Internet Appendix

"Endogenous Risk-Exposure and Systemic Instability" by Chong Shu

Section I of this Internet Appendix contains committed analyses. Section II contains proofs.

I. Omitted Analyses

I.1 Distribution of Primitive Asset and Systemic Defaults

Similar to the benchmark model, each bank selects its project's return in the good state, $\alpha = Z_i$. To capture the concept of risk-return trade-off, we assume that the probabilities of a successful project for the bank, $P_\alpha(Z_i)$ and $P_\beta(Z_i)$, decrease as Z_i increases. Consequently, the probability of the bank's project failing, represented by $P_\gamma(Z_i) \equiv 1 - P_\alpha(Z_i) - P_\beta(Z_i)$, increases with Z_i. This assumption implies that a bank is more likely to have a failed project if it chooses greater risk, irrespective of whether it is stand-alone or within a network. Under this new cash flow distribution, each bank's expected profit is calculated as follows:

$$
\begin{aligned}
\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big] = {} & P_\alpha(Z_i)\sum_{\boldsymbol{\omega}_{-i}}\Big[\underbrace{Z_i - v - \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\alpha}})\Big)}_{>0}\Big]\cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \\
& + P_\beta(Z_i)\sum_{\boldsymbol{\omega}_{-i}}\Big[\underbrace{\beta - v - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}})\Big)}_{>0 \text{ or } =0}\Big]^+\cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \\
& + P_\gamma(Z_i)\sum_{\boldsymbol{\omega}_{-i}}\Big[\underbrace{\gamma - v - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\gamma})\Big)}_{=0}\Big]^+\cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})
\end{aligned}
$$

The payment equilibrium that determines $d_i^*(\boldsymbol{\omega})$ is the same as in the benchmark model. The bank's profit is zero when its project fails (as indicated in the last line of the equation). In the contagion state (second line), its profit can be either positive or zero, depending on $\boldsymbol{\omega}_{-\boldsymbol{i}}$. In the non-contagion state (first line), its profit is always positive.

We observe that counterparty risk now affects each bank's expected payoff in two ways. In the non-contagion state, an increase in counterparty risk reduces the bank's profit because the bank is obliged to offer greater cross-subsidies to other banks. This effect is the same as the one in the benchmark model. Conversely, in the contagion state – where the bank has a successful project but faces potential contagion – a rise in counterparty risk increases the likelihood of the bank defaulting due to the contagion from other banks' losses. This effect is new here. The next proposition demonstrates that incorporating the potential for cascading defaults does not change the strategic complementarity result.

PROPOSITION I.1. *With the cash flow distribution in Equation 8, banks' choices of risk exposure* $\mathbf{Z}$ *are strategically complementary.*

In the non-contagion state, heightened counterparty risk increases the likelihood that each bank will have to cross-subsidize other banks. Proposition 1 has already demonstrated that these cross-subsidies distort a bank's upside payoff, thereby encouraging a riskier portfolio. Moreover, in the newly-introduced contagion state, increased counterparty risk heightens the bank's susceptibility to default due to contagion, further reducing its upside payoff. These two distortions together discourage the bank from prioritizing the probability of success when weighing risk and return. As a result, in response to increased counterparty risk, a bank will opt for riskier exposure.

I.2 Banks' Incentives to Form Links

To model this concept, let $c_i \in \mathbb{R}^+$ represent the charter value of bank i. The bank can preserve this charter value only if its depositors are fully paid, either through the bank's own projects or via cross-subsidies from other banks. The expected payoff for bank i can thus be rewritten as:

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\Big] = P(Z_i)\Big[Z_i - v_i - \mathcal{D}_i(\mathbf{Z}_{-i})\Big] + \underbrace{c_i - \Big[1 - P(Z_i)\Big]\Pr\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f\Big) \cdot c_i}_{\text{expected charter value}} \tag{I.1}$$

where $\mathcal{F}^-_{\boldsymbol{\omega}} \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) < v\}$ denotes the set of insolvent banks – those unable to fully reimburse their depositors, and $\Pr(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f)$ represents the probability that bank i is insolvent given that its project fails. The bank will lose its charter value under such circumstances.

We can confirm that Corollary 1 remains hold: connected banks choose greater risk exposure than stand-alone banks. Intuitively, there are two forces that drive a connected bank to take on higher risks: (i) a network risk-taking distortion, as seen in the benchmark model, and (ii) downside protection offered by the financial network's co-insurance. The second factor is introduced by the consideration of banks' charter values. Both forces reduce banks' interest in the probability of success when weighing the risk-return trade-off.

Let's now explore whether banks are incentivized to form interbank links when considering both charter values and network risk-taking distortion.

PROPOSITION I.2. *There exists* $\bar{c} \in \mathbb{R}^+$*, such that if* $\min\{c_i\} > \bar{c}$*, banks have incentives to form links.*

Banks need to weigh three factors when deciding to join a network: (i) the preservation of their charter values, (ii) the reduced upside payoff due to cross-subsidies, and (iii) the distortion of investment due to risk-taking externalities. On one hand, joining a network can offer banks protection against defaulting on deposits, thereby safeguarding their charter values. On the other hand, this same co-insurance obligates banks to cross-subsidize their failed peers, diminishing their own potential gains. Such effects can further distort investment decisions, leading to instability.

Proposition I.2 argues that banks have incentives to join an interbank network if they place significant value on protecting their charter values or if regulators are vigilant in putting insolvent banks into receivership. However, upon joining a financial network, banks inevitably face risk-taking externalities, and the findings from Proposition 1 to 3 become applicable.

I.3 Correlated Risk Exposure

Since the $\boldsymbol{\phi}_i$ set by each bank impacts both its cross-subsidies and its charter value, a trade-off emerges. On the one hand, choosing a low-correlation project better safeguards the bank's charter value. On the other hand, a low-correlation project means that when the bank's own project succeeds, other banks' projects are more likely to fail. This, in turn, obligates the bank to pay higher cross-subsidies. Although a closed-form solution for the correlated risk-taking equilibrium cannot be readily derived and depends on the functional form of $Pr(.)$, the following proposition offers insights into how the magnitude of the charter value can influence this trade-off.

PROPOSITION I.3. *In any regular networks,*

1. *If $\boldsymbol{c} = 0$, the risk exposure of every bank is perfectly correlated, i.e., $\phi^*_{ij} = 1$ for all $i, j \in \mathcal{N}$. In this case, there is no risk-taking distortion, and each bank will choose the same level of risk exposure as a stand-alone bank would.*

2. *There exists $\bar{c} \in \mathbb{R}^+$ such that if $\min\{c_i\} > \bar{c}$, $\phi^*_{ij} < 1$ for some $i, j \in \mathcal{N}$ and there is risk-taking distortion. Each bank will choose a higher exposure to risks than a stand-alone bank.*

The first part of the proposition states that if the charter value is negligible, connected banks choose to expose themselves to a single systemic risk. Without the threat of losing the charter value, each bank correlates its project outcomes with other connected banks. By doing so, it will face no distortion in its upside payoff and hence will enjoy a higher expected profit. In this case, banks are better off netting out their interbank liabilities. This result predicts that a systemic crisis can endogenously evolve in a connected banking system when there is an insufficient threat of losing charter values. It comports with the empirical findings of International Monetary Fund (2009) and Bisias et al. (2012), which demonstrate a high level of distress dependence among major banks before the 2008 financial crisis, coinciding with an unprecedented level of interconnectedness in the banking system.

While the finding aligns with the predictions of collective moral hazard models such as Farhi and Tirole (2012), the underlying mechanism is different, leading to distinct policy implications. For instance, traditional collective moral hazard models suggest that bailouts generally encourage risk-taking. In contrast, the risk-taking externality model presented here implies that a bailout of interbank exposures alone could discourage both the size and correlation of bank risks. Further discussion on this topic will be presented in section 6.2.

Part 2 of the proposition demonstrates that if banks sufficiently care about their charter values, they will opt for projects with weaker correlations. This is because, in this scenario, banks would prioritize the protection of charter values provided by the financial network, leading to diversification in their project choices. Due to the imperfect correlation, the distortion in risk-taking exists, and the results from Propositions 1 to 3 continue to hold.

I.4 Endogenizing Deposit Rates

Suppose banks are connected in a regular network where $\bar{d}_i = \bar{d}$. The equilibrium $(d_i^*(\boldsymbol{\omega}), \boldsymbol{v}^*, \boldsymbol{Z}^*)$ is characterized by

$$\begin{aligned} d_i^*(\boldsymbol{\omega}) &= \left\{ \min \Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \boldsymbol{Z}) + e_i(\omega_i, Z_i) - v_i^*, \bar{d}, \Big] \right\}^+ && \forall i \in \mathcal{N}, \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \\ Z_i^* &= \underset{Z_i}{\operatorname{argmax}}\, \mathbb{E}\Big[\Pi_i^B(\boldsymbol{\omega}; \boldsymbol{Z}, v_i^*) \Big] && \forall i \in \mathcal{N} \\ 0 &= -M_i + \beta \cdot \mathbb{E}\Big[u\Big(\Pi_i^D(\boldsymbol{\omega}; v_i^*, \boldsymbol{Z}^*) \Big) \Big] && \forall i \in \mathcal{N} \end{aligned}$$

where $\Pi_i^B(\boldsymbol{\omega})$ represents the payoff to bank i in state $\boldsymbol{\omega}$, as described by Equation 3. $\Pi_i^D(\boldsymbol{\omega})$ denotes the payoff to bank i's depositors. When there is deposit insurance, the government guarantees the return to depositors, making $\Pi_i^D(\boldsymbol{\omega}) = v_i^*$ for all $\boldsymbol{\omega}$. Without deposit insurance, the payoff to depositors can be calculated as

$$\Pi_i^D(\boldsymbol{\omega}) = \min \Big[v_i^*, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) \Big]$$

It's worth noting that $\Pi_i^D(\boldsymbol{\omega})$ is a function of $\boldsymbol{\Theta}$ and $\boldsymbol{Z}^*$, meaning that depositors are aware of the network structure and perfectly anticipate banks' optimal risk exposure. The following proposition demonstrates banks' equilibrium risk-taking behaviors under various assumptions.

PROPOSITION I.4. *In any regular network,*

(a) If deposit insurance exists, a connected bank will choose higher risk exposure compared to a stand-alone bank.

(b) If there is no deposit insurance and depositors are risk-neutral, a connected bank will choose the same risk exposure as a stand-alone bank.

(c) If there is no deposit insurance and depositors are risk-averse, a connected bank will choose higher risk exposure compared to a stand-alone bank.

Part (a) of the proposition examines financial systems where the government fully insures deposits. In this case, depositors are insensitive to the banks' financial structures, and deposit rates remain constant. The equilibrium becomes the same as the one in the benchmark model, and the results from Propositions 1 to 3 continue to hold. Specifically, banks in a connected financial system will choose higher risk exposure compared to stand-alone banks.

The second part of the proposition demonstrates that, in the absence of deposit insurance, the risk exposure choices made by banks are identical – regardless of whether they are connected or stand-alone – when depositors are risk-neutral. To understand why this result differs from Corollary 1 in cases where deposit rates are exogenous, it is important to recognize that the network risk-taking distortion underlying Corollary 1 arises from the cross-subsidies flowing from successful banks to the depositors of failed banks. These cross-subsidies, in turn, make the depositors of connected banks feel more co-insured through interbank connections, making them demand a lower deposit rate. Both the reduced deposit rates and the cross-subsidies affect the upside payoffs of connected banks. Their opposing effects on banks' risk-taking incentives cancel each other out, as the risk-taking distortion is precisely driven by the transfer of wealth from successful banks to the depositors of failed banks.

On the other hand, if depositors are risk-averse, they won't value the co-insurance provided by the financial network as much as the decreased expected return from the bank. This means that while depositors offer connected banks lower deposit rates, this reduction won't entirely neutralize the effects of distortionary cross-subsidies on the bank's risk-taking incentives. As a result, the last part of the proposition demonstrates that banks in connected financial networks will still opt for higher levels of risk compared to stand-alone banks when there is no deposit insurance and if depositors are risk adverse.

I.5 Effect of Regulations on Asymmetric or Nonregular Networks

Consider a core-periphery network with $N = 5$ peripheral banks. Each peripheral bank holds both a debt and a claim, each of size $\bar{d}$, with the core. The extension discussed in Section 5.1 can be applied to a core-periphery network, so we assume that each bank can fulfill its total liabilities if its project succeeds. For the exposition, further assume that $\bar{d}$ is smaller than v_p, the deposit amount of each peripheral bank.

At the initial date, the core bank chooses its risk exposure, Z_c, and each peripheral bank i chooses its risk exposure, Z_i, simultaneously. The expected profit of the core bank and each peripheral bank is

$$\mathbb{E}\Big[\Pi_c(\boldsymbol{\omega};\boldsymbol{Z})\Big] = P(Z_c)\sum_{m=1}^{N}(Z_c - v_c - m\cdot\bar{d})\cdot\ \Pr(m \text{ peripheral banks fail})$$

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big] = P(Z_i)\sum_{m=1}^{N-1}\Big\{\big(1-P(Z_c)\big)\Big(Z_i - v_p + \Big(\frac{(N-m)\cdot\bar{d} - v_c}{N}\Big)^{+} - \bar{d}\Big) + P(Z_c)(Z_i - v_p)\Big\} \cdot \Pr(m \text{ other peripheral banks fail})$$

where $\Pr(m \text{ peripheral banks fail}) = \binom{N}{m}\big(1-P(Z_i)\big)^m\big(P(Z_i)\big)^{N-m}$. From here, we can derive the network risk-taking distortion for the core and each peripheral bank as

$$\mathcal{D}_c(\mathbf{Z_i}) = \sum_{m=1}^{N} m \cdot \bar{d} \cdot \text{Pr (}m\text{ peripheral banks fail)}$$

$$\mathcal{D}_i(Z_c, \mathbf{Z_{-i}}) = \sum_{m=1}^{N-1} \left[\bar{d} - \left(\frac{(N-m)\cdot \bar{d} - v_c}{N}\right)^+\right] \cdot \Big(1 - P(Z_c)\Big) \cdot \text{ Pr (}m\text{ other peripheral banks fail)}$$

The game is supermodular, and the banks' choices of risk exposure are strategically complementary. We observe that the core bank's risk exposure, Z_c, directly influences each periphery bank i's risk-taking distortion. In contrast, a peripheral bank's risk exposure, Z_{-i}, affects other periphery banks' risk-taking distortion only if the core bank fails. This observation suggests that a prudential policy would be more effective in reducing the financial system's overall risk exposure if targeted at the core bank rather than at individual peripheral banks.

For the line network shown in Figure 5(b), assume that an external firm owes Bank 1 a debt of $\bar{d}$, and Bank 5 owes an external firm a debt of $\bar{d}$. For notational simplicity, let $P(Z_0)$ denote the probability that Bank 1's claims will be paid. The probabilities of Banks 2 through 5 having their claims paid are determined in equilibrium. The expected profit for each bank is thus given by:

$$P(Z_i)(Z_i - v) - P(Z_i) \cdot \big(1 - P(Z_{i-1})\big) \cdot \bar{d}$$

We can now observe that guaranteeing Bank 1's liabilities reduces the risk-taking distortion for every other bank $j \in 2, 3, 4, 5$ due to the reduction in counterparty risk Z_{j-1}. In contrast, guaranteeing Bank 5's liabilities has no impact on the risk-taking incentives of any other bank in the system.

Now, consider the network structure shown in Figure 5(c), where each bank holds identical amounts of deposits and interbank liabilities and has the same number of direct counterparties. The only distinction is how banks are connected within different clusters of the network, labeled as L and R. For expository purposes, let's again assume $\bar{d} < v$. In the absence of regulation, each bank's equilibrium risk exposure is identical, which is

$$\tilde{Z} = \underset{Z}{\operatorname{argmax}}\, P(Z)(Z - v) - P(Z) \cdot \big[1 - P(\tilde{Z})\big] \cdot \bar{d}$$

Suppose the government can guarantee the liabilities of either bank L_1 or bank R_1. If it chooses to guarantee bank L_1, then bank L_2 will choose a lower level of risk exposure, and banks R_1, R_2, and R_3 will continue to choose $\tilde{Z}$. Conversely, if the government decides to guarantee bank R_1, then bank R_2 will choose a lower level of risk exposure, similar to bank L_1 in the previous scenario; bank R_3 will choose the risk exposure that is between $\tilde{Z}$ and that of R_2; banks L_1 and L_2 will continue to choose $\tilde{Z}$. Thus, we can see that guaranteeing bank R_1 results in lower endogenous risk exposure in the financial system, even though it requires the same resources as guaranteeing bank L_1.

II. Proofs

LEMMA 2

From the assumption $\underline{Z} \geqslant v_i + \bar{d}_i$, a successful bank's interbank payment is $\bar{d}_i$, independent of its choice of risk exposure Z_i. A failed bank's cash flow that will contribute to the interbank payment system is $e_i = 0$, also independent of its choice of risk exposure. Reordering Equation 2 gives us,

$$\begin{aligned} d_i^*(\boldsymbol{\omega};\mathbf{Z}) &= \bar{d}_i && \forall \omega_i = s \\ d_i^*(\boldsymbol{\omega};\mathbf{Z}) &= \left\{ \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\mathbf{Z}) - v_i \right\}^+ && \forall \omega_i = f \end{aligned} \tag{I.2}$$

We can see that the vector of risk exposure $\mathbf{Z}$ does not enter the system of equations. As a result, the fixed point $(d_1^*(\boldsymbol{\omega}), ...d_N^*(\boldsymbol{\omega}))$ is constant in $\mathbf{Z}$. □

AUXILIARY LEMMA

The payment vector $\boldsymbol{d}^*$ is weakly increasing in any bank's cash flow $\tilde{e}_j$. In particular, $\boldsymbol{d}^*(\boldsymbol{\omega})$ is higher when any bank's project succeeds ($\omega_j = s$) compared with when it fails ($\omega_j = f$).

Proof. The above lemma is identical to Eisenberg and Noe (2000) Lemma 5. The payment equilibrium (Equation 2) is a fixed point solution of a function $\boldsymbol{d}^* = \Phi(\boldsymbol{d}^*; \tilde{e}_j)$. Since both min and max operators preserve monotonicity, Φ is increasing in $\tilde{e}_j$. By monotone selection theorem (Milgrom and Roberts (1990), Theorem 1), the fixed point $\boldsymbol{d}^*$ is increasing in $\tilde{e}_j$.

□

PROPOSITION 1

Let's first prove the strategic complementarity result for regular networks, then proceed to nonregular networks. For regular networks, the expected payoff is

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\Big] = P(Z_i)\cdot\big(Z_i - v_i\big) - P(Z_i)\cdot\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}})$$

The first- and second-order conditions of the expected payoff are

$$\begin{aligned} F(Z_i;\mathbf{Z}_{-\boldsymbol{i}}) &= P'(Z_i)(Z_i - v_i) + P(Z_i) - P(Z_i)'\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}}) = 0 \\ S(Z_i;\mathbf{Z}_{-\boldsymbol{i}}) &= P''(Z_i)(Z_i - v_i) + 2P'(Z_i) - P(Z_i)''\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}}) < 0 \end{aligned}$$

From assumption 1, we obtain $S(Z_i;\mathbf{Z}_{-\boldsymbol{i}}) < 0$. Taking the total derivative of the FOC, we have

$$\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}})} = -\frac{\partial F(\widehat{Z}_i;\mathbf{Z}_{-\boldsymbol{i}})/\partial\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}})}{\partial F(\widehat{Z}_i;\mathbf{Z}_{-\boldsymbol{i}})/\partial Z_i} = \frac{P'(\widehat{Z}_i)}{S(\widehat{Z}_i;\mathbf{Z}_{-\boldsymbol{i}})} > 0 \tag{I.3}$$

The inequality implies that any factor increasing $\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}})$ will also increase bank i's optimal risk exposure $\widehat{Z}_i$. To examine the impact of bank m's risk exposure Z_m on $\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}})$, let's alter it from Z_m to Z'_m where $Z'_m > Z_m$. Let $\mathbf{Z'}_{-\boldsymbol{i}}$ represent the new risk-exposure vector that differs from $\mathbf{Z}_{-\boldsymbol{i}}$ only in Z_m. We have

$$
\begin{aligned}
&\mathcal{D}_i(\boldsymbol{Z'_{-i}}) - \mathcal{D}_i(\boldsymbol{Z_{-i}}) \\
&\quad = \sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[P(Z'_m)\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}})\Big) + \Big(1 - P(Z'_m)\Big)\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}})\Big)\Big] \\
&\qquad - \sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[P(Z_m)\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}})\Big) + \Big(1 - P(Z_m)\Big)\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}})\Big)\Big] \\
&\quad = \sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Big(P(Z'_m) - P(Z_m)\Big)\Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}})\Big)\Big] \geqslant 0
\end{aligned}
\tag{I.4}
$$

With slight abuse of notation, let $\boldsymbol{\omega_{-i-m}}$ denote a vector of $\boldsymbol{\omega}$ excluding the elements i and m, let $\boldsymbol{\omega^{m=s}}$ denote a vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = s$ and $\omega_i = s$, and let $\boldsymbol{\omega^{m=f}}$ denote a vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = f$ and $\omega_i = s$. The final inequality is derived from the auxiliary lemma. It signifies that bank i's risk-taking distortion is increasing in bank m's risk exposure. Combining Equation I.3 and Equation I.4, we have

$$
\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}Z_{-i}} = \frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}_i(\boldsymbol{Z_{-i}})} \frac{\mathrm{d}\mathcal{D}_i(\boldsymbol{Z_{-i}})}{\mathrm{d}Z_{-i}} > 0 \quad \forall i \quad \text{and} \quad -i
$$

As a result, bank i will choose greater risk exposure in response to an increase in its counterparty risk.

For nonregular networks, bank i's expected payoff is:

$$
\begin{aligned}
\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big] &= P(Z_i)\cdot(Z_i - v_i) - P(Z_i)\cdot\mathcal{D}_i(\boldsymbol{Z_{-i}}) \\
&\quad + \Big(1 - P(Z_i)\Big)\sum_{\boldsymbol{\omega_{-i}}}\Big[0 - v_i - \Big(d_i^*(\boldsymbol{\omega^{i=f}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=f}})\Big)\Big]^+ \cdot \Pr(\boldsymbol{\omega_{-i}})
\end{aligned}
$$

First, we partition the upside payoff into two categories: (i) scenarios where bank i doesn't default even when it fails, and (ii) scenarios where bank i defaults upon failure but survives upon success. Let's denote $\boldsymbol{\Omega^{f+}_{-i}}$ as the set of counterparty states $\boldsymbol{\omega_{-i}}$ in which bank i doesn't default even when its project fails (case i). Here, superscript f indicates that bank i fails its project, while the superscript "+" indicates that bank i does not default. Formally,

$$
\boldsymbol{\Omega^{f+}_{-i}} \equiv \Big\{\boldsymbol{\omega_{-i}}\Big|\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=f}}) - d_i^*(\boldsymbol{\omega^{i=f}}) - v_i > 0\Big\}
$$

, where $\boldsymbol{\omega^{i=f}} \equiv (\omega_1 ..., \omega_{i-1}, f, \omega_{i+1}, ...\omega_N)$ is a vector that appends bank i's failure to other banks' states of nature $\boldsymbol{\omega_{-i}}$. Because $d_i^*(\boldsymbol{\omega})$ is constant in $\boldsymbol{Z}$ for all $\boldsymbol{\omega}$, $\boldsymbol{\Omega^{f+}_{-i}}$ is constant in $\boldsymbol{Z}$. We can hence rewrite bank i's expected profit as

$$
\begin{aligned}
\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big] = \quad & P(Z_i)\cdot(Z_i - v_i) \quad + \\
& P(Z_i) \sum_{\boldsymbol{\omega_{-i}}\notin\boldsymbol{\Omega^{f+}_{-i}}} -\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=s}})\Big)\cdot\Pr(\boldsymbol{\omega_{-i}}) \quad + \\
& P(Z_i) \sum_{\boldsymbol{\omega_{-i}}\in\boldsymbol{\Omega^{f+}_{-i}}} -\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=s}})\Big)\cdot\Pr(\boldsymbol{\omega_{-i}}) \quad + \\
& \Big(1 - P(Z_i)\Big) \sum_{\boldsymbol{\omega_{-i}}\in\boldsymbol{\Omega^{f+}_{-i}}} \Big[0 - v_i - \Big(d_i^*(\boldsymbol{\omega^{i=f}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=f}})\Big)\Big]\cdot\Pr(\boldsymbol{\omega_{-i}})
\end{aligned}
$$

The first line represents the expected profit of a stand-alone bank. The second and third lines divide bank i's cross-subsidies (which can potentially be negative) based on whether $\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}$. The final line represents bank i's expected downside payoff, which is specific to nonregular networks.

Equation 2, when combined with the definition of $\boldsymbol{\Omega}_{-i}^{f+}$, suggests that $d_i^*(\boldsymbol{\omega}^{i=f}) = \bar{d}_i$ if $\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}$. Intuitively, because bank profit is junior to interbank debt, bank i will pay its interbank debt in full if it does not default. This also implies $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=f}) = \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})$ for all $\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}$, because bank i's state of nature becomes irrelevant for the payment equilibrium. With these results, we can derive the following cross-partial derivatives, which are essential for the strategic complementarity result.

$$\begin{aligned}\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big]}{\mathrm{d}Z_i\,\mathrm{d}Z_m} = & \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Big\{P(Z_i)\cdot Z_i\Big\}}_{=0} \\ & + \frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Big\{P(Z_i)\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i-\Big(\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})\Big\} \\ & + \underbrace{\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Bigg\{P(Z_i)\sum_{\boldsymbol{\omega}_{-i}\in\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i-\Big(\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i}) \\ &\qquad + \Big(1-P(Z_i)\Big)\sum_{\boldsymbol{\omega}_{-i}\in\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i-\Big(\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})\Bigg\}\end{aligned}}_{=0}\end{aligned} \tag{I.5}$$

Because $\sum_{\boldsymbol{\omega}_{-i}\in\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i-\Big(\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})$ is constant with respect to Z_i, the final two lines cancel each other out. Intuitively, in a scenario where bank i yields a positive profit irrespective of the success or failure of its project, its interbank claims become an independent asset that doesn't impact its risk-taking incentive. This is what Galeotti et al. (2020) referred to as pure externalities. As a result,

$$\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big]}{\mathrm{d}Z_i\,\mathrm{d}Z_m} = -P'(Z_i)\cdot\underbrace{\frac{\mathrm{d}}{\mathrm{d}Z_m}\Big\{\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}_{-i}^{f+}}\Big[v_i+\Big(\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})\Big\}}_{>0} > 0$$

The proof that the second part is greater than zero is similar the regular network case above (i.e., Equation I.4) with a minor modification to account for the fact that the counterparty risk Z_m can increase the probability of bank i defaulting (i.e., $\boldsymbol{\omega}_{-i} \notin \boldsymbol{\Omega}_{-i}^{f+}$). This further increases bank i's incentive to take greater risks. To formally prove this, let's first define $\mathcal{D}_i^*(\boldsymbol{Z}_{-i})$ as

$$\mathcal{D}_i^*(\boldsymbol{Z}_{-i}) \equiv \sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}_{-i}^{f+}}\Big[v_i+\Big(\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})$$

To account for the effect of Z_m on the probability of bank i defaulting, let's partition the distortion into two cases: (i) bank i, when failed, will default regardless of bank m's project; and (ii) bank i, when failed, will default if bank m fails and survive if bank m succeeds.

$$
\begin{aligned}
&\mathcal{D}_i^*(\boldsymbol{Z_{-i}}) \\
&= \underbrace{\sum_{\substack{\boldsymbol{\omega}_{-i}^{m=s}\notin\boldsymbol{\Omega}_{-i}^{f+}\\ \boldsymbol{\omega}_{-i}^{m=f}\notin\boldsymbol{\Omega}_{-i}^{f+}}} \Pr(\boldsymbol{\omega_{-i-m}})\Big[\Big(1-P(Z_m)\Big)\Big(v_i+\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{\overset{i=s}{m=f}})\Big)+P(Z_m)\Big(v_i+\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{\overset{i=s}{m=s}})\Big)\Big]}_{\boldsymbol{\omega_{-i-m}}\text{ is such that bank } i\text{, if failed, will default regardless of } \omega_m=s \text{ or } \omega_m=f} \\
&+ \underbrace{\sum_{\substack{\boldsymbol{\omega}_{-i}^{m=s}\in\boldsymbol{\Omega}_{-i}^{f+}\\ \boldsymbol{\omega}_{-i}^{m=f}\notin\boldsymbol{\Omega}_{-i}^{f+}}} \Pr(\boldsymbol{\omega_{-i-m}})\Big[\Big(1-P(Z_m)\Big)\Big(v_i+\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{\overset{i=s}{m=f}})\Big)\Big]}_{\boldsymbol{\omega_{-i-m}}\text{ is such that bank } i\text{, if failed, will default when } \omega_m=f \text{ and not default when } \omega_m=s.}
\end{aligned}
$$

, where $\boldsymbol{\omega}_{-i}^{m=s}$ ($N-1$ length) is the appended vector of $\boldsymbol{\omega_{-i-m}}$ ($N-2$ length) and $\omega_m=s$; and $\boldsymbol{\omega}^{\overset{i=s}{m=s}}$ (N length) is the appended vector of $\boldsymbol{\omega_{-i-m}}$, $\omega_m=s$, and $\omega_i=s$. Let $Z'_m>Z_m$, and let $\boldsymbol{Z'_{-i}}$ denotes the vector that differs from $\boldsymbol{Z_{-i}}$ only in Z_m.

Now, by applying the same technique as in the regular case (Equation I.4), we obtain:

$$
\begin{aligned}
&\mathcal{D}_i^*(\boldsymbol{Z'_{-i}})-\mathcal{D}_i^*(\boldsymbol{Z_{-i}}) = \\
&\underbrace{\sum_{\substack{\boldsymbol{\omega}_{-i}^{m=s}\notin\boldsymbol{\Omega}_{-i}^{f+}\\ \boldsymbol{\omega}_{-i}^{m=f}\notin\boldsymbol{\Omega}_{-i}^{f+}}} \Pr(\boldsymbol{\omega_{-i-m}})\Big[\Big(P(Z'_m)-P(Z_m)\Big)\Big(\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{\overset{i=s}{m=f}})-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{\overset{i=s}{m=s}})\Big)\Big]}_{>0\text{ (proof is identical to Equation I.4)}} \\
&+\sum_{\substack{\boldsymbol{\omega}_{-i}^{m=s}\in\boldsymbol{\Omega}_{-i}^{f+}\\ \boldsymbol{\omega}_{-i}^{m=f}\notin\boldsymbol{\Omega}_{-i}^{f+}}} \Pr(\boldsymbol{\omega_{-i-m}})\underbrace{\Big(P(Z_m)-P(Z'_m)\Big)}_{>0}\underbrace{\Big(v_i+\bar{d}_i-\sum_j\theta_{ij}d_j^*(\boldsymbol{\omega}^{\overset{i=s}{m=f}})\Big)}_{>0}
\end{aligned}
$$

The first line of the equation, analogous to the standard case, indicates that Z_m amplifies the upside distortion of bank i. The second line indicates that Z_m can elevate the likelihood of bank i defaulting, further augmenting its risk-taking distortion.

Thus, we have established $\mathrm{d}^2\mathbb{E}[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})]/\,\mathrm{d}Z_i\,\mathrm{d}Z_m>0$. The game is supermodular and $\boldsymbol{Z}$ is strategically complementary. □

PROPOSITION 2

Let's first establish the existence of the payment equilibrium and then establish the existence of the Nash equilibrium.

The payment equilibrium, for any state of nature, corresponds to the fixed-point solution of a system of equations, as represented by Equation I.2. Let this fixed point be denoted as $\boldsymbol{d}^*=\Phi(\boldsymbol{d}^*)$, where Φ is a continuous mapping with a convex and compact domain $[0,\bar{d}]^N$. Utilizing the Brouwer fixed point theorem, the payment equilibrium, the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega};\boldsymbol{Z})$ is guaranteed to exist for all $\boldsymbol{\omega}$ and $\boldsymbol{Z}$ (Eisenberg and Noe, 2001). This affirms the existence of the payment equilibrium for all $\boldsymbol{\omega}$ and $\boldsymbol{Z}$.

For the existence of the Nash equilibrium, as per Proposition 1, we have $\mathrm{d}\widehat{Z}_i/Z_{-i}\geqslant 0$ for all i and $-i$. This suggests that the Nash equilibrium characterizes a supermodular game in both regular and non-regular networks. Moreover, the domain of the risk-exposure vector, $[\underline{Z},\overline{Z}]^N$, forms a complete lattice.

Invoking Tarski's theorem, a fixed-point solution to the first-order conditions $F(Z_i^*; \boldsymbol{Z}_{-i}^*) = 0$ is ensured to exist. The equilibrium risk exposure is this fixed point. □

COROLLARY 1

First, we will establish the result for regular networks and then address nonregular networks. Let Z^r represent the equilibrium risk exposure of a bank within a regular network, and Z^s represent that of an identical stand-alone bank that netted interbank connections. Formally, these are determined by their respective first-order conditions, as follows:

$$
\begin{aligned}
&P'(Z^r)(Z^r - v_i) + P(Z^r) - P'(Z^r)\mathcal{D}_i(\boldsymbol{Z}_{-i}^r) = 0 \\
&P'(Z^s)(Z^s - v_i) + P(Z^s) = 0
\end{aligned}
$$

By Equation I.3, $\mathrm{d}Z^N/\mathrm{d}\mathcal{D}_i(\boldsymbol{Z}_{-i}^r) > 0$. We also know that the distortion $\mathcal{D}_i(\boldsymbol{Z}^N)$ is positive because $P(Z_j) < 1$ for all Z_j. Therefore, $Z^r > Z^s$.

For nonregular networks, Let Z_i^n represent the equilibrium risk exposure of bank i, and Z_i^s represent that of the same bank that has netted interbank connections. The first order condition for this bank, which is similar to Equation I.5, is the following.

$$
\frac{\mathrm{d}}{\mathrm{d}Z_i}\Big\{P(Z_i)\cdot Z_i\Big\} + \frac{\mathrm{d}}{\mathrm{d}Z_i}\left\{P(Z_i)\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i - \Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot \Pr(\boldsymbol{\omega}_{-i})\right\} = 0
$$

The first order condition for the stand-alone bank that netted interbank connections is

$$
\frac{\mathrm{d}}{\mathrm{d}Z_i}\Big\{P(Z_i)\cdot Z_i\Big\} + \frac{\mathrm{d}}{\mathrm{d}Z_i}\left\{P(Z_i)\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i - \Big(\bar{d}_i - \sum_j \theta_{ij} \bar{d}_j\Big)\Big]\cdot \Pr(\boldsymbol{\omega}_{-i})\right\} = 0
$$

Since $\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i - \Big(\bar{d}_i - \sum_j \theta_{ij}\bar{d}_j\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})$ is bigger than $\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}_{-i}^{f+}}\Big[-v_i - \Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot \Pr(\boldsymbol{\omega}_{-i})$, we have $Z_i^n > Z_i^s$. □

LEMMA 3

For any state of nature $\boldsymbol{\omega}$, suppose that there exist two vectors, $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$, such that $d_i^*(\boldsymbol{\omega}) = \{a_i(\boldsymbol{\omega})\bar{d} - b_i(\boldsymbol{\omega})v\}^+$. By definition, they should satisfy Equation I.2. After plugging $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ into Equation I.2, we have $(a_i, b_i) = (1, 0)\ \forall\omega_i = s$, and

$$
\begin{aligned}
d_i^*(\boldsymbol{\omega}) &= \left\{\sum_{\omega_j = s}\theta_{ij}\bar{d} + \sum_{j\in\mathcal{F}_{\boldsymbol{\omega}}^+}\theta_{ij}\Big(a_j(\boldsymbol{\omega})\bar{d} - b_j(\boldsymbol{\omega})v\Big) - v\right\}^+ \\
&= \left\{\Big(\sum_{j\in\mathcal{F}_{\boldsymbol{\omega}}^+}\theta_{ij}a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s}\theta_{ij}\Big)\bar{d} - \Big(\sum_{j\in\mathcal{F}_{\boldsymbol{\omega}}^+}\theta_{ij}b_j(\boldsymbol{\omega}) + 1\Big)v\right\}^+ \qquad \forall\omega_i = f
\end{aligned}
$$

, where $\mathcal{F}_{\boldsymbol{\omega}}^+ \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v \geqslant 0\}$. The set contains failed banks that are solvent (i.e., being able to fulfill their deposits). Similarly, denote $\mathcal{F}_{\boldsymbol{\omega}}^- \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v < 0\}$ as the set of insolvent failed banks, and $\mathcal{S}_\omega \equiv \{i : \omega_i = s)$ as the set of successful banks. By the conjecture, we need $\forall \omega_i \in f$,

$$a_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}^+_\omega} \theta_{ij} a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s} \theta_{ij} \tag{I.6}$$

$$b_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}^+_\omega} \theta_{ij} b_j(\boldsymbol{\omega}) + 1 \tag{I.7}$$

Since the RHS of the above equations is increasing in $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ respectively, the fixed points exist by Tarski's theorem. The conjecture is hence verified. Let's rewrite the above equations in matrix form for banks in $\mathcal{F}^+_\omega$.

$$\boldsymbol{a}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{a}_+(\boldsymbol{\omega}) + \boldsymbol{\Theta}_{+s}\mathbb{1}_s \tag{I.8}$$

$$\boldsymbol{b}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{b}_+(\boldsymbol{\omega}) + \mathbb{1}_+ \tag{I.9}$$

, where $\boldsymbol{a}_+(\boldsymbol{\omega})$ and $\boldsymbol{b}_+(\boldsymbol{\omega})$ are truncated vectors of $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ with only the rows that belong to $\mathcal{F}^+_\omega$. Similarly, $\boldsymbol{\Theta}_{++}$ is a truncated matrix of $\boldsymbol{\Theta}$ with only the rows and columns that belong to $\mathcal{F}^+_\omega$, and $\boldsymbol{\Theta}_{+s}$ is the truncated matrix of $\boldsymbol{\Theta}$ where each row belongs to $\mathcal{F}^+_\omega$ and each column belongs to $\mathcal{S}$. $\mathbb{1}_+$ and $\mathbb{1}_s$ are column vectors of ones with appropriate dimension. Note that $\boldsymbol{\Theta}_{++}$, $\boldsymbol{\Theta}_{+s}$, $\mathbb{1}_+$, and $\mathbb{1}_s$ are all state-contingent. To conserve space, I suppress their underscript $\boldsymbol{\omega}$.

By the Markovian property of $\boldsymbol{\Theta}$ (row sum equals to one), we have $\boldsymbol{\Theta}_{++}\mathbb{1}_+ + \boldsymbol{\Theta}_{+-}\mathbb{1}_- + \boldsymbol{\Theta}_{+s}\mathbb{1}_s = \mathbb{1}_+$. As a result, Equation I.8 implies that

$$\boldsymbol{a}_+(\boldsymbol{\omega}) = (\boldsymbol{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\boldsymbol{\Theta}_{+s}\mathbb{1}_s < \mathbb{1}_+ \tag{I.10}$$

Plugging $(\boldsymbol{a}_+, \boldsymbol{b}_+)$ into the network risk-taking distortion, we can rewrite $\mathcal{D}_i(\mathbf{Z}_{-i})$ in matrix form as

$$\begin{aligned}\mathcal{D}_i(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\bar{d} - \Big(\Theta_{is}\mathbb{1}_s\bar{d} + \Theta_{i+}(\boldsymbol{a}_+\bar{d} - \boldsymbol{b}_+v) + \Theta_{i-}\cdot 0\Big)\Big] \\ &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{i+}\Big((\mathbb{1}_+ - \boldsymbol{a}_+)\bar{d} + \boldsymbol{b}_+v\Big) + \boldsymbol{\Theta}_{i-}\mathbb{1}_-\bar{d} + \boldsymbol{\Theta}_{is}\mathbb{1}_s\cdot 0\Big]\end{aligned} \tag{I.11}$$

Each part of Equation I.11 has a clear interpretation: $\boldsymbol{\Theta}_{i+}\big((\mathbb{1}_+ - \boldsymbol{a}_+)\bar{d} + \boldsymbol{b}_+v\big)$ represents bank i's subsidies to solvent failed banks, $\boldsymbol{\Theta}_{i-}\mathbb{1}_-\bar{d}$ represents bank i's subsidies to insolvent failed banks, and $\boldsymbol{\Theta}_{is}\mathbb{1}_s\cdot 0$ represents bank i's subsidies to successful banks, which is zero.

To prove Lemma 3, let's compare three financial networks with the same topology $\boldsymbol{\Theta}$ but different interbank liabilities $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$, with $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \xi$. To prove monotonicity and concavity, we need to show $\mathcal{D}^3_i(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^2_i(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^1_i(\mathbf{Z}_{-i})$ and $\mathcal{D}^2_i(\mathbf{Z}_{-i}) - \mathcal{D}^1_i(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^3_i(\mathbf{Z}_{-i}) - \mathcal{D}^2_i(\mathbf{Z}_{-i})$ with inequality occurring somewhere.

Note that the set $\mathcal{F}^+_\omega \equiv \{i : \omega_i = f, a_i\bar{d} - b_iv \geqslant 0\}$ is a function of $\bar{d}$. We hence denote $\mathcal{F}^+_1(\boldsymbol{\omega})$, $\mathcal{F}^+_2(\boldsymbol{\omega})$, and $\mathcal{F}^+_3(\boldsymbol{\omega})$ as the sets of solvent failed bank in state $\boldsymbol{\omega}$ for network $(\bar{d}_1, \boldsymbol{\Theta})$, $(\bar{d}_2, \boldsymbol{\Theta})$, and $(\bar{d}_3, \boldsymbol{\Theta})$, respectively. From the monotone selection theorem (see auxiliary lemma), we have $d^{3*}_i(\boldsymbol{\omega}) \geqslant d^{2*}_i(\boldsymbol{\omega}) \geqslant d^{1*}_i(\boldsymbol{\omega}), \forall i \in \mathcal{N}$ and $\boldsymbol{\omega} \in \Omega$. That implies $\mathcal{F}^+_1(\boldsymbol{\omega}) \subseteq \mathcal{F}^+_2(\boldsymbol{\omega}) \subseteq \mathcal{F}^+_3(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$. Intuitively, a larger $\bar{d}$ can make more failed banks solvent. This is the essence of the concavity result.

Let's consider the following four cases: (1) $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$. (2) $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (3) $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (4) $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

Case I: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$.

From Equations I.8 and I.9, it's easy to see that $\boldsymbol{a}_+^{\mathbf{1}} = \boldsymbol{a}_+^{\mathbf{2}} = \boldsymbol{a}_+^{\mathbf{3}}$ and $\boldsymbol{b}_+^{\mathbf{1}} = \boldsymbol{b}_+^{\mathbf{2}} = \boldsymbol{b}_+^{\mathbf{3}}$. We also have $\boldsymbol{\Theta}_{\boldsymbol{i}+}$, $\mathbb{1}_+$, $\boldsymbol{\Theta}_{\boldsymbol{i}-}$, and $\mathbb{1}_-$ unchanged across the three networks. Therefore,

$$\mathcal{D}_i^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big] > 0$$
$$\mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big] > 0$$

The last inequality comes from Equation I.10. With $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \xi$, we have $\mathcal{D}_i^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) = \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) > 0$. Intuitively, this case implies that the network's risk-taking is linearly increasing in $\bar{d}$, provided the change in $\bar{d}$ does not cause additional insolvent banks to become solvent.

Case II: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

Let's first compare the distortion when the interbank liabilities are $\bar{d}_2$ and $\bar{d}_1$. Because $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega})$, some otherwise insolvent failed banks for $(\bar{d}_1, \boldsymbol{\Theta})$ become solvent for $(\bar{d}_2, \boldsymbol{\Theta})$ in some state of nature $\boldsymbol{\omega}$. Denote these banks as $t_1, t_2, ..., t_T$, where $T \geqslant 1$. From the continuity of the payment equilibrium in terms of $\bar{d}$ (see Equation 2), there exist values $\bar{d}_1 < \tilde{d}_1 < \tilde{d}_2 ... < ... < \tilde{d}_S < \bar{d}_2$ (where $1 \leqslant S \leqslant T$) such that when the interbank liabilities are $\bar{d} = \tilde{d}_s$, some bank t_t is exactly solvent, or formally $\tilde{a}_t(\boldsymbol{\omega})\tilde{d}_s - \tilde{b}_t(\boldsymbol{\omega})v = 0$. In other words, this bank t is solvent when $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1})$ and insolvent when $\bar{d} \in (\tilde{d}_{s-1}, \tilde{d}_s]$ respectively. Denote $\widetilde{\mathcal{D}}_i^s(\mathbf{Z}_{-\boldsymbol{i}})$ as the network risk-taking distortion at those cut-offs $\tilde{d}_s$. We have

$$\begin{aligned}
\mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) - \widetilde{\mathcal{D}}_i^S(\mathbf{Z}_{-\boldsymbol{i}}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\mathbf{2}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\mathbf{2}})(\bar{d}_2 - \tilde{d}_S) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\mathbf{2}}\mathbb{1}_-(\bar{d}_2 - \tilde{d}_S)\Big] > 0 \\
\widetilde{\mathcal{D}}_i^{s+1}(\mathbf{Z}_{-\boldsymbol{i}}) - \widetilde{\mathcal{D}}_i^s(\mathbf{Z}_{-\boldsymbol{i}}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}_+^{\boldsymbol{s}})(\tilde{d}_{s+1} - \tilde{d}_s) + \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}^{\boldsymbol{s}}\mathbb{1}_-(\tilde{d}_{s+1} - \tilde{d}_s)\Big] > 0 \\
\widetilde{\mathcal{D}}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\mathbf{1}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\mathbf{1}})(\tilde{d}_1 - \bar{d}_1) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\mathbf{1}}\mathbb{1}_-(\tilde{d}_1 - \bar{d}_1)\Big] > 0
\end{aligned} \tag{I.12}$$

Each column of $\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\mathbf{1}}$ corresponds to a solvent failed bank if $\bar{d} \in [\bar{d}_1, \tilde{d}_1]$. Each column of $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}$ corresponds to a solvent failed bank if $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1}]$. Each column of $\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\mathbf{2}}$ corresponds to a solvent failed bank if $\bar{d} \in [\tilde{d}_S, \bar{d}_2]$. The same notation applies to $\tilde{\boldsymbol{a}}_+$ and $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}$ as well. These are state-contingent, and to conserve space, the state is suppressed.

The above inequalities imply that $\mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) > \widetilde{\mathcal{D}}_i^S(\mathbf{Z}_{-\boldsymbol{i}}) > ... > \widetilde{\mathcal{D}}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) > \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}})$. The monotonicity result follows. To prove concavity, we observe that $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}\mathbb{1}_+ + \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}^{\boldsymbol{s}}\mathbb{1}_- = \boldsymbol{\Theta}_{\boldsymbol{i}f}\mathbb{1}_f$ for all s and $\boldsymbol{\omega}$. By definition, $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}$ and $\tilde{\boldsymbol{a}}_+^{\boldsymbol{s}}$ are sub-matrix of $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s+1}}$ and $\tilde{\boldsymbol{a}}_+^{\boldsymbol{s+1}}$ respectively. Hence, we have

$$\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}_+^{\boldsymbol{s}}) + \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}^{\boldsymbol{s}}\mathbb{1}_- > \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s+1}}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}_+^{\boldsymbol{s+1}}) + \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}^{\boldsymbol{s+1}}\mathbb{1}_- \qquad \forall s = 1, ..., S-1$$

After summing every difference in Equation I.12 and replacing the entire right-hand side with the first line (i.e., the smallest value), we have:

$$\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) > \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}_{i+}^{\mathbf{2}}(\mathbb{1}_+ - \mathbf{a}_+^{\mathbf{2}})(\bar{d}_2 - \bar{d}_1) + \mathbf{\Theta}_{i-}^{\mathbf{2}}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

Since $\mathcal{F}_2^+(\omega) = \mathcal{F}_3^+(\omega)$, we have the following identity:

$$\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) = \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}_{i+}^{\mathbf{2}}(\mathbb{1}_+ - \mathbf{a}_+^{\mathbf{2}})(\bar{d}_3 - \bar{d}_2) + \mathbf{\Theta}_{i-}^{\mathbf{2}}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$

From here, $\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) < \mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i})$, and the concavity follows.

Case III: $\mathcal{F}_1^+(\omega) = \mathcal{F}_2^+(\omega) \subset \mathcal{F}_3^+(\omega)$ for some ω.

The proof is identical to Case II, with a slight twist. Instead of replacing the entire RHS of Equation I.12 with the first line, we replace it with the last line. Hence, we obtain:

$$\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) < \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}_{i+}^{\mathbf{2}}(\mathbb{1}_+ - \mathbf{a}_+^{\mathbf{2}})(\bar{d}_3 - \bar{d}_2) + \mathbf{\Theta}_{i-}^{\mathbf{2}}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) = \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}_{i+}^{\mathbf{2}}(\mathbb{1}_+ - \mathbf{a}_+^{\mathbf{2}})(\bar{d}_2 - \bar{d}_1) + \mathbf{\Theta}_{i-}^{\mathbf{2}}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity results follow.

Case IV: $\mathcal{F}_1^+(\omega) \subset \mathcal{F}_2^+(\omega) \subset \mathcal{F}_3^+(\omega)$ for some ω.

The proof is a combination of Case II and Case III:

$$\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) < \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}_{i+}^{\mathbf{2}}(\mathbb{1}_+ - \mathbf{a}_+^{\mathbf{2}})(\bar{d}_3 - \bar{d}_2) + \mathbf{\Theta}_{i-}^{\mathbf{2}}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) > \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}_{i+}^{\mathbf{2}}(\mathbb{1}_+ - \mathbf{a}_+^{\mathbf{2}})(\bar{d}_2 - \bar{d}_1) + \mathbf{\Theta}_{i-}^{\mathbf{2}}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity results follow.

To summarize all four cases, because $\mathcal{F}_1^+(\omega) \subseteq \mathcal{F}_2^+(\omega) \subseteq \mathcal{F}_3^+(\omega)$ for all $\omega \in \Omega$, Cases I-IV (or some combination of them) exhaust all the possibilities. Intuitively, the proof shows that the network risk-taking distortion is increasing in $\bar{d}$, but at a slower rate. This is because the change of $\bar{d}$ makes some insolvent banks solvent, and this decreases the marginal effect of $\bar{d}$. □

PROPOSITION 3

From Lemma 1, the Nash Equilibrium for risk exposure $\mathbf{Z}^*$ is part of a supermodular game. According to Lemma 3, the bank's expected profit exhibits an increasing difference with respect to Z_i and $\bar{d}$. As a result, the Pareto-dominant equilibrium risk exposure increases with $\bar{d}$ (see Milgrom and Roberts (1990), Theorem 6). Additionally, the network risk-taking distortion $\mathcal{D}_i(\mathbf{Z}_{-i})$ is concave in $\bar{d}$, as observed in Lemma 3. In fact, if $\bar{d}$ exceeds $N \cdot v$, it ceases to affect both the distortion and the equilibrium risk exposure. Proposition 5 will discuss this finding in greater detail. □

PROPOSITION 4

Let us calculate the network distortion in the two types of networks separately, and then compare their sizes.

(a) Complete Network

In a complete network, failed banks are either all solvent or all insolvent. Let us solve the payment equilibrium (Equations I.6 and I.7) in those two cases.

1. For $\boldsymbol{\omega}$ where $\mathcal{F}^+(\boldsymbol{\omega}) = \mathcal{F}(\boldsymbol{\omega})$ (i.e., failed banks are solvent),
   If $\omega_i = f$, then $a_i(\boldsymbol{\omega}) = 1$ and $b_i(\boldsymbol{\omega}) = 1/(1 - \sum_{j \in \mathcal{F}_\omega} \theta_{ij})$.
2. For $\boldsymbol{\omega}$ where $\mathcal{F}^+(\boldsymbol{\omega}) = \varnothing$ (i.e., failed banks are insolvent),
   If $\omega_i = f$, then $a_i(\boldsymbol{\omega}) = \sum_{\omega_j = s} \theta_{ij}$ and $b_i(\boldsymbol{\omega}) = 1$.

By definition, a bank is solvent if $a_i\bar{d} - b_i v \geqslant 0$. Plugging in the solution from case 1, we have $\mathcal{F}(\boldsymbol{\omega})^+ = \mathcal{F}(\boldsymbol{\omega})$ if and only if $\bar{d} \geqslant 1/(1 - \sum_{j \in \mathcal{F}_\omega} \theta_{ij}) \cdot v$. We can therefore solve the payment equilibrium as:

$$d_i^C(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - v/(1 - \sum_{j \in \mathcal{F}_\omega} \theta_{ij})\right)^+ & \forall \quad \omega_i = f \quad \& \quad \bar{d} \geqslant 1/(1 - \sum_{j \in \mathcal{F}_\omega} \theta_{ij}) \cdot v \\ 0 & \text{if otherwise} \end{cases}$$

The third line is zero because When $\bar{d} < v/(1 - \sum_{j \in \mathcal{F}_\omega} \theta_{ij})$, insolvent failed banks pay zero (as in the above case two). So, we can collapse the second and the third lines.

$$d_i^C(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - \frac{1}{\sum_{\omega_j \in s} \theta_{ij}} v\right)^+ & \forall \quad \omega_i = f \end{cases}$$

, where $1/\sum_{\omega_j = s} \theta_{ij} = (N-1)$ / (# of successful banks). From here, we can rewrite the network risk-taking distortion as

$$\begin{aligned} \mathcal{D}_i^C(\mathbf{Z}_{-i}) &= \sum_{m=1}^{N-1} \Bigg(\bar{d} - \underbrace{\left(\bar{d} - \frac{N-1}{N-m} v\right)^+ \cdot \frac{m}{N-1}}_{\text{payment from failed banks}} - \underbrace{\bar{d} \cdot \frac{N-1-m}{N-1}}_{\substack{\text{payment from} \\ \text{successful banks}}}\Bigg) \cdot \Pr(m \text{ banks failed}) \\ &= \sum_{m=1}^{N-1} \min\left(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N-1}\right) \cdot \Pr(m \text{ banks failed}) \end{aligned} \tag{I.13}$$

, where

$$\Pr(m \text{ banks failed}) = \binom{N-1}{m} \Big(1 - P(Z_{-i})\Big)^m \Big(P(Z_{-i})\Big)^{N-1-m}$$

(b) Ring Network

For a failed bank, there are three scenarios: (1) its debtor succeeds, (2) its debtor fails but is solvent, and (3) its debtor fails and is insolvent. Let's solve the payment equilibrium (Equations I.6 and I.7) in these three types of states of nature.

1. $i \in \mathcal{F}$ and $\omega_{i-1} \in \mathcal{S}(\boldsymbol{\omega})$,

$$a_i(\boldsymbol{\omega}) = 1 \text{ and } b_i(\boldsymbol{\omega}) = 1.$$

2. $i \in \mathcal{F}$ and $\omega_{i-1} \in \mathcal{F}^{+}(\boldsymbol{\omega})$,

$$a_i(\boldsymbol{\omega}) = a_{i-1}(\boldsymbol{\omega}) \text{ and } b_i(\boldsymbol{\omega}) = b_{i-1}(\boldsymbol{\omega}) + 1.$$

3. $i \in \mathcal{F}$ and $\omega_{i-1} \in \mathcal{F}^{-}(\boldsymbol{\omega})$,

$$a_i(\boldsymbol{\omega}) = 0 \text{ and } b_i(\boldsymbol{\omega}) = 1.$$

By induction, we have

$$d_i^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - K_i(\boldsymbol{\omega})v\right)^{+} & \forall \quad \omega_i = f \end{cases} \tag{I.14}$$

, where $K_i(\boldsymbol{\omega}) \equiv \min\{o : \omega_{i-o} = s\}$ is the number of failed debtors in the chain before reaching the first successful bank. Conditioning on m banks failing, the total interbank payment received by bank i is

$$\sum_j \theta_{ij} d_j^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \text{w.p.} \quad \binom{N-2}{N-2-m} \Big/ \binom{N-1}{m} \\ \left(\bar{d} - v\right)^{+} & \text{w.p.} \quad \binom{N-3}{N-2-m} \Big/ \binom{N-1}{m} \\ \ldots\ldots & \\ \left(\bar{d} - mv\right)^{+} & \text{w.p.} \quad \binom{N-2-m}{N-2-m} \Big/ \binom{N-1}{m} \end{cases} \tag{I.15}$$

Equation I.15 has a clear interpretation. The first line corresponds to the scenario where i's direct debtor succeeds. In this case, bank i will receive an interbank payment of $\bar{d}$. Conditioning on m banks failing, the probability of this scenario is $\binom{N-2}{N-2-m} \Big/ \binom{N-1}{m}$. Similarly, the second line corresponds to the scenario where i's direct debtor fails but its debtor's debtor succeeds. In this case, bank i will receive an interbank payment of $(\bar{d} - v)^{+}$. The probability of this scenario is $\binom{N-3}{N-2-m} \Big/ \binom{N-1}{m}$. The same logic applies until all m banks fail. It is easy to confirm by the Hockey-stick identity (Lemma I.A) that the total probability in Equation I.15 is one. Taking the expectation, the network risk-taking distortion of a ring network is

$$\mathcal{D}_i^R(\boldsymbol{Z}_{-\boldsymbol{i}}) = \sum_{m=1}^{N-1} \left[\bar{d} - \sum_{l=0}^{m} \left(\bar{d} - lv\right)^{+} \binom{N-2-l}{N-2-m} \Big/ \binom{N-1}{m} \right] \cdot \Pr(m \text{ banks failed})$$

Now, let's finally compare $\mathcal{D}_i^R(\boldsymbol{Z}_{-\boldsymbol{i}})$ with $\mathcal{D}_i^C(\boldsymbol{Z}_{-\boldsymbol{i}})$.

$$\begin{aligned} \mathcal{D}_i^R(\boldsymbol{Z}_{-\boldsymbol{i}}) \leqslant & \sum_{m=1}^{N-1} \left[\bar{d} - \left(\sum_{l=0}^{m} \left(\bar{d} - lv\right) \binom{N-2-l}{N-2-m} \Big/ \binom{N-1}{m} - \bar{d} \cdot \frac{N-1-m}{N-1} \right)^{+} - \bar{d} \cdot \frac{N-1-m}{N-1} \right] \\ & \cdot \Pr(m \text{ banks failed}) \qquad \textbf{(\textit{By Lemma I.B})} \\ = & \sum_{m=1}^{N-1} \left(\bar{d} - \left(\bar{d} - \frac{N-1}{N-m} v \right)^{+} \cdot \frac{m}{N-1} - \bar{d} \cdot \frac{N-1-m}{N-1} \right) \cdot \Pr(m \text{ banks failed}) \qquad \textbf{(\textit{By Lemma I.A})} \\ = & \mathcal{D}_i^C(\boldsymbol{Z}_{-\boldsymbol{i}}) \end{aligned}$$

Finally, by the monotone selection theorem, the equilibrium risk exposure of banks in a complete network is larger than that of banks in a ring network. □

PROPOSITION 5

Part (a): Let's first rewrite the risk-taking distortion of each bank in a ring network as a function of N. By using Equation I.14, we can derive each bank's risk-taking distortion as

$$\mathcal{D}_i^R(N) = \mathbb{E}\Big[\bar{d} - \left(\bar{d} - K^N \cdot v\right)^+\Big]$$

, where $K^N \equiv \min\{o : \omega_{i-o} = s\}$ represents the number of failed debtors in the chain before reaching the first successful bank. It is a random variable with the parameter N. Intuitively, this symbolizes the propagation of negative shocks that must be covered by the focal bank i.

Now compare the two random variables K^{N+1} and K^N in terms of first-order stochastic dominance. For any t, we have $\Pr(K^{N+1} \geqslant t) \geqslant \Pr(K^N \geqslant t)$. This is because K^{N+1} includes one more trial than K^N. This establishes that K^{N+1} stochastically dominates K^N. Define the function $g(x) \equiv \bar{d} - \left(\bar{d} - xv\right)^+$. The function is increasing with respect to x. Then, by Lemma I.C, $\mathcal{D}_i^R(N+1) > \mathcal{D}_i^R(N)$.

Similarly, for a bank in a complete network with N banks, its risk-taking distortion is

$$\mathcal{D}_i^C(N) = \sum_{m=1}^{N-1} \min\left(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N-1}\right) \cdot \binom{N-1}{m} \left(1 - P(Z_{-i})\right)^m \left(P(Z_{-i})\right)^{N-1-m}$$

We want to compare $\mathcal{D}_i^C(N)$ with $\mathcal{D}_i^C(N+1)$. Note that there are one more item in $\mathcal{D}_i^C(N+1)$ inside the summation compared to $\mathcal{D}_i^C(N)$. As a result, we cannot compare the size item by item within the summation. Instead, let's rewrite the distortion as

$$\begin{aligned}
\mathcal{D}_i^C(N) &= \sum_{m=1}^{N-1} \min\left(\frac{m \cdot v}{N-m} \cdot \frac{N-1}{m}, \bar{d}\right) \cdot \frac{m}{N-1} \cdot \binom{N-1}{m} \left(1 - P(Z_{-i})\right)^m \left(P(Z_{-i})\right)^{N-1-m} \\
&= \sum_{m=1}^{N-1} \min\left(\frac{(N-1) \cdot v}{N-m}, \bar{d}\right) \cdot \binom{N-2}{m-1} \left(1 - P(Z_{-i})\right)^m \left(P(Z_{-i})\right)^{N-1-m} \\
&= \left(1 - P(Z_{-i})\right) \cdot \sum_{j=0}^{n} \min\left(\frac{(n+1) \cdot v}{n-j+1}, \bar{d}\right) \cdot \binom{n}{j} \left(1 - P(Z_{-i}\right)^j \left(P(Z_{-i})\right)^{n-j} \\
&= \left(1 - P(Z_{-i}\right) \mathbb{E}\Big[\min\left(\frac{(n+1) \cdot v}{n+1-X}, \bar{d}\right)\Big]
\end{aligned}$$

, where $X_n \sim \text{Binomial}(n, 1 - P(Z_{-i}))$. In the second to the last line, we have replaced $j = m - 1$ and $n = N - 2$. Since there is one more trial with the random variable X_{n+1} compared to X_n, the latter first-order stochastically dominates the former. Define function $g(x) \equiv \min\left(\frac{(n+1) \cdot v}{n+1-x}, \bar{d}\right)$. The function is increasing with respect to x. From Lemma I.C, $\mathbb{E}[g(X_{n+1})] \geqslant \mathbb{E}[g(X_n)]$ with strict inequality if $\bar{d} > v$. As a result, $\mathcal{D}_i^C(N+1) \geqslant \mathcal{D}_i^C(N)$.

Part (b): Let $\widetilde{\mathbf{\Theta}}$ denote the largest path-connected sub-network of $\mathbf{\Theta}$ where bank i belongs to. Suppose when $\bar{d} = \bar{d}_1$, all failed banks in this sub-network are solvent in any state of nature. This means $\widetilde{\mathbf{\Theta}}_{++}\mathbb{1}_+ + \widetilde{\mathbf{\Theta}}_{+s}\mathbb{1}_s = \mathbb{1}_+$. As a result, Equation I.10 becomes $\boldsymbol{a}_+(\boldsymbol{\omega}) = (\boldsymbol{I}_+ - \widetilde{\mathbf{\Theta}}_{++})^{-1}\widetilde{\mathbf{\Theta}}_{+s}\mathbb{1}_s = \mathbb{1}_+$ for all $\boldsymbol{\omega}$. If this is the case, Equation I.11 implies $\mathcal{D}_i(\mathbf{Z}_{-i}; \bar{d}_2) - \mathcal{D}_i(\mathbf{Z}_{-i}; \bar{d}_1) = 0$ for all $\bar{d}_2 > \bar{d}_1$.

To show the upper bound exists, it remains to prove that $\bar{d}_1$ exits: i.e., there exists a $\bar{d}_1$ such that all failed banks in $\widetilde{\mathbf{\Theta}}$ are solvent in any state of nature. Because $\widetilde{\mathbf{\Theta}}$ is path-connected by construction, there is a chain $\{j, a, b, c, ..., i\}$ from any failed bank j to the successful bank i. Then consider

$$\bar{d}_j^{max} = \frac{1}{\widetilde{\theta}_{bc}}(\frac{1}{\widetilde{\theta}_{ab}}(\frac{1}{\widetilde{\theta}_{ja}} \cdot v_j + v_a) + v_b) + v_c + ...$$

Clearly, $\bar{d}_j^{max}$ is finite because the network is path-connected ($\widetilde{\theta}_{ja}$, $\widetilde{\theta}_{ab}$, $\widetilde{\theta}_{bc}$.. are all strictly positive). Suppose $\bar{d} = \bar{d}_j^{max}$, then even when any bank outside this chain failed and insolvent (i.e., unable to contribute to the chain), bank j can fulfill its deposits and become solvent. Intuitively, that means $\bar{d}$ is so large that bank i can itself bail out bank j even though they may not be directly connected. Then let's define

$$\bar{d}^{max} = \max_j \bar{d}_j^{max}$$

When $\bar{d}_1 = \bar{d}^{max}$, then in any state of nature, all failed banks are solvent. This shows that the network risk-taking distortion is bounded from above.

To find the lowest upper bound for network structure $\boldsymbol{\Theta}$, note that $\mathcal{D}_i(\boldsymbol{Z_{-i}})$ reaches the maximum when every failed bank is solvent in all possible states of nature. In this case, we can rewrite failed banks' equilibrium payment (Equation I.2) as

$$\boldsymbol{d_f^*}(\boldsymbol{\omega}) = \boldsymbol{\Theta_{ff}} \boldsymbol{d_f^*}(\boldsymbol{\omega}) + \boldsymbol{\Theta_{fs}} \mathbb{1}_s \bar{d} - \mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

It implies

$$\boldsymbol{d_f^*}(\boldsymbol{\omega}) = (\mathbf{I}_f - \boldsymbol{\Theta_{ff}})^{-1}(\boldsymbol{\Theta_{fs}} \mathbb{1}_s \bar{d} - \mathbb{1}_f v) = \mathbb{1}_f \bar{d} - (\mathbf{I}_f - \boldsymbol{\Theta_{ff}})^{-1} \mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

The interbank payments received by the successful banks are

$$\boldsymbol{\Theta_{sf}} \boldsymbol{d_f^*}(\boldsymbol{\omega}) + \boldsymbol{\Theta_{ss}} \mathbb{1}_s \bar{d} = \mathbb{1}_s \bar{d} - \boldsymbol{\Theta_{sf}}(\mathbf{I}_f - \boldsymbol{\Theta_{ff}})^{-1} \mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

That means successful banks' network distortion vector in state $\boldsymbol{\omega}$ is $\vec{\mathcal{D}}(\boldsymbol{\omega}) = \boldsymbol{\Theta_{sf}}(\mathbf{I}_f - \boldsymbol{\Theta_{ff}})^{-1} \mathbb{1}_f v$. By the network symmetry, the expected distortion conditional on the set $\boldsymbol{f}$ fails will be the ratio of column sum of $\vec{\mathcal{D}}(\boldsymbol{\omega})$ and the number of columns. That is

$$\mathbb{E}[\mathcal{D}_i^{max} | \text{set } f \text{ fails}] = \frac{\mathbb{1}_s' \boldsymbol{\Theta_{sf}}(\mathbf{I}_f - \boldsymbol{\Theta_{ff}})^{-1} \mathbb{1}_f v}{\mathbb{1}_s' \mathbb{1}_s} = \frac{\mathbb{1}_f' \mathbb{1}_f}{\mathbb{1}_s' \mathbb{1}_s} v$$

Then a bank's unconditional expected network distortion is $\sum_{\boldsymbol{f}} \frac{\mathbb{1}_f' \mathbb{1}_f}{\mathbb{1}_s' \mathbb{1}_s} v \cdot \Pr(\mathcal{F} = \boldsymbol{f})$. Again due to the symmetry, the permutation among the failed banks is irrelevant. Therefore, the maximum network risk-taking distortion is

$$\mathcal{D}_i^{max}(\boldsymbol{Z_{-i}}) = \sum_{f=1}^{N-1} \frac{f}{N-f} \cdot v \cdot \binom{N-1}{f} \Big[P(Z_{-i})\Big]^{N-1-f} \Big[1 - P(Z_{-i})\Big]^f$$

The expression has a clear interpretation. Suppose that f banks fail their projects in some state. The amount of money that needs to be cross-subsidized in the network is $f \cdot v$, which is the size of failed banks' deposits. If the network is symmetric, each successful bank will cross-subsidize $f \cdot v/(N-f)$. The probability with which f banks fail is $\binom{N-1}{f}[P(Z_{-i})]^{N-1-f}[1 - P(Z_{-i})]^f$. It's easily to verify that Equation I.13 is identical to $\mathcal{D}_i^{max}(\boldsymbol{Z_{-i}})$ if $\bar{d}$ is sufficiently large.

By binomial theorem, we can rewrite the above equation as

$$\mathcal{D}_i^{max}(\boldsymbol{Z_{-i}}) = \frac{1 - P(Z_{-i}) - [1 - P(Z_{-i})]^N}{P(Z_{-i})} \cdot v$$

It is immediate that $\mathrm{d}\mathcal{D}_i^{max}(\boldsymbol{Z_{-i}}) / \mathrm{d}N > 0$. □

PROPOSITION 6

Denote the central clearing counterparty (CCP) as bank 0. Because the CCP has no outside liabilities, it's always solvent. The payment equilibrium when m banks fail can be represented as

$$\begin{aligned} d_s^* &= \bar{d} \\ d_f^* &= (d_0^*/N - v)^+ \\ d_0^* &= (N-m) \cdot d_s^* + m \cdot d_f^* \end{aligned}$$

The solution to the above fixed point is

$$d_i^{CCP}(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - \frac{N}{N-m} v\right)^+ & \forall \quad \omega_i = f \end{cases}$$

As a result, the risk-taking distortion of a bank in any financial network with a CCP is

$$\begin{aligned} \mathcal{D}_i^{CCP}(\boldsymbol{Z_{-i}}) &= \sum_{m=1}^{N-1} \Bigg(\bar{d} - \underbrace{\left(\bar{d} - \frac{N}{N-m} v\right)^+ \cdot \frac{m}{N}}_{\text{payment from failed banks}} - \underbrace{\bar{d} \cdot \frac{N-m}{N}}_{\substack{\text{payment from} \\ \text{successful banks}}} \Bigg) \cdot \Pr(m \text{ banks failed}) \\ &= \sum_{m=1}^{N-1} \min\left(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N} \right) \cdot \Pr(m \text{ banks failed}) \qquad \text{(I.16)} \end{aligned}$$

Compare Equation I.16 with Equation I.13, we have $\mathcal{D}_i^{CCP}(\boldsymbol{Z_{-i}}; \bar{d}) = \mathcal{D}_i^{C}(\boldsymbol{Z_{-i}}; \frac{N-1}{N}\bar{d})$. As a result, the risk-taking equilibrium of a network with a CCP is equivalent to that of a complete network with $(\frac{N-1}{N}\bar{d}, \boldsymbol{\Theta^C})$. □

PROPOSITION 7

Part (a): The proof is identical to that of Lemma 6 in Eisenberg and Noe (2001), and hence it is omitted here to conserve space. Intuitively, the right-hand side of Equation 2 is monotonic in λ. As a result, the fixed-point solution is also monotonic in λ.

Part (b): From Part (a), we know that $\min(d_i^C, d_i^R) \leqslant d_i^\lambda \leqslant \max(d_i^C, d_i^R)$. We also know that when $d_i^* > 0$, bank i must be solvent. That implies

$$\min\left(\mathbb{1}\left[i \in \mathcal{F}_C^+\right], \mathbb{1}\left[i \in \mathcal{F}_R^+\right]\right) \leqslant \mathbb{1}\left[i \in \mathcal{F}_\lambda^+\right] \leqslant \max\left(\mathbb{1}\left[i \in \mathcal{F}_C^+\right], \mathbb{1}\left[i \in \mathcal{F}_R^+\right]\right) \quad \forall i, \boldsymbol{\omega}$$

, where $\mathbb{1}\left[i \in \mathcal{F}_C^+\right]$ is the indicator function for bank i being solvent if it fails, and the subscript denotes that the bank is in a complete network.

Part (c): To prove that $\mathcal{D}_i^\lambda(\boldsymbol{Z}_{-\boldsymbol{i}})$ can be smaller than the minimum of $\mathcal{D}^R(\boldsymbol{Z}_{-\boldsymbol{i}})$ and $\mathcal{D}^C(\boldsymbol{Z}_{-\boldsymbol{i}})$, it suffices to use an example. Consider the following parameters: $\bar{d} = 2.5$, $v = 1$, $N = 4$, $\lambda = 0.5$, and $P(Z_j) = 0.5, \forall j \neq i$. Then, we can calculate the network distortion for bank i in the three types of networks as follows:

$$\mathcal{D}^R(\boldsymbol{Z}_{-\boldsymbol{i}}) = \Pr(m=0)\cdot 0 + \Pr(m=1)\cdot\frac{1}{3} + \Pr(m=2)\cdot 1 + \Pr(m=3)\cdot 2.5 = \frac{13}{16}$$
$$\mathcal{D}^\lambda(\boldsymbol{Z}_{-\boldsymbol{i}}) = \Pr(m=0)\cdot 0 + \Pr(m=1)\cdot\frac{1}{3} + \Pr(m=2)\cdot 1 + \Pr(m=3)\cdot\frac{43}{18} = \frac{115}{144}$$
$$\mathcal{D}^C(\boldsymbol{Z}_{-\boldsymbol{i}}) = \Pr(m=0)\cdot 0 + \Pr(m=1)\cdot\frac{1}{3} + \Pr(m=2)\cdot 1 + \Pr(m=3)\cdot 2.5 = \frac{13}{16}$$

, where m represents the number of banks that fail their projects. The Online Appendix provides the details of the calculations for each number in the above three equations. In this case, $\mathcal{D}^\lambda(\boldsymbol{Z}_{-\boldsymbol{i}}) < \min\left(\mathcal{D}^R(\boldsymbol{Z}_{-\boldsymbol{i}}), \mathcal{D}^C(\boldsymbol{Z}_{-\boldsymbol{i}})\right)$. From the monotone selection theorem, the equilibrium risk-taking of each bank in a λ-network is less than that in both a complete network and a ring network.

Intuitively, the lower distortion of the bank in the λ-network results from scenarios where all banks except bank i fail. When bank i is in the λ-network, it is not completely exposed to the risks of its direct neighbors, as it would be in a ring network. As a result, the distortion can be smaller than that of a bank in a ring network. □

PROPOSITION 8

Part (a): The proof is similar to that of lemma 3. Let $\boldsymbol{\Theta}$ denote the largest path-connected network that encompasses both bank i and bank j. In any state of nature $\boldsymbol{\omega}$, the payment vector for solvent failed banks is $\boldsymbol{d}_+^* = \boldsymbol{\Theta}_{++}\cdot\boldsymbol{d}_+^* + \boldsymbol{\Theta}_{+\boldsymbol{s}}\cdot\mathbb{1}_s\cdot\bar{d} - \mathbb{1}_+\cdot v + \boldsymbol{\epsilon}_+$, or

$$\boldsymbol{d}_+^* = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}}\cdot\mathbb{1}_s\cdot\bar{d} - \mathbb{1}_+\cdot v + \boldsymbol{\epsilon}_+)$$

To conserve space, I suppress the state $\boldsymbol{\omega}$ in $\boldsymbol{d}_+^*(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{++}(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{+\boldsymbol{s}}(\boldsymbol{\omega})$, $\mathbb{1}_s(\boldsymbol{\omega})$ and $\boldsymbol{\epsilon}_+(\boldsymbol{\omega})$. We can again rewrite bank i's network risk-taking distortion in a matrix form as

$$\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}}\Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+\bar{d} - \boldsymbol{d}_+^*) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-\bar{d}\Big] \tag{I.17}$$

Compare three financial systems with different sizes of equity for bank j: ϵ_{j1}, ϵ_{j2}, ϵ_{j3} where $\epsilon_{j3} - \epsilon_{j2} = \epsilon_{j2} - \epsilon_{j1} = \xi$. Denote $\boldsymbol{\epsilon_1}/\boldsymbol{\epsilon_2}/\boldsymbol{\epsilon_3}$ as the vectors of banks' equities with the j's item being $\epsilon_{j1}/\epsilon_{j2}/\epsilon_{j3}$. We need to consider the following four cases.

Case I: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$.

For all $\boldsymbol{\omega}$, $\boldsymbol{d}_+^*$ is linearly increasing in ϵ_j. This is because $\boldsymbol{d}_+^{\boldsymbol{3}*} - \boldsymbol{d}_+^{\boldsymbol{2}*} = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\epsilon}_{3+} - \boldsymbol{\epsilon}_{2+})$ equals $\boldsymbol{d}_+^{\boldsymbol{2}*} - \boldsymbol{d}_+^{\boldsymbol{1}*} = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\epsilon}_{2+} - \boldsymbol{\epsilon}_{1+})$. From here, we know that the network risk-taking distortion is linearly decreasing in ϵ_j.

$$\mathcal{D}_i^3(\boldsymbol{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^2(\boldsymbol{Z}_{-\boldsymbol{i}}) = \mathcal{D}_i^2(\boldsymbol{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\boldsymbol{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}}\Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\boldsymbol{d}_+^{\boldsymbol{1}*} - \boldsymbol{d}_+^{\boldsymbol{2}*})\Big] \leqslant 0$$

Case II: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

We first compare the equity $\boldsymbol{\epsilon_2}$ with $\boldsymbol{\epsilon_1}$. The condition $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega})$ implies that, in $\boldsymbol{\omega}$, some otherwise insolvent failed banks for $(\bar{d}, \boldsymbol{\Theta}; \boldsymbol{\epsilon_1})$ become solvent for $(\bar{d}, \boldsymbol{\Theta}; \boldsymbol{\epsilon_2})$. Denote those banks as bank $t_1, t_2, ..., t_T$, where $T \geqslant 1$. Because payment equilibrium is continuous in ϵ_j (Equation 9), there exists $\epsilon_{j1} < \tilde{\epsilon}_{j_1} < \tilde{\epsilon}_{j_2} ... < ... < \tilde{\epsilon}_{j_S} < \epsilon_{j2}$ (where $1 \leqslant S \leqslant T$) between ϵ_{j1} and ϵ_{j2}, such that when $\epsilon_j = \tilde{\epsilon}_{j_s}$, some banks are exactly solvent (i.e., just able to fulfill their deposits). In other words, those margin banks are solvent when $\epsilon_j \in (\tilde{\epsilon}_{j_s}, \tilde{\epsilon}_{j_{s+1}})$ and insolvent when $\epsilon_j \in (\tilde{\epsilon}_{j_{s-1}}, \tilde{\epsilon}_{j_s})$ respectively. Denote $\widetilde{\mathcal{D}}_i^s(\boldsymbol{Z_{-i}})$ as the network risk-taking distortion when $\epsilon_j = \tilde{\epsilon}_{j_s}$. We have

$$\begin{aligned}
\mathcal{D}_i^2(\boldsymbol{Z_{-i}}) - \widetilde{\mathcal{D}}_i^S(\boldsymbol{Z_{-i}}) &= \sum_{\boldsymbol{\omega_{-i}}} \Pr(\boldsymbol{\omega_{-i}}) \Big[\boldsymbol{\Theta}_{i+}^{\mathbf{2}} (\tilde{\boldsymbol{d}}_+^{\mathbf{S}*} - \boldsymbol{d}_+^{\mathbf{2}*}) \Big] \leqslant 0 \\
\widetilde{\mathcal{D}}_i^{s+1}(\boldsymbol{Z_{-i}}) - \widetilde{\mathcal{D}}_i^s(\boldsymbol{Z_{-i}}) &= \sum_{\boldsymbol{\omega_{-i}}} \Pr(\boldsymbol{\omega_{-i}}) \Big[\widetilde{\boldsymbol{\Theta}}_{i+}^{\boldsymbol{s}} (\tilde{\boldsymbol{d}}_+^{\boldsymbol{s}*} - \tilde{\boldsymbol{d}}_+^{\boldsymbol{s}+\mathbf{1}*}) \Big] \leqslant 0 \qquad \forall s = 1, ..., S-1 \qquad \text{(I.18)} \\
\widetilde{\mathcal{D}}_i^1(\boldsymbol{Z_{-i}}) - \mathcal{D}_i^1(\boldsymbol{Z_{-i}}) &= \sum_{\boldsymbol{\omega_{-i}}} \Pr(\boldsymbol{\omega_{-i}}) \Big[\boldsymbol{\Theta}_{i+}^{\mathbf{1}} (\boldsymbol{d}_+^{\mathbf{1}*} - \tilde{\boldsymbol{d}}_+^{\mathbf{1}*}) \Big] \leqslant 0
\end{aligned}$$

Summing the above equations, we can find that $\mathcal{D}_i^2(\boldsymbol{Z_{-i}}) - \mathcal{D}_i^1(\boldsymbol{Z_{-i}}) \leqslant 0$. Using the technique in case I, we can find that $\mathcal{D}_i^3(\boldsymbol{Z_{-i}}) - \mathcal{D}_i^2(\boldsymbol{Z_{-i}}) \leqslant 0$. Hence the monotonicity result follows. The proof of concavity is identical to the proof of Lemma 3.

Case III: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$: The proof is identical to proof of Case II except we use the technique of Equation I.18 to prove $\mathcal{D}_i^3(\boldsymbol{Z_{-i}}) - \mathcal{D}_i^2(\boldsymbol{Z_{-i}}) \leqslant 0$.

Case IV: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$: The proof is a combination of case 2 and case 3.

Because $\mathcal{F}_1^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_2^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$, Case I to IV or some combination of them exhaust all the possibilities. □

Part (b): Taking the partial derivative of bank i's expected payoff $\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}, \boldsymbol{\epsilon})\Big]$ respect to Z_i and ϵ_i:

$$\frac{\mathrm{d}^2 \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}, \boldsymbol{\epsilon})\Big]}{\mathrm{d}Z_i \, \mathrm{d}\epsilon_i} = P'(Z_i) < 0$$

Similarly,

$$\frac{\mathrm{d}^2 \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}, \boldsymbol{\epsilon})\Big]}{\mathrm{d}Z_i \, \mathrm{d}\epsilon_j} = -P'(Z_i) \cdot \frac{\mathrm{d}\mathcal{D}_i(\boldsymbol{Z_{-i}}; \boldsymbol{\epsilon})}{\mathrm{d}\epsilon_j} < 0$$

The above two equations imply that $\mathbb{E}[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}, \boldsymbol{\epsilon})]$ exhibits a decreasing difference in Z_i and ϵ_i and also a decreasing difference in Z_i and ϵ_j. By the Monotone Selection Theorem, the equilibrium Z_i^* is decreasing in ϵ_i and ϵ_j. □

COROLLARY 2

The effect of an increase in v_j is equivalent to a decrease in ϵ_j on the payment equilibrium. As a result, by adapting Part (b) of Proposition 8, we can derive that an increase in v_j leads to an increase in $\mathcal{D}_i(\boldsymbol{Z_{-i}})$, thereby resulting in greater equilibrium risk exposure Z_i^*. $\square$

PROPOSITION 9

The payment vector for failed but solvent banks is

$$\boldsymbol{d}_+^*(\boldsymbol{\omega};\tau) = \begin{cases} (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(\tau - v)) & \text{if} \quad \#\{l|\omega_l = f\} \geqslant \eta \\ (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(0 - v)) & \text{otherwise} \end{cases}$$

The first line corresponds to the state of nature where a bailout occurs. The second line corresponds to the other cases. Compare two bailout amount τ_1 and τ_2 with $\tau_2 - \tau_1 = \xi > 0$. We again have two cases: (1) $\mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_1^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$. (2) $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. Let $\mathcal{B}(\boldsymbol{\omega})$ denote the bailout indicator $\mathbb{1}\big[\#\{l|\omega_l = f\} > \eta\big]$. Since $\eta < N$, $\Pr\big(\mathcal{B}(\boldsymbol{\omega}) = 1\big) > 0$.

For Case I,

$$\boldsymbol{d}_+^*(\boldsymbol{\omega};\tau_2) - \boldsymbol{d}_+^*(\boldsymbol{\omega};\tau_1) = \mathcal{B}(\boldsymbol{\omega})(\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\mathbb{1}_+\xi \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega}$$

From Equation I.17,

$$\begin{aligned} &\mathcal{D}_i(\boldsymbol{Z_{-i}};\tau_2) - \mathcal{D}_i(\boldsymbol{Z_{-i}};\tau_1) \\ &\qquad = \sum_{\boldsymbol{\omega_{-i}}} \Pr(\boldsymbol{\omega_{-i}})\Big[\boldsymbol{\Theta_{i+}}(\boldsymbol{d_+^{1*}} - \boldsymbol{d_+^{2*}})\Big] = \sum_{\boldsymbol{\omega_{-i}}} -\mathcal{B}(\boldsymbol{\omega}^{i=s})\Pr(\boldsymbol{\omega_{-i}})\Big[\boldsymbol{\Theta_{i+}}(\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\mathbb{1}_+\xi\Big] < 0 \end{aligned}$$

The proof of case 2 is identical to case 2 of lemma 3 or proposition 8.(a), hence omitted here to avoid repetition. From here, we proved that $\mathcal{D}_i(\boldsymbol{Z_{-i}})$ is decreasing in τ. If $\tau \leqslant v$, a failed bank earns zero profit, and this will preserve $\mathbb{E}\big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\big] = P(Z_i)\big(Z_i - v - \mathcal{D}_i(\boldsymbol{Z_{-i}},\tau)\big)$. As a result $\mathbb{E}\big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z},\tau)\big]$ exhibits a decreasing difference in Z_i and τ. From the Monotone Selection Theorem, banks' equilibrium risk exposure is decreasing in τ. $\square$

PROPOSITION I.1

With the cash flow distribution in Equation 8, Lemma 2 still holds: the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega};\boldsymbol{Z})$ is constant in the risk exposure vector $\boldsymbol{Z}$. To see why, reordering $d_1^*(\boldsymbol{\omega};\boldsymbol{Z}), ...d_N^*(\boldsymbol{\omega};\boldsymbol{Z})$ in Equation 2 gives us,

$$\begin{aligned} &d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) = \bar{d} && \forall \omega_i = \text{good} \\ &d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) = \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\boldsymbol{Z}) + \beta - v && \forall \omega_i = \text{middle} \\ &d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) = \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\boldsymbol{Z}) + \gamma - v\Big)^+ && \forall \omega_i = \text{bad} \end{aligned}$$

As a result, the fixed point solution $\boldsymbol{d}^*(\boldsymbol{\omega};\boldsymbol{Z})$ is constant in $\boldsymbol{Z}$.

From the payment equilibrium (Equation 2), we can find that the expected profit in the contagion state $\big[\beta - v - \big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i=\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i=\beta}})\big)\big]^+ = \beta - v - \big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i=\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i=\beta}})\big)$. Intuitively, $d_i^*(\boldsymbol{\omega}^{\boldsymbol{i=\beta}})$ will adjust endogenously in the payment equilibrium such that the previous expression is always non-negative. Define

$$\mathcal{D}_\alpha(\boldsymbol{Z}_{-\boldsymbol{i}}) \equiv \sum_{\boldsymbol{\omega}_{-i}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\alpha}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})$$
$$\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}}) \equiv \sum_{\boldsymbol{\omega}_{-i}} \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})$$

Proposition 1 proved $\mathrm{d}\mathcal{D}_\alpha(\boldsymbol{Z}_{-\boldsymbol{i}})/\,\mathrm{d}Z_m > 0$. To find the sign of $\mathrm{d}\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}})/\,\mathrm{d}Z_m$, let's examine $\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}})$ when we change Z_m to Z'_m with $Z'_m > Z_m$. Let $\boldsymbol{Z'}_{-\boldsymbol{i}}$ denote the new risk-exposure vector that differs from $\boldsymbol{Z}_{-\boldsymbol{i}}$ only in Z_m. We have

$$\begin{aligned}
&\mathcal{D}_\beta(\boldsymbol{Z'}_{-\boldsymbol{i}}) - \mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}}) \\
&= \sum_{\boldsymbol{\omega}_{-i-m}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}) \cdot \bigg\{ P_\alpha(Z'_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big) \\
&\qquad + P_\beta(Z'_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}})\Big) + P_\gamma(Z'_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma})\Big) \bigg\} \\
&\qquad - \sum_{\boldsymbol{\omega}_{-i-m}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}) \cdot \bigg\{ P_\alpha(Z_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big) \\
&\qquad\quad + P_\beta(Z_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}})\Big) + P_\gamma(Z_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma})\Big) \bigg\} \\
&= \sum_{\boldsymbol{\omega}_{-i-m}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}) \cdot \\
&\qquad \bigg\{ \underbrace{\Big(P_\alpha(Z'_m) - P_\alpha(Z_m) \Big)}_{<0} \cdot \underbrace{\Big[\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big) - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma})\Big) \Big]}_{<0} \\
&\qquad + \underbrace{\Big(P_\beta(Z'_m) - P_\beta(Z_m) \Big)}_{<0} \cdot \underbrace{\Big[\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}})\Big) - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma})\Big) \Big]}_{<0} \bigg\} \\
&> 0
\end{aligned}$$

The inequality $\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big) - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\gamma})\Big) < 0$ means that bank i's cross-subsidies to other banks are smaller when bank m succeeds compared with when it fails. To see why this is true, define $H(\omega_m) \equiv d_i^*(\boldsymbol{\omega}_{-m};\omega_m) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}_{-m};\omega_m)$. From Equation 1, $H(\omega_m) = \min[\beta - v, \bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}_{-m};\omega_m)]$. By auxiliary lemma, we find $H(\omega_m = \alpha) < H(\omega_m = \gamma)$ and $H(\omega_m = \beta) < H(\omega_m = \gamma)$. From here, we proved $\mathrm{d}\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}})/\,\mathrm{d}Z_m > 0$. Finally,

$$\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big]}{\mathrm{d}Z_i\,\mathrm{d}Z_m} = \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Big\{P_\alpha(Z_i)\Big(Z_i - v - \mathcal{D}_\alpha(\boldsymbol{Z}_{-\boldsymbol{i}})\Big)\Big\}}_{>0\text{, by proposition 1}} + P'_\beta(Z_i)\Big(-\frac{\mathrm{d}\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}})}{\mathrm{d}Z_m}\Big) > 0$$

The game is supermodular and hence $\boldsymbol{Z}$ is strategically complementary. □

PROPOSITION I.2

From bank i's expected payoff, it will prefer to join the network $(\bar{d}, \Theta)$ (over stand-alone) if

$$P(Z_i^*)\Big[Z_i^* - v - \mathcal{D}_i(\boldsymbol{Z}_{-i}^*)\Big] + c_i - \Big[1 - P(Z_i^*)\Big]\Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f\Big) \cdot c_i > P(Z_i^{**})\Big[Z_i^{**} - v + c_i\Big] \tag{I.19}$$

where Z_i^* is equilibrium risk-taking of a bank in the network: $Z_i^* = \operatorname{argmax} \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}^*)\Big]$, and Z_i^{**} is the optimal risk-taking of a stand-alone bank: $Z_i^{**} = \operatorname{argmax} P(Z_i)(Z_i - v + c_i)$. From definition of the Nash equilibrium, the LHS of Equation I.19 is greater than

$$A \equiv P(Z_i^{**})\Big[Z_i^{**} - v - \mathcal{D}_i(\boldsymbol{Z}_{-i}^*)\Big] + c_i - \Big[1 - P(Z_i^{**})\Big]\Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f\Big) \cdot c_i$$

Define the RHS of Equation I.19 as $B \equiv P(Z_i^{**})\Big[Z_i^{**} - v + c_i\Big]$

$$A - B = \Big[1 - P(Z_i^{**})\Big]\Big[1 - \Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f\Big)\Big] \cdot c_i - P(Z_i^{**}) \cdot \mathcal{D}_i(\boldsymbol{Z}_{-i}^*)$$

If $\bar{d} > v$, $\Pr(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f) < 1$. This means that it is possible that bank i's deposits get fully fulfilled from counterparties' cross subsidies. Since Z^* and Z^{**} are bounded, there exists $\bar{c} \in \mathbb{R}^+$ such that if all $c_i > \bar{c}$, $A - B > 0$. □

PROPOSITION I.3

The introduction of the dependence matrix $\boldsymbol{\Phi}_i$ does not alter the independence of the payment vector $\boldsymbol{d}^*$ from the risk vector $\boldsymbol{Z}$ or the correlation matrix $\boldsymbol{\Phi}$. Lemma 1 and Lemma 2 continue to hold.

Let's first consider the case where there is no charter value: $\boldsymbol{c} = 0$. If we compare bank i's expected profit when it chooses between ϕ_{ij} and $\widetilde{\phi}_{ij}$ with $\widetilde{\phi}_{ij} > \phi_{ij}$, we have

$$\begin{aligned}
\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \widetilde{\boldsymbol{\phi}}_{\boldsymbol{ij}})\Big] - \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \boldsymbol{\phi}_{\boldsymbol{ij}})\Big] = \\
- \sum_{\boldsymbol{\omega}_{-i-j}} \Big(\bar{d} - \sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,j=s}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = s) \cdot P(Z_j) \cdot (\widetilde{\phi}_{ij} - \phi_{ij}) \\
+ \sum_{\boldsymbol{\omega}_{-i-j}} \Big(\bar{d} - \sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,j=f}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = f) \cdot P(Z_j) \cdot (\widetilde{\phi}_{ij} - \phi_{ij})
\end{aligned}$$

Suppose $\phi_{j,k}^* = 1$ for all $k \neq i$. That implies $\Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = s) = 1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is s. Similarly, $\Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = f) = 1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is f.

By Auxiliary Lemma in the appendix above, $\sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,-i=s}}) \geqslant \sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,-i=f}})$. This implies bank i's expected profit is increasing in its project's dependence ϕ_{ij} with other banks. Therefore, for all $\boldsymbol{Z}$, bank i's choices of conditional dependence with bank j won't deviate from $\phi_{i,j}^* = 1$. With perfect correlation, the network risk-taking distortion disappears: $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}}^*, \mathbb{1}) = 0$ for all $\boldsymbol{Z}_{-\boldsymbol{i}}^*$. Hence, the equilibrium is characterized by

$$\begin{aligned}
&\phi_{ij}^* = 1 \quad \forall i, j \in \mathcal{N} \\
&P'(Z_i^*)(Z_i^* - v) + P(Z_i^*) = 0 \quad \forall i \in \mathcal{N}
\end{aligned}$$

And $\rho^*_{ij} = 1$ for all i, j. Intuitively, given that banks do not prioritize charter values, there are no advantages to diversifying their projects. Consequently, they opt for perfectly correlated projects to protect themselves from paying cross-subsidies.

Now, let's consider a scenario where $\boldsymbol{c} > 0$. Let's first prove that $\boldsymbol{\Phi}^* = 1$ (i.e. maximum correlation) is not an equilibrium if c is sufficiently large. To see this, consider a situation where bank i deviates in its strategy such that $\boldsymbol{\phi}_i = 0$ and $\widetilde{Z}_i$ is the optimal risk-exposure following $\boldsymbol{\phi}_i = 0$. In essence, think of a case where bank i formulates a strategy that is uncorrelated with any other bank.

With this deviation of strategy, bank i's profit becomes

$$A \equiv P(\widetilde{Z}_i)\Big[\widetilde{Z}_i - v - \mathcal{D}_i(\mathbf{Z}^*_{-i})\Big] + c_i - \Big[1 - P(\widetilde{Z}_i)\Big]\Pr\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f\Big) \cdot c_i$$

, which is bigger than the following equation from the definition of $\widetilde{Z}_i$

$$P(Z^*)\Big[Z^* - v - \mathcal{D}_i(\mathbf{Z}^*_{-i})\Big] + c_i - \Big[1 - P(Z^*)\Big]\Pr\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f\Big) \cdot c_i$$

In contrast, if the strategy $\boldsymbol{\Phi}^* = 1$ remained unchanged, the expected profit is simply

$$B \equiv P(Z^*_i)\Big[Z^*_i - v + c_i\Big]$$

Then $A > B$ if,

$$\Big[1 - P(Z^*_i)\Big]\Big[1 - \Pr\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f\Big)\Big] \cdot c_i - P(Z^*_i) \cdot \mathcal{D}_i(\mathbf{Z}^*_{-i}) > 0$$

The rest of the proof is similar to Proposition I.2. Since Z^* and Z^{**} are bounded, there exists $\bar{c} \in \mathbb{R}^+$ such that if $c_i > \bar{c}$, $A - B > 0$.

As a result, $\boldsymbol{\Phi} = 1$ can not be an equilibrium. All this implies that the optimal strategy involves $\phi^*_{i,j} < 1$ for some i, j if the c_i is sufficiently large. In other words, banks seek to diversify their projects so that they can enjoy the co-insurance benefits provided by the financial network. For the bank that $\boldsymbol{\phi}^*_i < 1$, the optimal risk-taking Z^*_i will be greater than a stand-alone bank because the risk-taking distortion exists. □

PROPOSITION I.4

Part (a): Suppose there is deposit insurance, $\Pi^D_i(\boldsymbol{\omega}) = v^*_i$ for all $\boldsymbol{\omega}$. The equilibrium condition that determines the deposit rate is

$$\beta \cdot u(v^*_i) - M = 0$$

Or, $v^*_i = u^{-1}\big(M/\beta\big)$. Plugging it into bank i's optimization, we have

$$P'(Z^*_i)\Big(Z^*_i - u^{-1}\Big(\frac{M}{\beta}\Big) - \mathcal{D}_i(\mathbf{Z}^*_{-\boldsymbol{i}})\Big) + P(Z^*_i) = 0$$

If the bank is stand-alone, then its optimization becomes

$$P'(Z^{S.A}_i)\Big(Z^{S.A}_i - u^{-1}\Big(\frac{M}{\beta}\Big)\Big) + P(Z^{S.A}_i) = 0$$

Using the same argument as in Equation I.3, we can deduce $Z^*_i > Z^{S.A}_i$.

Part (b) and (c): Suppose there is no deposit insurance. The *expected return* to bank i's depositors' is

$$\mathbb{E}\Big[\Pi_i^D(\boldsymbol{\omega}; v_i^*, \boldsymbol{Z}_i^*)\Big] = \mathbb{E}\Bigg\{\min\Big[v_i^*, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}_i^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega})\Big]\Bigg\}$$

$$= \underbrace{\mathbb{E}\Big\{e_i(\boldsymbol{Z}_i^*, \boldsymbol{\omega})\Big\}}_{=P(Z_i^*)\cdot Z_i^*} + \underbrace{\mathbb{E}\Big\{\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega})\Big\}}_{=0} - \underbrace{\mathbb{E}\Big\{\Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}_i^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) - v_i^*\Big)^+\Big\}}_{=P(Z_i^*)(Z_i^* - v_i^*) - P(Z_i^*)\mathcal{D}_i(\boldsymbol{Z}_{-i}^*)}$$

$$= P(Z_i^*) \cdot v_i^* + P(Z_i^*) \cdot \mathcal{D}_i(\boldsymbol{Z}_{-i}^*)$$

The second line follows the first line because for all $x, y \in \mathbb{R}$, $\min(x, y) = y - (y - x)^+$. From the symmetry assumption, $\mathbb{E}[d_j^*(\boldsymbol{\omega})] = \mathbb{E}[d_i^*(\boldsymbol{\omega})]$, $\forall i \neq j$. Hence $\mathbb{E}[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega})] = 0$.

It's worth noting that the second term $P(Z_i^*) \cdot \mathcal{D}_i(\boldsymbol{Z}_{-i}^*)$ is precise what bank i's shareholders lose due to the cross-subsidies. Intuitively, the risk-taking distortion stems precisely from the transfer of wealth from successful banks to failed banks' depositors, which, as a result, shows up in depositors' payoff. Also note that the above formula represents the expected return to bank i's depositors, not the expected utility. If depositors are risk-averse, the utility function $u(.)$ is concave. As a result, we have the following inequality.

$$\mathbb{E}\Big[u\Big(\Pi_i^D(\boldsymbol{\omega}; v_i^*, \boldsymbol{Z}^*)\Big)\Big] < u\Big(\mathbb{E}\Big[\Pi_i^D(\boldsymbol{\omega}; v_i^*, \boldsymbol{Z}_i^*)\Big]\Big) = u\Big(P(Z_i^*) \cdot (v_i^* + \mathcal{D}_i(\boldsymbol{Z}_{-i}^*))\Big) \quad \text{(I.20)}$$

Plugging the above inequality into the equilibrium condition, we have

$$P(Z_i^*) \cdot \Big(v_i^* + \mathcal{D}_i(Z_i^*)\Big) > u^{-1}\Big(\frac{M}{\beta}\Big) \quad \text{(I.21)}$$

Bank i's risk exposure Z_i^* is the result of its optimization. Explicitly,

$$P'(Z_i^*) \cdot \Big(Z_i^* - v_i^* - \mathcal{D}_i(\boldsymbol{Z}_{-i}^*)\Big) + P(Z_i^*) = 0 \quad \text{(I.22)}$$

Equations I.21 and I.22 jointly determine banks' equilibrium risk exposure as

$$P'(Z_i^*) \cdot \Big[Z_i^* - \frac{u^{-1}\big(\frac{M}{\beta}\big)}{P(Z_i^*)}\Big] + P(Z_i^*) < 0 \quad \text{(I.23)}$$

It is easy to see that for a stand-alone bank, the first-order condition is

$$P'(Z_i^{S.A}) \cdot \Big[Z_i^{S.A} - \frac{u^{-1}\big(\frac{M}{\beta}\big)}{P(Z_i^{S.A})}\Big] + P(Z_i^{S.A}) = 0$$

From here, we can deduce that $Z_i^* > Z_i^{S.A}$ because the first-order condition is a decreasing function of Z. Intuitively, even though depositors will reward connected banks by demanding a lower deposit rate, v_i^*, this decrease in the deposit rate will not completely offset the effect of the distortionary cross-subsidies, $\mathcal{D}_i(\boldsymbol{Z}_{-i}^*)$. This is because risk-averse depositors do not value the co-insurance benefit as highly as they do the increased expected return from the cross-subsidies.

Now suppose there is no deposit insurance and depositors are risk-neutral. The inequality sign in Equation I.20 becomes equality, and so do the following inequalities in I.21 and I.23. From here, we can see that the first-order condition that determines bank i's risk-taking incentives is identical if it is connected versus or it is stand-alone. □

Auxiliary Lemmas

LEMMA I.A [Hockey-stick Identity]. *For all $n > r$, we have*

$$\text{(i)} \quad \sum_{l=r}^{n} \binom{l}{r} = \binom{n+1}{r+1} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{l=r}^{n} \binom{l}{r}(n-l) = \binom{n+1}{r+1}\frac{n-r}{r+2}$$

PROOF.

We proceed by induction. For an initial $n = r+1$, we have

$$\text{(i)} \quad \binom{r}{r} + \binom{r+1}{r} = \binom{r+2}{r+1} \qquad \text{and} \qquad \text{(ii)} \quad \binom{r}{r} * 1 + \binom{r+1}{r} * 0 = \binom{r+2}{r+1} * \frac{1}{r+2} = 1$$

Now suppose the two equality holds for $n = k$. Then For $n = k+1$, we have

$$\text{(i)} \quad \sum_{l=r}^{k+1} \binom{l}{r} = \sum_{l=r}^{k} \binom{l}{r} + \binom{k+1}{r} = \binom{k+1}{r+1} + \binom{k+1}{r} = \binom{k+2}{r+1}$$

$$\text{(ii)} \quad \sum_{l=r}^{k+1} \binom{l}{r}(k+1-l) = \sum_{l=r}^{k} \binom{l}{r}(k+1-l) = \binom{k+1}{r+1}\frac{k-r}{r+2} + \binom{k+1}{r+1} = \binom{k+2}{r+1}\frac{k+1-r}{r+2}$$

Q.E.D by induction. □

LEMMA I.B [Triangle Inequality]. *For any sequence $\{A_i\}$ and $B \in \mathbb{R}$ with $B < \max_i(A_i)$, we have*

$$\sum_i \Big(A_i\Big)^+ \geqslant \Big(\sum_i A_i - B\Big)^+ + B$$

PROOF.

Without loss of generality, let $A_0 = \max_i(A_i)$

$$\sum_i \Big(A_i\Big)^+ - B = \sum_{i \neq 0} \Big(A_i\Big)^+ + (A_0 - B)^+ \geqslant \Big(\sum_i A_i - B\Big)^+$$

□

LEMMA I.C [First order Stochastic Dominance]. *If random variable X has first-order stochastic dominance over random variable Y, then $\mathbb{E}[\phi(X)] \geqslant \mathbb{E}[\phi(Y)]$ for all increasing functions ϕ.*