Online appendix for "Selling bonus actions in video games"

Appendix A: Derivation of the four selling strategies for casual games

We consider casual games for which $\beta_H + \beta_L$ is sufficiently high. Specifically, we assume $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$. Below, we characterize the optimal prices and revenue under the four selling strategies (pure advance, pure spot, regular hybrid, and reverse hybrid). The optimal revenues under each selling strategy are denoted as Π^A, Π^S, Π^H, and Π^{RH} respectively. Without causing confusion, we denote the optimal prices as p_A^* and p_S^* without specifying the selling strategies. We let $\epsilon = \beta_H - \beta_L$.

Reverse HAS strategy

As we assume $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$, Lemma 2 implies that a reverse HAS strategy does not exist for casual games. That is, the firm can never set prices p_A and p_S such that low-skill players prefer buying in the spot but high-skill players prefer buying in advance. In this case, we simply let $\Pi^{RH} = 0$. ■

PAS strategy

Lemma A.1 *For casual games, if the firm commits to selling bonus actions only before the attempt, the optimal advance purchase price is*

$$p_A^* = \begin{cases} (1-\beta_L)(\beta_L P_B + v), & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L \\ (1-\beta_H)(\beta_H P_B + v), & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L. \end{cases}$$

The corresponding optimal revenue is

$$\Pi^A = \begin{cases} (1-\beta_L)(\beta_L P_B + v)N_L, & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L \\ (1-\beta_H)(\beta_H P_B + v)(N_H+N_L), & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L. \end{cases}$$

Proof of Lemma A.1: When the firm commits to selling bonus actions only before the attempt, a type i player will purchase bonus actions in the advance sales market if and only if $p_A \leq (1-\beta_i)(\beta_i P_B + v)$. The assumption $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$ results in $(1-\beta_L)(\beta_L P_B + v) \geq (1-\beta_H)(\beta_H P_B + v)$. As a result, the firm's optimization problem is given by

$$\max_{p_A \geq 0} \Pi(p_A) = \begin{cases} p_A(N_H + N_L), & \text{if } p_A \leq (1-\beta_H)(\beta_H P_B + v), \\ p_A N_L, & \text{if } (1-\beta_H)(\beta_H P_B + v) < p_A \leq (1-\beta_L)(\beta_L P_B + v), \\ 0, & \text{if } p_A > (1-\beta_L)(\beta_L P_B + v). \end{cases} \quad \text{(A.1)}$$

The firm's revenue is a piece-wise linear increasing function. Thus, the optimal price p_A^* is either $(1-\beta_L)(\beta_L P_B + v)$ or $(1-\beta_H)(\beta_H P_B + v)$, depending on whichever leads to a higher revenue. Therefore, it suffices to compare the revenues under these two candidate prices. We have

$$\Pi\left((1-\beta_L)(\beta_L P_B + v)\right) = (1-\beta_L)(\beta_L P_B + v)N_L,$$

$$\Pi\left((1-\beta_H)(\beta_H P_B+v)\right)=(1-\beta_H)(\beta_H P_B+v)(N_H+N_L).$$

Their difference is equivalent to

$$\begin{aligned}&\Pi\left((1-\beta_L)(\beta_L P_B+v)\right)-\Pi\left((1-\beta_H)(\beta_H P_B+v)\right)\\ &=(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]N_L-(1-\beta_H)(v+\beta_H P_B)N_H,\end{aligned}$$

from which we conclude that $\Pi((1-\beta_L)(\beta_L P_B+v))-\Pi((1-\beta_H)(\beta_H P_B+v))\geq 0$ if and only if $N_H\leq\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$.

As a result, the optimal advance purchase price will be

$$p_A^*=\begin{cases}(1-\beta_L)(\beta_L P_B+v), & \text{if } N_H\leq\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L\\ (1-\beta_H)(\beta_H P_B+v), & \text{if } N_H>\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L.\end{cases}$$

Following (A.1), the corresponding optimal revenue will be

$$\Pi^A=\begin{cases}(1-\beta_L)(\beta_L P_B+v)N_L, & \text{if } N_H\leq\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L,\\ (1-\beta_H)(\beta_H P_B+v)(N_H+N_L), & \text{if } N_H>\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L.\end{cases}$$

■

Regular HAS strategy

Lemma A.2 *For casual games, if the firm adopts the regular HAS strategy (that induces low-skilled players purchase before the attempt but high-skilled players purchase after failing the attempt), the optimal spot price is* $p_S^*=\begin{cases}\frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } v+(\beta_H-3\delta)P_B<0\\ v+(\beta_H-\delta)P_B, & \text{if } v+(\beta_H-3\delta)P_B\geq 0\end{cases}$, *and the optimal advance purchase price is*

$$p_A^*=\begin{cases}(1-\beta_L)\left(v+\beta_L P_B-\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\right), & \text{if } v+(\beta_H-3\delta)P_B<0,\\ (1-\beta_L)(v+\beta_L P_B), & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \epsilon\geq 2\delta,\\ (1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}], & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \epsilon<2\delta.\end{cases}$$

The corresponding optimal revenue is

$$\begin{aligned}&\Pi^H\\ &=\begin{cases}(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H+(1-\beta_L)[v+\beta_L P_B-\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v+(\beta_H-3\delta)P_B<0,\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H+(1-\beta_L)(v+\beta_L P_B)N_L, & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \epsilon\geq 2\delta,\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H+(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]N_L, & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \epsilon<2\delta.\end{cases}\end{aligned}$$

Proof of Lemma A.2: We solve the problem backwards. The firm first determines the price p_S to maximize its revenue in the spot market where only high-skill players will make purchases. We denote the firm's spot market revenue as Π_S. The firm's optimization problem in the spot market is given by

$$\max_{p_S\geq 0}\Pi_S(p_S)=p_S N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B-p_S\geq 0)]$$

$$= \begin{cases} p_S N_H(1-\beta_H), & \text{if } p_S \le v + (\beta_H - \delta)P_B \\ p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}, & \text{if } v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B \\ 0 & \text{if } p_S > v + (\beta_H + \delta)P_B. \end{cases}$$

Clearly, $\Pi_S(p_S)$ is continuous. When $p_S \le v + (\beta_H - \delta)P_B$, $\Pi_S(p_S)$ increases in p_S. When $v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$, $\Pi_S(p_S)$ is a concave quadratic function of p_S. We have

$$\frac{d\Pi_S(p_S)}{dp_S} = \frac{d\left(p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}\right)}{dp_S} = N_H(1-\beta_H)\frac{v + (\beta_H + \delta)P_B - 2p_S}{2\delta P_B}.$$

In particular, at $p_S = v + (\beta_H + \delta)P_B$, we obtain $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H + \delta)P_B} = -N_H(1-\beta_H)\frac{v + (\beta_H + \delta)P_B}{2\delta P_B} < 0$. At $p_S = v + (\beta_H - \delta)P_B$, we obtain $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H - \delta)P_B} = -N_H(1-\beta_H)\frac{v + (\beta_H - 3\delta)P_B}{2\delta P_B}$ which can be positive or negative.

If $v + (\beta_H - 3\delta)P_B \ge 0$, implying $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H - \delta)P_B} \le 0$, we can conclude that $\Pi_S(p_S)$ increases in p_S when $p_S \le v + (\beta_H - \delta)P_B$, and $\Pi_S(p_S)$ decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be $p_S^* = v + (\beta_H - \delta)P_B$.

If $v + (\beta_H - 3\delta)P_B < 0$, implying $\frac{d\Pi_S}{dp_S}|_{p_S = v + (\beta_H - \delta)P_B} > 0$, we can conclude that $\Pi_S(p_S)$ increases in p_S when $p_S \le v + (\beta_H - \delta)P_B$, and $\Pi_S(p_S)$ first increases and then decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition $\frac{d\Pi_S(p_S)}{dp_S} = 0$. That is, $p_S^* = \frac{v + (\beta_H + \delta)P_B}{2}$.

Given the optimal spot price p_S^*, the firm determines p_A to maximize its revenue from low-skill players in the advance sales market. We denote the firm's revenue in the advance sales market as Π_A. Thus, the optimization problem is given by

$$\begin{aligned} &\max_{p_A \ge 0} \Pi_A(p_A) = p_A N_L \\ &\qquad \text{s.t.} \quad p_A \le (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]\} \\ &\qquad\qquad\quad p_A > (1-\beta_H)\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+]\}. \end{aligned}$$

Following Lemma 2, since we assume $\beta_L + \beta_H \ge 1 - \frac{v}{P_B}$, there must exist p_A satisfying $(1-\beta_H)\{v + \beta_H P_B - \mathbb{E}[(v + \alpha_H P_B - p_S^*)^+] < p_A \le (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]\}$. To maximize its revenue, the firm should set p_A as high as possible. Therefore, the optimal advance purchase price should be $p_A = (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+]$. More specifically, given $p_S^* = \begin{cases} \frac{v + (\beta_H + \delta)P_B}{2}, & \text{if } v + (\beta_H - 3\delta)P_B < 0 \\ v + (\beta_H - \delta)P_B, & \text{if } v + (\beta_H - 3\delta)P_B \ge 0 \end{cases}$, we are able to derive

$$\begin{aligned} p_A^* &= (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v + \alpha_L P_B - p_S^*)^+] \\ &= \begin{cases} (1-\beta_L)\left(v + \beta_L P_B - \frac{[v + (2\beta_L + \delta - \beta_H)P_B]^2}{16\delta P_B}\right), & \text{if } v + (\beta_H - 3\delta)P_B < 0, \\ (1-\beta_L)(v + \beta_L P_B), & \text{if } v + (\beta_H - 3\delta)P_B \ge 0 \text{ and } \beta_H - \beta_L \ge 2\delta, \\ (1-\beta_L)[v + \beta_L P_B - \frac{(2\delta - \beta_H + \beta_L)^2 P_B}{4\delta}], & \text{if } v + (\beta_H - 3\delta)P_B \ge 0 \text{ and } \beta_H - \beta_L < 2\delta. \end{cases} \end{aligned}$$

Finally, the corresponding optimal revenue is

$$\Pi^H = p_A^* N_L + p_S^* N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S^* \geq 0)]. \quad \blacksquare$$

PSS strategy

Lemma A.3 *For casual games, if the firm commits to selling bonus actions only after the attempts fails, the optimal spot price is*

$$p_S^* = \begin{cases} v+(\beta_L-\delta)P_B, & \text{if } N_H \leq r_1 N_L \\ \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}, & \text{if } r_1 N_L < N_H \leq r_2 N_L \\ v+(\beta_H-\delta)P_B, & \text{if } r_2 N_L < N_H < r_3 N_L \\ \frac{(1-\beta_H)N_H[v+(\beta_H+\delta)P_B]+(1-\beta_L)N_L[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}, & \text{if } N_H \geq r_3 N_L, \end{cases}$$

where the three thresholds r_1, r_2, and r_3 are defined in Table 3. The corresponding optimal revenue is

$$\Pi^S = \begin{cases} [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)], & \text{if } N_H \leq r_1 N_L \\ \frac{[2(1-\beta_H)\delta N_H P_B + (1-\beta_L)N_L(v+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}, & \text{if } r_1 N_L < N_H < r_2 N_L \\ [v+(\beta_H-\delta)P_B]N_H(1-\beta_H), & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon \geq 2\delta \\ [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}] & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon < 2\delta, \\ \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]} & \text{if } N_H \geq r_3 N_L. \end{cases}$$

	Suppose $\epsilon = \beta_H - \beta_L < 2\delta$		
	r_1	r_2	r_3
If $v+(2\beta_H-\beta_L-3\delta)P_B \leq 0$	0	0	0
If $v+(\beta_H-3\delta)P_B$ $\leq 0 < v+(2\beta_H-\beta_L-3\delta)P_B$	0	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{(3\delta-\beta_H)P_B-v}$
If $v+(\beta_L-3\delta)P_B$ $\leq 0 < v+(\beta_H-3\delta)P_B$	0	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$	∞
If $0 < v+(\beta_L-3\delta)P_B$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$	∞
	Suppose $\epsilon = \beta_H - \beta_L \geq 2\delta$		
	r_1	r_2	r_3
If $v+(\beta_L-3\delta)P_B > 0$, and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$	∞
If $v+(\beta_L-3\delta)P_B > 0$, and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} > \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$	∞
If $v+(\beta_L-3\delta)P_B \leq 0$,	0	$\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$	∞

Table 3 **Thresholds r_1, r_2, and r_3 for the determining the optimal spot price for hardcore games**

Proof of Lemma A.3: When the firm commits to selling bonus actions only in the spot market, its optimization problem is given by

$$\max_{p_S \geq 0} \Pi(p_S) = p_S\{N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S \geq 0)] + N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B - p_S \geq 0)]\}.$$

Since we assume α_i follows a uniform distribution $U[\beta_i - \delta, \beta_i + \delta]$ for $i = H, L$ and $\beta_L < \beta_H$, the firm's revenue $\Pi(p_S)$ will be a piece-wise continuous function. We consider two scenarios: (I) Suppose $\beta_H - \beta_L < 2\delta$, which is equivalent to $\beta_H - \delta < \beta_L + \delta$. Then the support of α_H has overlap with that of α_L; (II) Suppose $\beta_H - \beta_L \geq 2\delta$, which is equivalent to $\beta_H - \delta \geq \beta_L + \delta$. Then, the support of α_H does not overlap with that of α_L.

We start with Scenario (I). We explicitly express the firm's revenue $\Pi(p_S)$ to be

$$
\begin{aligned}
\Pi(p_S) &= p_S\{N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S \geq 0)] + N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B - p_S \geq 0)]\} \\
&= \begin{cases}
p_S\{N_H(1-\beta_H) + N_L(1-\beta_L)\}, & \text{if } p_S \leq v + (\beta_L - \delta)P_B \\
p_S\{N_H(1-\beta_H) + N_L(1-\beta_L)\frac{(\beta_L + \delta - \frac{p_S - v}{P_B})}{2\delta}\}, & \text{if } v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B \\
p_S\{N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta} & \\
\qquad + N_L(1-\beta_L)\frac{(\beta_L + \delta - \frac{p_S - v}{P_B})}{2\delta}\}, & \text{if } v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B \\
p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}, & \text{if } v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B \\
0, & \text{if } p_S > v + (\beta_H + \delta)P_B.
\end{cases}
\end{aligned}
\tag{A.2}
$$

It is straightforward to see that the first piece of $\Pi(p_S)$ when $p_S \leq v + (\beta_L - \delta)P_B$ is a linear increasing function of p_S, while the rest three pieces when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$ are concave quadratic functions of of p_S.

To determine the monotonicity of $\Pi(p_S)$, we would like to investigate its first-order derivative at the kink points. Because $\Pi(p_S)$ may not be smooth, we denote $\frac{d\Pi_S(p_S^0+)}{dp_s}$ as the right derivative when p_S approaches to p_S^0 from the right, and $\frac{d\Pi_S(p_S^0-)}{dp_s}$ as the left derivative when p_S approaches to p_S^0 from the left. We have

$$
\begin{aligned}
\frac{d\Pi_S([v + (\beta_L - \delta)P_B]-)}{dp_s} &= N_H(1-\beta_H) + N_L(1-\beta_L), \\
\frac{d\Pi_S([v + (\beta_L - \delta)P_B]+)}{dp_s} &= N_H(1-\beta_H) - N_L(1-\beta_L)\frac{v + (\beta_L - 3\delta)P_B}{2\delta P_B}, \\
\frac{d\Pi_S([v + (\beta_H - \delta)P_B]-)}{dp_s} &= N_H(1-\beta_H) - N_L(1-\beta_L)\frac{v + (2\beta_H - \beta_L - 3\delta)P_B}{2\delta P_B}, \\
\frac{d\Pi_S([v + (\beta_H - \delta)P_B]+)}{dp_s} &= -N_H(1-\beta_H)\frac{v + (\beta_H - 3\delta)P_B}{2\delta P_B} - N_L(1-\beta_L)\frac{v + (2\beta_H - \beta_L - 3\delta)P_B}{2\delta P_B}, \\
\frac{d\Pi_S([v + (\beta_L + \delta)P_B]-)}{dp_s} &= -N_H(1-\beta_H)\frac{v + (2\beta_L + \delta - \beta_H)P_B}{2\delta P_B} - N_L(1-\beta_L)\frac{v + (\beta_L + \delta)P_B}{2\delta P_B}, \\
\frac{d\Pi_S([v + (\beta_L + \delta)P_B]+)}{dp_s} &= -N_H(1-\beta_H)\frac{v + (2\beta_L + \delta - \beta_H)P_B}{2\delta P_B}, \\
\frac{d\Pi_S([v + (\beta_H + \delta)P_B]-)}{dp_s} &= -N_H(1-\beta_H)\frac{v + (\beta_H + \delta)P_B}{2\delta P_B}.
\end{aligned}
$$

We make several observations. First, $\frac{d\Pi_S([v+(\beta_L - \delta)P_B]-)}{dp_s} \geq 0$, meaning that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_L - \delta)P_B$. Second, Scenario (I) assumes $\beta_H - \beta_L < 2\delta$, we obtain $(2\beta_L +$

$\delta - \beta_H) = (\beta_L - \delta) + (\beta_L + 2\delta - \beta_H) > 0$. Therefore, $\frac{d\Pi_S([v+(\beta_H+\delta)P_B]-)}{dp_s} < 0$, $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]+)}{dp_s} < 0$, and $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} < 0$, which implies that $\Pi(p_S)$ decreases in p_S when $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. Furthermore, we have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$.

When $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, there are four possible cases:

(I-a): Suppose $v + (\beta_L - 3\delta)P_B \leq v + (\beta_H - 3\delta)P_B \leq v + (2\beta_H - \beta_L - 3\delta)P_B \leq 0$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$, $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} > 0$, and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} > 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, it first increases and then decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, and it decreases in p_S when $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, which is equivalent to

$$\frac{d\Pi(p_S)}{dp_S} = N_H(1 - \beta_H)\frac{v + (\beta_H + \delta)P_B - 2p_S}{2\delta P_B} + N_L(1 - \beta_L)\frac{v + (\beta_L + \delta)P_B - 2p_S}{2\delta P_B} = 0.$$

And we solve $p_S^* = \frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]+(1-\beta_L)[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}$.

(I-b): Suppose $v + (\beta_L - 3\delta)P_B \leq v + (\beta_H - 3\delta)P_B \leq 0 < v + (2\beta_H - \beta_L - 3\delta)P_B$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$. However, $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$ and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$ (satisfying $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$) can be positive or negative.

(I-b-1): Suppose $N_H \geq N_L \left(\frac{1-\beta_L}{1-\beta_H}\right) \frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{[(3\delta-\beta_H)P_B-v]}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} \geq 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, it first increases and then decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$, and it decreases in p_S when $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the same as (I-a), that is $p_S^* = \frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]+(1-\beta_L)[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}$.

(I-b-2): Suppose $N_H \leq N_L \left(\frac{1-\beta_L}{1-\beta_H}\right) \frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $0 \geq \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_L - \delta)P_B$, it first increases and then decreases in p_S when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$, and it decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$, which is equivalent to

$$\frac{d\Pi(p_S)}{dp_S} = N_H(1 - \beta_H) + N_L(1 - \beta_L)\frac{v + (\beta_L + \delta)P_B - 2p_S}{2\delta P_B} = 0. \tag{A.3}$$

And we solve $p_S^* = \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

(I-b-3): Suppose $N_L \left(\frac{1-\beta_L}{1-\beta_H}\right) \frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B} < N_H < N_L \left(\frac{1-\beta_L}{1-\beta_H}\right) \frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{(3\delta-\beta_H)P_B-v}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} > 0 > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, and it decreases in p_S when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$. As a result, the optimal spot price should be $p_S^* = v + (\beta_H - \delta)P_B$.

(I-c): Suppose $v+(\beta_L-3\delta)P_B \le 0 < v+(\beta_H-3\delta)P_B \le v+(2\beta_H-\beta_L-3\delta)P_B$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$ and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} < 0$. But $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$ can be positive or negative.

(I-c-1): Suppose $N_H \ge N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} \ge 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_H-\delta)P_B$.

(I-c-2): Suppose $N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} < 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, it first increases and then decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition (A.3), that is $p_S^* = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

(I-d): Suppose $0 < v+(\beta_L-3\delta)P_B \le v+(\beta_H-3\delta)P_B \le v+(2\beta_H-\beta_L-3\delta)P_B$.

We obtain $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s} < 0$. However, $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}$ and $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$ (satisfying $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$) can be positive or negative.

(I-d-1): Suppose $N_H \ge N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s} \ge 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_H-\delta)P_B$.

(I-d-2): Suppose $N_H \le N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B}$.

We have $0 \ge \frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, and it decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_L-\delta)P_B$.

(I-d-3): Suppose $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B} < N_H < \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s} > 0 > \frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, it first increases and then decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be the unique solution of the first-order condition (A.3), that is $p_S^* = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

Finally, in Scenario (I) with $\beta_H-\beta_L < 2\delta$, we define $r_1 = r_2 = r_3 = 0$ if $v+(\beta_L-3\delta)P_B \le v+(\beta_H-3\delta)P_B \le v+(2\beta_H-\beta_L-3\delta)P_B \le 0$. We define $r_1 = 0$, $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$, and $r_3 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{(3\delta-\beta_H)P_B-v}$ if $v+(\beta_L-3\delta)P_B \le v+(\beta_H-3\delta)P_B \le 0 < v+(2\beta_H-\beta_L-3\delta)P_B$. We define $r_1 = 0$, $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$, and $r_3 = \infty$ if $v+(\beta_L-3\delta)P_B \le 0 < v+(\beta_H-3\delta)P_B \le$

$v+(2\beta_H-\beta_L-3\delta)P_B$. We define $r_1=\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B}$, $r_2=\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(2\beta_H-\beta_L-3\delta)P_B]}{2\delta P_B}$, and $r_3=\infty$ if $0<v+(\beta_L-3\delta)P_B\le v+(\beta_H-3\delta)P_B\le v+(2\beta_H-\beta_L-3\delta)P_B$.

From the above analysis, we conclude that the optimal spot price p_S^* will be

$$p_S^*=\begin{cases} v+(\beta_L-\delta)P_B, & \text{if } N_H\le r_1N_L\\ \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}+\frac{v+(\beta_L+\delta)P_B}{2}, & \text{if } r_1N_L<N_H\le r_2N_L\\ v+(\beta_H-\delta)P_B, & \text{if } r_2N_L<N_H<r_3N_L\\ \frac{(1-\beta_H)N_H[v+(\beta_H+\delta)P_B]+(1-\beta_L)N_L[v+(\beta_L+\delta)P_B]}{2[(1-\beta_H)N_H+(1-\beta_L)N_L]}, & \text{if } N_H\ge r_3N_L. \end{cases}$$

■

Next, we consider Scenario (II). We explicitly express the firm's revenue $\Pi(p_S)$ to be

$$\begin{aligned}\Pi(p_S)&=p_S\{N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_HP_B-p_S\ge 0)]+N_L(1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_LP_B-p_S\ge 0)]\}\\ &=\begin{cases} p_S\{N_H(1-\beta_H)+N_L(1-\beta_L)\}, & \text{if } p_S\le v+(\beta_L-\delta)P_B\\ p_S\{N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(\beta_L+\delta-\frac{p_S-v}{P_B})}{2\delta}\}, & \text{if } v+(\beta_L-\delta)P_B<p_S\le v+(\beta_L+\delta)P_B\\ p_SN_H(1-\beta_H), & \text{if } v+(\beta_L+\delta)P_B<p_S\le v+(\beta_H-\delta)P_B\\ p_SN_H(1-\beta_H)\frac{(\beta_H+\delta-\frac{p_S-v}{P_B})}{2\delta}, & \text{if } v+(\beta_H-\delta)P_B<p_S\le v+(\beta_H+\delta)P_B\\ 0, & \text{if } p_S>v+(\beta_H+\delta)P_B. \end{cases}\end{aligned} \tag{A.4}$$

We apply a similar analysis as in Scenario (I). Specifically, we examine the first-order derivative at the kink points. We have

$$\begin{aligned}
\frac{d\Pi_S([v+(\beta_L-\delta)P_B]-)}{dp_s}&=N_H(1-\beta_H)+N_L(1-\beta_L),\\
\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}&=N_H(1-\beta_H)-N_L(1-\beta_L)\frac{v+(\beta_L-3\delta)P_B}{2\delta P_B},\\
\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}&=N_H(1-\beta_H)-N_L(1-\beta_L)\frac{v+(\beta_L+\delta)P_B}{2\delta P_B},\\
\frac{d\Pi_S([v+(\beta_L+\delta)P_B]+)}{dp_s}&=N_H(1-\beta_H),\\
\frac{d\Pi_S([v+(\beta_H-\delta)P_B]-)}{dp_s}&=N_H(1-\beta_H),\\
\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}&=-N_H(1-\beta_H)\frac{v+(\beta_H-3\delta)P_B}{2\delta P_B},\\
\frac{d\Pi_S([v+(\beta_H+\delta)P_B]-)}{dp_s}&=-N_H(1-\beta_H)\frac{v+(\beta_H+\delta)P_B}{2\delta P_B}.
\end{aligned}$$

We make several observations. First, $\Pi(p_S)$ increases in p_S when $p_S\le v+(\beta_L-\delta)P_B$ and when $v+(\beta_L+\delta)P_Bp_S\le v+(\beta_H-\delta)P_B$. Second, Scenario (II) assumes $\beta_H-\beta_L\ge 2\delta$ and we also assume $\beta_L\ge\delta$, we obtain $\beta_H\ge\beta_L+2\delta\ge 3\delta$. Hence, $\frac{d\Pi_S([v+(\beta_H-\delta)P_B]+)}{dp_s}<0$, implying that $\Pi(p_S)$ decreases in p_S when $v+(\beta_H-\delta)P_B<p_S\le v+(\beta_H+\delta)P_B$. In addition, we have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}>\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$.

When $v+(\beta_L-\delta)P_B<p_S\le v+(\beta_L+\delta)P_B$, there are two possible cases:

(II-a): Suppose $v+(\beta_L-3\delta)P_B \le 0$.

We obtain $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0$. But $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$ can be positive or negative.

(II-a-1): Suppose $N_H \ge N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} \ge 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As a result, the optimal spot price should be $p_S^* = v+(\beta_H+\delta)P_B$.

(II-a-2): Suppose $N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} < 0$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \le v+(\beta_L-\delta)P_B$, it first increases and then decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \le v+(\beta_L+\delta)P_B$, then it increases in p_S when $v+(\beta_L+\delta)P_B < p_S \le v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \le v+(\beta_H+\delta)P_B$. As we can see, $\Pi(p_S)$ has two peaks at $p_S = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$ and $p_S = v+(\beta_H-\delta)P_B$.

We compare the firm's revenues at these two peaks which are equal to

$$\Pi\left(v+(\beta_H-\delta)P_B\right) = [v+(\beta_H-\delta)P_B]N_H(1-\beta_H),$$
$$\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) = \frac{[2(1-\beta_H)\delta N_H P_B + (1-\beta_L)N_L(1+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}.$$

In particular, we investigate the ratio of the above revenues which can be simplified to be

$$\frac{\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right)}{\Pi\left(v+(\beta_H-\delta)P_B\right)} \tag{A.5}$$
$$= \frac{\delta P_B}{2[v+(\beta_H-\delta)P_B]}\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) + \frac{[v+(\beta_L+\delta)P_B]}{2[v+(\beta_H-\delta)P_B]} + \frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B[v+(\beta_H-\delta)P_B]}\left(\frac{N_L(1-\beta_L)}{N_H(1-\beta_H)}\right).$$

One can view (A.5) as a function of $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right)$. It can be easily verify that the ratio (A.5) decreases in $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right)$ when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) \le \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$. When $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) = \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$, we have (A.5) $= \frac{v+(\beta_L+\delta)P_B}{v+(\beta_H+\delta)P_B} \le 1$. Therefore, there exists a unique solution $\hat{x}$ satisfying $0 < \hat{x} < \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$ and

$$\frac{\delta P_B}{2[v+(\beta_H-\delta)P_B]}\hat{x} + \frac{[v+(\beta_L+\delta)P_B]}{2[v+(\beta_H-\delta)P_B]} + \frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B[v+(\beta_H-\delta)P_B]}\frac{1}{\hat{x}} = 1.$$

We solve out

$$\hat{x} = \frac{[v+(2\beta_H-\beta_L-3\delta)P_B] - 2\sqrt{(\beta_H-\beta_L-2\delta)P_B[v+(\beta_H-\delta)P_B]}}{2\delta P_B}.$$

Finally, when $N_H \le N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$, we have $\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) \ge \Pi\left(v+(\beta_H-\delta)P_B\right)$. As a result, the optimal spot price will be $p_S^* = \delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}$.

But when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$, we have $\Pi\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}\right) < \Pi\left(v+(\beta_H-\delta)P_B\right)$. As a result, the optimal spot price will be $p_S^* = v+(\beta_H-\delta)P_B$.

(II-b): Suppose $v+(\beta_L-3\delta)P_B>0$.

Then $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s}$ and $\frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$ can be positive or negative.

(II-b-1): Suppose $N_H \geq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > \frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s} \geq 0$. The result will be the same as (II-a-1).

(II-b-2): Suppose $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}\} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta w}$.

We have $\frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > 0 > \frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$. Similarly as (II-a-2), $\Pi(p_S)$ has two peaks at $p_S=\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}+\frac{v+(\beta_L+\delta)P_B}{2}$ and $p_S=v+(\beta_H-\delta)P_B$. We need to investigate the ratio of their corresponding revenues (A.5).

Previously, we have shown that (A.5) decreases in $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right)$ when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) \leq \frac{[v+(\beta_L+\delta)P_B]}{2\delta P_B}$. Furthermore, when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) \leq \hat{x}$, we have (A.5) ≥ 1; and when $\left(\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}\right) > \hat{x}$, we have (A.5) < 1.

What is left-over is to compare $\hat{x}$ with $\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. We are able to show that $\hat{x} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$ if and only if $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

(II-b-2.1): Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, whenever $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}\} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta w}$, it implies $N_H > N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$. Hence, we always have $\Pi_S\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}+\frac{v+(\beta_L+\delta)P_B}{2}\right) < \Pi_S\left(v+(\beta_H-\delta)P_B\right)$. As a result, the optimal spot price should be $p_S^*=v+(\beta_H-\delta)P_B$.

(II-b-2.2): Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} > \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}\} < N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$, we have $\Pi_S\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}+\frac{v+(\beta_L+\delta)P_B}{2}\right) \geq \Pi_S\left(v+(\beta_H-\delta)P_B\right)$. The optimal spot price should be $p_S^*=\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}+\frac{v+(\beta_L+\delta)P_B}{2}$. But when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x} < N_H < N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L+\delta)P_B]}{2\delta w}$, we have $\Pi_S\left(\delta P_B\frac{N_H(1-\beta_H)}{N_L(1-\beta_L)}+\frac{v+(\beta_L+\delta)P_B}{2}\right) < \Pi_S\left(v+(\beta_H-\delta)P_B\right)$. The optimal spot price should be $p_S^*=v+(\beta_H-\delta)P_B$.

(II-b-3): Suppose $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

We have $0 \geq \frac{d\Pi_S([v+(\beta_L-\delta)P_B]+)}{dp_s} > \frac{d\Pi_S([v+(\beta_L+\delta)P_B]-)}{dp_s}$. Thus, we conclude that $\Pi(p_S)$ increases in p_S when $p_S \leq v+(\beta_L-\delta)P_B$, it decreases in p_S when $v+(\beta_L-\delta)P_B < p_S \leq v+(\beta_L+\delta)P_B$, then it increases in p_S when $v+(\beta_L+\delta)P_B < p_S \leq v+(\beta_H-\delta)P_B$, and it decreases in p_S when $v+(\beta_H-\delta)P_B < p_S \leq v+(\beta_H+\delta)P_B$. Hence, $\Pi(p_S)$ has two peaks at $p_S=v+(\beta_L-\delta)P_B$ and $p_S=v+(\beta_H-\delta)P_B$.

We compare the firm's revenues at these two peaks which are equal to

$$\Pi\left(v+(\beta_L-\delta)P_B\right)=[v+(\beta_L-\delta)P_B]\{N_H(1-\beta_H)+N_L(1-\beta_L)\},$$

$$\Pi\left(v+(\beta_L-\delta)P_B\right)=[v+(\beta_H-\delta)P_B]N_H(1-\beta_H).$$

It is easy to see that $\Pi(v+(\beta_L-\delta)P_B) \geq \Pi(v+(\beta_H-\delta)P_B)$ if and only if $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$. However, we need to compare the two thresholds $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$ and $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. It turns out that $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$ may be greater or less than $\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

(II-b-3.1) Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B]} \geq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, whenever $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$, it implies that $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B}{(\beta_H-\beta_L)P_B}$. Hence, we always have $\Pi(v+(\beta_L-\delta)P_B) \geq \Pi(v+(\beta_H-\delta)P_B)$. As a result, the optimal spot price should be $p_S^* = v+(\beta_L-\delta)P_B$.

(II-b-3.2) Suppose $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} < \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$.

In this case, when $N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B}{(\beta_H-\beta_L)P_B}$, we have $\Pi(v+(\beta_L-\delta)P_B) \geq \Pi(v+(\beta_H-\delta)P_B)$. The optimal spot price should be $p_S^* = v+(\beta_L-\delta)P_B$. But when $N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B}{(\beta_H-\beta_L)P_B} < N_H \leq N_L\left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$, we have $\Pi(v+(\beta_L-\delta)P_B) < \Pi(v+(\beta_H-\delta)P_B)$. The optimal spot price should be $p_S^* = v+(\beta_H-\delta)P_B$.

Finally, in Scenario (II) with $\beta_H-\beta_L \geq 2\delta$, we define $r_1 = r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B}$ if $v+(\beta_L-3\delta)P_B \geq 0$ and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} \leq \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. We define $r_1 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$ and $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$ if $v+(\beta_L-3\delta)P_B \geq 0$ and $\frac{[v+(\beta_L-\delta)P_B]}{(\beta_H-\beta_L)P_B} > \frac{[v+(\beta_L-3\delta)P_B]}{2\delta P_B}$. We define $r_1 = 0$ and $r_2 = \left(\frac{1-\beta_L}{1-\beta_H}\right)\hat{x}$ if $v+(\beta_L-3\delta)P_B < 0$. We define $r_3 = \infty$.

From the analysis above, we conclude that the optimal spot price p_S^* will be

$$p_S^* = \begin{cases} v+(\beta_L-\delta)P_B, & \text{if } N_H \leq r_1 N_L \\ \delta P_B \frac{N_H(1-\beta_H)}{N_L(1-\beta_L)} + \frac{v+(\beta_L+\delta)P_B}{2}, & \text{if } r_1 N_L < N_H < r_2 N_L \\ v+(\beta_H-\delta)P_B, & \text{if } r_2 N_L \leq N_H < r_3 N_L. \end{cases}$$

■

Lastly, the firm's optimal revenue follows from (A.2) and (A.4). ■

Appendix B: Derivation of the four selling strategies for hardcore games

We consider hardcore games for which $\beta_H+\beta_L$ is relatively low. Specifically, we assume $\beta_L+\beta_H < 1-\frac{v}{P_B}$. Below, we characterize the optimal prices and revenue under the four selling strategies (pure advance, pure spot, regular hybrid, and reverse hybrid). Without causing confusion, we denote the optimal revenues under each selling strategy as Π^A, Π^S, Π^H, and Π^{RH} respectively. We denote the optimal prices as p_A^* and p_S^* without specifying the selling strategies. Recall that $\epsilon = \beta_H-\beta_L$.

PSS strategy

Notice that in the proof of Lemma A.3, the assumption $\beta_L+\beta_H \geq 1-\frac{v}{P_B}$ does not play a role at all. In other words, whether $\beta_H+\beta_L$ is greater or less than $(1-\frac{v}{P_B})$ does not have an impact on a pure spot strategy. Therefore, the optimal spot price and revenue Π^S will be the same as Lemma A.3. ■

Regular HAS strategy

Lemma A.4 *For hardcore games, the optimal regular HAS strategy exists (i.e., there exist p_A and p_S satisfying (5)-(7)) if and only if one of the following conditions holds:*

(1): $v+(\beta_H-3\delta)P_B \geq 0$, $\beta_H-\beta_L \geq 2\delta$, and $(1-\beta_H)[v+(\beta_H-\delta)P_B] < (1-\beta_L)(v+\beta_L P_B)$;

(2): $v+(\beta_H-3\delta)P_B \geq 0$, $\beta_H-\beta_L < 2\delta$, and $(1-\beta_H)[v+(\beta_H-\delta)P_B] < (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]$;

(3): $v+(\beta_H-3\delta)P_B < 0$, and $(1-\beta_H)[v+\beta_H P_B - \frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}] < (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}]$.

Suppose one of the three conditions hold and the optimal regular HAS strategy exists. The optimal spot price is $p_S^* = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } v+(\beta_H-3\delta)P_B<0 \\ v+(\beta_H-\delta)P_B, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \end{cases}$, and the optimal advance purchase price is

$$p_A^* = \begin{cases} (1-\beta_L)\left(v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\right), & \text{if } v+(\beta_H-3\delta)P_B<0, \\ (1-\beta_L)(v+\beta_L P_B), & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta, \\ (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}], & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta. \end{cases}$$

The corresponding optimal revenue is

$$\Pi^H$$
$$= \begin{cases} (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v+(\beta_H-3\delta)P_B<0, \\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta, \\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta. \end{cases}$$

Proof of Lemma A.4: The proof for the optimal spot price p_S^* is the same as Lemma A.2. But when $\beta_L+\beta_H < 1-\frac{v}{P_B}$, there may not exist any p_A satisfying the IC constraints $(1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+] < p_A \leq (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\}$.

Suppose $v+(\beta_H-3\delta)P_B \geq 0$. Then the optimal spot price is $p_S^* = v+(\beta_H-\delta)P_B$. We have

$$(1-\beta_H)\left\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\} = (1-\beta_H)[v+(\beta_H-\delta)P_B],$$
$$(1-\beta_L)\left\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\right\} = \begin{cases} (1-\beta_L)(v+\beta_L P_B), & \text{if } \beta_H-\beta_L \geq 2\delta, \\ (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}], & \text{if } \beta_H-\beta_L < 2\delta. \end{cases}$$

Suppose $v+(\beta_H-3\delta)P_B < 0$. Then the optimal spot price is $p_S^* = \frac{v+(\beta_H+\delta)P_B}{2}$. In addition, we have $v+(\beta_L-\delta)P_B \leq v+(\beta_H-\delta)P_B \leq \frac{v+(\beta_H+\delta)P_B}{2} \leq v+(\beta_L+\delta)P_B \leq v+(\beta_H+\delta)P_B$. Thus,

$$(1-\beta_H)\left\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\} = (1-\beta_H)[v+\beta_H P_B - \frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}],$$
$$(1-\beta_L)\left\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\right\} = (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}].$$

For the existence of the optimal regular HAS strategy, equivalently the existence of p_A satisfying $(1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+] < p_A \leq (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\}$, we have to require $(1-\beta_H)\{v+\beta_H P_B-\mathbb{E}[(v+\alpha_H P_B-p_S^*)^+]<(1-\beta_L)\{v+\beta_L P_B-\mathbb{E}[(v+\alpha_L P_B-p_S^*)^+]\}$. Specifically, $(1-\beta_H)[v+(\beta_H-\delta)P_B]<(1-\beta_L)(v+\beta_L P_B)$ when $v+(\beta_H-3\delta)P_B\geq 0$ and $\beta_H-\beta_L\geq 2\delta$; or $(1-\beta_H)[v+(\beta_H-\delta)P_B]<(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]$ when $v+(\beta_H-3\delta)P_B\geq 0$ and $\beta_H-\beta_L<2\delta$; or $(1-\beta_H)[v+\beta_H P_B-\frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}]<(1-\beta_L)[v+\beta_L P_B-\frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}]$ when $v+(\beta_H-3\delta)P_B<0$.

Finally, if there exists a feasible p_A satisfying the IC constraints, then the optimal advance purchase price should be $p_A^*=(1-\beta_L)\{v+\beta_L P_B-\mathbb{E}[(v+\alpha_L P_B-p_S^*)^+]\}$ which is the same as Lemma A.2. And the corresponding optimal revenue will be the same as Lemma A.2 as well. ■

PAS strategy

Lemma A.5 *For hardcore games, if the firm commits to selling bonus actions only before the attempt, the optimal advance purchase price is*

$$p_A^*=\begin{cases}(1-\beta_L)(v+\beta_L P_B), & \text{if } N_H\leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L\\ (1-\beta_H)(v+\beta_H P_B), & \text{if } N_H> \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L.\end{cases}$$

The corresponding optimal revenue is

$$\Pi^A=\begin{cases}(1-\beta_L)(v+\beta_L P_B)(N_H+N_L), & \text{if } N_H\leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L\\ (1-\beta_H)(v+\beta_H P_B)N_H, & \text{if } N_H> \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L.\end{cases}$$

Proof of Lemma A.5:

Recall that when the firm commits to selling bonus actions only before the attempt, a type i player will purchase bonus actions in the advance sales market if and only if $p_A\leq(1-\beta_i)(\beta_i P_B+v)$. For hardcore games, we assume $\beta_L+\beta_H<1-\frac{v}{P_B}$, resulting in $(1-\beta_L)(\beta_L P_B+v)<(1-\beta_H)(\beta_H P_B+v)$. As a result, the firm's optimization problem is given by

$$\max_{p_A\geq 0}\Pi(p_A)=\begin{cases}p_A(N_H+N_L), & \text{if } p_A\leq(1-\beta_L)(\beta_L P_B+v),\\ p_A N_H, & \text{if } (1-\beta_L)(\beta_L P_B+v)<p_A\leq(1-\beta_H)(\beta_H P_B+v),\\ 0, & \text{if } p_A>(1-\beta_H)(\beta_H P_B+v).\end{cases} \tag{A.6}$$

As we can see, the optimal price p_A^* is either $(1-\beta_L)(\beta_L P_B+v)$ or $(1-\beta_H)(\beta_H P_B+v)$, depending on whichever leads to a higher revenue. We compare the revenues under these two candidate prices. We have

$$\Pi\left((1-\beta_L)(\beta_L P_B+v)\right)=(1-\beta_L)(\beta_L P_B+v)(N_H+N_L),$$
$$\Pi\left((1-\beta_H)(\beta_H P_B+v)\right)=(1-\beta_H)(\beta_H P_B+v)N_H.$$

Their difference is equivalent to

$$\Pi\left((1-\beta_L)(\beta_L P_B+v)\right)-\Pi\left((1-\beta_H)(\beta_H P_B+v)\right)$$

$$= (1-\beta_L)(\beta_L P_B + v)N_L - (\beta_H - \beta_L)[(1-\beta_H-\beta_L)P_B - v]N_H.$$

Notice that $\beta_L + \beta_H < 1 - \frac{v}{P_B}$ is equivalent to $(1-\beta_H-\beta_L)P_B - v > 0$. We conclude that $\Pi((1-\beta_L)(\beta_L P_B + v)) - \Pi((1-\beta_H)(\beta_H P_B + v)) \geq 0$ if and only if $N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B - v]} N_L$.

As a result, the optimal advance purchase price is

$$p_A^* = \begin{cases} (1-\beta_L)(v+\beta_L P_B), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L \\ (1-\beta_H)(v+\beta_H P_B), & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L. \end{cases}$$

Following (A.6), the corresponding optimal revenue will be

$$\Pi^A = \begin{cases} (1-\beta_L)(v+\beta_L P_B)(N_H+N_L), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L \\ (1-\beta_H)(v+\beta_H P_B)N_H, & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L. \end{cases}$$

$\blacksquare$

Reverse HAS strategy

Lemma A.6 *For hardcore games, the optimal reverse HAS strategy exists (i.e., there exist p_A and p_A satisfying (8)-(10)) if and only if (1) $\beta_L \leq (1-\beta_H) - \frac{v}{P_B}$; and (2) $\beta_L < 3\delta - \frac{v}{P_B}$; and one of the following conditions holds:*

(3.1) $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0$ *and* $(1-\beta_L)\left[v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}$.

Or,

(3.2) $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$ *and* $(1-\beta_L)\left[v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$.

Suppose the conditions hold and the optimal reverse HAS strategy exists. The optimal spot price is $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$, and the optimal advance purchase price is

$$p_A^* = \begin{cases} (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \textit{if } v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0, \\ (1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}], & \textit{if } v + (2\beta_H - \beta_L - 3\delta)P_B < 0. \end{cases}$$

The corresponding optimal revenue is

$$\Pi^{RH} = \begin{cases} N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \textit{if } v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0, \\ N_H(1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \textit{if } v + (2\beta_H - \beta_L - 3\delta)P_B < 0. \end{cases}$$

Proof of Lemma A.6:

First of all, Lemma 2 implies that for the existence of the optimal reverse HAS strategy, we must require $v + (\beta_H + \beta_L - 1)P_B < 0$.

Next, we solve the problem backwards. The firm first determines the price p_S to maximize its revenue in the spot market where only low-skill players will make purchases. Recall that we denote the firm's spot market revenue as Π_S. Thus, the firm's optimization problem is given by

$$\max_{p_S \geq 0} \Pi_S(p_S) = p_S N_L (1-\beta_L)\mathbb{E}[\mathbb{1}(v + \alpha_L P_B - p_S \geq 0)]$$

$$= \begin{cases} p_S N_L(1-\beta_L), & \text{if } p_S \le v + (\beta_L - \delta)P_B, \\ p_S N_L(1-\beta_L)\frac{(\beta_L+\delta-\frac{p_S-v}{P_B})}{2\delta}, & \text{if } v+(\beta_L-\delta)P_B < p_S \le v+(\beta_L+\delta)P_B, \\ 0, & \text{if } p_S > v+(\beta_L+\delta)P_B. \end{cases}$$

The analysis for the optimal spot price p_S^* will be the same as Lemma A.2, except that we change the subscript from H to L. We conclude that if $v + (\beta_L - 3\delta)P_B \ge 0$, the optimal spot price is $p_S^* = v + (\beta_L - \delta)P_B$. If $v + (\beta_L - 3\delta)P_B < 0$, the optimal spot price is $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$.

Given the optimal spot price p_S^*, the firm determines p_A to maximize its revenue from high-skill players in the advance sales market. Recall that Π_A represents the firm's revenue in the advance sales market. Therefore, the firm's optimization problem is given by

$$\begin{aligned} \max_{p_A \ge 0} \; & \Pi_A(p_A) = p_A N_H \\ \text{s.t.} \quad & p_A > (1-\beta_L)\{v + \beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\} \\ & p_A \le (1-\beta_H)\{v + \beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\}. \end{aligned}$$

We examine the existence of p_A satisfying $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < p_A \le (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$.

Suppose $v + (\beta_L - 3\delta)P_B \ge 0$. Then, the optimal spot price is $p_S^* = v + (\beta_L - \delta)P_B$. We have

$$\begin{aligned} (1-\beta_L)\left\{v + \beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\right\} &= (1-\beta_L)[v + (\beta_L - \delta)P_B], \\ (1-\beta_H)\left\{v + \beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\} &= (1-\beta_H)[v + (\beta_L - \delta)P_B]. \end{aligned}$$

We obtain $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] \ge (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. Hence, there cannot exist any p_A satisfying the IC constraints $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < p_A \le (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. That is, for the existence of the optimal reverse HAS strategy, we must also require $v + (\beta_L - 3\delta)P_B < 0$, equivalently $\beta_L < 3\delta - \frac{v}{P_B}$.

Suppose $v + (\beta_L - 3\delta)P_B < 0$. Then, the optimal spot price is $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$. We have

$$(1-\beta_L)\left\{v + \beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]\right\} = (1-\beta_L)\left[v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right],$$

and

$$\begin{aligned} & (1-\beta_H)\left\{v + \beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\} \\ & = \begin{cases} (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \text{if } \frac{v+(\beta_L+\delta)P_B}{2} \le v + (\beta_H - \delta)P_B, \\ (1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right], & \text{if } \frac{v+(\beta_L+\delta)P_B}{2} > v + (\beta_H - \delta)P_B. \end{cases} \end{aligned}$$

For the existence of the optimal reverse HAS strategy, equivalently the existence of p_A satisfying $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < p_A \le (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$, we have to require $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+] < (1-\beta_H)\{v+\beta_H P_B -$

$\mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. Specifically, $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}$ when $\frac{v+(\beta_L+\delta)P_B}{2} \le v+(\beta_H-\delta)P_B$ (which is also equivalent to $v+(2\beta_H-\beta_L-3\delta)P_B \ge 0$); or $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$ when $\frac{v+(\beta_L+\delta)P_B}{2} > v+(\beta_H-\delta)P_B$ (which is also equivalent to $v+(2\beta_H-\beta_L-3\delta)P_B < 0$).

Finally, we summarize the conditions needed to ensure the existence of the optimal reverse HAS strategy: (1) $\beta_L \le (1-\beta_H) - \frac{v}{P_B}$; and (2) $\beta_L < 3\delta - \frac{v}{P_B}$; and (3) $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}$ when $v+(2\beta_H-\beta_L-3\delta)P_B \ge 0$, or $(1-\beta_L)\left[v+\beta_L P_B - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}\right] < (1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$ when $v+(2\beta_H-\beta_L-3\delta)P_B < 0$.

If there exists a feasible p_A satisfying the IC constraints, then the optimal spot price must be $p_S^* = \frac{v+(\beta_L+\delta)P_B}{2}$, and the optimal advance purchase price should be $p_A^* = (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]$. More specifically, we obtain

$$
\begin{aligned}
p_A^* &= (1-\beta_H)\left\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S^*)^+]\right\}\\
&= \begin{cases} (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B \ge 0,\\ (1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right], & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B < 0. \end{cases}
\end{aligned}
$$

The corresponding optimal revenue is equal to

$$
\begin{aligned}
\Pi^{RH} &= p_A^* N_H + p_S^* N_L (1-\beta_L)\mathbb{E}[\mathbb{1}(v+\alpha_L P_B - p_S^* \ge 0)]\\
&= \begin{cases} N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B \ge 0,\\ N_H(1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B < 0. \end{cases}
\end{aligned}
$$

■

Appendix C: Technical proofs for the results in the main paper

Proof of Lemma 1

The proof follows the utility functions (U_i^A, U_i^{NA}, and u_i^S) and the IC and IR constraints. ■

Proof of Lemma 2

We consider the difference $U_i^A - U_i^{NA}$ under a HAS strategy that is equal to

$$
\begin{aligned}
U_i^A - U_i^{NA} &= \{\beta_i P_N + (1-\beta_i)(\beta_i P_B + v) - p_A\} - \{\beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+]\}\\
&= (1-\beta_i)(\beta_i P_B + v) - (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+] - p_A.
\end{aligned}
$$

We define $\Delta U_i(p_S) = (1-\beta_i)(\beta_i P_B + v) - (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S)^+]$. More specifically,

$$
\Delta U_i(p_S) = \begin{cases} (1-\beta_i)p_S & \text{if } p_S \le v+(\beta_i-\delta)P_B\\ (1-\beta_i)[v+\beta_i P_B - \frac{(v+(\beta_i+\delta)P_B - p_S)^2}{4\delta P_B}] & \text{if } v+(\beta_i-\delta)P_B < p_S < v+(\beta_i+\delta)P_B\\ (1-\beta_i)(v+\beta_i P_B) & \text{if } v+(\beta_i+\delta)P_B \le p_S. \end{cases}
$$

Lemma 1 states that a type i will purchase bonus actions in the advance sales market if and only if $p_A \leq \Delta U_i(p_S)$. In the following, we want to prove that $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ for all p_S if and only if $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$.

First of all, suppose $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ for all p_S. Especially when $p_S \geq v + (\beta_H + \delta)P_B \geq v + (\beta_L + \delta)P_B$, we have

$$\Delta U_L(p_S) - \Delta U_H(p_S) = (1-\beta_L)(v + \beta_L P_B) - (1-\beta_H)(v + \beta_H P_B) = (\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]. \tag{A.7}$$

Therefore, $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ implies $[v + (\beta_H + \beta_L - 1)P_B] \geq 0$, which is equivalent to $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$.

Next, suppose $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$. We would like to show $\Delta U_H(p_S) \leq \Delta U_L(p_S)$ for all p_S. Clearly, when $p_S \leq v + (\beta_L - \delta)P_B \leq v + (\beta_H - \delta)P_B$, we obtain $\Delta U_H(p_S) = (1-\beta_H)p_S \leq \Delta U_L(p_S) = (1-\beta_L)p_S$. Besides, when $p_S \geq v + (\beta_H + \delta)P_B \geq v + (\beta_L + \delta)P_B$, given that $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$, we know from (A.7) that $\Delta U_H(p_S) = (1-\beta_H)(v + \beta_H P_B) \leq \Delta U_L(p_S) = (1-\beta_L)(v + \beta_L P_B)$.

The left-over case is when $v + (\beta_L - \delta)P_B < p_S < v + (\beta_H + \delta)P_B$. We examine the difference $\Delta U_L(p_S) - \Delta U_H(p_S)$. It is straightforward to verify that $\Delta U_L(p_S) - \Delta U_H(p_S)$ is a continuous function of p_S. Moreover, its first-order derivative is equal to

$$\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$$

$$= \begin{cases} -(1-\beta_H) + (1-\beta_L)\frac{v + (\beta_L + \delta)P_B - p_S}{2\delta P_B}, & \text{if } v + (\beta_L - \delta)P_B < p_S \leq \min\{v + (\beta_L + \delta)P_B, v + (\beta_H - \delta)P_B\} \\ -(1-\beta_H), & \text{if } v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B \\ (\beta_H - \beta_L)\frac{v + (\beta_H + \beta_L + \delta - 1)P_B - p_S}{2\delta P_B}, & \text{if } v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B \\ -(1-\beta_H)\frac{v + (\beta_H + \delta)P_B - p_S}{2\delta P_B}, & \text{if } \max\{v + (\beta_L + \delta)P_B, v + (\beta_H - \delta)P_B\} < p_S \leq v + (\beta_H + \delta)P_B. \end{cases}$$

Note that when $\beta_H - \beta_L \geq 2\delta$, the case $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$ cannot happen. When $\beta_H - \beta_L < 2\delta$, the case $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$ cannot happen. Thus, the derivative $\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$ has only three pieces as p_S increases from $v + (\beta_L - \delta)P_B$ to $v + (\beta_H + \delta)P_B$.

The derivative $\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$ is continuous in p_S. Moreover, as p_S increases from $v + (\beta_L - \delta)P_B$ to $v + (\beta_H + \delta)P_B$, the derivative $\frac{d(\Delta U_L(p_S) - \Delta U_H(p_S))}{dp_S}$ is first positive and then becomes negative. It means that the difference $\Delta U_L(p_S) - \Delta U_H(p_S)$ first increases in p_S and then decreases in p_S when $v + (\beta_L - \delta)P_B < p_S < v + (\beta_H + \delta)P_B$.

To sum up, we know that the difference $\Delta U_L(p_S) - \Delta U_H(p_S)$ is continuous, first increasing in p_S and then decreasing in p_S. In addition, at $p_S = v + (\beta_L - \delta)P_B$ and $p_S = v + (\beta_H + \delta)P_B$, we have $\Delta U_L(p_S) - \Delta U_H(p_S) \geq 0$. Therefore, we can conclude that $\Delta U_L(p_S) - \Delta U_H(p_S) \geq 0$ whenever $v + (\beta_L - \delta)P_B < p_S < v + (\beta_H + \delta)P_B$.

Above, we have proven that $\Delta U_L(p_S) - \Delta U_H(p_S) \geq 0$ for all p_S, if and only if $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$. Since $U_H^A - U_H^{NA} = \Delta U_H(p_S) - p_A$ and $U_L^A - U_L^{NA} = \Delta U_L(p_S) - p_A$, we finish the proof of Lemma 2.

■

Proof of Corollary 1

Under a PAS strategy, $U_i^A - U_i^{NA} = (1-\beta_i)(\beta_i P_B + v) - p_A$. Corollary 1 comes from Equation (A.7). ∎

Proof of Proposition 1

Following Lemma A.2, we have

$$\Pi^H = \begin{cases} (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v+(\beta_H-3\delta)P_B<0,\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L, & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \epsilon\geq 2\delta\\ (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)[v+\beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \epsilon< 2\delta. \end{cases}$$

For sake of presentation, we denote the three expressions of Π^H as Π^{H1}, Π^{H2}, and Π^{H3} respectively.

Following Lemma A.3, we have

$$\Pi^S = \begin{cases} [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)], & \text{if } N_H\leq r_1 N_L\\ \frac{[2(1-\beta_H)\delta N_H P_B + (1-\beta_L)N_L(v+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}, & \text{if } r_1 N_L < N_H < r_2 N_L\\ [v+(\beta_H-\delta)P_B]N_H(1-\beta_H), & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon\geq 2\delta\\ [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}] & \text{if } r_2 N_L \leq N_H < r_3 N_L \text{ and } \epsilon< 2\delta,\\ \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]} & \text{if } N_H\geq r_3 N_L, \end{cases}$$

where the thresholds r_1, r_2, and r_3 are given in Table 3. Notice that Π^S is a piece-wise function with at most four pieces. We denote the four pieces of Π^S to be Π^{S1}, Π^{S2}, Π^{S31} (when $\epsilon\geq 2\delta$) or Π^{S32} (when $\epsilon<2\delta$), and Π^{S4}. Besides, it is straightforward to verify that Π^S is continuous in N_H.

We would like to prove $\Pi^H\geq\Pi^S$ for all N_H and N_L. To do so, we first make several observations.

(O1) $\Pi^{H2}\geq\Pi^{S1}$ for all N_H and N_L. Because $\Pi^{H2} = (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L$ and $\Pi^{S1} = [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)]$.

(O2) $\Pi^{H2}\geq\Pi^{S31}$ for all N_H and N_L. Because $\Pi^{H2} = (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + (1-\beta_L)(v+\beta_L P_B)N_L$ and $\Pi^{S31} = [v+(\beta_H-\delta)P_B]N_H(1-\beta_H)$.

(O3) $\Pi^{H1}\geq\Pi^{S31}$ for all N_H and N_L. We have $\Pi^{H1} = (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L$ and $\Pi^{S31} = [v+(\beta_H-\delta)P_B]N_H(1-\beta_H)$. Both can be viewed as linear functions of N_H. Clearly, the intercept of Π^{H1} is higher than that of Π^{S31}. It suffices to prove the slope of Π^{H1} is also higher than that of Π^{S31}. We have

$$(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} - (1-\beta_H)[v+(\beta_H-\delta)P_B] = (1-\beta_H)\frac{[v+(\beta_H-3\delta)P_B]^2}{8\delta P_B}\geq 0.$$

(O4) $\Pi^{H1}\geq\Pi^{S2}$ at $N_H=0$ if $v+(\beta_L-3\delta)P_B<0$. We have $\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$ and $\Pi^{S2}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. Therefore, we obtain

$$\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - [v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$$

$$<\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}-[v+\beta_L P_B-\frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}]=\frac{(3v+3\beta_L P_B-\delta P_B)(v+\beta_L P_B-3\delta P_B)}{16\delta P_B}.$$

If $v+(\beta_L-3\delta)P_B<0$ and we also have $3v+3\beta_L P_B-\delta P_B>0$, we finally obtain that $\Pi^{H1}\geq\Pi^{S2}$ at $N_H=0$.

(O5) $\Pi^{H2}\geq\Pi^{S2}$ at $N_H=0$ if $v+(\beta_L-3\delta)P_B<0$. We have seen that $\Pi^{S2}|_{N_H=0}=N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. And $\Pi^{H2}|_{N_H=0}=N_L(1-\beta_L)(v+\beta_L P_B)$. Then, we have

$$\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}-(v+\beta_L P_B)=\frac{(v+\beta_L P_B)^2-6\delta P_B(v+\beta_L P_B)+(\delta P_B)^2}{8\delta P_B}.$$

Notice that the quadratic function $x^2-6xy+y^2$ is negative when $y\leq x<3y$. Therefore, if $v+(\beta_L-3\delta)P_B<0$, equivalently $v+\beta_L P_B<3\delta P_B$, and we also have $v+\beta_L P_B\geq\delta P_B$, we conclude that $\Pi^{H2}\geq\Pi^{S2}$ at $N_H=0$.

(O6) $\Pi^{H3}\geq\Pi^{S1}$ for all N_H and N_L if $\beta_H-\beta_L<2\delta$. We have $\Pi^{H3}=(1-\beta_H)[v+(\beta_H-\delta)P_B]N_H+(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]N_L$ and $\Pi^{S1}=[v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)]$, both of which are linear functions of N_H. Clearly, Π^{H3} has a higher slope than Π^{S1}. Moreover, the intercept of Π^{H3} satisfies

$$\begin{aligned}N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]&=N_L(1-\beta_L)[v+\beta_L P_B-\delta P_B+\frac{(\beta_H-\beta_L)(4\delta-\beta_H+\beta_L)}{4\delta}]\\&\geq N_L(1-\beta_L)[v+(\beta_L-\delta)P_B],\end{aligned}$$

where the inequality holds since $\beta_H-\beta_L<2\delta<4\delta$. Thus, Π^{H3} also has a higher intercept than Π^{S1}. We conclude that if $\beta_H-\beta_L<2\delta$, $\Pi^{H3}\geq\Pi^{S1}$ for all N_H and N_L.

(O7) $\Pi^{H3}\geq\Pi^{S32}$ for all N_H and N_L. We have $\Pi^{H3}=(1-\beta_H)[v+(\beta_H-\delta)P_B]N_H+(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]N_L$ and $\Pi^{S32}=[v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}]$. Notice that Π^{H3} and Π^{S32}, as functions of N_H, have the same slope. Their intercepts satisfy

$$\begin{aligned}&N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]-N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}]\\=&N_L(1-\beta_L)(\beta_H-\beta_L)\frac{[2v+(\beta_H+\beta_L-2\delta)P_B]}{4\delta}\geq 0,\end{aligned}$$

where the inequality holds because $\beta_H>\beta_L\geq\delta$. Therefore, we conclude that $\Pi^{H3}\geq\Pi^{S32}$ for all N_H and N_L.

(O8) $\Pi^{H3}\geq\Pi^{S2}$ at $N_H=0$ if $v+(\beta_L-3\delta)P_B<0\leq v+(\beta_H-3\delta)P_B$ and $\beta_H-\beta_L<2\delta$. We have $\Pi^{H3}|_{N_H=0}=N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]$. When $v+(\beta_H-3\delta)P_B\geq 0$, we obtain

$$\frac{[v+(2\beta_L+\delta-\beta_H)P_B]}{4\delta}-\frac{(2\delta-\beta_H+\beta_L)P_B}{2\delta}=\frac{v+(\beta_H-3\delta)P_B}{4\delta}\geq 0$$

In addition, when $\beta_H-\beta_L<2\delta$, we have $\frac{[v+(2\beta_L+\delta-\beta_H)P_B]}{4\delta}\geq\frac{(2\delta-\beta_H+\beta_L)P_B}{2\delta}>0$, resulting in

$$\Pi^{H3}|_{N_H=0}=N_L(1-\beta_L)[v+\beta_L P_B-\frac{(2\delta-\beta_H+\beta_L)^2P_B}{4\delta}]$$

$$>\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$$
$$>\Pi^{S2}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B},$$

where the last inequality comes from (O4) and we assume $v+(\beta_L-3\delta)P_B<0$.

(O9) Π^{S2} is a convex quadratic function of N_H. And it increases in N_H whenever $N_H \geq 0$. This can be easily seen from the definition of Π^{S2}.

(O10) Π^{S4} is a convex function of N_H. And it increases in N_H whenever $N_H \geq 0$. This is because $\Pi^{S4} = \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]}$. We are able to show

$$\frac{\partial^2\Pi^{S4}}{\partial N_H^2} = \frac{(1-\beta_H)^2(\beta_H-\beta_L)^2(1-\beta_L)^2N_L^2P_B}{4\delta[(1-\beta_H)N_H+(1-\beta_L)N_L]^3} > 0,$$
$$\frac{\partial\Pi^{S4}}{\partial N_H}|_{N_H=0} = \frac{(1-\beta_H)[v+(\beta_L+\delta)P_B][v+(2\beta_H-\beta_L+\delta)P_B]}{8\delta P_B} > 0,$$
$$\lim_{N_H\to\infty}\frac{\partial\Pi^{S4}}{\partial N_H} = \frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} > 0.$$

(O11) $\Pi^{H1} \geq \Pi^{S4}$ for all N_H and N_L if $v+(\beta_L-3\delta)P_B<0$. Recall that $\Pi^{H1} = (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L$ is a linear function of N_H. Its slope is equal to $(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}$, implying that $\frac{\partial\Pi^{S4}}{\partial N_H} \leq \frac{\partial\Pi^{H1}}{\partial N_H}$ for all N_H. In addition, at $N_H=0$, we have $\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$ and $\Pi^{S4}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. In (O4), we have already shown that $N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}] \geq N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$ if $v+(\beta_L-3\delta)P_B<0$. From above, we can conclude that $\Pi^{H1} \geq \Pi^{S4}$ for all N_H and N_L when $v+(\beta_L-3\delta)P_B<0$.

(O12) $\Pi^{H1} \geq \Pi^{S32}$ for all N_H and N_L if $v+(\beta_L-3\delta)P_B<0$. Recall that $\Pi^{H1} = (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L$ and $\Pi^{S32} = [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}]$. Both are linear functions of N_H. We first compare their slopes and we achieve

$$(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} - (1-\beta_H)[v+(\beta_H-\delta)P_B] = (1-\beta_H)\frac{[v+(\beta_H-3\delta)P_B]^2}{8\delta P_B} \geq 0.$$

That is, Π^{H1} has a higher slope than Π^{S32}. Then, we compare their intercepts which are given by $\Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]$ and $\Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}$. Notice that

$$\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - [v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta} = \frac{[v+(2\beta_H-\beta_L-3\delta)P_B]^2}{8\delta P_B} \geq 0$$

which implies that

$$\Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}$$

$$\leq \Pi^{S2}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$$
$$\leq \Pi^{H1}|_{N_H=0} = N_L(1-\beta_L)[v+\beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}].$$

The last inequality comes from (O4) and we assume $v+(\beta_L-3\delta)P_B<0$. Finally, we can conclude that $\Pi^{H1}\geq\Pi^{S32}$ for all N_H and N_L if $v+(\beta_L-3\delta)P_B<0$.

Given the above observations, we are ready to prove $\Pi^H\geq\Pi^S$ for all N_H and N_L. According to Lemma A.2 and Lemma A.3, we prove the result case by case.

We start with the case with $\epsilon=\beta_H-\beta_L\geq 2\delta$.

(C1.1): If $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$, then $\Pi^H=\Pi^{H2}$ and $\Pi^S=\begin{cases}\Pi^{S1}, & \text{if } N_H\leq r_1N_L,\\ \Pi^{S2}, & \text{if } r_1N_L<N_H<r_2N_L,\\ \Pi^{S31}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O1) and (O2), we know that $\Pi^H\geq\Pi^S$ when $N_H\leq r_1N_L$ and $N_H\geq r_2N_L$. In particular, $\Pi^H\geq\Pi^S$ at $N_H=r_1N_L$ and $N_H=r_2N_L$. Given that Π^S is continuous and $\Pi^S=\Pi^{S2}$ is convex when $r_1N_L<N_H<r_2N_L$, we further conclude that $\Pi^H\geq\Pi^S$ when $r_1N_L<N_H<r_2N_L$. In summary, we have shown $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon\geq 2\delta$ and $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$.

(C1.2): If $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$, then $\Pi^H=\Pi^{H2}$ and $\Pi^S=\begin{cases}\Pi^{S2}, & \text{if } 0\leq N_H<r_2N_L,\\ \Pi^{S31}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Similarly as above, (O2) indicates that $\Pi^H\geq\Pi^S$ when $N_H\geq r_2N_L$. In particular, $\Pi^H\geq\Pi^S$ at $N_H=r_2N_L$. Moreover, (O5) indicates that $\Pi^H\geq\Pi^S$ when $N_H=0$, which implies $\Pi^H\geq\Pi^S$ when $0\leq N_H<r_2N_L$. In summary, we have shown $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon\geq 2\delta$ and $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$.

(C1.3): If $0>v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B$, then $\Pi^H=\Pi^{H1}$ and $\Pi^S=\begin{cases}\Pi^{S2}, & \text{if } 0\leq N_H<r_2N_L,\\ \Pi^{S31}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O3) and (O4) and a similar argument as (C1.2), we conclude that $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon\geq 2\delta$ and $0>v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B$.

Above, we have finished the proof for the case with $\epsilon=\beta_H-\beta_L\geq 2\delta$. Next, we consider the case with $\epsilon=\beta_H-\beta_L<2\delta$.

(C2.1): If $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$, then $\Pi^H=\Pi^{H3}$ and $\Pi^S=\begin{cases}\Pi^{S1}, & \text{if } N_H\leq r_1N_L,\\ \Pi^{S2}, & \text{if } r_1N_L<N_H<r_2N_L,\\ \Pi^{S32}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O6) and (O7) and a similar argument as (C1.1), we conclude that $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon<2\delta$ and $v+(\beta_H-3\delta)P_B\geq v+(\beta_L-3\delta)P_B\geq 0$.

(C2.2): If $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$, then $\Pi^H=\Pi^{H2}$ and $\Pi^S=\begin{cases}\Pi^{S2}, & \text{if } 0\leq N_H<r_2N_L,\\ \Pi^{S32}, & \text{if } r_2N_L\leq N_H<\infty.\end{cases}$ Following (O7) and (O8) and a similar argument as (C1.2), we conclude that $\Pi^H\geq\Pi^S$ for all N_H and N_L if $\epsilon<2\delta$ and $v+(\beta_H-3\delta)P_B\geq 0>v+(\beta_L-3\delta)P_B$.

(C2.3): If $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0 > v + (\beta_H - 3\delta)P_B$, then $\Pi^H = \Pi^{H1}$ and $\Pi^S = \begin{cases} \Pi^{S2}, & \text{if } 0 \leq N_H < r_2 N_L, \\ \Pi^{S32}, & \text{if } r_2 N_L \leq N_H < r_3 N_L \\ \Pi^{S4}, & \text{if } r_3 N_L \leq N_H < \infty. \end{cases}$ Following (O11), (O12), and (O4), we conclude that $\Pi^H \geq \Pi^S$ for all N_H and N_L if $\epsilon < 2\delta$ and $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0 > v + (\beta_H - 3\delta)P_B$.

(C2.4): If $0 > v + (2\beta_H - \beta_L - 3\delta)P_B \geq v + (\beta_H - 3\delta)P_B$, then $\Pi^H = \Pi^{H1}$ and $\Pi^S = \Pi^{S4}$. Following (O11), we conclude that $\Pi^H \geq \Pi^S$ for all N_H and N_L if $\epsilon < 2\delta$ and $0 > v + (2\beta_H - \beta_L - 3\delta)P_B \geq v + (\beta_H - 3\delta)P_B$.

In conclusion, we have discussed all possible cases and shown $\Pi^H > \Pi^S$ for all N_H and N_L. ■

Proof of Theorem 1

For casual games, we assume $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$. Lemma 2 implies that the reverse HAS strategy does not exist and Proposition 1 further indicates that the PSS strategy is dominated and can never be optimal. As a result, the optimal selling strategy must be either the PAS strategy or the regular HAS strategy. We compare the firm's revenue under the PAS strategy and the regular HAS strategy which are equal to

$$\Pi^A = \begin{cases} (1-\beta_L)(\beta_L P_B + v)N_L, & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L \\ (1-\beta_H)(\beta_H P_B + v)(N_H + N_L), & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L. \end{cases}$$

$$\Pi^H = \begin{cases} (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} N_H + (1-\beta_L)[v + \beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v + (\beta_H - 3\delta)P_B < 0, \\ (1-\beta_H)[v + (\beta_H - \delta)P_B]N_H + (1-\beta_L)(v + \beta_L P_B)N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0,\ \epsilon \geq 2\delta \\ (1-\beta_H)[v + (\beta_H - \delta)P_B]N_H + (1-\beta_L)[v + \beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0,\ \epsilon < 2\delta. \end{cases}$$

Notice that the firm's revenue under the regular HAS strategy Π^H can be viewed as a linear function of N_H, whereas the firm's the firm's revenue under the PAS strategy Π^A can be viewed as a piece-wise linear function of N_H.

We start with the case that $v + (\beta_H - 3\delta)P_B \geq 0$ and $\beta_H - \beta_L \geq 2\delta$. We consider the difference $\Pi^A - \Pi^H$ which can be simplified to

$$\Pi^A - \Pi^H = \begin{cases} -(1-\beta_H)[v + (\beta_H - \delta)P_B]N_H, & \text{if } N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L, \\ \delta P_B(1-\beta_H)N_H - (\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]N_L, & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L. \end{cases}$$

Clearly, $\Pi^A \leq \Pi^H$ when $N_H \leq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L$. When $N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} N_L$, we can see from above that $\Pi^A \geq \Pi^H$ if and only if $N_H \geq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B} N_L$. Lastly, we compare $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$ with $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$. Obviously, $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B} > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$. Therefore, when $v + (\beta_H - 3\delta)P_B \geq 0$ and $\beta_H - \beta_L \geq 2\delta$, we conclude that $\Pi^A \geq \Pi^H$ if and only if $N_H \geq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B} N_L$; otherwise, $\Pi^A < \Pi^H$.

Next, we consider the case that $v + (\beta_H - 3\delta)P_B \geq 0$ and $\beta_H - \beta_L < 2\delta$. Similarly as above, we investigate the difference $\Pi^A - \Pi^H$ which is equal to

$\Pi^A - \Pi^H$

$$= \begin{cases} -(1-\beta_H)[v+(\beta_H-\delta)P_B]N_H + \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}N_L, & \text{if } N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L \\ \delta P_B(1-\beta_H)N_H - \{\frac{4\delta(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}\}N_L, & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L. \end{cases}$$

When $N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \le \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}N_L$. When $N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \ge \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 N_L$.

However, we need to compare $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$ with $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$. We are able to show that $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]} < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$ if and only if $\left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2 < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$. In addition, we need to compare $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ with $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$. We can achieve that $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}$ if and only if $\left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2 < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$. Note that $\left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2$ can be greater or less than $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$.

Suppose $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} \le (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$. In this case, we have $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} \le \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, which implies that $\Pi^A \ge \Pi^H$ for all N_H and N_L. Suppose $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} > (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$. In this case, we have $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} > \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$. As a result, from the above discussion, we can conclude that $\Pi^A \ge \Pi^H$ when $N_H \le \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}N_L$ and when $N_H \ge \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 N_L$. And $\Pi^A < \Pi^H$ when $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}N_L < N_H < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2 N_L$.

Finally, we examine the case that $v+(\beta_H-3\delta)P_B<0$. We obtain

$$\Pi^A - \Pi^H$$

$$= \begin{cases} -(1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}N_L, & \text{if } N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L, \\ (1-\beta_H)\frac{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + \left\{-(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B] + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\right\}N_L, & \\ & \text{if } N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L. \end{cases}$$

When $N_H \le \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \le \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}N_L$. We have shown earlier in the proof of Proposition 1 that $8\delta P_B(v+\beta_H P_B) - [v+(\beta_H+\delta)P_B]^2 > 0$. Thus, when $N_H > \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, we obtain that $\Pi^A \ge \Pi^H$ if and only if $N_H \ge \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}N_L$.

Furthermore, we can show that if $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 < \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$, meaning that $\Pi^H > \Pi^A$ at $N_H = \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)}N_L$, it implies that $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2} < \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} < \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$. As a result, from the above discussion, we conclude that $\Pi^A \ge \Pi^H$ when $N_H \le \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}N_L$

and when $N_H \geq \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}N_L$. And $\Pi^A < \Pi^H$ when $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}N_L < N_H < \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}N_L$.

Instead, if $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 \geq \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$, we have $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2} \geq \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)(v+\beta_H P_B)} \geq \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, implying that $\Pi^A \geq \Pi^H$ for all N_H and N_L.

In summary, we define $\underline{n}$ and $\bar{n}$ as follows:

$$\underline{n} = \begin{cases} \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}, & \text{if } v+(\beta_H-3\delta)P_B<0 \text{ and } \left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 < \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}, \\ \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}, & \text{if } v+(\beta_H-3\delta)P_B\geq 0,\ \beta_H-\beta_L<2\delta, \text{ and} \\ & \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} > (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2, \\ 0, & \text{otherwise.} \end{cases} \tag{A.8}$$

$$\bar{n} = \begin{cases} \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}, & \text{if } v+(\beta_H-3\delta)P_B<0 \text{ and} \\ & \left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 < \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}, \\ \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2, & \text{if } v+(\beta_H-3\delta)P_B\geq 0,\ \beta_H-\beta_L<2\delta, \text{ and} \\ & \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} > (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2, \\ \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}, & \text{if } v+(\beta_H-3\delta)P_B\geq 0 \text{ and } \beta_H-\beta_L\geq 2\delta, \\ 0, & \text{otherwise.} \end{cases} \tag{A.9}$$

We have prove that $\Pi^A \geq \Pi^H$ if and only if $\frac{N_H}{N_L} \leq \underline{n}$ or $\frac{N_H}{N_L} \geq \bar{n}$ while $\Pi^A < \Pi^H$ if and only if $\underline{n} < \frac{N_H}{N_L} < \bar{n}$. ■

Proof of Theorem 2

To prove the theorem, we show the following results:

(1): The PAS strategy dominates the reverse HAS strategy (if exists), i.e., $\Pi^A \geq \Pi^{RH}$.

(2) The PAS strategy dominates the regular HAS strategy (if exists), i.e., $\Pi^A \geq \Pi^H$.

(3) The PAS strategy dominates the pure spot strategy, i.e., $\Pi^A \geq \Pi^S$.

Following Lemma A.5, we have

$$\Pi^A = \begin{cases} (1-\beta_L)(v+\beta_L P_B)(N_H+N_L), & \text{if } N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L \\ (1-\beta_H)(v+\beta_H P_B)N_H, & \text{if } N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L. \end{cases}$$

Notice that Π^A can be viewed as a piece-wise linear function of N_H. For sake of demonstration, we denote the two pieces of Π^A as Π^{A1} and Π^{A2}. Besides, it is straightforward to verify that Π^A is continuous in N_H.

We start with (1) and prove $\Pi^A \geq \Pi^{RH}$ (if Π^{RH} exists and is positive). Following Lemma A.6, we have

$$\Pi^{RH} = \begin{cases} N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B\geq 0, \\ N_H(1-\beta_H)\left[v+\beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B<0. \end{cases}$$

From the proof of Lemma A.5, we know that $\Pi^A \geq \Pi^{A1} = (1-\beta_L)(v+\beta_L P_B)(N_H+N_L)$ for all N_H and N_L and the strict inequality holds when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L$. Since we want to prove $\Pi^A \geq \Pi^{RH}$ for all N_H and N_L, it suffices to prove $\Pi^{A1} \geq \Pi^{RH}$ for all N_H and N_L.

Suppose $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0$. Then, we have $\Pi^{RH} = N_H(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} + N_L \frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. The difference $\Pi^{RH} - \Pi^{A1}$ is given by

$$\begin{aligned}
\Pi^{RH} - \Pi^{A1} = N_H &\left\{(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} - (1-\beta_L)(v+\beta_L P_B)\right\} \\
&+ N_L \left\{\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - (1-\beta_L)(v+\beta_L P_B)\right\},
\end{aligned}$$

which is a linear function of N_L. Its slope satisfies

$$\begin{aligned}
\left\{(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2} - (1-\beta_L)(v+\beta_L P_B)\right\} &\leq \left\{(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]}{2} - (1-\beta_L)(v+\beta_L P_B)\right\} \\
&= -\frac{1}{2}(1-\beta_L)[v+(\beta_L-\delta)P_B] < 0.
\end{aligned}$$

And its intercept can be simplified to be

$$\left\{\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - (1-\beta_L)(v+\beta_L P_B)\right\} = (1-\beta_L)\frac{(v+\beta_L P_B)^2 - 6\delta P_B(v+\beta_L P_B) + (\delta P_B)^2}{8\delta P_B}.$$

For the existence of the optimal reverse HAS strategy, we need to assume $v + (\beta_L - 3\delta)P_B < 0$. As a result, we have $\delta P_B \leq v + \beta_L P_B < 3\delta P_B$, which implies $(v+\beta_L P_B)^2 - 6\delta P_B(v+\beta_L P_B) + (\delta P_B)^2 < 0$. In conclusion, the slope and the intercept of $\Pi^{RH} - \Pi^{A1}$ are both negative. Hence, when $v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0$, we have $\Pi^{RH} < \Pi^{A1} \leq \Pi^A$ for all N_H and N_L.

Similarly as above, when $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$, we have $\Pi^{RH} = N_H(1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] + N_L \frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. The difference $\Pi^{RH} - \Pi^{A1}$ is equal to

$$\begin{aligned}
\Pi^{RH} - \Pi^{A1} = N_H &\left\{(1-\beta_H)\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] - (1-\beta_L)(v+\beta_L P_B)\right\} \\
&+ N_L \left\{\frac{(1-\beta_L)[v+(\beta_L+\delta)P_B]^2}{8\delta P_B} - (1-\beta_L)(v+\beta_L P_B)\right\}.
\end{aligned}$$

We have already shown that, as a linear function of N_H, the difference $\Pi^{RH} - \Pi^{A1}$ has a negative intercept. Since we assume $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$, we achieve

$$\frac{\partial}{\partial \beta_H}\left\{v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right\} = -\frac{v+(2\beta_H-\beta_L-3\delta)P_B}{4\delta} > 0.$$

Thus, the term $[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}]$ increases in β_H. Given that $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$, equivalently $\beta_H < \frac{(3\delta+\beta_L)P_B - v}{2P_B}$, we obtain

$$\left[v + \beta_H P_B - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}\right] \leq \left[v + \left(\frac{(3\delta+\beta_L)P_B - v}{2P_B}\right)P_B - \frac{[v+(2+\left(\frac{(3\delta+\beta_L)P_B-v}{2P_B}\right)-\beta_L+\delta)P_B]^2}{16\delta P_B}\right]$$

$$= \frac{v + (\beta_L + \delta)P_B}{2}.$$

Finally, the slope satisfies

$$\begin{aligned}
&\left\{(1-\beta_H)\left[v + \beta_H P_B - \frac{[v + (2\beta_H - \beta_L + \delta)P_B]^2}{16\delta P_B}\right] - (1-\beta_L)(v + \beta_L P_B)\right\} \\
&\le \left\{(1-\beta_H)\left[\frac{v + (\beta_L + \delta)P_B}{2}\right] - (1-\beta_L)(v + \beta_L P_B)\right\} \\
&\le \left\{(1-\beta_L)\left[\frac{v + (\beta_L + \delta)P_B}{2}\right] - (1-\beta_L)(v + \beta_L P_B)\right\} = -\frac{1}{2}(1-\beta_L)[v + (\beta_L - \delta)P_B] < 0.
\end{aligned}$$

Thus, the slope is also negative. Hence, when $v + (2\beta_H - \beta_L - 3\delta)P_B < 0$, we conclude $\Pi^{RH} < \Pi^{A1} \le \Pi^A$ for all N_H and N_L.

In summary, we have shown $\Pi^{RH} < \Pi^{A1} \le \Pi^A$ for all N_H and N_L. ■

Next, we prove (2) $\Pi^A \ge \Pi^H$ (if Π^H exists and is positive. Following Lemma A.4, we have

$$\Pi^H = \begin{cases} (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{8\delta P_B}N_H + (1-\beta_L)[v + \beta_L P_B - \frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}]N_L, & \text{if } v + (\beta_H - 3\delta)P_B < 0, \\ (1-\beta_H)[v + (\beta_H - \delta)P_B]N_H + (1-\beta_L)(v + \beta_L P_B)N_L, & \text{if } v + (\beta_H - 3\delta)P_B \ge 0,\ \epsilon \ge 2\delta, \\ (1-\beta_H)[v + (\beta_H - \delta)P_B]N_H + (1-\beta_L)[v + \beta_L P_B - \frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}]N_L, & \text{if } v + (\beta_H - 3\delta)P_B \ge 0,\ \epsilon < 2\delta. \end{cases}$$

As in the proof of Proposition 1, we denote the three expressions of Π^H as Π^{H1}, Π^{H2}, and Π^{H3}, all of which are linear functions of N_H.

Clearly, Π^{H1}, Π^{H2}, and Π^{H3} have intercepts no greater than that of Π^{A1}. In addition, as shown in the proof of Proposition 1, we obtain that Π^{H1}, Π^{H2}, and Π^{H3} have smaller slopes than Π^{A2}. Below, we want to show that Π^{H1}, Π^{H2}, and Π^{H3} have smaller slopes than Π^{A1} as well.

According to Lemma A.4, the optimal regular HAS strategy exists if $v + (\beta_H - 3\delta)P_B \ge 0$, $\beta_H - \beta_L \ge 2\delta$, and $(1-\beta_H)[v + (\beta_H - \delta)P_B] < (1-\beta_L)(v + \beta_L P_B)$. Therefore, we obtain that Π^{H2} has a smaller slope than Π^{A1}; Or if $v + (\beta_H - 3\delta)P_B \ge 0$, $\beta_H - \beta_L \ge 2\delta$, and $(1-\beta_H)[v + (\beta_H - \delta)P_B] < (1-\beta_L)(v + \beta_L P_B)$, from which we know Π^{H3} has a smaller slope than Π^{A1}; Or if $v + (\beta_H - 3\delta)P_B < 0$, and $(1-\beta_H)[v + \beta_H P_B - \frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}] < (1-\beta_L)[v + \beta_L P_B - \frac{[v+(2\beta_L-\beta_H+\delta)P_B]^2}{16\delta P_B}]$. Given that $v + (\beta_H - 3\delta)P_B < 0$, we have $v + \beta_H P_B - \frac{3[v+(\beta_H+\delta)P_B]^2}{16\delta P_B} = \frac{-[3(v+\beta_H P_B)\delta P_B][v+\beta_H P_B - 3\delta P_B]}{16\delta P_B} > 0$. As a result,

$$\begin{aligned}
&(1-\beta_H)\frac{[v + (\beta_H + \delta)P_B]^2}{8\delta P_B} \\
\le&(1-\beta_H)\frac{[v + (\beta_H + \delta)P_B]^2}{8\delta P_B} + (1-\beta_H)[v + \beta_H P_B - \frac{3[v + (\beta_H + \delta)P_B]^2}{16\delta P_B}] + (1-\beta_L)\frac{[v + (2\beta_L - \beta_H + \delta)P_B]^2}{16\delta P_B} \\
=&(1-\beta_H)[v + \beta_H P_B - \frac{[v + (\beta_H + \delta)P_B]^2}{16\delta P_B}] + (1-\beta_L)\frac{[v + (2\beta_L - \beta_H + \delta)P_B]^2}{16\delta P_B} \\
<&(1-\beta_L)[v + \beta_L P_B).
\end{aligned}$$

We conclude that Π^{H1} has a smaller slope than Π^{A1}.

According to the above discussion, for $j = 1, 2, 3$, we have shown that $\Pi^A|_{N_H=0} = \Pi^{A_1}|_{N_H=0}$ is greater than $\Pi^{Hj}|_{N_H=0}$. In addition, Π^{A1} has a higher slope than Π^{Hj}. Therefore, when $N_H \le \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L$, we always have $\Pi^A = \Pi^{A1} \ge \Pi^{Hj}$. Moreover, Π^{A2} has a higher slope than Π^{Hj}. By continuity, we also know that $\Pi^A|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L} = \Pi^{A_1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L} = \Pi^{A_2}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L}$ is greater than $\Pi^{Hj}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L}$. Thus, when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]} N_L$, we always have $\Pi^A = \Pi^{A2} \ge \Pi^{Hj}$.

In summary, we have shown $\Pi^H \le \Pi^A$ for all N_H and N_L. ■

Finally, we prove (3) $\Pi^A \ge \Pi^S$. Following Lemma A.3, we have

$$\Pi^S = \begin{cases} [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)], & \text{if } N_H \le r_1 N_L \\ \frac{[2(1-\beta_H)\delta N_H P_B+(1-\beta_L)N_L(v+(\beta_L+\delta)P_B)]^2}{8(1-\beta_L)\delta N_L P_B}, & \text{if } r_1 N_L < N_H < r_2 N_L \\ [v+(\beta_H-\delta)P_B]N_H(1-\beta_H), & \text{if } r_2 N_L \le N_H < r_3 N_L \text{ and } \epsilon \ge 2\delta \\ [v+(\beta_H-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)\frac{(2\delta+\beta_L-\beta_H)}{2\delta}] & \text{if } r_2 N_L \le N_H < r_3 N_L \text{ and } \epsilon < 2\delta, \\ \frac{\{N_H(1-\beta_H)[v+(\beta_H+\delta)P_B]+N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]\}^2}{8\delta P_B[N_H(1-\beta_H)+N_L(1-\beta_L)]} & \text{if } N_H \ge r_3 N_L, \end{cases}$$

where the three thresholds r_1, r_2, and r_3 are defined in Table 3. As before, we denote the four pieces of Π^S as Π^{S1}, Π^{S2}, Π^{S31} (when $\epsilon \ge 2\delta$) or Π^{S32} (when $\epsilon < 2\delta$), and Π^{S4}.

We make the following observations:

(B1) $\Pi^{A1} \ge \Pi^{S1}$ for all N_H and N_L. Because $\Pi^{A1} = (1-\beta_L)(v+\beta_L P_B)(N_H+N_L)$ and $\Pi^{S1} = [v+(\beta_L-\delta)P_B][N_H(1-\beta_H)+N_L(1-\beta_L)]$.

(B2) $\Pi^{A2} \ge \Pi^{S31}$ for all N_H and N_L. Because $\Pi^{A2} = (1-\beta_H)(v+\beta_H P_B)N_H$ and $\Pi^{S31} = (1-\beta_H)[v+(\beta_H-\delta)P_B]N_H$.

(B3) $\Pi^{A1} \ge \Pi^{S2}$ at $N_H = 0$ if $v+(\beta_L-3\delta)P_B < 0$. Note that $\Pi^{A1}|_{N_H=0} = N_L(1-\beta_L)(v+\beta_L P_B)$ and $\Pi^{S2}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{8\delta P_B}$. The proof is the same as (O5) in the proof of Proposition 1.

(B4) $\frac{\partial \Pi^{A2}}{\partial N_H} > \frac{\partial \Pi^{S4}}{\partial N_H}$ for all N_H if $v+(\beta_H-3\delta)P_B < 0$. We have shown in (O10) that Π^{S4} is a convex function of N_H. And we are able to show that when $v+(\beta_H-3\delta)P_B < 0$,

$$\lim_{N_H\to\infty} \frac{\partial \Pi^{S4}}{\partial N_H} = \frac{(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}{8\delta P_B} < (1-\beta_H)(v+\beta_H P_B) = \frac{\partial \Pi^{A2}}{\partial N_H},$$

which implies $\frac{\partial \Pi^{A2}}{\partial N_H} > \frac{\partial \Pi^{S4}}{\partial N_H}$ for all N_H.

(B5) $\Pi^{A1} \ge \Pi^{S4}$ for all N_H and N_L if $v+(2\beta_H-\beta_L-3\delta)P_B \le 0$. First, at $N_H = 0$, we have $\Pi^{A1}|_{N_H=0} = (1-\beta_L)(v+\beta_L P_B)N_L$ and $\Pi^{S4}|_{N_H=0} = N_L(1-\beta_L)\frac{[v+(\beta_L+\delta P_B)]^2}{8\delta P_B}$. In the proof of Proposition 1, we have shown $(v+\beta_L P_B) \ge \frac{[v+(\beta_L+\delta P_B)]^2}{8\delta P_B}$. Thus, $\Pi^{A1}|_{N_H=0} \ge \Pi^{S4}|_{N_H=0}$. The condition $v+(2\beta_H-\beta_L-3\delta P_B \le 0$ is equivalent to $v+(2\beta_H-\beta_L+\delta P_B \le 4\delta P_B$. Therefore,

$$\frac{\partial \Pi^{S4}}{\partial N_H}|_{N_H=0} = \frac{(1-\beta_H)[v+(\beta_L+\delta)P_B][v+(2\beta_H-\beta_L+\delta)P_B]}{8\delta P_B}$$

$$\leq \frac{(1-\beta_H)[v+(\beta_L+\delta)P_B]4\delta P_B}{8\delta P_B} = \frac{(1-\beta_H)[v+(\beta_L+\delta)P_B]}{2} \leq (1-\beta_L)(v+\beta_L P_B).$$

Given that Π^{S4} is convex, we can conclude from the above analysis that $\Pi^{A1} \geq \Pi^{S4}$ for all N_H and N_L if $v+(2\beta_H-\beta_L-3\delta)P_B \leq 0$.

(B6) $\Pi^{A1} \geq \Pi^{S32}$ at $N_H = 0$ and $N_H = \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. First, $\Pi^{A1}|_{N_H=0} = N_L(1-\beta_L)(v+\beta_L P_B)$ and $\Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta-\beta_H+\beta_L)}{2\delta}$. Recall that we define $\epsilon = \beta_H - \beta_L$. We further have $[v+(\beta_H-\delta)P_B] \geq (\beta_H-\beta_L)P_B = \epsilon P_B$. Finally, we achieve

$$\begin{aligned}
2\delta(v+\beta_L P_B) - [v+(\beta_H-\delta)P_B](2\delta-\beta_H+\beta_L) &= 2\delta[(v+\beta_L P_B) - v - (\beta_H-\delta)P_B] + \epsilon[v+(\beta_H-\delta)P_B] \\
&= 2\delta(\delta-\epsilon)P_B + \epsilon[v+(\beta_H-\delta)P_B] \\
&\geq 2\delta(\delta-\epsilon)P_B + \epsilon^2 P_B = P_B(2\delta^2 - 2\delta\epsilon + \epsilon^2) \geq 0.
\end{aligned}$$

Equivalently, we have shown $\Pi^{A1}|_{N_H=0} = N_L(1-\beta_L)(v+\beta_L P_B) \geq \Pi^{S32}|_{N_H=0} = N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta-\beta_H+\beta_L)}{2\delta}$.

Next, at $N_H = \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we have

$$\begin{aligned}
\Pi^{A1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} &= (1-\beta_H)(v+\beta_H P_B)\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L \\
\Pi^{S32}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} &= (1-\beta_H)[v+(\beta_H-\delta)P_B]\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L \\
&\quad + N_L(1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}].
\end{aligned}$$

Their difference is equal to

$$\begin{aligned}
&\Pi^{A1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} - \Pi^{S32}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} \\
=&(1-\beta_H)\delta P_B\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L - (1-\beta_L)[v+(\beta_H-\delta)P_B]\frac{(2\delta+\beta_L-\beta_H)}{2\delta}N_L.
\end{aligned}$$

By reorganizing the terms, we can show

$$(1-\beta_H)\delta P_B(v+\beta_L P_B)(2\delta) - [v+(\beta_H-\delta)P_B](2\delta+\beta_L-\beta_H)(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v] \geq 0.$$

As a result, we conclude $\Pi^{A1}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} \geq \Pi^{S32}|_{N_H=\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L}$. It further implies that $\Pi^{A1} \geq \Pi^{S32}$ when $N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$ because both Π^{A1} and Π^{S32} are linear functions.

Given the above observations, we are able to prove $\Pi^A \geq \Pi^S$ for all N_H and N_L. We prove the result case by case.

We start with the case when $\epsilon = \beta_H - \beta_L \geq 2\delta$. According to Lemma A.3, $\Pi^S = \begin{cases} \Pi^{S1}, & \text{if } N_H \leq r_1 N_L, \\ \Pi^{S2}, & \text{if } r_1 N_L < N_H < r_2 N_L, \\ \Pi^{S31}, & \text{if } r_2 N_L \leq N_H < \infty. \end{cases}$ if $v+(\beta_L-3\delta)P_B \geq 0$. Or $\Pi^S = \begin{cases} \Pi^{S2}, & \text{if } 0 \leq N_H < r_2 N_L, \\ \Pi^{S31}, & \text{if } r_2 N_L \leq N_H < \infty. \end{cases}$ if

$v+(\beta_L-3\delta)P_B<0$. Following (B1), (B2) and (B4), we can conclude that $\Pi^A \geq \Pi^S$ for all N_H and N_L if $\epsilon \geq 2\delta$.

Next, suppose $\epsilon=\beta_H-\beta_L<2\delta$. According to Lemma A.3, if $v+(\beta_L-3\delta)P_B \geq 0$, then $\Pi^S = \begin{cases} \Pi^{S1}, & \text{if } N_H \leq r_1 N_L, \\ \Pi^{S2}, & \text{if } r_1 N_L < N_H < r_2 N_L, \\ \Pi^{S32}, & \text{if } r_2 N_L \leq N_H < \infty. \end{cases}$. Following (B1), we know that $\Pi^A \geq \Pi^S$ when $N_H \leq r_1 N_L$. (B6) further implies that $\Pi^{A1} \geq \Pi^{S32}$ when $N_H \leq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$ and $\Pi^{A2} \geq \Pi^{S32}$ when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. Therefore, $\Pi^A=\max\{\Pi^{A1},\Pi^{A2}\} \geq \Pi^{S32}$ when $r_2 N_L \leq N_H < \infty$, which also implies that $\Pi^A \geq \Pi^{S2}$ when $r_1 N_L < N_H < r_2 N_L$. In conclusion, $\Pi^A \geq \Pi^S$ for all N_H and N_L if $\epsilon < 2\delta$ and $v+(\beta_L-3\delta)P_B \geq 0$. Similarly as the proof of Proposition 1, given (B3)-(B6), we are able to show that $\Pi^A \geq \Pi^S$ for all N_H and N_L if $\epsilon<2\delta$ and $v+(\beta_L-3\delta)P_B<0$. Concerning the length of the appendix, we do not repeat the detailed analysis.

In conclusion, from the above analysis, we have shown $\Pi^A \geq \Pi^S$ for all N_H and N_L. ■

Proof of Proposition 2

First of all, we show that $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} - (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ decreases in δ when $2\delta > \beta_H-\beta_L$. It is because its derivative satisfies

$$\frac{\partial}{\partial\delta}\left\{\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} - \left(\frac{2\delta-\beta_H+\beta_L}{2\delta}\right)^2\right\}$$
$$= -\frac{(\beta_H-\beta_L)[2\delta(v+\beta_H P_B)-(\beta_H-\beta_L)(1-\beta_L)P_B]}{2(1-\beta_L)\delta^3 P_B} < 0,$$

where the last inequality comes from the facts that $2\delta > \beta_H-\beta_L$ and $(v+\beta_H P_B)-(1-\beta_L)P_B = v+(\beta_H+\beta_L-1)P_B \geq v+(2\beta_L-1)P_B > 0$. We further obtain that $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 = \left(1-\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)^2$ increases in δ.

As a result, when δ increases from 0, Theorem 1 indicates that $\underline{n}$ is first zero. When δ is sufficiently large, $\underline{n}$ becomes positive. In particular, $\underline{n}$ is first equal to $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, and when δ is even large, $\underline{n}$ is finally equal to $\left(\frac{1-\beta_L}{2(1-\beta_H)}\right)\left(1-\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)^2$.

We already have that when $\underline{n} = \left(\frac{1-\beta_L}{2(1-\beta_H)}\right)\left(1-\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)^2$, it increases in δ. When $\underline{n} = \frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, its first-order derivative is given by

$$\frac{\partial \underline{n}}{\partial\delta} = \frac{\partial}{\partial\delta}\left\{\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}\right\} = \frac{P_B(1-\beta_L)(2\delta-\beta_H+\beta_L)[(\beta_H-\beta_L+2\delta)v+(\beta_H^2-\beta_H\beta_L+2\beta_L\delta)P_B]}{4\delta^2(1-\beta_H)[v+(\beta_H-\delta)P_B]^2}.$$

We obtain $\frac{\partial \underline{n}}{\partial\delta} > 0$ since $\beta_H > \beta_L$ and $2\delta > \beta_H-\beta_L$.

In summary, we have proven that whenever $\underline{n}$ is positive, it increases in δ. ■

Next, Theorem 1 indicates that as δ increases from 0, $\bar{n}$ is first equal to $\bar{n} = \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$, then $\bar{n} = \frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)} - \frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$; when δ is quite large, $\bar{n}$ is equal to $\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, and finally $\bar{n}$ becomes zero. Clearly, when $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$, it decreases in δ.

When $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, it suffices to prove $(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ decreasing in δ under the case that $2\delta>\beta_H-\beta_L$. In fact, we have

$$\frac{\partial\left\{(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2\right\}}{\partial\delta}=\frac{(\beta_H-\beta_L)(2\delta-\beta_H+\beta_L)}{2\delta^3}>0.$$

Thus, when $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, it will also decrease in δ.

Finally, when $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, its derivative can be simplified to be

$$\begin{aligned}
&\frac{\partial\bar{n}}{\partial\delta}\\
&=\frac{2P_B(1-\beta_L)[v+(\beta_H+\delta)P_B][v+(2\beta_L+\delta-\beta_H)P_B]}{(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}^2}\times\\
&\left\{\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)[v+(\beta_H-\delta)P_B]-(\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B}-1)(\beta_H-\beta_L)P_B\right\}.
\end{aligned}$$

In this case we have $v+(\beta_H-3\delta)P_B<0$ which implies $\beta_H<3\delta<2\beta_L+\delta$. Hence, it suffices to examine the sign of the following term:

$$\left\{\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)[v+(\beta_H-\delta)P_B]-(\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B}-1)(\beta_H-\beta_L)P_B\right\}. \tag{A.10}$$

Clearly, we have $\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B}>1$. If $\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)\leq 0$, we can easily obtain $\frac{\partial\bar{n}}{\partial\delta}<0$. Suppose $\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)>0$. Following the proof of Theorem 1, $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$ if $v+(\beta_H-3\delta)P_B<0$ and $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2<\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$. As a result, we have

$$\begin{aligned}
\frac{\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}}{\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}}&=\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H+\delta)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B](v+\beta_H P_B)}\\
&>\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}-\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right)\\
&<\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)-\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)\\
&=\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)\left(\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right).
\end{aligned}$$

Therefore, if $\left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B} - \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}\right) > 0$, we achieve

$$\text{(A.10)} < \left\{ \left(\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B}\right)\left(\frac{2(\beta_H-\beta_L)P_B}{v+(\beta_H+\delta)P_B}\right)[v+(\beta_H-\delta)P_B] - \left(\frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B} - 1\right)(\beta_H-\beta_L)P_B \right\}. \tag{A.11}$$

Since $\delta \le \beta_H < 3\delta$, we further have $\frac{v+\beta_H P_B}{v+(\beta_H+\delta)P_B} < \frac{4(v+\beta_H P_B)}{v+(\beta_H+\delta)P_B} - 1$ and $\frac{2[v+(\beta_H-\delta)P_B]}{v+(\beta_H+\delta)P_B} < 1$, from which we conclude (A.11) < 0, implying that $\frac{\partial \bar{n}}{\partial \delta} < 0$.

In summary, we have proven that $\frac{\partial \bar{n}}{\partial \delta} < 0$ whenever $\bar{n} > 0$, i.e., $\bar{n}$ decreases in δ whenever it is positive. ■

Proof of Proposition 3

First, we argue that when $v + (\beta_H - 3\delta)P_B < 0$, $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2 - \frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}$ decreases in β_H. Clearly, $\left(\frac{v+(2\beta_L+\delta-\beta_H)P_B}{v+(\beta_H+\delta)P_B}\right)^2$ decreases in β_H. In addition, we have

$$\frac{\partial\left\{\frac{2(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_L)(v+\beta_H P_B)}\right\}}{\partial \beta_H} = \frac{2\left\{[v+(2\beta_H-1)P_B](v+\beta_H P_B) - (\beta_H-\beta_L)P_B[v+(\beta_H+\beta_L-1)P_B]\right\}}{(1-\beta_L)(v+\beta_H P_B)^2} > 0,$$

where the inequality results from the facts that $v+(2\beta_H-1)P_B \ge (\beta_H-\beta_L)P_B$ and $v+\beta_H P_B \ge v+(\beta_H+\beta_L-1)P_B$. Next, we argue that whenever $v+(\beta_H-3\delta)P_B \ge 0$ and $\beta_H - \beta_L < 2\delta$, $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)} - (\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ increases in β_H. From the above analysis, we can easily obtain $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B][v+(\beta_H-\delta)P_B]}{(1-\beta_L)\delta P_B(v+\beta_H P_B)}$ increases in β_H, whereas $(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$ decreases in β_H when $\beta_H - \beta_L < 2\delta$.

As a result, when β_H increases from $\beta_H = \beta_L$, $\underline{n}$ is first zero. Then it equal to $\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}$. As β_H keeps increasing, $\underline{n}$ becomes $\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$. Finally, when β_H is sufficiently large, $\underline{n}$ drops to 0.

When $\underline{n} = \frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}$, we obtain

$$\begin{aligned}\frac{\partial \underline{n}}{\partial \beta_H} = &\frac{(1-\beta_L)[v+(\beta_H+\delta)P_B][v+(2\beta_L+\delta-\beta_H)P_B]}{2\{(1-\beta_H)[v+(\beta_H+\delta)P_B]^2\}^2}\times \\ &\{[v+(2\beta_L+\delta-\beta_H)P_B][v+(3\beta_H+\delta-2)P_B] - 2P_B(1-\beta_H)[v+(\beta_H+\delta)P_B]\}.\end{aligned}$$

The following term determines the sign of $\frac{\partial \underline{n}}{\partial \beta_H}$:

$$[v+(2\beta_L+\delta-\beta_H)P_B][v+(3\beta_H+\delta-2)P_B] - 2P_B(1-\beta_H)[v+(\beta_H+\delta)P_B]. \tag{A.12}$$

Clearly, if $[v+(3\beta_H+\delta-2)P_B] \le 0$, (A.12) is negative, thus we obtain $\frac{\partial \underline{n}}{\partial \beta_H} \le 0$. In addition, if $[v+(3\beta_H+\delta-2)P_B] > 0$, we have $v+(2\beta_L+\delta-\beta_H)P_B \le v+(\beta_H+\delta)P_B$ and

$$[v+(3\beta_H+\delta-2)P_B] - 2P_B(1-\beta_H) = v+(\beta_H-3\delta)P_B + 4(\beta_H+\delta-1)P_B \le 0,$$

where the inequality results from $v+(\beta_H-3\delta)P_B<0$ and $\beta_H+\delta\le 1$. Hence, (A.12), meaning that $\frac{\partial \underline{n}}{\partial \beta_H}\le 0$. In conclusion, whenever $\underline{n}=\frac{(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)[v+(\beta_H+\delta)P_B]^2}$, $\underline{n}$ decreases in β_H.

When $\underline{n}=\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, we have

$$\frac{\partial \underline{n}}{\partial \beta_H}=\frac{(1-\beta_L)P_B(2\delta-\beta_H+\beta_L)}{4\delta(1-\beta_H)^2[v+(\beta_H-\delta)P_B]^2}\times\{-2(1-\beta_H)[v+(\beta_H-\delta)P_B]+(2\delta-\beta_H+\beta_L)[v+(2\beta_H-\delta-1)P_B]\}.$$

Similarly, it suffices to investigate the sign of the following term:

$$-2(1-\beta_H)[v+(\beta_H-\delta)P_B]+(2\delta-\beta_H+\beta_L)[v+(2\beta_H-\delta-1)P_B]. \tag{A.13}$$

If $[v+(2\beta_H-\delta-1)P_B]\le 0$, (A.13) is negative, thus we obtain $\frac{\partial \underline{n}}{\partial \beta_H}\le 0$. In addition, if $[v+(2\beta_H-\delta-1)P_B]>0$, we have $v+(2\beta_H-\delta-1)P_B\le v+(\beta_H-\delta)P_B$ and $2(1-\beta_H)-(2\delta-\beta_H+\beta_L)=2-\beta_H-\beta_L-2\delta\ge 0$ since $\beta_L+\delta\le\beta_H+\delta\le 1$. Hence, (A.13) is negative, and we achieve $\frac{\partial \underline{n}}{\partial \beta_H}\le 0$. In conclusion, whenever $\underline{n}=\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)^2P_B}{4\delta(1-\beta_H)[v+(\beta_H-\delta)P_B]}$, $\underline{n}$ decreases in β_H.

In summary, we have proven that whenever $\underline{n}$ is positive. it decreases in β_H. ■

Now, we consider $\bar{n}$. Following Theorem 1, when β_H increases from $\beta_H=\beta_L$, $\bar{n}$ is first zero. Then it equal to $\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$. As β_H keeps increasing, $\bar{n}$ is equal to $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, finally it becomes $\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$.

When $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{(1-\beta_H)\delta P_B}$, it is straightforward to see that $\bar{n}$ increase in β_H. Second, when $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, requiring $\beta_H-\beta_L<2\delta$, we have

$$\frac{\partial\left(\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2\right)}{\partial\beta_H}=-\frac{(1-\beta_L)(2\delta-\beta_H+\beta_L)(2-\beta_H-\beta_L-2\delta)}{4(1-\beta_H)^2\delta^2}<0.$$

We conclude that whenever $\bar{n}=\frac{(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]}{\delta P_B(1-\beta_H)}-\frac{(1-\beta_L)}{(1-\beta_H)}(\frac{2\delta-\beta_H+\beta_L}{2\delta})^2$, $\bar{n}$ increases in β_H.

Finally, when $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, we can easily see that the numerator, $16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2$, increases in β_H. The denominator satisfies

$$\begin{aligned}&\frac{\partial\{(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}{\partial\beta_H}\\ &=(\delta P_B)^2-6(\delta P_B)(v+\beta_H P_B)+6(\delta P_B)(1-\beta_H)P_B+(v+\beta_H P_B)^2-2(1-\beta_H)P_B(v+\beta_H P_B),\end{aligned}\tag{A.14}$$

which can be viewed as a convex quadratic function of δP_B. The axis of symmetry is given by $\delta P_B=3[v+\beta_H P_B-(1-\beta_H)P_B]>0$. When $\bar{n}=\frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, we must have $\frac{v+\beta_H P_B}{3}<\delta P_B\le v+\beta_H P_B$. We want to show (A.14) is negative whenever $\frac{v+\beta_H P_B}{3}<\delta P_B\le v+\beta_H P_B$. It suffices to check the sign at the two boundary points $\delta P_B=\frac{v+\beta_H P_B}{3}$ and $\delta P_B=v+\beta_H P_B$. In particular, we have

$$(\text{A.14})|_{\delta P_B=v+\beta_H P_B}=-4(v+\beta_H P_B)[v+(2\beta_H-1)P_B]<0,$$

$$\text{(A.14)}|_{\delta P_B = \frac{v+\beta_H P_B}{3}} = -\frac{8}{9}(v+\beta_H P_B)^2 < 0.$$

where the first inequality results from $v+(2\beta_H-1)P_B \geq v+(\beta_H+\beta_L-1)P_B \geq 0$. Hence, we obtain (A.14) is negative when $\frac{v+\beta_H P_B}{3} < \delta P_B \leq v+\beta_H P_B$, meaning that the denominator of $\bar{n}$ decreases in β_H. Thus, whenever $\bar{n} = \frac{16\delta P_B(\beta_H-\beta_L)[v+(\beta_H+\beta_L-1)P_B]-(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]^2}{2(1-\beta_H)\{8\delta P_B(v+\beta_H P_B)-[v+(\beta_H+\delta)P_B]^2\}}$, $\bar{n}$ increases in β_H.

In summary, we have proven that whenever $\bar{n}$ is positive. it increases in β_H. ■

Proof of Proposition 4

First, we derive the total player welfare under each of the selling strategies for casual games. We define the total player welfare as $PW = N_H U_H + N_L U_L$, where U_H is the utility of a high-type player and U_L is the utility of a low-type player. We use superscripts (A, S, and H) to denote the PAS, PSS, and regular HAS strategies respectively.

According to Lemma A.1 and the discussion in Section 3, under the PAS strategy, a type i player receives utilities that follow

$$U_H = \beta_H P_N \quad \text{and} \quad U_L = \begin{cases} \beta_L P_N, & \text{if } p_A^* = (1-\beta_L)(v+\beta_L P_B), \\ \beta_L P_N + (\beta_H - \beta_L)[v+(\beta_H+\beta_L-1)P_B], & \text{if } p_A^* = (1-\beta_H)(v+\beta_H P_B). \end{cases}$$

Under the regular HAS strategy, a type i player receives utilities that follow

$$U_H = \begin{cases} \beta_H P_N + (1-\beta_H)\delta P_B, & \text{if } v+(\beta_H-3\delta)P_B \geq 0, \\ \beta_H P_N + (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}, & \text{if } v+(\beta_H-3\delta)P_B < 0, \end{cases}$$

$$U_L = \begin{cases} \beta_L P_N, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon \geq 2\delta \\ \beta_L P_N + (1-\beta_L)\frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}, & \text{if } v+(\beta_H-3\delta)P_B \geq 0 \text{ and } \epsilon < 2\delta \\ \beta_L P_N + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}, & \text{if } v+(\beta_H-3\delta)P_B < 0. \end{cases}$$

Notice that under the optimal regular HAS strategy, low-type players receive the same utility from purchasing in advance and in the spot market. Thus, the above utility functions can be derived from $\beta_i P_N + (1-\beta_i)\mathbb{E}[(\alpha_i P_B + v - p_S^*)^+]$. In other words, under the regular HAS strategy, the utility functions can be expressed as $U_H = \beta_H P_N + (1-\beta_H)\mathbb{E}[(\alpha_H P_B + v - p_S^*)^+]$ and $U_L = \beta_L P_N + (1-\beta_L)\mathbb{E}[(\alpha_L P_B + v - p_S^*)^+]$.

Under the PSS strategy, a type i player receives utilities that should be given by

$$U_H = \beta_H P_N + (1-\beta_H)\mathbb{E}[(\alpha_H P_B + v - p_S^*)^+] \quad \text{and} \quad U_L = \beta_L P_N + (1-\beta_L)\mathbb{E}[(\alpha_L P_B + v - p_S^*)^+]$$

where

$$\mathbb{E}[(\alpha_i P_B + v - p_S^*)^+] = \begin{cases} \beta_i P_B + v - p_S^*, & p_S^* \leq (\beta_i - \delta)P_B + v \\ \frac{[v+(\beta_i+\delta)P_B - p_S]^2}{4\delta P_B}, & (\beta_i-\delta)P_B + v < p_S^* < (\beta_i+\delta)P_B + v \\ 0, & p_S^* \geq (\beta_i+\delta)P_B + v. \end{cases}$$

Following Lemma A.2 and Lemma A.3, we can easily see that the regular HAS strategy charges a higher spot price p_S^* than the PSS strategy. Combining with the above analysis, we conclude that

U_H and U_L will be higher under the PSS strategy than the regular HAS strategy, implying that the total welfare under the regular HAS strategy will be smaller than that under the PSS strategy. Therefore, when $\underline{n}N_L < N_H < \bar{n}N_L$, although the regular HAS strategy maximizes the firm's profit, the total player welfare will be higher under the PSS strategy.

Second, when $N_H \geq \bar{n}N_L$, the total welfare under the PAS strategy is equal to

$$PW^A = \beta_H P_N N_H + \{\beta_L P_N + (\beta_H - \beta_L)[v + (\beta_H + \beta_L - 1)P_B]\beta_L P_N\}\, N_L,$$

whereas the total welfare under the HAS strategy is equal to

$$PW^H = \begin{cases} [\beta_H P_N + (1-\beta_H)\delta P_B]N_H + \beta_L P_N N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0,\ \epsilon \geq 2\delta, \\ [\beta_H P_N + (1-\beta_H)\delta P_B]N_H + \{\beta_L P_N + (1-\beta_L)\frac{(2\delta-\beta_H+\beta_L)^2 P_B}{4\delta}\}N_L, & \text{if } v + (\beta_H - 3\delta)P_B \geq 0,\ \epsilon < 2\delta, \\ [\beta_H P_N + (1-\beta_H)\frac{[v+(\beta_H+\delta)P_B]^2}{16\delta P_B}]N_H + \{\beta_L P_N + (1-\beta_L)\frac{[v+(2\beta_L+\delta-\beta_H)P_B]^2}{16\delta P_B}\}N_L, & \text{if } v + (\beta_H - 3\delta)P_B < 0. \end{cases}$$

Following the definition of $\bar{n}$, we obtain that $PW^H > PW^A$ when $N_H \geq \bar{n}N_L$. That is, although the PAS strategy results in a higher firm's profit, the regular HAS strategy results in a higher player welfare, further implying from above that the PSS strategy results in the highest player welfare.

Lastly, when $N_H \leq \underline{n}N_L$, the PAS strategy is optimal. The corresponding total player welfare is equal to $PW^A = \beta_H P_N N_H + \beta_L P_N N_L$. Clearly, $PW^A < PW^S$. That is, the PSS strategy results in a higher player welfare than the PAS strategy,

In conclusion, we have proven that the PSS strategy leads to maximal player welfare. Thus, for casual games, there cannot exist a selling strategy that leads to both the highest firm's profit and the highest players' welfare. ■

Proof of Proposition 5

We derive the total player welfare under each of the selling strategies for hardcore games. First, the total welfare under the PSS strategy, denoted as PW^S, is the same as the one in the proof of Proposition 4. In addition, the total welfare under the regular HAS strategy (if exists), denoted as PW^H, is also the same as the one in the proof of Proposition 4.

Following Lemma A.5, under the PAS strategy, a type i player receives utilities that are given by

$$U_H = \begin{cases} \beta_H P_N, & \text{if } p_A^* = (1-\beta_H)(v + \beta_H P_B), \\ \beta_H P_N + (\beta_H - \beta_L)[(1-\beta_H-\beta_L)P_B - v], & \text{if } p_A^* = (1-\beta_L)(v+\beta_L P_B), \end{cases} \quad \text{and} \quad U_L = \beta_L P_N.$$

Under the reverse HAS strategy (if exists), a type i player receives utilities that follow:

$$U_H = \begin{cases} \beta_H P_N + (1-\beta_H)\frac{[v+(2\beta_H-\beta_L-\delta)P_B]}{2}, & \text{if } v + (2\beta_H - \beta_L - 3\delta)P_B \geq 0, \\ \beta_H P_N + (1-\beta_H)\frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}], & \text{if } v + (2\beta_H - \beta_L - 3\delta)P_B < 0. \end{cases}$$

$$U_L = \beta_L P_N + (1-\beta_L)\frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}.$$

We start by showing if the regular or reverse HAS strategy exists, it leads to a higher player welfare than the PAS strategy. That is, $PW^H > PW^A$ and $PW^{RH} > PW^A$. Note that the utility of a low-type player is $U_L = \beta_L P_N$ under the PAS strategy and it is smaller than that under the regular or reverse HAS strategy. In addition, when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, the utility of a high-type player is $U_H = \beta_H P_N$ under the PAS strategy which is also smaller than that under the regular or reverse HAS strategy. Hence, when $N_H > \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we have $PW^A < PW^H$ and $PW^A < PW^{RH}$.

When $N_H \le \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, the utility of a high-type player is given by $U_H = \beta_H P_N + (1-\beta_H)(v+\beta_H P_B) - p_A^*$. In order to prove $PW^A < PW^H$ and $PW^A < PW^{RH}$, it suffices to prove the advance sale price p_A^* is highest under the PAS strategy. If $N_H \le \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, under the PAS strategy, we have $p_A^* = (1-\beta_L)(v+\beta_L P_B)$. Following Lemma A.4, it is straightforward to see that p_A^* is higher under the PAS strategy than under the regular HAS strategy. Therefore, we conclude $PW^A < PW^H$. Under the reverse HAS strategy, we have $p_A^* = \begin{cases}(1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}, & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B \ge 0,\\ (1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}], & \text{if } v+(2\beta_H-\beta_L-3\delta)P_B < 0.\end{cases}$ Clearly, $(1-\beta_L) \ge (1-\beta_H)$ and $(v+\beta_L P_B) > \frac{[v+(\beta_L+\delta)P_B]}{2}$ since $\beta_L \ge \delta$. Moreover,

$$\frac{\partial[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}]}{\partial \beta_H} = \frac{-[v+(2\beta_H-\beta_L-3\delta)P_B]}{4\delta}.$$

Therefore, if $v+(2\beta_H-\beta_L-3\delta)P_B < 0$, under the reverse HAS strategy, $p_A^* = (1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}]$ satisfying

$$\begin{aligned}(1-\beta_H)[(v+\beta_H P_B) - \frac{[v+(2\beta_H-\beta_L+\delta)P_B]^2}{16\delta P_B}] &< (1-\beta_H)[(v+\beta_L P_B) - \frac{[v+(2\beta_L-\beta_L+\delta)P_B]^2}{16\delta P_B}]\\ &= (1-\beta_H)[(v+\beta_L P_B) - \frac{[v+(\beta_L+\delta)P_B]^2}{16\delta P_B}]\\ &< (1-\beta_H)\frac{[v+(\beta_L+\delta)P_B]}{2}\\ &< (1-\beta_L)(v+\beta_L P_B).\end{aligned}$$

As a result, p_A^* is higher under the PAS strategy than under the reverse HAS strategy. We conclude $PW^{RH} > PW^A$.

So far, we have shown $PW^H > PW^A$ and $PW^{RH} > PW^A$ when the regular or reverse HAS strategy exists. Finally, we prove that when neither the regular nor the reverse hybrid exists, under certain conditions, there is a threshold $\bar{t} < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}$ such that $PW^A > PW^S$ if $\bar{t}N_L < N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$.

When $N_H \geq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, following Lemma A.5, we have $p_A^* = (1-\beta_H)(v+\beta_H P_B)$ under the PAS strategy, resulting in

$$PW^A = \beta_H P_N N_H + \beta_L P_N N_L < PW^S.$$

That is, whenever $N_H \geq \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, the PSS strategy will lead to a higher player welfare than the PAS strategy.

When $N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, we achieve

$$\begin{aligned} PW^A - PW^S =& (\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B - v]N_H \\ & - (1-\beta_H)\mathbb{E}[(\alpha_H P_B + v - p_S^*)^+]N_H - (1-\beta_L)\mathbb{E}[(\alpha_L P_B + v - p_S^*)^+]N_L, \end{aligned}$$

which can be viewed as a linear function of N_H. Clearly, at $N_H = 0$, $(PW^A - PW^S)|_{N_H=0} < 0$. However, as N_H approaches to $\frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$, it is possible that $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$. For example, consider an instance with $P_N = 3$, $P_B = 2$, $\beta_H = 0.3$, $\beta_L = 0.1$ and $\delta = 0.02$. One can verify that $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$ in this case. For sake of the appendix length, we do not present the algebra.

As a result, if $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} \leq 0$, it implies $PW^A - PW^S < 0$ whenever $N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. But if $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$, we can conclude that there exists a threshold $\bar{t}$ such that $PW^A > PW^S$ if $\bar{t}N_L < N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$.

In summary, we have proven that if the optimal HAS strategy exists, it results in a higher player welfare than the PAS strategy. That is, $PW^H > PW^A$ and $PW^{RH} > PW^A$. Therefore, the PAS strategy yields the firm it's highest profit but player welfare is not maximized. If the optimal HAS strategy does not exists, when $(PW^A - PW^S)|_{N_H \to \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L} > 0$, there exists a threshold $\bar{t}$ such that $PW^A > PW^S$ if $\bar{t}N_L < N_H < \frac{(1-\beta_L)(v+\beta_L P_B)}{(\beta_H-\beta_L)[(1-\beta_H-\beta_L)P_B-v]}N_L$. That is, the PAS strategy is a win-win strategy for the firm and players when the ratio N_H/N_L is moderate. ■

Proof of Proposition 6

Suppose the firm charges a personalized price in the spot market. The PAS strategy shuts down the spot market, thus the optimal PAS strategy is not affected. The optimal revenue under PAS strategy is given in Lemma A.1 for casual games and in Lemma A.5 for hardcore games.

If the firm adopts a PSS strategy, it chooses $p_S(\alpha)$ to maximize its profit from the spot market. Note that the player utility from purchasing bonus actions in the spot market is given by $u^S = v + \alpha P_B - p_S$. As a result, the optimal price that the firm can charge is $p_S^*(\alpha) = v + \alpha P_B$. The corresponding optimal revenue is

$$\Pi^{S,ps} = N_H(1-\beta_H)\mathbb{E}[p_S^*(\alpha_H)] + N_L(1-\beta_L)\mathbb{E}[p_S^*(\alpha_L)] = N_H(1-\beta_H)(v+\beta_H P_B) + N_L(1-\beta_L)(v+\beta_L P_B).$$

If the firm adopts a regular HAS strategy, following the same argument above, the optimal spot price that the firm can charge is $p_S^*(\alpha_H) = v + \alpha_H P_B$. The advance purchase price p_A must satisfy the constraints (6) and (7). Hence, the optimal advance purchase price is $p_A^* = (1-\beta_L)(v+\beta_L P_B)$. The corresponding optimal revenue is

$$\Pi^{H,ps} = N_H(1-\beta_H)\mathbb{E}[p_S^*(\alpha_H)] + N_L p_A^* = N_H(1-\beta_H)(v+\beta_H P_B) + N_L(1-\beta_L)(v+\beta_L P_B).$$

Similarly, if the firm adopts a reverse HAS strategy, the optimal spot price that the firm can charge is $p_S^*(\alpha_L) = v + \alpha_L P_B$. The advance purchase price p_A must satisfy the constraints (9) and (10). Hence, the optimal advance purchase price is $p_A^* = (1-\beta_H)(v+\beta_H P_B)$. The corresponding optimal revenue is

$$\Pi^{RH,ps} = N_H p_A^* + N_L(1-\beta_L)\mathbb{E}[p_S^*(\alpha_L)] = N_H(1-\beta_H)(v+\beta_H P_B) + N_L(1-\beta_L)(v+\beta_L P_B).$$

Finally, for casual games, we assume $\beta_L + \beta_H \geq 1 - \frac{v}{P_B}$. It implies that $(1-\beta_L)(v+\beta_L P_B) \geq (1-\beta_H)(v+\beta_H P_B)$. Therefore, we conclude that $\Pi^{S,ps} = \Pi^{H,ps} \geq \Pi^A$. For hardcore games, we assume $\beta_L + \beta_H < 1 - \frac{v}{P_B}$. It implies that $(1-\beta_L)(v+\beta_L P_B) < (1-\beta_H)(v+\beta_H P_B)$. Therefore, we conclude that $\Pi^{S,ps} = \Pi^{RH,ps} \geq \Pi^A$. ■

Lemma A.7 *For casual games, if the firm commits prices that induce low-skilled players to purchase before the attempt but high-skilled players to purchase after failing the attempt, the optimal spot price is*

$$p_S^* = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } \epsilon \geq 2\delta \text{ and } v+(\beta_H-3\delta)P_B < 0, \\ \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B] + N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}, & \text{if } \epsilon < 2\delta \text{ and } v+(\beta_H-3\delta)P_B < \frac{N_L(1-\beta_L)}{N_H(1-\beta_H)}(\beta_L+2\delta-\beta_H)P_B, \\ v+(\beta_H-\delta)P_B, & \text{otherwise.} \end{cases}$$

The optimal advance purchase prices satisfies $p_A^* = (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^*)^+]$.

<u>Proof of Lemma A.7:</u> The firm's optimization problem is given by

$$\max_{p_A \geq 0, p_S \geq 0} \Pi(p_A, p_S) := p_A N_L + p_S N_H(1-\beta_H)\mathbb{E}[\mathbb{1}(v+\alpha_H P_B - p_S \geq 0)] \quad \text{(A.15)}$$

$$\text{s.t.} \quad p_A \leq (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S)^+]\} \quad \text{(A.16)}$$

$$p_A > (1-\beta_H)\{v+\beta_H P_B - \mathbb{E}[(v+\alpha_H P_B - p_S)^+]\}. \quad \text{(A.17)}$$

The objective function (A.15) increases in p_A. Hence, p_A must reach the upperbound in (A.16) at optimum. We replace p_A by the upperbound and the objective function becomes a function of p_S that is

$$\text{(A.15)} = N_L(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S)^+]\} + p_S N_H(1-\beta_H)\mathbb{E}[1(v+\alpha_H P_B - p_S \geq 0)].$$

Suppose $\beta_H - \delta \geq \beta_L + \delta$, we have

$$\text{(A.15)} = \begin{cases} N_L(1-\beta_L)p_S + p_S N_H(1-\beta_H), & \text{if } p_S \leq v + (\beta_L - \delta)P_B, \\ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H), & \text{if } v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H), & \text{if } v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}, & \text{if } v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B), & \text{if } p_S > v + (\beta_H + \delta)P_B. \end{cases}$$

We can easily see that (A.15) increases in p_S when $p_S \leq v + (\beta_L - \delta)P_B$ and $v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B$. Furthermore, we obtain

$$\begin{aligned} &\frac{d}{dp_S}\left\{N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H)\right\} \\ &= \frac{N_L(1-\beta_L)[v + (\beta_L+\delta)P_B - p_S]}{2\delta P_B} + N_H(1-\beta_H), \end{aligned} \tag{A.18}$$

which is positive when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$. Thus, (A.15) increases in p_S when $v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B$.

Lastly, when $v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B$, we have

$$\frac{d}{dp_S}\left\{N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}\right\} = \frac{N_H(1-\beta_H)[v + (\beta_H+\delta)P_B - 2p_S]}{2\delta P_B} \tag{A.19}$$

In particular,

$$\begin{aligned} \text{(A.19)}|_{p_S = v + (\beta_H - \delta)P_B} &= -\frac{N_H(1-\beta_H)[v + (\beta_H - 3\delta)P_B]}{2\delta P_B}, \\ \text{(A.19)}|_{p_S = v + (\beta_H + \delta)P_B} &= -\frac{N_H(1-\beta_H)[v + (\beta_H + \delta)P_B]}{2\delta P_B} < 0. \end{aligned}$$

In conclusion, if $v + (\beta_H - 3\delta)P_B \geq 0$, (A.15) increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$ and decreases in p_S when $p_S > v + (\beta_H - \delta)P_B$. So the optimal spot price is $p_S^* = v + (\beta_H - \delta)P_B$. If $v + (\beta_H - 3\delta)P_B < 0$, (A.15) increases in p_S when $p_S \leq v + (\beta_H - \delta)P_B$, increases and then decreases in p_S when $p_S > v + (\beta_H - \delta)P_B$. The optimal spot price is solved from the first-order condition

$$\frac{d}{dp_S}\left\{N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}\right\} = \frac{N_H(1-\beta_H)[v + (\beta_H+\delta)P_B - 2p_S]}{2\delta P_B} = 0,$$

which results in $p_S^* = \frac{v + (\beta_H + \delta)P_B}{2}$.

Suppose $\beta_H - \delta < \beta_L + \delta$, we have

$$\text{(A.15)} = \begin{cases} N_L(1-\beta_L)p_S + p_S N_H(1-\beta_H), & \text{if } p_S \leq v + (\beta_L - \delta)P_B, \\ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H), & \text{if } v + (\beta_L - \delta)P_B < p_S \leq v + (\beta_H - \delta)P_B, \\ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v+(\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} & \\ \qquad + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}, & \text{if } v + (\beta_H - \delta)P_B < p_S \leq v + (\beta_L + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B) + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{P_B})}{2\delta}, & \text{if } v + (\beta_L + \delta)P_B < p_S \leq v + (\beta_H + \delta)P_B, \\ N_L(1-\beta_L)(v + \beta_L P_B). & \text{if } p_S > v + (\beta_H + \delta)P_B. \end{cases}$$

Clearly, (A.15) increases in p_S when $p_S \le v + (\beta_L - \delta)P_B$. The above analysis further shows that (A.15) increases in p_S when $v + (\beta_L - \delta)P_B < p_S \le v + (\beta_H - \delta)P_B$.

Following the above analysis, when $v + (\beta_L + \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$, the derivative of the objective function is given by (A.19). And we have

$$\text{(A.19)}|_{p_S = v+(\beta_L+\delta)P_B} = -\frac{N_H(1-\beta_H)[v + (2\beta_L + \delta - \beta_H)P_B]}{2\delta P_B} < 0,$$

$$\text{(A.19)}|_{p_S = v+(\beta_H+\delta)P_B} = -\frac{N_H(1-\beta_H)[v + (\beta_H + \delta)P_B]}{2\delta P_B} < 0.$$

The first inequality comes from the assumption $\beta_H - \delta < \beta_L + \delta$, equivalently $\beta_H < \beta_L + 2\delta < 2\beta_L + \delta$. Therefore, (A.15) decreases in p_S when $v + (\beta_L + \delta)P_B < p_S \le v + (\beta_H + \delta)P_B$.

Finally, when $v + (\beta_H - \delta)P_B < p_S \le v + (\beta_L + \delta)P_B$, we have

$$\frac{d}{dp_S}\left\{ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v + (\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{p_B})}{2\delta} \right\}$$
$$= \frac{N_L(1-\beta_L)[v + (\beta_L+\delta)P_B - p_S]}{2\delta P_B} + \frac{N_H(1-\beta_H)[v + (\beta_H+\delta)P_B - 2p_S]}{2\delta P_B}. \tag{A.20}$$

In particular,

$$\text{(A.20)}|_{p_S = v+(\beta_H-\delta)P_B} = \frac{N_L(1-\beta_L)(\beta_L + 2\delta - \beta_H)P_B}{2\delta P_B} - \frac{N_H(1-\beta_H)[v + (\beta_H - 3\delta)P_B]}{2\delta P_B},$$

$$\text{(A.20)}|_{p_S = v+(\beta_L+\delta)P_B} = -\frac{N_H(1-\beta_H)[v + (2\beta_L + \delta - \beta_H)P_B]}{2\delta P_B} < 0.$$

In conclusion, if $N_L(1-\beta_L)(\beta_L + 2\delta - \beta_H)P_B \le N_H(1-\beta_H)[v + (\beta_H - 3\delta)P_B]$, (A.15) increases in p_S when $p_S \le v + (\beta_H - \delta)P_B$ and decreases in p_S when $p_S > v + (\beta_H - \delta)P_B$. So the optimal spot price is $p_S^* = v + (\beta_H - \delta)P_B$. If $N_L(1-\beta_L)(\beta_L + 2\delta - \beta_H)P_B > N_H(1-\beta_H)[v + (\beta_H - 3\delta)P_B]$, (A.15) increases in p_S when $p_S \le v + (\beta_H - \delta)P_B$, increases and then decreases in p_S when $p_S > v + (\beta_H - \delta)P_B$. The optimal spot price is solved from the first-order condition

$$0 = \frac{d}{dp_S}\left\{ N_L(1-\beta_L)\{v + \beta_L P_B - \frac{[v + (\beta_L+\delta)P_B - p_S]^2}{4\delta P_B}\} + p_S N_H(1-\beta_H)\frac{(\beta_H + \delta - \frac{p_S - v}{p_B})}{2\delta} \right\}$$
$$= \frac{N_L(1-\beta_L)[v + (\beta_L+\delta)P_B - p_S]}{2\delta P_B} + \frac{N_H(1-\beta_H)[v + (\beta_H+\delta)P_B - 2p_S]}{2\delta P_B},$$

which results in $p_S^* = \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B] + N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}$.

To sum up, following the analysis above, we conclude that the optimal spot price under commitment is

$$p_S^* = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } \epsilon \ge 2\delta \text{ and } v + (\beta_H - 3\delta)P_B < 0, \\ \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B] + N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}, & \text{if } \epsilon < 2\delta \text{ and } v + (\beta_H - 3\delta)P_B < \frac{N_L(1-\beta_L)}{N_H(1-\beta_H)}(\beta_L + 2\delta - \beta_H)P_B, \\ v + (\beta_H - \delta)P_B, & \text{otherwise.} \end{cases}$$

■

Proof of Proposition 7

Following Lemma A.2 and Lemma A.7, $p_A^{*,dynamic} = (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^{*,dynamic})^+]$ and $p_A^{*,commit} = (1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S^{*,commit})^+]$. Notice that the function $(1-\beta_L)\{v+\beta_L P_B - \mathbb{E}[(v+\alpha_L P_B - p_S)^+]$ increases in p_S. As a result, it suffices to prove $p_S^{*,commit} \geq p_S^{*,dynamic}$.

We have

$$p_S^{*,dynamic} = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } v+(\beta_H-3\delta)P_B < 0, \\ v+(\beta_H-\delta)P_B, & \text{if } v+(\beta_H-3\delta)P_B \geq 0. \end{cases}$$

and

$$p_S^{*,commit} = \begin{cases} \frac{v+(\beta_H+\delta)P_B}{2}, & \text{if } \epsilon \geq 2\delta \text{ and } v+(\beta_H-3\delta)P_B < 0, \\ \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}, & \text{if } \epsilon < 2\delta \text{ and } v+(\beta_H-3\delta)P_B < \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B}{N_H(1-\beta_H)}, \\ v+(\beta_H-\delta)P_B, & \text{otherwise.} \end{cases}$$

When $\epsilon \geq 2\delta$, we obtain that $p_S^{*,dynamic} = p_S^{*,commit}$ and thereby $p_A^{*,dynamic} = p_A^{*,commit}$.

When $\epsilon < 2\delta$, we have $p_S^{*,dynamic} = p_S^{*,commit} = v+(\beta_H-\delta)P_B$ if $v+(\beta_H-3\delta)P_B \geq \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B}{N_H(1-\beta_H)}$. If $0 \leq v+(\beta_H-3\delta)P_B < \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B}{N_H(1-\beta_H)}$, $p_S^{*,dynamic} = v+(\beta_H-\delta)P_B$ and $p_S^{*,commit} = \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}$ from which we obtain

$$p_S^{*,commit} - p_S^{*,dynamic} = \frac{N_L(1-\beta_L)(\beta_L+2\delta-\beta_H)P_B - N_H(1-\beta_H)[v+(\beta_H-3\delta)P_B]}{N_L(1-\beta_L)+2N_H(1-\beta_H)} > 0.$$

Lastly, if $v+(\beta_H-3\delta)P_B < 0$, $p_S^{*,dynamic} = \frac{v+(\beta_H+\delta)P_B}{2}$ and $p_S^{*,commit} = \frac{N_L(1-\beta_L)[v+(\beta_L+\delta)P_B]+N_H(1-\beta_H)[v+(\beta_H+\delta)P_B}{N_L(1-\beta_L)+2N_H(1-\beta_H)}$. We assume that $\epsilon = \beta_H - \beta_L < 2\delta$ which implies $\beta_H < \beta_L + 2\delta \leq 2\beta_L + \delta$. Thus,

$$p_S^{*,commit} - p_S^{*,dynamic} = \frac{N_L(1-\beta_L)[v+(2\beta_L+\delta-\beta_H)P_B]}{2[N_L(1-\beta_L)+2N_H(1-\beta_H)]} > 0.$$

In conclusion, we have shown $p_S^{*,commit} \geq p_S^{*,dynamic}$ and correspondingly $p_A^{*,commit} \geq p_A^{*,dynamic}$. ■