we consider a scenario in which multiple design options exist for each product. In this experiment, we concurrently determine the eligibility for return and the price discount applicable to the products. Each design embodies a distinct combination of return eligibility and discount level. We designate $\{\gamma_k\}_{k=1}^{K}$ to represent the set of discount levels, with K being the total number of discount levels. We posit that $\gamma_k \in (0,1]$ for all $k = 1,\ldots,K$. For instance, implementing a discount level $\gamma_k = 0.8$ equates to offering a 20% discount. As each discount level can be associated with two return options, we have a total of $D = 2K$ designs. The total number of designs $D \in \{2,4,6,8\}$ can be adjusted by manipulating the proposed discount levels, as depicted in Table EC.8.

For each product $i \in \mathcal{N}$, we let the original price $p_i^{\max}$ be its maximum price in the dataset. We let design $d \in \{1,2,\ldots,K\}$ represent the non-returnable version of the product with each discount level, so $p_{id} = p_i^{\max}\gamma_d$ for all $i \in \mathcal{N}$. Similarly, we let design $d \in \{K+1, K+2,\ldots,2K\}$ represent the returnable version of the product with different discount levels, where $p_{id} = p_i^{\max}\gamma_{d-K}$ for all $i \in \mathcal{N}$. We randomly sample the initial inventory level C_i from the uniform distribution over $\{10,12,14,16,20\}$ and let $C_i^d = C_i$ for all $d \in \{1,\ldots,D\}$ and $i \in \mathcal{N}$.

D	Return Eligibility	Price Discount
2	{Yes, No}	{100%}
4	{Yes, No}	{100%, 90%}
6	{Yes, No}	{100%, 90%, 80%}
8	{Yes, No}	{100%, 90%, 80%, 70%}

Table EC.8 Experiment Setup of Multiple Designs

For simplicity of the experiment, we focus on the setting with homogeneous customer arrivals. Recall that our estimation procedure yields the nominal utility $\hat{u}_i^t \geq 0$ for each item and the price sensitivity $\hat{\beta}^t$ for each $t \in \mathcal{T}$, which is utilized in Section 6. The net utility of item i given price p is assumed to be $\hat{u}_i^t - \hat{\beta}^t p$. In this experiment, we select the first period $t = 0$ for all arriving customers to construct our experimental instances. Following this set up, we assume that v_{id} is the preference weight of the d-th design of item i at each period and that the customers' choice decisions follow the MNL choice model ϕ such that the choice probability under assortment $A \subseteq \mathcal{N}$ is:

$$\phi_i(A\,|\,\boldsymbol{x}) \;=\; \frac{\sum_{d\in\mathcal{D}} v_{id}x_{id}}{v_0 \;+\; \sum_{j\in A}\sum_{h\in\mathcal{D}} v_{jh}x_{jh}}\mathbb{1}[i\in A], \quad \forall i\in\mathcal{N}.$$

Similar to our discussion in 6, we assume that the estimated utility parameters correspond to the basic design. Therefore, we set $v_{id} = \exp(\hat{u}_i^0 - \hat{\beta}^0 p_{id})$ for all $d \in \{1,2,\ldots,K\}$. We assume that the nominal utility of the non-returnable version of each product is $\eta\hat{u}_i^t$, where η is generated randomly uniformly over $[0,1]$. In particular, η represents the discount of the nominal utility when not being allowed to return the product. Thus, $v_{id} = \exp(\eta \hat{u}_i^0 - \hat{\beta}^0 p_{id})$ for all $d \in \{K+1, K+2, \dots, 2K\}$. It remains to generate the preference weight for the no-purchase option. Let α_0 denote the no-purchase probability when we offer a fully returnable design and full assortment for arrival t. We randomly sample α_0 from $\{0.25, 0.75\}$. The preference weight of the no-purchase option is therefore $v_0^t = \alpha_0/(1-\alpha_0) \cdot \sum_{i \in \mathcal{N}} v_{id}$. We set the return likelihood parameters $\sigma_{\mathrm{R}} = 0.001$ and generate a sequence of $T = \sigma_{\mathrm{L}} \sum_{i \in N} C_{i1}$ customers for $\sigma_{\mathrm{L}} \in \{1.0, 1.2, 1.4\}$. For each value of (D, σ_{L}), a total of 50 problem instances is generated.

Number of Designs D	Load Parameter σ_{L}	Offline Effectiveness $\mathsf{Eff}_{\text{off}}(\boldsymbol{x}_{\mathsf{REV}})$
2	1.0	99.3
2	1.2	99.2
2	1.4	99.6
4	1.0	98.1
4	1.2	97.2
4	1.4	96.8
6	1.0	96.2
6	1.2	94.4
6	1.4	92.8
8	1.0	95.7
8	1.2	91.3
8	1.4	90.0

Table EC.9 Offline effectiveness of revenue-based rounding with multiple designs

Table EC.9 demonstrates the high effectiveness of the revenue-based rounding method, even when handling a large number of designs. With an empirical performance exceeding 90%, it significantly surpasses the theoretical lower bound. Moreover, we consistently observe that the REV approach outperforms both NoC and NoR when considering multiple designs. These findings emphasize the robustness of the revenue-based rounding method in real-world datasets, showcasing its applicability across a wider range of scenarios.

Appendix M: Further Integration of the Joined Decisions: A Coordinate Descent Algorithm

Recall that in this paper we have developed a two-step framework to solve the JPDAO problem (Theorem 2.1). In the first step, we obtain an offline design decision $\boldsymbol{x}^H$ by approximately solving the problem $\max_{\boldsymbol{x} \in \mathcal{X}} \mathsf{UB}(\boldsymbol{x})$ using the revenue-based rounding algorithm (Section 4). In the second step, we compute $\boldsymbol{\pi}^H$ by finding an approximate solution to $\max_{\boldsymbol{\pi} \in \Pi} J(\boldsymbol{x}^H, \boldsymbol{\pi})$ using the value function approximation method (Section 5), where for notation simplicity we define

$$J(\boldsymbol{x}, \boldsymbol{\pi}) \;=\; \mathbb{E}\Bigg[\sum_{t \in \mathcal{T}} R_t^{\boldsymbol{\pi}}(\boldsymbol{x})\Bigg]$$

as the expected revenue from a joint decision $(\boldsymbol{x}, \boldsymbol{\pi})$. One might argue that our solution $(\boldsymbol{x}^H, \boldsymbol{\pi}^H)$ is not fully integrated, because, upon a closer look, one can observe that no information regarding $\boldsymbol{\pi}^H$ is incorporated into the computation of $\boldsymbol{x}^H$. In other words, there is no "feedback loop" from $\boldsymbol{\pi}^H$ to $\boldsymbol{x}^H$. In this section, we present an initial exploration towards fully integrating $\boldsymbol{x}$ and $\boldsymbol{\pi}$.

On a high level, the JPDAO problem can be regarded as a joint optimization

$$\max_{(\boldsymbol{x},\boldsymbol{\pi})\in\mathcal{X}\times\Pi} J(\boldsymbol{x}, \boldsymbol{\pi})$$

with two sets of decision variables $\boldsymbol{x}$ and $\boldsymbol{\pi}$. Naturally, the well known coordinate descent procedure presents the blueprint for solving problem of this type as follows. We first initialize an offline decision $\boldsymbol{x}^{(0)}$. Then in each iteration $k = 1, \ldots, K$, we compute

$$\boldsymbol{\pi}^{(k)} \;=\; \arg\max_{\pi\in\Pi} J(\boldsymbol{x}^{(k-1)}, \boldsymbol{\pi}) \quad \text{and} \quad \boldsymbol{x}^{(k)} \;=\; \arg\max_{\boldsymbol{x}\in\mathcal{X}} J(\boldsymbol{x}, \boldsymbol{\pi}^{(k)}), \tag{M.1}$$

where K is a pre-set upper bound on the number of iterations. Conceptually, the techniques that we developed in this paper can be regarded as a single incomplete iteration in the the above procedure. Namely, we provide an initial solution $\boldsymbol{x}^{(0)} = \boldsymbol{x}^H$. Then, we computation $\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^H$ by finding an approximate solution to $\max_\pi J(\boldsymbol{x}^{(0)}, \pi)$. If we keep iterating the aforementioned procedure, we might hope to achieve a tighter integration of offline and online decisions.

The coordinate descent algorithm requires us to solve both problems in (M.1) in each iteration k. While there are various heuristics to solve the first problem in (M.1), including the value function approximation method we develop in Section 5 with a provable performance guarantee, it is challenging to solve $\max_{\boldsymbol{x}\in\mathcal{X}} J(\boldsymbol{x}, \boldsymbol{\pi}^{(k)})$. A notable reason is that $J(\boldsymbol{x}, \boldsymbol{\pi}^{(k)})$ depends on the offline decision $\boldsymbol{x}$ in an extremely complex and non-linear way. The CDLP upper bound, $\mathsf{UB}(\boldsymbol{x})$, which we use as surrogate problem of $\max_{\boldsymbol{\pi}} J(\boldsymbol{x}, \boldsymbol{\pi})$ in our approximation framework proposed in Section 2.2, does not contain any information about the specific choice of policy $\boldsymbol{\pi}$, so it is not applicable in the coordinate descent algorithm. However, motivated by our theoretical development, we next propose the following remedy to infuse information about the policy into the CDLP upper bound as an approximation to $J(\boldsymbol{x}, \boldsymbol{\pi}^{(k)})$ to apply the coordinate descent framework.

Given any $\boldsymbol{x} \in \mathcal{X}$, we obtain greedy policy $\boldsymbol{\pi}(\boldsymbol{x})$ using the approach in Section 5. We can run simulation experiments to get an *empirical estimate* of the probability of purchasing each item during each period by the policy $\boldsymbol{\pi}(\boldsymbol{x})$ given $\boldsymbol{x}$. Let $\hat{\boldsymbol{\mu}} = (\hat{\mu}_i^t : i \in \mathcal{N}, t \in \mathcal{T})$ denote the empirical estimate, where $\hat{\mu}_i^t$ represent the frequency of offering item i during period t by the policy across all simulation runs. We can embed $\hat{\boldsymbol{\mu}}$ into $\mathsf{UB}(\boldsymbol{x})$ to construct a CDLP that captures the specific information of the policy. In particular, we define

$$\mathsf{UB}(\boldsymbol{x}; \hat{\boldsymbol{\mu}}) \;=\; \max_{\boldsymbol{z}} \quad \sum_{t\in\mathcal{T}} \sum_{S\in\mathcal{F}} z^t(S) \sum_{i\in\mathcal{N}} \phi_i^t(S \,|\, \boldsymbol{x})\, r_i(\boldsymbol{x})$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{S\in\mathcal{F}} z^{\tau}(S)\phi_i^{\tau}(S\,|\,\boldsymbol{x})\alpha_i(t-\tau\,|\,\boldsymbol{x}) \;\le\; C_i(\boldsymbol{x}) && \forall t\in\mathcal{T}, i\in\mathcal{N},\\
& \sum_{S\in\mathcal{F}} z^t(S) \;\le\; 1 && \forall t\in\mathcal{T},\\
& z^t(S) \;\ge\; 0 && \forall t\in\mathcal{T}, S\in\mathcal{F}\;,\\
& \left|\sum_{S\in\mathcal{F}:i\in S} z^t(S)\phi_i^t(S\,|\,\boldsymbol{x}) - \hat{\mu}_i^t\right| \;\le\; \epsilon && \forall t\in\mathcal{T}, i\in\mathcal{N} \qquad \text{(M.2)}
\end{aligned}$$

In essence, $\mathsf{UB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$ is the restricted version of $\mathsf{UB}(\boldsymbol{x})$ with an additional constraint (M.2), which requires that the probability of offering each item at each period given $\boldsymbol{z}$ deviate from the empirical estimation $\hat{\mu}_i^t$ for at most ϵ. We let $\epsilon > 0$ be a tolerance parameter. Since $\hat{\boldsymbol{\mu}}$ is obtained through simulation by running $\boldsymbol{\pi}(\boldsymbol{x})$, we hope to use (M.2) to make sure that the feasible $\boldsymbol{z}$ incorporates information regarding $\boldsymbol{\pi}(\boldsymbol{x})$. Therefore, we regard $\max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{UB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$ as a proxy to $\max_{\boldsymbol{x}\in\mathcal{X}} J(\boldsymbol{x},\boldsymbol{\pi}(\boldsymbol{x}))$. In a similar vein, we define

$$\begin{aligned}
\mathsf{VB}(\boldsymbol{x};\hat{\boldsymbol{\mu}}) = \max_{\boldsymbol{y}\ge\boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) r_{id}\\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^{\tau}(\boldsymbol{A})\psi_{id}^{\tau}(\boldsymbol{A})\alpha_{id}(t-\tau) \;\le\; C_{id}x_{id} && \forall\; i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T},\\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \;\le\; 1 && \forall\; t\in\mathcal{T}\;.\\
& \left|\sum_{d\in\mathcal{D}_i}\sum_{\boldsymbol{A}\in\mathcal{G}:i^d\in\boldsymbol{A}} y^t(\boldsymbol{A})\psi_{id}^t(\boldsymbol{A}) - \hat{\mu}_i^t\right| \;\le\; \epsilon && \forall\; i\in\mathcal{N}, t\in\mathcal{T} \qquad \text{(M.3)}
\end{aligned}$$

Following the proof of Theorem 3.5, we note that $\mathsf{VB}(\boldsymbol{x};\hat{\boldsymbol{\mu}}) = \mathsf{UB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$. We use $\mathsf{UB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$ and $\mathsf{VB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$ to fill in the missing piece in (M.1) of the coordinate descent procedure. One caveat is that $\mathsf{VB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$ is not always feasible due to the noise in the estimator $\hat{\boldsymbol{\mu}}$ in constraint (M.3). Therefore, for computational purposes, one can consider adding the violation of constraint (M.3) as a penalty to the objective function:

$$\begin{aligned}
\mathsf{VB}_{\lambda}(\boldsymbol{x};\hat{\boldsymbol{\mu}}) = \max_{\boldsymbol{y}\ge\boldsymbol{0}} \quad & \sum_{t\in\mathcal{T}}\sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} \psi_{id}^t(\boldsymbol{A}) r_{id} - \lambda\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\left|\sum_{\boldsymbol{A}\in\mathcal{G}:i^d\in\boldsymbol{A}} y^t(\boldsymbol{A})\psi_{id}^t(\boldsymbol{A}) - \mu_i^t\right|\\
\text{s.t.} \quad & \sum_{\tau=1}^{t}\sum_{\boldsymbol{A}\in\mathcal{G}} y^{\tau}(\boldsymbol{A})\psi_{id}^{\tau}(\boldsymbol{A})\alpha_{id}(t-\tau) \;\le\; C_{id}x_{id} && \forall\; i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T},\\
& \sum_{\boldsymbol{A}\in\mathcal{G}} y^t(\boldsymbol{A}) \;\le\; 1 && \forall\; t\in\mathcal{T}\;.
\end{aligned}$$

In particular, we remove (M.3) and add it as a penalty into the objective function with penalty weight $\lambda > 0$. We treat λ as a hyperparameter, which can be determined by a grid search in practice.

In summary, based on the idea of coordinate descent, we develop an iterative algorithm to better integrate offline design and online assortment policy, as stated in the following. We let K represent the limit on the number of iterations.

Approximate Coordinate Descent Algorithm

Step 1 (Initialization): Initialize $\boldsymbol{x}^{(0)}$ by solving $\max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{VB}(\boldsymbol{x})$ approximately using the revenue-based rounding method.

Step 2 (Iteration): For $k=1,\dots,K$ do:

- **2.1 (Online Policy Optimization)** Compute online policy $\pi^{(k)}$ by approximately solving $\arg\max_{\boldsymbol{\pi}\in\Pi} J(\boldsymbol{x}^{(k-1)},\boldsymbol{\pi})$ with the value function approximation.
- **2.2 (Simulation and Estimation)** Estimate $\hat{\boldsymbol{\mu}}^{(k)}$ by simulating the online assortment optimization problem with policy $\boldsymbol{\pi}^{(k)}$ under design $\boldsymbol{x}^{(k-1)}$.
- **2.3 (Offline Design Optimization)** Compute $\boldsymbol{x}^{(k)}$ by solving $\max_{\boldsymbol{x}\in\mathcal{X}} \mathsf{VB}(\boldsymbol{x};\hat{\boldsymbol{\mu}}^{(k)})$ approximately with the revenue-based rounding method.

Step 3 (Output): Return $\left(\boldsymbol{x}^{(K)},\boldsymbol{\pi}^{(K)}\right)$.

We note that again K represents the iteration limits. We also note that the previous construction of $\mathsf{UB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$ and $\mathsf{VB}(\boldsymbol{x};\hat{\boldsymbol{\mu}})$ requires that the CDLP captures the probability of offering each product at each period by the target policy.

M.1. Numerical Results

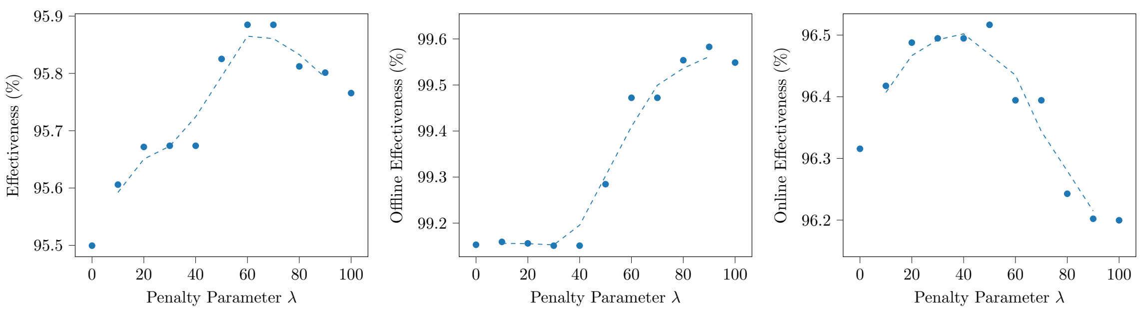
We note that our proposal here is heuristic in nature. Unfortunately, we are not able to provide a theoretical performance guarantee for the iterative algorithm. In the following, we resort to numerical experiments to examine the empirical performance of the iterative algorithm.

We randomly generate 50 problem instances with load parameter $\sigma_L=1.2$ with the same specification as described in Section 6. We adopt the sales-based construction for computational convenience, similar to our discussion in Section 6 as follows:

$$
\begin{aligned}
W_\lambda^{\mathsf{LP}}(\boldsymbol{x};\hat{\boldsymbol{w}}) = \max_{\boldsymbol{w}} \quad & \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} r_{id} w_{id}^t - \lambda \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} \left| \sum_{d\in\mathcal{D}_i} w_{id}^t - \hat{w}_i^t \right| \\
\text{s.t.} \quad & \sum_{\tau=1}^{t} \alpha_{id}(t-\tau) w_{id}^\tau \le C_{id} x_{id} \quad \text{and} \quad w_{id}^t \le v_{id}^t w_0^t \qquad \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}, \\
& w_0^t + \sum_{i\in\mathcal{N}}\sum_{d\in\mathcal{D}_i} w_{id}^t \le 1 \qquad \forall\, t\in\mathcal{T}, \\
& w_0^t \ge 0,\ w_{id}^t \ge 0, \qquad \forall\, i\in\mathcal{N}, d\in\mathcal{D}_i, t\in\mathcal{T}.
\end{aligned}
$$

We run the approximately coordinate descent algorithm with $K=5$ iterations and penalty parameter $\lambda\in\{0,10,\dots,130\}$ for each problem instance. Particularly, $\lambda=0$ represents our original method without integration. We still focus on the effectiveness metrics, similar to our experiments in Section 6. For each λ, we record the maximum overall effectiveness achieved among the K iterations for every problem instance. The results are plotted in Fig. EC.1. We use dots to represent the result for each λ value. We also plot a smoothed curve with a moving average window of three for better visualization.

Figure EC.1 Overall effectiveness and Decomposition to offline and online effectiveness



(a) Overall (b) Offline (c) Online

Figure EC.1a displays the average maximum overall effectiveness across 50 instances for different penalty parameters. The results indicate an initial performance improvement of the iterative algorithm with increasing penalty parameter values. However, this improvement eventually plateaus and reaches a point of diminishing returns. Approximately half of the instances exhibit enhanced performance as a result of the iterative algorithm. Within this subset, the algorithm achieves a maximum improvement of 3.23% and a mean improvement of 1.35% in the effectiveness. To further analyze the effectiveness, we decompose it into offline effectiveness and conditional online effectiveness, as shown in Figure EC.1b and Figure EC.1c, respectively. This decomposition is performed at the iteration that achieves the maximum effectiveness for each instance. Notably, we observe that the quality of offline and online decisions change in opposite directions. The primary driver for the observed performance improvement appears to be the offline decision.

The following observations can be drawn from this experiment. Firstly, the introduction of online policy data into the computation of offline decisions via the approximate coordinate descent algorithm bolsters the joint method's effectiveness. This is primarily achieved through an enhancement in the effectiveness of the offline decision, influenced by the feedback from the online decision. This method shows promise. Secondly, careful hyperparameter tuning is necessary to optimize performance improvement. More specifically, it is noted that the increased effectiveness in offline operations comes at the cost of conditional online effectiveness when the hyperparameter is large. This initial exploration of integrating joint decisions with the coordinate descent algorithm indeed suggests potential, but a deeper understanding of the underlying theory is required. This calls for future research to be dedicated to this direction.

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