Approximation Algorithms for Dynamic Inventory Management on Networks: Electronic Companion

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Appendix A: Proofs of Technical Lemmas

In this section we state and prove a few lemmas used for various proofs in the rest of the paper.

Lemma 3. *For any lead time $L \geq 0$ and any feasible policy π we have*

$$C(\pi) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_i c_i I_i^T + \sum_{t=1}^{T}\left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right].$$

Proof. By repeated application of the inventory evolution equation (1) we have for each T and i

$$\sum_{t=1-L}^{T-L} z_i^t = I_i^T - I_i^0 + \sum_{t=1}^{T}\sum_k a_{ik} x_k^t = I_i^T + \sum_{t=1}^{T}\sum_k a_{ik} x_k^t,$$

where the second equality follows from $I_i^0 = 0$. In words, this identity states that the cumulative ordering quantity for each resource equals the current inventory plus the cumulative fulfillment quantity for activities using this resource. This implies the expression inside the expectation in the objective (3) is equal to

$$\sum_{t=1}^{T}\left(\sum_i (h_i I_i^t + c_i z_i^{t-L}) + \sum_j p_j B_j^t + \sum_k q_k x_k^t\right),$$

$$= \sum_i c_i I_i^T + \sum_{t=1}^{T}\left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t\right).$$

Q.E.D.

Lemma 4. *For any lead time $L \geq 0$ and any feasible policy π we have*

$$C(\pi) \geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right].$$

Proof. Follows directly from $I_i^T \geq 0$ and Lemma 3. Q.E.D.

Lemma 5. *For any lead time $L \geq 0$, if a policy π has backlog for any j such that $\liminf_{T\to\infty} \mathbb{E}\left[B_j^T\right]/T > 0$, then $C(\pi) = \infty$.*

Proof. If $\liminf_{T\to\infty} \mathbb{E}\left[B_j^T\right]/T > 0$, then for some $\epsilon > 0$, there exists τ such that for all $T \geq \tau$ we have $\mathbb{E}\left[B_j^T\right]/T \geq \epsilon$, or equivalently, for all $t \geq \tau$ we have $\mathbb{E}\left[B_j^t\right] \geq \epsilon t$. Thus, since $\mathbb{E}\left[B_j^t\right] \geq 0$, for $T \geq \tau$ we have

$$\sum_{t=1}^{T}\mathbb{E}\left[B_j^t\right] \geq \sum_{t=\tau}^{T}\mathbb{E}\left[B_j^t\right] \geq \epsilon\sum_{t=\tau}^{T} t = \frac{\epsilon}{2}\left(T(T+1) - \tau(\tau-1)\right),$$

which implies that

$$C(\pi) \geq \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[B_j^t\right] \geq \limsup_{T\to\infty}\frac{\epsilon}{2}\left(T+1-\frac{\tau(\tau-1)}{T}\right) = \infty.$$

Q.E.D.

Recall

$$g_j^0 = \min_{k\in\mathcal{N}(j)}\left\{q_k + \sum_i a_{ik} c_i\right\}.$$

Proposition 2. *For any lead time $L \geq 0$ and any feasible policy π we have the following lower bound on the long-run average cost of π:*

$$\sum_j g_j^0 \mu_j \leq C(\pi).$$

Proof. We consider two cases. First, if there exists j such that $\liminf_{T\to\infty} \mathbb{E}\left[B_j^T\right]/T > 0$, then by Lemma 5 we have $C(\pi) = \infty$ and the claim holds.

Otherwise consider the case if $\liminf_{T\to\infty} \mathbb{E}\left[B_j^T\right]/T = 0$ for all j. Since the system starts with zero backlog, at time T the cumulative demand for each product equals the current backlog plus the cumulative fulfillment quantity for activities fulfilling this product (which can be verified by repeated applications of the backlog evolution equation (2)):

$$\sum_{t=1}^{T} D_j^t = B_j^T + \sum_{t=1}^{T}\sum_k r_{kj} x_k^t, \ \forall j$$

which by Lemma 4 implies

$$\begin{aligned} C(\pi) &\geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_j\sum_k r_{kj}(q_k + \sum_i c_i a_{ik}) x_k^t\right)\right], \\ &\geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\sum_j g_j^0 \sum_k r_{kj} x_k^t\right], \\ &\geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_j \left(\sum_{t=1}^{T} g_j^0 D_j^t - g_j^0 B_j^T\right)\right], \\ &= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\sum_j g_j^0 D_j^t\right] - \liminf_{T\to\infty}\sum_j g_j^0 \frac{\mathbb{E}\left[B_j^T\right]}{T}, \\ &= \sum_j g_j^0 \mu_j. \end{aligned}$$

Q.E.D.

Appendix B: Proof of Theorem 1

Proof of Theorem 1. Following Lemma 4 we will prove the result by proving the following claim

$$\underline{G}(\beta,\delta,\rho) \leq \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right]$$

We first observe that by rearranging the sum and noting $x_k^s = 0$ for all $s \leq 0$ and all k, we have

$$\sum_{t=1}^{T} x_k^t \geq \frac{1}{L+1}\sum_{t=1}^{T}\sum_{s=t-L}^{t} x_k^s, \tag{26}$$

for all activities k and all time horizons $T \geq 1$. Further, for each j, time horizon T, and $\rho \in [0,1]$ we have

$$\sum_{t=1}^{T} B_j^t = (1-\rho)\sum_{t=1}^{T} B_j^t + \rho\left(\sum_{t=1}^{T} B_j^{t-L-1} + \sum_{t=T-L}^{T} B_j^t\right) \geq \sum_{t=1}^{T}\left[(1-\rho)B_j^t + \rho B_j^{t-L-1}\right],$$

where the equality follows from $B_j^t = 0$ for $t \leq 0$ and the inequality follows from $B_j^t \geq 0$. Thus, by Lemma 4 and (26), the long-run average cost of any policy is lower bounded by

$$\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\sum_i h_i I_i^t + \sum_j p_j(\rho B_j^{t-L-1} + (1-\rho)B_j^t) + \frac{1}{L+1}\sum_k (q_k + \sum_i c_i a_{ik})\sum_{s=t-L}^{t} x_k^s\right]$$

We next make a distinction between the portion of an activity that fulfills backlog vs. new demand in a period. In particular, for each activity k in each period t, let $x_k^t = x_k^{t,D} + x_k^{t,B}$, where $x_k^{t,D}$ represents the fulfillment of period t demand (i.e., $D_{j(k)}^t$) and $x_k^{t,B}$ represents the fulfillment of existing backlog (i.e., $B_{j(k)}^{t-1}$). Since a feasible policy must maintain $\sum_k r_{kj} x_k^t \le B_j^{t-1} + D_j^t$, it is always possible to break x_k^t down into $x_k^{t,D}$ and $x_k^{t,B}$ satisfying the following inequalities

$$\sum_k r_{kj} x_k^{t,D} \le D_j^t, \ \forall j, \tag{27}$$

$$\sum_k r_{kj} x_k^{t,B} \le B_j^{t-1}, \ \forall j, \tag{28}$$

and so we use these constraints and $x_k^t = x_k^{t,D} + x_k^{t,B}$ without loss of generality.

Then, observe that the constraints $I_i^t \ge 0, B_j^t \ge 0$ and evolution equations (1) and (2) imply that

$$I_i^t = I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s - \sum_{s=t-L}^{t} \sum_k a_{ik} x_k^s \ge 0, \ \forall i \tag{29}$$

$$B_j^t = B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} x_k^s \ge 0, \ \forall j \tag{30}$$

Thus, we can lower bound the holding and activity costs for any $\beta, \delta \in [0,1]$ as follows

$$\begin{aligned}
&\sum_{t=1}^{T} \left[\sum_i h_i I_i^t + \frac{1}{L+1} \sum_k (q_k + \sum_i c_i a_{ik}) \sum_{s=t-L}^{t} x_k^s \right], \\
&= \sum_{t=1}^{T} \left[\sum_i h_i \left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s \right) + \sum_k (Q_k - H_k) \sum_{s=t-L}^{t} (x_k^{s,D} + x_k^{s,B}) \right], \\
&\ge \sum_{t=1}^{T-1} \left[\sum_i h_i \left(\beta \left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s \right) + (1-\delta)(1-\beta) \left(I_i^{t-L} + \sum_{s=t-2L+1}^{t-L+1} z_i^s \right) \right) \right] \\
&\quad + \sum_{t=1}^{T-1} \left[\sum_k (Q_k - H_k) \sum_{s=t-L}^{t} (x_k^{s,D} + \delta x_k^{s,B} + (1-\delta) x_k^{s+1,B}) \right], \\
&\ge \sum_{t=1}^{T-1} \left[\sum_i h_i \left(\beta \left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s \right) + (1-\delta)(1-\beta) \sum_{s=t-L+1}^{t+1} \sum_k a_{ik} x_k^{s,B} \right) \right] \\
&\quad + \sum_{t=1}^{T-1} \left[\sum_k (Q_k - H_k) \sum_{s=t-L}^{t} (x_k^{s,D} + \delta x_k^{s,B} + (1-\delta) x_k^{s+1,B}) \right], \\
&= \sum_{t=1}^{T-1} \left[\sum_i h_i \beta \left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s \right) \right] \\
&\quad + \sum_{t=1}^{T-1} \left[\sum_k (Q_k - H_k) \sum_{s=t-L}^{t} (x_k^{s,D} + \delta x_k^{s,B}) + \sum_k (Q_k - \beta H_k) \sum_{s=t-L+1}^{t+1} (1-\delta) x_k^{s,B} \right],
\end{aligned}$$

where the first inequality follows from shifting the terms in the sum by one and dropping first and last positive elements (and noting that $1 \ge 1-\delta$), and the second inequality follows from (29). Next, for any $\delta \in [0,1]$ we lower bound the backlog cost as follows

$$\begin{aligned}
&\sum_{t=1}^{T} \sum_j p_j B_j^t, \\
&= \sum_{t=1}^{T} \sum_j p_j \left[B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (x_k^{s,D} + x_k^{s,B}) \right],
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{t=1}^{T-L-1} \sum_j p_j \delta \left(B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (x_k^{s,D} + x_k^{s,B}) \right) \\
&\quad + \sum_{t=1}^{T-L-1} \sum_j p_j (1-\delta) \left(B_j^t + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (x_k^{s,D} + x_k^{s+1,B}) \right), \\
&= \sum_{t=1}^{T-L-1} \sum_j p_j \delta \left(B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (x_k^{s,D} + x_k^{s,B}) \right) \\
&\quad + \sum_{t=1}^{T-L-1} \sum_j p_j (1-\delta) \left(B_j^{t-L-1} + 2 \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (2x_k^{s,D} + x_k^{s,B} + x_k^{s+1,B}) \right),
\end{aligned}
$$

where the inequality follows from shifting terms in the sum and dropping positive terms. Thus, we have the following lower bound on costs over T periods

$$
\begin{aligned}
&\sum_{t=1}^{T} \left(\sum_i h_i I_i^t + \sum_j p_j (\rho B_j^{t-L-1} + (1-\rho) B_j^t) + \frac{1}{L+1} \sum_k (q_k + \sum_i c_i a_{ik}) \sum_{s=t-L}^{t} x_k^s \right), \\
&\geq \sum_{t=1}^{T-L-1} \left[\sum_i h_i \beta \left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s \right) + \sum_j p_j \rho B_j^{t-L-1} \right] \\
&\quad + \sum_{t=1}^{T-L-1} \left[\sum_k (Q_k - H_k) \sum_{s=t-L}^{t} (x_k^{s,D} + \delta x_k^{s,B}) + \sum_k (Q_k - \beta H_k) \sum_{s=t-L+1}^{t+1} (1-\delta) x_k^{s,B} \right], \\
&\quad + \sum_{t=1}^{T-L-1} \sum_j (1-\rho) p_j \delta \left(B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (x_k^{s,D} + x_k^{s,B}) \right) \\
&\quad + \sum_{t=1}^{T-L-1} \sum_j (1-\rho) p_j (1-\delta) \left(B_j^{t-L-1} + 2 \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (2x_k^{s,D} + x_k^{s,B} + x_k^{s+1,B}) \right),
\end{aligned}
$$

and we complete the proof by proving the following inequality holds for each t

$$
\begin{aligned}
&\mathbb{E} \left[\sum_i h_i \beta \left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s \right) + \sum_j p_j \rho B_j^{t-L-1} \right] \\
&\quad + \mathbb{E} \left[\sum_k (Q_k - H_k) \sum_{s=t-L}^{t} (x_k^{s,D} + \delta x_k^{s,B}) + \sum_k (Q_k - \beta H_k) \sum_{s=t-L+1}^{t+1} (1-\delta) x_k^{s,B} \right] \\
&\quad + \mathbb{E} \left[\sum_j (1-\rho) p_j \delta \left(B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (x_k^{s,D} + x_k^{s,B}) \right) \right] \\
&\quad + \mathbb{E} \left[\sum_j (1-\rho) p_j (1-\delta) \left(B_j^{t-L-1} + 2 \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t} \sum_k r_{kj} (2x_k^{s,D} + x_k^{s,B} + x_k^{s+1,B}) \right) \right], \\
&\geq \underline{G}(\beta, \delta, \rho),
\end{aligned}
$$

since, following the above arguments, this implies the long-run average cost is larger than

$$
\limsup_{T \to \infty} \frac{T-L-1}{T} \underline{G}(\beta, \delta, \rho) = \underline{G}(\beta, \delta, \rho).
$$

To proceed, let $\mathcal{F}_{t-L-1} = \sigma\{\mathbf{D}^1, \ldots, \mathbf{D}^{t-L-1}, \mathbf{x}^1, \ldots, \mathbf{x}^{t-L-1}, \mathbf{z}^1, \ldots, \mathbf{z}^{t-L-1}\}$ denote the σ-algebra representing the information set containing all the realized demand and decisions made before the start of period $t-L$.

Thus we have $\mathbb{E}\left[I_i^{t-L-1}|\mathcal{F}_{t-L-1}\right] = I_i^{t-L-1}\ \forall i$, $\mathbb{E}\left[z_i^s|\mathcal{F}_{t-L-1}\right] = z_i^s\ \forall s \le t-L$ and $\forall i$, and $\mathbb{E}\left[B_j^{t-L-1}|\mathcal{F}_{t-L-1}\right] = B_j^{t-L-1}\ \forall j$. Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sum_i h_i\beta\left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s\right) + \sum_j p_j\rho B_j^{t-L-1}\middle|\mathcal{F}_{t-L-1}\right] \\
&\quad + \mathbb{E}\left[\sum_k (Q_k - H_k)\sum_{s=t-L}^{t}(x_k^{s,D} + \delta x_k^{s,B}) + \sum_k (Q_k - \beta H_k)\sum_{s=t-L+1}^{t+1}(1-\delta)x_k^{s,B}\middle|\mathcal{F}_{t-L-1}\right] \\
&\quad + \mathbb{E}\left[\sum_j (1-\rho)p_j\delta\left(B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t}\sum_k r_{kj}(x_k^{s,D} + x_k^{s,B})\right)\middle|\mathcal{F}_{t-L-1}\right] \\
&\quad + \mathbb{E}\left[\sum_j (1-\rho)p_j(1-\delta)\left(B_j^{t-L-1} + 2\sum_{s=t-L}^{t} D_j^s - \sum_{s=t-L}^{t}\sum_k r_{kj}(2x_k^{s,D} + x_k^{s,B} + x_k^{s+1,B})\right)\middle|\mathcal{F}_{t-L-1}\right], \\
&\ge \sum_i h_i\beta\left(I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s\right) + \sum_j p_j\rho B_j^{t-L-1} + \mathbb{E}\left[G\left(\mathbf{I}^{t-L-1} + \sum_{s=t-2L}^{t-L}\mathbf{z}^s, \mathbf{B}^{t-L-1}, \beta, \delta, \rho\middle|\mathcal{D}^t\right)\middle|\mathcal{F}_{t-L-1}\right] \\
&\ge \underline{G}(\beta, \delta, \rho),
\end{aligned}
$$

where the first inequality follows from letting $x_k^l = x_k^{t-L-1+l,D}$ and $w_k^l = x_k^{t-L-1+l,B}$ for each k and $l = 1, \ldots, L+1$, $w_k^{L+2} = \mathbb{E}\left[x_k^{t+1,B}|\mathcal{F}_{t-1}, \mathbf{D}^t\right]$ for each k (where the expectation, which is implicitly taken over the random variable $\mathbf{D}^{t+1}$, is necessary because $x_k^{t+1,B}$ is adapted to the additional information in $\mathbf{D}^{t+1}$, but the feasible space of of the second stage LP should be measurable with respect to just $\mathbf{D}^t$), $y_j^l = D_j^{t-L-1+l} - \sum_k r_{kj}(x_k^l + w_k^l)$ for each j and $l = 2, \ldots, L+1$, $y_j^0 = B_j^{t-L-1} - \sum_k r_{kj} w_k^1$ for each j, $y_j^1 = D_j^{t-L} - \sum_k r_{kj} x_k^1$ for each j, and $y_j^{L+2} = B_j^{t-L-1} + \sum_{s=t-L}^{t} D_j^s - \sum_{l=1}^{L+1}\sum_k r_{kj}(x_k^{s,D} + x_k^{s,B}) - \sum_k r_{kj} x_k^{t+1,B}$, and noting that this constitutes a feasible solution for the second stage LP $G\left(\mathbf{I}^{t-L-1} + \sum_{s=t-2L}^{t-L}\mathbf{z}^s, \mathbf{B}^{t-L-1}, \beta, \delta, \rho\middle|\mathcal{D}^t\right)$ with an objective value equal to the left hand side of the inequality. We argue feasibility for the LP $G\left(\mathbf{I}^{t-L-1} + \sum_{s=t-2L}^{t-L}\mathbf{z}^s, \mathbf{B}^{t-L-1}, \beta, \delta, \rho\middle|\mathcal{D}^t\right)$ as follows: the first constraint follows from (29), the second through fifth constraints follow by definition of the LP variables above (substituting the $\mathbf{y}^l$ variable definitions for $l \le L+1$ into the fifth constraint gives exactly the definition of $\mathbf{y}^{L+2}$), the sixth and seventh constraints follow from (28) (a straightforward induction argument shows $B_j^{t-L-1+l} = \sum_{l'=1}^{l} y_j^{l'}$ for all $1 \le l \le L+1$), and the eighth constraint follows from (27). Then the second inequality above follows from substituting S_i for $I_i^{t-L-1} + \sum_{s=t-2L}^{t-L} z_i^s$ and B_j for B_j^{t-L-1} and noting the resulting expression gives a feasible cost for the stochastic program in (5) because $\mathcal{D}^t$ has the same distribution as $\mathcal{D}$. Then, the proof is complete by taking another expectation on each side of this inequality. Q.E.D.

Appendix C: Proofs for Section 4

Before proving Proposition 1, we first state a lemma claiming the optimal solution of the stochastic program (9) is obtained by a finite vector $\mathbf{S}$. The proof, which we include for completeness, follows closely that of similar result in Reiman and Wang (2015).

Lemma 6. *The minimum of* (9) *is obtained at a finite solution, i.e.,* $S_i^V < \infty$ *for all* i*.*

Proof. Observe that $\mathbf{S} = 0$, $\mathbf{x} = 0$, and $\mathbf{y} = \mathbf{D}$ is a feasible solution for (9) giving objective value $\sum_j (p_j + g_j^V)\mathbb{E}[D_j]$, and thus we must have $\sum_j (p_j + g_j^V)\mathbb{E}[D_j] \ge \underline{C}^V$. Further, note that we have $\sum_k \left(q_k + \sum_i a_{ik} c_i\right) x_k^V(\mathbf{D}) + \sum_j (p_j + g_j^V) y_j^V(\mathbf{D}) \ge 0$ for all $\mathbf{D}$, which implies that

$$
\underline{C}^V \ge \mathbb{E}\left[\sum_i h_i (S_i^V - \sum_k a_{ik} x_k^V(\mathbf{D}))\right],
$$

$$\geq \mathbb{E}\left[h_i\left(S_i^V - \sum_k a_{ik} x_k^V(\mathbf{D})\right)\right],$$
$$\geq h_i\left(S_i^V - \sum_k a_{ik}\mathbb{E}\left[D_{j(k)}\right]\right),$$

where the second inequality follows from the first constraint of the LP defining $F^V(\mathbf{S}|\mathbf{D})$, and the final inequality follows because $x_k^V(\mathbf{D}) \leq D_{j(k)}$ by the second constraint of the LP defining $F^V(\mathbf{S}|\mathbf{D})$ and the fact that all $x_k^V(\mathbf{D}) \geq 0$ and $y_j^V(\mathbf{D}) \geq 0$. Thus, combining this with $\sum_j (p_j + g_j^V)\mathbb{E}[D_j] \geq \underline{C}^V$ gives

$$S_i^V \leq \sum_k a_{ik}\mathbb{E}\left[D_{j(k)}\right] + \frac{\sum_j (p_j + g_j^V)\mathbb{E}[D_j]}{h_i} < \infty.$$

Q.E.D.

Proof of Proposition 1. Consider any feasible ABBS policy with backlog assignment matrix V, and let this policy's base-stock levels be denoted by S_i for all i and fulfillment decisions by x_k^t for all k,t. We first show that this policy maintains an inventory level below S_i in each period, i.e., $I_i^t \leq S_i$ for all i,t, and that we also have

$$z_i^t = S_i - I_i^{t-1} + \sum_k a_{ik} v_{kj(k)} B_{j(k)}^{t-1} \tag{31}$$

for all i,t, i.e., (6) can drop the positive part function without loss of generality. We show the claim $I_i^t \leq S_i$ by induction, and (31) will follow as a byproduct. For $t = 0$, it is clearly true that $I_i^0 \leq S_i$ because we assume the system start with no inventory, $I_i^0 = 0$. Then for the inductive step assume for $t \geq 0$ that $I_i^t \leq S_i$ and consider $t+1$. Since $I_i^t \leq S_i$, by (6) we have

$$z_i^{t+1} = \left(S_i - I_i^t + \sum_k a_{ik} v_{kj(k)} B_{j(k)}^t\right)^+,$$
$$= S_i - I_i^t + \sum_k a_{ik} v_{kj(k)} B_{j(k)}^t,$$

where the second equality follows from $S_i - I_i^t \geq 0$ and $\sum_k a_{ik} v_{kj(k)} B_{j(k)}^t \geq 0$. Thus, it is clear that if $I_i^t \leq S_i$, then (31) follows for $t+1$, and thus (31) follows for all t once the induction is complete. To complete the induction, by the inventory evolution equation we have

$$I_i^{t+1} = I_i^t + z_i^{t+1} - \sum_k a_{ik} x_k^{t+1},$$
$$= S_i + \sum_k a_{ik}(v_{kj(k)} B_{j(k)}^t - x_k^{t+1}),$$
$$\leq S_i,$$

where the inequality follows because $v_{kj(k)} B_{j(k)}^t - x_k^{t+1} \leq 0$ for all k by (7).

The rest of the proof follows two steps: first we show that $\underline{C}^V$ provides a lower bound on the long-run average cost of any ABBS policy for matrix V, then we show that the policy described achieves this cost.

To prove the lower bound, we show that for any feasible ABBS policy with backlog assignment matrix V we have

$$\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right] \geq \sum_{t=1}^{T-1} \underline{C}^V = (T-1)\underline{C}^V, \tag{32}$$

and so by Lemma 4, the long-run average cost of such a policy is larger than

$$\limsup_{T\to\infty} \frac{T-1}{T}\underline{C}^V = \underline{C}^V.$$

Thus, we complete the proof that $\underline{C}^V$ is a lower bound by proving (32). To do so, for activity k in period t define

$$x_k^{t,D} = x_k^t - v_{kj(k)} B_{j(k)}^{t-1} \geq 0$$

as the quantity of activity k usage above the backlog fulfillment required by (7) (i.e., the quantity of activity k used to fill period t's demand). Further, note that

$$\sum_k (q_k + \sum_i c_i a_{ik}) v_{kj(k)} B_{j(k)}^t = \sum_k (q_k + \sum_i c_i a_{ik}) \sum_j v_{kj} B_j^t = \sum_j g_j^V B_j^t, \tag{33}$$

where the first equality follows from the fact that $v_{kj'} = 0$ for all $j' \neq j(k)$, and the second follows from the definition of g_j^V. Then, we can lower bound the sum inside the expectation operator on the left hand side of (32) as follows

$$\begin{aligned}
&\sum_{t=1}^{T} \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t \right), \\
&= \sum_{t=1}^{T} \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) (x_k^{t,D} + v_{kj(k)} B_{j(k)}^{t-1}) \right), \\
&= \sum_{t=1}^{T} \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,D} \right) + \sum_{t=0}^{T-1} \sum_j g_j^V B_{j(k)}^t, \\
&\geq \sum_{t=1}^{T-1} \left(\sum_i h_i I_i^t + \sum_j (p_j + g_j^V) B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,D} \right),
\end{aligned}$$

where the first equality follows by definition of $x_k^{t,D}$, the second equality from (33), and the inequality from dropping non-negative terms from each sum. We now complete the proof of (32) by showing that the expected value of each term in this sum is larger than $\underline{C}^V$. To see this, observe that for each t we have

$$\begin{aligned}
&\sum_i h_i I_i^t + \sum_j (p_j + g_j^V) B_j^t, \\
&= \sum_i h_i (I_i^{t-1} + z_i^t - \sum_k a_{ik} x_k^t) + \sum_j (p_j + g_j^V)(B_j^{t-1} + D_j^t - \sum_k r_{kj} x_k^t), \\
&= \sum_i h_i (S_i + \sum_k a_{ik} (v_{kj(k)} B_{j(k)}^{t-1} - x_k^t)) + \sum_j (p_j + g_j^V)(B_j^{t-1} + D_j^t - \sum_k r_{kj} x_k^t), \\
&= \sum_i h_i (S_i - \sum_k a_{ik} x_k^{t,D}) + \sum_j (p_j + g_j^V)(D_j^t - \sum_k r_{kj} x_k^{t,D}),
\end{aligned}$$

where the first equality follows from the evolution equations, the second equality follows from (31), and the final equality from the definition of $x_k^{t,D}$ and the fact that $\sum_k r_{kj} v_{kj} = 1$ for each j. Thus, taking expectations we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_i h_i I_i^t + \sum_j (p_j + g_j^V) B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,D} \right], \\
&= \sum_i h_i S_i + \mathbb{E}\left[\sum_j (p_j + g_j^V)(D_j^t - \sum_k r_{kj} x_k^{t,D}) + \sum_k (q_k + \sum_i a_{ik}(c_i - h_i)) x_k^{t,D} \right], \\
&\geq \sum_i h_i S_i + \mathbb{E}\left[F^V(\mathbf{S}|\mathbf{D}^t) \right], \\
&\geq \underline{C}^V,
\end{aligned}$$

where the first inequality follows because $x_k = x_k^{t,D}$ for all k and $y_j = D_j^t - \sum_k r_k j x_k^{t,D}$ provide a feasible solution for the LP defining $F^V(\mathbf{S}|\mathbf{D}^t)$ (because $x_k \geq 0$ is guaranteed by (7), $y_j \geq 0$ is guaranteed by $B_j^t \geq 0$ and the backlog evolution equation, and $\sum_k a_{ik} x_k \leq S_i$ is guaranteed by $I_i^t \geq 0$, the inventory evolution equation, and (31)), and the final inequality follows because $\mathbf{S}$ is feasible for (9). This completes the proof of (32) and thus also completes the proof that $\underset{\sim}{C}^V$ provides a lower bound on the long-run average cost of any ABBS policy for matrix V.

We now complete the proof by showing that a ABBS policy with base-stock levels $\mathbf{S}^V$ and fulfillment decisions $\mathbf{x}^{t,V}$ (from (10)) achieves a long-run average cost of at most $\underset{\sim}{C}^V$. To see this, observe that Lemma 3 implies the long-run average cost of this policy is

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_i c_i I_i^T + \sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,V}\right)\right],$$
$$\leq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_i c_i S_i^V + \sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,V}\right)\right],$$
$$= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,V}\right)\right],$$

where the inequality follows because $I_i^T \leq S_i$ for all i, T, and the equality follows from $S_i^V < \infty$ for all i by Lemma 6. Thus we complete the proof by showing that

$$\mathbb{E}\left[\sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,V}\right)\right] \leq \sum_{t=1}^T \underset{\sim}{C}^V = T\underset{\sim}{C}^V. \tag{34}$$

The sum inside the expectation on the left hand side of (34) can be bound as follows

$$\sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^{t,V}\right),$$
$$= \sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik})(x_k^V(\mathbf{D}^t) + v_{kj(k)} B_{j(k)}^{t-1})\right),$$
$$= \sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^V(\mathbf{D}^t)\right) + \sum_{t=0}^{T-1}\sum_j g_j^V B_j^t,$$
$$\leq \sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j (p_j + g_j^V) B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^V(\mathbf{D}^t)\right),$$

where the first equality follows by definition of $x_k^{t,V}$ in (10), the second equality from (33), and the inequality from adding in $\sum_j g_j^V B_j^T \geq 0$ and noting that $\sum_j g_j^V B_j^0 = 0$. We now complete the proof of (34) by showing that the expected value of each term in this sum is equal to $\underset{\sim}{C}^V$. To see this, observe that for each t have

$$\sum_i h_i I_i^t + \sum_j (p_j + g_j^V) B_j^t,$$
$$= \sum_i h_i (I_i^{t-1} + z_i^t - \sum_k a_{ik} x_k^{t,V}) + \sum_j (p_j + g_j^V)(B_j^{t-1} + D_j^t - \sum_k r_{kj} x_k^{t,V}),$$
$$= \sum_i h_i (S_i^V + \sum_k a_{ik}(v_{kj(k)} B_{j(k)}^{t-1} - x_k^{t,V})) + \sum_j (p_j + g_j^V)(B_j^{t-1} + D_j^t - \sum_k r_{kj} x_k^{t,V}),$$
$$= \sum_i h_i (S_i^V - \sum_k a_{ik} x_k^V(\mathbf{D}^t)) + \sum_j (p_j + g_j^V)(D_j^t - \sum_k r_{kj} x_k^V(\mathbf{D}^t)),$$

where the first equality follows from the evolution equations, the second equality follows from (31), and the final equality from the definition of $x_k^{t,V}$ in (10) and the fact that $\sum_k r_{kj} v_{kj} = 1$ for each j. Therefore, taking expectations we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_i h_i I_i^t + \sum_j (p_j + g_j^V) B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^V(\mathbf{D}^t)\right], \\
&= \sum_i h_i S_i^V + \mathbb{E}\left[\sum_j (p_j + g_j^V)(D_j^t - \sum_k r_{kj} x_k^V(\mathbf{D}^t)) + \sum_k (q_k + \sum_i a_{ik}(c_i - h_i)) x_k^V(\mathbf{D}^t)\right], \\
&= \sum_i h_i S_i^V + \mathbb{E}\left[F^V(\mathbf{S}^V|\mathbf{D}^t)\right], \\
&= \underline{C}^V,
\end{aligned}$$

where the second equality follows from noting that $y_j^V(\mathbf{D}^t) = D_j^t - \sum_k r_{kj} x_k^V(\mathbf{D}^t)$ for each j, and the third equality from the fact that $\mathbf{S}^V$ is the minimizer of (9). This completes the proof of (34), and thus also the proposition. Q.E.D.

Proof of Corollary 1. Given any backlog assignment matrix V, Proposition 1 implies the best ABBS policy for V has cost $\underline{C}^V$. By definition we have $g_j^V \geq g_j^{V^*}$ for all j. Therefore, we have the following lower bound on any ABBS policy

$$\underline{C}^V = \sum_i h_i S_i^V + \mathbb{E}\left[F^V(\mathbf{S}^V|\mathbf{D})\right] \geq \sum_i h_i S_i^V + \mathbb{E}\left[F^{V^*}(\mathbf{S}^V|\mathbf{D})\right] \geq \underline{C}^{V^*},$$

where the first inequality follows because $F^V(\mathbf{S}^V|\mathbf{D})$ and $F^{V^*}(\mathbf{S}^V|\mathbf{D})$ have the same constraint set (so an optimal solution for the former is feasible for the latter), and the second inequality follows because $\mathbf{S}^V$ is a feasible solution for the stochastic program (9) with backlog assignment matrix V^*. Thus, because the policy described for matrix V^* achieves cost $\underline{C}^{V^*}$ by Proposition 1, it is optimal within the class of ABBS policies. Q.E.D.

Proof of Corollary 2. The proof breaks the long-run average objective into two pieces corresponding to whether or not products (activities) are in $\mathcal{J}$ ($\mathcal{K}$). In particular, by Lemma 4 the cost of any feasible policy can be bound as follows

$$\begin{aligned}
C(\pi) &\geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right], \\
&= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_{j\in\mathcal{J}} p_j B_j^t + \sum_{k\in\mathcal{K}} (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right] \qquad (35) \\
&\quad + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^T \left(\sum_{j\notin\mathcal{J}} p_j B_j^t + \sum_{k\notin\mathcal{K}} (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right] \qquad (36)
\end{aligned}$$

We first show that (36) is lower bounded by $\sum_{j\notin\mathcal{J}} g_j^0 \mathbb{E}[D_j]$. To do so, consider two cases. First, if there exists $j \notin \mathcal{J}$ such that the policy's backlog satisfies $\liminf_{T\to\infty} \mathbb{E}\left[B_j^T\right]/T > 0$, then by Lemma 5 the policy has cost $C(\pi) = \infty$ and the proposition holds trivially. Otherwise consider the case if the policy satisfies $\liminf_{T\to\infty} \mathbb{E}\left[B_j^T\right]/T = 0$ for all $j \notin \mathcal{J}$. Then, since the system starts with zero backlog, observe that at time T the cumulative demand for each product must equal the current backlog plus the cumulative fulfillment quantity for activities fulfilling this product:

$$\sum_{t=1}^T D_j^t = B_j^T + \sum_{t=1}^T \sum_{k\in\mathcal{N}(j)} x_k^t, \ \forall j,$$

which implies we can lower bound (36) as follows

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_{j\notin\mathcal{J}}p_jB_j^t+\sum_{k\notin\mathcal{K}}(q_k+\sum_i c_ia_{ik})x_k^t\right)\right],\\
&=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_{j\notin\mathcal{J}}p_jB_j^t+\sum_{j\notin\mathcal{J}}\sum_{k\in\mathcal{N}(j)}(q_k+\sum_i c_ia_{ik})x_k^t\right)\right],\\
&\geq\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{j\notin\mathcal{J}}g_j^0\sum_{k\in\mathcal{N}(j)}x_k^t\right],\\
&=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{j\notin\mathcal{J}}g_j^0\left(\sum_{t=1}^{T}D_j^t-B_j^T\right)\right],\\
&=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{j\notin\mathcal{J}}g_j^0\sum_{t=1}^{T}D_j^t\right]-\liminf_{T\to\infty}\sum_{j\notin\mathcal{J}}g_j^0\frac{\mathbb{E}\left[B_j^T\right]}{T},\\
&=\sum_{j\notin\mathcal{J}}g_j^0\mathbb{E}\left[D_j\right].
\end{aligned}$$

We next complete the proof by showing (35) is larger than $\underline{C}^{\beta}(\mathcal{J})$. We observe that this bound can be interpreted as applying to a newsvendor network restricted to the products in $\mathcal{J}$ and the activities that serve them. Thus, for notational convenience in the rest of the proof, we drop the notation indicating dependence on the sets $\mathcal{J}$ and $\mathcal{K}$, with the understanding that we are focused on a system restricted to include only these products and activities. Specifically, fix $\mathcal{J}$ and let $\underline{C}^{\beta}=\underline{C}^{\beta}(\mathcal{J})$ and $F^{\beta}(\mathbf{S}|\mathbf{D})=F^{\beta}(\mathbf{S}|\mathbf{D},\mathcal{J})$. We then recall that the proof of Theorem 1 shows (35) is lower bounded by the SP $\underline{G}(\beta,0,0)$ (which we again assume denotes the SP for the problem restricted to the network on $\mathcal{J}$ and $\mathcal{K}$), which we in turn show is larger than $\underline{C}^{\beta}$ to complete the proof. To do so, first note that $\delta=\rho=0$ and $L=0$ imply the stochastic program $\underline{G}(\beta,0,0)$ can be simplified to

$$G^{\beta}(\mathbf{S},\mathbf{B}|\mathbf{D})=\left\{\begin{array}{ll}
\min & \sum_k\left((q_k+\sum_i a_{ik}(c_i-h_i))x_k^1+(q_k+\sum_i a_{ik}(c_i-\beta h_i))w_k^2\right)+\sum_j p_j(y_j^1+y_j^2)\\
\text{s.t.} & \sum_k a_{ik}(x_k^1+w_k^1)\leq S_i,\ \forall i\\
& \sum_k r_{kj}w_k^1+y_j^0=B_j,\ \forall j\\
& \sum_k r_{kj}x_k^1+y_j^1=D_j,\ \forall j\\
& \sum_k r_{kj}(x_k^1+w_k^1+w_k^2)+y_j^2=D_j+B_j,\ \forall j\\
& \mathbf{w}^l,\mathbf{x}^l,\mathbf{y}^l\geq 0,\ l\in\{0,1,2\}
\end{array}\right\}$$

$$\underline{G}^{\beta}=\min_{\mathbf{S}\geq 0,\mathbf{B}\geq 0}\sum_i\beta h_iS_i+\mathbb{E}\left[G^{\beta}(\mathbf{S},\mathbf{B}|\mathbf{D})\right],\tag{37}$$

We provide some intuition for this stochastic program. As in $\underline{G}(\beta,0,0)$, the variables $\mathbf{x}$ and $\mathbf{w}$ model fulfillment for newly arriving demand and existing backlog, respectively. The model considers fulfillment in two periods, denoted 1 and 2, with period 1 having both demand and backlog fulfillment, and period 2 having only backlog fulfillment. The variable $\mathbf{y}^0$ denotes existing backlog that is left as backlog in period 1, $\mathbf{y}^1$ denotes newly arriving demand that is left as backlog in period 1, and $\mathbf{y}^2$ denotes backlog from any source remaining in period 2.

The objective only counts costs from fulfilling period 1's demand ($\mathbf{x}^1$) and period 2's backlog ($\mathbf{w}^2$), as well as backlog costs from period 1 demand ($\mathbf{y}^1$) and any remaining period 2 backlog ($\mathbf{y}^2$). The costs are allocated in this way because the remaining costs are accounted for in a stochastic program for another period.

Then, observe that $G^\beta \geq C^\beta$ if $G^\beta(\mathbf{S},\mathbf{B}|\mathbf{D}) \geq F^\beta(\mathbf{S}|\mathbf{D})$ for all $\mathbf{S},\mathbf{B}$, and $\mathbf{D}$. To see that this is the case, consider an optimal solution for $G^\beta(\mathbf{S},\mathbf{B}|\mathbf{D})$ denoted $\mathbf{w}^{l*},\mathbf{x}^{l*},\mathbf{y}^{l*} \geq 0$ for $l \in \{0,1,2\}$, and observe that

$$\begin{aligned}
&\sum_k (q_k + \sum_i a_{ik}(c_i - \beta h_i))w_k^{2*} + \sum_j p_j y_j^{2*}, \\
&= \sum_j \left(\sum_{k\in\mathcal{N}(j)} (q_k + \sum_i a_{ik}(c_i - \beta h_i))w_k^{2*} + p_j y_j^{2*} \right), \\
&\geq \sum_j \left(g_j^\beta \sum_k r_{kj} w_k^{2*} + p_j y_j^{2*} \right), \\
&= \sum_j \left(g_j^\beta \sum_k r_{kj} w_k^{2*} + p_j y_j^{2*} + f_j^\beta (D_j + B_j - \sum_k r_{kj}(x_k^{1*} + w_k^{1*} + w_k^{2*}) - y_j^{2*}) \right), \\
&= \sum_j \left((g_j^\beta - f_j^\beta) \sum_k r_{kj} w_k^{2*} + (p_j - f_j^\beta) y_j^{2*} + f_j^\beta (y_j^{0*} + y_j^{1*}) \right), \\
&\geq \sum_j f_j^\beta y_j^{1*},
\end{aligned}$$

where the first equality follows because the sets $\mathcal{N}(j)$ form a partition of the set of activities, the first inequality follows from the definition of g_j^β, the second equality follows from the fourth constraint in the LP defining $G^\beta(\mathbf{S},\mathbf{B}|\mathbf{D})$ and the third equality follows from the second and third constraints in the same LP, and the final inequality follows since all decision variables are non-negative, $g_j^\beta, p_j \geq f_j^\beta$ by definition, and $f_j^\beta \geq 0$ by Condition 1. Thus, the final line implies that

$$G^\beta(\mathbf{S},\mathbf{B}|\mathbf{D}) \geq \sum_k (q_k + \sum_i a_{ik}(c_i - h_i))x_k^{1*} + \sum_j (p_j + f_j^\beta) y_j^{1*} \geq F^\beta(\mathbf{S}|\mathbf{D}),$$

since $\mathbf{x}^{1*},\mathbf{y}^{1*}$ constitute a feasible solution for $F^\beta(\mathbf{S}|\mathbf{D})$ as implied by the first and third constraints of $G^\beta(\mathbf{S},\mathbf{B}|\mathbf{D})$ and the fact that $\mathbf{w}^{1*} \geq 0$. Q.E.D.

Proof of Lemma 1. The proof follows from bounding costs incurred by products based on their membership in $\mathcal{J}^\beta$. First, for products in $\mathcal{J}^\beta$ we claim that

$$p_j + g_j^0 \leq (1+\beta)(f_j^\beta + p_j), \ \forall j \in \mathcal{J}^\beta. \tag{38}$$

To see this, observe that Condition 1 implies $q_k + \sum_i a_{ik}(c_i - \beta h_i) \geq (1-\beta)(q_k + \sum_i a_{ik} c_i)$ for all k. Therefore, for each j we have

$$g_j^\beta \geq (1-\beta) \min_{k\in\mathcal{N}(j)} \left\{ q_k + \sum_i a_{ik} c_i \right\} = (1-\beta) g_j^0.$$

Then, for $j \in \mathcal{J}^\beta$ we have $\beta g_j^0 \leq p_j$ so it follows that

$$p_j + g_j^0 = p_j + (1-\beta^2) g_j^0 + \beta^2 g_j^0 \leq p_j + (1+\beta) g_j^\beta + \beta p_j = (1+\beta)(g_j^\beta + p_j),$$

and thus (38) follows for j such that $g_j^\beta = f_j^\beta$. Otherwise, if $g_j^\beta \neq f_j^\beta$, then $f_j^\beta = p_j$, in which case

$$p_j + g_j^0 \leq \left(1 + \frac{1}{\beta}\right) p_j \leq (1+\beta) 2 p_j = (1+\beta)(f_j^\beta + p_j),$$

where the second inequality follows because $1+1/\beta \le 2(1+\beta)$ for $\beta \ge 1/2$. This completes the proof of (38). Next for products not in $\mathcal{J}^\beta$ we observe that

$$p_j + g_j^0 \le (1+\beta) g_j^0, \ \forall j \notin \mathcal{J}^\beta, \tag{39}$$

which follows directly from $p_j \le \beta g_j^0$ by the definition of $\mathcal{J}^\beta$. Then, let x_k^β for $k \in \mathcal{K}$ and y_j^β for $j \in \mathcal{J}^\beta$ denote an optimal solution to the LP defining $F^\beta(\mathbf{S}^\beta|\mathbf{D}, \mathcal{J}^\beta)$. Then, construct a feasible solution for the LP defining $F^{V^*}(\mathbf{S}^\beta|\mathbf{D})$ as follows:

$$x'_k = \begin{cases} x_k^\beta & k \in \mathcal{K} \\ 0 & k \notin \mathcal{K} \end{cases}, \qquad y'_j = \begin{cases} y_j^\beta & j \in \mathcal{J}^\beta \\ D_j & j \notin \mathcal{J}^\beta \end{cases}.$$

This solution inherits feasible for the LP defining $F^{V^*}(\mathbf{S}^\beta|\mathbf{D})$ from the LP defining $F^\beta(\mathbf{S}^\beta|\mathbf{D}, \mathcal{J}^\beta)$ for $k \in \mathcal{K}$ and $j \in \mathcal{J}^\beta$, and it is straightforward to observe that setting $x'_k = 0$ for $k \notin \mathcal{K}$ maintains the first constraint, while the second constraint is satisfied by $y'_j = D_j$ for $j \notin \mathcal{J}^\beta$. Thus, we have

$$\begin{aligned}
F^{V^*}(\mathbf{S}^\beta|\mathbf{D}) &\le \sum_k \left(q_k + \sum_i a_{ik}(c_i - h_i) \right) x'_k + \sum_j (p_j + g_j^0) y'_j, \\
&= \sum_{k \in \mathcal{K}} \left(q_k + \sum_i a_{ik}(c_i - h_i) \right) x_k^\beta + \sum_{j \in \mathcal{J}^\beta} (p_j + g_j^0) y_j^\beta + \sum_{j \notin \mathcal{J}^\beta} (p_j + g_j^0) D_j, \\
&\le (1+\beta) \left[\sum_{k \in \mathcal{K}} \left(q_k + \sum_i a_{ik}(c_i - h_i) \right) x_k^\beta + \sum_{j \in \mathcal{J}^\beta} (p_j + f_j^\beta) y_j^\beta + \sum_{j \notin \mathcal{J}^\beta} g_j^0 D_j \right], \\
&= (1+\beta) \left[F^\beta(\mathbf{S}^\beta|\mathbf{D}, \mathcal{J}^\beta) + \sum_{j \notin \mathcal{J}^\beta} g_j^0 D_j \right],
\end{aligned}$$

where the second inequality follows from (38), (39), and the fact that $q_k + \sum_i a_{ik}(c_i - h_i) \ge 0$ for all k by Condition 1. Q.E.D.

Proof of Theorem 2. By Corollary 1, the cost of the optimal ABBS policy is $\underset{\sim}{C}^{V^*}$, and for any $\beta \in [0.5, 1)$ we have

$$\begin{aligned}
\underset{\sim}{C}^{V^*} &\le \sum_i h_i S_i^\beta + \mathbb{E}\left[F^{V^*}(\mathbf{S}^\beta|\mathbf{D}) \right], \\
&\le \frac{1}{\beta} \sum_i \beta h_i S_i^\beta + (1+\beta) \left[\mathbb{E}\left[F^\beta(\mathbf{S}^\beta|\mathbf{D}, \mathcal{J}^\beta) \right] + \sum_{j \notin \mathcal{J}^\beta} g_j^0 \mu_j \right], \\
&\le \max\left(\frac{1}{\beta}, 1+\beta \right) C^*,
\end{aligned}$$

where the first inequality follows because $\mathbf{S}^\beta$ is feasible for (9), the second from Lemma 1, and the final inequality from Corollary 2. The function $\max\left(\frac{1}{\beta}, 1+\beta\right)$ is minimized when $1/\beta = 1+\beta$, which implies $\beta = (\sqrt{5}-1)/2 > 0.5$, giving an approximation factor of $(\sqrt{5}+1)/2 \approx 1.618$. Q.E.D.

Proof of Corollary 3. Letting $\beta = \delta = 1$ in the stochastic program (5) and noting $L = 0$, the result follows from Theorem 1 by noting $G(\mathbf{S}, \mathbf{B}, 1, 1, \rho|\mathcal{D})$ is equivalent to $F^\rho(\mathbf{S}, \mathbf{B}|\mathbf{D})$ by letting $x_k = x_k^1 + w_k^1$ and $y_j = y_j^0 + y_j^1$. Q.E.D.

Proof of Theorem 3. Let $\mathbf{S}^\rho$ and $\mathbf{B}^\rho$ denote an optimal solution to the stochastic program (12), and let $\mathbf{x}^\rho(\mathbf{D})$, $\mathbf{y}^\rho(\mathbf{D})$ denote an optimal solution to the LP defining $F^\rho(\mathbf{S}^\rho, \mathbf{B}^\rho|\mathbf{D})$. Then, define a solution for the LP defining $F^{V^*}(\mathbf{S}^\rho|\mathbf{D})$ as follows:

$$\begin{aligned} x'_k(\mathbf{D}) &= x^\rho_k(\mathbf{D}) \left(1 - \frac{B^\rho_{j(k)}}{\sum_{k'} r_{k'j(k)} x^\rho_{k'}(\mathbf{D})}\right)^+, \ \forall k, \\ y'_j(\mathbf{D}) &= D_j - \sum_k r_{kj} x'_k(\mathbf{D}), \ \forall j, \end{aligned}$$

where feasibility of this solution for the LP defining $F^{V^*}(\mathbf{S}^\rho|\mathbf{D})$ is inherited from feasibility of $\mathbf{x}^\rho(\mathbf{D})$, $\mathbf{y}^\rho(\mathbf{D})$ for the LP defining $F^\rho(\mathbf{S}^\rho, \mathbf{B}^\rho|\mathbf{D})$. Next we observe that, by definition we have

$$x^\rho_k(\mathbf{D}) \left(1 - \frac{B^\rho_{j(k)}}{\sum_{k'} r_{k'j(k)} x^\rho_{k'}(\mathbf{D})}\right) \le x'_k(\mathbf{D}) \le x^\rho_k(\mathbf{D}), \ \forall k. \tag{40}$$

Further, we claim that

$$y'_j(\mathbf{D}) \le y^\rho_j(\mathbf{D}), \ \forall j, \tag{41}$$

$$y'_j(\mathbf{D}) \le D_j, \ \forall j, \tag{42}$$

where (42) is clear by definition of $y'_j(\mathbf{D})$ and the fact that $x'_k(\mathbf{D}) \ge 0$, and (41) follows from observing that

$$\begin{aligned} y'_j(\mathbf{D}) &= D_j - \sum_k r_{kj} x'_k(\mathbf{D}), \\ &\le D_j - \sum_k r_{kj} x^\rho_k(\mathbf{D}) \left(1 - \frac{B^\rho_j}{\sum_{k'} r_{k'j} x^\rho_{k'}(\mathbf{D})}\right), \\ &= D_j + B^\rho_j - \sum_k r_{kj} x^\rho_k(\mathbf{D}), \\ &= y^\rho_j(\mathbf{D}), \end{aligned}$$

where the inequality follows from the lower bound in (40). Next we note that

$$\bar{h}_j \le \frac{1-\alpha}{\alpha} p_j, \ \forall j, \tag{43}$$

which follows from the fact that $\alpha(p_j + \bar{h}_j) \le p_j$ by the definition of α. Then, by Corollary 1, the cost of the optimal ABBS policy is $\underset{\sim}{C}^{V^*}$, we bound as follows

$$\begin{aligned} \underset{\sim}{C}^{V^*} &\le \sum_i h_i S^\rho_i + \mathbb{E}\left[F^{V^*}(\mathbf{S}^\rho|\mathbf{D})\right], \\ &\le \sum_i h_i S^\rho_i + \mathbb{E}\left[\sum_k \left(q_k + \sum_i a_{ik}(c_i - h_i)\right) x'_k(\mathbf{D}) + \sum_j (p_j + g^{V^*}_j) y'_j(\mathbf{D})\right], \\ &\le \sum_i h_i S^\rho_i + \mathbb{E}\left[\sum_k \left(q_k + \sum_i a_{ik} c_i\right) x^\rho_k(\mathbf{D}) - \sum_k \sum_i a_{ik} h_i x^\rho_k(\mathbf{D}) \left(1 - \frac{B^\rho_{j(k)}}{\sum_{k'} r_{k'j} x^\rho_{k'}(\mathbf{D})}\right)\right] \\ &\quad + \mathbb{E}\left[\sum_j \left(p_j y^\rho_j(\mathbf{D}) + g^{V^*}_j D_j\right)\right], \\ &= \mathbb{E}\left[\sum_i h_i \left(S^\rho_i - \sum_k a_{ik} x^\rho_k(\mathbf{D})\right) + \sum_k \left(q_k + \sum_i a_{ik} c_i\right) x^\rho_k(\mathbf{D}) + \sum_j p_j y^\rho_j(\mathbf{D})\right] \\ &\quad + \mathbb{E}\left[\sum_j \frac{B^\rho_j}{\sum_{k'} r_{k'j} x^\rho_{k'}(\mathbf{D})} \sum_k r_{kj} x^\rho_k(\mathbf{D}) \sum_i a_{ik} h_i\right] + \sum_j g^{V^*}_j \mu_j, \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mathbb{E}\left[\sum_i h_i \left(S_i^\rho - \sum_k a_{ik} x_k^\rho(\mathbf{D})\right) + \sum_k \left(q_k + \sum_i a_{ik} c_i\right) x_k^\rho(\mathbf{D}) + (1-\rho)\sum_j p_j y_j^\rho(\mathbf{D})\right]}{1-\rho} \\
&\quad + \sum_j \bar{h}_j B_j^\rho + \sum_j g_j^{V^*} \mu_j, \\
&\leq \frac{\sum_i h_i S_i^\rho + \mathbb{E}\left[F^\rho(\mathbf{S}^\rho, \mathbf{B}^\rho | \mathbf{D})\right]}{1-\rho} + \frac{1-\alpha}{\alpha\rho}\rho \sum_j p_j B_j^\rho + \sum_j g_j^{V^*} \mu_j, \\
&\leq \max\left(\frac{1}{1-\rho}, \frac{1-\alpha}{\alpha\rho}\right) \underline{C}^\rho + \sum_j g_j^{V^*} \mu_j, \\
&\leq \left(1 + \max\left(\frac{1}{1-\rho}, \frac{1-\alpha}{\alpha\rho}\right)\right) C^*,
\end{aligned}
$$

where the first inequality follows because $\mathbf{S}^\rho$ is a feasible solution for the stochastic program (9), the second inequality follows because $\mathbf{x}'(\mathbf{D}), y'(\mathbf{D})$ is a feasible solution to the LP defining $F^{V^*}(\mathbf{S}^\rho|\mathbf{D})$, the third inequality from the bounds in (40), (41), and (42), the fourth inequality from the fact that $S_i^\rho - \sum_k a_{ik} x_k^\rho(\mathbf{D}) \geq 0$ for all i and $\sum_i a_{ik} h_i \leq \underline{h}_{j(k)}$ for all k, the fifth from (43), the sixth from the definition of $\underline{C}^\rho$, and the final inequality from the lower bounds in Corollary 3 and Proposition 2. Then the result follows from letting $\rho = 1-\alpha$. Q.E.D.

Appendix D: Proofs for Lead Times in Section 5

Proof of Lemma 2. First, we assume without loss of generality that during periods $t \leq 0$ all quantities are 0 (inventories, backlogs, fulfillment quantities, etc.) except the order quantities. We then observe that in period $1-L$, (19) implies the order quantity $z_i^{1-L} = S_i$ for all i since all inventory and backlog levels and prior order quantities are zero. Then, for all periods $1-L < t \leq 0$ (19) implies $z_i^t = 0$ since all inventory and backlog levels are still 0, and $\sum_{s=t-L}^{t-1} z_i^s = z_i^{1-L} = S_i$.

We will show the claimed identities by induction on the period t. We start by verifying the claims for the base case $t = 1$. For each i we have

$$
\begin{aligned}
I_i^1 &= I_i^0 + z_i^{1-L} - \sum_k a_{ik} x_k^1, \\
&= S_i - \sum_k a_{ik} (\tilde{X}_k^1 + \tilde{B}_k^{1-L-1}) \\
&= S_i - \sum_k a_{ik} (\hat{B}_k^0 + \hat{D}_k^1 - \hat{B}_k^1) \\
&= S_i - \sum_k a_{ik} \left(\sum_{s=1-L}^{1} \hat{D}_k^s - \hat{B}_k^1 \right),
\end{aligned}
$$

where the first equality is the evolution equation (1), the second follows from $I_i^0 = 0$ and the fulfillment policy (17), the third follows from $\tilde{\mathbf{B}}^s = 0$ for $s \leq 0$ and the evolution equation (14) for $\hat{B}_k^t$, and the final equality follows since $\hat{D}_k^s = 0$ for $s \leq 0$.

Further, for each i and $1-L < t \leq 1$ we have

$$
\begin{aligned}
z_i^t &= \left(S_i - I_i^{t-1} - \sum_{s=t-L}^{t-1} z_i^s + \sum_k a_{ik} \bar{B}_k^{t-1} \right)^+ \\
&= 0 \\
&= \sum_k a_{ik} \left(\hat{D}_k^{t-1} + \tilde{B}_k^{t-1} \right),
\end{aligned}
$$

where the first equality is the order policy (19), the second follows from $\mathbf{I}^{t-1} = 0$, $\mathbf{B}^{t-1} = 0$, and $\sum_{s=t-L}^{t-1} z_i^s = S_i$, and the third from $\hat{\mathbf{D}}^{t-1} = 0$ and $\tilde{\mathbf{B}}^{t-1} = 0$ for $t \leq 1$.

Also, for each j we have

$$\begin{aligned}\bar{B}_k^1 &= \bar{B}_k^0 + \hat{D}_k^1 + \tilde{B}_k^1 - x_k^1, \\ &= \hat{B}_k^1 + \tilde{X}_k^1 - \hat{B}_k^0 + \tilde{B}_k^1 - \tilde{X}_k^1 - \tilde{B}_k^{1-L-1}, \\ &= \hat{B}_k^1 + \sum_{s=1-L}^{1} \tilde{B}_k^s,\end{aligned}$$

where the first equality is the backlog evolution equation (18) for $\bar{B}_k^t$, the second follows from $\bar{B}_k^0 = 0$, the fulfillment policy (17), and the evolution equation (14) for $\hat{B}_k^t$, and the final equality follows from the fact that $\tilde{\mathbf{B}}^s = 0$ for $s \le 0$ and $\hat{B}_k^0 = 0$. This completes the proof of the base case $t = 1$.

Then, for any $t \ge 2$, assume the claim is true for all $s \le t-1$ and consider period t. For each i we have

$$\begin{aligned}I_i^t &= I_i^{t-1} + z_i^{t-L} - \sum_k a_{ik} x_k^t \\ &= S_i - \sum_k a_{ik} \left(\sum_{s=t-L-1}^{t-1} \hat{D}_k^s - \hat{B}_k^{t-1} \right) + \sum_k a_{ik} \left(\hat{D}_k^{t-L-1} + \tilde{B}_k^{t-L-1} \right) - \sum_k a_{ik} (\tilde{X}_k^t + \tilde{B}_k^{t-L-1}) \\ &= S_i - \sum_k a_{ik} \left(\sum_{s=t-L}^{t} \hat{D}_k^s - \hat{B}_k^t \right),\end{aligned}$$

where the first inequality is the evolution equation (1), the second follows from the induction hypothesis and the fulfillment policy (17), and the final follows from the backlog evolution equation (14) for $\hat{B}_k^t$.

Further, for each i we have

$$\begin{aligned}z_i^t &= \left(S_i - I_i^{t-1} - \sum_{s=t-L}^{t-1} z_i^s + \sum_k a_{ik} \bar{B}_k^{t-1} \right)^+ \\ &= \left(\sum_k a_{ik} \left(\sum_{s=t-L-1}^{t-1} \hat{D}_k^s - \hat{B}_k^{t-1} \right) - \sum_{s=t-L-1}^{t-2} \sum_k a_{ik} \left(\hat{D}_k^s + \tilde{B}_k^s \right) + \sum_k a_{ik} \left(\hat{B}_k^{t-1} + \sum_{s=t-L-1}^{t-1} \tilde{B}_k^s \right) \right)^+ \\ &= \sum_k a_{ik} \left(\hat{D}_k^{t-1} + \tilde{B}_k^{t-1} \right),\end{aligned}$$

where the first equality is the ordering policy (19), the second follows from the inductive hypothesis, and the third follows from canceling terms and observing the resulting expression is non-negative.

Finally, for each k we have

$$\begin{aligned}\bar{B}_k^t &= \bar{B}_k^{t-1} + \hat{D}_k^t + \tilde{B}_k^t - x_k^t \\ &= \hat{B}_k^{t-1} + \sum_{s=t-L-1}^{t-1} \tilde{B}_k^s + \hat{D}_k^t + \tilde{B}_k^t - \tilde{X}_k^t - \tilde{B}_k^{t-L-1} \\ &= \hat{B}_k^t + \sum_{s=t-L}^{t} \tilde{B}_k^s,\end{aligned}$$

where the first equality is the backlog evolution equation (18) for $\bar{B}_k^t$, the second follows from the inductive hypothesis and the ordering policy (17), and the third follows from the backlog evolution equation (14) for $\hat{B}_k^t$. This completes the induction step and the proof of the claimed identities.

Next, we demonstrate feasibility of the policy. It is clear that the policy is non-anticipating since we assume $\tilde{\mathbf{X}}^t$, $\tilde{\mathbf{B}}^t$, and $\hat{\mathbf{D}}^t$ are, and that the order and fulfillment quantities are non-negative. It thus remains to show that the backlog and inventory state variables remain non-negative under this policy. To see that the backlog levels remain non-negative, note that the third identity implies

$$B_j^t = \sum_k r_{kj} \bar{B}_k^t = \sum_k r_{kj} \left(\hat{B}_k^t + \sum_{s=t-L}^{t} \tilde{B}_j^s \right),$$

so that $B_j^t \geq 0$ if we have $\hat{B}_k^t \geq 0$, as $\tilde{B}_j^s \geq 0$ by definition. But $\hat{B}_k^t \geq 0$ is guaranteed by the evolution equation (14) and the constraint (15). Then, for the inventory levels, by the first identity we have

$$\begin{aligned} I_i^t &= S_i - \sum_k a_{ik} \left(\sum_{s=t-L}^{t} \hat{D}_k^s - \hat{B}_k^t \right) \\ &= S_i - \sum_k a_{ik} \left(\sum_{s=t-L}^{t-1} \hat{D}_k^s + \tilde{X}_k^t - \hat{B}_k^{t-1} \right) \\ &\geq 0, \end{aligned}$$

where the second equality follows from the backlog evolution equation (14) for $\hat{B}_k^t$, and the inequality follows from constraint (16). Q.E.D.

For notational convenience, let for each product j, let $\breve{B}_j^t = \sum_k r_{kj} \tilde{B}_k^t$ denote the period t backlog for product j that is resigned to waiting for replenishment. Then observe that by (22), for each j we have

$$\breve{B}_j^t = \sum_k r_{kj} v_{kj}^* \left(D_j^t - \sum_{k'} r_{k'j} \tilde{X}_{k'}^t \right) = D_j^t - \sum_k r_{kj} \tilde{X}_k^t,$$

where the second equality follows because $\sum_k r_{kj} v_{kj}^* = 1$ for each j.

Proof of Corollary 4. First observe that $\beta = \delta = 1$ and $\rho = 0$ simplifies the stochastic program (5) to the following

$$G^L(\mathbf{S}, \mathbf{B} | \tilde{\mathbf{D}}^{L+1}) = \left\{ \begin{array}{ll} \min_{\mathbf{x}, \mathbf{y} \geq 0} & \sum_k u_k^L x_k + \sum_j p_j y_j \\ \text{s.t.} & \sum_k a_{ik} x_k \leq S_i, \ \forall i \\ & \sum_k r_{kj} x_k + y_j = B_j + \tilde{D}_j^{L+1}, \ \forall j \end{array} \right\}$$

$$\underline{G}^L = \min_{\mathbf{S}, \mathbf{B} \geq 0} \sum_i h_i S_i + \mathbb{E}\left[G^L(\mathbf{S}, \mathbf{B} | \tilde{\mathbf{D}}^{L+1}) \right], \tag{44}$$

by letting $x_k = \sum_{l=1}^{L+1} \left(x_k^l + w_k^l \right)$ and $y_j = \sum_{l=0}^{L+1} y_j^l$. Then, by Theorem 1, the result follows by showing $\underline{C}^L \leq \underline{G}^L$ (i.e., the optimum in (44) is obtained by $\mathbf{B} = 0$), and we further observe that this follows if $G^L(\mathbf{S}, \mathbf{B} | \tilde{\mathbf{D}}^{L+1}) \geq F^L(\mathbf{S} | \tilde{\mathbf{D}}^{L+1})$ for all $\mathbf{S}, \mathbf{B}$, and $\tilde{\mathbf{D}}^{L+1}$. To show this, for a given $\mathbf{S}, \mathbf{B}$, and $\tilde{\mathbf{D}}^{L+1}$, let x_k^* and y_j^* denote an optimal solution to the LP defining $G^L(\mathbf{S}, \mathbf{B} | \tilde{\mathbf{D}}^{L+1})$, and let

$$\begin{aligned} x'_k &= x_k^* \min \left(1, \frac{\tilde{D}_j^{L+1}}{\sum'_k r_{k'j} x_{k'}^*} \right), \ \forall k, \\ y'_j &= \tilde{D}_j^{L+1} - \sum_k r_{kj} x'_k, \ \forall j, \end{aligned}$$

denote a feasible solution to $F^L(\mathbf{S} | \tilde{\mathbf{D}}^{L+1})$, where feasibility for the first constraint follows from feasibility of x_k^* for the LP defining $G^L(\mathbf{S}, \mathbf{B} | \tilde{\mathbf{D}}^{L+1})$ and the fact that $x'_k \leq x_k^*$, and non-negativity for y'_j follows from observing that

$$\begin{aligned} y'_j &= \tilde{D}_j^{L+1} - \sum_k r_{kj} x_k^* \min \left(1, \frac{\tilde{D}_j^{L+1}}{\sum'_k r_{k'j} x_{k'}^*} \right) \\ &= \left(\tilde{D}_j^{L+1} - \sum_k r_{kj} x_k^* \right)^+ \\ &\geq 0. \end{aligned}$$

Then we note that $y_j^* \geq y_j'$ for all j, which follows from the above observation that $y_j' = \left(\tilde{D}_j^{L+1} - \sum_k r_{kj} x_k^*\right)^+$ and the fact that $y_j^* \geq 0$ and

$$y_j^* = B_j + \tilde{D}_j^{L+1} - \sum_k r_{kj} x_k^* \geq \tilde{D}_j^{L+1} - \sum_k r_{kj} x_k^*.$$

Then we have

$$G^L(\mathbf{S},\mathbf{B}|\tilde{\mathbf{D}}^{L+1}) = \sum_k u_k^L x_k^* + \sum_j p_j y_j^* \geq \sum_k u_k^L x_k' + \sum_j p_j y_j' \geq F^L(\mathbf{S}|\tilde{\mathbf{D}}^{L+1}),$$

where the first inequality follows from $x_k^* \geq x_k'$ and $u_k^L \geq 0$ (i.e., Condition 2) for all k and $y_j^* \geq y_j'$ for all j, and the second inequality follows from the feasibility of x_k' and y_j' for the LP defining $F^L(\mathbf{S}|\tilde{\mathbf{D}}^{L+1})$. Q.E.D.

Lemma 7. *An ABBS policy with finite* $\mathbf{S}$ *that uses* $\hat{\mathbf{D}}^t$ *and* $\tilde{\mathbf{B}}^t$ *defined by* (21) *and* (22) *has long-run average cost less than*

$$\sum_i h_i S_i + \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[\sum_k u_k^L \sum_{s=t-L}^{t} \tilde{X}_k^s + \sum_j \left(g_j^{V^*} \check{B}_j^{t-L-1} + p_j \sum_{s=t-L}^{t} \check{B}_j^s\right)\right]$$

Proof. First, note that by rearranging the sum we have

$$\sum_{t=1}^{T} \tilde{X}_k^t \leq \frac{1}{L+1} \sum_{t=1}^{T+L} \sum_{s=t-L}^{t} \tilde{X}_k^s,$$

for all activities k and all time horizons $T \geq 1$. Therefore, we have

$$\begin{aligned}
\sum_{t=1}^{T} \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t &= \sum_{t=1}^{T} \sum_k (q_k + \sum_i c_i a_{ik}) (\tilde{X}_k^t + \tilde{B}_k^{t-L-1}) \\
&= \sum_{t=1}^{T} \left(\sum_k (q_k + \sum_i c_i a_{ik}) \tilde{X}_k^t + \sum_j g_j^{V^*} \check{B}_j^{t-L-1} \right) \\
&\leq \frac{1}{L+1} \sum_{t=1}^{T+L} \sum_{s=t-L}^{t} \sum_k (q_k + \sum_i c_i a_{ik}) \tilde{X}_k^s + \sum_{t=1}^{T} \sum_j g_j^{V^*} \check{B}_j^{t-L-1},
\end{aligned}$$

where the first equality follows from (17). Next, letting $\bar{g}_j = \max_{k\in\mathcal{N}(j)} \{q_k + \sum_i c_i a_{ik}\}$, observe that

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\frac{1}{L+1} \sum_{t=1}^{T+L} \sum_{s=t-L}^{t} \sum_k (q_k + \sum_i c_i a_{ik}) \tilde{X}_k^s \right] \\
&\leq \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\frac{1}{L+1} \sum_{t=T+1}^{T+L} \sum_{s=t-L}^{t} \sum_j \bar{g}_j \sum_k r_{kj} \tilde{X}_k^s + \frac{1}{L+1} \sum_{t=1}^{T} \sum_{s=t-L}^{t} \sum_k (q_k + \sum_i c_i a_{ik}) \tilde{X}_k^s \right] \\
&\leq \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\frac{1}{L+1} \sum_{t=T+1}^{T+L} \sum_{s=t-L}^{t} \sum_j \bar{g}_j D_j^s + \frac{1}{L+1} \sum_{t=1}^{T} \sum_{s=t-L}^{t} \sum_k (q_k + \sum_i c_i a_{ik}) \tilde{X}_k^s \right] \\
&= \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\sum_j \bar{g}_j \mu_j + \frac{1}{L+1} \sum_{t=1}^{T} \sum_{s=t-L}^{t} \sum_k (q_k + \sum_i c_i a_{ik}) \tilde{X}_k^s \right] \\
&= \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\frac{1}{L+1} \sum_{t=1}^{T} \sum_{s=t-L}^{t} \sum_k (q_k + \sum_i c_i a_{ik}) \tilde{X}_k^s \right],
\end{aligned}$$

where the second inequality follows from constraint (15), the fact that $\sum_k r_{kj}\hat{D}_k^t \le D_j^t$, and the fact that $\hat{B}_k^{t-1}=0$, and the final equality follows from the finiteness of mean demand. This implies that

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_i h_i I_i^t+\sum_k (q_k+\sum_i c_i a_{ik})x_k^t\right)\right]\\
&\le \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_i h_i I_i^t+\frac{1}{L+1}\sum_{s=t-L}^{t}\sum_k (q_k+\sum_i c_i a_{ik})\tilde{X}_k^s+\sum_j g_j^{V^*}\check{B}_j^{t-L-1}\right)\right]\\
&= \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_i h_i S_i+\sum_k \frac{q_k+\sum_i a_{ik}(c_i-(L+1)h_i)}{L+1}\sum_{s=t-L}^{t}\tilde{X}_k^s+\sum_j g_j^{V^*}\check{B}_j^{t-L-1}\right)\right]\\
&= \sum_i h_i S_i+\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_k u_k^L\sum_{s=t-L}^{t}\tilde{X}_k^s+\sum_j g_j^{V^*}\check{B}_j^{t-L-1}\right)\right],
\end{aligned}$$

where the first equality follows by the first identity of Lemma 2 and the facts that $\tilde{X}_k^t=\hat{D}_k^t$ and $\hat{B}_k^t=0$. Further, we also observe that the first identity of Lemma 2 implies that $I_i^t\le S_i$ for all i and all t, so that

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_i c_i I_i^t\right]\le \limsup_{T\to\infty}\frac{1}{T}\sum_i c_i S_i=0,$$

since $\mathbf{S}$ is finite. Thus, by Lemma 3, the long-run average cost is

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_i c_i I_i^T+\sum_{t=1}^{T}\left(\sum_i h_i I_i^t+\sum_j p_j B_j^t+\sum_k (q_k+\sum_i c_i a_{ik})x_k^t\right)\right]\\
&\le \sum_i h_i S_i+\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_k u_k^L\sum_{s=t-L}^{t}\tilde{X}_k^s+\sum_j\left(g_j^{V^*}\check{B}_j^{t-L-1}+p_j B_j^t\right)\right)\right]\\
&= \sum_i h_i S_i+\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\left(\sum_k u_k^L\sum_{s=t-L}^{t}\tilde{X}_k^s+\sum_j\left(g_j^{V^*}\check{B}_j^{t-L-1}+p_j\sum_{s=t-L}^{t}\check{B}_j^s\right)\right)\right],
\end{aligned}$$

where the equality follows from the third identity in Lemma 2. Q.E.D.

Lemma 8. *The random variables* $\tilde{X}_k^t$ *defined by Algorithm 1 are non-anticipating, non-negative, and, together with* $\hat{\mathbf{D}}^t\ge 0$ *and* $\tilde{\mathbf{B}}^t\ge 0$ *defined in* (21) *and* (22), *feasible for constraints* (15) *and* (16).

Proof. It is clear that $\tilde{\mathbf{X}}^t$ is non-anticipating because Algorithm 1 only uses information available at time t, and non-negativity follows by the non-negativity of W_k^t. Then, it is clear that (15) is satisfied because $\hat{D}_k^t=\tilde{X}_k^t$ and $\hat{B}_k^t=0$ for all k and t. Further, (16) is guaranteed for each i by the definition of $\tilde{X}_k^t$ in Step 4 in Algorithm 1 because $\hat{D}_k^t=\tilde{X}_k^t$ and $\hat{B}_k^t=0$ for all k and t. Thus, it remains to show that $\tilde{\mathbf{B}}^t\ge 0$, which by (22) is equivalent to

$$\sum_k r_{kj}\tilde{X}_k^t\le D_j^t,$$

for each j. To see this, first observe that for each j

$$\sum_k r_{kj}x_k(\mathbf{D}^t+\tilde{\mathbf{d}}^L)=D_j^t+\tilde{d}_j^L-y_j(\mathbf{D}^t+\tilde{\mathbf{d}}^L)\le D_j^t+\tilde{d}_j^L,$$

where the first equality follows from the second constraint of the LP defining $F^L(\mathbf{S}^L|\tilde{\mathbf{D}}^{L+1})$, and the inequality follows from $y_j(\mathbf{D}^t+\tilde{\mathbf{d}}^L)\ge 0$. We then have for each j

$$\sum_k r_{kj}\tilde{X}_k^t\le\sum_k r_{kj}W_k^t=\frac{D_j^t}{D_j^t+\tilde{d}_j^L}\sum_k r_{kj}x_k(\mathbf{D}^t+\tilde{\mathbf{d}}^L)\le D_j^t.$$

Q.E.D.

Recall that $\bar{x}_k = \mathbb{E}\left[x_k(\tilde{\mathbf{D}}^{L+1})\right]$ and $\bar{y}_j = \mathbb{E}\left[y_j(\tilde{\mathbf{D}}^{L+1})\right]$ denote the expected fulfillment quantity for activity k and the expected shortage for product j, respectively, in the optimal solution of the stochastic program (20).

Lemma 9. *For each k and t the W_k^t variables defined in Algorithm 1 satisfy $\tilde{X}_k^t \leq W_k^t$ and*

$$\mathbb{E}\left[W_k^t\right] = \frac{\bar{x}_k}{L+1}$$

Proof. It is clear by the definition in Step 4 of Algorithm 1 that $\tilde{X}_k^t \leq W_k^t$. Then, for the expectation bound, recall that $\mathbf{D}^l$ for $l = 1, \ldots, L+1$ are i.i.d. with the same distribution as $\mathbf{D}^t$ for all t. Then, W_k^t has the same distribution as

$$Z_k^l = \frac{x_k\left(\sum_{l'=1}^{L+1} \mathbf{D}^{l'}\right) D_{j(k)}^l}{\sum_{l'=1}^{L+1} D_{j(k)}^{l'}}$$

for any $l = 1, \ldots, L+1$. Therefore, we have

$$(L+1)\mathbb{E}\left[W_k^t\right] = \mathbb{E}\left[\sum_{l=1}^{L+1} Z_k^l\right] = \mathbb{E}\left[\sum_{l=1}^{L+1} \frac{x_k\left(\sum_{l'=1}^{L+1} \mathbf{D}^{l'}\right) D_{j(k)}^l}{\sum_{l'=1}^{L+1} D_{j(k)}^{l'}}\right] = \mathbb{E}\left[x_k\left(\sum_{l'=1}^{L+1} \mathbf{D}^{l'}\right)\right] = \bar{x}_k,$$

where the last equality follows because $\sum_{l'=1}^{L+1} \mathbf{D}^{l'}$ has the same distribution as $\tilde{\mathbf{D}}^{L+1}$. This implies the claim. Q.E.D.

Lemma 10. *For base-stock levels $(\eta+1)\mathbf{S}^L$ for any $\eta \geq 0$, using the $\tilde{X}_k^t$ variables defined in Algorithm 1 implies that for each j and t we have*

$$\mathbb{E}\left[\check{B}_j^t\right] \leq \frac{m}{\eta}\mu_j + \frac{\bar{y}_j}{L+1}.$$

Proof. We first claim that for any k and t, if $\sum_{s=t-L}^{t-1} \sum_{k'} a_{ik'} \tilde{X}_{k'}^s \leq \eta S_i^L$ for all $i \in \mathcal{S}(k)$, then we have $\tilde{X}_k^t = W_k^t$. To see this, observe that for all k we have $D_{j(k)}^t \leq D_{j(k)}^t + \tilde{d}_{j(k)}^L$, which implies that

$$\sum_k a_{ik} W_k^t = \sum_k a_{ik} \frac{x_k(\mathbf{D}^t + \tilde{\mathbf{d}}^L) D_{j(k)}^t}{D_{j(k)}^t + \tilde{d}_{j(k)}^L} \leq \sum_k a_{ik} x_k(\mathbf{D}^t + \tilde{\mathbf{d}}^L) \leq S_i^L,$$

where the final inequality follows from the first constraint of the LP defining $F^L(\mathbf{S}^L | \mathbf{D}^t + \tilde{\mathbf{d}}^L)$. Thus, if $\sum_{s=t-L}^{t-1} \sum_{k'} a_{ik'} \tilde{X}_{k'}^s \leq \eta S_i^L$ for all $i \in \mathcal{S}(k)$ then for each $i \in \mathcal{S}(k)$ we have

$$\sum_k a_{ik} W_k^t + \sum_{s=t-L}^{t-1} \sum_{k'} a_{ik'} \tilde{X}_{k'}^s \leq (\eta+1) S_i^L,$$

implying that the minimum in Step 4 of Algorithm 1 is attained by W_k^t (since $\tilde{X}_k^t$ is initialized to zero before this step and so is 0 on the right hand expression of the assignment), and so we have $\tilde{X}_k^t = W_k^t$.

Next, observe that if $\sum_{s=t-L}^{t-1} \sum_{k'} a_{ik'} W_{k'}^s \leq \eta S_i^L$ for all $i \in \mathcal{S}(k)$, then because $\tilde{X}_k^s \leq W_{k'}^s$ by definition, we arrive at the same conclusion that $\tilde{X}_k^t = W_k^t$. Thus, letting $\mathscr{A}_k$ denote the event that $\sum_{s=t-L}^{t-1} \sum_{k'} a_{ik'} W_{k'}^s \leq \eta S_i^L$ for all $i \in \mathcal{S}(k)$, and recalling that $\tilde{X}_k^t \geq 0$, by the law of total expectation we have

$$\mathbb{E}\left[\tilde{X}_k^t\right] \geq \mathbb{P}(\mathscr{A}_k) \mathbb{E}\left[W_k^t | \mathscr{A}_k\right] = \mathbb{P}(\mathscr{A}_k) \mathbb{E}\left[W_k^t\right] = \mathbb{P}(\mathscr{A}_k) \frac{\bar{x}_k}{L+1},$$

where the first equality follows because $W_{k'}^s$ for all k' and all $s \leq t-1$ (and hence $\mathscr{A}_k$) is independent of W_k^t, and the second follows from Lemma 9.

Then, for each j let $\zeta_j = \min_{k\in\mathcal{N}(j)} \mathbb{P}(\mathscr{A}_k)$, and we have by the definition of $\tilde{B}_j^t$ that

$$\begin{aligned}
\mathbb{E}\left[\check{B}_j^t\right] &= \mathbb{E}\left[D_j^t - \sum_k r_{kj}\tilde{X}_k^t\right] \\
&= \mu_j - \sum_k r_{kj}\mathbb{E}\left[\tilde{X}_k^t\right] \\
&\le \mu_j - \zeta_j \sum_k r_{kj}\frac{\bar{x}_k}{L+1} \\
&= (1-\zeta_j)\mu_j + \zeta_j\left(\mu_j - \sum_k r_{kj}\frac{\bar{x}_k}{L+1}\right) \\
&= (1-\zeta_j)\mu_j + \zeta_j\frac{\bar{y}_j}{L+1} \\
&\le (1-\zeta_j)\mu_j + \frac{\bar{y}_j}{L+1},
\end{aligned}$$

where the second to last line follows from the second constraint in the LP defining $F^L(\mathbf{S}^L|\tilde{\mathbf{D}}^{L+1})$, and the last line follows from $\zeta_j \le 1$. Thus, the result will follow from an upper bound on $1-\zeta_j$, which we can rewrite as

$$1-\zeta_j = 1 - \min_{k\in\mathcal{N}(j)} \mathbb{P}(\mathscr{A}_k) = \max_{k\in\mathcal{N}(j)}(1-\mathbb{P}(\mathscr{A}_k)) = \max_{k\in\mathcal{N}(j)} \mathbb{P}\left(\mathscr{A}_k^C\right),$$

where $\mathscr{A}_k^C$ denotes the complement of $\mathscr{A}_k$. We thus focus on upper bounding the probability of $\mathscr{A}_k^C$, which is the event that $\sum_{s=t-L}^{t-1}\sum_{k'} a_{ik'}W_{k'}^s > \eta S_i^L$ for any $i \in \mathcal{S}(k)$, or letting $Z_i^s = \sum_{k'} a_{ik'}W_{k'}^s$, we can write the event as

$$\mathscr{A}_k^C = \bigcup_{i\in\mathcal{S}(k)} \sum_{s=t-L}^{t-1} Z_i^s > \eta S_i^L.$$

Therefore, for each k we have

$$\begin{aligned}
\mathbb{P}\left(\mathscr{A}_k^C\right) &\le \sum_{i\in\mathcal{S}(k)} \mathbb{P}\left(\sum_{s=t-L}^{t-1} Z_i^s > \eta S_i^L\right) \\
&\le \sum_{i\in\mathcal{S}(k)} \frac{\mathbb{E}\left[\sum_{s=t-L}^{t-1} Z_i^s\right]}{\eta S_i^L} \\
&= \sum_{i\in\mathcal{S}(k)} \frac{\frac{L}{L+1}\sum_k' a_{ik'}\bar{x}_{k'}}{\eta S_i^L} \\
&\le \sum_{i\in\mathcal{S}(k)} \frac{1}{\eta} \\
&\le \frac{m}{\eta},
\end{aligned}$$

where the first inequality is a union bound, the second is Markov's, the equality follows from Lemma 9, the second to last inequality follows from the first constraint of the LP defining $F^L(\mathbf{S}^L|\tilde{D}^{L+1})$, and the last inequality follows from the definition of m. Q.E.D.

For the following proof, let C^{L*} denote the optimal long-run average cost for the problem with lead time L.

Proof of Theorem 4. Consider an ABBS policy using base-stock levels $(\eta+1)\mathbf{S}^L$ and the $\tilde{X}_k^t$ variables defined in Algorithm 1. Then by Lemma 7 the cost of this policy is less than

$$\sum_i h_i(\eta+1)S_i^L + \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^T \mathbb{E}\left[\sum_k u_k^L \sum_{s=t-L}^t \tilde{X}_k^s + \sum_j\left(g_j^{V^*}\check{B}_j^{t-L-1} + p_j\sum_{s=t-L}^t \check{B}_j^s\right)\right]$$

$$
\begin{aligned}
&\leq \sum_i h_i(\eta+1)S_i^L + \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_k u_k^L \sum_{s=t-L}^{t}\frac{\bar{x}_k}{L+1} + \sum_j\left(g_j^{V^*}\mu_j + p_j\sum_{s=t-L}^{t}\left(\frac{m}{\eta}\mu_j + \frac{\bar{y}_j}{L+1}\right)\right)\\
&= \sum_i h_i(\eta+1)S_i^L + \sum_k u_k^L\sum_{s=t-L}^{t}\bar{x}_k + \sum_j\left(g_j^{V^*}\mu_j + p_j\left(\frac{m(L+1)}{\eta}\mu_j + \bar{y}_j\right)\right)\\
&= \underline{C}^L + \sum_j g_j^{V^*}\mu_j + \eta\sum_i h_i S_i^L + \frac{m(L+1)}{\eta}\sum_j p_j\mu_j\\
&\leq 2C^{L*} + \left(\eta + \frac{m\gamma(L+1)}{\eta}\right)C^{L*},
\end{aligned}
$$

where the first inequality follows from Lemmas 9 and 10 and the fact that $\tilde{B}_j^{t-L-1} = D_j^{t-L-1} - \sum_k r_{kj}\tilde{X}_k^{t-L-1} \leq D_j^{t-L-1}$, and the final inequality follows from the lower bounds on optimal cost in Corollary 4 and Proposition 2 and the fact that $\sum_i h_i S_i^L \leq \underline{C}^L$ (since $F^L(\mathbf{S}^L|\tilde{D}^{L+1}) \geq 0$ by Condition 2). The result then follows by letting $\eta = \sqrt{m\gamma(L+1)}$ to minimize the bound. Q.E.D.

Next, a key ingredient for proving our approximation guarantee is the following bound, which follows from Chebychev's inequality.

Lemma 11. *For a random variable X with $\mu = \mathbb{E}[X] < \infty$, $\sigma^2 = Var(X) < \infty$, and any $a > 1$,*

$$\mathbb{E}\left[(X - a\mu)^+\right] \leq \frac{\sigma^2}{(a-1)\mu}.$$

Proof. Writing the expectation as the integral of the survival function, we have

$$
\begin{aligned}
\mathbb{E}\left[(X-a\mu)^+\right] &= \int_0^\infty \mathbb{P}\left((X-a\mu)^+ \geq t\right)dt\\
&\leq \int_0^\infty \mathbb{P}\left(|X-\mu| \geq (a-1)\mu + t\right)dt\\
&\leq \int_0^\infty \frac{\sigma^2}{((a-1)\mu+t)^2}dt\\
&= \frac{\sigma^2}{(a-1)\mu},
\end{aligned}
$$

where the last inequality is Chebyshev's. Q.E.D.

Using Lemma 11, we also prove the following lemma.

Lemma 12. *Let $\sigma_j^2 = Var(D_j)$ denote the variance of the one-period demand for product j. Then for each k and t, we have*

$$\hat{B}_k^t = \left(\sum_{s=t-L}^{t}\hat{D}_k^s - U_k\right)^+, \qquad \textit{and} \qquad \mathbb{E}\left[\hat{B}_k^t\right] \leq \frac{\phi_k\sigma_{j(k)}^2}{\left(\frac{U_k}{\bar{x}_k} - 1\right)\mu_{j(k)}}.$$

Proof. We first observe that by the evolution equation (14) for $\hat{B}_k^t$ and (25) we have

$$\hat{B}_k^{t-1} - \hat{B}_k^t + \hat{D}_k^t = \tilde{X}_k^t = \min\left(\hat{D}_k^t + \hat{B}_k^{t-1}, U_k + \hat{B}_k^{t-1} - \sum_{s=t-L}^{t-1}\hat{D}_k^s\right),$$

which after cancelling and rearranging terms gives the characterization of $\hat{B}_k^t$. Then, we bound the expectation as follows

$$
\begin{aligned}
\mathbb{E}\left[\hat{B}_k^t\right] &= \phi_k\mathbb{E}\left[\left(\sum_{s=t-L}^{t}D_{j(k)}^s - \frac{U_k}{\phi_k}\right)^+\right]\\
&= \phi_k\mathbb{E}\left[\left(\tilde{D}_{j(k)}^{L+1} - \frac{(L+1)\mu_{j(k)}U_k}{\bar{x}_k}\right)^+\right]
\end{aligned}
$$

$$\leq \frac{\phi_k \mathrm{Var}(\tilde{D}_{j(k)}^{L+1})}{\left(\frac{U_k}{\bar{x}_k} - 1\right)(L+1)\mu_{j(k)}}$$
$$= \frac{\phi_k \sigma_{j(k)}^2}{\left(\frac{U_k}{\bar{x}_k} - 1\right)\mu_{j(k)}},$$

where the second equality follows because $\sum_{s=t-L}^{t} D_{j(k)}^s$ has the same distribution as $\tilde{D}_{j(k)}^{L+1}$, the first inequality follows from Lemma 11 and the fact that $\mathbb{E}\left[\tilde{D}_{j(k)}^{L+1}\right] = (L+1)\mu_{j(k)}$, and the final equality follows from $\mathrm{Var}(\tilde{D}_{j(k)}^{L+1}) = (L+1)\sigma_{j(k)}^2$ by independence. Q.E.D.

Proof of Theorem 5. Assuming a feasible ABBS policy, we first prove a series of cost bounds. Then we specify a specific policy, show it is feasible, and put the cost bounds together to prove the approximation factor.

To begin, by the first identity in Lemma 2, for each i and t we have

$$\begin{aligned} I_i^t &= S_i - \sum_k a_{ik}\left(\sum_{s=t-L}^{t} \hat{D}_k^s - \hat{B}_k^t\right) \\ &= S_i - \sum_k a_{ik} \min\left(\sum_{s=t-L}^{t} \hat{D}_k^s, U_k\right) \\ &\leq S_i, \end{aligned}$$

where the second equality follows from Lemma 12, and the inequality holds because both $\hat{D}_k^s, U_k \geq 0$. Therefore, we have

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_i c_i I_i^T\right] \leq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_i c_i S_i\right] = 0. \tag{45}$$

Now, we claim that for each k and t,

$$\mathbb{E}\left[\sum_{s=t-L}^{t} \tilde{X}_k^s\right] = \bar{x}_k, \tag{46}$$

To see this, rearrange the evolution equation (14) to obtain

$$\tilde{X}_k^t = \hat{B}_k^{t-1} - \hat{B}_k^t + \hat{D}_k^t,$$

which implies

$$\sum_{s=t-L}^{t} \tilde{X}_k^s = \sum_{s=t-L}^{t} (\hat{B}_k^{s-1} - \hat{B}_k^s + \hat{D}_k^s) = \hat{B}_k^{t-L-1} - \hat{B}_k^t + \sum_{s=t-L}^{t} \hat{D}_k^s. \tag{47}$$

Therefore, taking expectations, for any t we have

$$\begin{aligned} \mathbb{E}\left[\sum_{s=t-L}^{t} \tilde{X}_k^s\right] &= \mathbb{E}\left[\hat{B}_k^{t-L-1} - \hat{B}_k^t + \sum_{s=t-L}^{t} \hat{D}_k^s\right] \\ &= \mathbb{E}\left[\left(\sum_{s=t-2L-1}^{t-L-1} \hat{D}_k^s - U_k\right)^+\right] - \mathbb{E}\left[\left(\sum_{s=t-L}^{t} \hat{D}_k^s - U_k\right)^+\right] + \mathbb{E}\left[\sum_{s=t-L}^{t} \hat{D}_k^s\right] \\ &= (L+1)\phi_k \mu_{j(k)} \\ &= \bar{x}_k, \end{aligned}$$

where the second line follows by the characterization of $\hat{B}_k^t$ in Lemma 12, the third because $\sum_{s=t-2L-1}^{t-L-1} \hat{D}_k^s$ has the same distribution as $\sum_{s=t-L}^{t} \hat{D}_k^s$, and the fourth from the definition of ϕ_k.

Next, we claim that for each t

$$\mathbb{E}\left[\sum_i h_i I_i^t\right] \leq \sum_i h_i \left(S_i - \sum_k a_{ik}\bar{x}_k\right) + \sum_i h_i \sum_k a_{ik} \frac{\phi_k \sigma_{j(k)}^2}{\left(\frac{U_k}{\bar{x}_k} - 1\right)\mu_{j(k)}}. \tag{48}$$

To see this, observe that by the first identity in Lemma 2, we have

$$\begin{aligned}
\sum_i h_i I_i^t &= \sum_i h_i \left(S_i - \sum_k a_{ik}\left(\sum_{s=t-L}^{t} \hat{D}_k^s - \hat{B}_k^t\right)\right) \\
&= \sum_i h_i \left(S_i - \sum_k a_{ik}\left(\sum_{s=t-L}^{t} \hat{D}_k^s + \hat{B}_k^{t-L-1} - \hat{B}_k^t\right)\right) + \sum_i h_i \sum_k a_{ik}\hat{B}_k^{t-L-1} \\
&= \sum_i h_i \left(S_i - \sum_k a_{ik} \sum_{s=t-L}^{t} \tilde{X}_k^s\right) + \sum_i h_i \sum_k a_{ik}\hat{B}_k^{t-L-1},
\end{aligned}$$

where the third line follows from (47). Then (48) holds by (46) and the bound in Lemma 12.

Next, we claim that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\sum_k (q_k + \sum_i c_i a_{ik})x_k^t\right] \leq \frac{1}{L+1}\sum_k (q_k + \sum_i c_i a_{ik})\bar{x}_k + \sum_j g_j^{V^*}\mu_j, \tag{49}$$

To see this, note that for each t

$$\begin{aligned}
\sum_k (q_k + \sum_i c_i a_{ik})x_k^t &= \sum_k (q_k + \sum_i c_i a_{ik})\left(\tilde{X}_k^t + \tilde{B}_k^{t-L-1}\right) \\
&= \sum_k (q_k + \sum_i c_i a_{ik})\tilde{X}_k^t + \sum_k (q_k + \sum_i c_i a_{ik}) v_{kj(k)}^* \psi_{j(k)} D_{j(k)}^{t-L-1} \\
&= \sum_k (q_k + \sum_i c_i a_{ik})\tilde{X}_k^t + \sum_j g_j^{V^*}\psi_j D_j^{t-L-1},
\end{aligned}$$

where the first equality follows from (17) and the rest from definition. Then, observe that by rearranging the sum we have

$$\sum_{t=1}^{T} \tilde{X}_k^t \leq \frac{1}{L+1}\sum_{t=1}^{T+L}\sum_{s=t-L}^{t}\tilde{X}_k^s,$$

for all activities k and all time horizons $T \geq 1$. Therefore, we have

$$\begin{aligned}
\sum_{t=1}^{T}\sum_k (q_k + \sum_i c_i a_{ik})x_k^t &= \sum_{t=1}^{T}\sum_k (q_k + \sum_i c_i a_{ik})\tilde{X}_k^t + \sum_j g_j^{V^*}\psi_j D_j^{t-L-1} \\
&\leq \frac{1}{L+1}\sum_{t=1}^{T+L}\sum_{s=t-L}^{t}\sum_k (q_k + \sum_i c_i a_{ik})\tilde{X}_k^s + \sum_{t=1}^{T}\sum_j g_j^{V^*}\psi_j D_j^{t-L-1},
\end{aligned}$$

and by (46) we have

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\frac{1}{L+1}\sum_{t=1}^{T+L}\sum_{s=t-L}^{t}\sum_k (q_k + \sum_i c_i a_{ik})\tilde{X}_k^s\right] \\
&= \limsup_{T\to\infty}\frac{T+L}{T}\frac{1}{L+1}\sum_k (q_k + \sum_i c_i a_{ik})\bar{x}_k \\
&= \frac{1}{L+1}\sum_k (q_k + \sum_i c_i a_{ik})\bar{x}_k,
\end{aligned}$$

which implies

$$
\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{k}(q_k+\sum_i c_i a_{ik})x_k^t\right]\\
&\leq \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\frac{1}{L+1}\sum_{t=1}^{T+L}\sum_{s=t-L}^{t}\sum_{k}(q_k+\sum_i c_i a_{ik})\tilde{X}_k^s+\sum_{t=1}^{T}\sum_j g_j^{V^*}\psi_j D_j^{t-L-1}\right]\\
&=\frac{1}{L+1}\sum_k(q_k+\sum_i c_i a_{ik})\bar{x}_k+\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\sum_j g_j^{V^*}\psi_j D_j^{t-L-1}\right]\\
&\leq \frac{1}{L+1}\sum_k(q_k+\sum_i c_i a_{ik})\bar{x}_k+\sum_j g_j^{V^*}\mu_j,
\end{aligned}
$$

where the last line follows from $\psi_j \leq 1$, completing the proof of (49).

Then, our final cost bound, for each t, is

$$
\mathbb{E}\left[\sum_j p_j B_j^t\right]\leq \sum_j p_j\left(\sum_k r_{kj}\frac{\phi_k\sigma_j^2}{\left(\frac{U_k}{\bar{x}_k}-1\right)\mu_j}+\bar{y}_j\right), \tag{50}
$$

which follows from

$$
\begin{aligned}
\mathbb{E}\left[\sum_j p_j B_j^t\right]&=\mathbb{E}\left[\sum_j p_j\sum_k r_{kj}\left(\hat{B}_k^t+\sum_{s=t-L}^{t}\tilde{B}_k^s\right)\right]\\
&=\mathbb{E}\left[\sum_j p_j\sum_k r_{kj}\left(\hat{B}_k^t+\sum_{s=t-L}^{t}v_{kj(k)}^*\psi_{j(k)}D_{j(k)}^s\right)\right]\\
&=\sum_j p_j\left(\sum_k r_{kj}\mathbb{E}\left[\hat{B}_k^t\right]+(L+1)\psi_j\mu_j\right)\\
&\leq \sum_j p_j\left(\sum_k r_{kj}\frac{\phi_k\sigma_j^2}{\left(\frac{U_k}{\bar{x}_k}-1\right)\mu_j}+\bar{y}_j\right),
\end{aligned}
$$

where the first line follows from the fact that $B_j^t=\sum_k r_{kj}\bar{B}_k^t$ and the third identity in Lemma 2, and the final line follows from the bound in Lemma 12 and the definition of ψ_j.

Now we specify the ABBS policy, which under (23), (24), and (25) simply requires specifying the S_i and U_k for all i and k. For some $\theta>0$, let $S_i=(\theta+1)S_i^L$ for each i and $U_k=(\theta+1)\bar{x}_k$ for each k. By Lemma 2, the policy is feasible if $\tilde{\mathbf{X}}^t$, $\tilde{\mathbf{B}}^t$, and $\hat{\mathbf{D}}^t$ are non-anticipating and satisfy constraints (13), (15), and (16). These vectors are non-anticipating because each of (23), (24), and (25) only use information available at time t. Further, constraint (13) is satisfied because for each j

$$
\sum_k r_{kj}(\hat{D}_k^t+\tilde{B}_k^t)=\sum_k r_{kj}(\phi_k D_j^t+v_{kj}^*\psi_j D_j^t)=D_j^t\left(\sum_k r_{kj}\phi_k+\psi_j\right)=D_j^t,
$$

where the final equality is guaranteed by taking an expectation over the second constraint of the LP defining $F^L(\mathbf{S}|\tilde{\mathbf{D}}^{L+1})$ and the definition of ϕ_k and ψ_j. Also, constraint (15) is satisfied by the definition of $\tilde{X}_k^t$ in (25), while constraint (16) is also satisfied because by (25) we have

$$
\begin{aligned}
\sum_k a_{ik}\tilde{X}_k^t&\leq \sum_k a_{ik}\left(U_k+\hat{B}_k^{t-1}-\sum_{s=t-L}^{t-1}\hat{D}_k^s\right)\\
&=(1+\theta)\sum_k a_{ik}\bar{x}_k-\sum_k a_{ik}\left(\sum_{s=t-L}^{t-1}\hat{D}_k^s-\hat{B}_k^{t-1}\right)\\
&\leq (1+\theta)S_i^L-\sum_k a_{ik}\left(\sum_{s=t-L}^{t-1}\hat{D}_k^s-\hat{B}_k^{t-1}\right),
\end{aligned}
$$

where the final inequality follows because $\sum_k a_{ik}\bar{x}_k \le S_i^L$ by taking an expectation over the first constraint of the LP defining $F^L(\mathbf{S}|\tilde{\mathbf{D}}^{L+1})$. This completes the proof of feasibility.

Then by Lemma 3, the long-run average cost is

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_i c_i I_i^T + \sum_{t=1}^T \left(\sum_i h_i I_i^t + \sum_j p_j B_j^t + \sum_k (q_k + \sum_i c_i a_{ik}) x_k^t\right)\right] \\
&\le \sum_i h_i \left(S_i - \sum_k a_{ik}\bar{x}_k\right) + \sum_i h_i \sum_k a_{ik} \frac{\phi_k \sigma_{j(k)}^2}{\left(\frac{U_k}{\bar{x}_k} - 1\right)\mu_{j(k)}} + \sum_j p_j \left(\sum_k r_{kj} \frac{\phi_k \sigma_j^2}{\left(\frac{U_k}{\bar{x}_k} - 1\right)\mu_j} + \bar{y}_j\right) \\
&\quad + \frac{1}{L+1}\sum_k (q_k + \sum_i c_i a_{ik})\bar{x}_k + \sum_j g_j^{V^*}\mu_j \\
&= (\theta+1)\sum_i h_i S_i^L + \sum_k u_k^L \bar{x}_k + \sum_j p_j \bar{y}_j + \sum_j g_j^0 \mu_j + \sum_j \sum_k r_{kj}\left(p_j + \sum_i a_{ik} h_i\right)\frac{\phi_k \sigma_j^2}{\theta \mu_j} \\
&\le (\theta+1)\left(\sum_i h_i S_i^L + \sum_k u_k^L \bar{x}_k + \sum_j p_j \bar{y}_j\right) + \sum_j g_j^0 \mu_j + \sum_j \left(p_j + \bar{h}_j\right)\frac{\sigma_j^2}{\theta\mu_j} \\
&= (\theta+1)\underline{C}^L + \left(1 + \frac{\sum_j \left(p_j + \bar{h}_j\right)\mu_j \nu_j^2}{\theta \sum_j g_j^0 \mu_j}\right)\sum_j g_j^0 \mu_j \\
&\le \left(2 + \theta + \frac{\sum_j \left(p_j + \bar{h}_j\right)\mu_j \nu_j^2}{\theta \sum_j g_j^0 \mu_j}\right) C^{L*},
\end{aligned}$$

where the first inequality follows from (45), (48), (49), and (50), the second inequality follows from $\sum_k r_{kj}\phi_k \le 1$, $\sum_i a_{ik}h_i \le \bar{h}_j$ for all $k \in \mathcal{N}(j)$, and the fact that $\sum_k u_k^L \bar{x}_k + \sum_j p_j \bar{y}_j \ge 0$, and the final inequality from Corollary 4 and Proposition 2. The result then follows from letting

$$\theta = \sqrt{\frac{\sum_j \left(p_j + \bar{h}_j\right)\mu_j \nu_j^2}{\sum_j g_j^0 \mu_j}}.$$

Q.E.D.

Appendix E: Simulation Details

In this appendix we provide more details on some of the simulation studies performed in the main body.

E.1. Details on Example 1

Here we provide a detailed explanation of the calculation of the optimal policy cost in Example 1. The costs and demand distributions are given in Example 1, and for completeness we explicitly characterize the network structure with the following matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

where we number the three activities as follows: activity 1 uses resource 1 to fulfill product 1, activity 2 uses resource 1 to fulfill product 2, and activity 3 uses resource 2 to fulfill product 2. Given these inputs, we next aim to solve the dynamic program minimizing long-run average cost in this system.

We will solve this dynamic program using the theory of finite state/action average cost MDPs. From Bertsekas (2012) Proposition 7.4.1, for such an MDP with states i, actions u, transition probabilities $p_{ij}(u)$ and single period cost function $g(i,u)$, the Bellman optimality equations are

$$\lambda + h(i) = \min_{u \in U(i)} \left[g(i,u) + \sum_j p_{ij}(u) h(j)\right], \ \forall i,$$

where λ represents the long-run average optimal cost, $h(i)$ represents the value function in state i, and $U(i)$ represents the set of actions available in state i. Using the standard linear programming approach, the optimal value of λ can then be solved for with the following linear program

$$\max_{\lambda,\mathbf{h}} \lambda$$
$$\text{s.t. } \lambda + h_i \leq g(i,u) + \sum_j p_{ij}(u)h_j, \ \forall i, u \in U(i).$$

To solve this LP, we must then ascertain all possible state-action pairs, transition probabilities, and costs. To maintain a finite state and action space, we discretize both to the integers, and also derive upper bounds on the reachable states/actions of an optimal policy. Since the products that resource 1 serves can have at most 2 units of total demand in a period, an optimal policy will always maintain $I_1^t \leq 2$, and by a similar argument we have $I_2^t \leq 1$.

For backlogs in the system, we first note that an optimal policy will always order a total quantity of resources at least as large as the number of backlogs waiting in the system in each period. We then observe that the backlog for product 1 can be at most 2, i.e., $B_1^t \leq 2$. To see this, note that for a policy to obtain $B_1^t > 2$ it must first reach $B_1^t = 2$, since at most one new backlog can arrive in a period. If a policy ever reaches $B_1^t = 2$, then to minimize cost it must necessarily order 2 units of resource 1 to fulfill this backlog in the next period, and at most one unit of this resource can be diverted to fill product 2 demand in the next period (since at most one new unit of product 2 demand can arrive, and any existing backlog from the prior period would have been ordered for separately to minimize costs). Thus, there is at least one unit of resource 1 available in the next period to clear a backlog, so since at most one new unit of demand can arrive, the backlog cannot rise above 2. Similarly, we argue that backlog for product 2 can be at most 3, i.e., $B_2^t \leq 3$, since if a policy reaches $B_2^t = 3$, then at least 3 units of resources are ordered in the next period, and at most 1 of these will be ordered for resource 1 (since resource 1 is more expensive, and at most 1 new unit of product 1 demand will arrive in the next period for which the resource 1 unit could be profitably diverted). Thus, there are at least 2 units of resource 2 available to clear product 2 backlog in the next period, so that it cannot rise above 3. Thus, the state space is restricted to $I_1^t \in \{0,1,2\}$, $I_2^t \in \{0,1\}$, $B_1^t \in \{0,1,2\}$, and $B_2^t \in \{0,1,2,3\}$, for 72 total states.

Then, we observe that the ordering quantities z_i^t for each i, should never rise above 7, since that is the total amount of backlog plus new demand that can be seen in a period. Further, the fulfillment quantities x_k^t for each k should never rise above 4, since that is the highest amount of backlog plus new demand that can be seen in a period for any individual product. Thus, the action space is restricted to $z_i^t \in \{0,\dots,7\}$ for all i and $x_k^t \in \{0,\dots,4\}$ for all k, for a total of 8,000 possible actions.

This leads to a universe of 576,000 possible state-action pairs, most of which, however, are not feasible given the constraints of the problem. Thus, for each state we iterate through each action and let those actions remain that are both feasible for the problem constraints, and that lead to an allowable state in the following period. This leaves a total of 548 total state-action pairs, for which we solve the LP above. The transition probabilities are calculated from the Bernoulli distribution parameters, and the costs are determined from the state, action, and cost information. We solve the resulting LP in Gurobi 10.0 and obtain an optimal objective of 0.936.

E.2. Benchmark Policy

We compare to a benchmark from Govindarajan et al. (2021b), who define a heuristic base-stock policy for an omni-channel fulfillment problem. We adapt their heuristic for setting base-stock levels to our Simple Fulfillment Network setting of Section 4.2.3 in Algorithm 2 below by using just the portion for online fulfillment centers (their full algorithm also includes a portion for brick-and mortar stores). We note that Govindarajan et al.

(2021b) consider a model with lost sales, and thus do not need to map backlogs to activities, so we adapt their policy by simply mapping all backlogs to their lowest cost activity. To state the algorithm, we define the aggregate cost parameters across the network as

$$\bar{p} = \sum_j p_j/N, \qquad \bar{h} = \sum_i h_i/M, \qquad \underline{q} = \min_k q_k.$$

Algorithm 2 Base-Stock Levels Procedure (Govindarajan et al., 2021b)

1: Calculate total base-stock level, S^{TOT}, s.t. $\mathbb{P}(D^{\text{TOT}} \le S^{\text{TOT}}) = \frac{\bar{p}-\underline{q}}{\bar{h}+\bar{p}-\underline{q}}$, where $D^{\text{TOT}} = \sum_j D_j$
2: Set $S_i = 0$ for all i, and rem $= \lfloor S^{\text{TOT}} \rfloor$
3: **while** rem > 0 **do**
4: Calculate marginal cost for all i: $\text{MC}_i = (\bar{h} + \bar{p} - \underline{q})\mathbb{P}(D_i \le S_i) - (\bar{p} - \underline{q})$
5: Choose $i^* \in \arg\min_i \text{MC}_i$
6: Update $S_{i^*} \leftarrow S_{i^*} + 1$ and rem $\leftarrow$ rem $- 1$
7: **end while**

Given the base-stock levels set by Algorithm 2, Govindarajan et al. (2021b) follow a base-stock ordering policy, and use the following LP to determine the fulfillment quantities x_k^t in each period t

$$\begin{aligned} \min \ & \sum_k (q_k - p_{j(k)}) x_k^t \\ \text{s.t.} \ & \sum_k a_{ik} x_k^t \le I_i^{t-1} + z_i^t, \ \forall i \\ & \sum_k r_{kj} x_k^t \le D_j^t + B_j^{t-1}, \ \forall j \\ & x_k^t \ge 0, \ \forall k. \end{aligned}$$

E.3. ABBS Policy Comparisons, Longer Lead Times, and LP-based Fulfillment

In this section we further compare our ABBS policies developed in Section 5 over longer lead times and also benchmark against an alternative heuristic fulfillment policy.

E.3.1. LP-based Fulfillment. The policy development in Section 5 is aimed at producing a policy class whose evolution can be tracked (and optimized) over time in a way that allows proving a theoretical bound. Critical to this development is implementing a fulfillment policy that mimics the stochastic program lower bound in order to control fulfillment costs. As an alternative to this approach, here we simply seek a heuristic that can leverage current state information to execute smart fulfillment decisions in each period. Perhaps the most natural such policy is one that minimizes current fulfillment and backlog costs through the solution of an allocation LP. While such a policy is myopic, it may still perform well as it adapts well to current state information.

In particular, in this section we follow the same ordering strategy as Section 5: set base-stock levels using stochastic program (20) and assign backlogs to their lowest cost activity for calculating inventory positions. But we add the following simple LP-based fulfillment decision: in period t, given the ending inventory and

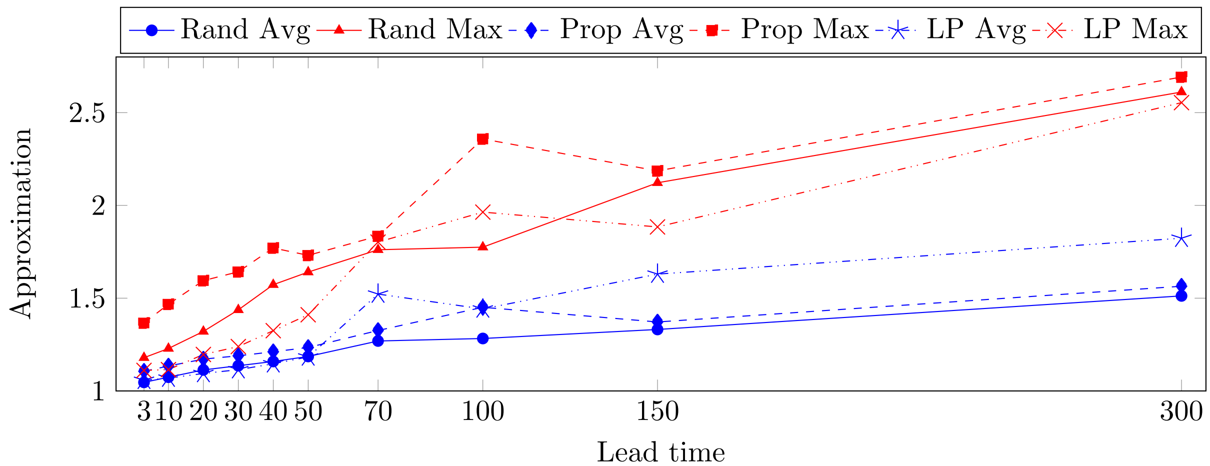

Figure 5 **Approximation factors of ABBS policies compared to LP-based fulfillment as lead time grows**

backlog state from period $t-1$, I_i^{t-1} $\forall i$, B_j^{t-1} $\forall j$, and the arriving orders z_i^{t-L} $\forall i$, we solve the following LP to determine the fulfillment decision x_k^t $\forall k$:

$$\mathbf{x}^t \in \arg\min_{\mathbf{x}\geq 0} \left\{ \begin{aligned} &\min_{\mathbf{y}\geq 0} \ \sum_k q_k x_k + \sum_j p_j y_j \\ &\text{s.t.} \ \sum_k a_{ik} x_k \leq I_i^{t-1} + z_i^{t-L}, \ \forall i \\ &\qquad \sum_k r_{kj} x_k + y_j = B_j^{t-1} + D_j^t, \ \forall j \end{aligned} \right\},$$

in other words, we make a myopic fulfillment decision to minimize the current fulfillment and backlog costs.

E.3.2. Further Lead Time Simulations We simulate the LP-based fulfillment policy for the same 108 problem instances as tested in Section 5.3 across several lead times. We further extend the simulation out to longer lead times here in order to assess the relative performance of the policies as the lead time grows. The results are compared to our ABBS policies in Figure 5, which shows that the LP-based fulfillment policy performs similarly to our ABBS policies as the lead time grows. In particular, while the average case performance of the LP-based policy is slightly worse than the ABBS policies for larger lead time values, the worst case performance of the of the LP-based policy tends to be better than the ABBS policies for all but a few intermediate lead time values. That said, as the lead time gets larger, the performance of all policies degrades at a similar rate, both on average and in the worst case. This suggests that either our policies or lower bound are somewhat loose as the lead time grows, and implies an important direction for future work to improve our analysis: can the lower bound be tightened or the policy be improved? This serves to further illustrate the point made at the end of Section 5, that our ABBS policy framework and examples (i.e., randomized and proportional fulfillment policies) are meant to illustrate the possibilities for extending base-stock policies to networks with lead-times; there may exist significant improvements upon these specific policies and we leave this open as an interesting avenue for future research.

E.4. Heuristic for Individual Lead Times

In specifying our base model in Section 2 we assumed that each resource has a common lead time of L periods. From a practical perspective, it would be beneficial to specify a model that instead has an individual lead time for each resource, e.g., an order for resource i takes L_i periods to arrive, so that evolution equation (1) becomes

$$I_i^t = I_i^{t-1} + z_i^{t-L_i} - \sum_k a_{ik} x_k^t, \ \forall i.$$

We call the model with this evolution equation, and all else the same as Section 2, the individual lead time model. In this section we provide more detail on the challenges of analyzing the individual lead time model, while also providing a reasonable heuristic extension of our stochastic programming approach to this setting.

First, we observe that the individual lead time model is a generalization of the notoriously difficult dual sourcing problem (see e.g., Xin and Goldberg, 2018 for a discussion of the analytical challenges). To see this correspondence, consider a simple network with a single product and two resources that each can fulfill demand for the product. One resource has a longer lead time and the other a shorter, with the shorter lead time resource having a higher ordering cost, while both resources have the same holding and fulfillment costs. With this setup, the two resources mimic the two suppliers in the classic dual sourcing problem. Thus, an approximation for the individual lead time newsvendor network model would imply an approximation for the dual sourcing problem.

Further, we observe that, while a theoretical performance bound may be difficult, perhaps a lower bound stochastic program may still be derived and used as a heuristic in the individual lead time problem. Techniques to derive such stochastic program lower bounds exist for the special case of ATO systems, see Reiman et al. (2023), however they are quite computationally intensive, as they involve a multi-stage stochastic program, with a separate stage for each unique lead time in the system. This degrades the computational complexity quickly, since for more than a handful of lead times the optimization problem again essentially boils down to solving a dynamic program, which is what our stochastic programming approach is attempting to avoid by providing a simpler optimization problem to establish a good policy. Therefore, in what follows we propose a simple two-stage stochastic program meant to approximate the individual lead time problem in a way that can provide a hueristic base-stock policy for implementation.

To specify the stochastic program, let $L_k = \frac{\sum_i a_{ik} L_i}{\sum_i a_{ik}}$ denote the average lead time for resources used in activity k. Then, let $L_j = \left\lceil \frac{\sum_k r_{kj} L_k}{\sum_k r_{kj}} \right\rceil$ denote the average lead time of resources that could be used in fulfilling product j, rounded up to the nearest integer. Then let $\mathscr{D}$ denote a random demand vector with the j^{th} element determined by a cumulative sum of $L_j + 1$ periods of random demand: $\mathscr{D}_j = \sum_{l=1}^{L_j+1} D_j^l$, and define the following stochastic program:

$$\mathscr{F}(\mathbf{S}|\mathscr{D}) = \begin{Bmatrix} \min\limits_{\mathbf{x},\mathbf{y}\geq 0} & \sum\limits_k \left(\frac{1}{L_k+1} \left(q_k + \sum\limits_i a_{ik} c_i \right) - \sum\limits_i a_{ik} h_i \right) x_k + \sum\limits_j p_j y_j \\ \text{s.t.} & \sum\limits_k a_{ik} x_k \leq S_i, \ \forall i \\ & \sum\limits_k r_{kj} x_k + y_j = \mathscr{D}_j, \ \forall j \end{Bmatrix}$$

$$\min_{\mathbf{S}\geq 0} \sum_i h_i S_i + \mathbb{E}\left[\mathscr{F}(\mathbf{S}|\mathscr{D})\right], \tag{51}$$

This stochastic program does not provide a lower bound on the optimal value of the individual lead time model because it considers demand for product j over $L_j + 1$ periods, essentially assuming a policy cannot influence replenishment for resources serving this product over that time frame, while in the true system some resources will have less than this average lead time and therefore can be replenished sooner (indeed this observation is precisely why Reiman et al. (2023) require a multi-stage stochastic program to derive a valid lower bound). However, we propose this stochastic program as a reasonable approximation to expected cost in the system, as the average lead time may provide a good proxy for the true demand that needs to be planned for with each order decision.

Table 4 Average and worst case ratio of ABBS Performance relative to SP (51)

Instance Type	Average	Worst Case
E-commerce	1.37	1.89
Production	2.21	2.75
Overall	1.79	2.75

The optimal **S** solution of (51) can be implemented as a heuristic base-stock policy in the individual lead time setting in conjunction with an ABBS fulfillment policy as described in Section 5. The main adjustment needed is to the backlog assignment mapping, which should now take into account the fact that different activities may take longer to be replensished due to the individual lead times. To account for this, we propose the following adjustment to the "minimum cost activity" selection rule proposed in the main body of the paper: for product j map all backlog to an activity

$$k \in \arg\min_{k \in \mathcal{N}(j)} \left\{ q_k + \sum_i a_{ik} c_i + p_j L_k \right\},$$

where we add the term $p_j L_k$ to penalize the activity for longer average lead time. This backlog assignment rule completes a description of our heuristic ABBS policy for individual lead times.

As an example, we implement the randomized ABBS policy from Section 5.2.1 on the same 108 problem instances simulated in Section 5.3, with lead times of 20, 30, or 40 randomly assigned to each resource. The results of this simulation are summarized in Table 4, where the ratios are reported relative to the stochastic program objective (51) (as opposed to a lower bound on optimal in earlier simulations). As expected, this simulation shows the performance of the heuristic degrades when including individual lead times, but it may provide a reasonable starting point for a practical policy implementation in real world settings.