Appendices

A. Justifications of Key Assumptions

A.1. Degradation Cost

Almost all batteries (e.g., lithium-ion) suffer a degradation process, and thus, should be replaced after some amount of time/work. Considering that more than 90% of the newly installed large-scale batteries in 2024 are lithium-ion batteries and the fact that lithium-ion batteries are predicted to remain dominant for the near future (EIA 2021b, MordorIntelligence 2022), this study provides general lessons on energy stacking with BESS.

A.1.1. Key Stress Factors The process of degradation depends on numerous factors, which have been extensively studied in the engineering literature. The three critical factors are temperature, charging/discharging rate (C-rate), and depth-of-discharge (DOD) (Millner 2010, Kalogirou 2017, Xu et al. 2018a).

Two of these factors are highly interrelated and controllable: temperature and C-rate. The temperature at which the battery operates significantly impacts degradation and also influences batteries' safety (Plett 2015). Though C-rate has been well documented to also increase degradation, Lam and Bauer (2012) and Tanim et al. (2018) report in cell-level experimental studies that C-rates showed only a minor impact on degradation when the cell temperature is well controlled, implying that C-rate indirectly accelerates degradation by increasing cell temperature due to Ohmic (resistive) heating. With the design of battery management systems (BMS) and the development of thermal/cooling systems (e.g., liquid-based coolants), the impact of temperature and therefore C-rates can be minimized, especially for stationary batteries that we consider in this work.[33] Given our domain of application (large-scale battery deployments, which typically deploy BMSs), we restrict our focus to DOD, which we describe next.

A.1.2. Degradation Function Several engineering papers have experimentally tested the effect of DOD on degradation, most often by relating the expected number of cycles N_i that a battery can provide before it is completely degraded if operated at DOD level δ_i ($0 \leq \delta_i \leq 1$). This is called the Wöhler curve or the SN curve showing the relationship between the stress level (S) and the number of cycles (N); see Sauer and Wenzl (2008) and Ecker et al. (2014) for two representative studies of this form. Though many functional forms for this relationship have been explored, the most commonly adopted form assumes

$$N_i = \left(\frac{1}{z\delta_i}\right)^{\beta}, \tag{A.1}$$

[33] Controlling temperature and the effect of C-rate is less straightforward for mobile batteries used in electric vehicles (EVs), though active cooling systems are also used in most EVs.

where $z > 0$ and $\beta > 1$ are constants that do not depend on the DOD. Rewriting (A.1), the cost of a single cycle of DOD δ_i can be characterized by the degradation cost function $\Phi(\delta_i) : [0,1] \to \mathbb{R}$ (Xu et al. 2017):

$$\Phi(\delta_i) = \underbrace{c_{cell} \cdot E}_{battery\ price} \cdot \frac{1}{N_i} = c_{cell} \cdot E \cdot (z\delta_i)^{\beta}. \tag{A.2}$$

For notational simplicity, we define DOD level $u_i = \delta_i \cdot E$ as the actual energy charged/discharged in a cycle (MWh). Accordingly, we redefine the degradation cost function $\Phi(u_i) : [0, E] \to \mathbb{R}$:

$$\Phi(u_i) = c_{cell} \cdot E \cdot z^{\beta} \cdot (\frac{u_i}{E})^{\beta} \tag{A.3}$$

$$= \frac{c_{cell} \cdot E \cdot z^{\beta}}{E^{\beta}} \cdot u_i^{\beta} = \frac{c_0^b}{E^{\beta-1}} \cdot u_i^{\beta}, \tag{A.4}$$

where

- $c_0^b := c_{cell} \cdot z^{\beta}$ is the battery degradation coefficient and represents the cost of operating a battery with capacity $E = 1$ MWh for a single full cycle at DOD $u = 1$ MWh,
- z (unitless) is the longevity coefficient from the Wöhler curve,
- β (unitless) is the degradation growth exponent from the Wöhler curve,
- c_{cell} ($/MWh) is the battery cell price per unit of energy capacity, and
- E (MWh) is the battery capacity.

Note that z and $\beta > 1$ are E-invariant properties of the battery that depend on the battery's manufacturer and chemistry (e.g., lithium iron phosphate), and can be fitted based on lab experiments (e.g., accelerated degradation test).

In this paper, we assume $\beta = 2$ based on the evaluations of several empirical papers (Pourmousavi et al. 2012, Koller et al. 2013, Laresgoiti et al. 2015, Xu et al. 2018b,a). This particular degradation model is also called the *quadratic* model (Xu et al. 2018a). In practice, $\beta > 2$ and $1 < \beta < 2$ were also reported depending on the combination of a battery's chemistry and manufacturer; see Table A.1. In Appendix D, we numerically evaluate the sensitivity of our results to the quadratic degradation assumption used in the main analysis by comparing different exponents $\beta \in [1.5, 2.5]$.

Another degradation model introduced in the literature is an exponential model as follows:

$$\Phi(u_i) = c_1^b u_i \exp(c_2^b u_i),$$

where $c_i^b, i = 1, 2$ again depend on a battery's characteristics (Millner 2010, Xu et al. 2018a). We performed numerical experiments with a class of exponential models. While the value of the degradation function influences the profits, the structural results hold unchanged.

Our model is intended to describe the mechanism behind simultaneous stacking and to show how stacking balances potentially increased utilization (and corresponding degradation costs) and

Reference	β	Battery type	Manufacturer
PowerTechSystems 2021	2.3	LiFePO4 (16000mAh)	Power Tech Systems
PowerTechSystems 2021	1.9	LiFePO4	Power Tech Systems
Broussely et al. 2005	2.1	Li-ion	Saft Batteries
Pourmousavi et al. 2012	2.0	Li-ion	Saft Batteries
Peterson et al. 2010	1.5	LiFePO4 (ANR26650M1 cell)	A123 Systems
Zhang et al. 2015	1.7	Li/MoS2	-

Table A.1 Degradation functions in the industry/literature

revenues. There are several additional cost factors that we do not consider such as fixed costs for financing, personnel, land, electrical equipment (e.g., transformers, battery management system), and HVAC (heating, ventilation, and air conditioning) that are incurred every time period. EIA (2020) provides overall capital cost estimates including these factors.

A.1.3. Rainflow-Counting Algorithm Definition A shows the rainflow-counting algorithm (Rychlik 1987, Xu et al. 2017):

Definition A. *Let* $C = (c_1, c_2, ..., c_{|C|})$ *denote the state of charge of the battery across time, with the* c_i *values capturing* all *local minima and maxima.*

Algorithm 1: Rainflow-counting algorithm

Input: *Battery SOC profile* C

Output: *A set of full cycles* $U = \{u_i\}$ *and half cycles* $\tilde{U} = \{\tilde{u}_i\}$

```
1  U = ∅
2  while |C| ≥ 4 do
3      Calculate Δc_1 = |c_1 − c_2|, ··· , Δc_i = |c_i − c_{i+1}|, ··· , Δc_{|C|−1} = |c_{|C|−1} − c_{|C|}|.
4      if there exists some 2 ≤ i ≤ |C| − 2 such that Δc_{i−1} ≥ Δc_i and Δc_{i+1} ≥ Δc_i then
5          C := (c_1, ··· , c_{i−1}, c_{i+2}, ··· , c_{|C|}), and U := U ∪ {Δc_i}.
6      else
7          Go to line 10
8      end
9  end
10 Ũ = {|c_1 − c_2|, ··· , |c_{|C|−1} − c_{|C|}|}.
```

The battery degradation cost of C *is defined as* $\frac{c_0^b}{E}\sum_{u_i \in U} u_i^2 + \frac{c_0^b}{2E}\sum_{\tilde{u}_i \in \tilde{U}} \tilde{u}_i^2$ *based on eq.* (1) *with* $\beta = 2$.

A.2. Energy Neutrality of Regulation Signals

Many ISO/RTOs have introduced an energy-neutrality commitment for their regulation signals. However, the level of guarantee may differ across types of signals and across ISOs. For example, PJM initially introduced strict energy neutrality for the regD signal within 15 minutes. Since, at times, this strict-neutrality of the regD signal dispatched a measurable amount of energy in an undesirable direction, PJM switched to *conditional* energy neutrality within 30 minutes in 2017.[34] Also, different ISOs use different signals for regulation: ISO-NE uses continuous and trinary signals, while PJM uses its regD and regA signals. More than 80% of PJM's regD and ISO-NE's continuous signals restored their initial SOC within 30 minutes, whereas it takes on average more than an hour for ISO-NE's energy-neutral trinary signal to restore a battery's initial SOC 80% of the time. In our work, we adjust our definition of time periods T_L, T_H, and T to model different levels of energy-neutrality of regulation signals. Appendix D.1 illustrates regulation signals and the corresponding net energy change.

A.3. Market Clearing Mechanics and Market Structure

In this work, we consider day-ahead (forward) energy and regulation markets. In this section, we provide details on bidding and commitment decisions.

A.3.1. Scheduling/Bidding/Market Clearing Process In most ISO/RTOs in the US, energy markets consist of two markets: (1) a day-ahead market and (2) a real-time market. Frequency regulation is procured by the ISO/RTO through an ancillary services market, which consists of either two markets (e.g., NYISO) or a single real-time market (e.g., PJM). Below we briefly explain bidding and market clearing processes and discuss commitment/settlement rules based on PJM's Manual 11 (Energy & Ancillary Services Market Operations), Manual 12 (Balancing Operations), and Manual 28 (Operating Agreement Accounting), but other ISO/RTOs share very similar processes (PJM 2022a,b, 2023, NYISO 2021, 2023).

Battery participation models: ISO/RTOs match supply with demand by choosing the least expensive resource available (aka "economic dispatch"). ISO/RTOs do this by allowing market participants (e.g., batteries) to either (1) self-schedule (aka, batteries bid/commit service quantities and accept any cleared market prices; also called the "must-run" mode), or (2) bid a curve (aka, pooled-scheduled or ISO-scheduled mode), specifying a list of price ($/MW or $/MWh) and quantity (MW or MWh) pairs over multiple segments (e.g., 10 segments in CAISO/PJM).[35]

(1) In the day-ahead market (forward market), service prices are cleared on an hourly basis. Bidding and market clearing processes happen a day before the operating day, and ISOs post the market clearing results. The resources cleared in the market commit to following the day-ahead schedule and have no ability to react to uncertainty, except for additionally participating in real-time markets to respond to real-time price spikes and/or purposely deviating from the

[34] Prior to 2017, PJM (2017) biased the regA signal to overcorrect the regD signal to achieve energy neutrality. Since 2017, however, regA and regD signals work together so that energy neutrality is only supported for regD resources when there are available regA resources with extra capacity (*conditional energy neutrality*).

[35] FERC 841 ensures that a battery can bid curves as both a seller and a buyer.

schedule/signal requested by the grid and incurring the associate penalties, e.g., Cheng and Powell (2016).

(2) The real-time (energy and ancillary) market is a spot market where prices are cleared every 5 minutes. Bidding for the real-time market starts after the day-ahead results are posted, and bids can be updated up to an hour prior to the operating hour (65 minutes prior in PJM, 75 minutes prior in NYISO).

For regulation, resource owners must provide hourly capability (above and below regulation basepoint, MW). The battery is paid for each unit of committed capacity for each hour (regulation capability credit) and receives an additional credit (regulation performance credit) based on the actual amount of work. Oftentimes, the mileage (measured as the sum of the total amount of charging and the total amount of discharging requested for a given time period) or their derivatives are used to measure the amount of work requested of a resource following the regulation signal.

A resource-specific performance score is then calculated based on how accurately the battery followed the signal. For instance, PJM calculates a resource's performance score by combining accuracy (the correlation between the signal and the unit's response), delay, and precision (instantaneous error between the signal and the unit's response). This performance score works as a multiplier to determine the unit's performance credit. Batteries generally perform well; e.g., in 2022, 77% of batteries (which follow the regD signal) had average performance scores within the 0.91 – 1.00 range, compared with 25% of slower-responding regA resources (MonitoringAnalytics 2022).

A.3.2. Market-Related Assumptions in the Paper

In this work, we consider day-ahead markets and do not consider real-time markets. This assumption is largely motivated by a desire to retain a parsimonious model for a battery operator, coupled with the fact that the literature provides mixed messages about the importance of real-time arbitrage for a battery's overall profitability (e.g., Krishnamurthy et al. 2017, Löhndorf and Wozabal 2023).

Though we opt to defer focus on stacking in the real-time energy market to future work, we note that preliminary numerical experiments (results not shown) suggest that the key lessons continue to hold (i.e., the inclusion of the real-time markets does not lead to different lessons from what we establish with day-ahead markets), and the benefits of stacking versus standalone services can actually be expanded in this setting compared to what we report in this work with the day-ahead market alone. For instance, we can obtain higher S ratios, which means that simultaneous stacking may become more attractive relative to standalone services when the battery can participate in both day-ahead and real-time energy markets. This is because, under recent grid/market conditions, regulation is significantly more profitable for batteries than arbitrage, and including real-time arbitrage improves the profitability of stacking (which provides both services) versus its strongest competitor (regulation-only).

In both energy and regulation markets, we assume that the battery is self-scheduling (the battery bids/commits service quantities and accepts any cleared market prices) based on empirical evidence. According to MonitoringAnalytics (2022), 96.5% (98.7%) of all regulation MW clearing PJM's regD regulation market had an effective price of $0.00 in 2022 (2021) from MWs with zero offers plus self-scheduled MWs. Similarly, ISO-NE's 2023 regulation offer dataset (ISO-NE 2023) showed that almost all batteries make zero-price offers, which again confirms that they are effectively self-scheduling.[36]

A.4. Additional Literature Review

A.4.1. Inventory Theory versus Economics of Batteries Storage of energy in batteries brings parallels to inventory management, as buying batteries could be treated as capacity management and storing energy as inventory management. The conventional inventory problem focuses on physical storage and typically considers three types of costs: a purchase cost, a holding cost, and a shortage/penalty cost (Scarf 1959). This framework has been extended in multiple dimensions including stochastic settings (e.g., Karlin 1960), interdependent multi-echelon systems (e.g., Clark and Scarf 1960), and capacitated settings (e.g., Kapuscinski and Tayur 1998).

In the operations management (OM) literature, one of the key elements related to storage is the tradeoffs between the cost of holding inventory and the cost of ordering frequency or running out of inventory. In battery settings, holding costs are almost negligible.[37] Instead, batteries lose some fraction of energy during charging/discharging, which incurs an additional cost to compensate for the loss (energy replenishment cost). Loss of energy bears a resemblance to yield models in OM such as Yano and Lee (1995). However, yield models generally apply to a production process, not to a storage process, and the results cannot be translated to losses connected with the injection and discharge of energy.

Furthermore, the most significant cost driver for batteries is degradation (batteries become practically useless after some volume of "use"), which is a key element of our model. Degradation is, however, somewhat complicated. It is based on netted flows in opposite directions and has a

[36] Regulation prices are mainly determined by other resources such as natural power plants. MonitoringAnalytics (2022) reported that 67.1% of regulation was settled by natural gas and hydro resources, while 24.6% was settled by batteries. These empirical evidences show that battery operators in day-ahead markets bid based on expected values (historical prices).

[37] The concept corresponding to holding cost would be batteries' *self-discharge* or *leakage*, a phenomenon in which the stored energy is reduced without any connection to external circuits. Lithium-ion batteries are known to lose 2–7% of their stored energy per month (Johnson and White 1998, MPower 2007).

nonlinear relationship with the amount a battery is charged or discharged,[38] which is challenging to capture in an inventory model. While convex holding cost can be incorporated into inventory models, it does not seem possible to apply these techniques to the change of inventory level (which is analogous to the battery charge/discharge amount). Clearly multiple of these path-dependent complexities in the cost function are not present in the inventory literature, yet they are central to the profitability of battery operations. The only OM papers that bear some partial methodological or structural similarity are the ones explicitly analyzing energy markets with storage such as Secomandi (2010).

Uncertainty plays a key role in classical inventory models such as the newsvendor model. In the newsvendor model, uncertainty in the demand realization incurs the key costs of the model through overage and underage costs, and the optimal order level is selected to balance these two costs. The ordering decision interacts critically with the demand uncertainty, because if we over-order then we must salvage the extra items (or incur holding costs in multi-period models), while if we under-order then we cannot meet all our demand. Our optimal objective value is critically linked to the amount of uncertainty we have – we will have much higher expected cost with additional uncertainty in the demand.

In our battery stacking application there are two main sources of uncertainty: 1) uncertainty around the specific realization of the regulation signal that we will face, and 2) uncertainty around the prices for the arbitrage and regulation service that we will receive as price-takers in the day-ahead market. The first source of uncertainty (regulation signal uncertainty) is much less critical than demand uncertainty in a typical inventory model. Under demand uncertainty, we may under-order and not be able to meet the actual demand, incurring large costs due to unfulfilled demand. While penalties for not following the regulation signal also exist for the frequency regulation service, it is quite uncommon for a battery to be unable to provide the requested regulation due to the energy neutrality of the regulation signal. We can see evidence of this in the near-perfect performance scores that batteries obtain when providing the regulation service (see Appendix A.3.1 for details). Since batteries are generally able to follow the regulation signal closely regardless of the signal realization, the key impact of regulation signal uncertainty would be to add some day-to-day variability in the degradation of a battery, an effect that would balance out over time (days with higher-than-expected degradation balance days with lower-than-expected degradation). A similar story exists for our second source of uncertainty (price uncertainty). While our decision-maker is a price-taker in the day-ahead market and faces uncertainty in the daily price of arbitrage

[38] Interestingly, as we describe in Section 3.2, some literature in the engineering area focuses on evaluation and estimation of these nonlinearities.

and regulation, these uncertainties are expected to balance out over time (days with higher-than-expected prices balance days with lower-than-expected prices). Unlike a newsvendor model where the expected cost objective function grows as demand uncertainty grows, we would not necessarily expect our expected profit to differ under low-uncertainty versus high-uncertainty scenarios for our prices and regulation signal realizations.

In short, standard inventory models do not capture critical aspects of battery systems. In this work, we capture the key aspects of batteries in our modeling (see Section 3.2 and Appendix A.1), and they drive many of our findings in this work.

B. Proofs of Statements (Main)

Proof of Proposition 1

Let $\underline{c}$ be the minimum SOC experienced by the battery while providing regulation. By energy neutrality, each incidence of SOC $\underline{c}$ will be followed by another incidence of SOC $\underline{c}$ within two time units. Consider a sequence c_i of the local minima and maxima of the battery SOC, for $i = 1, ..., K$. Consequently, the sign of $(c_{i+1} - c_i)$ alternates and $c_1 = c_K = \underline{c}$. Let the u_i's be the sizes of cycles identified by the rainflow-counting algorithm. It is easy to argue that $\sum_{i=1}^{K-1} |c_i - c_{i+1}| = 2\sum_i u_i$.[39]

If we are providing r units of frequency regulation, energy neutrality in two periods implies that the maximum SOC $\bar{c}$ satisfies $\bar{c} \leq \underline{c} + r$, and therefore that all cycles returned by the rainflow-counting algorithm cannot exceed size r. This bound on the cycle size in turn yields a bound on the total degradation cost, as the total absolute SOC change cannot exceed Tr when providing r units of regulation for T time periods:

$$\begin{aligned} Deg &= \frac{c_0^b}{E}\sum_i u_i^2 \leq \frac{c_0^b}{E} r \sum_i u_i = \frac{c_0^b r}{2E}\left(\sum_{i=1}^{K-1} |c_i - c_{i+1}|\right) \\ &\leq \frac{c_0^b r^2 T}{2E}. \end{aligned}$$

The signal alternating +1 and -1 every one time unit yields $T/2$ cycles of size r, meaning it achieves the maximum possible degradation cost of $\frac{c_0^b r^2 T}{2E}$. □

Proof of Proposition 2 (Cost-saving)

Proposition 2 follows trivially from degradation function (4). □

Proof of Lemma 1 (Equal excess regulation)

Assume that A is chosen. At optimum we clearly have $r_L \geq a_L$ and $r_H \geq a_H$, so maximizing (5) is equivalent to maximizing

$$(p^a + 2p^r)A + p^r x_L T_L + p^r x_H T_H - \frac{c_0^b}{E}[A^2 + 2A \cdot \max(x_L, x_H) + \frac{T_L}{2}x_L^2 + \frac{T_H}{2}x_H^2].$$

At the optimal point, if x_L and x_H are not identical, (due to the symmetry of the expression) let $x_L < x_H$. Then $\partial\Pi/\partial x_L = T_L(p^r - \frac{c_0^b}{E}x_L) = 0$ and $\partial\Pi/\partial x_H = T_H(p^r - \frac{c_0^b}{E}(\frac{2A}{T_H} + x_H)) = 0$. This immediately implies a contradiction, $x_L \geq x_H$. Thus, at optimality we must have $x = x_L = x_H$.

Expressing (5) with x gives

$$(p^a + 2p^r)A + p^r xT - \frac{c_0^b}{E}[A^2 + 2Ax + \frac{T}{2}x^2],$$

[39] While Definition A allows for half cycles and full cycles to be returned by the rainflow-counting algorithm, for simplicity, we assume here that only full cycles are returned. The argument can trivially be extended to the case with both full and half cycles.

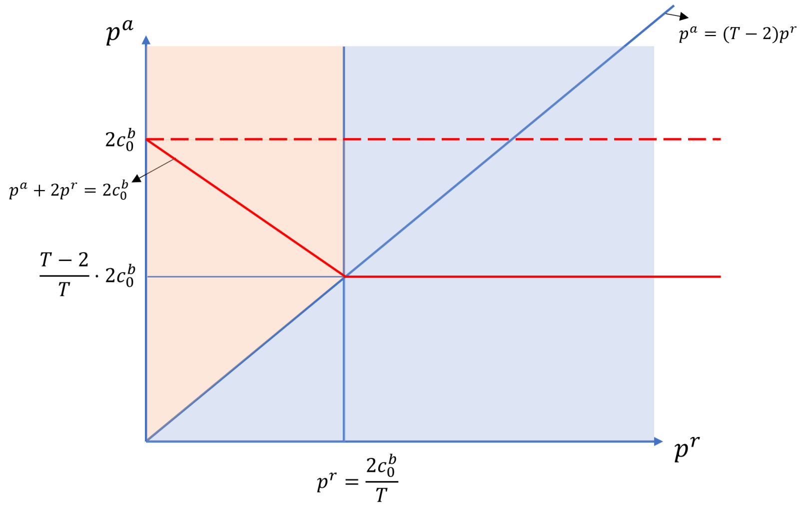


Figure B.1 Illustration of Theorem 1 – The shaded region denotes the optimal excess regulation: $x^* = 0$ (orange) and $x^* > 0$ (blue). Energy capacity is binding on the red line and above for stacking (solid) and standalone arbitrage (dashed).

and the first-order condition becomes $x = \frac{p^r E}{c_0^b} - \frac{2A}{T}$. Since x must be non-negative at the optimal point, we conclude that $x_L^* = x_H^* = (\frac{p^r E}{c_0^b} - \frac{2A}{T})^+$. □

Proof of Theorem 1 (Optimal stacking)

From Lemma 1, we have expressed optimal excess regulation x^* as a function of arbitrage quantity A, yielding profit function

$$\Pi(A) = \begin{cases} p^a A + \frac{(p^r)^2 TE}{2c_0^b} - \frac{c_0^b (T-2)}{TE} A^2, & \text{if } A < \frac{T p^r E}{2c_0^b}, \text{ and} \\ (p^a + 2p^r)A - \frac{c_0^b}{E} A^2, & \text{otherwise.} \end{cases}$$

When $T > 2$, $\partial\Pi/\partial A$ is continuous, piecewise linear, and decreasing, with a single change of slope when $A = \frac{T p^r E}{2c_0^b}$.

We first optimize A^*_{FOC}, the optimal arbitrage when ignoring the energy constraint. If $A < \frac{p^r TE}{2c_0^b}$, then $\partial\Pi/\partial A = p^a - \frac{2c_0^b (T-2)A}{TE}$. We conclude that the profit will be maximized at some $A < \frac{p^r TE}{2c_0^b}$ if and only if $p^a < (T-2)p^r$, in which case $A^*_{FOC} = \frac{p^a TE}{2c_0^b (T-2)}$. Otherwise, $\frac{\partial \Pi}{\partial A} = p^a + 2p^r - 2\frac{c_0^b}{E} A$, so $A^*_{FOC} = \frac{(p^a + 2p^r)E}{2c_0^b}$. Since $\frac{\partial \Pi}{\partial A}$ is decreasing, we conclude that $A^* = \min(A^*_{FOC}, E)$. The corresponding r_L^*, r_H^*, x^*, and Π^*_{stk} follow immediately from Lemma 1.

□

Figure B.1 summarizes the structure of the optimal stacking as service prices change.

Proof of Proposition 3 (Supermodularity)

We defined $f(A,R,p^a,p^r,-c_0^b)$ to be the optimal profit of delivering A total amount of arbitrage and R total amount of regulation across a given day when the price/degradation parameter set is given as $(p^a,p^r,-c_0^b)$:

$$f(A,R,p^a,p^r,-c_0^b)=\max_{r_L,r_H,x_L,x_H}\ p^aA+p^rR-\frac{c_0^b}{E}[A^2+2A\max(x_L,x_H)+\frac{T_L}{2}x_L^2+\frac{T_H}{2}x_H^2]$$
$$\begin{aligned} s.t.\ & a_i=\frac{A}{T_i},\ i=L,H\\ & R=r_LT_L+r_HT_H\\ & x_i=(r_i-a_i)^+,\ i=L,H\\ & A\le E,\\ & A,r_L,r_H\ge 0.\end{aligned}$$

By Lemma 1, $x_L^*=x_H^*=\frac{(R-2A)^+}{T}$, and we have

$$f(A,R,p^a,p^r,-c_0^b)=\begin{cases} p^aA+p^rR-\frac{c_0^b}{E}A^2, & \text{if } R\le 2A, \text{ and}\\ p^aA+p^rR-\frac{c_0^b}{E}[A^2\cdot\frac{T-2}{T}+\frac{R^2}{2T}], & \text{if } R>2A.\end{cases}$$

We first note that f is defined over a lattice since (1) $R,p^a,p^r,c_0^b\in[0,\infty)$, and $A\in[0,E]$ are lattices and (2) the direct product of lattices is a lattice. We then show the supermodularity of f by showing that the function has increasing differences on each pair of variables using Topkis (1978), Theorem 3.2.

For all variable pairs except $(x_i,x_j)=(R,-c_0^b)$ and $(A,-c_0^b)$, f is twice differentiable in x_i,x_j with $\frac{\partial^2 f}{\partial x_i\partial x_j}\ge 0$ everywhere, establishing increasing differences. We now establish increasing differences for $(A,-c_0^b)$ and $(R,-c_0^b)$.

First, we show increasing differences in $(A,-c_0^b)$. Consider $A'\ge A$ and $-c_0^{b'}\ge -c_0^b$. Further, we assume $A'\ge R/2$ and $A\le R/2$ (if not, second derivative arguments establish increasing differences). Writing $\frac{R}{2}=A+\delta_1,\ A'=A+\delta_2$ for some $\delta_2\ge\delta_1\ge 0$ leads to

$$\begin{aligned} & f(A',-c_0^{b'})+f(A,-c_0^b)-f(A',-c_0^b)-f(A,-c_0^{b'})\\ =& (c_0^b-c_0^{b'})[(A')^2-A^2\cdot\frac{T-2}{T}-\frac{R^2}{2T}]\\ =& (c_0^b-c_0^{b'})[\delta_2(2A+\delta_2)-\frac{2}{T}\delta_1(2A+\delta_1)]\ge 0,\end{aligned}$$

so we have increasing differences in $(A,-c_0^b)$.

Second, we show increasing differences in $(R,-c_0^b)$. Consider $R'\ge R$ and $-c_0^{b'}\ge -c_0^b$. Further, we assume $R'\ge 2A$ and $R\le 2A$ (if not, second derivative arguments establish increasing differences). Then

$$f(R',-c_0^{b'})+f(R,-c_0^b)-f(R',-c_0^b)-f(R,-c_0^{b'})=\frac{(c_0^b-c_0^{b'})}{2T}[(R')^2-(2A)^2]\ge 0,$$

so we have increasing differences in $(R, -c_0^b)$.

Next, we show that $\Pi^*(p^a, p^r, -c_0^b) = \max_{A,R} f(A, R, p^a, p^r, -c_0^b)$ is supermodular in $(p^a, p^r, -c_0^b)$. Since the supermodularity is preserved under the maximization operator from Theorem 2.7.6 of Topkis (1998), the result immediately follows.

Lastly, we show the complementarity between decision variables and parameters, i.e., (A^*, R^*) increases in $(p^a, p^r, -c_0^b)$. From Theorem 2.8.1 of Topkis (1998), the result immediately follows. □

Proof of Proposition 4 (Superlinear gain)

For completeness, we first present the closed-form expression of $\Pi^*_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}]$:

$$\Pi^*_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}] = \begin{cases} \frac{p^r[2p^a - p^r(T-2)]E}{2c_0^b}, & \text{if } p^r \le \frac{p^a}{T-2},\ \frac{p^a+2p^r}{2} < c_0^b \\ \frac{(p^a)^2 E}{2c_0^b(T-2)}, & \text{if } p^r \ge \frac{p^a}{T-2},\ \frac{p^a T}{2(T-2)} < c_0^b \\ (p^a + 2p^r)E - c_0^b E - (\frac{(p^a)^2 E}{4c_0^b} + \frac{(p^r)^2 T E}{2c_0^b}), & \text{if } p^r \le \frac{p^a}{T-2},\ \frac{p^a}{2} \le c_0^b \le \frac{p^a+2p^r}{2} \\ p^a E - \frac{(p^a)^2 E}{4c_0^b} - \frac{T-2}{T} c_0^b E, & \text{if } p^r \ge \frac{p^a}{T-2},\ \frac{p^a}{2} \le c_0^b \le \frac{p^a T}{2(T-2)} \\ \frac{p^r E}{2c_0^b}(4c_0^b - p^r T), & \text{if } p^r \le \frac{p^a}{T-2},\ \frac{p^a}{2} \ge c_0^b, \text{ and} \\ \frac{2}{T} c_0^b E, & \text{if } p^r \ge \frac{p^a}{T-2},\ \frac{p^a}{2} \ge c_0^b. \end{cases}$$

Next, we will prove the superlinear gain result, $\Pi^*_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}] > 0$. When $p^r \le \min(\frac{p^a}{T-2}, \frac{2c_0^b}{T})$, we can lower-bound stacking's profit with $\tilde{\Pi}^{(1)}_{stk}$, which we define to be the profit of stacking when it uses arbitrage amount A^*_{arb} and the maximum free regulation ($R = 2A^*_{arb}$). Then

$$\begin{aligned} \Pi^*_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}] &\ge \tilde{\Pi}^{(1)}_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}] \\ &= \begin{cases} \frac{p^r[2p^a - p^r T]E}{2c_0^b} \ge \frac{(p^r)^2(T-4)E}{2c_0^b} > 0, & \text{if } \frac{p^a}{2} < c_0^b, \text{ and} \\ \frac{p^r E}{2c_0^b}[4c_0^b - p^r T] > 0, & \text{if } \frac{p^a}{2} \ge c_0^b. \end{cases} \end{aligned}$$

When $p^r \ge \min(\frac{p^a}{T-2}, \frac{2c_0^b}{T})$, we can lower-bound stacking's profit with $\Pi^{(2)}_{stk}$, which we define to be the profit of stacking when it uses arbitrage amount A^*_{arb} and regulation amount $R^*_{reg} = \frac{p^r T E}{c_0^b}$, distributed between the low- and high-price regions to ensure equal excess regulation amounts. Then

$$\begin{aligned} \Pi^*_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}] &\ge \tilde{\Pi}^{(2)}_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}] \\ &= \frac{2c_0^b}{TE}(A^*_{arb})^2 > 0. \end{aligned}$$

□

Proof of Theorem 2 (Upper bound on stacking performance)

We first consider the case when energy capacity is not binding at optimality for arbitrage-only or stacking. Then:

(a) The ratio of stacking profit to arbitrage profit, $\frac{\Pi^*_{stk}}{\Pi^*_{arb}}$, is monotonically increasing in $\frac{p^r}{p^a}$:

$$\frac{\Pi^*_{stk}}{\Pi^*_{arb}} = \begin{cases} \frac{T}{T-2} + 2T(\frac{p^r}{p^a})^2, & \text{if } p^a \leq (T-2)p^r, \text{ and} \\ (1+2\frac{p^r}{p^a})^2, & \text{if } p^a \geq (T-2)p^r. \end{cases} \tag{B.1}$$

(b) The ratio of stacking profit to frequency-regulation profit, $\frac{\Pi^*_{stk}}{\Pi^*_{reg}}$, is monotonically increasing in $\frac{p^a}{p^r}$:

$$\frac{\Pi^*_{stk}}{\Pi^*_{reg}} = \begin{cases} 1 + \frac{1}{2(T-2)}(\frac{p^a}{p^r})^2, & \text{if } p^a \leq (T-2)p^r, \text{ and} \\ \frac{1}{2T}(\frac{p^a}{p^r}+2)^2, & \text{if } p^a \geq (T-2)p^r. \end{cases} \tag{B.2}$$

The two ratios are equal when $\Pi^*_{arb} = \Pi^*_{reg}$, or $p^a = \sqrt{2T}p^r$. Since $T \geq 6$, then $\sqrt{2T}p^r \leq (T-2)p^r$, and the maximum S value of $S = \frac{2T-2}{T-2}$ follows by evaluating (B.1) and (B.2) at $p^a = \sqrt{2T}p^r$.

We now proceed to show that $\frac{2T-2}{T-2}$ is the upper bound for energy-binding cases. When energy capacity is binding for stacking but not for standalone arbitrage or when $\Pi^*_{reg} > \Pi^*_{arb}$, the S ratio is upper-bounded by the S ratio under no energy constraint, meaning $S \leq \frac{2T-2}{T-2}$. As a result, the remainder of this proof will focus on the case when $\Pi^*_{arb} > \Pi^*_{reg}$ ($p^aE - c^b_0E > \frac{(p^r)^2TE}{2c^b_0}$) and when standalone arbitrage is energy-constrained ($p^a \geq 2c^b_0$). Then Theorem 1 states that

$$\Pi^*_{stk} = \begin{cases} \frac{(p^r)^2TE}{2c^b_0} + p^aE - c^b_0(1-\frac{2}{T})E, & \text{if } p^r > \frac{2c^b_0}{T}, \text{ and} \\ (p^a+2p^r)E - c^b_0E, & \text{otherwise.} \end{cases}$$

If $p^r \leq \frac{2c^b_0}{T}$, then

$$\begin{aligned} \frac{\Pi^*_{stk}}{\Pi^*_{arb}} = \frac{(p^a+2p^r)E - c^b_0E}{p^aE - c^b_0E} &\leq \frac{p^aE + \frac{4c^b_0E}{T} - c^b_0E}{p^aE - c^b_0E} \\ &\leq \frac{1}{1-\frac{c^b_0}{p^a}} \ (\because T \geq 4) \\ &\leq 2 \ (\because p^a \geq 2c^b_0). \end{aligned}$$

If $p^r > \frac{2c^b_0}{T}$, then

$$\begin{aligned} \frac{\Pi^*_{stk}}{\Pi^*_{arb}} &= \frac{\frac{(p^r)^2TE}{2c^b_0} + p^aE - c^b_0(1-\frac{2}{T})E}{p^aE - c^b_0E} \\ &< \frac{2(p^aE - c^b_0E) + \frac{2c^b_0}{T}E}{p^aE - c^b_0E} = 2 + \frac{\frac{2c^b_0}{T}}{p^a - c^b_0} \ (\because \Pi^*_{reg} < \Pi^*_{arb}) \\ &\leq 2 + \frac{2}{T} \ (\because p^a \geq 2c^b_0) \\ &< 2 + \frac{2}{T-2} = \frac{2T-2}{T-2}. \end{aligned}$$

□

Proof of Theorem 3 (Improved stopping threshold)

We derive the stopping threshold for each operation and then we can easily compare the stopping thresholds between stacking and standalone services and complete the proof.

First, we derive the stopping threshold for standalone arbitrage. Recall that providing arbitrage with efficiency η is equivalent to providing arbitrage with perfect efficiency, low-period price $\underline{p}' = \frac{\underline{p}}{\eta^2}$, and degradation coefficient $c_0^{b\prime} = \frac{c_0^b}{\eta^2}$. As derived at the beginning of Section 5, it is optimal to provide arbitrage as long as $\bar{p} > \underline{p}'$, or, equivalently, whenever $\eta \in (\eta_{arb}, 1]$, where $\eta_{arb} = \sqrt{\frac{\underline{p}}{\bar{p}}}$ is the stopping threshold for standalone arbitrage.

Next, consider stacking arbitrage and regulation. At optimality, we must have $r_L \geq a_L$ and $r_H \geq a_H$. Defining $A = T_H a_H$ to be the amount of arbitrage sold to the grid in the high-price period and $X = T_L(r_L - a_L) + T_H(r_H - a_H)$ to be the amount of excess regulation sold to the grid, the energy balance equations (17)-(18) imply $T_L a_L = \frac{A}{\eta^2} + \frac{X}{2}(\frac{1}{\eta^2} - 1)$ (the amount of arbitrage energy we must buy in the low-price period $T_L a_L$ is a linear function of A and X). Then we have $\Pi_{stk} = \tilde{p}^a A + \tilde{p}^x X - Deg(D, x_L, x_H)$, for constants $\tilde{p}^a = \bar{p} - \frac{\underline{p}}{\eta^2} + p^r(1 + \frac{1}{\eta^2})$ and $\tilde{p}^x = \frac{p^r}{2}(\frac{1}{\eta^2} + 1) - \frac{\underline{p}}{2}(\frac{1}{\eta^2} - 1)$.

For stacking with sufficiently small A, $X = O(\epsilon)$, we have degradation of order $O(\epsilon^2)$. Therefore, we will always provide at least a bit of service if either $\tilde{p}^a > 0$ or $\tilde{p}^x > 0$. When $\underline{p} \leq p^r$, then $\tilde{p}^a, \tilde{p}^x > 0$, so we always provide service regardless of η. Otherwise, $\tilde{p}^a > 0$ is equivalent to $\eta > \eta_{stk} := \sqrt{\frac{(\underline{p} - p^r)^+}{\bar{p} + p^r}}$ and $p^x > 0$ is equivalent to $\eta > \eta_{reg} := \sqrt{\frac{(\underline{p} - p^r)^+}{\underline{p} + p^r}}$. Since $\eta_{stk} \leq \eta_{reg}$, we provide stacked service whenever $\eta > \eta_{stk}$.

For regulation-only service, we require $A = 0$. Therefore, similar reasoning shows that we provide service if $\eta > \eta_{reg}$.

Proof of Lemma 2 (Excess regulation: Bounded power)

We prove the statement by contradiction, in all cases showing that an adjustment to the regulation bids can result in an increased profit while still respecting the power and energy constraints. Suppose $x_H^* > x_L^*$. Then $r_H^* > 0$ and $a_L^* + r_L^* < a_H^* + r_H^*$, meaning there's excess power capacity in the low-price period, so it is feasible to decrease r_H^* by some sufficiently small $\epsilon > 0$ and to increase r_L^* by $\epsilon T_H/T_L$. The net changes in regulation revenues between the low- and high-price periods cancel out, while the adjustment reduces degradation costs by $\frac{c_0^b}{E}(2A^* + T_H(x_H^* - x_L^*))\epsilon + O(\epsilon^2) > 0$, a contradiction to optimality.

Similarly, if $r_L^* + a_L^* > r_H^* + a_H^*$, then $r_L^* - a_L^* > r_H^* - a_H^*$ and, thus, $x_L^* > x_H^*$ or $x_L^* = x_H^* = 0$. If $x_L^* > x_H^*$, then for sufficiently small $\epsilon > 0$ it is feasible to increase r_H^* by ϵ and to decrease r_L^* by $\epsilon T_H/T_L$, yielding identical revenue but reducing degradation costs by $\frac{c_0^b}{E} T_H(2A^*/T_L + x_L^* - x_H^*)\epsilon + O(\epsilon^2) > 0$. If $x_L^* = x_H^* = 0$, then it is feasible to increase r_H^* by some small $\epsilon > 0$, increasing profit by $p^r T_H \epsilon + O(\epsilon^2) > 0$. Thus, both cases yield a contradiction to optimality. □

Proof of Theorem 4 (Upper bound on stacking's performance: Bounded power)

We first state the optimal single-service solutions:

$$r^*_{reg} = \begin{cases} \frac{p^r E}{c_0^b}, & \text{if } P > \frac{p^r E}{c_0^b}, \text{ and} \\ P, & \text{otherwise,} \end{cases}$$

$$\Pi^*_{reg} = \begin{cases} \frac{(p^r)^2 TE}{2c_0^b}, & \text{if } P > \frac{p^r E}{c_0^b}, \text{ and} \\ p^r PT - \frac{c_0^b TP^2}{2E}, & \text{otherwise,} \end{cases}$$

$$A^*_{arb} = \begin{cases} \frac{p^a E}{2c_0^b}, & \text{if } p^a < 2\frac{c_0^b}{E} \cdot \min(PT_H, E), \text{ and} \\ \min(PT_H, E), & \textit{otherwise}, \end{cases}$$

$$\Pi^*_{arb} = \begin{cases} \frac{(p^a)^2 E}{4c_0^b}, & \text{if } p^a < 2\frac{c_0^b}{E} \cdot \min(PT_H, E), \text{ and} \\ p^a \cdot \min(PT_H, E) - \frac{c_0^b}{E}[\min(PT_H, E)]^2, & \text{otherwise.} \end{cases}$$

We next introduce and prove two technical lemmas (Lemma B.1 and Lemma B.2). Then we prove the main result.

Lemma B.1. *If* $P \le \frac{p^r E}{c_0^b}$*, then* $\Pi^*_{stk} \le \Pi^*_{arb} + \Pi^*_{reg}$.

We first show that power is binding in both time periods for stacking when $P \le \frac{p^r E}{c_0^b}$ (regulation-only is power binding). Suppose power is not binding in the low-price period at the optimality under $P \le \frac{p^r E}{c_0^b}$. Since $x^*_L \ge x^*_H$ (Lemma 2), we can increase the objective (5) by increasing r_L as follows:

$$\frac{\partial \Pi}{\partial r_L} = \begin{cases} p^r T_L - \frac{c_0^b}{E}[2A^* + T_L(r^*_L - a^*_L)] = T_L[p^r - \frac{c_0^b}{E}[r^*_L + a^*_L]], & \text{if } r^*_L \ge a^*_L, \text{ and} \\ p^r T_L, & \text{if } r^*_L < a^*_L. \end{cases}$$

The marginal profit change is always positive as $r^*_L + a^*_L < P$ and $P \le \frac{p^r E}{c_0^b}$, which is a contradiction.

Since power is binding in both time periods, stacking can be expressed as a single-variable optimization problem — given arbitrage amount A we have $r_i = P - \frac{A}{T_i}$ and $x_i = (P - \frac{2A}{T_i})^+$ for $i = L, H$. Define $\Pi_{stk}(A) = (p^a - 2p^r)A + p^r PT - Deg(A, x_L, x_H)$ to be the profit of stacking with A fixed, define $\Pi^*_{reg} = p^r PT - \frac{c_0^b TP^2}{2E}$ to be the optimal profit of regulation-only, and define $\Pi_{arb}(A) = p^a A - \frac{c_0^b}{E}A^2$ to be the profit of arbitrage-only with A fixed. Then we have

$$\begin{aligned} \Pi_{arb}(A) + \Pi^*_{reg} - \Pi_{stk}(A) &= 2p^r A - \frac{c_0^b TP^2}{2E} + \frac{c_0^b}{E}[2Ax_L + \frac{T_L}{2}x_L^2 + \frac{T_H}{2}x_H^2] \\ &\ge \frac{c_0^b}{E}[2AP - \frac{TP^2}{2} + 2Ax_L + \frac{T_L}{2}x_L^2 + \frac{T_H}{2}x_H^2], \end{aligned}$$

where the inequality is from $p^r \ge P\frac{c_0^b}{E}$. Let $f(A) = 2AP - \frac{TP^2}{2} + 2Ax_L + \frac{T_L}{2}x_L^2 + \frac{T_H}{2}x_H^2$.

$$f'(A) = \begin{cases} 2P, & \text{if } A > \frac{PT_L}{2} \ (x_L = x_H = 0), \\ 2P - \frac{4A}{T_L}, & \text{if } \frac{PT_H}{2} \le A \le \frac{PT_L}{2} \ (x_L > x_H = 0), \text{ and} \\ 4A(\frac{1}{T_H} - \frac{1}{T_L}), & \text{if } A < \frac{PT_H}{2} \ (x_H > 0), \end{cases}$$

so $f(A)$ is increasing in A from an initial value of 0 at $A = 0$. Defining A^* to be the profit-maximizing arbitrage amount under stacking, $\Pi^*_{stk} = \Pi^*_{stk}(A^*) \le \Pi^*_{reg} + \Pi^*_{arb}(A^*) \le \Pi^*_{reg} + \Pi^*_{arb}$. □

Lemma B.2. *If* $P > \frac{p^r E}{c_0^b}$ *and* $p^a \geq 2\frac{c_0^b}{E} \cdot \min(PT_H, E)$*, then* $\frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})} \leq \frac{2T-2}{T-2}$.

Proof. From $P > \frac{p^r E}{c_0^b}$, we know that frequency regulation-only is not power binding at optimality. Therefore, when $\Pi^*_{reg} \geq \Pi^*_{arb}$, the result that $\frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})} \leq \frac{2T-2}{T-2}$ follows from Theorem 2. The remainder of this proof will focus on the case when $\Pi^*_{arb} > \Pi^*_{reg}$. We consider the two subcases of (1) $PT_H \leq E$ and (2) $E \leq PT_H$.

(1) $PT_H \leq E$: In this case, $\Pi^*_{reg} = \frac{(p^r)^2 TE}{2c_0^b}$ and $\Pi^*_{arb} = p^a(PT_H) - \frac{c_0^b}{E}(PT_H)^2$. First, we establish that stacking is power-binding in the high-price period. From Observation 3, unconstrained stacking has a strictly larger A^* (and therefore strictly higher power use in both time periods) than unconstrained arbitrage-only. Since arbitrage-only is power-binding when $p^a \geq 2\frac{c_0^b}{E} PT_H$, stacking must be as well.

Next, we establish that $x^*_H = 0$. Suppose not ($x^*_H > 0$). Given high-price period power use $2a^*_H + x^*_H = P$, $A^* < \frac{PT_H}{2}$. Then consider increasing A by some small $\epsilon > 0$ and decreasing r_L by $\frac{\epsilon}{T_L}$ and r_H by $\frac{\epsilon}{T_H}$, which does not affect the power usage in either time period. This implies a $\frac{2\epsilon}{T_L}$ decrease in x_L and a $\frac{2\epsilon}{T_H}$ decrease in x_H. Then the marginal profit change becomes

$$\begin{aligned}
[p^a - 2p^r - 2\frac{c_0^b}{E}(A(1-\frac{2}{T_L}) - x_H)]\epsilon + O(\epsilon^2) &> \frac{c_0^b}{E}\epsilon[2PT_H - 2A(1 - \frac{2}{T_L} + \frac{2}{T_H})] + O(\epsilon^2) \\
&> \frac{c_0^b}{E}\epsilon PT_H[2 - (1 - \frac{2}{T_L} + \frac{2}{T_H})] + O(\epsilon^2) \\
&= \frac{c_0^b}{E}\epsilon PT_H[1 + \frac{2}{T_L} - \frac{2}{T_H}] + O(\epsilon^2),
\end{aligned}$$

where the first inequality is from $\frac{p^r E}{c_0^b} < P < \frac{p^a E}{2c_0^b T_H}$ and the second inequality is from $A^* < \frac{PT_H}{2}$. This is positive for sufficiently small $\epsilon > 0$, a contradiction of optimality.

Next, we establish that if $x^*_L = x^*_H = 0$, then $\Pi^*_{stk} \leq 2\Pi^*_{arb}$:

$$\begin{aligned}
\Pi^*_{stk} &= p^a A^* + p^r r^*_L T_L + p^r r^*_H T_H - \frac{c_0^b}{E}(A^*)^2 \\
&\leq p^a A^* - \frac{c_0^b}{E}(A^*)^2 + 2p^r A^* \ (\because r^*_i \leq \frac{A^*}{T_i}, i = L, H) \\
&\leq p^a(PT_H) - \frac{c_0^b}{E}(PT_H)^2 + 2p^r(PT_H) \ (\because p^a \geq 2\frac{c_0^b}{E}PT_H \geq 2\frac{c_0^b}{E}A^*) \\
&< \Pi^*_{arb} + 2\frac{c_0^b}{E}P^2 T_H \leq \Pi^*_{arb} + \frac{c_0^b}{E}(PT_H)^2 \leq 2\Pi^*_{arb}.
\end{aligned}$$

Last, we establish that if $x^*_L > x^*_H = 0$, then $\Pi^*_{stk} \leq \Pi^*_{arb} + \Pi^*_{reg}$. We first note that power is not binding in the low-price period for stacking when $P > \frac{p^r E}{c_0^b}$. Suppose this is not true ($x^*_L + \frac{2A^*}{T_L} = P$). Then we can increase the profit by decreasing r_L (and x_L) by a unit as the marginal profit change is $-p^r T_L - \frac{c_0^b}{E}[-2A - T_L x_L] = T_L[\frac{c_0^b}{E}P - p^r] > 0$. Thus, the power constraint in the low-period is always inactive ($r_L + a_L < P$). For given A, the optimal x_L can be found by taking the first-order condition in terms of x_L and setting it equal to zero, which leads to $x_L^*(A) = \frac{p^r E}{c_0^b} - \frac{2A}{T_L}$ and $r_L^*(A) = \frac{p^r E}{c_0^b} - \frac{A}{T_L}$. Thus, the $x_L^* > 0$ region only happens when $p^r > \frac{2c_0^b A^*}{T_L E}$. Noting that $r_H^* = P - a_H^*$ when $x_H^* = 0$, then

$$\begin{aligned}\Pi_{stk}^* &= \Pi_{reg}^* + p^a A^* - \frac{c_0^b}{E}(A^*)^2 + \frac{2c_0^b}{T_L E}(A^*)^2 + p^r(PT_H - 2A^* - \frac{p^r E}{2c_0^b}T_H) \\ &< \Pi_{reg}^* + (p^a + \frac{2c_0^b PT_H}{T_L E})A^* - \frac{c_0^b}{E}(1 + \frac{2T}{T_L^2})(A^*)^2,\end{aligned}$$

where the strict inequality comes from the fact that $p^r(PT_H - 2A^* - \frac{p^r E}{2c_0^b}T_H)$ is strictly decreasing in p^r (since $x_H^* = 0$ implies $A^* \geq \frac{PT_H}{2}$), so we obtain an upper bound by plugging in $p^r = \frac{2c_0^b A^*}{T_L E}$. All that remains is to prove that $(p^a + \frac{2c_0^b PT_H}{T_L E})(A^*) - (1 + \frac{2T}{T_L})(A^*)^2 \leq \Pi_{arb}^*$. Using $p^a \geq 2\frac{c_0^b}{E}PT_H$, the first-order condition of $f(A) = (p^a + \frac{2c_0^b PT_H}{T_L E})A - (1 + \frac{2T}{T_L})A^2$ implies that $f(A)$ is maximized at $B = \min(PT_H, \bar{B})$ for some $\bar{B} \geq \frac{PT_H(T_L^2 + T_L)}{T_L^2 + 2T} \geq \frac{PT_H}{2}$. This leads to

$$\begin{aligned}\Pi_{stk}^* &< \Pi_{reg}^* + (p^a + \frac{2c_0^b PT_H}{T_L E})B - \frac{c_0^b}{E}(1 + \frac{2T}{T_L^2})B^2 \\ &\leq \Pi_{reg}^* + \Pi_{arb}^* + \frac{2c_0^b PT_H}{T_L E}B - \frac{c_0^b}{E}\frac{2T}{T_L^2}B^2.\end{aligned}$$

Since $\frac{2c_0^b PT_H}{T_L E}B - \frac{c_0^b}{E}\frac{2T}{T_L^2}B^2$ is decreasing in B for $B \geq \frac{PT_H}{2}$ and is negative when evaluated at lower bounds $B = PT_H$ and $B = \frac{PT_H(T_L^2 + T_L)}{T_L^2 + 2T}$, we conclude that $\Pi_{stk}^* < \Pi_{reg}^* + \Pi_{arb}^*$.

(2) $E \leq PT_H$: In this case, standalone arbitrage is energy constrained. Let $\tilde{\Pi}_{stk}^*$ be the optimal stacking profit under $P = \infty$. Then $\frac{\Pi_{stk}^*}{\Pi_{arb}^*} \leq \frac{\tilde{\Pi}_{stk}^*}{\Pi_{arb}^*} \leq \frac{2T-2}{T-2}$, where the second inequality is immediate from the proof of Theorem 2.

□

Lastly, we prove the main result.

When $p^r \geq \frac{c_0^b}{E}P$ or $p^r < \frac{c_0^b}{E}P, p^a \geq 2\frac{c_0^b}{E} \cdot \min(PT_H, E)$, then Lemmas B.1 and B.2 combined imply that $\frac{\Pi_{stk}^*}{\max(\Pi_{arb}^*, \Pi_{reg}^*)} \leq \frac{2T-2}{T-2}$. When $p^r < \frac{c_0^b}{E}P$ and $p^a < 2\frac{c_0^b}{E} \cdot \min(PT_H, E)$, then arbitrage-only and regulation-only are not power/energy constrained at optimality, so $\frac{\Pi_{stk}^*}{\max(\Pi_{arb}^*, \Pi_{reg}^*)} \leq \frac{2T-2}{T-2}$ immediately follows from Theorem 2. □

C. Generalizations/Extensions and Proofs

C.1. Extension 1: Correlated Prices

C.1.1. Empirical Evidence Many ISO/RTOs publish their operational datasets (e.g., market-clearing results). From two different ISOs (PJM and ISO-NE), we obtained three types of data for period July 2020 - June 2021: (1) day-ahead hourly energy prices, (2) 5-min based regulation prices (the sum of capacity price (\$/MWh) + performance price (\$/MW) × mileage (ΔMW)), and (3) regulation signal. We calibrate the model parameters from the data as follows: Daily peak prices $\bar{p}$ are obtained by computing the 2-hour period with the highest average price each day. Daily off-peak prices $\underline{p}$ are an average of the remaining 22 hours. The 5-min based regulation prices are averaged on an hourly basis since both ISOs make capacity commitments on an hourly basis and at least an hour ahead of time. Daily $\bar{p}^r$ and $\underline{p}^r$ are obtained using the same 2-hour and 22-hour periods identified in each day when computing $\bar{p}$ and $\underline{p}$. This gives 365 data points (1 year) for $(\underline{p}, \underline{p}^r)$ and $(\bar{p}, \bar{p}^r)$. Then the base regulation price p_0^r and the linear coefficient k are fitted via linear regression. For ISO-NE we have $k = 0.23$ and $p_0^r = 10.99$, and for PJM we have $k = 0.29$ and $p_0^r = 12.65$.

C.1.2. Optimal Stacking: Correlated Prices

Optimal structure. Unlike the uncorrelated setting, we might expect maintaining unequal excess regulation ($x_H^* > x_L^*$) to be optimal when k is sufficiently large (such that regulation during the high-price period is attractive enough). The following theorem states the optimal actions and the corresponding profit.

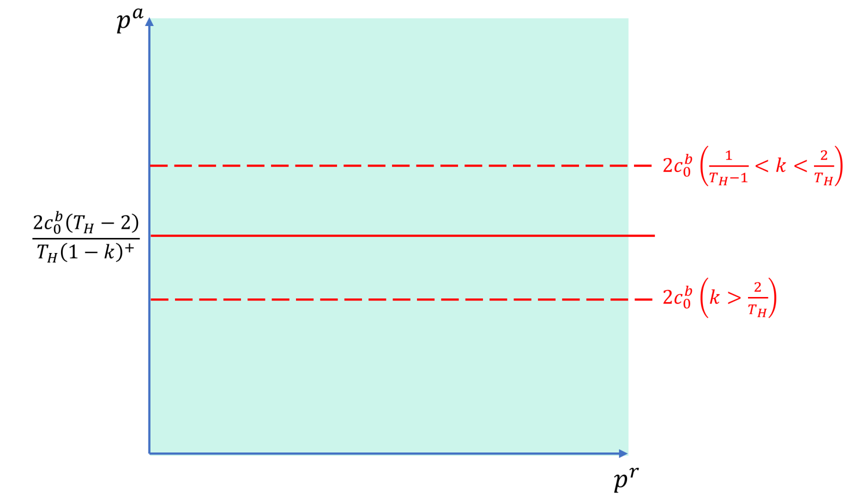
Theorem C.1 (Optimal stacking: Correlated prices)**.** *With correlated prices, the optimal arbitrage and regulation quantities and profit are as follows:*

Case	$x_H^* > x_L^* > 0$	$x_H^* = x_L^* = 0$	$x_H^* = x_L^* > 0$
cond.	$k > \frac{1}{T_H - 1}$ *or* $p^a > \frac{2c_0^b}{kT_H}$	$p^r \le \min(\frac{1+\frac{k(T_L-T_H)}{T}}{T-2}p^a, \frac{2c_0^b}{T})$	*otherwise*
A^*	$\min(\frac{T_H p^a (1-k)^+}{2c_0^b (T_H-2)}E, E)$	$\min(\frac{(1+\frac{k(T_L-T_H)}{T})p^a + 2p^r}{2c_0^b}E, E)$	$\min(\frac{p^a[T+k(T_L-T_H)]}{2c_0^b(T-2)}E, E)$
R^*	$\frac{p^r TE}{c_0^b}$	$2A^*$	$\frac{p^r TE}{c_0^b}$
Π^*	$\frac{T(p^r)^2 + \frac{k^2 T_L T_H}{T}(p^a)^2}{2c_0^b}E + (1-k)p^a A^*$ $-\frac{c_0^b}{E}(1 - \frac{2}{T_H})(A^*)^2$	$[(1 + \frac{k(T_L-T_H)}{T})p^a + 2p^r]A^*$ $-\frac{c_0^b}{E}(A^*)^2$	$(1 + \frac{(T_L-T_H)k}{T})p^a A^* + \frac{TE}{2c_0^b}(p^r)^2$ $-\frac{c_0^b}{E}(1 - \frac{2}{T})(A^*)^2$

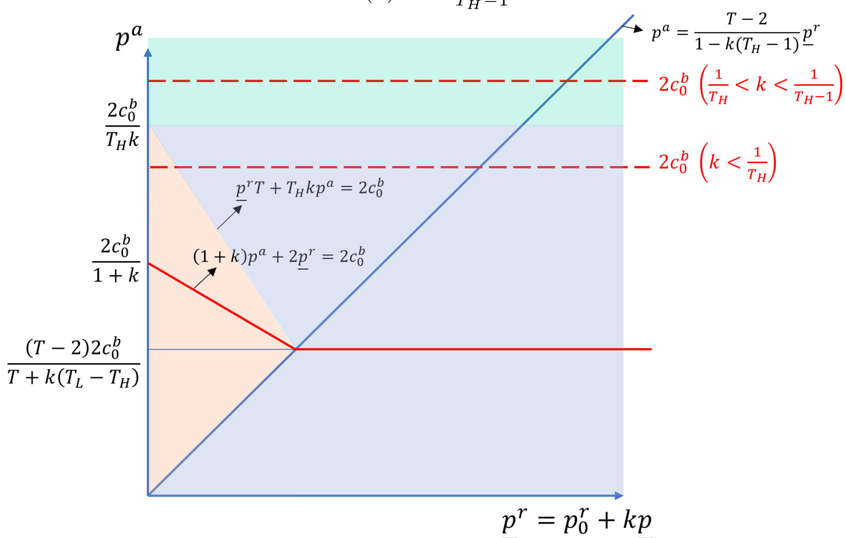
where $p^r := \frac{T_L \underline{p}^r + T_H \bar{p}^r}{T}$ *is defined to be the average regulation price throughout the day.*

Figure C.1 summarizes the structure of the optimal stacking.

Proof. First, we establish that $x_H^* \ge x_L^*$. If this were not the case ($x_L^* > x_H^*$), then we could decrease r_L^* by some sufficiently small $\epsilon > 0$ and increase r_H^* by $\frac{T_L \epsilon}{T_H}$, yielding a profit change of at least $((\bar{p}^r - \underline{p}^r)T_L + \frac{c_0^b}{E}(2A + T_L(x_L^* - x_H^*))\epsilon + O(\epsilon^2)$. This achieves a profit improvement for sufficiently small $\epsilon > 0$, a contradiction of optimality.



(a) $k > \frac{1}{T_H-1}$



(b) $k \leq \frac{1}{T_H-1}$

Figure C.1 Illustration of Theorem C.1 – The shaded region denotes the optimal excess regulation: $x_H^* > x_L^* > 0$ (green), $x_H^* = x_L^* = 0$ (orange), and $x_H^* = x_L^* > 0$ (blue). Energy capacity is binding on the red line and above for stacking (solid) and standalone arbitrage (dashed).

Let $x_H - x_L \coloneqq \delta \geq 0$. Then we can rewrite the objective function as

$$\Pi = (\bar{p} - \underline{p})A + \underline{p}^r(x_L + a_L)T_L + \bar{p}^r(x_L + \delta + a_H)T_H - \frac{c_0^b}{E}[A^2 + 2A(x_L + \delta) + \frac{T_L}{2}x_L^2 + \frac{T_H}{2}(x_L + \delta)^2].$$

For fixed A and x_L the first-order conditions indicate an optimal δ value of

$$\delta^*(A, x_L) = (\frac{\bar{p}^r E}{c_0^b} - \frac{2A}{T_H} - x_L)^+,$$

which can be used to derive the optimal profit for fixed A and x_L:

$$\Pi(A,x_L)=\begin{cases}((p^a+\bar{p}^r+\underline{p}^r)A-\frac{c_0^b}{E}A^2)+(\underline{p}^rT_L+\bar{p}^rT_H-2\frac{c_0^b}{E}A)x_L-\frac{c_0^bT}{2E}x_L^2, \text{ if } \frac{2A}{T_H}+x_L\geq\frac{\bar{p}^rE}{c_0^b}\ (\delta^*=0), \text{ and}\\ (\frac{(\bar{p}^r)^2T_HE}{2c_0^b}+(1-k)p^aA-\frac{c_0^b}{E}(1-\frac{2}{T_H})A^2)+\underline{p}^rT_Lx_L-\frac{c_0^bT_L}{2E}x_L^2, \text{ if } \frac{2A}{T_H}+x_L<\frac{\bar{p}^rE}{c_0^b}\ (\delta^*>0).\end{cases}$$

Next, we derive the optimal x_L value given fixed A. When $A\geq\frac{\bar{p}^rT_HE}{2c_0^b}$, then $\delta^*=0$ regardless of the value of x_L. The first-order condition immediately yields $x_L^*=\frac{(\underline{p}^rT_L+\bar{p}^rT_H-2\frac{c_0^b}{E}A)^+}{\frac{c_0^b}{E}T}$. Otherwise, $\delta^*>0$ for $x_L<\frac{\bar{p}^rE}{c_0^b}-\frac{2A}{T_H}$ and $\delta^*=0$ otherwise. In this case, $\Pi(A,x_L)$ is a strictly concave function of x_L with a continuous, piecewise linear, decreasing derivative that takes value

$$\Pi'(A,\frac{\bar{p}^rE}{c_0^b}-\frac{2A}{T_H})=\frac{2c_0^bAT_L}{T_HE}-kp^aT_L.$$

We conclude that $\delta^*>0$ whenever $\Pi'(A,\frac{\bar{p}^rE}{c_0^b}-\frac{2A}{T_H})<0$ $(A<\frac{kp^aT_HE}{2c_0^b})$, in which case $x_L^*=\frac{\underline{p}^rE}{c_0^b}$. Otherwise, $\delta^*=0$, in which case $x_L^*=\frac{\underline{p}^rT_L+\bar{p}^rT_H-2\frac{c_0^b}{E}A}{\frac{c_0^b}{E}T}$.

Taken together, these results yield the optimal profit for fixed A:

$$\Pi(A)=\begin{cases}\frac{(\underline{p}^r)^2T_LE}{2c_0^b}+\frac{(\bar{p}^r)^2T_HE}{2c_0^b}+(1-k)p^aA-\frac{c_0^b}{E}(1-\frac{2}{T_H})A^2, & \text{if } A<\frac{kp^aT_HE}{2c_0^b}\ (x_L^*>0,\delta^*>0),\\ \frac{(\underline{p}^rT_L+\bar{p}^rT_H)^2E}{2c_0^bT}+(1+\frac{(T_L-T_H)k}{T})p^aA-\frac{c_0^b}{E}(1-\frac{2}{T})A^2, & \text{if } \frac{kp^aT_HE}{2c_0^b}\leq A<\frac{\bar{p}^rT_H+\underline{p}^rT_LE}{2c_0^b}\ (x_L^*>0,\delta^*=0), \text{ and}\\ (p^a+\bar{p}^r+\underline{p}^r)A-\frac{c_0^b}{E}A^2, & \text{if } A\geq\frac{\bar{p}^rT_H+\underline{p}^rT_LE}{2c_0^b}\ (x_L^*=0,\delta^*=0).\end{cases}$$

$\Pi(A)$ is a strictly concave function with a continuous, piecewise linear, decreasing derivative that takes values

$$\Pi'(\frac{kp^aT_HE}{2c_0^b})=p^a(1-k(T_H-1)) \text{ and}$$
$$\Pi'(\frac{\bar{p}^rT_H+\underline{p}^rT_LE}{2c_0^b})=p^a-(T_H-1)\bar{p}^r-(T_L-1)\underline{p}^r.$$

Then the optimal solution satisfies the following:

- $A^*<\frac{kp^aT_HE}{2c_0^b}$ whenever $\Pi'(\frac{kp^aT_HE}{2c_0^b})<0$ (equivalently, $k>\frac{1}{T_H-1}$) or $E<\frac{kp^aT_HE}{2c_0^b}$,
- $A^*\geq\frac{\bar{p}^rT_H+\underline{p}^rT_LE}{2c_0^b}$ whenever $\Pi'(\frac{\bar{p}^rT_H+\underline{p}^rT_LE}{2c_0^b})\geq 0$ and $E\geq\frac{(\bar{p}^rT_H+\underline{p}^rT_L)E}{2c_0^b}$, and
- $\frac{kp^aT_HE}{2c_0^b}\leq A^*<\frac{(\bar{p}^rT_H+\underline{p}^rT_L)E}{2c_0^b}$ otherwise.

The optimal A^* values follow immediately from the first-order conditions of the various regions of $\Pi(A)$ and the energy constraint E.

□

Synergistic behavior. Following Theorem C.1, let $p^r:=\frac{T_L\underline{p}^r+T_H\bar{p}^r}{T}$ denote the weighted average of the regulation prices. Consistent with Observation 3, A^* (R^*) increases in p^r (p^a) whenever $x_H^*=x_L^*=0$ (free regulation-only region). However, the threshold for how high the arbitrage price needs to be to trigger this free regulation-only region is relaxed whenever $k>0, T_L>T_H$ compared to the base model.

Observation C.1. *The criterion for the free regulation-only region is relaxed whenever $k > 0, T_L > T_H$ compared to the base model.*

The synergistic behavior is further expanded under correlated prices: for fixed p^a, A^* increases in k (or $\bar{p}^r - \underline{p}^r$; the gap between the regulation prices) whenever $x_H^* = x_L^*$. That is, as long as the gap $\bar{p}^r - \underline{p}^r$ is not too high ($k \le \frac{1}{T_H - 1}$, $p^a \le \frac{2c_0^b}{kT_H}$), the additional margin from the higher regulation price as k is increased leads to a higher arbitrage quantity, which is used to maximize the cost saving benefit from equal excess regulation. When $\bar{p}^r - \underline{p}^r$ is high enough, we opt to maintain unequal excess $x_H^* > x_L^* > 0$, which reduces cost-savings (see Proposition 2) and causes A^* to no longer increase in k for fixed p^a.

We now establish whether other key properties from the base model hold under correlated prices:

Corollary C.1. *With the correlated prices, the following hold for stacking:*

- $x_H^* = x_L^*$ *when* $k \le \frac{1}{T_H - 1}$ *and* $p^a \le \frac{2c_0^b}{kT_H}$,
- $A_{stk}^* > A_{arb}^*$ *when* $k < \frac{2}{T_H}$ *and* $p^a < 2c_0^b$,
- $\frac{R^*}{A^*} \ge 2$,
- $R_{stk}^* > R_{reg}^*$ *when* $p^a[1 + k(\frac{T_L - T_H}{T})] > (T-2)\underline{p}^r$, $\frac{2c_0^b}{T} > \underline{p}^r$,
- *Positive cost-saving synergy when* $k < \frac{5T_L + T_H}{(4T_H - 3)T_L + T_H}$ *and* $p^a < \frac{(T_H + 5T_L)c_0^b}{2T_L T_H k}$, *and*
- *Superlinear gains when (i)* $k \le \frac{1}{T_H}$, $p^a \le \frac{2c_0^b}{T_H k}$ *or (ii)* $\frac{1}{T_H} < k < \frac{T_L - T_H + \sqrt{T^2 - 2T}}{2T_L T_H - T}$, $p^a <$
$$\begin{cases} \frac{B_1 + \sqrt{B_1^2 - A_1 C_1}}{A_1}, & \underline{p}^r \le \underline{p}_1^r \\ \frac{B_2 + \sqrt{B_2^2 - A_2 C_2}}{A_2}, & \underline{p}_1^r \le \underline{p}^r \le \frac{2c_0^b}{T}\left[\frac{T(T_H k^2 - T_H k + 1) - T_H k\sqrt{T(T - 2T_L T_H)k^2 + 2T(T_L - T_H)k + 2T}}{T + 2T_L T_H k^2}\right] \\ 2c_0^b\left[\frac{[T + (T_L - T_H)k] + \sqrt{T(T - 2T_L T_H)k^2 + 2T(T_L - T_H)k + 2T}}{T + 2T_L T_H k^2}\right], & \text{otherwise,} \end{cases}$$
where $A_1 = k[(2T_H - 1)k - 2]$, $B_1 = 2\underline{p}^r[1 - k(T_H - 1)]$, $C_1 = 2(T-2)(\underline{p}^r)^2$, $A_2 = 1 + 2T_H k^2$, $B_2 = 2c_0^b(1+k) - 2T_H k\underline{p}^r$, $C_2 = 2T(\underline{p}^r)^2 + 4(c_0^b)^2 - 8c_0^b\underline{p}^r$, *and* $\underline{p}_1^r$ *is the regulation price satisfying* $p^a = \frac{B_1 + \sqrt{B_1^2 - A_1 C_1}}{A_1}$ *and* $(1+k)p^a + 2\underline{p}^r = 2c_0^b$ *when* $k > \frac{2}{2T_H - 1}$ *and* $\underline{p}_1^r = 0$ *otherwise.*

Proof. (i) The first property can be immediately checked from Theorem C.1.

(ii) $A_{stk}^* > A_{arb}^*$: If $p^a \ge 2c_0^b$ ($A_{arb}^* = E$) or $k \ge \frac{2}{T_H}$ ($A_{stk}^* = \min(\frac{cp^a E}{2c_0^b}, E)$ for some $c \le 1$), then the property does not hold. Otherwise, $A_{stk}^* = \min(\frac{cp^a E}{2c_0^b}, E)$ for some $c > 1$ and $A_{arb}^* = \frac{p^a E}{2c_0^b} < E$, so the property holds.

(iii) $\frac{R^*}{A^*} \ge 2$: Stacking always uses the full free regulation, which implies $\frac{R^*}{A^*} \ge 2$.

(iv) $R_{stk}^* > R_{reg}^*$: When (1) $x_H^* > x_L^* > 0$ or (3) $x_H^* = x_L^* > 0$, $R_{stk}^* = R_{reg}^*$. When (2) $x_H^* = x_L^* = 0$, the inequality holds if and only if $p^a > (T_L - 1)\underline{p}^r + (T_H - 1)\bar{p}^r$ (or $p^a[1 + k(\frac{T_L - T_H}{T})] > (T-2)\underline{p}^r$) when $A_{stk}^* = \frac{p^a + \underline{p}^r + \underline{p}^r}{2c_0^b}E < E$, and it holds if and only if $2c_0^b > T\underline{p}^r$ when $A_{stk}^* = E$ ($\frac{p^a + \underline{p}^r + \underline{p}^r}{2c_0^b}E \ge E$).

(v) Cost-saving synergy: From Proposition 2, we have a positive cost saving for $k \le \frac{1}{T_H - 1}$, $p^a \le \frac{2c_0^b}{kT_H}$ because $x_L^* = x_H^*$. For $k > \frac{1}{T_H - 1}$ or $p^a > \frac{2c_0^b}{kT_H}$ ($x_H^* > x_L^* > 0$), $x_L^* = \frac{\underline{p}^r E}{c_0^b}$, $x_H^* = \frac{\bar{p}^r E}{c_0^b} - \frac{2A^*}{T_H}$. We have a positive cost saving if and only if $\frac{a_L^* + a_H^*}{2} > x_H^* - x_L^*$. This inequality is equivalent to $k <$

$\frac{5T_L+T_H}{(4T_H-3)T_L+T_H}$ if $\frac{T_H p^a(1-k)^+}{2(T_H-2)} < c_0^b$ $(A^* < E)$ and $\frac{2T_H T_L p^a k}{T_H+5T_L} < c_0^b$ if $c_0^b \le \frac{T_H p^a(1-k)}{2(T_H-2)}$ $(A^* = E)$. Combining the conditions yields $k < \frac{5T_L+T_H}{(4T_H-3)T_L+T_H}$ and $p^a < \frac{(T_H+5T_L)c_0^b}{2T_L T_H k}$.

(vi) $\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$: From Theorem C.1, it is straightforward to derive $\Delta\Pi := \Pi^*_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}]$. When $x^*_H > x^*_L > 0$ ($k > \frac{1}{T_H-1}$ or $p^a > \frac{2c_0^b}{kT_H}$), $\Delta\Pi$ can be written in the form $c_1(p^a)^2 + c_2 p^a + c_3$, defined over some interval for p^a. Then it is easy to check that the quadratic function has negative values at the boundaries, concluding that we never have superlinear gains.

When $x^*_H = x^*_L$ ($k \le \frac{1}{T_H-1}$, $p^a \le \frac{2c_0^b}{kT_H}$), $\Delta\Pi$ can be written in the form $c_1(p^a)^2 + c_2 p^a \underline{p}^r + c_3(\underline{p}^r)^2 + c_4 p^a + c_5 \underline{p}^r + c_6$. Establishing cases with superlinear profit gain requires careful applications of the quadratic formula to establish the range of p^a where the property holds. Somewhat tedious enumeration of cases for the optimal solution yields the conditions in Corollary C.1.

□

Benefit of stacking. We now derive the upper bound on stacking's performance under correlated prices. In fact, we can show that correlated prices almost always (whenever $T_L > T_H$) have some set of prices $(\underline{p}, \bar{p}, p_0^r, k)$ that improve over the best achievable S under the base model. For analytical simplicity, we focus on $T_H \ge 4$ ($\frac{1}{T_H-1} < \frac{1}{\sqrt{2T_H}}$), but the case of $T_H = 2$ can be done similarly (in this case, $\frac{1}{T_H-1} > \frac{1}{\sqrt{2T_H}}$). We will first assume that energy is non-binding (Lemma C.1, Theorem C.2, and Corollary C.2) and then study the effect of binding energy constraint on S ratios (Corollary C.3).

Lemma C.1. *In the correlated price setting with non-binding energy capacity, ratios* $\frac{\Pi^*_{arb}}{\Pi^*_{stk}}$, $\frac{\Pi^*_{arb}}{\Pi^*_{reg}}$, *and* $\frac{\Pi^*_{stk}}{\Pi^*_{reg}}$ *are increasing in* $\bar{p}$.

Proof. Let $p^a := \bar{p} - \underline{p}$, $\underline{p}^r := p_0^r + k\underline{p}$, and $\alpha = \frac{T+2k(T_L-T_H)+k^2(T-2T_L T_H)}{4c_0^b(T-2)} E > 0$. Then we obtain the following:

$$\Pi^*_{arb} = \frac{(p^a)^2 E}{4c_0^b},$$

$$\Pi^*_{reg} = \frac{T_L(\underline{p}^r)^2 + T_H(\underline{p}^r + kp^a)^2}{2c_0^b} E,$$

$$\Pi^*_{stk} = \begin{cases} \Pi^*_{reg} + \frac{T_H((1-k)^+)^2}{4c_0^b(T_H-2)}(p^a)^2 E, & \text{if Case 1 } (x^*_H > x^*_L > 0), \\ \frac{E}{c_0^b}(\underline{p}^r + \frac{1+k}{2}p^a)^2, & \text{if Case 2 } (x^*_H = x^*_L = 0), \text{ and} \\ \Pi^*_{reg} + \alpha(p^a)^2, & \text{if Case 3 } (x^*_H = x^*_L > 0). \end{cases}$$

Here, cases 1 through 3 correspond to the three columns of the statement of the optimal stacking solution in Theorem C.1. The expressions for Π^*_{reg} and all three cases of Π^*_{stk} are of the form $a + bp^a + c(p^a)^2$ for constants $a, b, c > 0$, so it is immediate that Π^*_{arb}/Π^*_{reg} and Π^*_{arb}/Π^*_{stk} are strictly increasing in p^a. Similarly, for cases 1 and 3 we have $\Pi^*_{stk}/\Pi^*_{reg} = 1 + d(p^a)^2/\Pi^*_{reg}$ for some constant $d \ge 0$; the fact that $\Pi^*_{reg} = a + bp^a + c(p^a)^2$ for constants $a, b, c > 0$ implies that Π^*_{stk}/Π^*_{reg} is increasing in p^a for these two cases (strictly increasing except when $k \ge 1$ in Case 1).

For Case 2, we can establish that Π^*_{stk}/Π^*_{reg} is strictly increasing in p^a via differentiation, noting from the statement of Theorem C.1 that $x^*_H = x^*_L = 0$ requires $k \le \frac{1}{T_H-1} \le 1$:

$$\frac{\partial(\Pi^*_{stk}/\Pi^*_{reg})}{\partial p^a} = \frac{\frac{E^2}{2(c_0^b)^2}(\underline{p}^r + \frac{1+k}{2}p^a)}{(\Pi^*_{reg})^2}\left(T(\underline{p}^r)^2 + k(\underline{p}^r)^2(T_L - T_H) + T_H k p^a \underline{p}^r \underbrace{(1-k)}_{\ge 0}\right) > 0.$$

□

Using Lemma C.1, we can also derive the relative performance of stacking when energy capacity is not binding.

Theorem C.2 (Upper bound on stacking's performance: Correlated prices)**.** *For fixed $T_L, T_H \ge 4, \underline{p}, p_0^r, k$, and non-binding energy capacity, $S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$ is up-down in $\bar{p}$; its maximum value is the value of $\frac{\Pi^*_{stk}}{\Pi^*_{reg}}$, evaluated at*

$$\bar{p} = \bar{p}_1 := \begin{cases} \frac{2T_H p_0^r k + \underline{p} + \underline{p}^r\sqrt{2T - 4k^2 T_L T_H}}{1-2k^2 T_H}, & \text{if } k \le \frac{1}{\sqrt{2T_H}}, \\ \infty, & \text{otherwise,} \end{cases}$$

achieving value

$$\max S = \begin{cases} 1 + \frac{T + 2k(T_L - T_H) - k^2(2T_L T_H - T)}{T-2}, & \text{if } k \le \frac{1}{T_H - 1}, \\ 1 + \frac{T_H(1-k)^2}{T_H - 2}, & \text{if } \frac{1}{T_H-1} \le k \le \frac{1}{\sqrt{2T_H}}, \text{ and} \\ 1 + \frac{((1-k)^+)^2}{2(T_H-2)k^2}, & \text{if } k \ge \frac{1}{\sqrt{2T_H}}. \end{cases}$$

This maximum value is up-down in k and maximized at $k = \frac{T_L - T_H}{2T_L T_H - T}$. For fixed T_L and T_H, S achieves a maximum value of $\max S = \frac{2T - 2 + \frac{(T_L - T_H)^2}{2T_L T_H - T}}{T-2}$.

Proof. For fixed $T_L, T_H, \underline{p}, p_0^r$ and k, we know from Lemma C.1 that $\frac{\Pi^*_{arb}}{\Pi^*_{reg}}$ is increasing in $\bar{p}$ from a minimum value of 0 for $\bar{p} = \underline{p}$ up to a limiting value of $\frac{1}{2T_H k^2}$ as $\bar{p} \to \infty$. Therefore, when $k \ge \frac{1}{\sqrt{2T_H}}$, regulation-only is the best competitor; otherwise, there is a unique $\bar{p}$ at which the two competitors have equal profits, which can be easily computed as $\bar{p} = \bar{p}_1$.

When $k \le \frac{1}{T_H-1}$, then by Lemma C.1 S is up-down in $\bar{p}$, with a maximum value at $\bar{p} = \bar{p}_1$ (or equivalently, $\frac{p^a}{\underline{p}^r} = \frac{2T_H k + \sqrt{2T - 4k^2 T_H T_L}}{1 - 2k^2 T_H}$), which is the price at which $\Pi^*_{arb} = \Pi^*_{reg}$. At this point, stacking always has $x^*_H = x^*_L > 0$ (that is, $\frac{p^a}{\underline{p}^r} < \frac{T-2}{1-k(T_H-1)}$). To see this, we first note that $k \le \frac{1}{T_H-1} < \frac{T_H-1}{2T_H}$, which implies that

$$2k^2 T_H < k(T_H - 1).$$

Further, $2T_H k + \sqrt{2T - 4k^2 T_H T_L} < T - 2$ is equivalent to $f(T_H, T_L, k) := 4k^2 T_H T - 4T_H(T-2)k + (T-2)^2 - 2T > 0$. Then f decreases in k (for fixed T_L and T_H) and increases in T_L (for fixed T_H and k). Thus

$$f(T_H, T_L, k) \ge f(T_H, T_H, \frac{1}{T_H - 1}) = [T_H - (T_H-1)^2][2T_H - (T_H - 1)^2] > 0.$$

Then combining these leads to $\frac{2T_H k+\sqrt{2T-4k^2T_HT_L}}{1-2k^2T_H} \leq \frac{T-2}{1-k(T_H-1)}$.

Then from the argument at the beginning of the proof of Lemma C.1, we have:

$$\Pi^*_{stk} = \Pi^*_{reg} + \Pi^*_{arb} \cdot \frac{T + 2k(T_L - T_H) - k^2(2T_LT_H - T)}{T-2},$$

which implies that

$$\max S = 1 + \frac{T + 2k(T_L - T_H) - k^2(2T_LT_H - T)}{T-2}.$$

Since $\max S$ is up-down in k, it is maximized at $k = \frac{T_L - T_H}{2T_LT_H - T} (< \frac{1}{T_H - 1})$, achieving the grand maximum ratio:

$$\max S = \frac{2T - 2 + \frac{(T_L - T_H)^2}{2T_LT_H - T}}{T-2}.$$

When $\frac{1}{T_H-1} < k < \frac{1}{\sqrt{2T_H}}$, then again S is up-down in $\bar{p}$, with a maximum value at $\bar{p} = \bar{p}_1$. At this point, we have $x^*_H > x^*_L > 0$ from Theorem C.1. From the argument at the beginning of the proof of Lemma C.1, we have:

$$\Pi^*_{stk} = \Pi^*_{reg} + \Pi^*_{arb} \cdot \frac{T_H(1-k)^2}{(T_H - 2)},$$

and therefore S achieves maximum value $1 + \frac{T_H(1-k)^2}{T_H-2}$, which is decreasing in k.

When $k \geq \frac{1}{\sqrt{2T_H}}$, the best competitor is always regulation-only; since $\frac{\Pi^*_{stk}}{\Pi^*_{reg}}$ is increasing in $\bar{p}$ from Lemma C.1, we know it is maximized at $\bar{p} = \infty$. Π^*_{stk} is in Case 1, so S is maximized at value $1 + \frac{((1-k)^+)^2}{2(T_H-2)k^2}$, which is decreasing in k.

□

Theorem C.2 implies that the relative performance of stacking can improve under correlated prices compared to the uncorrelated setting ($\max S \to 2$ as $T \to \infty$).

Corollary C.2. *In the limit as $T \to \infty$, keeping T_H fixed,* $\max S \to 2 + \frac{1}{2T_H - 1}$.

It can be easily checked that Corollary C.2 remains consistent with $T_H = 2$ (with the same maximizer $k = \frac{T_L - T_H}{2T_LT_H - T}$, $\max S \to 2 + \frac{1}{2T_H-1}$).

Theorem C.2 and Corollary C.2 assume non-binding energy capacity. With binding energy capacity, we can show that the maximum S ratio is still upper bounded by $\frac{2T-2+\frac{(T_L-T_H)^2}{2T_LT_H-T}}{T-2}$.

Corollary C.3 (Upper bound on stacking's performance: Correlated prices)**.** *$maxS \leq \frac{2T-2+\frac{(T_L-T_H)^2}{2T_LT_H-T}}{T-2}$.*

Proof. When energy capacity is binding for stacking but not for standalone arbitrage ($p^a < 2c^b_0$) or when $\Pi^*_{arb} < \Pi^*_{reg}$, the S ratio is upper-bounded by the S ratio under no energy constraint. Also, if $x^*_H > x^*_L > 0$ at optimality, then Corollary C.1 implies that $\Delta\Pi = \Pi^*_{stk} - [\Pi^*_{arb} + \Pi^*_{reg}] < 0$, concluding that the S ratio should decrease and become smaller than 2. Moreover, for $\frac{1}{T_H} \le k < \frac{1}{T_H - 1}$, $p^a \ge 2c_0^b$ implies $x_H^* > x_L^* > 0$. As a result, the remainder of this proof will focus on the following two cases when standalone arbitrage is energy-constrained ($p^a \ge 2c_0^b$), $k \le \frac{1}{T_H}$, and $\Pi_{arb}^* > \Pi_{reg}^*$ ($p^a - c_0^b > \frac{(\underline{p}^r)^2 T_L + (\bar{p}^r)^2 T_H}{2c_0^b}$):

- $x_H^* = x_L^* = 0$: if $k < \frac{1}{T_H}, \max(2c_0^b, c_0^b + \frac{T_L(\underline{p}^r)^2 + T_H(\bar{p}^r)^2}{2c_0^b}) \le p^a \le \frac{2c_0^b - T\underline{p}^r}{T_H k}$),
- $x_H^* = x_L^* > 0$: if $k < \frac{1}{T_H}, \max(2c_0^b, \frac{2c_0^b - T\underline{p}^r}{T_H k}, c_0^b + \frac{T_L(\underline{p}^r)^2 + T_H(\bar{p}^r)^2}{2c_0^b}) \le p^a \le \frac{2c_0^b}{T_H k}$.

In the first case, we claim

$$\frac{\Pi_{stk}^*}{\Pi_{arb}^*} = \frac{(p^a + \bar{p}^r + \underline{p}^r)E - c_0^b E}{p^a E - c_0^b E} = 1 + \frac{2\underline{p}^r + kp^a}{p^a - c_0^b} \le 2,$$

which is equivalent to show that

$$2\underline{p}^r + c_0^b \le (1-k)p^a.$$

From $p^a \le \frac{2c_0^b - T\underline{p}^r}{T_H k}$, it suffices to show $(1 - k + \frac{2T_H k}{T})p^a \ge \frac{T+4}{T} c_0^b$, which holds as follows:

$$\begin{aligned}
(1 - k + \frac{2T_H k}{T})p^a &\ge (1 - k + \frac{2T_H k}{T}) \cdot 2c_0^b = \frac{T - (T_L - T_H)k}{T} \cdot 2c_0^b \ (\because p^a \ge 2c_0^b) \\
&> \frac{T - (T_L - T_H)\frac{1}{T_H}}{T} \cdot 2c_0^b = \frac{TT_H - (T_L - T_H)}{TT_H} \cdot 2c_0^b \ (\because k < \frac{1}{T_H}) \\
&\ge \frac{T+4}{T} c_0^b \ (\because T_H \ge 2).
\end{aligned}$$

In the second case, we claim

$$\frac{\Pi_{stk}^*}{\Pi_{arb}^*} = \frac{\frac{(\underline{p}^r T_L + \bar{p}^r T_H)^2 E}{2c_0^b T} + (1 + \frac{(T_L - T_H)k}{T})p^a E - c_0^b(1 - \frac{2}{T})E}{p^a E - c_0^b E} \le \frac{2T - 2 + \frac{(T_L - T_H)^2}{2T_L T_H - T}}{T - 2},$$

which is equivalent to show that

$$\frac{(T\underline{p}^r + T_H k p^a)^2}{2c_0^b T} \le \frac{p^a[T^2 + k(2T_L - 2T_H - T_L^2 + T_H)^2 + \frac{T(T_L - T_H)^2}{2T_L T_H - T}]}{(T-2)T} + \frac{c_0^b[-T^2 - 2T + 4 - \frac{T(T_L - T_H)^2}{2T_H T_L - T}]}{T(T-2)}.$$

Since the left-hand-side is increasing in $\underline{p}^r$ for fixed p^a, while the right-hand-side is independent of $\underline{p}^r$, it suffices to show that the inequality holds at $\underline{p}^r = \frac{-kT_H p^a + \sqrt{(kT_H p^a)^2 - T[T_H(kp^a)^2 - 2p^a c_0^b + 2(c_0^b)^2]}}{T}$ (when $\Pi_{arb}^* = \Pi_{reg}^*$), which yields a quadratic inequality in terms of p^a:

$$T_L T_H k^2 (p^a)^2 + 2c_0^b[\frac{2T - k(T_L^2 - T_H^2 - 2T_L + 2T_H) + \frac{T(T_L - T_H)^2}{2T_H T_L - T}}{T - 2}]p^a - 2(c_0^b)^2 \frac{4T - 4 + \frac{T(T_L - T_H)^2}{2T_L T_H - T}}{T - 2} \ge 0.$$

Then it is easy to algebraically check that the quadratic inequality holds for $2c_0^b \le p^a \le \frac{2c_0^b}{T_H k}$.

Combining these results conclude that the maximum ratio of S under a binding energy capacity is bounded by $\frac{2T - 2 + \frac{(T_L - T_H)^2}{2T_L T_H - T}}{T - 2}$, which is the grand maximum from the non-binding case. □

C.2. Extension 2: Round-Trip Energy Efficiency (η^2)

In this section, we first derive the optimal actions and stopping threshold for standalone regulation with energy loss. Then we derive the optimal stacking actions.

As established in the proof of Theorem 3, the profit from the grid can be rewritten as a function of $A = T_H a_H$ and $X = T_L d_L + T_H d_H$, where $d_L := r_L - a_L$ and $d_H := r_H - a_H$ are the excess regulation from the grid's perspective in the low-price and high-price periods. Then the profit from the grid can be rewritten as $\tilde{p}^a A + \tilde{p}^x X$, for $\tilde{p}^a = \bar{p} - \frac{\underline{p}}{\eta^2} + p^r(1 + \frac{1}{\eta^2})$ and $\tilde{p}^x = \frac{p^r}{2}(\frac{1}{\eta^2} + 1) - \frac{\underline{p}}{2}(\frac{1}{\eta^2} - 1)$.

While it is straightforward that the degradation cost to provide some amount A of arbitrage and no excess regulation is always $\frac{c_0^b}{E} A^2/\eta^2$, the degradation cost to provide some positive amount of excess regulation $X > 0$ varies depending on how that excess regulation is distributed between the two periods.

Lemma C.2. *For fixed $A \geq 0$ and $X > 0$, the optimal degradation cost $deg(A,X)$ to provide this amount of service is:*

	$x_L > x_H = 0$	$x_L > x_H > 0$	$x_L = x_H > 0$
cond.	$\eta < \eta^*, X \leq c_2 A$	$\eta < \eta^*, X > c_2 A$	$\eta \geq \eta^*$
d_H^*	0	$\frac{T_H}{2T_L c_3}(X - c_2 A)$	X/c_1
$deg(A,X)$	$k_1 A^2 + k_2 AX + k_3 X^2$	$k_4 A^2 + k_5 AX + k_6 X^2$	$k_7 A^2 + k_8 AX + k_9 X^2$

for $\eta^* := \sqrt{1 - 2/T_L}$, $c_1 = \eta^2 T_L + T_H$, $c_2 = \frac{T_L(1-\eta^2)-2}{\eta^2}$, $c_3 = \frac{T_H^2}{4\eta^2} - \frac{T_H^2}{2} + \frac{T_H^2\eta^2}{4} - \frac{T_H^2}{2\eta^2 T_L} + \frac{T_H^2}{T_L} + \frac{T_H\eta^2}{2} > 0$,[40] $k_1 = \frac{c_0^b}{\eta^2 E}$, $k_2 = \frac{2c_0^b}{\eta^2 T_L E}$, $k_3 = \frac{c_0^b}{2\eta^2 T_L E}$, $k_4 = \frac{T_H^2(T_L-2) + T_H T_L^2 \eta^4}{2\eta^4 c_3 T_L^2} \cdot \frac{c_0^b}{E}$, $k_5 = \frac{T_H^2(T_L-2)(1/\eta^2-1) + 2T_H T_L \eta^2}{2\eta^2 c_3 T_L^2} \cdot \frac{c_0^b}{E}$, $k_6 = \frac{T_H^2(T_L-2)(1/\eta-\eta)^2 + 2T_H T_L \eta^2}{8\eta^2 c_3 T_L^2} \cdot \frac{c_0^b}{E}$, $k_7 = \frac{c_0^b}{\eta^2 E}$, $k_8 = \frac{2\eta^2 + T_H(1-\eta^2)}{\eta^2 c_1} \cdot \frac{c_0^b}{E}$, *and* $k_9 = \frac{T_H^2(1-\eta^2)^2 + 2\eta^2 T_H(2-\eta^2) + 2\eta^4 T_L}{4\eta^2 c_1^2} \cdot \frac{c_0^b}{E}$. *All k_i constants are strictly positive.*

Proof. The degradation cost function is defined as:

$$\frac{c_0^b}{E}\left(D^2 + 2D\max(x_L, x_H) + \frac{T_L}{2}x_L^2 + \frac{T_H}{2}x_H^2\right).$$

Note that $x_L = d_L/\eta$ and $x_H = d_H \eta$, and further that $d_L = \frac{X - T_H d_H}{T_L}$, due to the fact that we need to provide the full excess regulation X across the two regions. $x_L \geq x_H$ when $d_H \leq X/c_1$, while $x_L < x_H$ when $d_H > X/c_1$. Noting that $D = \frac{T_H}{2} d_H(\frac{1}{\eta} - \eta) + \frac{A}{\eta}$, the degradation can be expressed as a function of d_H. When $d_H > X/c_1$ and therefore $x_H > x_L$, degradation costs are always increasing in d_H:

$$deg(d_H) = \frac{c_0^b}{E}\left(D^2 + 2Dx_H + \frac{T_L}{2}x_L^2 + \frac{T_H}{2}x_H^2\right),$$

[40] $c_3 = \frac{T_H^2}{4}(1/\eta - \eta)^2 + \frac{T_H^2}{T_L}(1 - \frac{1}{2\eta^2}) + \frac{T_H \eta^2}{2}$. This is clearly positive as long as $1 - \frac{1}{2\eta^2} \geq 0$. If $1 - \frac{1}{2\eta^2} < 0$, then c_3 is minimized at $T_L = T_H$, taking value $c_3 = \frac{T_H^2}{4}(1/\eta - \eta)^2 + \frac{T_H}{2}(\eta^2 + 2 - 1/\eta^2) = \frac{T_H}{4}((T_H - 2)(1/\eta - \eta)^2 + 4\eta^2) > 0$.

$$\begin{aligned} deg'(d_H) &= \frac{c_0^b}{E}\Big(2D(D'+x_H')+2D'x_H+T_Lx_Lx_L'+T_Hx_Hx_H'\Big)\\ &= \frac{c_0^b}{E}\Big(2D(D'+x_H')+\frac{T_H}{\eta}(x_H-x_L)\Big)>0. \end{aligned}$$

As a result, we focus on optimizing over $d_H \in [0, X/c_1]$, where $x_L \le x_H$. In this region, the degradation takes the form

$$\begin{aligned} deg(d_H) &= \frac{c_0^b}{E}\Big(D^2+2Dx_L+\frac{T_L}{2}x_L^2+\frac{T_H}{2}x_H^2\Big)\\ &= \frac{c_0^b}{E}\Big((\frac{A^2}{\eta^2}+\frac{2AX}{\eta^2 T_L}+\frac{X^2}{2\eta^2 T_L})+\underbrace{\frac{T_H}{T_L}(c_2A-X)}_{:=\beta}d_H+c_3d_H^2\Big). \end{aligned}$$

The degradation function is strictly convex in d_H, meaning the degradation cost will be minimized at endpoint $d_H = X/c_1$ iff the degradation function is non-increasing at that point. The derivative at that point is

$$\frac{c_0^b}{E}\Big(\beta+\frac{2Xc_3}{c_1}\Big)=\frac{c_0^b}{E}\underbrace{\Big(\frac{T_H^2}{2c_1}X(\frac{1}{\eta^2}-1)+\frac{T_H}{\eta^2}A\Big)}_{>0}\Big(1-\frac{2}{T_L}-\eta^2\Big).$$

We conclude that $d_H^* = X/c_1$ iff $\eta \ge \eta^* := \sqrt{1-2/T_L}$.

When $\eta < \eta^*$, then $d_H^* = 0$ iff the degradation cost is non-decreasing in d_H at $d_H = 0$. This is equivalent to $\beta \ge 0$, or $X \le c_2 A$. Otherwise, d_H^* lies between 0 and X/c_1; the first-order condition tells us in this case that $d_H^* = -\frac{\beta}{2c_3}$. The k_i constants follow by evaluating $deg(d_H^*)$ for each of the three cases; by inspection all are strictly positive. □

C.2.1. Optimal Standalone Regulation with Energy Loss

From Lemma C.2, we state the optimal standalone regulation under energy loss.

Theorem C.3. *Optimal standalone regulation under energy loss provides no service for* $\eta < \eta_{reg} := \sqrt{\frac{(\underline{p}-p^r)^+}{\underline{p}+p^r}}$, *and otherwise satisfies the following:*

	$x_L^* = x_H^* > 0$	$x_L^* > x_H^* > 0$
cond.	$\eta^* \le \eta \le 1$	$\eta_{reg} \le \eta < \eta^*$
X^*	$\tilde{p}^x/(2k_9)$	$\tilde{p}^x/(2k_6)$

for $\eta^* := \sqrt{1-2/T_L}$ *and for constants* $k_6, k_9 > 0$ *from the statement of Lemma C.2.*

Proof. For standalone regulation ($A = 0$), Lemma C.2 establishes that the optimal degradation cost to provide fixed excess regulation quantity X is $deg(0, X) = cX^2$, where constant c takes value k_9 for $\eta \ge \eta^*$ (the $x_L = x_H = 0$ case) and takes value k_6 for $\eta < \eta^*$ (the $x_L > x_H > 0$ case). Then the optimal profit for fixed X is $\Pi(X) = \tilde{p}^x X - cX^2$, where $\tilde{p}^x \ge 0$ iff $\eta \ge \eta_{reg}$ (see Theorem 3). Then from the first-order conditions $X^* = (\tilde{p}^x)^+/(2c)$. □

C.2.2. Optimal Stacking with Energy Loss

Optimal structure. We first state the optimal structure of stacking under energy loss.

Theorem C.4. *Optimal stacking under energy loss provides no service* ($A^* = X^* = 0$) *for* $\eta < \eta_{stk} := \sqrt{\frac{(\underline{p}-p^r)^+}{\overline{p}+p^r}}$, *provides free regulation only* ($A^* = \frac{\eta^2 \tilde{p}^a E}{2c_0^b}$, $X^* = 0$) *for* $\eta_{stk} \le \eta < \eta_{reg} := \sqrt{\frac{(\underline{p}-p^r)^+}{\underline{p}+p^r}}$, *and otherwise satisfies the following:*

	(1a) $x_L^* = x_H^* > 0,\ A^* > 0$	*(1b)* $x_L^* = x_H^* > 0,\ A^* = 0$
cond.	$\eta \ge \eta^*,\ d_1\tilde{p}^x < p^a \le d_2\tilde{p}^x$	$\eta \ge \eta^*,\ p^a \le d_1\tilde{p}^x$
A^*	$\min(\frac{c_1^2E^2}{2(T-2)}(2k_9\tilde{p}^a - k_8\tilde{p}^x)/(c_0^b)^2, \eta E)$	0
X^*	$(\tilde{p}^x - k_8A^*)/(2k_9)$	$\tilde{p}^x/(2k_9)$

	(2a) $x_L^* > x_H^* > 0,\ A^* > 0$	*(2b)* $x_L^* > x_H^* > 0,\ A^* = 0$
cond.	$\eta < \eta^*,\ d_3\tilde{p}^x < p^a \le d_4\tilde{p}^x$	$\eta < \eta^*,\ p^a \le d_3\tilde{p}^x$
A^*	$\min(\frac{\eta^2 c_3 T_L^2 E^2}{T_H(T_L-2)}(2k_6\tilde{p}^a - k_5\tilde{p}^x)/(c_0^b)^2, \eta E)$	0
X^*	$(\tilde{p}^x - k_5A^*)/(2k_6)$	$\tilde{p}^x/(2k_6)$

	(3) $x_L^* > x_H^* = 0$	*(4)* $x_L^* = x_H^* = 0$
cond.	$\eta < \eta^*, d_4\tilde{p}^x < p^a \le d_5\tilde{p}^x$	$\tilde{p}^x < \begin{cases} \min(p^a/d_2, \eta E k_8), & \text{if } \eta \ge \eta^* \\ \min(p^a/d_5, \eta E k_2), & \text{if } \eta < \eta^* \end{cases}$
A^*	$\min(\frac{\eta^2 T_L E}{2(T_L-2)}(\tilde{p}^a - 2\tilde{p}^x)/c_0^b, \eta E)$	$\min(\eta^2\tilde{p}^aE/(2c_0^b), \eta E)$
X^*	$(\tilde{p}^x - k_2A^*)/(2k_3)$	0

$$\text{for } \eta^* := \sqrt{1-2/T_L},\ d_1 = \frac{2T_H\eta^2(1-\eta^2)(T-2)}{T_H^2(1-\eta^2)^2 + 2\eta^2T_H(2-\eta^2) + 2\eta^4T_L},\ d_2 = \begin{cases} \frac{2\eta^2(T-2)}{2\eta^2+T_H(1-\eta^2)} & \text{if } \eta E k_8 \le \tilde{p}^x \\ \infty & \text{if } \eta E k_8 > \tilde{p}^x \end{cases},$$

$$d_3 = \frac{2T_H(T_L-2)(\eta^2-\eta^4)}{2T_L\eta^4 + T_H(T_L-2)(1-\eta^2)^2},\ d_4 = \begin{cases} \frac{2\eta^2}{1-\eta^2} & \text{if } \tilde{p}^x \le \frac{c_0^b(T_L-2)(1-\eta^2)}{\eta^3 T_L} \\ \infty & \text{if } \tilde{p}^x > \frac{c_0^b(T_L-2)(1-\eta^2)}{\eta^3T_L} \end{cases},$$

$$d_5 = \begin{cases} T_L - 2 & \text{if } \tilde{p}^x \le \eta E k_2 \\ \infty & \text{if } \eta E k_2 \le \tilde{p}^x \le \frac{c_0^b(T_L-2)(1-\eta^2)}{\eta^3T_L} \end{cases}, \text{ and constants } k_i \text{ and } c_i \text{ from Lemma C.2.}$$

Proof. As established in Theorem 3, $\tilde{p}^a, \tilde{p}^x < 0$ when $\eta < \eta_{stk}$, so no service will be provided in this case. Further, when $\eta_{stk} \le \eta < \eta_{reg}$ then $\tilde{p}^a \ge 0, \tilde{p}^x < 0$, so only free regulation will be used ($X^* = 0$). As a result, we focus the proof of this theorem on the case where $\eta \ge \eta_{reg}$, meaning $\tilde{p}^a, \tilde{p}^x \ge 0$.

In the case of $\eta \ge \eta^*$, Lemma C.2 establishes that $x_L^* = x_H^*$, and the optimal degradation cost is $k_7A^2 + k_8AX + k_9X^2$ for $k_7, k_8, k_9 > 0$, which yields profit function

$$\Pi(A, X) = (\tilde{p}^aA - k_7A^2) + (\tilde{p}^x - k_8A)X - k_9X^2.$$

For fixed A, it is immediate that $X^* = (\tilde{p}^x - k_8A)^+/(2k_9)$, yielding the following optimal profit as a function of A:

$$\Pi(A) = \begin{cases} \frac{(\tilde{p}^x)^2}{4k_9} + (\tilde{p}^a - \frac{k_8\tilde{p}^x}{2k_9})A + (\frac{k_8^2}{4k_9} - k_7)A^2, & \text{if } A < \tilde{p}^x/k_8, \text{ and} \\ \tilde{p}^aA - k_7A^2, & \text{if } A \ge \tilde{p}^x/k_8. \end{cases}$$

Noting that $\frac{k_8^2}{4k_9} - k_7 = \frac{-(T-2)}{2k_9 c_1^2}(\frac{c_0^b}{E})^2 < 0$, we conclude that $\Pi(A)$ is a strictly concave function with a continuous, piecewise linear, decreasing derivative. Given the energy constraint $A \leq \eta E$ and derivatives

$$\Pi'(0) = \tilde{p}^a - (2+d_1)\tilde{p}^x \text{ and}$$
$$\Pi'(\tilde{p}^x/k_8) = \tilde{p}^a - (2+d_2)\tilde{p}^x,$$

it is clear that $A^* = 0$ when $\tilde{p}^a \leq (2+d_1)\tilde{p}^x$, $A^* \in (0, \tilde{p}^x/k_8)$ when $(2+d_1)\tilde{p}^x < \tilde{p}^a$ and $\tilde{p}^a < (2+d_2)\tilde{p}^x$ or $\eta E \leq \tilde{p}^x/k_8$, and $A^* \geq \tilde{p}^x/k_8$ and therefore $X^* = 0$ when $\tilde{p}^a \geq (2+d_2)\tilde{p}^x$ and $\eta E \geq \tilde{p}^x k_8$. Noting that $\tilde{p}^a = 2\tilde{p}^x + p^a$, condition $\tilde{p}^a \leq (2+c)\tilde{p}^x$ is equivalent to $p^a \leq c\tilde{p}^x$ for any c. In each region, the optimal A^* follows immediately from the first-order conditions and the energy constraint $A \leq \eta E$.

In the case of $\eta < \eta^*$ (which implies $T_L > 2$), Lemma C.2 establishes two separate cases for $x_L^* > x_H^*$, which differ in whether $x_H^* = 0$. These cases combine to yield profit function

$$\Pi(A,X) = \begin{cases} (\tilde{p}^a A - k_1 A^2) + (\tilde{p}^x - k_2 A)X - k_3 X^2, \text{ if } X \leq c_2 A, \text{ and} \\ (\tilde{p}^a A - k_4 A^2) + (\tilde{p}^x - k_5 A)X - k_6 X^2, \text{ if } X > c_2 A. \end{cases}$$

For fixed A, $\Pi(A,X)$ is a strictly concave function of X with a continuous, piecewise linear, decreasing derivative, with derivative value $\tilde{p}^x - \frac{c_0^b (T_L-2)(1-\eta^2)}{\eta^4 T_L E} A$ at $X = c_2 A$. From this characterization, it follows that $X^* = (\tilde{p}^x - k_2 A)^+/(2k_3)$ when $\tilde{p}^x - \frac{c_0^b (T_L-2)(1-\eta^2)}{\eta^4 T_L E} A \leq 0$, and otherwise $X^* = (\tilde{p}^x - k_5 A)/(2k_6)$. This yields the following optimal profit as a function of A:

$$\Pi(A) = \begin{cases} \frac{(\tilde{p}^x)^2}{4k_6} + (\tilde{p}^a - \frac{k_5 \tilde{p}^x}{2k_6})A + (\frac{k_5^2}{4k_6} - k_4)A^2, & \text{if } A < \frac{\eta^4 T_L E}{c_0^b (T_L-2)(1-\eta^2)}\tilde{p}^x, \\ \frac{(\tilde{p}^x)^2}{4k_3} + (\tilde{p}^a - \frac{k_2 \tilde{p}^x}{2k_3})A + (\frac{k_2^2}{4k_3} - k_1)A^2, & \text{if } \frac{\eta^4 T_L E}{c_0^b (T_L-2)(1-\eta^2)}\tilde{p}^x \leq A < \tilde{p}^x/k_2, \text{ and} \\ \tilde{p}^a A - k_1 A^2, & \text{if } A \geq \tilde{p}^x/k_2. \end{cases}$$

Noting that $\frac{k_2^2}{4k_3} - k_1 = \frac{-(T_L-2)}{\eta^2 T_L} \cdot \frac{c_0^b}{E} < 0$ and $\frac{k_5^2}{4k_6} - k_4 = \frac{-2(T_L-2)}{T_H (T_L-2)(1/\eta - \eta)^2 + 2T_L \eta^2} \cdot \frac{c_0^b}{E} < 0$, $\Pi(A)$ is a strictly concave function with a continuous, piecewise linear, decreasing derivative. Given the energy constraint and derivatives

$$\Pi'(0) = \tilde{p}^a - (2+d_3)\tilde{p}^x,$$
$$\Pi'(\frac{\eta^4 T_L E}{c_0^b (T_L-2)(1-\eta^2)}\tilde{p}^x) = \tilde{p}^a - (2+d_4)\tilde{p}^x, \text{ and}$$
$$\Pi'(\tilde{p}^x/k_2) = \tilde{p}^a - (2+d_5)\tilde{p}^x,$$

it is clear that $A^* = 0$ when $p^a \leq d_3 \tilde{p}^x$, $A^* \in (0, \frac{\eta^4 T_L E}{c_0^b (T_L-2)(1-\eta^2)}\tilde{p}^x)$ and therefore $x_L^* > x_H^* > 0$ when $d_3 \tilde{p}^x < p^a$ and $p^a < d_4 \tilde{p}^x$ or $\eta E \leq \frac{\eta^4 T_L E}{c_0^b (T_L-2)(1-\eta^2)}\tilde{p}^x$, $A^* \in [\frac{\eta^4 T_L E}{c_0^b (T_L-2)(1-\eta^2)}\tilde{p}^x, \tilde{p}^x/k_2)$ and therefore $x_L^* > x_H^* = 0$ when $d_4 \tilde{p}^x \leq p^a$, $\eta E \geq \frac{\eta^4 T_L E}{c_0^b (T_L-2)(1-\eta^2)}\tilde{p}^x$, and $p^a < d_5 \tilde{p}^x$ or $\eta E \leq \tilde{p}^x/k_2$, and $A^* \geq \tilde{p}^x/k_2$ and therefore $x_L^* = x_H^* = 0$ when $p^a \geq d_5 \tilde{p}^x$ and $\eta E > \tilde{p}^x/k_2$. In each region, the optimal A^* follows immediately from the first-order conditions and the energy constraint $A \leq \eta E$.

□

Note that $d_1 \le \bar{d}$ when $\eta \ge \eta^*$ and $d_2 \le \bar{d}$ when $\eta \le \eta^*$. We now prove that the most of the main properties remain consistent with the round-trip energy efficiency $\eta^2 < 1$ ($\frac{R^*}{A^*} := \infty$ for $A^* = 0$).

Corollary C.4. *With the round-trip energy efficiency $\eta^2 \in [0,1)$, the following hold for stacking:*

- R^* *increases in* p^a *when* $\tilde{p}^x < \begin{cases} \min(p^a/d_2, \eta E k_8), & \eta \ge \eta^* \\ \min(p^a/d_5, \eta E k_2), & \eta < \eta^* \end{cases}$, $p^a \le -\underline{p}(1 - \frac{1}{\eta^2}) - p^r(1 + \frac{1}{\eta^2}) + 2c_0^b/\eta$,
- A^* *increases in* p^r *when* $\tilde{p}^x < \begin{cases} \min(p^a/d_2, \eta E k_8), & \eta \ge \eta^* \\ \min(p^a/d_5, \eta E k_2), & \eta < \eta^* \end{cases}$, $p^a \le -\underline{p}(1 - \frac{1}{\eta^2}) - p^r(1 + \frac{1}{\eta^2}) + 2c_0^b/\eta$,
- $\frac{R^*}{A^*} \ge 2$,
- $x_L^* = x_H^*$ *when* $\eta \ge \eta^*$ *or* $\tilde{p}^x < \min(p^a/d_5, \eta E k_2)$,
- $A^* > 0$ *when* $\eta > \eta^*, p^a > d_1 \tilde{p}^x$ *or* $\eta < \eta^*, p^a > d_3 \tilde{p}^x$,
- $A^*_{stk} > A^*_{arb}$ *when* $\eta_{stk} < \eta < \eta_{arb}$ *or* $\frac{\bar{p}\eta^2 - \underline{p}}{2c_0^b} < \eta$, $p^a > \begin{cases} \max(\bar{p}_1 - \underline{p}, d_1 \tilde{p}^x), & \eta > \eta^* \\ \max(\bar{p}_2 - \underline{p}, d_2 \tilde{p}^x), & \eta < \eta^* \end{cases}$, *where*

$$\bar{p}_1 = \frac{(\frac{1}{\eta^2} - \eta^2)T_H(T-2)p^r - [(\frac{1}{\eta} - \eta)^2 T_H(T_L - 2) - 4T_H(1-\eta^2) - 2T\eta^2 + 2T - 4]\underline{p}}{2[2\eta^2 - 2T_H(\eta^2 - 1) + \frac{T_H^2}{2}(\frac{1}{\eta} - \eta)^2]},$$

$$\bar{p}_2 = \frac{(\frac{1}{\eta^2} - \eta^2)T_H(T_L - 2)p^r - 2[T_L - 2 - T_L\eta^2]\underline{p}}{4\eta^2 + (T_L - 2)T_H(\frac{1}{\eta} - \eta)^2},$$

- $R^*_{stk} > R^*_{reg}$ *when* $\tilde{p}^x < \begin{cases} \min(p^a[\frac{c_0^b}{2k_9\eta^2 E} - 2]^{-1}, 4k_9\eta E), & \eta \ge \eta^* \\ \min(p^a[\frac{c_0^b}{2k_6\eta^2 E} - 2]^{-1}, 4k_6\eta E), & \eta < \eta^* \end{cases}$,
- *Positive cost-saving synergy when one of the following holds:*

  —$\eta \ge \eta^*, d_1\tilde{p}^x < p^a$

  —$\eta < \eta^*, \tilde{p}^x < \min(p^a/d_5, \eta E k_2)$

  —$\eta < \eta^*, d_3\tilde{p}^x < p^a < d_4\tilde{p}^x, \eta^2 T + 4T_H\eta + T_H(T_L - 2)\eta_1(\frac{\eta_1}{2} - 2) > 0$

  —$\eta < \eta^*, d_4\tilde{p}^x < p^a \le d_5\tilde{p}^x$, *and* $\frac{-s_1 s_2}{s_1 s_3 + s_4} p' < p^a < \frac{2(T_L - 2)c_0^b}{\eta T_L}$ *or* $s_5 > s_6 p', p^a > \frac{2(T_L-2)c_0^b}{\eta T_L}$

  where $s_1 = \frac{\eta^2 T_L E}{2c_0^b(T_L - 2)}, s_2 = \frac{\eta_2 E}{8c_0^b}(\eta_1 - 4), s_3 = \frac{\eta(\eta+4)E}{4c_0^b(T_L-2)}, s_4 = \frac{T_H}{2}(\frac{\eta^2 T_L E}{2c_0^b T_H(T_L - 2)})^2, s_5 = 4c_0^b(T\eta + 4T_H), s_6 = T_L T_H \eta_2(4 - \eta_1)$, *and* $p' = p^r - \frac{\eta_1}{\eta_2}\underline{p}$.

Proof. Clearly, the first two properties only holds when energy capacity is not binding. Also, from (17) – (18), $R^* = (A^* + \frac{X^*}{2}) \cdot (1 + \frac{1}{\eta^2})$. Then those properties are direct from restating the unconstrained optimal A^* and R^* for each region in the form $c_1\bar{p} + c_2\underline{p} + c_3 p^r$ and signing the region-specific coefficients:

- $A^*_{1a} = [\frac{T\eta^2 + 2T_H\eta_1\eta + \frac{T_H^2}{2}\eta_1^2}{2(T-2)\frac{c_0^b}{E}}]\bar{p} + [\frac{\eta_1^2 T_H E}{4c_0^b} - \frac{T\eta^2 + 2T_H\eta_1\eta + \frac{T_H^2}{2}\eta_1^2}{2(T-2)\frac{c_0^b}{E}}]\underline{p} + [-\frac{\eta_1\eta_2 T_H E}{4c_0^b}]p^r$ $(c_1 > 0, c_2 < 0, c_3 < 0)$ and $R^*_{1a} = \frac{[-T_H\eta_1\eta_2]\bar{p} + [-T_L\eta_1\eta_2]\underline{p} + [\eta_2^2 T]p^r}{4\frac{c_0^b}{E}}$ $(c_1 < 0, c_2 < 0, c_3 > 0)$;
- $A^*_{1b} = 0$ and $R^*_{1b} = -\frac{\eta_1(\frac{1+\eta^2}{2})^2(T_L + \frac{T_H}{\eta^2})^2}{\frac{c_0^b}{E}\eta_2[\frac{(T_H\eta_1)^2}{2} + 2T_H\eta_1\eta + T\eta^2]}\underline{p} + \frac{(\frac{1+\eta^2}{2})^2(T_L + \frac{T_H}{\eta^2})^2}{c^b[\frac{(T_H\eta_1)^2}{2} + 2T_H\eta_1\eta + T\eta^2]}p^r$ $(c_1 = 0, c_2 < 0, c_3 > 0)$;
- $A^*_{2a} = [\frac{T_L\eta^2 E}{2c_0^b(T_L - 2)} + \frac{\eta_1^2 T_H E}{4c_0^b}]\bar{p} + [-\frac{T_L\eta^2 E}{2c_0^b(T_L-2)}]\underline{p} + [\frac{-\eta_1\eta_2 T_H E}{4c_0^b}]p^r$ $(c_1 > 0, c_2 < 0, c_3 < 0)$ and $R^*_{2a} = \frac{[-T_H\eta_1\eta_2]\bar{p} + [-T_L\eta_1\eta_2]\underline{p} + [\eta_2^2 T]p^r}{4\frac{c_0^b}{E}}$ $(c_1 < 0, c_2 < 0, c_3 > 0)$;

- $A_{2b}^*=0$ and $R_{2b}^*=-\frac{\eta_1\eta_2 T_L E}{4c_0^b}[\frac{\frac{T_H T_L}{2}\eta_1^2+T\eta^2-T_H\eta_1^2}{\frac{T_H T_L}{2}\eta_1^2+T_L\eta^2-T_H\eta_1^2}]\underline{p}+\frac{\eta_2^2 T_L E}{4c_0^b}[\frac{\frac{T_H T_L}{2}\eta_1^2+T\eta^2-T_H\eta_1^2}{\frac{T_H T_L}{2}\eta_1^2+T_L\eta^2-T_H\eta_1^2}]p^r$ $(c_1=0,c_2<0,c_3>0)$;
- $A_3^*=\frac{\eta^2 T_L E}{2(T_L-2)c_0^b}(\bar{p}-\underline{p})$ and $R_3^*=\frac{\eta^2 T_L E}{4c_0^b}[-(\frac{1}{\eta^2}-1)\underline{p}+p^r(1+\frac{1}{\eta^2})](1+\frac{1}{\eta^2})$ $(A_3^*: c_1>0,c_2<0,c_3=0$ and $R_3^*: c_1=0,c_2<0,c_3>0)$;
- $A_4^*=\frac{\eta^2\bar{p}-\underline{p}+(1+\eta^2)p^r}{2\frac{c_0^b}{E}}$ $(c_1>0,c_2<0,c_3>0)$ and $R_4^*=\frac{\eta^2\bar{p}-\underline{p}+(1+\eta^2)p^r}{2\frac{c_0^b}{E}}(1+\frac{1}{\eta^2})$ $(c_1>0,c_2<0,c_3>0)$,

where $\eta_1=\frac{1}{\eta}-\eta$ and $\eta_2=\frac{1}{\eta}+\eta$.

Then A^* (R^*) increases in p^r (p^a) if and only if the energy capacity is not binding and we are in Region 4, i.e., $\tilde{p}^x<\begin{cases}\min(p^a/d_2,\eta E k_8), & \eta\ge\eta^*\\ \min(p^a/d_5,\eta E k_2), & \eta<\eta^*\end{cases}$, $p^a\le-\underline{p}(1-\frac{1}{\eta^2})-p^r(1+\frac{1}{\eta^2})+2c_0^b/\eta$.

From Theorem C.4, $x_L^*=x_H^*$ and $A^*>0$ are direct. Also, $R^*=(A^*+\frac{X^*}{2})\cdot(1+\frac{1}{\eta^2})$ and $\frac{R^*}{A^*}\ge 1+\frac{1}{\eta^2}\ge 2$, which implies that the ratio always strictly exceeds 2 when $\eta<1$.

$A_{stk}^*>A_{arb}^*=\min(\frac{\bar{p}\eta^2-\underline{p}}{2c_0^b}E,\eta E)$: Clearly, this property only holds when standalone arbitrage is unconstrained ($A_{stk}^*<\eta E$, or equivalently, $\frac{\bar{p}\eta^2-\underline{p}}{2c_0^b}<\eta$). When stacking is energy-constrained and standalone arbitrage is not, the property always holds. Thus, we now focus on unconstrained solutions for both stacking and standalone arbitrage (A_{stk}^*, $A_{arb}^*<\eta E$). It is obvious that $A_4^*\ge A_{arb}^*$ holds. Also, $A_3^*>A_{arb}^*\Leftrightarrow 2\eta^2\bar{p}>[\eta^2 T_L-(T_L-2)]\underline{p}$ always holds since $\eta^2 T_L-(T_L-2)<0$ when $\eta<\eta^*$. We now see the remaining cases (1a) and (2a). For Case (1a) ($\eta\ge\eta^*, d_1\tilde{p}^x<p^a$ are required), $A_{1a}^*>A_{arb}^*\Leftrightarrow 2[2\eta^2+2T_H\eta_1\eta+\frac{T_H^2}{2}\eta_1^2]\bar{p}+[\eta_1^2 T_H(T_L-2)-4T_H\eta_1\eta-2T\eta^2+2T-4]\underline{p}>\eta_1\eta_2 T_H(T-2)p^r$. For Case (2a) ($\eta<\eta^*, d_3\tilde{p}^x<p^a$ are required), $A_{2a}^*>A_{arb}^*\Leftrightarrow[4\eta^2+(T_L-2)T_H(\frac{1}{\eta}-\eta)^2]\bar{p}+2[T_L-2-T_L\eta^2]\underline{p}>(\frac{1}{\eta^2}-\eta^2)T_H(T_L-2)p^r$. Combining these, we have necessary and sufficient conditions for $A_{stk}^*>A_{arb}^*$.

$R_{stk}^*>R_{reg}^*$: When $\eta\ge\eta^*$, we compare R_{1a}^* and R_4^* with $R_{reg}^*=R_{1b}^*$ (subscripts represent corresponding cases). Then $R_{stk}^*>R_{reg}^*$ holds if and only if

$$R_{1a}^*>R_{reg}^*\Leftrightarrow(A_{1a}^*+\frac{X_{1a}^*}{2})(1+\frac{1}{\eta^2})>\frac{\tilde{p}^x}{4k_9}(1+\frac{1}{\eta^2})\Leftrightarrow 4k_9>k_8,$$
$$R_4^*>R_{reg}^*\Leftrightarrow A_4^*>\frac{\tilde{p}^x}{4k_9}\Leftrightarrow\min(\frac{\eta^2\tilde{p}^a E}{2c_0^b},\eta E)>\frac{\tilde{p}^x}{4k_9}.$$

It can be checked that $4k_9\le k_8$, so the first set is empty. Since $4k_9\eta E<k_8\eta E$, the property holds for $\eta>\eta^*$ if and only if $\tilde{p}^a>\frac{c_0^b\tilde{p}^x}{2\eta^2 E k_9}$, $\tilde{p}^x<4k_9\eta E$.

When $\eta_{reg}\le\eta<\eta^*$, we compare R_{2a}^*, R_3^*, and R_4^* with $R_{reg}^*=R_{2b}^*$ and obtain the following:

$$R_{2a}^*>R_{reg}^*\Leftrightarrow(A_{2a}^*+\frac{X_{2a}^*}{2})(1+\frac{1}{\eta^2})>\frac{\tilde{p}^x}{4k_6}(1+\frac{1}{\eta^2})\Leftrightarrow 4k_6>k_5,$$
$$R_3^*>R_{reg}^*\Leftrightarrow A_3^*(1-\frac{k_2}{4k_3})>\frac{\tilde{p}^x}{4}(\frac{1}{k_6}-\frac{1}{k_3}),$$
$$R_4^*>R_{reg}^*\Leftrightarrow A_4^*>\frac{\tilde{p}^x}{4k_6}\Leftrightarrow\min(\frac{\eta^2\tilde{p}^a E}{2c_0^b},\eta E)>\frac{\tilde{p}^x}{4k_6}.$$

It can be checked that the first two sets are empty. Since $4k_6\eta E<k_2\eta E$, the property holds for $\eta<\eta^*$ if and only if $\tilde{p}^a>\frac{c_0^b\tilde{p}^x}{2\eta^2 E k_6}$, $\tilde{p}^x<4k_6\eta E$.

Positive cost saving: Clearly, we have positive cost-saving synergy for $\eta \geq \eta^*, p^a > d_1\tilde{p}^x$. There is no cost-saving synergy in Cases (1b) and (2b) since standalone regulation is optimal.

We now check the remaining of Cases (2a) and (3). For each case, we can state the optimal actions in terms of p', a_H^*, and $A^* = T_H \cdot a_H^*$. From Proposition 2, then the cost saving for Case (2a) becomes:

$$Cost - saving = \frac{c_0^b}{E} \cdot A^* \cdot [\frac{a_L^* + a_H^*}{2} - (x_L^* - x_H^*)] \tag{C.1}$$
$$= \underbrace{\frac{c_0^b}{E} A^* \cdot C_{2a} \cdot (a_H^* + \frac{\eta_1\eta_2 E}{4c_0^b} p')}_{>0} \cdot R_{2a}, \tag{C.2}$$

where $C_{2a} > 0$ is some positive constant and $R_{2a} := \eta^2 T + 4T_H\eta + T_H(T_L - 2)\eta_1(\frac{\eta_1}{2} - 2)$. Thus, we have positive cost saving if and only if $R_{2a} > 0$.

Similarly, the cost saving for Case (3) becomes:

$$Cost - saving = \frac{c_0^b}{E} \cdot [A^*(\frac{a_L^*}{2} - x_L^*) + \frac{T_H}{2}(r_H^*)^2] \tag{C.3}$$
$$= \underbrace{\frac{c_0^b}{E} \cdot C_3}_{>0} \cdot \underbrace{[a_H^* - \frac{T_L\eta}{T\eta + 4T_H} \cdot \frac{\eta_2(4-\eta_1)E}{4c_0^b} p']}_{:=R_3}, \tag{C.4}$$

where $C_3 > 0$ is some positive constant. Then signing R_3 leads to:

$$\text{sgn}(R_3) > 0 \Leftrightarrow \begin{cases} p^a > \frac{-s_1 s_2}{s_1 s_3 + s_4} p', & p^a < \frac{2(T_L-2)c_0^b}{\eta T_L} \\ s_5 > s_6 p', & p^a \geq \frac{2(T_L-2)c_0^b}{\eta T_L}, \end{cases} \tag{C.5}$$

where $s_1 = \frac{\eta^2 T_L E}{2c_0^b(T_L-2)}, s_2 = \frac{\eta_2 E}{8c_0^b}(\eta_1 - 4), s_3 = \frac{\eta(\eta+4)E}{4c_0^b(T_L-2)}, s_4 = \frac{T_H}{2}(\frac{\eta^2 T_L E}{2c_0^b T_H(T_L-2)})^2, s_5 = 4c_0^b(T\eta + 4T_H)$, and $s_6 = T_L T_H \eta_2(4 - \eta_1)$.

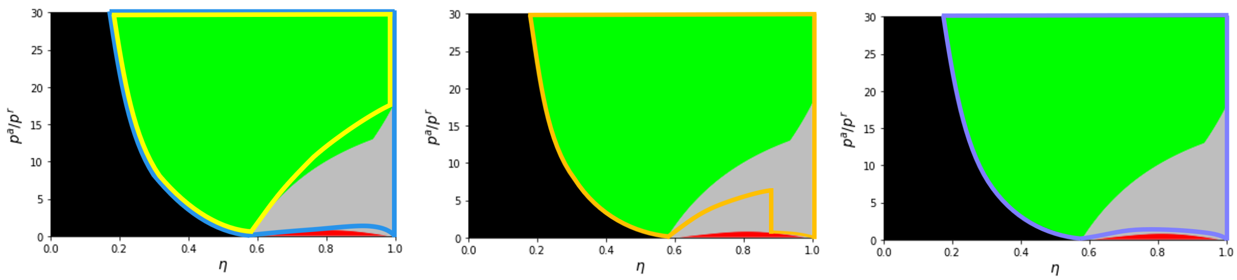
□

Synergistic behavior. Note that the synergistic behaviors under stacking can be dramatically expanded for $\eta < 1$. For example, consistent with the base model, the first two properties in Corollary C.4 hold when stacking is in the "free regulation-only region" (when the price of arbitrage is sufficiently high compared to the price of regulation). While under the base model this region only occurs when $\frac{p^a}{p^r} \geq T - 2$, the threshold for how high the arbitrage price needs to be to trigger this region actually gets looser as the battery gets less efficient when $p^r < \underline{p}$. Eventually ($\eta_{stk} < \eta \leq \eta_{reg}$), stacking always uses free regulation only regardless of the price of arbitrage.

Observation C.2. *If $\eta_{stk} < \eta \leq \eta_{reg}$, then $x_L^* = x_H^* = 0$ (free regulation-only region). That is, all the properties in Corollary C.4 hold.*

Figure 8 illustrates the expanded synergy when stacking is combined with inefficient batteries.

Benefit of stacking. Theorem 3 highlights that stacking can significantly enhance profitability with an inefficient battery, and the relative benefit of stacking, $S = \frac{\Pi_{stk}^*}{\max(\Pi_{arb}^*, \Pi_{reg}^*)}$, is unbounded: $\max S = \infty$.



(a) $A^*_{stk} > A^*_{arb}, R^*_{stk} > R^*_{reg}$ (b) Positive cost-saving (c) $\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$

Figure C.2 Illustration of Corollary C.4: (a) $A^*_{stk} > A^*_{arb}$ (blue), $R^*_{stk} > R^*_{reg}$ (yellow), (b) positive cost-saving (orange), (c) $\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$ (purple). Adapted from Figure 8 (green: free regulation; black: no battery use; red: regulation-only; gray: excess regulation).

C.3. Extension 3: Non-Uniform Regulation Signal

C.3.1. Optimal Stacking: Trinary Signal (+1, 0, -1)

As stated in Section 6.3, we assume the regulation price $p^r(\alpha)$ to be an increasing function of the dispatch-to-contract ratio α. For simplicity in exposition, we will omit the dependency when no confusion arises. We note that structural properties of optimal stacking remain unchanged when the regulation price is a function of α since we can derive optimal actions given p^a and p^r. All that changes is comparative statics on α, which we will state later. For simplicity, we present comparative statics under $p^r = c_1$ and $p^r = c_2 \cdot \alpha$.

Optimal structure. Similar to Lemma 1, it is easy to verify that $x^* := x^*_L = x^*_H$ for given arbitrage amount A, and the optimal actions are as follows:

Theorem C.5 (Optimal stacking: Trinary signal)**.** *With a trinary signal* $(+1, 0, -1)$ *and the dispatch-to-contract ratio* α*, the optimal arbitrage and regulation quantities and profit in the non-binding power setting are as follows:*

cond.	$p^a \le 2(\frac{1}{\alpha}-1)p^r$	$2(\frac{1}{\alpha}-1)p^r \le p^a$ *and* $\{p^a \le \frac{T-2\alpha}{\alpha}p^r$ *or* $p^r \ge \frac{2c_0^b\alpha}{T}\}$	$p^a \ge \frac{T-2\alpha}{\alpha}p^r,\ p^r \le \frac{2c_0^b\alpha}{T}$
A^*	0	$\min(\frac{T(p^a-2(\frac{1}{\alpha}-1)p^r)}{2c_0^b(T-2)}E, E)$	$\min(\frac{p^a+2p^r}{2c_0^b}E, E)$
R^*	$\frac{p^rT}{c_0^b\alpha^2}E$	$\frac{p^rTE}{c_0^b\alpha^2} - \frac{2(1-\alpha)}{\alpha}A^*$	$2A^*$
x^*	$\frac{p^rE}{c_0^b\alpha}$	$\frac{p^rE}{c_0^b\alpha} - \frac{2A^*}{T}$	0
Π^*	$\frac{(p^r)^2TE}{2c_0^b\alpha^2}$	$p^aA^* - 2p^rA^*(\frac{1}{\alpha}-1) + \frac{T(p^r)^2E}{2c_0^b\alpha^2} - \frac{c_0^b}{E}(A^*)^2(1-\frac{2}{T})$	$p^aA^* + 2p^rA^* - \frac{c_0^b}{E}(A^*)^2$

Figure C.3 summarizes the structure of the optimal stacking.

Proof. From $x^* = x^*_L = x^*_H$, the profit function can be restated:

$$\Pi(A,x) = p^aA + 2p^rA + \frac{xT}{\alpha}p^r - \frac{c_0^b}{E}[A^2 + 2Ax + \frac{T}{2}x^2].$$

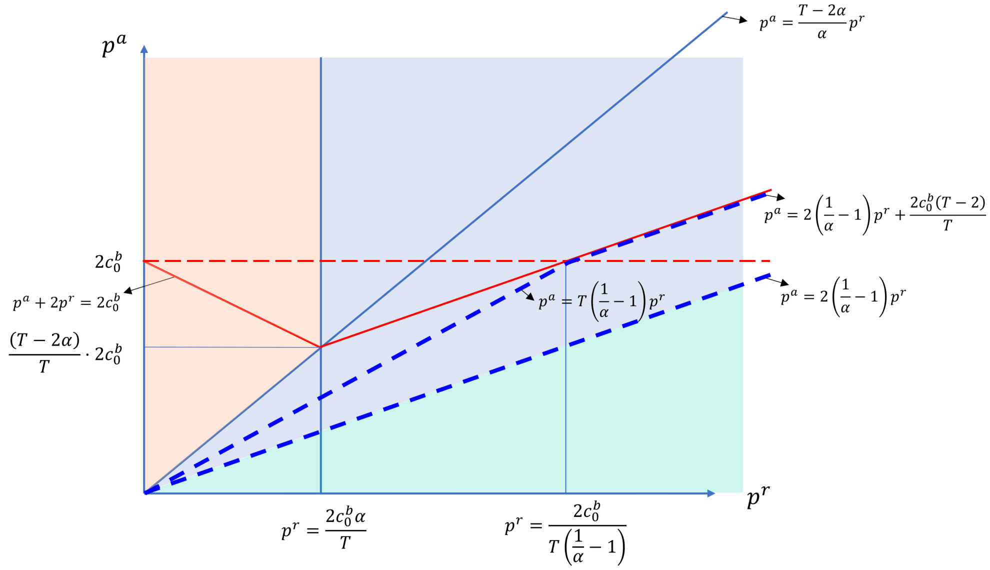


Figure C.3 Illustration of Theorem C.5 – Colors denote discrete regions: $A^* = 0$ (green), $A^* > 0,\ x^* = 0$ (orange), and $A^* > 0,\ x^* > 0$ (blue). Energy capacity is binding on the red line and above for stacking (solid) and standalone arbitrage (dashed). The subregion within the blue dashed lines denotes where $A^*_{stk} < A^*_{arb}$.

For fixed A, it is immediate that $x^* = (\frac{p^r E}{\alpha c_0^b} - \frac{2A}{T})^+$,yielding the following optimal profit as a function of A:

$$\Pi(A) = \begin{cases} p^a A + 2p^r A - \frac{c_0^b}{E} A^2, & \text{if } A > \frac{Tp^r E}{2\alpha c_0^b}, \text{ and} \\ p^a A - 2p^r A(\frac{1}{\alpha} - 1) + \frac{T(p^r)^2 E}{2c_0^b \alpha^2} - \frac{c_0^b}{E} A^2 (1 - \frac{2}{T}), & \text{if } A < \frac{Tp^r E}{2\alpha c_0^b}. \end{cases}$$

Thus, $\Pi(A)$ is a strictly concave function with a continuous, piecewise linear, decreasing derivative. Given derivatives

$$\Pi'(0) = p^a - 2(\frac{1}{\alpha} - 1)p^r \text{ and}$$
$$\Pi'(\frac{Tp^r E}{2\alpha c_0^b}) = p^a - \frac{T - 2\alpha}{\alpha} p^r,$$

it is clear that $A^* = 0$ when $p^a \leq 2(\frac{1}{\alpha} - 1)p^r$ (or $\alpha \leq \frac{2p^r}{p^a + 2p^r}$) and $A^* \geq \frac{Tp^r E}{2\alpha c_0^b}$ and therefore $x^* = 0$ when $p^a \geq \frac{T-2\alpha}{\alpha} p^r$ (or $\alpha \geq \frac{Tp^r}{p^a + 2p^r}$).

□

From Theorem C.5, we immediately check the most of the properties still hold as follows ($\frac{R^*}{A^*} := \infty$ for $A^* = 0$).

Corollary C.5. *With the dispatch-to-contract ratio $\alpha \leq 1$, the following hold for stacking:*

- *R^* increases in p^a when $\frac{T-2\alpha}{\alpha} p^r \leq p^a < 2c_0^b - 2p^r$,*
- *A^* increases in p^r when $\frac{T-2\alpha}{\alpha} p^r \leq p^a < 2c_0^b - 2p^r$,*

- *Stacking always uses equal excess regulation:* $x_L^* = x_H^*$,
- $\frac{R^*}{A^*} \geq 2$,
- *Positive cost-saving synergy when* $p^a > \frac{2(1-\alpha)}{\alpha} p^r$,
- $A_{stk}^* > A_{arb}^*$ *when* $\frac{T(1-\alpha)}{\alpha} p^r < p^a < 2c_0^b$,
- $A_{stk}^* > 0$ *when* $p^a > \frac{2(1-\alpha)}{\alpha} p^r$,
- $R_{stk}^* > R_{reg}^*$ *when* $(\frac{T}{\alpha^2} - 2)p^r < p^a$ *and* $\frac{p^r T}{2\alpha^2} < c_0^b$,
- *Superlinear gains* $(\Pi_{stk}^* > \Pi_{arb}^* + \Pi_{reg}^*)$ *when*

(i) $\frac{T - \sqrt{T^2 - 2T}}{2} \leq \alpha < 1$, $p^a > p_\alpha^a = \begin{cases} \frac{(1-\alpha)[T + \sqrt{T(T-2)}]}{\alpha} p^r, & p^r \leq \frac{2c_0^b \alpha (T-2)}{(1-\alpha)T[T-2+\sqrt{T(T-2)}]}, \\ 2c_0^b - \sqrt{8c_0^b[\frac{c_0^b}{T} - p^r(\frac{1}{\alpha} - 1)]}, & \frac{2c_0^b \alpha (T-2)}{(1-\alpha)T[T-2+\sqrt{T(T-2)}]} \leq p^r \leq \frac{c_0^b}{T(\frac{1}{\alpha} - 1)}, \end{cases}$

or (ii) $1/2 \leq \alpha < \frac{T - \sqrt{T^2 - 2T}}{2}$, $p^a > (\frac{T}{2\alpha^2} - 1)p^r$, $p^r \leq \frac{2c_0^b \alpha}{T}$,

or (iii) $\alpha < 1/2$, $p^a > p_\alpha^a = \begin{cases} (\frac{T}{2\alpha^2} - 1)p^r, & p^r \leq \frac{4\alpha^2 c_0^b}{T + 2\alpha^2}, \\ 2c_0^b - \sqrt{\frac{8(\alpha c_0^b)^2}{T} - \frac{2T}{\alpha^2}(p^r - \frac{2\alpha^2 c_0^b}{T})^2}, & \frac{4\alpha^2 c_0^b}{T + 2\alpha^2} \leq p^r \leq \frac{4\alpha^2 c_0^b}{T}. \end{cases}$

Proof. Theorem C.5 immediately implies that the first five properties still hold.

$A_{stk}^* > A_{arb}^*$: This requires $A_{arb}^* < E$, which is equivalent to $\frac{p^a}{2} < c_0^b$. This holds when $x^* = 0$. When $x^* = \frac{E}{c_0^b}(\frac{T-2\alpha}{\alpha(T-2)} p^r - \frac{p^a}{T-2})$, it happens if and only if $p^a > \frac{T(1-\alpha)}{\alpha} p^r$, or equivalently, $\alpha > \frac{Tp^r}{p^a + Tp^r}$.

$A_{stk}^* > 0$: This holds if and only if $p^a > 2(\frac{1}{\alpha} - 1)p^r$, or equivalently, $\alpha > \frac{2p^r}{p^a + 2p^r}$.

$R_{stk}^* > R_{reg}^*$: When $x^* > 0$, $R_{stk}^* \leq R_{reg}^*$. When $x^* = 0$, $R_{stk}^* > R_{reg}^*$ holds if and only if $p^a > p^r(\frac{T}{\alpha^2} - 2)$, or equivalently, $\alpha > \sqrt{\frac{Tp^r}{p^a + 2p^r}}$ when $A_{stk}^* < E$ or $c_0^b > \frac{p^r T}{2\alpha^2}$ when $A_{stk}^* = E$. This leads to (i) $(\frac{T}{\alpha^2} - 2)p^r < p^a < 2c_0^b - 2p^r$ or (ii) $p^a \geq \max(\frac{T-2\alpha}{\alpha} p^r, 2c_0^b - 2p^r)$ and $\frac{p^r T}{2\alpha^2} < c_0^b$, which can be simplified as $(\frac{T}{\alpha^2} - 2)p^r < p^a$ and $\frac{p^r T}{2\alpha^2} < c_0^b$.

$\Pi_{stk}^* > \Pi_{arb}^* + \Pi_{reg}^*$: From Theorem C.5, it is straightforward to derive $\Delta\Pi := \Pi_{stk}^* - [\Pi_{arb}^* + \Pi_{reg}^*]$, which can be written in the form $c_1(p^a)^2 + c_2 p^a p^r + c_3(p^r)^2 + c_4 p^a + c_5 p^r + c_6$. Then the conditions for superlinear profit gains in Corollary C.5 can be derived from the enumeration of solutions for the quadratic formulas.

□

Synergistic behavior. From Theorem C.5, the condition for the $x^* = 0$ region requires the conditions $p^a \geq \frac{T-2\alpha}{\alpha} p^r$ and $p^r \leq \frac{2c_0^b \alpha}{T}$, which implies that the synergistic behavior is still present for sufficiently large α. For instance, consistent with Observation 3, the optimal arbitrage and regulation amounts are both monotonically increasing in p^a and p^r when $x_L^* = x_H^* = 0$. However, it requires a stronger condition than that of the base model ($\alpha = 1$) under both constant and linear regulation price assumptions. This is because regulation becomes more attractive than arbitrage under the non-uniform signal.

Recall that under the base model, at least a small amount of arbitrage is always beneficial, since the sum of increased revenue and cost-saving for regulation is larger than the increased cost for arbitrage (Observation 4). However, when the dispatch-to-contract ratio α is sufficiently low ($p^a \leq 2(\frac{1}{\alpha}-1)p^r$), the cost-saving from regulation becomes marginal, making it optimal to provide regulation alone ($A^*=0$). Note that under the constant price assumption, the benefits of stacking consistently diminish (i.e., $2(\frac{1}{\alpha}-1)p^r$ approaches infinity) as α approaches zero. In contrast, under the linear price assumption, stacking can still be beneficial as long as $p^a > 2(\frac{1}{\alpha}-1)p^r = 2(1-\alpha)c_2$, since the regulation price also diminishes when α approaches zero.

Benefit of stacking. When energy capacity is not binding ($p^a < 2c_0^b - 2p^r$ or $p^a < \min(2(\frac{1}{\alpha} - 1)p^r + \frac{2c_0^b(T-2)}{T}, 2c_0^b)$), $\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$ holds for $\alpha \in [1/2+\epsilon, 1]$ for some small $\epsilon = \frac{T-1-\sqrt{T^2-2T}}{2} > 0$. When $2(\frac{1}{\alpha}-1)p^r \leq p^a \leq \frac{T-2\alpha}{\alpha}p^r$, $\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$ holds for $\alpha \in [1/2+\epsilon, 1]$ for some small $\epsilon = \frac{T-2}{[T-2+\sqrt{T^2-2T}]^2} > 0$. Thus, roughly speaking, the superlinear profit improvement still holds for $\alpha > 1/2$ with sufficiently large p^a.

Now we investigate stacking's performance compared to standalone services. First, with regulation being the only service, the (unconstrained) optimal amount of frequency regulation and the corresponding profit can be easily determined: $r^*_\alpha = \frac{p^r E}{c_0^b \alpha^2}$, $\Pi^*_{reg,\alpha} = \frac{(p^r)^2 TE}{2c_0^b\alpha^2}$. Then we can compare stacking's profitability as follows:

Corollary C.6 (Upper bound on stacking's performance: Trinary signal)**.** *Given dispatch-to-contract ratio α and non-binding energy capacity, $S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$ is as follows:*

$$S = \begin{cases} 1, & \text{if } \frac{p^a}{p^r} \leq 2(\frac{1}{\alpha}-1), \\ \frac{T+2\alpha^2-4\alpha}{T-2} - \frac{2(1-\alpha)\alpha}{T-2}\cdot\frac{p^a}{p^r} + \frac{1}{2(T-2)}(\frac{\alpha p^a}{p^r})^2, & \text{if } 2(\frac{1}{\alpha}-1) \leq \frac{p^a}{p^r} \leq \frac{\sqrt{2T}}{\alpha}, \\ \frac{T}{T-2} - \frac{4T(1-\alpha)}{\alpha(T-2)}\cdot\frac{p^r}{p^a} + \frac{2T(T+2\alpha^2-4\alpha)}{\alpha^2(T-2)}(\frac{p^r}{p^a})^2, & \text{if } \frac{\sqrt{2T}}{\alpha} \leq \frac{p^a}{p^r} \leq \frac{T-2\alpha}{\alpha}, \text{ and} \\ 1 + 4\frac{p^r}{p^a} + 4(\frac{p^r}{p^a})^2, & \text{if } \frac{p^a}{p^r} \geq \frac{T-2\alpha}{\alpha}. \end{cases}$$

Similar to the result of the base model, $S \in [1, \frac{2T+2\alpha^2-4\alpha-2(1-\alpha)\sqrt{2T}}{T-2}]$, where $\max S = \frac{2T+2\alpha^2-4\alpha-2(1-\alpha)\sqrt{2T}}{T-2}$ is achieved at $\frac{p^a}{p^r(\alpha)} = \frac{\sqrt{2T}}{\alpha}$, which is increasing in α. However, with sufficiently large T, the decrease becomes negligible ($\max S \approx 2$). Since in the base model, the maximizer is achieved at $\frac{p^a}{p^r} = \sqrt{2T}$, the non-uniform signal may shift the peak either to the left or right (depending on the functional form of $p^r(\alpha)$). Recognizing the relationship between p^r and α may emphasize stacking's benefits — Considering the recent grid/market conditions, where frequency regulation is more profitable than arbitrage, the assumption that regulation price being a function of α (e.g., linear price) may reduce frequency regulation's attractiveness, making arbitrage more comparable to regulation (recall that the ratio $S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$) improve as the two standalone services become more comparable and is maximized at $\Pi^*_{arb} = \Pi^*_{reg}$).

Below we consider the effect of binding energy capacity. Corollary C.6 assumes non-binding energy capacity. In the presence of binding energy capacity, we can show that the S ratio is still upper bounded by the grand maximum from the unbinding case ($\alpha = 1$).

Theorem C.6. *Given dispatch-to-contract ratio α and T, $S \le \frac{2T-2}{T-2}$.*

Proof. When E is binding for stacking but not for standalone arbitrage or when $\Pi^*_{reg} > \Pi^*_{arb}$, the S ratio is upper-bounded by the S ratio under no energy constraint. Further, if $A^*_{stk} < A^*_{arb} = E$ and $x^* > 0$ (or $\max(2c_0^b, 2(\frac{1}{\alpha}-1)p^r) \le p^a \le \frac{2c_0^b(T-2)}{T} + 2(\frac{1}{\alpha}-1)p^r$), then Corollary C.5 implies that we never have superlinear profit gain ($\Pi^*_{stk} < \Pi^*_{arb} + \Pi^*_{reg}$), concluding that the S ratio is always strictly smaller than 2. We now focus on the two following cases where energy capacity is binding for both standalone arbitrage and stacking and $\Pi^*_{arb} > \Pi^*_{reg}$ ($p^a > c_0^b + \frac{(p^r)^2 T}{2c_0^b\alpha^2}$):

$$\Pi^*_{stk} = \begin{cases} p^aE + 2p^rE - c_0^bE, & p^a \ge 2c_0^b,\ p^r \le \frac{2c_0^b\alpha}{T}, \\ p^aE - 2p^rE(\frac{1}{\alpha}-1) + \frac{T(p^r)^2E}{2c_0^b\alpha^2} - c_0^bE(1-\frac{2}{T}), & p^a \ge \max(2c_0^b, \frac{2c_0^b(T-2)}{T} + 2(\frac{1}{\alpha}-1)p^r),\ p^r > \frac{2c_0^b\alpha}{T} \end{cases}$$

If $p^a \ge 2c_0^b,\ p^r \le \frac{2c_0^b\alpha}{T}$ ($x^* = 0$), then

$$\begin{aligned} \frac{\Pi^*_{stk}}{\Pi^*_{arb}} &\le \frac{p^aE + \frac{4c_0^bE\alpha}{T} - c_0^bE}{p^aE - c_0^bE}\ (\because p^r \le \frac{2c_0^b\alpha}{T}) \\ &\le \frac{1}{1-\frac{c_0^b}{p^a}} \\ &\le 2\ (\because p^a \ge 2c_0^b). \end{aligned}$$

If $p^a \ge \max(2c_0^b, \frac{2c_0^b(T-2)}{T} + 2(\frac{1}{\alpha}-1)p^r),\ p^r > \frac{2c_0^b\alpha}{T}$ ($x^* > 0$), then

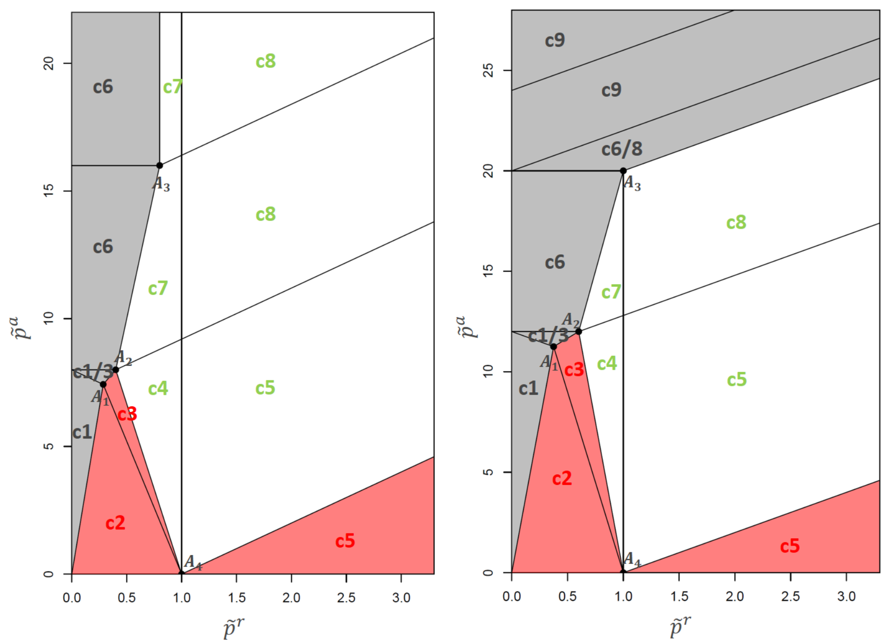
$$\begin{aligned} \frac{\Pi^*_{stk}}{\Pi^*_{arb}} &\le \frac{2(p^aE - c_0^bE) + \frac{2c_0^bE}{T} - 2p^rE(\frac{1}{\alpha}-1)}{p^aE - c_0^bE}\ (\because \frac{(p^r)^2TE}{2c_0^b\alpha^2} < p^aE - c_0^bE) \\ &= 2 + \frac{2}{T}[\frac{c_0^bE - p^rE(\frac{1}{\alpha}-1)T}{p^aE - c_0^bE}] \\ &\le 2 + \frac{2}{T}[\frac{c_0^bE}{p^aE - c_0^bE}] \le 2 + \frac{2}{T}\ (\because p^a \ge 2c_0^b) \\ &< \frac{2T-2}{T-2}. \end{aligned}$$

Combining the above results concludes the proof. □

Though the grand maximum ($S = \frac{2T-2}{T-2}$ when $\alpha = 1$) cannot be improved under the binding energy capacity, the relative benefit of stacking over the best standalone for certain α can improve.

Observation C.3. *$S(\alpha) > \frac{2T+2\alpha^2-4\alpha-2(1-\alpha)\sqrt{2T}}{T-2}$ can happen where $\frac{2T+2\alpha^2-4\alpha-2(1-\alpha)\sqrt{2T}}{T-2}$ is the maximum S ratio under non-binding energy capacity in Corollary C.6.*

This situation arises when $A^*_{stk} = \frac{TE(p^a-2(\frac{1}{\alpha}-1)p^r)}{2c_0^b(T-2)} < A^*_{arb} = E$ — the limited capacity restricts standalone arbitrage (which outperforms standalone regulation, i.e., $\Pi^*_{arb} > \Pi^*_{reg}$), while stacking remains unaffected by the energy constraint due to its ability to balance the quantities of the two services ($A^*_{stk} = \frac{T(p^a-2(\frac{1}{\alpha}-1)p^r)}{2c^b(T-2)}$). Consequently, this leads to an augmentation in the S ratio. Numerically, the increase in the S ratio (i.e., $S(\alpha) - \frac{2T+2\alpha^2-4\alpha-2(1-\alpha)\sqrt{2T}}{T-2}$) can be as high as 4%.



(a) $T_L \geq 2T_H$ (b) $T_L \leq 2T_H$

Figure C.4 Illustration of Lemma C.3. The colored regions denote $x_L^* = x_H^* > 0$ (pink), $x_L^* = x_H^* = 0$ (gray), and $x_L^* > x_H^*$ (white). $A_1 = (\frac{T_H}{T}, \frac{T_H(T-2)}{T})$, $A_2 = (\frac{T_H}{T_L}, T_H)$, $A_3 = \begin{cases} (\frac{2T_H}{T_L}, 2T_H), & T_L \geq 2T_H \\ (1, T_L), & T_L \leq 2T_H \end{cases}$, $A_4 = (1,0)$ where $T_L = 20, T_H = 8$ and $T_L = 20, T_H = 12$ are used for (a) and (b), respectively.

C.4. Extension 4: Stacking when Power is Binding

C.4.1. Optimal Stacking: Power is Binding

Below we provide a detailed description of the comparative statics. We normalize service prices by the degradation coefficient c_0^b and the energy-to-power ratio. A.1.2).

Lemma C.3. *Define $\tilde{p}^a := \frac{p^a E}{c_0^b P}$ and $\tilde{p}^r := \frac{p^r E}{c_0^b P}$ to be the prices of the services, normalized by c_0^b and energy-to-power ratio. Then for a fixed arbitrage amount A, the optimal profit of the stacked model with power constraints becomes:*

$$\Pi(A) = \begin{cases} (p^a + 2p^r)A - \frac{c_0^b}{E}A^2, & \text{if } (c1) \\ \frac{T(p^r)^2 E}{2c_0^b} + p^a A - \frac{c_0^b}{E}(1 - \frac{2}{T})A^2, & \text{if } (c2) \\ TP(p^r - \frac{\frac{c_0^b}{E}P}{2}) + (p^a + \frac{2T_L(\frac{c_0^b}{E}P - p^r)}{T_H})A - \frac{c_0^b}{E}(1 + \frac{2(T_L - T_H)}{T_H^2})A^2, & \text{if } (c3) \\ (\frac{(p^r)^2 T_L E}{2c_0^b} + p^r T_H P - \frac{c_0^b T_H P^2}{2E}) + (p^a - 2p^r + 2\frac{c_0^b}{E}P)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L} + \frac{2}{T_H})A^2, & \text{if } (c4) \\ TP(p^r - \frac{\frac{c_0^b}{E}P}{2}) + (p^a - 2p^r + 2\frac{c_0^b}{E}P)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L} + \frac{2}{T_H})A^2, & \text{if } (c5) \\ p^r P T_H + p^a A - \frac{c_0^b}{E}A^2, & \text{if } (c6) \\ (p^r P T_H + \frac{T_L (p^r)^2 E}{2c_0^b}) + (p^a - 2p^r)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L})A^2, & \text{if } (c7) \\ (p^r P T - \frac{c_0^b P^2 T_L}{2E}) + (p^a - 2p^r)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L})A^2, & \text{if } (c8) \\ p^r P T + (p^a - 2p^r)A - \frac{c_0^b}{E}A^2, & \text{if } (c9) \end{cases}$$

where conditions (c1) – (c9) are defined as follows:

- *Region 1: $0 \leq A \leq \frac{PT_H}{2}$:*

— *(c1):* $\tilde{p}^r \le \frac{T_H}{T}, A \ge \frac{Tp^rE}{2c_0^b}$,

— *(c2):* $\tilde{p}^r \le \frac{T_H}{T}, A < \frac{Tp^rE}{2c_0^b}$ *or* $\frac{T_H}{T} < \tilde{p}^r < 1, A < \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T)}$,

— *(c3):* $\frac{T_H}{T} < \tilde{p}^r \le \frac{T_H}{T_L}, A \ge \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T)}$ *or* $\frac{T_H}{T_L} < \tilde{p}^r < 1, \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T)} \le A \le \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T_L)}$,

— *(c4):* $\frac{T_H}{T_L} < \tilde{p}^r < 1, A > \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T_L)}$,

— *(c5):* $\tilde{p}^r \ge 1$,

- *Region 2:* $\frac{PT_H}{2} < A \le \min(PT_H, \frac{PT_L}{2})$:

— *(c6):* $\tilde{p}^r \le \frac{T_H}{T_L}$ *or* $\frac{T_H}{T_L} < \tilde{p}^r < \min(\frac{2T_H}{T_L}, 1), A \ge \frac{T_L p^r E}{2c_0^b}$,

— *(c7):* $T_L > 2T_H, \frac{2T_H}{T_L} \le \tilde{p}^r < 1$ *or* $\frac{T_H}{T_L} < \tilde{p}^r < \min(\frac{2T_H}{T_L}, 1), A < \frac{T_L p^r E}{2c_0^b}$,

— *(c8):* $\tilde{p}^r \ge 1$,

- *Region 3:* $\frac{PT_L}{2} < A \le PT_H$:

— *(c9):* $T_L < 2T_H$.

The optimal solutions under conditions $(c1)$ – $(c9)$ *follow the properties listed in Table C.1.*

All proofs of results stated in Appendix C.4.1 are provided in Appendix C.4.2. From Lemma C.3, Figure C.4 is derived. Table C.1 provides specific conditions required for the key properties to hold.

Properties	Regions[†]								
	$(c1)$	$(c2)$	$(c3)$	$(c4)$	$(c5)$	$(c6)$	$(c7)$	$(c8)$	$(c9)$
$x_L^* = x_H^* > 0$	×	✓	$\tilde{p}^r > \frac{T_H}{T_L}$ or $A < \frac{PT_H}{2}$	×	×	×	×	×	×
$x_L^* = x_H^* = 0$	✓	×	$\tilde{p}^r \le \frac{T_H}{T_L}$ and $A = \frac{PT_H}{2}$	×	×	✓	×	$2T_H \ge T_L$ and $A = \frac{PT_L}{2}$	✓
P unconstrained in both periods	$A < \frac{PT_H}{2}$	✓	×	×	×	×	×	×	×
P constrained in both periods	×	×	×	×	✓	$T_L \le 2T_H$ and $A = \frac{PT_L}{2}$	×	✓	✓
Uses all available free regulation	✓	✓	✓	✓	✓	×	×	×	×
$\frac{\partial^2 \Pi(A)}{\partial p^r \partial A} > 0$	✓	×	×	×	×	×	×	×	×
$\frac{\partial R^*}{\partial A} > 0$	✓	×	×	×	×	×	×	×	×
$\frac{R^*}{A} \ge 2$	✓	✓	✓	✓	✓	×	$4A \le \frac{T_L p^r E}{c_0^b} + PT_H$	$T_L \ge 3T_H$ or $4A \le PT$	×
$\Pi'_{stk}(A) > \Pi'_{arb}(A)$[‡]	✓	✓	✓	×	×	×	×	×	×
$R^*_{stk} > R^*_{reg}$	$A > \frac{Tp^rE}{2c_0^b}$	×	×	×	×	$\tilde{p}^r < \frac{T_H}{T}$	×	×	$A < \frac{PT}{2}(1-\tilde{p}^r)$ and $\tilde{p}^r < \frac{T_H}{T}$
Cost-saving synergy	✓	✓	✓	$T_H > \frac{3T}{8}$ or $A < \frac{2T_L T_H P(1-\tilde{p}^r)}{3T_L - 5T_H}$	$T_H > \frac{3T}{8}$	✓	always for $T_H > \frac{3T}{8}$; otherwise $A > \frac{P\left(1+\tilde{p}^r+\sqrt{(1+\tilde{p}^r)^2 - \frac{5T_H+T_L}{T_L}}\right)}{\frac{5}{T_L}+\frac{1}{T_H}}$	always for $T_H > \frac{3T}{8}$; never for $T_H < \frac{2T}{7}$; otherwise $A > \frac{P(2+\sqrt{3-\frac{5T_H}{T_L}})}{\frac{5}{T_L}+\frac{1}{T_H}}$	✓

Table C.1 Properties of the optimal power-constrained solution for fixed arbitrage amount A when $T_L > T_H$.

†: Regions $(c1)$ – $(c9)$ refer to the conditions defined in the statement of Lemma C.3.

‡: Here we define $\Pi_{arb}(A) = p^a A - \frac{c_0^b}{E} A^2$ to be the profit from providing A units of arbitrage and no regulation.

Observation C.4 (Guaranteed cost-saving synergy with binding P)**.** *According to Proposition C.1, a limited power capacity may lead to negative cost-saving synergy when excess regulation values* *are highly imbalanced:* $x_L^* > x_H^*$*. We read from Table C.1 that such an imbalance will never occur if the high-price period is sufficiently long relative to the low-price period (*$T_H > \frac{3T}{8}$*).*

Proposition C.1 states properties of the optimal solution with a power limit P but no energy limit E.

Proposition C.1. *At optimality, the stacked model with finite power capacity* P *but no energy limit* E *satisfies the following when* $T_L > T_H$*:*

- *Optimal excess regulation:*
  - — *Positive excess regulation in equal quantities between the two time periods (*$x_H^* = x_L^* > 0$*,* $A^* > 0$*) is optimal iff*

$$\tilde{p}^a < \begin{cases} (T-2)\tilde{p}^r, & \textit{if } \tilde{p}^r \le \frac{T_H}{T}, \\ \frac{2T_L}{T_H}\tilde{p}^r + (T_H - 2), & \textit{if } \frac{T_H}{T} < \tilde{p}^r \le \frac{T_H}{T_L}, \textit{and} \\ \frac{T_L T_H}{T_L - T_H}(1-\tilde{p}^r), & \textit{if } \frac{T_H}{T_L} < \tilde{p}^r < 1, \end{cases}$$

  - — *Free regulation only (*$x_H^* = x_L^* = 0$*) is optimal iff*

$$\tilde{p}^a \ge \begin{cases} (T-2)\tilde{p}^r, & \textit{if } \tilde{p}^r \le \frac{T_H}{T}, \\ \frac{2T_L}{T_H}\tilde{p}^r + (T_H - 2), & \textit{if } \frac{T_H}{T} < \tilde{p}^r \le \frac{T_H}{T_L}, \\ T_L\tilde{p}^r, & \textit{if } \frac{T_H}{T_L} < \tilde{p}^r < \min(\frac{2T_H}{T_L}, 1), \textit{and} \\ 2\tilde{p}^r + T_L - 2, & \textit{if } \tilde{p}^r \ge 1, T_L \le 2T_H, \end{cases}$$

  - — *Regulation-only (*$r_H^* = r_L^* = P,\ A^* = 0$*) is optimal iff* $\tilde{p}^r \ge 1,\ \tilde{p}^a \le 2(\tilde{p}^r - 1)$*.*
  - — *In all other cases, the stacked model satisfies* $x_L^* > x_H^*$*.*
- *Power constraints:*
  - — *Stacking is power unconstrained in both time periods iff*

$$\tilde{p}^a < \begin{cases} T_H - 2\tilde{p}^r, & \textit{if } \tilde{p}^r \le \frac{T_H}{T}, \textit{and} \\ \frac{(T-2)T_H}{T_L}(1-\tilde{p}^r), & \textit{if } \frac{T_H}{T} < \tilde{p}^r < 1, \end{cases}$$

  - — *Stacking is power constrained in both time periods either when* $\tilde{p}^r \ge 1$ *or when* $\tilde{p}^r < 1$*,* $\tilde{p}^a \ge T_L$*, and* $T_L \le 2T_H$*.*
  - — *Stacking uses all available free regulation (*$r_L^* \ge a_L^*, r_H^* \ge a_H^*$*) iff*

$$\tilde{p}^a \le \begin{cases} T_H, & \textit{if } \tilde{p}^r \le \frac{T_H}{T_L}, \textit{and} \\ 2\tilde{p}^r + T_H(1 - \frac{2}{T_L}), & \textit{if } \tilde{p}^r > \frac{T_H}{T_L}. \end{cases}$$

- *Key properties:*
  - — A^* *(*R^**) is strictly increasing in* p^r *(*p^a*) when* $\tilde{p}^r < \frac{T_H}{T}$*,* $(T-2)\tilde{p}^r \le \tilde{p}^a \le -2\tilde{p}^r + T_H$*,*
  - — $A^*_{stk} > A^*_{arb}$ *when* $\tilde{p}^a < \min(T_H, \frac{T_L T_H}{T_L - T_H}(1-\tilde{p}^r))$*,*
  - — $R^*_{stk} > R^*_{reg}$ *when* $\begin{cases} \tilde{p}^r < \min(\frac{\tilde{p}^a}{T-2}, \frac{T_H}{T}), T_L \ge 2T_H, \textit{ or} \\ \tilde{p}^r < \min(\frac{\tilde{p}^a}{T-2}, \frac{T_H}{T}, \frac{T-\tilde{p}^a}{T-2}), T_L < 2T_H, \textit{ or} \\ \frac{T-\tilde{p}^a}{T-2} \le \tilde{p}^r < \frac{T_L - T_H}{T}, T_L < 2T_H, \end{cases}$

— $\frac{R^*}{A^*} \geq 2$ *when* $\tilde{p}^a \leq \tilde{p}^a_1 = \begin{cases} T_H, & \text{if } \tilde{p}^r < \frac{T_H}{T_L} \\ \frac{T_L+2}{2}\tilde{p}^r + \frac{T_H(T_L-2)}{2T_L}, & \text{if } \frac{T_H}{T_L} \leq \tilde{p}^r < 1 \\ 2\tilde{p}^r + \frac{T}{2}(1-\frac{2}{T_L}), & \text{if } \tilde{p}^r \geq 1, T_L < 3T_H \\ \infty, & \text{if } \tilde{p}^r \geq 1, T_L \geq 3T_H, \end{cases}$

— $A^* > 0$ *when* $\tilde{p}^a > 2(\tilde{p}^r - 1)$,

— *Positive cost-saving synergy occurs when*

$$\begin{cases} T_L < \frac{5T_H}{3}, \text{or} \\ \tilde{p}^a \geq 2\tilde{p}^r + \frac{2T_H(T_L-2)}{5T_H+T_L} \cdot \max(2+\sqrt{\frac{3T_L-5T_H}{T_L}}, 1+\tilde{p}^r+\sqrt{(1+\tilde{p}^r)^2 - \frac{5T_H+T_L}{T_L}}), \frac{5T_H}{3} \leq T_L \leq \frac{5T_H}{2}, \text{ or} \\ \tilde{p}^a \geq 2\tilde{p}^r + \frac{2T_H(T_L-2)}{5T_H+T_L}(1+\tilde{p}^r+\sqrt{(1+\tilde{p}^r)^2 - \frac{5T_H+T_L}{T_L}}), \tilde{p}^r \leq \frac{5T_H}{2T_L}, T_L \geq \frac{5T_H}{2}, \text{ or} \\ \tilde{p}^a \leq \frac{4T_LT_H+2T}{3T_L-5T_H}(1-\tilde{p}^r), T_L \geq \frac{5T_H}{3}, \end{cases}$$

— $\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$ *when* $T_L \geq 2T_H$, $\tilde{p}^a < \begin{cases} \infty, & \tilde{p}^r \leq \frac{2T_H}{T} \\ \tilde{p}^a_{(c7)}, & \frac{2T_H}{T} \leq \tilde{p}^r \leq \frac{T_H}{T_L}(\frac{2+\sqrt{T_LT_H-2T_H+2T_L}}{T_H+2}), \text{ or} \\ \tilde{p}^a_{(c4)}, & \frac{T_H}{T_L}(\frac{2+\sqrt{T_LT_H-2T_H+2T_L}}{T_H+2}) \leq \tilde{p}^r < 1 \end{cases}$

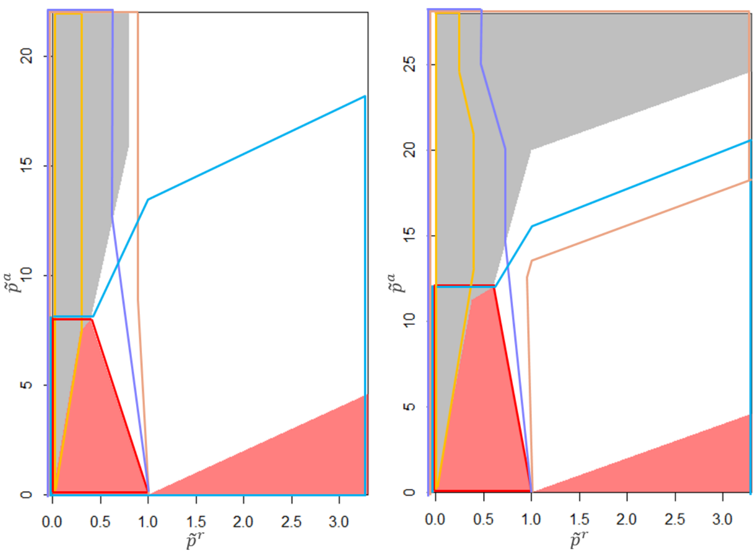
$$T_L < 2T_H,\ \tilde{p}^a < \begin{cases} \infty, & \tilde{p}^r \leq \frac{2(T_L-T_H)}{T} \\ 2\tilde{p}^r + 2T_H - \sqrt{8\tilde{p}^rT_H - 4T\tilde{p}^r + 2T(\tilde{p}^r)^2}, & \frac{2(T_L-T_H)}{T} \leq \tilde{p}^r \leq \frac{2(T_L-T_H)}{T-2} \\ T - (\frac{T}{2}-1)\tilde{p}^r, & \frac{2(T_L-T_H)}{T-2} \leq \tilde{p}^r \leq \frac{2T_H}{T+2} \\ \tilde{p}^a_{(c6/8)}, & \frac{2T_H}{T+2} \leq \tilde{p}^r \leq \frac{2T_H}{T} \\ \tilde{p}^a_{(c7)}, & \frac{2T_H}{T} \leq \tilde{p}^r \leq \frac{T_H}{T_L}(\frac{2+\sqrt{T_LT_H-2T_H+2T_L}}{T_H+2}) \\ \tilde{p}^a_{(c4)}, & \frac{T_H}{T_L}(\frac{2+\sqrt{T_LT_H-2T_H+2T_L}}{T_H+2}) \leq \tilde{p}^r < 1, \end{cases}$$

where

$$\tilde{p}^a_{(c6/8)} = T_L + \sqrt{\frac{2T_H^2}{T} - 2T(\tilde{p}^r - \frac{T_H}{T})^2},$$
$$\tilde{p}^a_{(c7)} = T_L\tilde{p}^r - \sqrt{(T_L-2)(T(\tilde{p}^r)^2 - 2T_H\tilde{p}^r)},\ \text{and}$$
$$\tilde{p}^a_{(c4)} = \frac{T_LT_H}{T_L-T_H}(1-\tilde{p}^r)(1+\frac{\sqrt{T_LT_H+2T_L-2T_H}}{T_L}).$$

Based on Proposition C.1, Figure C.5 represents regions where each synergistic property holds for the case when energy is not binding for stacking. Most of the synergistic properties (e.g., positive cost-saving) continue to hold even when power is binding. Some properties hold more broadly under wide peaks ($T_L \leq 2T_H$) than under narrow peaks ($T_L \geq 2T_H$). For instance, the regions where stacking achieves positive cost savings are expanded; Observation C.4 states that stacking always results in a positive cost saving when $T_H > \frac{3T}{8}$.

Naturally, the optimal stacking solution characterized by Proposition C.1 may select an infeasible amount of arbitrage for finite energy capacity E ($A^* > E$). However, since A^* is monotone increasing in $\tilde{p}^a$ for fixed $\tilde{p}^r$, it is always possible in such situations to select some $\tilde{p}^a_E < \tilde{p}^a$ such that the optimal solution to the energy-constrained problem with normalized price of arbitrage $\tilde{p}^a$ exactly matches the optimal solution to the energy-unconstrained problem considered in Proposition C.1 with normalized price of arbitrage $\tilde{p}^a_E$. Proposition C.2 formalizes this notion.



(a) $T_L \geq 2T_H$ (b) $T_L \leq 2T_H$

Figure C.5 Illustration of Proposition C.1. The shaded regions denote $x_L^* = x_H^* > 0$ (pink), $x_L^* = x_H^* = 0$ (gray), and $x_L^* > x_H^*$ (white). The subregions within colored lines denote where each property holds: $A_{stk}^* > A_{arb}^*$ (red), $R_{stk}^* > R_{reg}^*$ (yellow), $\frac{R^*}{A^*} \geq 2$ (blue), positive cost-saving (orange), and superlinear profit (purple).

Proposition C.2 (Equivalence of the optimal structure)**.** *The optimal solution of a problem with power constraint P, energy constraint E, normalized regulation price $\tilde{p}^r$, and normalized arbitrage price $\tilde{p}^a$ is exactly the same as the optimal solution to a problem with power constraint P, no enforced energy constraint, normalized regulation price $\tilde{p}^r$, and normalized arbitrage price $\min(\tilde{p}^a, \tilde{p}^a_E)$, for some $\tilde{p}^a_E$ that depends on E/P and $\tilde{p}^r$.*
In particular, when $E/P \geq T_H$, then

$$\tilde{p}^a_E = \infty.$$

When $\frac{T_L}{2} \leq E/P < T_H$ and $T_L < 2T_H$, then

$$\tilde{p}^a_E = 2\tilde{p}^r + 2E/P.$$

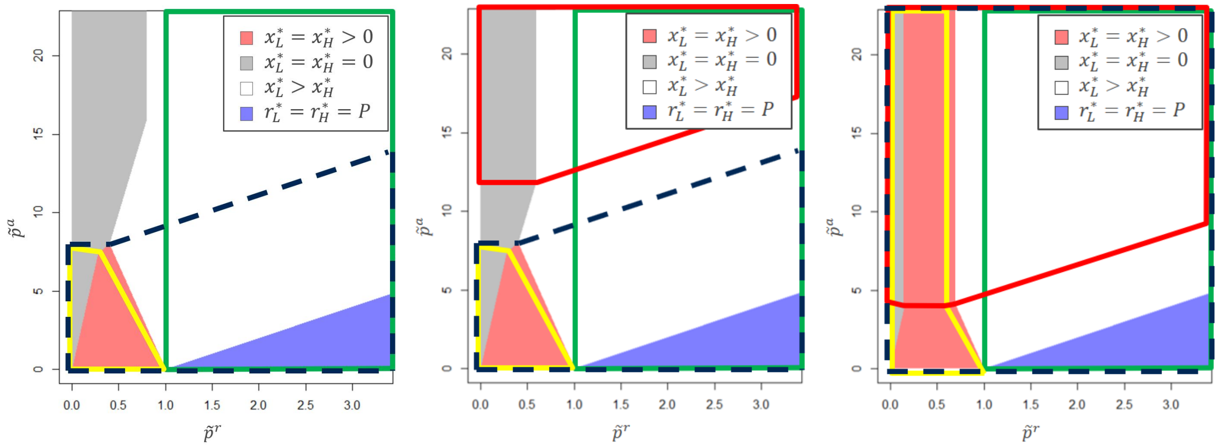
When $\frac{T_H}{2} \leq E/P < \min(\frac{T_L}{2}, T_H)$, then

$$\tilde{p}^a_E = \begin{cases} 2E/P, & \text{if } 0 \leq \tilde{p}^r \leq \frac{2}{T_L} E/P, \text{ and} \\ 2\tilde{p}^r + 2(1 - \frac{2}{T_L})E/P, & \text{if } \tilde{p}^r \geq \frac{2}{T_L} E/P. \end{cases}$$

Finally, when $E/P < \frac{T_H}{2}$, then

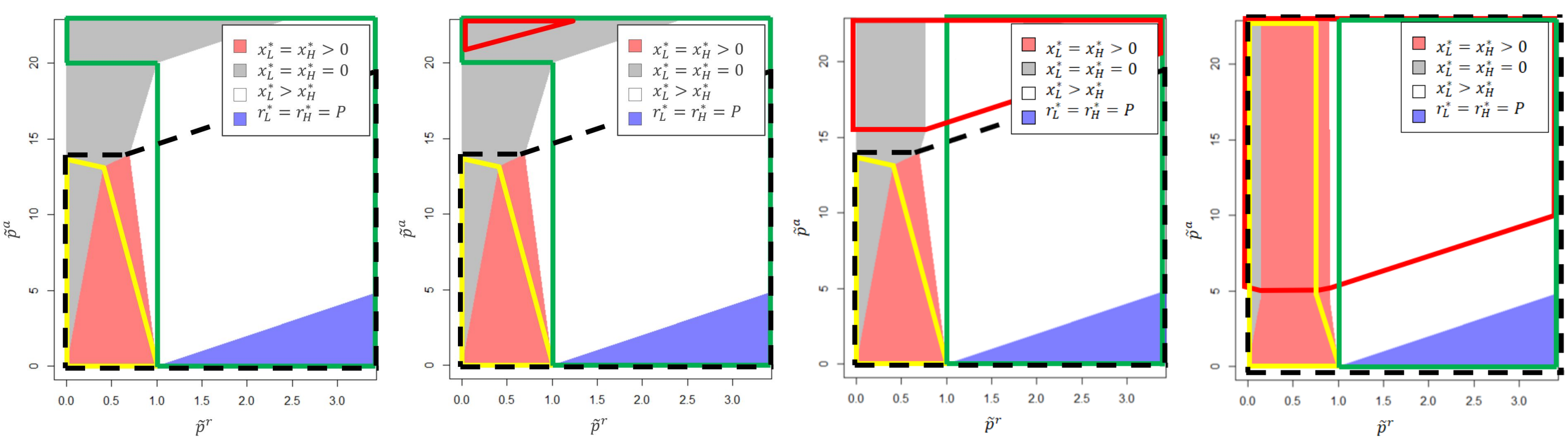
$$\tilde{p}^a_E = \begin{cases} 2E/P - 2\tilde{p}^r, & \text{if } 0 \leq \tilde{p}^r \leq \frac{2}{T} E/P, \\ 2(1 - \frac{2}{T})E/P, & \text{if } \frac{2}{T} E/P \leq \tilde{p}^r \leq 1 - \frac{2T_L}{T_H T} E/P, \\ 2(1 + \frac{2(T_L - T_H)}{T_H^2})E/P - \frac{2T_L(1-\tilde{p}^r)}{T_H}, & \text{if } 1 - \frac{2T_L}{T_H T} E/P \leq \tilde{p}^r \leq 1 - \frac{2(T_L - T_H)}{T_L T_H} E/P, \text{ and} \\ 2(1 - \frac{2}{T_L} + \frac{2}{T_H})E/P - 2(1 - \tilde{p}^r), & \text{if } \tilde{p}^r \geq 1 - \frac{2(T_L - T_H)}{T_L T_H} E/P. \end{cases}$$

Figures C.6 – C.7 characterize optimal battery operations such as when $x_L^* = x_H^*$ vs. $x_L^* > x_H^*$ (etc.) as a function of $\tilde{p}^a$ and $\tilde{p}^r$ for $T_L \geq 2T_H$ and $T_L < 2T_H$, respectively.[41]



(a) $E/P > T_H$ (b) $\frac{T_H}{2} < E/P \leq T_H$ (c) $E/P \leq \frac{T_H}{2}$

Figure C.6 Illustration of optimal battery operations ($T_L \geq 2T_H$). Power is unconstrained within the yellow line, power is constrained in both time periods within the green line, and otherwise, power is constrained only in the high-price period. Energy is constrained within the red line. Maximum free regulation is used in both time periods within the black dashed area. The blue area denotes regulation alone ($A^* = 0$). $T_L = 20$ and $T_H = 8$ are used.



(a) $E/P > T_H$ (b) $\frac{T_L}{2} < E/P \leq T_H$ (c) $\frac{T_H}{2} < E/P \leq \frac{T_L}{2}$ (d) $E/P \leq \frac{T_H}{2}$

Figure C.7 Illustration of the optimal battery operation ($T_L < 2T_H$). Power is unconstrained within the yellow line, power is constrained in both time periods within the green line, and otherwise, power is constrained only in the high-price period. Energy is constrained within the red line. Maximum free regulation is used in both time periods within the black dashed area. The blue area denotes regulation alone ($A^* = 0$). $T_L = 20$ and $T_H = 14$ are used.

[41] Appendix C.4.3 describes comparative statics with respect to other key parameters (e.g., p^a and c^b).

Next, we state the implications of limited power/energy capacity. Clearly, both energy and power capacity affect the optimal structure and the benefit of stacking (e.g., synergistic properties). Figure C.7 illustrates the optimal structure of different energy-to-power ratios for $T_L < 2T_H$, where the bottom of the red outlined region marks $\tilde{p}^a_E$ for a given $\tilde{p}^r$. The key properties of stacking established in Proposition C.1 (e.g., the optimal solution structure, $\frac{R^*}{A^*} \geq 2$, positive cost-saving) then immediately follow by reading the relevant property for $\tilde{p}^a_E$ instead of $\tilde{p}^a$. Further, Figure C.8 illustrates the effect of limited energy capacity by comparing $E/P < \frac{T_H}{2}$ and $E/P > T_H$. Some key observations are as follows:

- A binding energy capacity may limit or expand some of the synergistic behaviors. As shown in Figure C.8, the range for $R^*_{stk} > R^*_{reg}$ becomes smaller under limited energy capacity, while the ranges for $A^*_{stk} > A^*_{arb}$ and positive cost-saving become expanded.
- Relative benefits of stacking can be stronger under a binding energy capacity: Interestingly, stacking's performance relative to standalone services can be stronger under a binding energy capacity as opposed to non-binding capacity. This is exemplified in the expanded regions where stacking achieves superlinear profit improvement: the purple regions ($\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$) are larger when $E/P < \frac{T_H}{2}$ compared to the unconstrained setting as shown in Figure C.8. Intuitively, when arbitrage prices are high, the limited energy capacity constrains standalone arbitrage more than it constrains stacking since stacking has the option to shift to the regulation service.

C.4.2. Proof of Lemma C.3 and Propositions C.1 – C.2

We will use regions and notations introduced in Lemma C.3.

Proof of Lemma C.3:

We begin by deriving $\Pi(A)$, the optimal profit as a function of fixed arbitrage amount A, for different ranges of A values. Given this characterization, we determine the conditions that yield the various properties of interest.

Region 1: $0 \leq A \leq PT_H/2$

For $A \leq PT_H/2$, both the low-price and high-price periods have sufficient power to provide the full amount of free regulation; at optimality both will do so. Defining the total amount of excess regulation sold to the grid as $X := T_L x_L + T_H x_H$ and observing that we sell $R = 2A + X$ total regulation, note that the battery generates a profit from the grid of $(p^a + 2p^r)A + p^r X$. It is straightforward that the degradation-minimizing way to provide X amount of excess regulation is to provide it evenly between the two periods ($x_L = x_H = X/T$) if doing so does not violate the power constraint ($X \leq T(P - 2a_H)$), and it's otherwise optimal to provide the maximum physically

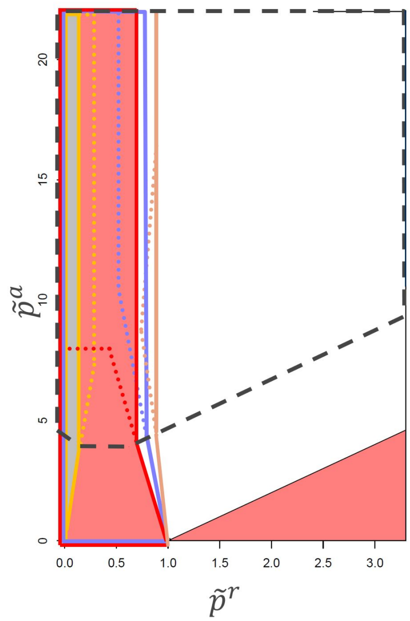


Figure C.8 Illustration of Propositions C.1 – C.2 ($T_L \geq 2T_H$). The limited energy capacity shifts the range where each of the properties holds, as seen by comparing the regions in solid lines ($E/P < \frac{T_H}{2}$) and the regions in dotted lines (unconstrained setting). Colors denote properties $A^*_{stk} > A^*_{arb}$ (red), $R^*_{stk} > R^*_{reg}$ (yellow), positive cost-saving (orange), and superlinear profit (purple). Energy is constrained within the black dashed line for the $E/P < \frac{T_H}{2}$ case.

possible excess regulation in the high-price period and the remainder in the low-price period ($x_H = P - 2a_H,\ x_L = X/T_L - T_H P/T_L + 2a_L$). Allocating the excess regulation in this way yields the following optimal profit for fixed A and X:

$$\Pi(A,X) = \begin{cases} c_1 + (p^r - \frac{2c_0^b A}{TE})X - \frac{c_0^b}{2TE}X^2, & \text{if } 0 \leq X \leq T(P - \frac{2A}{T_H}) \\ c_2 + (p^r - \frac{c_0^b(4A - T_H P)}{T_L E})X - \frac{c_0^b}{2T_L E}X^2, & \text{if } T(P - \frac{2A}{T_H}) \leq X \leq TP - 4A \end{cases}, \text{ for}$$

$$c_1 = (p^a + 2p^r)A - \frac{c_0^b}{E}A^2 \text{ and}$$

$$c_2 = (p^a + 2p^r)A - \frac{c_0^b}{E}[A^2(1 + \frac{6}{T_L} + \frac{2}{T_H}) + \frac{T_H T P^2}{2T_L} - \frac{2P(2T_H + T_L)}{T_L}A].$$

For fixed A, $\Pi(A,X)$ is a continuous, concave function. Its derivative is a piecewise linear, strictly decreasing function with a discontinuity at $X = T(P - 2a_H)$:

$$\Pi'(A,0) = p^r - \frac{2c_0^b A}{TE},$$

$$\Pi'_-(A, T(P - 2a_H)) = p^r - \frac{c_0^b}{E}P + 2\frac{c_0^b}{E}A(\frac{1}{T_H} - \frac{1}{T})),$$

$$\Pi'_+(A, T(P - 2a_H)) = p^r - \frac{c_0^b}{E}P + 2\frac{c_0^b}{E}A(\frac{1}{T_H} - \frac{1}{T_L})), \text{ and}$$

$$\Pi'(A, TP - 4A) = p^r - \frac{c_0^b}{E}P.$$

When $p^r \geq \frac{c_0^b}{E}P$ ($\tilde{p}^r \geq 1$), then $X^* = TP - 4A$ and the maximum possible regulation is provided in both time periods; otherwise, a smaller amount is used. Using $0 \leq A \leq PT_H/2$, we have $p^r - \frac{c_0^b PT_H}{TE} \leq \Pi'(A,0) \leq p^r$, $p^r - \frac{c_0^b}{E}P \leq \Pi'_-(A, T(P-2a_H)) \leq p^r - \frac{c_0^b PT_H}{TE}$, and $p^r - \frac{c_0^b}{E}P \leq \Pi'_+(A, T(P-2a_H)) \leq p^r - \frac{c_0^b PT_H}{T_L E}$.

Signing $\Pi'(A,X)$ at these points yields a complete characterization of X^* given a fixed A:

$$X^* = \begin{cases} 0 & \text{if } (c1): \ \tilde{p}^r \leq \frac{T_H}{T}, A \geq \frac{Tp^r E}{2c_0^b}, \\ \frac{p^r TE}{c_0^b} - 2A & \text{if } (c2): \ \tilde{p}^r \leq \frac{T_H}{T}, A < \frac{Tp^r E}{2c_0^b} \text{ or } \frac{T_H}{T} < \tilde{p}^r < 1, A < \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T)}, \\ T(P - 2\frac{A}{T_H}) & \text{if } (c3): \ \frac{T_H}{T} < \tilde{p}^r \leq \frac{T_H}{T_L}, A \geq \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T)} \text{ or } \frac{T_H}{T_L} < \tilde{p}^r < 1, \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T)} \leq A \leq \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T_L)}, \\ \frac{p^r T_L E}{c_0^b} + PT_H - 4A & \text{if } (c4): \ \frac{T_H}{T_L} < \tilde{p}^r < 1, A > \frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T_L)}, \\ TP - 4A & \text{if } (c5): \ \tilde{p}^r \geq 1 \end{cases}$$

Each of these conditions has a natural interpretation. In $(c1)$, only free regulation is used in both time periods ($x_L^* = x_H^* = 0$), and neither time period has power binding for $A < PT_H/2$. In $(c2)$, an equal, positive amount of excess regulation is used in both time periods ($x_L^* = x_H^* > 0$), and neither time period has power binding. In $(c3)$, an equal, positive amount of excess regulation is used in both time periods ($x_L^* = x_H^* > 0$), and power is binding in the high-price time period (but not in the low-price period, unless $T_L = T_H$ or $A = 0$). In $(c4)$, an unequal amount of excess regulation is used ($x_L^* > x_H^*$), and power is binding in the high-price period but not in the low-price period. In $(c5)$, the maximum physically possible excess regulation is provided, resulting in binding power in both time periods (in this case $x_L^* > x_H^*$, except when $T_L = T_H$ or $A = 0$).

Given this characterization, we can state $\Pi(A)$, the optimal profit for fixed A:

$$\Pi(A) = \begin{cases} (p^a + 2p^r)A - \frac{c_0^b}{E}A^2, & \text{if } (c1), \\ \frac{T(p^r)^2 E}{2c_0^b} + p^a A - \frac{c_0^b}{E}(1 - \frac{2}{T})A^2, & \text{if } (c2), \\ TP(p^r - \frac{c_0^b P}{2E}) + (p^a + \frac{2T_L(\frac{c_0^b}{E}P - p^r)}{T_H})A - \frac{c_0^b}{E}(1 + \frac{2(T_L - T_H)}{T_H^2})A^2, & \text{if } (c3), \\ (\frac{(p^r)^2 T_L E}{2c_0^b} + p^r T_H P - \frac{c_0^b T_H P^2}{2E}) + (p^a - 2p^r + 2\frac{c_0^b}{E}P)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L} + \frac{2}{T_H})A^2, & \text{if } (c4), \\ TP(p^r - \frac{c_0^b P}{2E}) + (p^a - 2p^r + 2\frac{c_0^b}{E}P)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L} + \frac{2}{T_H})A^2, & \text{if } (c5). \end{cases}$$

Region 2: $PT_H/2 < A \leq \min(PT_H, PT_L/2)$

In this region, the high-price time period does not have enough power to use all of its free regulation, while the low-price period has enough power for all of its free regulation. At optimality both will use as much free regulation as possible. Then $r_L = a_L + x_L$ and $r_H = P - a_H$; if we define $X = T_L x_L$ to be the amount of excess regulation sold to the grid, then $R = T_L r_L + T_H r_H = X + PT_H$. This means that the total profit from the grid is $p^a A + p^r(X + PT_H)$. The optimal profit for fixed A and $0 \leq X \leq PT_L - 2A$ immediately follows:

$$\Pi(A,X) = (p^r PT_H + p^a A - \frac{c_0^b}{E}A^2) + (p^r - \frac{2c_0^b A}{T_L E})X - \frac{c_0^b}{2T_L E}X^2.$$

For fixed A, $\Pi(A,X)$ is concave, with derivatives $\Pi'(A,0) = p^r - \frac{2c_0^b A}{T_L E}$ and $\Pi'(A, PT_L - 2A) = p^r - \frac{c_0^b}{E}P$. From $PT_H/2 < A \leq \min(PT_H, PT_L/2)$, we have $\max(p^r - \frac{2c_0^b PT_H}{T_L E}, p^r - \frac{c_0^b}{E}P) \leq \Pi'(A,0) < p^r - \frac{c_0^b PT_H}{T_L E}$. Signing $\Pi'(A,X)$ at these points yields a complete characterization of X^* given a fixed A:

$$X^* = \begin{cases} 0, & \text{if } (c6): \tilde{p}^r \leq \frac{T_H}{T_L} \text{ or } \frac{T_H}{T_L} < \tilde{p}^r < \min(\frac{2T_H}{T_L}, 1), A \geq \frac{T_L p^r E}{2c_0^b}, \\ \frac{T_L p^r E}{c_0^b} - 2A, & \text{if } (c7): T_L > 2T_H, \frac{2T_H}{T_L} \leq \tilde{p}^r < 1 \text{ or } \frac{T_H}{T_L} < \tilde{p}^r < \min(\frac{2T_H}{T_L}, 1), A < \frac{T_L p^r E}{2c_0^b}, \\ PT_L - 2A, & \text{if } (c8): \tilde{p}^r \geq 1. \end{cases}$$

Again, each condition has a natural interpretation. In $(c6)$, only free regulation is used in both time periods ($x_L^* = x_H^* = 0$), and the high-price period has power binding (the low-price period is not power binding for $A < TP_L/2$). In $(c7)$, excess regulation is provided in the low-price period alone ($x_L^* > x_H^* = 0$); the high-price period has power binding but the low-price period does not. In $(c8)$, the maximum physically possible excess regulation is provided ($x_L^* > x_H^* = 0$), resulting in binding power in both time periods.

Given this characterization, we can state $\Pi(A)$, the optimal profit for fixed A:

$$\Pi(A) = \begin{cases} p^r PT_H + p^a A - \frac{c_0^b}{E}A^2, & \text{if } (c6), \\ (p^r PT_H + \frac{T_L (p^r)^2 E}{2c_0^b}) + (p^a - 2p^r)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L})A^2, & \text{if } (c7), \\ (p^r PT - \frac{c_0^b P^2 T_L}{2E}) + (p^a - 2p^r)A - \frac{c_0^b}{E}(1 - \frac{2}{T_L})A^2, & \text{if } (c8). \end{cases}$$

Region 3: $PT_L/2 < A \leq PT_H$

In this region (c9), which only occurs when $T_L < 2T_H$, the battery does not have sufficient power to use all available free regulation in either time period. At optimality, the battery will use its full power to provide free regulation, meaning $r_L = P - a_L$, $r_H = P - a_H$, and $R = PT - 2A$. This means the total profit from the grid is $(p^a - 2p^r)A + p^r PT$, and the optimal profit for fixed A is

$$\Pi(A) = p^r PT + (p^a - 2p^r)A - \frac{c_0^b}{E}A^2, \qquad \text{if } (c9).$$

Given the optimal X^* and R^* under fixed A for conditions $(c1)$ – $(c9)$, it is straightforward to establish the properties listed in Table C.1.

□

Proof of Proposition C.1:

Optimal excess regulation and power constraints

(1) Properties of the optimal solution: $\tilde{p}^r \leq \frac{T_H}{T}$

When $\tilde{p}^r \leq \frac{T_H}{T}$, $\Pi(A)$ follows condition $(c2)$ for $0 \leq A < \frac{Tp^r E}{2c_0^b}$, condition $(c1)$ for $\frac{Tp^r E}{2c_0^b} \leq A \leq \frac{PT_H}{2}$, condition $(c6)$ for $\frac{PT_H}{2} < A \leq \min(PT_H, \frac{PT_L}{2})$, and condition $(c9)$ for $T_L < 2T_H, \frac{PT_L}{2} < A \leq PT_H$.

By inspection, $\Pi(A)$ is a concave function with a piecewise linear, decreasing derivative that has discontinuities at $A=\frac{PT_H}{2}$ and $A=\frac{PT_L}{2}$. Further,

$$\begin{aligned}
\Pi'(0) &= p^a,\\
\Pi'(\frac{Tp^rE}{2c_0^b}) &= p^a-(T-2)p^r,\\
\Pi'_-(\frac{PT_H}{2}) &= p^a+2p^r-\frac{c_0^b}{E}PT_H,\\
\Pi'_+(\frac{PT_H}{2}) &= p^a-\frac{c_0^b}{E}PT_H,\\
\Pi'_-(\frac{PT_L}{2}) &= p^a-\frac{c_0^b}{E}PT_L,\\
\Pi'_+(\frac{PT_L}{2}) &= p^a-2p^r-\frac{c_0^b}{E}PT_L, \text{ and}\\
\Pi'(PT_H) &= \begin{cases} p^a-2\frac{c_0^b}{E}PT_H, & T_L\ge 2T_H\\ p^a-2p^r-2\frac{c_0^b}{E}PT_H, & T_L<2T_H \end{cases}.
\end{aligned}$$

(2) Properties of the optimal solution: $\frac{T_H}{T}<\tilde{p}^r\le\frac{T_H}{T_L}$

When $\frac{T_H}{T}<\tilde{p}^r\le\frac{T_H}{T_L}$, $\Pi(A)$ follows condition ($c2$) for $0\le A<\frac{\frac{c_0^b}{E}P-p^r}{2\frac{c_0^b}{E}(1/T_H-1/T)}$, condition ($c3$) for $\frac{\frac{c_0^b}{E}P-p^r}{2\frac{c_0^b}{E}(1/T_H-1/T)}\le A\le\frac{PT_H}{2}$, condition ($c6$) for $\frac{PT_H}{2}<A\le\min(PT_H,\frac{PT_L}{2})$, and condition ($c9$) for $T_L<2T_H, \frac{PT_L}{2}<A\le PT_H$. By inspection, $\Pi(A)$ is a concave function with a piecewise linear, decreasing derivative that has discontinuities at $A=\frac{PT_H}{2}$ and $A=\frac{PT_L}{2}$. Further,

$$\begin{aligned}
\Pi'(0) &= p^a,\\
\Pi'(\frac{\frac{c_0^b}{E}P-p^r}{2\frac{c_0^b}{E}(1/T_H-1/T)}) &= p^a-\frac{(T-2)T_H}{T_L}(\frac{c_0^b}{E}P-p^r),\\
\Pi'_-(\frac{PT_H}{2}) &= p^a-\frac{2T_L}{T_H}p^r-\frac{c_0^b}{E}P(T_H-2),\\
\Pi'_+(\frac{PT_H}{2}) &= p^a-\frac{c_0^b}{E}PT_H, \text{ and}\\
\Pi'_-(\frac{PT_L}{2}) &= p^a-\frac{c_0^b}{E}PT_L.
\end{aligned}$$

$\Pi'_+(\frac{PT_L}{2})$ and $\Pi'(PT_H)$ are the same as the previous case ($\tilde{p}^r\le\frac{T_H}{T}$).

(3) Properties of the optimal solution: $\frac{T_H}{T_L}<\tilde{p}^r<1$

When $\frac{T_H}{T_L}<\tilde{p}^r<1$ (which implies $T_L>T_H$), $\Pi(A)$ follows condition ($c2$) for $0\le A<\frac{\frac{c_0^b}{E}P-p^r}{2\frac{c_0^b}{E}(1/T_H-1/T)}$, condition ($c3$) for $\frac{\frac{c_0^b}{E}P-p^r}{2\frac{c_0^b}{E}(1/T_H-1/T)}\le A\le\frac{\frac{c_0^b}{E}P-p^r}{2\frac{c_0^b}{E}(1/T_H-1/T_L)}$, and condition ($c4$) for $\frac{\frac{c_0^b}{E}P-p^r}{2\frac{c_0^b}{E}(1/T_H-1/T_L)}<A\le\frac{PT_H}{2}$. When $T_L\le 2T_H$ or $\frac{T_H}{T_L}<\tilde{p}^r<\frac{2T_H}{T_L}$, then $\Pi(A)$ follows condition ($c7$) for $\frac{PT_H}{2}<A<\frac{T_Lp^rE}{2c_0^b}$, condition ($c6$) for $\frac{T_Lp^rE}{2c_0^b}\le A\le\min(PT_H,\frac{PT_L}{2})$, and condition ($c9$) for $T_L<$

$2T_H, \frac{PT_L}{2} < A \leq PT_H$. Otherwise $\Pi(A)$ follows condition $(c7)$ for $\frac{PT_H}{2} < A \leq PT_H$. By inspection, $\Pi(A)$ is a concave function with a piecewise linear, decreasing derivative that has a discontinuity at $A = \frac{PT_L}{2}$. Further,

$$\begin{aligned}
\Pi'(0) &= p^a, \\
\Pi'\Big(\frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T)}\Big) &= p^a - \frac{(T-2)T_H}{T_L}\Big(\frac{c_0^b}{E}P - p^r\Big), \\
\Pi'\Big(\frac{\frac{c_0^b}{E}P - p^r}{2\frac{c_0^b}{E}(1/T_H - 1/T_L)}\Big) &= p^a - \frac{T_L T_H}{T_L - T_H}\Big(\frac{c_0^b}{E}P - p^r\Big), \\
\Pi'\Big(\frac{PT_H}{2}\Big) &= p^a - 2p^r - \frac{c_0^b}{E}PT_H\Big(1 - \frac{2}{T_L}\Big), \\
\Pi'\Big(\frac{T_L p^r E}{2c_0^b}\Big) &= p^a - T_L p^r, \text{ and} \\
\Pi'_-\Big(\frac{PT_L}{2}\Big) &= p^a - \frac{c_0^b}{E}PT_L.
\end{aligned}$$

$\Pi'_+(\frac{PT_L}{2})$ and $\Pi'(PT_H)$ are the same as the first case ($\tilde{p}^r \leq \frac{T_H}{T}$).

(4) Properties of the optimal solution: $\tilde{p}^r \geq 1$

When $\tilde{p}^r \geq 1$, $\Pi(A)$ follows condition $(c5)$ for $0 \leq A < \frac{PT_H}{2}$, condition $(c8)$ for $\frac{PT_H}{2} \leq A \leq \min(PT_H, \frac{PT_L}{2})$, and condition $(c9)$ for $T_L < 2T_H, \frac{PT_L}{2} < A \leq PT_H$. $\Pi(A)$ is a concave function with a piecewise linear, decreasing derivative that has a discontinuity at $A = \frac{PT_L}{2}$. Further,

$$\begin{aligned}
\Pi'(0) &= p^a - 2p^r + 2\frac{c_0^b}{E}P, \\
\Pi'\Big(\frac{PT_H}{2}\Big) &= p^a - 2p^r + \frac{c_0^b}{E}PT_H\Big(1 - \frac{2}{T_L}\Big), \text{ and} \\
\Pi'_-\Big(\frac{PT_L}{2}\Big) &= p^a - 2p^r - \frac{c_0^b}{E}P(T_L - 2).
\end{aligned}$$

$\Pi'_+(\frac{PT_L}{2})$ and $\Pi'(PT_H)$ are the same as the first case ($\tilde{p}^r \leq \frac{T_H}{T}$).

Key properties (Proposition C.1)

Most properties follow straightforwardly by combining the results in Table C.1 with the $\Pi'(A)$ results presented in the proof of Proposition C.1. For instance, we read from Table C.1 that stacking uses all available free regulation under conditions $(c1) - (c5)$. When $\tilde{p}^r \leq \frac{T_H}{T}$, this means the property holds for $A^* \leq \frac{PT_H}{2}$, and $\Pi'_+(\frac{PT_H}{2}) \leq 0$ gives the condition for the property to hold. Here, we present a few properties that don't follow immediately from the results in Table C.1.

Positive arbitrage amount ($A^* > 0$): Stacking uses a positive amount of arbitrage whenever $\Pi'(0) > 0$. Since $\Pi'(0) = p^a > 0$ for $\tilde{p}^r < 1$ and $\Pi'(0) = p^a - 2p^r + 2\frac{c_0^b}{E}P$ for $\tilde{p}^r \geq 1$, we conclude that stacking uses a positive amount of arbitrage whenever $\tilde{p}^a > 2(\tilde{p}^r - 1)$.

Superlinear profit gain ($\Pi^*_{stk} > \Pi^*_{arb} + \Pi^*_{reg}$): Superlinear profit gain holds whenever $\Pi(A)$ follows conditions $(c1)$ (with $A < \frac{PT_H}{2}$) or $(c2)$ due to the fact that the stacking solution is not power constrained in such situations (see Table C.1; as a result, superlinear profit follows by Proposition 4). Similarly, superlinear profit gain never holds when $\tilde{p}^r \geq 1$, as established in Lemma B.1.

To establish when superlinear profit gain holds in other scenarios, we exhaustively checked the property under each of the remaining conditions — $(c1)$ with $A = \frac{PT_H}{2}$, $(c3)$, $(c4)$, $(c6)$, $(c7)$, and $(c9)$ with $\tilde{p}^r < 1$. The $\Pi'(A)$ results established thus far show that either A^* occurs at a corner point (e.g., $A^* = \frac{PT_H}{2}$ for $\tilde{p}^r \leq \frac{T_H}{T}$), in which case it holds for a range of p^a and p^r values, or otherwise A^* is a linear combination of p^a and p^r, which can be derived by solving $\Pi'(A) = 0$. In either case, $\Pi^*_{stk} - (\Pi^*_{arb} + \Pi^*_{reg})$ can be written in the form $c_1(p^a)^2 + c_2 p^a p^r + c_3 (p^r)^2 + c_4 p^a + c_5 p^r + c_6$. Establishing cases with superlinear profit gain requires careful applications of the quadratic formula to establish the range of p^a where the property holds. Somewhat tedious enumeration of cases for the optimal solution yields the conditions in Proposition C.1. □

Proof of Proposition C.2:

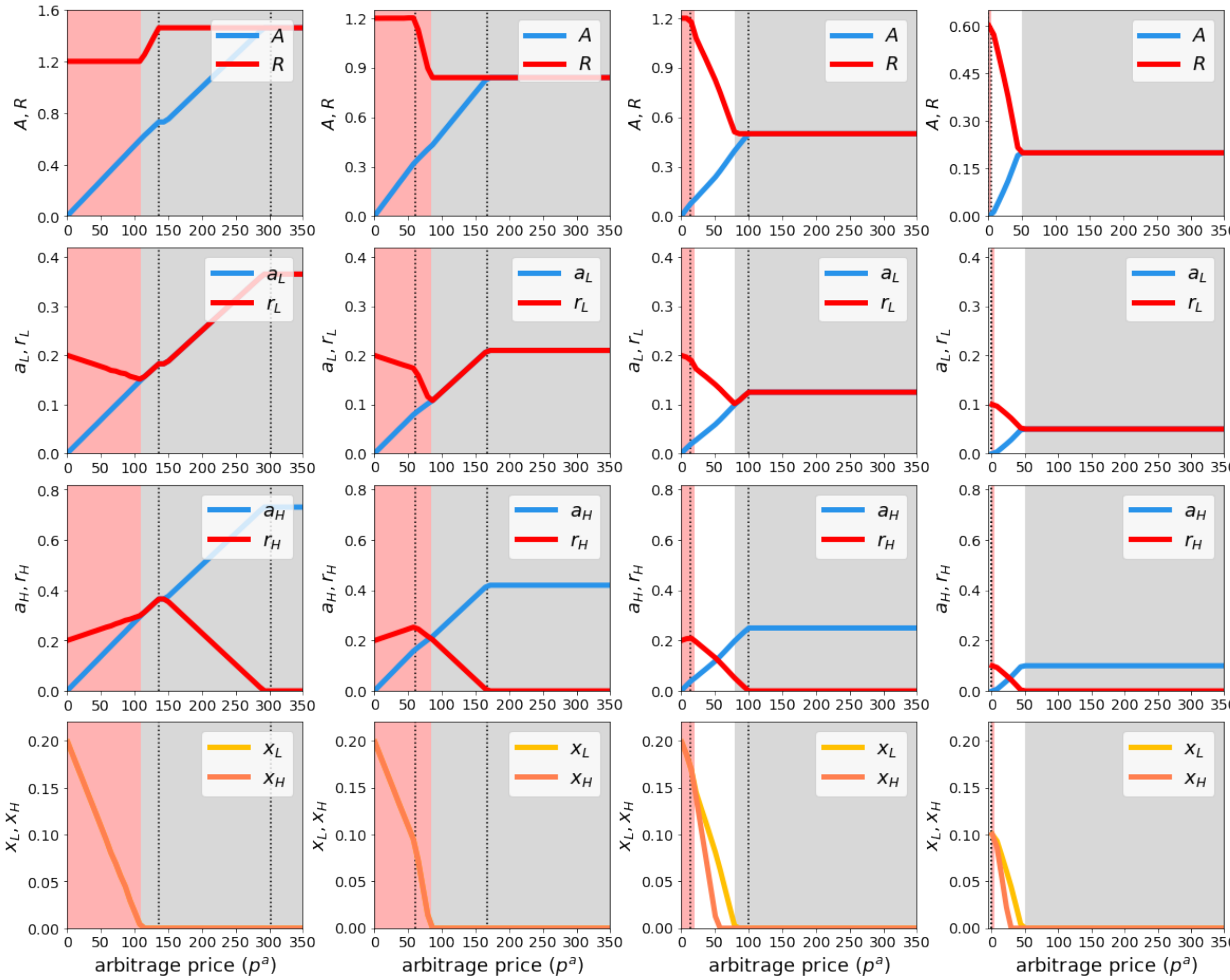
If $E \geq PT_H$, then the energy constraint is never binding, and $\tilde{p}^a_E = \infty$ can be used in the statement of Proposition C.2. Otherwise, $\Pi(E)$ can follow any of the nine conditions $(c1)$ – $(c9)$, and we obtain Proposition C.2's result by setting $\tilde{p}^a_E$ to be a $\tilde{p}^a$ value such that $\Pi'(A) > 0$ for $A < E$ and $\Pi'(A) < 0$ for $A > E$. □

C.4.3. Comparative Statics (Binding Power Capacity) Grid conditions evolve as technologies and policies change. For instance, MonitoringAnalytics (2022) reported that PJM's grid conditions have changed dramatically (the number of active battery storage projects in the PJM increased from 1 to 71 between 2014 and 2020) due to battery-favorable developments such as FERC (Federal Energy Regulatory Commission) Order No. 755. We characterize grid/market conditions using comparative statics, providing forward-looking guidance to decision makers about the effect of two key parameters, arbitrage price p^a and battery cell price c^b, on the optimal mix of arbitrage and regulation that they should provide with their battery.

p^a **effect.** The arbitrage price, p^a, has been changing over time, and these dynamics are expected to continue with increased renewable energy penetration.[42] Figure C.9 illustrates changes in the optimal mix as p^a increases. Depending on a battery's capacity, the optimal strategy can differ dramatically different from the non-binding setting (Figure 4a).

When $\frac{T_H}{T} \leq \tilde{p}^r \leq \frac{T_H}{T_L}$ (Figure C.9b), after the first dotted line ($a^*_H + r^*_H = P$), excess regulation starts to decrease more rapidly than before, which leads to sharp decreases in both r_L and r_H.

[42] California and Texas recently observed increases in p^a as solar generation shapes electricity load into a 'duck curve' — the wholesale energy price has higher prices at the beginning and end of the day and a sharp decrease in prices (including occasionally negative prices) in the middle of the day (Lazard 2021).

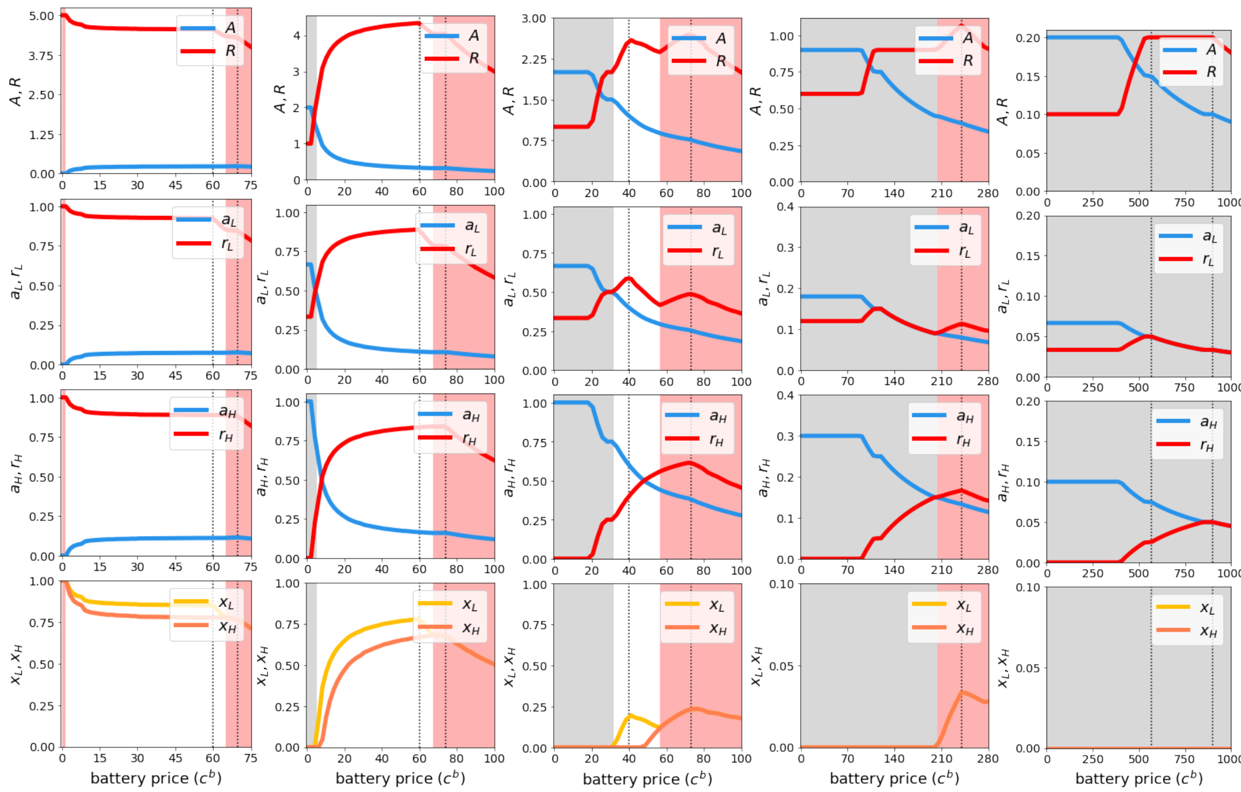


(a) $\tilde{p}^r \leq \frac{T_H}{T}$ (b) $\frac{T_H}{T} \leq \tilde{p}^r \leq \frac{T_H}{T_L}$ (c) $\frac{T_H}{T_L} \leq \tilde{p}^r \leq 1$ (d) $\tilde{p}^r \geq 1$

Figure C.9 Comparative statics (p^a effect). The backgrounds for $x_L^* = x_H^* > 0, x_L^* = x_H^* = 0$, and $x_L^* > x_H^* > 0$ are colored pink, gray, and white, respectively, and the dotted vertical lines show when the power capacity starts to bind in the high-price (left vertical line) and low-price (right vertical line) regions ($T_H = 8, T = 24, E = 2$).

Eventually, r_L starts to increase again when $x_L = x_H = 0$ since all frequency regulation is free, and the amount of arbitrage being provided is increasing. Meanwhile, r_H continues to decrease due to the fact that power is binding in the high-price period, meaning any increase in the arbitrage amount must be matched with a corresponding decrease in regulation.

When $\frac{T_H}{T_L} \leq \tilde{p}^r \leq 1$ (Figure C.9c), on the other hand, there exists a region of p^a at which maintaining unequal excess $x_L^* > x_H^*$ is optimal as shown in non-shaded/white region. This enables a slower decrease in r_L compared to the rate that it would have decreased if $x_L = x_H$ is maintained. Note that performing more regulation (compared to the quantity under $x_L = x_H$) while having unequal excess be clearly beneficial when the regulation price is sufficiently high and the peak is narrow ($\tilde{p}^r \geq \frac{T_H}{T_L}$).



(a) $p^a \le 2p^r$ (b) $2p^r \le p^a \le \frac{2T_L p^r}{T_H}$ (c) $\frac{2T_L p^r}{T_H} \le p^a \le T_L p^r$ (d) $T_L p^r \le p^a \le (T-2)p^r$ (e) $p^a \ge (T-2)p^r$

Figure C.10 Comparative statics (c_0^b effect). The backgrounds for $x_L^* = x_H^* > 0, x_L^* = x_H^* = 0$, and $x_L^* > x_H^* > 0$ are colored pink, gray, and white, respectively, and the dotted vertical lines show when the power capacity starts to bind in the high-price (left vertical line) and low-price (right vertical line) regions.

c_0^b **effect.** Recall that battery price and degradation coefficient (c^b) are proportional. The effect of changing c^b is very relevant, as battery prices have decreased precipitously in recent years. Figure C.10c illustrates how complicated changes in the optimal mix can be as the relationship between excess regulation evolves: r_L changes in an up-down-up-down manner; r_H goes up and down. When the battery price is very high, both arbitrage and regulation increase as c^b decreases (unconstrained solution). However, once c^b is low enough so that power P is binding in the high-price period (right vertical line in Figure C.10c), the optimal profit needs to be carefully adjusted, and it is optimal to maintain equal excess by reducing both x_L and x_H.[43] This highlights that maintaining equal or unequal excess regulation is determined by the full consideration of market conditions (region length: T_H, T_L and service prices: p^a, p^r) and battery specifications (c^b and E/P ratio).

C.4.4. Two Common Models in Literature Due to the iterative computations needed to compute degradation costs using the rainflow-counting algorithm, many papers have resorted to

[43] Note that r_L also should be decreased even though $r_L + a_L < P$ since the battery price is still burdensome. When c^b gets low enough, increasing r_L and having unequal excess becomes optimal as long as $r_L + a_L < P$. Once $r_L + a_L = P$, both regulation should be decreased in order to increase A.

simpler models to estimate degradation costs. Below we describe the two common models in the literature.

No-Degradation Model

Interestingly, most stacking papers ignore the impacts of usage patterns on the degradation cost of batteries. Instead, they consider a fixed capital cost (e.g., up-front investment cost) and assume that a battery's degradation cost is independent of its usage profile (Lazard 2021, Fitzgerald et al. 2015). However, models that ignore cost of degradation may easily overestimate the benefits of stacking, since they do not account for increased battery utilization under stacking.

The no-degradation model maximizes the stacked revenue without consideration of usage-based battery degradation. Then, the objective function can be formulated as follows:

$$\max_{A, r_L, r_H} \quad p^a A + p^r r_L T_L + p^r r_H T_H. \tag{C.6}$$

The optimal action under the no-degradation model can be easily derived:

$$A^* = \begin{cases} 0, & \text{if } p^a \leq 2p^r \\ \min(E, PT_H), & \text{if } p^a > 2p^r \end{cases},$$

$$r_i^* = P - a_i^* = \begin{cases} P, & \text{if } p^a \leq 2p^r \\ P - \frac{\min(E, PT_H)}{T_i}, & \text{if } p^a > 2p^r \end{cases} \text{ for } i = L, H.$$

Under the no-degradation model, we always operate a battery at its maximum power capacity and do either "no" arbitrage or "full" arbitrage. Unsurprisingly, this aggressive usage of batteries may lead to undesirable or even disastrous outcomes (we will see this).

Linear-Degradation Model

The linear-degradation model ($\beta = 1$) has been commonly used in the industry/literature to capture the tradeoff between increased revenue and degradation costs that comes from increased use of a battery. The obvious advantage of the linear-degradation model is its simplicity — under the rainflow-counting algorithm, this is equivalent to assuming that degradation is proportional to the total energy throughput (charging or discharging) of the battery (Hoke et al. 2011, Zhang et al. 2021). While a linear approximation of degradation is methodologically appealing (no need to rely on the iterative calculation of the rainflow-counting algorithm), it may easily lead to poor battery performance due to misrepresenting the actual (nonlinear) degradation process.

The objective function under the linear-degradation model is formulated as follows:

$$\max_{A, r_L, r_H} \quad p^a A + p^r r_L T_L + p^r r_H T_H - c^b_{lin}[A + \frac{T_L}{2} x_L + \frac{T_H}{2} x_H]. \tag{C.7}$$

where c^b_{lin} denotes the battery degradation coefficient under the linear degradation model.[44] Following Shi et al. (2017), the linearized degradation coefficient c^b_{lin} is fitted as follows:

$$c^b_{lin} = \frac{c_{cell} \times E}{Nl \times E} = \frac{c_{cell}}{Nl}, \tag{C.8}$$

where c_{cell} is the battery cell price (\$/MWh), $l \in [0,1]$ is the operable DOD range (as a fraction of the total energy capacity E), and N is the number of cycles that the battery can operate at the specified DOD range. Under the linear-degradation model, some of the dynamics are undesirable:

Lemma C.4 (Linear-degradation)**.** *The optimal regulation rates given a fixed arbitrage amount A satisfy the following:*

- *If $2p^r > c^b_{lin}$, $r^*_i(A) = P - a_i,\ i = L, H$.*
- *If $2p^r \le c^b_{lin}$, $r^*_i(A) = \min(P - a_i, a_i),\ i = L, H$. Thus, $x^*_L = x^*_H = 0$.*

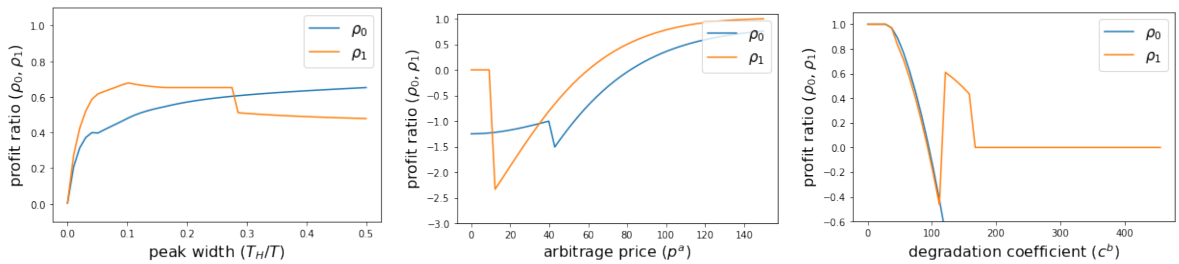
Proof. For each time period $i \in \{L, H\}$, $\frac{\partial \Pi}{\partial r_i} = \begin{cases} p^r T_i, & \text{if } r_i < a_i \\ (2p^r - c^b_{lin})\frac{T_i}{2}, & \text{if } r_i > a_i \end{cases}$. If $c^b_{lin} < 2p^r$, then it is always profitable to increase regulation, so we use the maximum physically possible regulation in each time period. Otherwise, only free regulation is profitable, so we use the maximum physically possible free regulation in each time period. □

As stated in Section 2, a significant portion of the existing literature disregards the cost of degradation or uses a simplified (linearized) degradation function, and thus fails to capture the true impact of stacking on batteries' degradation costs. As we shall see, these models rely heavily on a battery's power or energy capacity; since stacking usually leads to increased battery usage, disregarding or incorrectly modeling degradation costs may lead to overuse of batteries. In this section and later in Appendix D, we numerically illustrate the effect of misspecification of degradation costs on stacking's performance.

Figure C.11 quantifies the performance of the two misspecified models across various values for the peak width $\frac{T_H}{T}$, arbitrage price p^a, and degradation coefficient c^b. In the figure, $\rho_0 = \frac{\Pi^*_{no}}{\Pi^*_{quad}}$ and $\rho_1 = \frac{\Pi^*_{lin}}{\Pi^*_{quad}}$ denote the portion of the optimal stacking profit that is achieved in the misspecified models, where Π^*_{quad} is the optimal stacking profit under the correct model specification and Π^*_{lin} and Π^*_{no} are the (true) stacking profits under the optimal solutions to the two misspecified models.

As expected, ρ_0 can be negative when either c^b is high or service prices are low. Interestingly, it is possible that $\rho_0 > \rho_1$ or $\rho_0, \rho_1 < 0$. For example, when $2p^r \le c^b_{lin}$, the linear model may be too conservative and not use the battery at all, whereas the no-degradation model's (over-)optimism

[44] Typically, c^b_{lin} is calibrated over a limited SOC range where linear fitting works relatively well, but at the sacrifice of batteries' maximum usability. For example, Shi et al. (2017) assumes the linear degradation function by restricting the operable SOC region to 0–70% of energy capacity (the maximum DOD is upper bounded by 70% to avoid deep-cycle effects).



(a) peak width $\frac{T_H}{T}$ (b) arbitrage price p^a (c) degradation coefficient c_0^b

Figure C.11 Stacking performance under the two misspecified models ($\rho_0 = \frac{\Pi^*_{no}}{\Pi^*_{quad}}, \rho_1 = \frac{\Pi^*_{lin}}{\Pi^*_{quad}}$). (a): $E = 1, P = 2, p^r = 10, p^a = 80, c_0^b = 150, c_{lin}^b = 75$**, (b):** $E = 1, P = 2, T_H = 4, T = 10, p^r = 20, c_0^b = 150, c_{lin}^b = 75$**, (c):** $E = 1, P = 0.25, T_H = 12, T = 40, p^r = 10, p^a = 60$**, and** $l = 0.5$ **are used as parameters.**

lets the battery operate, generating positive profit in some cases. On the other hand, a c_{lin}^b that is too small underestimates the true battery degradation coefficient c^b, which may lead to overusage of a battery and $\rho_1 < 0$. In the engineering literature on stacking, as stated in Section 2.2, increased utilization and diversified revenues are often assumed to drive the economic feasibility of stacking. The comparative statics above highlight the importance of correctly characterizing the degradation costs, and that two common models from the industry/literature may result in a significant suboptimality.[45]

C.5. Details of Table 1

To summarize our modeling extensions, Table C.2 provides specific conditions (for simplicity, we exclusively compare cases when energy capacity is non-binding) required for the key properties established for the base model in Section 5 to continue to hold.

[45] Xu et al. (2018b) raises similar issues: the no-degradation model achieves negative profits (the battery is so overused that it must be replaced within 1-3 years), while the linear-degradation model tends to be too conservative, making the battery remain idle most of the time.

Category	Properties	Base model	Correlated prices	Inefficiency $\eta \in (\eta_{stk}, 1]$	Trinary signal	$P, E < \infty$
Synergy	A^* increases in p^r	$p^a \geq (T-2)p^r$	inc. in k $(k \leq \frac{1}{T_H-1})$, inc. in p_0^r $(k \leq \frac{p^a-(T-2)p_0^r}{(T_H-1)\bar{p}+(T_L-1)\underline{p}})$	$p^a \geq \bar{d}\tilde{p}^x$	$p^a \geq \frac{T-2\alpha}{\alpha}p^r$	$(T-2)\tilde{p}^r \leq \tilde{p}^a \leq -2\tilde{p}^r + T_H$, $\tilde{p}^r < \frac{T_H}{T}^{\star}$
	R^* increases in p^a	$p^a \geq (T-2)p^r$	$k \leq \frac{p^a-(T-2)p_0^r}{(T_H-1)\bar{p}+(T_L-1)\underline{p}}$	$p^a \geq \bar{d}\tilde{p}^x$, dec. in $\underline{p}$	$p^a \geq \frac{T-2\alpha}{\alpha}p^r$	$(T-2)\tilde{p}^r \leq \tilde{p}^a \leq -2\tilde{p}^r + T_H$, $\tilde{p}^r < \frac{T_H}{T}$
	$A^*_{stk} > A^*_{arb}$	always	$k < \frac{2}{T_H}$	$p^a \geq \bar{d}\tilde{p}^x$ or $\eta > \hat{\eta}_A, p^a \leq \bar{d}\tilde{p}^x$	$p^a > \frac{T(1-\alpha)}{\alpha}p^r$	Prop C.1 – C.2
	$R^*_{stk} > R^*_{reg}$	$p^a > (T-2)p^r$	$k \leq \frac{p^a-(T-2)p_0^r}{(T_H-1)\bar{p}+(T_L-1)\underline{p}}$	$p^a \geq \bar{d}\tilde{p}^x$ and $\eta > \eta^*$ or $\eta < \hat{\eta}_R$	$p^a > (\frac{T}{\alpha^2} - 2)p^r$	Prop C.1 – C.2
	$A^* > 0$	always	$k < 1$	$\eta \geq \eta^*, p^a \geq d_1\tilde{p}^x$ or $\eta < \eta^*, p^a \geq d_2\tilde{p}^x$	$p^a > \frac{2(1-\alpha)}{\alpha}p^r$	$\tilde{p}^a > 2(\tilde{p}^r - 1)$
Optimal structure	$x^*_L = x^*_H$	always	$k \leq \frac{1}{T_H-1}$	$\eta \geq \eta^*$ or $p^a \geq \bar{d}\tilde{p}^x$	always	Prop C.1 – C.2
	$\frac{R^*}{A^*}$ indpt. of c_0^b	always	always	always	always	dependent
	$\frac{R^*}{A^*} \geq 2$	always	always	always	always	Prop C.1 – C.2
Benefit	$\max S$ (large T)	2	$2 + \frac{1}{2T_H-1}$	∞	2	2
	Cost-saving synergy	always	$k < \frac{5T_L+T_H}{(4T_H-3)T_L+T_H}$	$\eta \geq \eta^*$ or $p^a \geq \bar{d}\tilde{p}^x$	$p^a > \frac{2(1-\alpha)}{\alpha}p^r$	Prop C.1 – C.2
	Superlinear profit	always	$k \leq \frac{2}{2T_H-1}$ or $k < \frac{T_L-T_H+\sqrt{T(T-2)}}{2T_LT_H-T}$, $p^a \leq \frac{B-\sqrt{B^2-AC}}{A}^{\dagger}$	suff. large p^a	$p^a > p^a_\alpha$	Prop C.1 – C.2

Table C.2 Key properties under the extensions considered (when c_0^b is sufficiently large).

†: See Corollary C.1 for the definition of constants A, B, and C;
‡: See the definition of constants from Theorem C.4 ($\bar{d}, d_1, d_2, d_3$, and η^*) and Corollary C.4 ($\hat{\eta}_A$, and $\hat{\eta}_R$);
*: $\tilde{p}^a = \frac{p^a E}{c_0^b P}$ and $\tilde{p}^r = \frac{p^r E}{c_0^b P}$ from Lemma C.3.

D. Numerical Results

In this section, we numerically check the sensitivity of our results to key modeling assumptions and evaluate the performance of stacking under various plausible situations, with the goal of establishing the realistically achievable benefits of stacking under current grid conditions and of confirming that the synergy properties predicted by our model continue to hold under more realistic data.

Section D.1 explains the operational dataset used for the numerical study. Section D.2 constructs comprehensive optimization problems for different operations (standalone services, time-exclusive stacking, and simultaneous stacking) and calibrates model parameters to reflect realistic grid/market circumstances and batteries' characteristics. Section D.3 reports the numerical results including illustrations of actual dispatching profiles (Section D.3.1) and extensive profitability analysis (Section D.3.2).

D.1. Dataset

D.1.1. Day-Ahead Energy/Regulation Price Dataset

Figure D.1 shows price patterns retrieved from the 2022 NYISO operational dataset (NYISO 2022). Note that the energy prices exhibit distinctive seasonality. During summer, energy prices

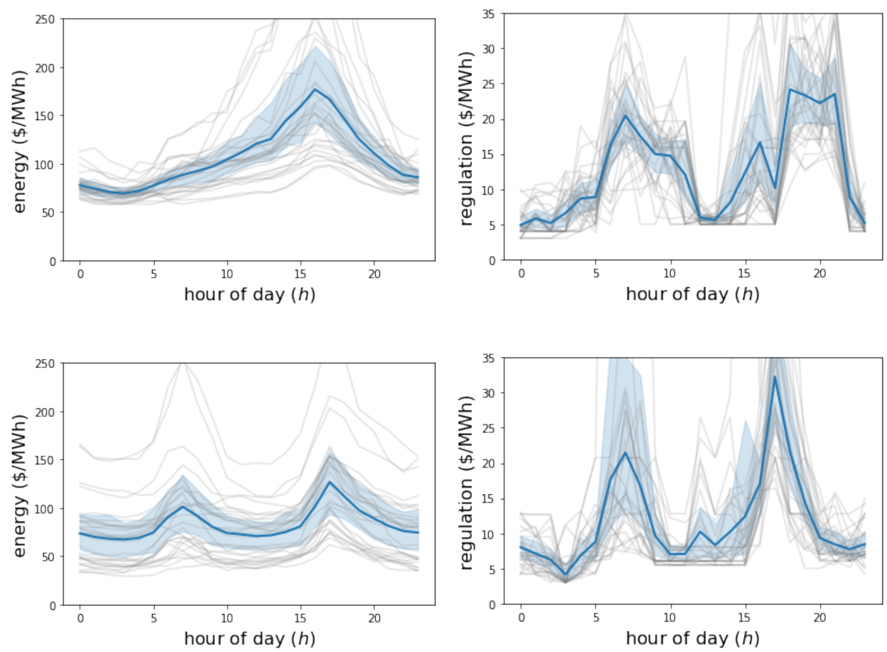


Figure D.1 Day-ahead (hourly) energy and regulation prices: (top) summer, (bottom) winter. Gray lines represent realized price patterns. The blue solid line and blue region specify the average prices and confidence interval, respectively (source: NYISO 2022).

tend to have a single peak per day quite consistently at 3 – 5 pm. During winter, energy prices tend to have two peaks consistently at 5 – 7 am and 5 – 7 pm. Despite the price uncertainty, the timing of peak price periods is quite predictable, which confirms the predictability of peak timing in Assumption 1. Regulation prices also have two peaks, which are correlated with price peaks during winter. Previous years (2019 – 2021) show similar patterns but have smaller price uncertainty compared to the year 2022. As detailed in Appendix D.2.2, we calibrated price parameters based on the historical price dataset.

D.1.2. Regulation Signal Dataset

As stated in Appendix A.2, the energy-neutrality guarantee level may differ across regulation signals and ISOs. Figure D.2 illustrates three different regulation signals for a single day from the ISO-NE simulated automatic generator control (AGC) dataset (ISO-NE 2014) and the PJM regulation signal dataset (Jan 1st, 2020; PJM 2020), along with the corresponding net energy change assuming a unit capacity commitment. As detailed in Appendix D.2.2, we calibrated the characteristics of regulation signals based on the historical signal dataset.

D.2. Hourly Optimization Models

In numerical experiments, we formulate and solve optimization problems for each operation (standalone services, time-exclusive stacking, and simultaneous stacking), using realistic grid/market

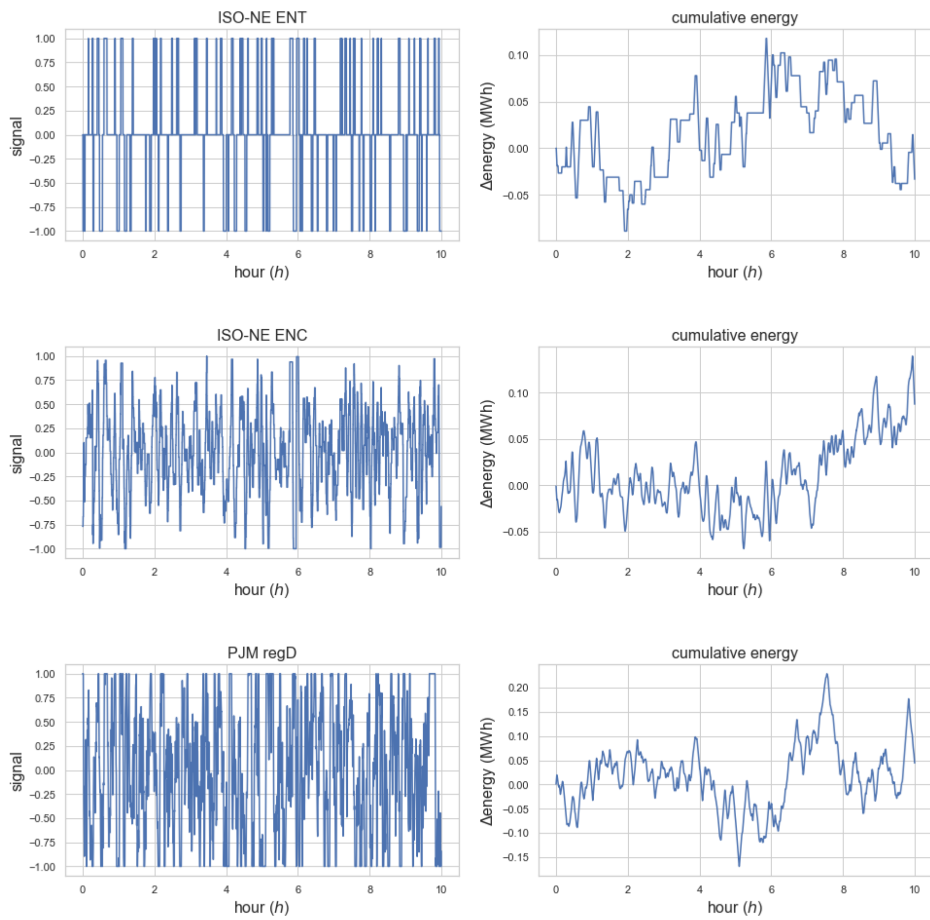


Figure D.2 Regulation signals and the corresponding net energy change under a unit capacity commitment: (top) ISO-NE trinary; (middle) ISO-NE continuous; (bottom) PJM regD.

parameters and battery characteristics. In Appendix D.2.1, we first formulate an hourly optimization problem with intra-day price non-stationarity (expected hourly service prices p_i^e for energy and p_i^r for regulation), dispatch-to-contract ratio α, energy-neutrality (we assume energy neutrality will be restored within $2\Delta^c$ time units), battery power P, battery energy E, and battery round-trip efficiency η^2. We then evaluate the performance of the optimized schedule using real grid data (price profiles and regulation signals).

D.2.1. Hourly Optimization Model Formulation for Simultaneous Stacking

$$\max_{a_i,r_i,I_i,J_i,S_{max},S_{min},D} \quad -\sum_{i=1}^{T_L} p_i^e a_i + \sum_{i=T_L+1}^{T} p_i^e a_i + \sum_{i=1}^{T} p_i^r r_i - \frac{c_0^b}{E}\left[D^2 + \sum_{i=1}^{T} \frac{(x_i\Delta^c)^2}{2\Delta^c}\right] \tag{D.1}$$

$$x_i = \begin{cases} \alpha(r_i - a_i)^+/\eta, & \forall i = 1,\cdots,T_L \\ \alpha(r_i - a_i)^+\eta, & \forall i = T_L+1,\cdots,T \end{cases} \tag{D.2}$$

$$S_1 = S_{T+1} \tag{D.3}$$

$$S_{i+1} = S_i \tag{D.4}$$

$$+\begin{cases} I_i[\frac{\alpha}{2}[a_i(1/\eta+\eta)-r_i(1/\eta-\eta)]+(1-\alpha)a_i\eta]+J_i[a_i\eta], & \forall i=1,\cdots,T_L \\ I_i[-\frac{\alpha}{2}[a_i(1/\eta+\eta)+r_i(1/\eta-\eta)]-(1-\alpha)a_i/\eta]+J_i[-a_i/\eta], & \forall i=T_L+1,\cdots,T \end{cases}$$

$$r_i - a_i \le MI_i,\ \forall i=1,\cdots,T \tag{D.5}$$

$$a_i - r_i \le MJ_i,\ \forall i=1,\cdots,T \tag{D.6}$$

$$I_i + J_i = 1,\ \forall i=1,\cdots,T \tag{D.7}$$

$$S_{min} \le S_i - x_i\Delta^c,\ \forall i=1,\cdots,T_L \tag{D.8}$$

$$S_{min} \le S_{i+1} - x_i\Delta^c,\ \forall i=T_L+1,\cdots,T \tag{D.9}$$

$$S_{max} \ge S_{i+1} + x_i\Delta^c,\ \forall i=1,\cdots,T_L \tag{D.10}$$

$$S_{max} \ge S_i + x_i\Delta^c,\ \forall i=T_L+1,\cdots,T \tag{D.11}$$

$$D \ge S_{max} - S_{min} \tag{D.12}$$

$$D \le E \tag{D.13}$$

$$a_i + r_i \le P,\ \forall i=1,\cdots,T_L \tag{D.14}$$

$$\frac{r_i - a_i}{\eta} \le P, \forall i=1,\cdots,T_L \tag{D.15}$$

$$\frac{a_i + r_i}{\eta} \le P,\ \forall i=T_L+1,\cdots,T \tag{D.16}$$

$$I_i \in \{0,1\},\ \forall i=1,\cdots,T \tag{D.17}$$

$$J_i \in \{0,1\},\ \forall i=1,\cdots,T \tag{D.18}$$

$$a_i \ge 0,\ \forall i=1,\cdots,T \tag{D.19}$$

$$r_i \ge 0,\ \forall i=1,\cdots,T. \tag{D.20}$$

We optimally decide hourly service rates for arbitrage (a_i) and regulation (r_i) that the battery commits to provide the grid for each time interval i, requiring that the arbitrage rate be used to fill the battery during low-price periods 1 through T_L, while arbitrage be used to discharge the battery during high-price periods T_L+1 through T. S_i, the SOC level at the beginning of time interval i, evolves following eq. (D.4), where M is some large constant and the binary variables I_i and J_i differentiate whether excess regulation is strictly positive or not (e.g., when $r_i > a_i$, then $I_i = 1,\ J_i = 0$). As in the main model, we assume the battery operates over multiple days, so we require the starting and ending SOC to be equal. Degradation is estimated using the $+1, 0, -1, 0, \cdots$ pattern from Section 6.3, with the time spent at each signal level as $\alpha\Delta^c, (1-\alpha)\Delta^c, \alpha\Delta^c, \cdots$. D represents the largest cycle. The energy constraint in eq. (D.13) is more general than eq. (8) from Section 6.4, capturing the potential expansion of the largest cycle due to positive excess regulation. Eqs. (D.14) – (D.16) guarantee that the services combined are deliverable without violating the battery's power capacity. Since decision variables are defined from the grid's perspective (Beil et al.

Type	Notation	Parameter	Value	Unit	Sources
Grid/market condition	p_i^e	energy hourly price $(i=1,\cdots,24)$	historical values	\$/MWh	NYISO (2022)
	p_i^r	regulation hourly price $(i=1,\cdots,24)$	historical values	\$/MWh	NYISO (2022)
	$2\Delta^c$	energy neutrality time period	1	hrs	PJM (2021), ISO-NE (2021), NYISO (2022)
	T_H	peak length	2	hrs	PJM (2021), ISO-NE (2021), NYISO (2022)
	T	total time length (winter/summer)	12 (w) / 24 (s)	hrs	PJM (2021), ISO-NE (2021), NYISO (2022)
	α	dispatch-to-contract ratio	0.6	-	PJM (2021), ISO-NE (2021), NYISO (2022)
Battery specification	β	degradation exponent	1.5–2.5‡	-	Table A.1
	c_0^b	degradation coefficient	150 (165)†	\$	Xu et al. (2018b), BloombergNEF (2022), Lazard (2021)
	c_{lin}^b	linearized degradation coefficient $(l=0.3)$	13.5 (14.9)†	\$/(MWh)	Table A.1, Perez et al. (2016)
	c_{lin}^b	linearized degradation coefficient $(l=0.5)$	37.5 (41.3)†	\$/(MWh)	Table A.1, Perez et al. (2016)
	c_{lin}^b	linearized degradation coefficient $(l=0.7)$	73.5 (80.9)†	\$/(MWh)	Table A.1, Shi et al. (2017)
	l	operable DOD range for linear degradation models	0.3–0.7	-	Perez et al. (2016), Shi et al. (2017)
	P	power capacity	0.5	MW	S&P Global (2019)
	E	energy capacity	1	MWh	S&P Global (2019)
	η^2	round-trip efficiency	0.8 (0.7)‡	-	Eftekhari (2017), EIA (2021a)

Table D.1 Summary of the model parameters used in Appendix D.

†: The degradation coefficients c_0^b and c_{lin}^b are fitted based on the cell price of 2022 (2020 for those in the parentheses);
‡: As a default, $\eta^2=0.8$ and $\beta=2$ are used, except when otherwise specified for the sensitivity analyses in Tables D.2 – D.3.

2015, Nguyen et al. 2017), the discharging (charging) rate is upper limited by $P\eta$ (P) after factoring in energy loss.

The benchmark models can be defined similarly with some adjustments below.

Time-exclusive stacking: Simultaneous stacking with eqs. (D.5) – (D.6) being replaced by:

$$a_i \le MJ_i,\ \forall i=1,\cdots,T, \tag{D.21}$$

$$r_i \le MI_i,\ \forall i=1,\cdots,T, \tag{D.22}$$

implying arbitrage and regulation can be only provided exclusively.

Regulation-Only: Time-exclusive stacking with additional constraints

$$a_i=0,\ \forall i=T_L+1,\cdots,T.$$

Note that we allow $a_i>0$ for $i\le T_L$ to allow energy replenishment to offset energy losses from providing regulation, as we did in Section 6.2.

Arbitrage-Only: Time-exclusive stacking with additional constraints

$$r_i=0,\ \forall i=1,\cdots,T.$$

D.2.2. Calibration of Model Parameters To calibrate our models, we used real-world historical data from the NYISO day-ahead energy and regulation hourly price dataset from Jan 01, 2019 to Dec 31, 2022 and from the ISO-NE trinary/continuous regulation signal dataset, which includes data from before (2019 – 2020) and during (2021 – 2022) the global energy crisis. Other model parameters are calibrated using real-world operational data from multiple ISO/RTOs in the United States. Using the battery specifications from Key Capture Energy's Bulk Energy Storage Market Bridge Incentive (MBI) program in New York, we consider a lithium-ion BESS with energy capacity $E = 1\text{MWh}$ and power capacity $P = 0.5\text{MW}$ (see also S&P Global 2019, EIA 2021b, KCE 2023 for typical grid-scale battery specifications). The round-trip efficiency is set to 80% (the average utility-scale battery's round-trip efficiency from EIA 2021a). For grid/market conditions, we take typical values reported or numerically fitted from the NYISO (as well as ISO-NE and PJM) operational datasets. The energy neutrality time period is set as $2\Delta^c = 1$ hour based on the empirically observed energy neutrality time in the ISO-NE trinary/continuous signals, as described in Appendix A.2. The peak length (two hours per peak) and timing are determined from the historical pattern in Figure D.1. Table D.1 summarizes all the model parameters used and the sources for our numerical study.

D.3. Results

D.3.1. Actual Dispatching Profiles and Capacity Utilization

Figure D.3 illustrates canonical examples of actual dispatching profiles under each operation (arbitrage only, regulation only, time-exclusive stacking, and simultaneous stacking) when optimizing using the models from Appendix D.2.1 and parameter estimates from Appendix D.2.2. For illustrative purposes, three different regulation signals are drawn for regulation-only and stacking. The overall performance over $N = 100$ randomly selected realizations of the regulation signal and price profiles is $S = \frac{\Pi_{stk}}{\max(\Pi_{arb}, \Pi_{reg})} = 1.23$ and $\frac{\Pi_{stk}}{\Pi_{te}} = 1.23$; for a fair comparison, the net energy change between the initial and final energy levels (due to the regulation signal not being exactly energy neutral throughout the course of the day) is credited if positive and charged if negative, according to NYISO's accounting manual NYISO (2023).

In Figure D.3, the histogram of DOD sizes (the right column) is obtained from the rainflow counting algorithm for the blue realization in the SOC plot (the left column). It shows that arbitrage-only has a single cycle (two half cycles of DOD = 6.1%); regulation-only has a single mid-sized cycle (10% < DOD < 20%) and many small cycles (DOD < 10%). Time-exclusive stacking coincides with regulation-only since regulation is much more attractive than arbitrage. On the other hand, simultaneous stacking has a large cycle (DOD = 27.1%) with many small-sized cycles (DOD < 10%). Note that simultaneous stacking provides much more service (and earns larger revenues) than other benchmark models, especially during the peak hours of periods 4 and 10, as shown in the second column. This is because simultaneous stacking benefits from cost-saving synergy by increasing both service rates at the same time.

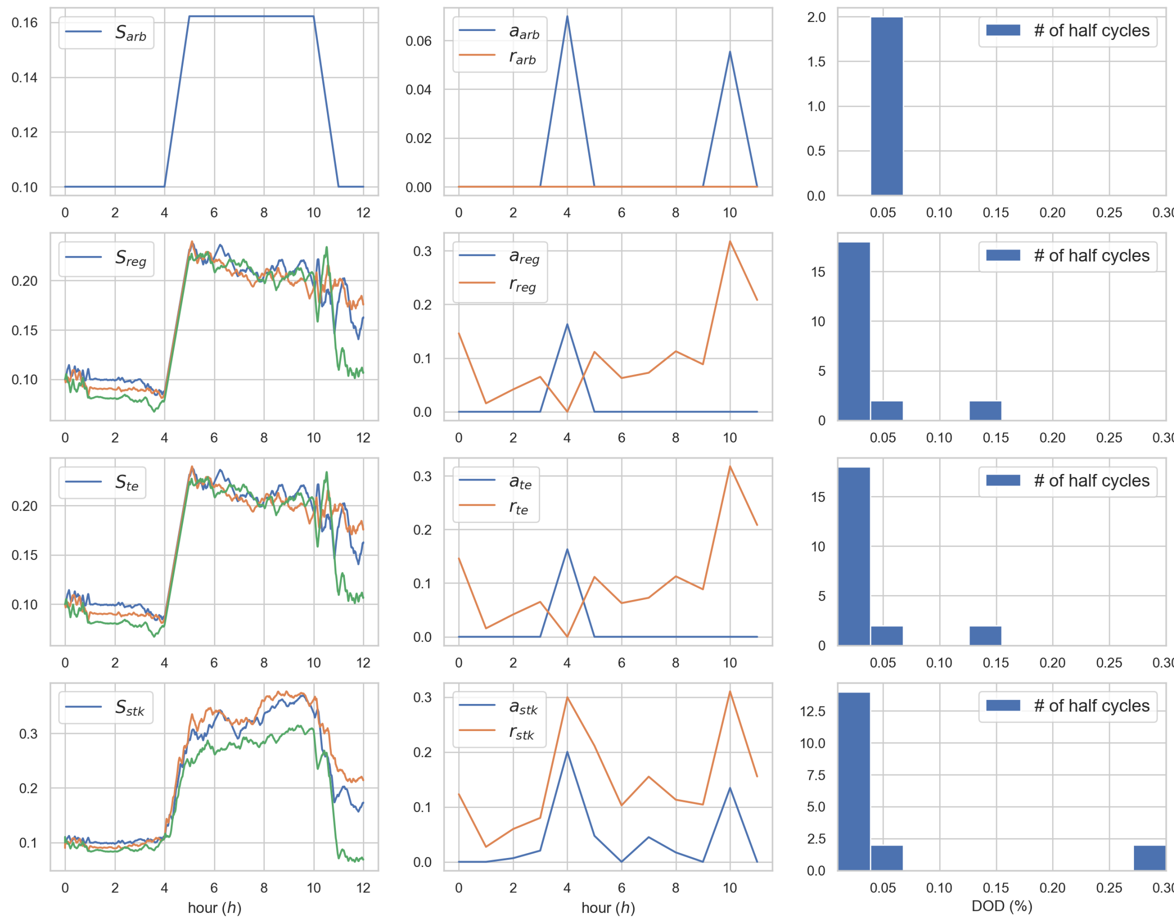


Figure D.3 Illustration of actual dispatching profile: each row sequentially represents arbitrage-only, regulation-only, time-exclusive stacking, and simultaneous stacking. The left, middle, and right columns represent the SOC levels (three realizations), provision rates for arbitrage/regulation, and the histogram of DOD sizes (the number of half cycles) from the rainflow counting algorithm for the blue realization of the regulation signal, respectively.

D.3.2. Sensitivity Analysis

We further perform an extensive sensitivity analysis over a range of parameters to evaluate stacking's performance. Tables D.2 (for $\eta^2 = 0.8$) and D.3 (for $\eta^2 = 0.7$) show stacking's performance under different scenarios that reflect alternative assumptions and parameters (e.g., different times of the year and degradation functions). For each operation, we report the average performance of over $N = 100$ realizations, where the realizations of service prices and regulation signals are randomly drawn from the historical data. The findings are five-fold. First and foremost, the key lessons of the analytical work (e.g., degradation/energy replenishment cost savings and superlinear profit improvement) continue to hold using realistic grid/market parameters and battery characteristics, even when the two-price assumption is relaxed to hourly service prices. Second, the overall performance of stacking relative to standalone service tends to improve when the cost of degradation is

Table D.2 Sensitivity analysis: NYISO dataset ($\eta^2 = 0.8$)

(a) 2021 – 2022 dataset, summer (one-peak)

β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	46.24	32.07	1.78	✓	✓	✓	✓	< 0	0.78	< 0	< 0
1.7	66.05	34.65	1.34	✓	✓	✓	✓	< 0	0.65	< 0	< 0
2.0	103.85	43.03	1.16	✓	✓	✓	✓	< 0	0.50	< 0	0.19
2.3	141.05	53.82	1.10	✓	✓	✓	✓	< 0	0.41	0.43	0.25
2.5	160.47	47.38	1.09	×	✓	✓	✓	< 0	0.36	0.62	0.27

(b) 2021 – 2022 dataset, winter (two-peak)

β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	67.23	44.52	2.43	✓	✓	✓	✓	< 0	0.58	< 0	< 0
1.7	84.13	38.34	1.64	✓	✓	✓	✓	< 0	0.42	< 0	< 0
2.0	119.07	36.42	1.28	✓	✓	✓	✓	< 0	0.30	< 0	0.11
2.3	153.43	40.78	1.15	✓	✓	✓	✓	< 0	0.23	0.48	0.14
2.5	171.67	36.70	1.12	✓	✓	✓	✓	< 0	0.22	0.70	0.15

(a) 2019 – 2020 dataset, summer (one-peak)

β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	3.20	2.68	2.49	✓	✓	✓	✓	< 0	< 0	< 0	< 0
1.7	6.30	3.38	1.98	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.0	13.40	5.02	1.42	✓	✓	✓	✓	< 0	0.26	< 0	0.18
2.3	22.00	7.13	1.25	×	✓	✓	✓	< 0	0.38	< 0	0.24
2.5	27.61	8.66	1.19	×	✓	✓	✓	< 0	0.37	< 0	0.36

(b) 2019 – 2020 dataset, winter (two-peak)

β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	5.65	4.78	2.20	✓	✓	✓	✓	< 0	< 0	< 0	< 0
1.7	9.80	4.92	2.10	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.0	18.20	5.22	1.84	×	✓	✓	✓	< 0	0.38	< 0	< 0
2.3	27.64	5.72	1.50	×	✓	✓	✓	< 0	0.60	< 0	< 0
2.5	33.52	6.30	1.39	×	✓	✓	✓	< 0	0.61	< 0	< 0

Note: The linearized coefficient c^b_{lin} is calibrated over three different operable DOD ranges ($l = 0.3, 0.5, 0.7$ representing 30%, 50%, and 70% of DOD, respectively). The hourly energy/regulation prices and regulation signals are randomly sampled ($N = 100$) for profitability analysis. $T_H = 2$ and $T = 12$ and 24 are used for winter and summer, respectively. $c^b_0 = 165$ and $c^b_0 = 150$ are used for degradation coefficients in 2019 – 2020 and 2021 – 2022, respectively, factoring in a drop in battery prices. Other parameters are summarized in Table D.1. Power and energy utilization are presented in Figure D.4.

less convex. Less convexity of the degradation function (when β is smaller) implies that frequency regulation carries a higher portion of the cost burden, as regulation cycles become more expensive. In such cases, stacking benefits more from cost-saving synergy. For example, when $\beta = 1.5$, the profit can be more than doubled ($S \approx 2.5$) compared to the profit of the best standalone service.

Table D.3 Profitability analysis: NYISO dataset ($\eta^2 = 0.7$)

(a) 2021 – 2022 dataset, summer (one-peak)											
β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb},\Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	27.55	28.84	2.14	✓	✓	✓	✓	< 0	< 0	< 0	< 0
1.7	39.47	37.27	1.70	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.0	64.76	46.31	1.21	✓	✓	✓	✓	< 0	0.25	< 0	< 0
2.3	90.95	56.32	1.14	✓	✓	✓	✓	< 0	0.51	< 0	< 0
2.5	103.41	49.13	1.13	✓	✓	✓	✓	< 0	0.52	< 0	< 0

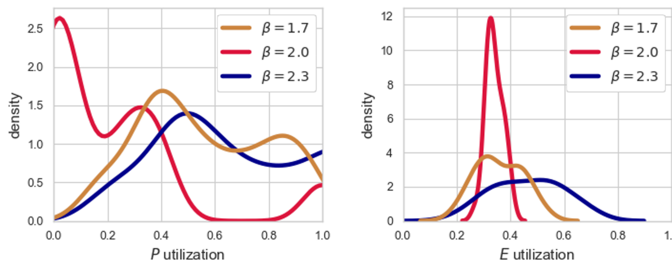
(b) 2021 – 2022 dataset, winter (two-peak)											
β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb},\Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	51.04	39.86	2.99	✓	✓	✓	✓	< 0	< 0	< 0	< 0
1.7	63.64	41.15	2.19	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.0	92.48	45.36	1.53	✓	✓	✓	✓	< 0	0.15	< 0	< 0
2.3	121.14	62.73	1.31	✓	✓	✓	✓	< 0	0.21	< 0	< 0
2.5	133.84	57.63	1.20	✓	✓	✓	✓	< 0	0.23	< 0	< 0

(a) 2019 – 2020 dataset, summer (one-peak)											
β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb},\Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	2.08	2.96	3.08	✓	✓	✓	✓	< 0	< 0	< 0	< 0
1.7	4.05	5.33	2.66	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.0	8.48	7.66	2.09	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.3	13.94	11.74	1.51	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.5	17.55	13.25	1.34	✓	✓	✓	✓	< 0	< 0	< 0	< 0

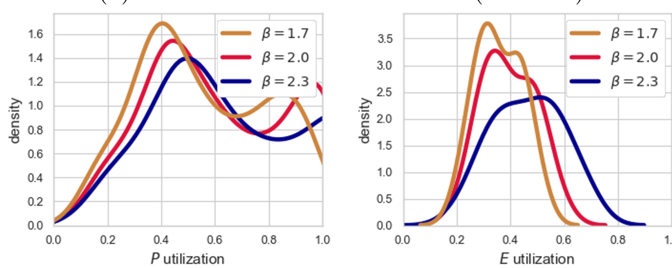
(b) 2019 – 2020 dataset, winter (two-peak)											
β	Π^*_{stk}	Cost-saving	$S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb},\Pi^*_{reg})}$	$A^*_{stk} > A^*_{arb}$	$R^*_{stk} > R^*_{reg}$	$\frac{R^*}{A^*} \geq 2$	Superlinear profit	ρ_0	ρ_1 ($l = 0.3$)	ρ_1 ($l = 0.5$)	ρ_1 ($l = 0.7$)
1.5	3.94	5.24	2.86	✓	✓	✓	✓	< 0	< 0	< 0	< 0
1.7	6.04	6.32	2.19	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.0	13.14	8.95	1.92	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.3	18.92	10.77	1.78	✓	✓	✓	✓	< 0	< 0	< 0	< 0
2.5	24.14	13.63	1.52	✓	✓	✓	✓	< 0	< 0	< 0	< 0

Note: The parameters are the same as those used in Table D.2.

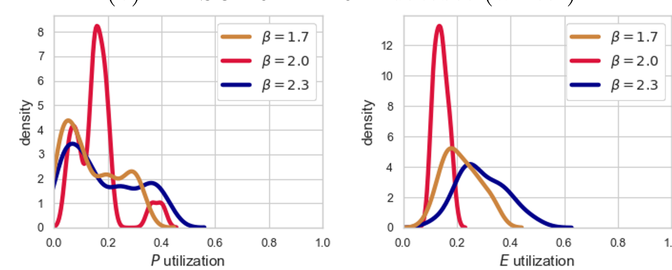
Third, having multiple peaks (per day) implies higher arbitrage rates, which increases the maximum free regulation. This implies that stacking can be more important under ISO/RTOs typically having multiple peaks such as CAISO and NYISO. Fourth, as battery efficiency decreases and service prices decrease, S ratios tend to increase, emphasizing the expanded benefit of stacking under such circumstances. From the point of view of the likely market trajectory, this dependency may be particularly relevant, as many used car batteries with lower efficiency could soon be available for second-life uses. Lastly, the values of ρ_0, ρ_1 imply that the misspecified models (no- and linear-degradation models) suffer severe suboptimality with real-world parameters, since they tend



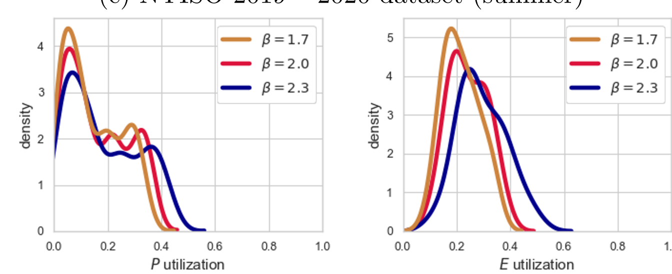
(a) NYISO 2021 – 2022 dataset (summer)



(b) NYISO 2021 – 2022 dataset (winter)



(c) NYISO 2019 – 2020 dataset (summer)



(d) NYISO 2019 – 2020 dataset (winter)

Figure D.4 **Power and energy utilization density plot for Table D.2 under different degradation exponents $\beta = 1.7, 2.0, 2.3$.**

to have larger DODs; thus, accurately capturing the nonlinearity in the degradation costs becomes crucial. It is also noteworthy that the heuristic DOD restrictions used in the linear-degradation model may still perform poorly.

Figure D.4 shows that power/energy utilization for Table D.2 with three different degradation exponents $\beta \in \{1.7, 2.0, 2.3\}$, which confirms that both constraints are not likely to be binding under the service prices experienced in 2019 – 2020. Similar findings are also reported in other numerical studies (e.g., Perez et al. 2016, Xu et al. 2018b). Xu et al. (2018b) demonstrate that providing frequency regulation at full capacity can result in very early battery retirement (within less than 3 years) and significant monetary loss, as the investment cost of the battery cannot be recovered.

As of 2021 – 2022, however, power capacity is from time to time (roughly 5% of the time) binding due to exceptional service prices due to the energy crisis. As the degradation function becomes more convex (i.e., has a larger β), we tend to provide more regulation. Intuitively, this is the situation where frequency regulation becomes less demanding in terms of degradation costs; thus, the overall power usage increases, which also increases the maximum DOD (energy utilization).

D.3.3. Managerial Implications

In this subsection, we will briefly discuss the potential managerial implications of this work for battery operators and practitioners. Specifically, we will address two key questions: (1) "Which ISO markets offer the greatest benefits for battery operators adopting stacking?" and (2) "Which types of batteries derive the greatest benefits from stacking in a given grid/market environment?".

For the first question, we analyze market environments with different characteristics such as varying peak widths (represented by $\frac{T_H}{T}$), energy-neutrality guarantees ($2\Delta^c$), and service price ratios ($\frac{p^a}{p^r}$), and we determine which ISO markets are most suitable for adopting the proposed stacking strategy, as measured by stacking's relative strength over standalone services. For the second question, we focus on identifying the relationship between battery type/characteristics and potential profit boost from stacking given some grid/market environment. Parameters such as longevity coefficient (z), round-trip efficiency (η^2), and power-to-energy ratio (P/E) play a crucial role in determining the batteries that are best suited for stacking.

By addressing these questions and considering various factors, we aim to provide valuable insights to battery operators and practitioners, enabling them to make informed decisions about whether to make battery management system (BMS) upgrades to enable stacking.

(1) Which ISO markets offer the greatest benefits for battery operators adopting stacking?

Even though the total capacity of grid-scale BESS continues to grow in the U.S. (and globally), regional patterns such as weather conditions and ISO/RTO market structures can have a significant impact on the profitability of a battery. In fact, EIA (2021b) reveals that the characteristics of grid-scale BESS in the U.S. vary significantly depending on their deployment locations. For instance, batteries located in CAISO and ISO-NE tend to have a lower power-to-energy ratio, typically consisting of 4-hour batteries. Conversely, batteries in PJM have a high power-to-energy ratio, with 1-hour batteries being the norm.[46]

[46] Later in our discussion, we will provide a rationale for this discrepancy by demonstrating that the PJM market design strongly favors batteries providing regulation services over other types of services. As a result, this market structure has incentivized investment in batteries with high power-to-energy ratios to provide regulation services effectively.

Determining which battery operators should consider adopting the stacking strategy based on the ISO in which they operate is a crucial question in practice, since adopting stacking may require a shift in BMS technology to a provider like E3 or Fluence.

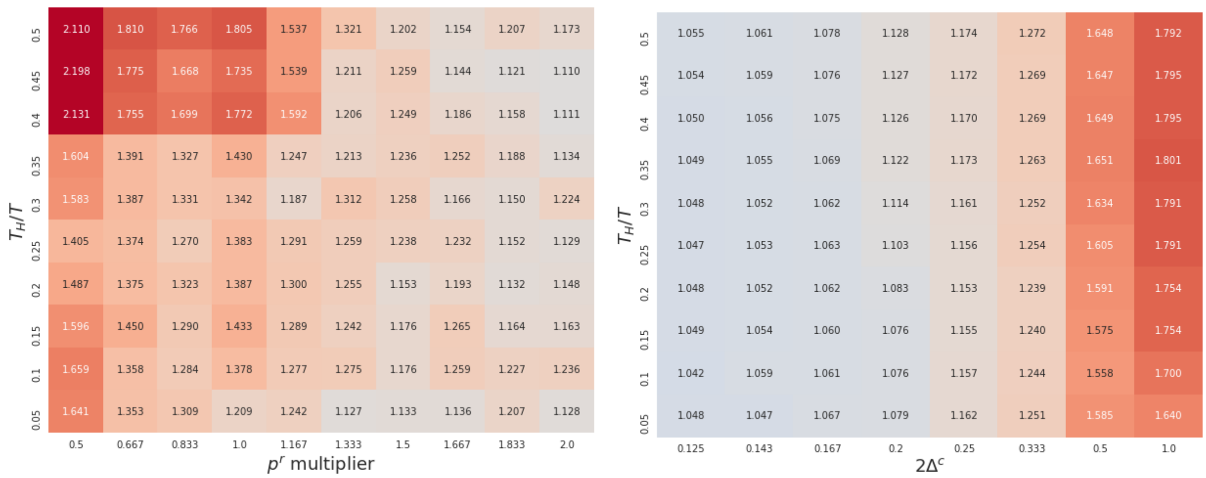
Figure D.5 shows stacking's relative profit $S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$ over various grid/market environments. Specifically, Figure D.5a varies the peak width $\frac{T_H}{T} \in [0.05, 0.5]$ and different service price ratios (by multiplying a constant to regulation prices p^r for fixed energy prices), while Figure D.5b varies the peak width and the energy-neutrality guarantee $2\Delta^c \in [0.125, 1]$. For each case, we report the average performance over $N = 100$ realizations, where the hourly energy/regulation prices and regulation signals are randomly sampled.

From Figure D.5a, stacking achieves the highest benefits in scenarios where the peak is wide (e.g., $\frac{T_H}{T} = 0.5$) and the regulation prices are lower than the current levels (e.g., p^r multiplier of 0.5).[47] This is because wide peaks enable the battery to better make use of cost-saving synergy, and reduced regulation prices relative to arbitrage prices (represented by the p^r multiplier being less than 1) make the profitability of providing regulation services more comparable to that of providing arbitrage services (recall that providing regulation is significantly more profitable compared to providing arbitrage services as of 2022).

From Figure D.5b, stacking yields greater benefits when the energy-neutrality guarantee is weaker (indicating a longer time to achieve energy neutrality while providing regulation services). This is because strong energy neutrality implies smaller regulation cycles, leading to lower degradation costs. Consequently, regulation services become even more favorable in comparison to arbitrage services, resulting in a smaller marginal benefit from stacking.

The obtained results provide insights about the ISO/RTOs where battery owners should put particular emphasis on implementing stacking. For instance, batteries located in PJM, where regulation-to-arbitrage service price ratios are relatively high and the energy-neutrality condition is more strict, may experience fewer benefits from stacking. According to MonitoringAnalytics (2022), PJM has observed high ratios of regulation prices to arbitrage prices. This is partly due to long-lasting market design failures, resulting in overpayment for regD resources compared to regA resources (MonitoringAnalytics 2022). Additionally, as stated in Appendix A.2, PJM initially enforced a strict energy neutrality requirement (15-minute neutrality; $2\Delta^c = 0.25$), which is more stringent than other ISO/RTOs. Even though the requirement has been recently relaxed to conditional neutrality within 30 minutes ($2\Delta^c \geq 0.5$), 80% of the time, energy neutrality is achieved

[47] To construct service prices for different peak widths based on the historical data, we interpolated the historical prices by increasing/decreasing the number of data points used for interpolation on the extended/shortened periods, respectively.



(a) $(\frac{T_H}{T}, p^r \text{multiplier})$ (b) $(\frac{T_H}{T}, 2\Delta^c)$

Figure D.5 Stacking's relative profit $S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$ **with various grid/market environments. The NYISO 2021 – 2022 dataset (winter) is used. Other parameters are summarized in Table D.1. The** p^r **multiplier in pane (a) operates by multiplying historical regulation prices by some constant value while keeping historical energy prices fixed. The energy neutrality guarantees** $2\Delta^c$ **in pane (b) is varied by assuming the regulation signal is updated every** $(4 \times \Delta)$ **seconds for** $\Delta = \{1/i | i = 1, \cdots, 8\}$ **while using the historical regulation signal.**

within 30 minutes from the PJM's regulation signal dataset (PJM 2020). On the other hand, batteries located in ISO-NE, where regulation-to-arbitrage service price ratios are relatively low and the energy-neutrality condition is less strict, may experience more benefits from stacking. The low regulation-to-arbitrage price ratio is partly due to price changes during the global energy crisis. According the ISO's annual report (ISO-NE 2022), ISO-NE observed average regulation capacity/service clearing prices of \$30.96/MW and \$0.27/mile in 2022, which are a 60.1% increase and a 28.7% increase from 2021, respectively, while the average day-ahead energy price was \$85.56/MWh in 2022, an 86% increase from 2021. Furthermore, the ISO-NE trinary signal dataset suggests weaker energy neutrality compared to PJM's regD signal, with energy neutrality achieved within 60 minutes 80% of the time (ISO-NE 2014).

These insights demonstrate how a battery's location may impact the attractiveness of implementing stacking. Batteries located in PJM, characterized by higher regulation prices and a historically strict energy-neutrality requirement, may experience limited benefits from stacking. Conversely, batteries located in ISO-NE, with lower regulation prices and a less stringent energy-neutrality condition, may enjoy greater advantages from the proposed stacking.

(2) Which types of batteries offer the greatest benefits from stacking?

Wood Mackenzie (2022a) reported a fragmented battery energy storage systems (BESS) market in 2020. The majority of the market share was occupied by NMC (Nickel-Manganese-Cobalt) batteries, followed by NCA (Nickel-Cobalt-Aluminum Oxide), LFP (Lithium-Iron-Phosphate), and other types of batteries. However, the report also projected significant changes in the market share distribution in the near future, indicating potential shifts in the dominance of different battery types.

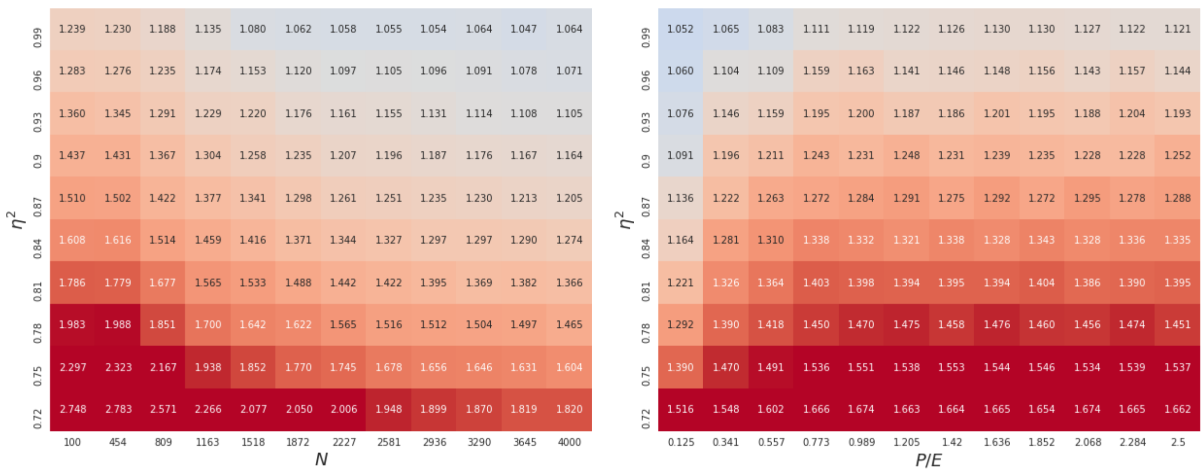
Additionally, it is anticipated that many electric vehicle (EV) batteries will retire from their initial use and become available as second-hand batteries after repurposing, as reported by Bloomberg (2018). It should be noted that repurposed batteries generally exhibit reduced longevity and lower round-trip efficiency compared to new batteries. These factors need to be considered when evaluating the performance and viability of repurposed batteries for various applications.

Determining which battery types should most prioritize stacking is a crucial question in practice for battery operators. Figure D.6 shows stacking's relative profit $S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$ for different battery types. Specifically, Figure D.6a varies the battery's remaining cycle count $N \in [100, 4000]$ and round-trip efficiency $\eta^2 \in [0.72, 1]$, while Figure D.6b varies power-to-energy ratio ($P/E \in [0.125, 2.5]$) and round-trip efficiency. We report the average performance over $N = 100$ realizations, where the hourly energy/regulation prices and regulation signals are randomly sampled. We use the 2023 average cell price of \$152 (BloombergNEF 2022) for a cell with E=1kWh, P=1kW, N=1000, and η^2=1.[48]

From Figure D.6a, batteries with lower efficiency and shorter life cycles, such as second-hand batteries, demonstrate the highest benefit from stacking. For instance, $S > 2.5$ is possible for a scenario with $N \leq 809$ and $\eta^2 = 0.72$. This finding is consistent with our analytical modeling, and explained by energy replenishment cost-saving (see Section 6.2). This finding has significant implications, particularly in the context of the fast-growing cumulative capacity of used EV batteries. Stacking can contribute not only to improving the profitability of second-hand batteries but also to reducing greenhouse gas emissions and maintaining a sustainable power grid, as highlighted by Wood Mackenzie (2022b).

Typical lithium-ion batteries, with cycle counts ranging between 500 – 2000 and round-trip efficiency of 0.8 – 0.9, exhibit significant benefits from stacking. The stacking benefit ratios (S) for such batteries fall within the range of [1.15,1.80]. For batteries with extended lifespans and high round-trip efficiency (e.g., 4000 cycles and $\eta^2 \geq 0.96$), the stacking benefit is reduced compared to other battery types. However, even in such cases, stacking still offers a minimum increase in profitability of 5%.

[48] We assume the cell price to be linear in power and energy, with the coefficient of power being roughly one-tenth of that of energy based on the empirical evidence on manufacturing costs (Sakti et al. 2015). For a given (P, E), we further assume that the cell price is linear in η^2 and $\log N$, based on the longevity (cycle counts) and efficiency of six typical Li-ion battery chemistries (Shepard 2022). The final expression used for cell price is $(c_1 E + c_2 P)(\eta^2 + c_3 \log N)$ for c_1=\$100/MWh, c_2=\$10/MW, and c_3=\$0.055 (unit: \$).



(a) (N, η^2) (b) $(P/E, \eta^2)$

Figure D.6 Stacking's relative profit $S = \frac{\Pi^*_{stk}}{\max(\Pi^*_{arb}, \Pi^*_{reg})}$ **with various battery types.** $\beta = 2, \eta^2 \in [0.72, 1]$**, and the NYISO 2021 – 2022 price dataset (winter) are used. Other parameters are summarized in Table D.1.** $N \in [100, 4000]$ **indicates the battery's remaining cycle count, while** $P/E \in [0.125, 2.5]$ **indicates the battery's power-to-energy ratio.**

From Figure D.6b, batteries with high power-to-energy ratios (i.e., fast batteries) and lower round-trip efficiency tend to exhibit higher S ratios. Conversely, when a battery has very high efficiency and a very low power-to-energy ratio, the stacking strategy only improves profitability by approximately 5 percent compared to the best standalone service.

Overall, we provide valuable insights to battery operators and practitioners, enabling them to make informed decisions about (a) whether to make battery management system (BMS) upgrades to enable stacking and (b) which battery types should most prioritize stacking. Beyond battery operators, others who may benefit from these insights are BMS companies, who could use them to target marketing efforts.

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