Appendix B: Extensions, additional results, and examples

This section includes the extensions discussed in the main text, additional examples, and the detail of examples.

B.1. Extensions and additional results

B.1.1. Extension of Algorithm 1 to constant number of user types Here, we show how Algorithm 1 extends to a setting with any constant $k > 2$ user types. We develop the algorithm for pro-rata rule and a similar algorithm works for user-centric. The problem of finding the optimal set with pro-rata revenue allocation rule becomes

$$\max_{\substack{J\subseteq\mathcal{M}\\ S\subseteq\{1,\dots,k\}\\ \beta\in[0,1]}} \left(\min_{i\in S}\lambda_i\sum_{j\in J}\pi_{ij}\right)\left(\sum_{i\in S}q_i\right)(1-\beta) \tag{60}$$

$$\text{s.t. } \beta\left(\min_{i\in S}\lambda_i\sum_{j\in J}\pi_{ij}\right)\left(\sum_{i\in S}q_i\right)\frac{\sum_{i\in S}q_i\lambda_i\pi_{ij}}{\sum_{i\in S}q_i\lambda_i\sum_{j\in J}\pi_{ij}}\geq R_j,\quad j\in J. \tag{61}$$

We solve the problem of choosing the optimal set of artists J and the optimal payout rate for a given set S which without loss of generality we assume is the entire set $\{1,\dots,k\}$. Problem (60) can then be solved by searching over 2^k subsets of $\{1,\dots,k\}$.

For $S=\{1,\dots,k\}$, suppose, without loss of generality, in the optimal solution J^* of problem (60), we have

$$1=\arg\min_{i\in\{1,\dots,k\}}\lambda_i\sum_{j\in J^*}\pi_{ij}.$$

We also let β^{pr} be the optimal β and

$$\alpha_i^{\text{pr}}=\frac{\lambda_i\sum_{j\in J^*}\pi_{ij}}{\lambda_1\sum_{j\in J^*}\pi_{1j}},\qquad \text{for } i=2,\dots,k,$$

be the ratio of type i users' utility to type 1 users' utility in the optimal solution. With these notations, we can write constraint (61) succinctly in terms of β^{pr} and α_i^{pr} as

$$\frac{R_j}{\sum_{i=1}^k q_i\lambda_i\pi_{ij}}\leq\frac{\beta^{\text{pr}}}{q_1+\sum_{i=2,\dots,k}q_i\alpha_i^{\text{pr}}},\qquad \text{for all } j\in J^*.$$

Motivated by this observation, for any β and $\boldsymbol{\alpha}=(\alpha_2,\dots,\alpha_k)$, we define the set $\mathcal{M}^{\text{pr}}(\beta,\boldsymbol{\alpha})$ as

$$\mathcal{M}^{\text{pr}}(\beta,\boldsymbol{\alpha})=\left\{j\in\mathcal{M}\ :\ \frac{R_j}{\sum_{i=1}^k q_i\lambda_i\pi_{ij}}\leq\frac{\beta}{q_1+\sum_{i=2,\dots,k}q_i\alpha_i}\right\}.$$

Therefore, knowing β^{pr} and $\boldsymbol{\alpha}^{\text{pr}}=(\alpha_2^{\text{pr}},\dots,\alpha_k^{\text{pr}})$, the optimal set of artists can be found by solving

$$\max_{J\subseteq\mathcal{M}^{\text{pr}}(\beta^{\text{pr}},\boldsymbol{\alpha}^{\text{pr}})}(1-\beta^{\text{pr}})\lambda_1\sum_{j\in J}\pi_{1j} \tag{62}$$

$$\text{s.t. } \lambda_1\sum_{j\in J}\pi_{1j}\leq\alpha_i^{\text{pr}}\lambda_i\sum_{j\in J}\pi_{ij},\qquad \text{for } i=2,\dots,k.$$

A similar argument to the one we used in developing and analyzing algorithm 1 shows this problem can be formulated as an instance of a knapsack problem (with multiple constraints). Therefore, for any constant k we can again find a PTAS of the optimal design within pro-rata class of revenue allocation strategies (and similarly within user-centric class of revenue allocation strategies).

B.1.2. Extension of Algorithm 2 to constant number of user types Here, we show that for any constant $k > 2$ user types, a similar algorithm finds a revenue allocation rule whose profit is at most $(k-1)\max_{j\in\mathcal{M}} R_j$ away from the optimal profit. First, note that, similar to Lemma 4, with k user types the problem can be formulated in terms of the sets of artists who join the platform and the set of user types who subscribe:

$$\max_{\substack{J\subseteq\mathcal{M}\\ S\subseteq\{1,\dots,k\}}} \left\{ \left(\min_{i\in S} \lambda_i \sum_{j\in J} \pi_{ij}\right)\left(\sum_{i\in S} q_i\right) - \sum_{j\in J} R_j \right\}. \tag{63}$$

We next show that for any $S\subseteq\{1,\dots,k\}$, a similar approach to Algorithm 2 finds a revenue allocation rule whose profit is at most $(k-1)\max_{j\in\mathcal{M}} R_j$ of the optimal profit. Without loss of generality, let us consider the subset $\{1,\dots,k\}$. Problem (63) becomes

$$\max_{J\subseteq\mathcal{M}} \left\{ \left(\min_{i\in S} \lambda_i \sum_{j\in J} \pi_{ij}\right)\left(\sum_{i\in S} q_i\right) - \sum_{j\in J} R_j \right\}.$$

whose linear programming relaxation becomes

$$\begin{aligned} \max_{z,x_1,\dots,x_m}\ & z - \sum_{j=1}^{m} x_j R_j \\ \text{s.t. }\ & z \le \sum_{j=1}^{m} x_j \lambda_1 \pi_{ij}, \quad i=1,\dots,k \\ & 0\le x_j \le 1, \quad j=1,\dots,m. \end{aligned} \tag{64}$$

The dual of problem (64), similar to the proof of Theorem 3, becomes

$$\begin{aligned} \min_{w_1,\dots,w_k,\mathbf{y}}\ & \sum_{j=1}^{m} y_j \\ \text{s.t. }\ & w_i \ge 0, \quad i=1,\dots,k \\ & \sum_{i=1}^{k} w_i = 1 \\ & \mathbf{y}\ge\mathbf{0} \\ & y_j \ge \sum_{i=1}^{k} w_i\lambda_i\pi_{ij} - R_j, \quad j=1,\dots,m. \end{aligned}$$

A similar approach to Algorithm 2 gives the desired approximation by defining the score of artist j for the tuple of dual variables $\mathbf{w}^{(t)} = (w_1^{(t)},\dots,w_k^{(t)})$ by $S_j(\mathbf{w}^{(t)}) = \sum_{i=1}^{k} w_i^{(t)}\lambda_i\pi_{ij} - R_j$.

B.1.3. The optimal revenue allocation strategy when artists multi-home Here, we consider an extension of our model in which artists can join the platform and offer their content through direct channels at the same time. Similar to our baseline model, we have a three stage game as follows:

1. In the first stage, the platform chooses the payment to artists as well as the subscription fee denoted by $p: 2^{\mathcal{M}}\to\mathbb{R}^m$ and $\text{FEE}: 2^{\mathcal{M}}\to\mathbb{R}$, respectively. The artists also choose the subscription fee of their direct channels denoted by $\mathbf{r}=(r_1,\dots,r_m)$.

2. In the second stage, given the tuple $(\lambda,\mathbf{q},\Pi)$ and the payment rules the artists decide whether they want to join the platform.

3. In the third stage, the users decide which subset of subscriptions they want to subscribe to. In particular, the users decide about subscribing to the bundle J on the platform as well as subscribing to direct channels of different artists.

The equilibrium of this game, similar to our baseline model, is obtained by backward induction. We next find the optimal revenue allocation rule. We have the following cases depending on the user types that subscribe to the platform:

1. Both users types subscribe: In this case, the revenue of the artists who have joined the platform from offering their content outside of the platform is zero. Therefore, the platform need to pay each artist $j \in J$, $\max_r \; r\; d(j,r)$ which is the same as our baseline model. Therefore, the platform's problem in this case becomes the same as (24) and Algorithm 2 finds a set of artists whose profit is within one artist from the optimal set.
2. Type 1 users subscribe to the platform: In this case the platform's optimal allocation rule will make all artists join the platform and the platform compensates each artist for the difference between the revenue she obtains by offering her content solely through a direct channel and offering her content both on the platform and through a direct channel.
3. Type 2 users subscribe to the platform: this case is similar to the previous case.
4. None of the user types subscribe to the platform: in this case, the platform's profit is zero.

The optimal revenue allocation strategy is the one with the maximum of the above four cases.

B.1.4. Comparing pro-rata and user-centric rules with different set of artists In Subsection 3.2, we compared pro-rata and user-centric rules when all artists are on the platform. Here, we compare these two rules, each with a different set of artists. In this regard, we let J^{pr} be the set of artists on the platform with pro-rata payment rule and J^{uc} be the set of artists on the platform with user-centric payment rule.

We next extend Propositions 2 and 3 to this setting.

Proposition 12. *Let $(J^{\text{pr}}, \beta^{\text{pr}})$ and $(J^{\text{uc}}, \beta^{\text{uc}})$ be the set of artists and payout rate with pro-rata and user-centric payments, respectively and suppose that both user types subscribe to the platform. For all $j \in J^{\text{pr}} \cap J^{\text{uc}}$, the ratio of artists' payment with pro-rata to their payment with user-centric is increasing in $\frac{\pi_{2j}}{\pi_{1j}}$.*

Proposition 12 is the analogue of Proposition 2 and establishes that, even with different set of artists for pro-rata and user-centric rules, artists who are predominantly listened by high volume users prefer the pro-rata allocation rule to the user-centric allocation rule. (The assumption that both user types subscribe to the platform is imposed because if a single user type subscribes, then pro-rata and user-centric rules become the same.)

Proposition 13. *Let $(J^{\text{pr}}, \textsc{Rev}^{\text{pr}})$ and $(J^{\text{uc}}, \textsc{Rev}^{\text{uc}})$ be the set of artists and the platform's revenue with pro-rata and user-centric payments, respectively and suppose that both user types subscribe to the platform. We also let*

$$y = \frac{q_1 \sum_{j \in J^{\text{pr}}} \pi_{1j} + q_2 \sum_{j \in J^{\text{pr}}} \pi_{2j}}{q_1 \lambda_1 \sum_{j \in J^{\text{pr}}} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J^{\text{pr}}} \pi_{2j}} \frac{\textsc{Rev}^{\text{uc}}}{\textsc{Rev}^{\text{pr}}}.$$

(a) If the pivot artist j for pro-rata belongs to J^{uc} and is such that $\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{q_1 y - q_1\lambda_1}{q_2\lambda_2 - q_2 y}$, then the profit of the platform with pro-rata revenue allocation rule is weakly higher than its profit with user-centric.

(b) If the pivot artist j for user-centric belongs to J^{pr} and is such that $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1 y - q_1\lambda_1}{q_2\lambda_2 - q_2 y}$, then the profit of the platform with user-centric revenue allocation rule is weakly higher than its profit with pro-rata.

The first part of Proposition 13 shows that if an artist who is predominantly listened by high volume users is the pivot artist for pro-rata, and is on the platform with user-c entric rule, then the platform prefers the pro-rata revenue allocation rule. Similarly, the second part of Proposition 13 shows that if an artist who is less likely to be listened to by users who have a higher usage rate is the pivot artist for user-centric, and is on the platform with pro-rata rule, then the platform prefers the user-centric rule.

Proof of Proposition 12: We let $\textsc{Rev}^{\mathrm{pr}}$ and $\textsc{Rev}^{\mathrm{uc}}$ denote the platform's revenue with pro-rata and user-centric rules. With pro-rata rule, the payment to artist $j \in J^{\mathrm{pr}}$ is

$$\beta^{\mathrm{pr}}\textsc{Rev}^{\mathrm{pr}}\frac{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}{q_1\lambda_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\lambda_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}.$$

With user-centric rule, the payment to artist $j \in J^{\mathrm{uc}}$ is

$$\beta^{\mathrm{uc}}\textsc{Rev}^{\mathrm{uc}}\frac{q_1\pi_{1j} + q_2\pi_{2j}}{q_1\sum_{j\in J^{\mathrm{uc}}}\pi_{1j} + q_2\sum_{j\in J^{\mathrm{uc}}}\pi_{2j}}$$

Therefore, the ratio of the revenue that artist $j \in J^{\mathrm{pr}} \cap J^{\mathrm{uc}}$ obtains with pro-rata versus user-cenrtric is

$$\frac{\beta^{\mathrm{pr}}\textsc{Rev}^{\mathrm{pr}}}{\beta^{\mathrm{uc}}\textsc{Rev}^{\mathrm{uc}}}\frac{q_1\lambda_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\lambda_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}{q_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}\frac{q_1\lambda_1 + q_2\lambda_2\frac{\pi_{2j}}{\pi_{1j}}}{q_1 + q_2\frac{\pi_{2j}}{\pi_{1j}}}.$$

The derivative of this ratio with respect to $\frac{\pi_{2j}}{\pi_{1j}}$ is

$$\frac{\beta^{\mathrm{pr}}\textsc{Rev}^{\mathrm{pr}}}{\beta^{\mathrm{uc}}\textsc{Rev}^{\mathrm{uc}}}\frac{q_1\lambda_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\lambda_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}{q_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}\frac{q_1q_2(\lambda_2 - \lambda_1)}{(q_1 + q_2\frac{\pi_{2j}}{\pi_{1j}})^2},$$

which is positive given $\lambda_2 \geq \lambda_1$. This shows that the ratio of artists' payment with pro-rata to their payment with user-centric is increasing in $\frac{\pi_{2j}}{\pi_{1j}}$. ■

Proof of Proposition 13: We let $\textsc{Rev}^{\mathrm{pr}}$ and $\textsc{Rev}^{\mathrm{uc}}$ denote the platform's revenue with pro-rata and user-centric rules. Letting j be the pivot artist for the pro-rata rule, we have

$$\beta^{\mathrm{pr}} = \frac{R_j}{\textsc{Rev}^{\mathrm{pr}}}\frac{q_1\lambda_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\lambda_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}.$$

For the user-centric rule, we must have

$$\beta^{\mathrm{uc}} \geq \frac{R_j}{\textsc{Rev}^{\mathrm{uc}}}\frac{q_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}{q_1\pi_{1j} + q_2\pi_{2j}}.$$

Therefore, we obtain $\beta^{\mathrm{uc}} \geq \beta^{\mathrm{pr}}$ if

$$\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{q_1 y - q_1\lambda_1}{q_2\lambda_2 - q_2 y}$$

where

$$y = \frac{q_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}{q_1\lambda_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j} + q_2\lambda_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}}\frac{\textsc{Rev}^{\mathrm{uc}}}{\textsc{Rev}^{\mathrm{pr}}}.$$

Similarly, letting j be the pivot artist for the user-centric rule, we have

$$\beta^{\text{uc}} = \frac{R_j}{\text{REV}^{\text{uc}}} \frac{q_1 \sum_{j \in J^{\text{pr}}} \pi_{1j} + q_2 \sum_{j \in J^{\text{pr}}} \pi_{2j}}{q_1 \pi_{1j} + q_2 \pi_{2j}}.$$

For the pro-rata rule, we must have

$$\beta^{\text{pr}} \geq \frac{R_j}{\text{REV}^{\text{pr}}} \frac{q_1 \lambda_1 \sum_{j \in J^{\text{pr}}} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J^{\text{pr}}} \pi_{2j}}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}}.$$

Therefore, we obtain $\beta^{\text{pr}} \geq \beta^{\text{uc}}$ if

$$\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1 y - q_1 \lambda_1}{q_2 \lambda_2 - q_2 y}$$

where

$$y = \frac{q_1 \sum_{j \in J^{\text{pr}}} \pi_{1j} + q_2 \sum_{j \in J^{\text{pr}}} \pi_{2j}}{q_1 \lambda_1 \sum_{j \in J^{\text{pr}}} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J^{\text{pr}}} \pi_{2j}} \frac{\text{REV}^{\text{uc}}}{\text{REV}^{\text{pr}}}.$$

This completes the proof. ■

B.1.5. Platform's monopoly and the existence of core Here, we derive conditions guaranteeing the emergence of a monopolist platform, i.e., conditions that no set $J \subseteq \mathcal{M}$ of artists would be better off creating a competing platform offering the content from artists in J. As suggested by the last statement of Corollary 1, neither pro-rata nor user-centric allocation can guarantee this. Thus, in this section we allow for an arbitrary payment rule. In particular, we let $f : 2^{\mathcal{M}} \to \mathbb{R}$ be a mapping from the set of artists who form a coalition and move to another platform to their maximum total collected subscription fee. That is

$$f(J) = \max_{S \subseteq \{1,2\}} \left\{ \left(\min_{i \in S} \lambda_i \sum_{j \in J} \pi_{ij} \right) \left(\sum_{i \in S} q_i \right) \right\}.$$

Note that the platform can sustain its monopoly if and only if there exists no coalition J of artists that generate a higher value for themselves on a competing platform offering artists $j \in J$, than what they collectively generate on the platform offering all artists. In other words, the platform monopoly emerges if and only if the *core* of the function $f(\cdot)$, defined next, is non-empty.

Definition 4 (Core). The vector $\mathbf{x} = (x_1, \ldots, x_m)$ forms the core for the function $f(\cdot)$ if we have

$$f(J) \leq \sum_{j \in J} x_j, \text{ for all } J \subseteq \mathcal{M},$$
$$f(\mathcal{M}) = \sum_{j \in \mathcal{M}} x_j.$$

The main result of our section provides a necessary condition in terms of the ratios $\frac{\pi_{1j}}{\pi_{2j}}$ for the existence of an allocation rule that belongs to the core.

Theorem 4. *For a given $\boldsymbol{\lambda}$ and $\mathbf{q}$, if $\left(\max_{j \in \mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) \leq \frac{\lambda_1}{q_2 \lambda_2}$ then core is non-empty.*

Theorem 4 again indicates importance of the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ for our analysis. It states that the core is non-empty if there are no artists who are considerably more popular with users who have high usage rate (i.e., type 2 users). This is because such artists have a potential to attract those users who have higher value for listening (joining some platform) than type 1 users. In the proof of Theorem 4, presented next, we also show that if

$\left(\max_{j\in\mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) > \frac{\lambda_1}{q_2\lambda_2}$, then one can choose other parameters of $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$ so that the core does not exist. This is because if this condition does not hold, there exists a subset of artists to which only type 1 users would subscribe and there exists another subset of artists to which only type 2 users would subscribe. If these two subsets of artists form a coalition J, i.e., offer their content on the competing platform and move together, then can get both type of users to subscribe to the new platform offering artists from J, resulting in a higher collective revenue for those artists than on the platform offering all artists.

Proof of Theorem 4: We first show that if $\frac{\pi_{21}}{\pi_{11}} \le \frac{\lambda_1}{q_2\lambda_2}$, then $x_j = \lambda_1\pi_{1j}$ for all $j\in\mathcal{M}$ defines a core. Consider set $J\subseteq\mathcal{M}$ and suppose these artists deviate altogether. We next list the possible revenues they can obtain and show that $\sum_{j\in J} x_j$ is (weakly) larger than that. We let $\pi_{iJ} = \sum_{j\in J}\pi_{ij}$ for $i=1,2$.

- $\frac{\pi_{2J}}{\pi_{1J}} > \frac{\lambda_1}{\lambda_2 q_2}$: This cannot happen as we have $\frac{\pi_{2j}}{\pi_{1j}} \le \frac{\lambda_1}{\lambda_2 q_2}$ for all j.
- $\frac{\lambda_1}{\lambda_2} < \frac{\pi_{2J}}{\pi_{1J}} \le \frac{\lambda_1}{\lambda_2 q_2}$: In this case the revenue of the artist in J is they deviate becomes $\lambda_1\pi_{1J}$ which is equal to $\sum_{j\in J} x_j$.
- $\frac{\lambda_1 q_1}{\lambda_2} < \frac{\pi_{2J}}{\pi_{1J}} \le \frac{\lambda_1}{\lambda_2}$: In this case the revenue of the artist in J is they deviate becomes $\lambda_2\pi_{2J}$ which is (weakly) smaller than $\lambda_1\pi_{1J} = \sum_{j\in J} x_j$.
- $\frac{\pi_{2J}}{\pi_{1J}} \le \frac{\lambda_1 q_1}{\lambda_2}$: In this case the revenue of the artist in J is they deviate becomes $\lambda_1 q_1\pi_{1J}$ which is (weakly) smaller than $\lambda_1\pi_{1J} = \sum_{j\in J} x_j$.

We next show that if $\frac{\pi_{21}}{\pi_{11}} > \frac{\lambda_1}{q_2\lambda_2}$ then there exists $\frac{\pi_{2j}}{\pi_{1j}}$ for which the platform cannot sustain the artists. In particular, we consider only two artists one in the set X and one in the set W. In particular, we let $\pi_{21} = 1-\pi_{11} = a$ and $\pi_{12} = 1-\pi_{22} = b$ and shows that there exists a and b such that the revenue of the platform is smaller than the sum of the individual revenues of these two artists. To have $1\in X$ and $2\in W$ the constraints on a and b become

$$1-a \le \frac{\lambda_2}{\lambda_1} q_2 b, \quad q_1 a \ge (1-b)\frac{\lambda_2}{\lambda_1}. \tag{65}$$

The platform cannot sustain both artists on it if we have

$$\lambda_1 < \lambda_2 q_2 b + q_1\lambda_1 a. \tag{66}$$

To satisfy (65), we let

$$a = \max\left\{(1-b)\frac{\lambda_2}{\lambda_1 q_1}, 1 - b\frac{\lambda_2 q_2}{\lambda_1}\right\},$$

which is less than 1 given

$$(1-b)\frac{\lambda_2}{\lambda_1 q_1} \le 1. \tag{67}$$

Plugging a into (66), we see that this inequality holds if we have

$$1-(1-b)\frac{\lambda_2}{\lambda_1} \le \frac{\lambda_2 q_2}{\lambda_1} b, \quad 1 - q_1(1 - b\frac{\lambda_2 q_2}{\lambda_1}) \le \frac{\lambda_2 q_2}{\lambda_1} b. \tag{68}$$

The inequalities given in (67) and (68) hold if we have

$$\frac{\lambda_2}{\lambda_1} \ge \max\left\{1 + \frac{q_1}{q_2}, \frac{1-q_1 q_2}{q_2}\right\},$$

completing the proof. ■

B.1.6. Usage-based services Here, we extend our baseline model and analysis in a setting in which the platform offers a usage-based service. The payment to the artists is the same as our baseline model. The charged fee on the user side is a function $\textsc{Rate}: 2^{\mathcal{M}} \to \mathbb{R}$ where $\textsc{Rate}(J)$ determines the per consumption fee that users need to pay to get access to the content of artists in the set J. With this definition, the utility of a type i user from subscription becomes

$$\lambda_i \sum_{j\in J} \pi_{ij} - \lambda_i \sum_{j\in J} \pi_{ij} \textsc{Rate}(J).$$

Lemma 8. *For any artist $j \in \mathcal{M}$, the optimal usage fee in going solo (i.e., offering the content directly to users) is* 1 *and revenue R'_j for artist j becomes $q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}$*

In the next proposition we compare different allocation rules when we have a usage-based service.

Proposition 14. *For any set $J \subseteq \mathcal{M}$, the optimal payment to artists for pro-rata rule, user-centric rule, and arbitrary rule coincide. Moreover, the platform's profit (with all three rules) is always zero.*

Proof of Proposition 14: Consider a platform with an arbitrary allocation rule. The optimal usage fee is 1, both user types subscribe, and the revenue of the platform becomes

$$\textsc{Rev} = q_1\lambda_1 \sum_{j\in J} \pi_{1j} + q_2\lambda_2 \sum_{j\in J} \pi_{2j}.$$

The optimal payment to artist $j \in J$ with an arbitrary allocation rule is equal to $R'_j = q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}$. Therefore, the platform's profit is zero. We next describe pro-rata and user-centric rules.

Pro-rata: Let β be the platform payout rate. Pro-rata payment to artist $j \in J$ is

$$\beta \ \textsc{Rev} \ \frac{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}{q_1\lambda_1 \sum_{j\in J}\pi_{1j} + q_2\lambda_2\sum_{j\in J}\pi_{2j}} = \beta \ \left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}\right).$$

This implies that the optimal platform's payout rate becomes

$$\beta^{\text{pr}} = \max_{j\in J}\left\{\frac{R'_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}\right\} \overset{(i)}{=} 1,$$

where (i) follow from Lemma 8. This further establishes that the platform's profit with pro-rata rule is zero.

User-centric: Let β be the platform payout rate. User-centric payment to artist $j \in J$ is

$$\beta\left(q_1\lambda_1 \sum_{j\in J}\pi_{1j}\frac{\lambda_1\pi_{1j}}{\sum_{j\in J}\lambda_1\pi_{1j}} + q_2\lambda_2\sum_{j\in J}\pi_{2j}\frac{\lambda_2\pi_{2j}}{\sum_{j\in J}\lambda_2\pi_{2j}}\right) = \beta \ \left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}\right).$$

This implies that the optimal platform's payout rate becomes

$$\beta^{\text{uc}} = \max_{j\in J}\left\{\frac{R'_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}\right\} \overset{(i)}{=} 1,$$

where (i) follow from Lemma 8. This further establishes that the platform's profit with user-centric rule is zero. ■

Notice that if the outside option is given by Lemma 8, then for any set $J \subseteq \mathcal{M}$, the platform's profit with all the three allocation rules is zero and the platform can bring all artists on it. Our algorithms, however, work for any outside option that the artists may have. We next establish how we can modify our algorithms to find the optimal set of artists when for any given outsider options of artists denoted by R'_j for $j \in \mathcal{M}$.

Optimal set of artists with usage-based service and an arbitrary allocation rule: With an arbitrary revenue allocation rule, the platform's problem becomes

$$\max_{J \subseteq \mathcal{M}} q_1 \lambda_1 \sum_{j \in J} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J} \pi_{1j} - \sum_{j \in J} R'_j$$

and therefore, the optimal set of artists is given by

$$J^* = \{ j \in \mathcal{M} \ : \ q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{1j} \geq R'_j \}.$$

Optimal set of artists with usage-based service and a pro-rata/user-centric allocation rule: With both pro-rata and user-centric rules, the platform's problem becomes

$$\max_{J \subseteq \mathcal{M}, \beta} (1-\beta) \left(q_1 \lambda_1 \sum_{j \in J} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J} \pi_{2j} \right)$$
$$\text{s.t. } \beta \left(q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j} \right) \geq R'_j \quad \text{for all } j \in J.$$

In order to solve this problem we first sort the values $\frac{R'_j}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}}$. Without loss of generality, let us assume

$$\frac{R'_1}{q_1 \lambda_1 \pi_{11} + q_2 \lambda_2 \pi_{21}} \leq \cdots \leq \frac{R'_m}{q_1 \lambda_1 \pi_{1m} + q_2 \lambda_2 \pi_{2m}}.$$

The optimal set of artists does not include any artist j for which

$$\frac{R'_j}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \geq 1.$$

For some $\bar{j} \in \mathcal{M}$, we let $\mathcal{M}' = \{1, \ldots, \bar{j}\}$ be the set of artists j for which $\frac{R'_j}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} < 1$ (if $\mathcal{M}' = \emptyset$, then the optimal set of artists is an empty set). Now notice that if the optimal set of artists includes $j' \leq \bar{j}$ then it must include all artists $j \leq j'$ as well. Therefore, the optimal set of artists is of the form $\{1, \ldots, j\}$ for some $j \in \mathcal{M}'$. In particular, the optimal set of artists is $\{1, \ldots, j^*\}$ where

$$j^* \in \arg\max_{j \in \mathcal{M}'} \left(1 - \frac{R'_j}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \right) \left(q_1 \lambda_1 \sum_{\ell=1}^{j} \pi_{1\ell} + q_2 \lambda_2 \sum_{\ell=1}^{j} \pi_{2\ell} \right).$$

Thus, we can find the optimal set of artists in time $O(m \log m)$.

B.1.7. Discriminatory subscription fees Here, we extend our baseline model and analysis in a setting in which the platform offers a subscription fee for each of the user types. The payment to the artists is the same as our baseline model. The charged fee on the user side are two functions $\text{FEE}_i : 2^{\mathcal{M}} \to \mathbb{R}$, for $i \in \{1, 2\}$, where $\text{FEE}_i(J)$ determines the subscription fee that type i users need to pay to get access to the content of artists in the set J. With this definition, the utility of a type i user from subscription becomes

$$\lambda_i \sum_{j \in J} \pi_{ij} - \text{FEE}_i(J).$$

LEMMA 9. *For any artist $j \in \mathcal{M}$, the optimal discriminatory subscription fee in going solo (i.e., offering the content directly to users) is*

$$\text{FEE}_1(\{j\}) = \lambda_1 \pi_{1j} \quad \text{FEE}_2(\{j\}) = \lambda_2 \pi_{2j}.$$

In the next proposition we compare different allocation rules with discriminatory subscription fees.

PROPOSITION 15. *For any set $J \subseteq \mathcal{M}$, the optimal payment to artists for pro-rata rule, user-centric rule, and arbitrary rule coincide. Moreover, the platform's profit is zero.*

Proof of Proposition 15: Consider a platform with an arbitrary allocation rule. The optimal discriminatory subscription fee is

$$\text{FEE}_1(J) = \lambda_1 \sum_{j \in J} \pi_{1j} \qquad \text{and} \qquad \text{FEE}_2(J) = \lambda_2 \sum_{j \in J} \pi_{2j},$$

both user types subscribe, and the revenue of the platform becomes

$$\text{REV} = q_1 \text{FEE}_1(J) + q_2 \text{FEE}_2(J).$$

The optimal payment to artist $j \in J$ with an arbitrary allocation rule is equal to $R'_j = q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}$. Therefore, the platform's profit is zero. We next describe pro-rata and user-centric rules.

Pro-rata: Let β be the platform payout rate. Pro-rata payment to artist $j \in J$ is

$$\beta \ \text{REV} \ \frac{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}{q_1\lambda_1 \sum_{j \in J} \pi_{1j} + q_2\lambda_2 \sum_{j \in J} \pi_{2j}} = \beta \ \left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}\right).$$

This implies that the optimal platform's payout rate becomes

$$\beta^{\text{pr}} = \max_{j \in J} \left\{ \frac{R'_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}} \right\} \overset{(i)}{=} 1,$$

where (i) follow from Lemma 9. This further establishes that the platform's profit with pro-rata rule is zero.

User-centric: Let β be the platform payout rate. User-centric payment to artist $j \in J$ is

$$\beta \left(q_1 \text{FEE}_1(J) \frac{\lambda_1\pi_{1j}}{\sum_{j \in J} \lambda_1\pi_{1j}} + q_2 \text{FEE}_2(J) \frac{\lambda_2\pi_{2j}}{\sum_{j \in J} \lambda_2\pi_{2j}} \right) = \beta \ \left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}\right).$$

This implies that the optimal platform's payout rate becomes

$$\beta^{\text{uc}} = \max_{j \in J} \left\{ \frac{R'_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}} \right\} \overset{(i)}{=} 1,$$

where (i) follows from Lemma 9. This further establishes that the platform's profit with user-centric rule is zero. ■

The intuition of Proposition 15 is similar to that of Proposition 14: with discriminatory subscription fee, the cross-subsidization among high and low volume users goes away and therefore pro-rata and user-centric rules coincide.

We conclude by noting that a similar analysis to that of Propositions 15 shows that for any number of user types with discriminatory subscription fees, pro-rata and user-centric coincide. However, if the number of subscription fees is smaller than the number of types, the cross-subsidization implies that pro-rata and user-centric rules have different implications for both the platform and the artists, as we established in our baseline model.

B.1.8. Paying artists their marginal contribution to the platform's revenue and VCG Here we consider a payment rule that allocates to an artist j her contribution to the platform's revenue. We first characterize this payment and then show that it can be below the outside option of the artists, implying that the platform is not profitable with this payment rule.

Suppose Assumption 1 holds. The revenue of the platform when all artists are on the platform is λ_1. We take artist j moves out of the platform, and again find the platform's revenue. The payment to artist j is the former revenue minus the latter revenue. We next characterize this payment.

LEMMA 10. *Suppose Assumption 1 holds. We have the following cases:*

1. $\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[0, \frac{q_1\lambda_1}{\lambda_2}\right)$: *if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_1(1-\pi_{1j})$, only type 1 users subscribe to the platform, and platform's revenue becomes $q_1\lambda_1(1-\pi_{1j})$. Therefore, the payment to artist j is $\lambda_1 - q_1\lambda_1(1-\pi_{1j})$.*
2. $\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}\right)$: *if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_2(1-\pi_{2j})$, both type of users subscribe to the platform, and platform's revenue becomes $\lambda_2(1-\pi_{2j})$. Therefore, the payment to artist j is $\lambda_1 - \lambda_2(1-\pi_{2j})$.*
3. $\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2 q_2}\right)$: *if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_1(1-\pi_{1j})$, both type of users subscribe to the platform, and platform's revenue becomes $\lambda_1(1-\pi_{1j})$. Therefore, the payment to artist j is $\lambda_1 - \lambda_1(1-\pi_{1j})$.*
4. $\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2 q_2}, \infty\right)$: *if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_2(1-\pi_{2j})$, only type 2 users subscribe to the platform, and platform's revenue becomes $q_2\lambda_2(1-\pi_{2j})$. Therefore, the payment to artist j is $\lambda_1 - q_2\lambda_2(1-\pi_{2j})$.*

The proof of this lemma is similar to the proof of Lemma 1 and hence is omitted.

Comparing the payments characterized in Lemma 10 to the revenue of an artist in a solo market (characterized in Lemma 1), for π_{2j} and π_{1j} in the following regions the payment to the artists according to its marginal contribution to the revenue is smaller than the revenue of the solo market (see Figure 7):

$$\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{q_2\lambda_2}, \infty\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ or}$$
$$\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{q_2\lambda_2}, \infty\right) \text{ or}$$
$$\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ or}$$
$$\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[0, \frac{q_1\lambda_1}{\lambda_2}\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{q_2\lambda_2}, \infty\right).$$

The following is the VCG (Vickrey (1961), Clarke (1971), Groves (1973)) mechanism in our setting:

PROPOSITION 16. *Let $\mathbf{R} = (R_1, \ldots, R_m)$ be the vector of reported outside options. Then the pricing scheme*

$$p_j(\mathbf{R}) = \begin{cases} q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} & q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} \geq R_j \\ 0 & q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} < R_j. \end{cases}$$

incentivizes artists to report their true outside option truthfully and maximizes the welfare.

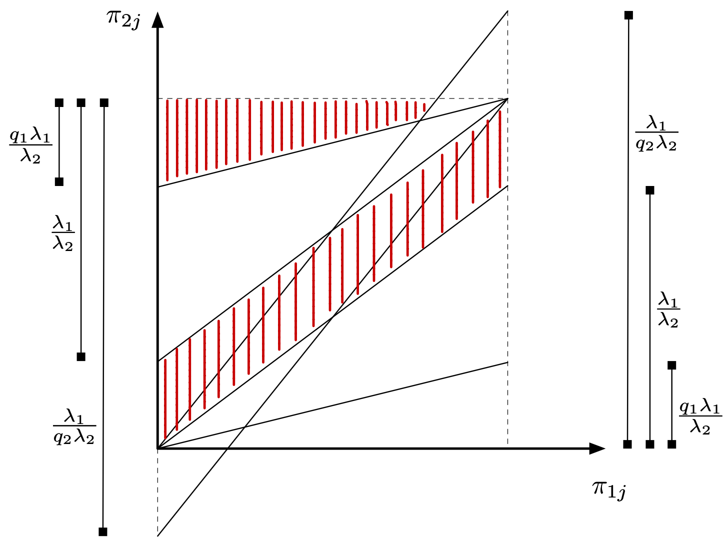


Figure 7 The pairs (π_{1j}, π_{2j}) for which the marginal contribution of artist j to platform's revenue is smaller than its revenue from a solo market.

Proof of Proposition 16: we first establish the functional form of the VCG payments and then simplify it.

For a vector of reported outside options $\mathbf{R}$, letting $a_j \in \{0,1\}$ for all $j \in \mathcal{M}$ denote whether artist j is on the platform or not, the VCG payment to each artist $\ell \in \mathcal{M}$ is given by

$$p_\ell(\mathbf{R}) = \left(\sum_{j \neq \ell} a_j(\mathbf{R})(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j) \right) + (q_1\lambda_1\pi_{1\ell} + q_1\lambda_1\pi_{1j\ell})a_\ell(\mathbf{R}) + h_\ell(\mathbf{R}_{-\ell}), \tag{69}$$

where $\mathbf{a}(\mathbf{R}) = (a_1(\mathbf{R}), \dots, a_m(\mathbf{R}))$ is

$$\mathbf{a}(\mathbf{R}) \in \arg\max_{\mathbf{a}\in\{0,1\}^m} \sum_{j=1}^{m} a_j(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j), \tag{70}$$

and

$$h_\ell(\mathbf{R}_{-\ell}) = -\max_{\mathbf{a}\in\{0,1\}^{m-1}} \sum_{j\neq\ell} a_j(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j). \tag{71}$$

This incentivizes the artists to report truthfully. We show that each artist $\ell \in \mathcal{M}$ prefers to report truthfully by comparing her utility when she reports truthfully and when she reports $R'_\ell \neq R_\ell$:

$$\begin{aligned} p_\ell(R'_\ell, \mathbf{R}_{-\ell}) + (1 - a_\ell(R'_\ell, \mathbf{R}_{-\ell}))R_\ell &= \sum_{j=1}^{m} a_j(R'_\ell, \mathbf{R}_{-\ell})\left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j\right) + R_\ell + h_\ell(\mathbf{R}_{-\ell}) \\ &\overset{(a)}{\leq} \sum_{j=1}^{m} a_j(R_\ell, \mathbf{R}_{-\ell})\left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j\right) + R_\ell + h_\ell(\mathbf{R}_{-\ell}) \\ &= p_\ell(R_\ell, \mathbf{R}_{-\ell}) + (1 - a_\ell(R_\ell, \mathbf{R}_{-\ell}))R_\ell \end{aligned}$$

where (a) follows from (70). We next simplify this payment rule. Notice that in (70) and (71), we have $a_j = 1$ if and only if $q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} \geq R_j$. Therefore, (69) simplifies to

$$p_\ell(\mathbf{R}) = \begin{cases} q_1\lambda_1\pi_{1\ell} + q_2\lambda_2\pi_{2\ell} & q_1\lambda_1\pi_{1\ell} + q_2\lambda_2\pi_{2\ell} \geq R_\ell \\ 0 & q_1\lambda_1\pi_{1\ell} + q_2\lambda_2\pi_{2\ell} < R_\ell. \end{cases}$$

This completes the proof. ■

B.1.9. Optimal additions to the set of artists on the platform Here, we show that Algorithm 2 readily extends and provides the same approximation of the optimal profit for a platform which is "locked in" (e.g., contractually) with a subset of artists already "established" on the platform and seeks to choose an optimal subset of "new" artists to add to its artists' portfolio. In particular, suppose set $\tilde{J}$ of artists already exists on the platform and the platform is choosing the optimal subset of $\mathcal{M} \setminus \tilde{J}$ to include. To solve this problem, we can update optimization (25) as follows

$$\begin{aligned} \max_{z,\ x_j\ :\ j \in \mathcal{M}\setminus\tilde{J}} \ & z - \sum_{j \in \mathcal{M}\setminus\tilde{J}} x_j R_j \\ \text{s.t. } & z \leq \sum_{j \in \mathcal{M}\setminus\tilde{J}} x_j \lambda_1 \pi_{1j} + \sum_{j \in \tilde{J}} \lambda_1 \pi_{1j} \\ & z \leq \sum_{j \in \mathcal{M}\setminus\tilde{J}} x_j \lambda_2 \pi_{2j} + \sum_{j \in \tilde{J}} \lambda_2 \pi_{2j} \\ & 0 \leq x_j \leq 1, \quad j \in \mathcal{M} \setminus \tilde{J}, \end{aligned}$$

and then solve it with an identical algorithm to 2 to obtain a solution that is integral except for one variable. Similar to Theorem 3, rounding up the fraction variable gives us a solution whose profit is at most $\max_{j\in\mathcal{M}} R_j$ away from the optimal profit.

B.1.10. Extension to costly service The results of Propositions 2 and 3 continue to hold in a setting with costly services. We next present the proof of Proposition 2 in this setting with costly services (and note that the proof of Proposition 3 extends similarly).

Let β^{pr} and β^{uc} be the β's with which pro-rata and user-centric rules can sustain all artists on the platform with pro-rata and user-centric allocation rules. For a pro-rata rule, the payment to artist j is

$$\beta^{\text{pr}} \textsc{Rev} \frac{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}}{q_1 \lambda_1 + q_2 \lambda_2}. \tag{72}$$

For a user-centric rule, the payment to artist j is

$$\beta^{\text{uc}} \textsc{Rev} (q_1 \pi_{1j} + q_2 \pi_{2j}) \tag{73}$$

Using equations (72) and (73), the ratio of the revenue that artist j obtains with pro-rata versus user-cenrtric is

$$\frac{\beta^{\text{pr}}}{\beta^{\text{uc}}(q_1\lambda_1 + q_2\lambda_2)} \frac{q_1 \lambda_1 \frac{\pi_{1j}}{\pi_{2j}} + q_2 \lambda_2}{q_1 \frac{\pi_{1j}}{\pi_{2j}} + q_2}.$$

Similar to the proof of Proposition 2, the ratio of artists' payment with pro-rata to their payment with user-centric is increasing in $\frac{\pi_{2j}}{\pi_{1j}}$.

Moreover, Algorithm 1 for finding the optimal pro-rata/user-cetric extend to this setting by noting that we developed them for arbitrary outside option of artists. More precisely, let us consider the algorithm for finding the optimal pro-rata rule. This algorithm continues to hold if we replace R_j by $\tilde{R}_j + c_j(\{1,2\})$ when the set $\{1,2\}$ of user types subscribe to the platform.

B.2. Examples

B.2.1. Introductory example with two artists We consider a setting with only two artists, $m=2$, and we also let $\pi_{11}=\pi_{22}=\pi$, i.e., each user type i listens to artist i's music with probability π and the other artist with probability $1-\pi$. In order to highlight how the usage rate of "fans" of an artist impact the payment of that artist, we further assume that π is sufficiently large. More precisely, we suppose $\frac{\pi}{1-\pi} \geq \max\{\frac{\lambda_2}{\lambda_1 q_1}, \frac{\lambda_1}{\lambda_2 q_2}\}$, which guarantees that user type i only subscribes to the platform if artist i is on the platform (as shown next). Additionally, we also suppose $\frac{\lambda_2}{\lambda_1} \leq \min\{\frac{1}{q_2}, \frac{1-\pi q_1}{\pi q_2}\}$, which guarantees platform sustainability, i.e., that both artists join the platform.

With these assumptions on user consumption we ensure that both artists join the platform and, consequently, that both user types subscribe. We next discuss artists' preferences over different payment allocation rules. We separate discussion in two cases:

1. $q_1 \leq q_2$: In this case, artist 2 whose fans have higher usage rate (recall that $\lambda_2 \geq \lambda_1$) has the majority of the users as her fans. We call artist 2 the "superstar" artist and artist 1 the "niche" artist. In this case, with both pro-rata and user-centric allocation rules, the platform pays the niche artist 1 more than what she could obtain through a direct channel. This is because of the positive externality of the presence of fans of the superstar artist on the platform as they listen to artist 1 with probability $1-\pi$. Furthermore, note that the niche artist 1 prefers the user-centric allocation rule to the pro-rata allocation rule (i.e., the payment to niche artist 1 with user-centric allocation rule is larger than her payment with pro-rata allocation rule) because fans of artist 1 have lower usage rate λ_1 and consequently, with unlimited consumption subscription fee model, end up paying more per unit of consumption than fans of artist 2. In Subsection 3.2.1, we show this result continues to hold with multiple artists.

2. $q_1 > q_2$: In this case, artist 2 preference depends on the ratio $\frac{\lambda_2}{\lambda_1}$. For $\frac{\lambda_2}{\lambda_1} \geq \frac{q_1}{q_2}$ (i.e., $\lambda_2 q_2 \geq \lambda_1 q_1$) artist 2 prefers the pro-rata allocation rule to the user-centric allocation rule. This is because the aggregate usage rate of fans of artist 2 is large enough that makes the pro-rata rule which is based on aggregate consumption more attractive. For $\frac{\lambda_2}{\lambda_1} < \frac{q_1}{q_2}$, however, artist 2 prefers user-centric allocation rule to pro-rata.

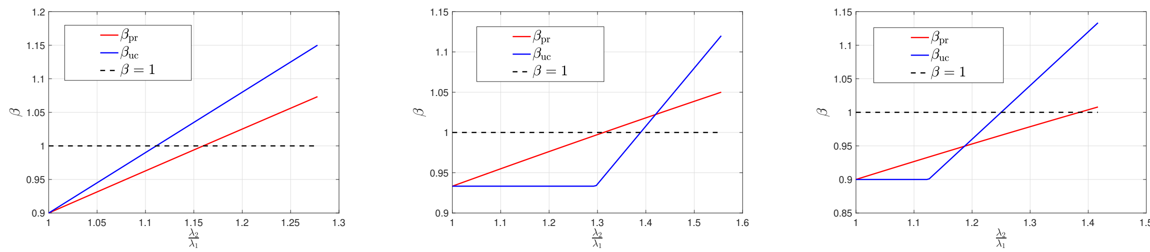
We next consider the platform's problem and characterize the preferred allocation rule of the platform. We denote by β^{pr} and β^{uc}, the smallest payout rate that guarantees the sustainability of the platform for pro-rata and user-centric allocation rules, respectively. Note that the platform's profit is $(1-\beta)R$ as any revenue-sharing rule mandates βR total payments to artists, hence the platform prefers the allocation rule that makes it profitable with β as small as possible. Also note that platform is not profitable if $\beta > 1$.

Proposition 17. *There exists $\bar{q} \geq \frac{1}{2}$ such that:*

(a) If $q_1 \leq q_2$, then $\beta^{\mathrm{pr}} < \beta^{\mathrm{uc}}$ and the platform's profit is higher with the pro-rata allocation than with the user-centric allocation.

(b) If $q_1 \geq q_2$ and $q_1 \geq \bar{q}$, then $\beta^{\mathrm{pr}} > \beta^{\mathrm{uc}}$ and the platform's profit is higher with the user-centric allocation than with the pro-rata allocation.

(c) If $q_1 \geq q_2$ and $q_1 \in [\frac{1}{2}, \bar{q}]$, then there exists $\bar{r} \geq 1$ such that (i) if $\frac{\lambda_2}{\lambda_1} \geq \bar{r}$, then $\beta^{\mathrm{pr}} < \beta^{\mathrm{uc}}$ and the platform's profit is higher with the pro-rata allocation than with the user-centric allocation, and (ii) if $\frac{\lambda_2}{\lambda_1} < \bar{r}$, then $\beta^{\mathrm{pr}} > \beta^{\mathrm{uc}}$ and the platform's profit is higher with the user-centric allocation than with the pro-rata allocation.



(a) $\beta^{\rm uc}$ and $\beta^{\rm pr}$ for $q_1 = 0.4$ (b) $\beta^{\rm uc}$ and $\beta^{\rm pr}$ for $q_1 = 0.7$ (c) $\beta^{\rm uc}$ and $\beta^{\rm pr}$ for $q_1 = 0.6$

Figure 8 Illustration of Proposition 17 for $\pi = 0.85$.

Part (a) of Proposition 17 shows that in the presence of superstar artist 2 (i.e., an artist whose fans are not only majority of platform users, $q_2 > q_1$, but also have a high consumption rate, $\lambda_2 > \lambda_1$), the pro-rata allocation rule is always better than user-centric allocation rule. This is because in such situations the pro-rata allocation, unlike the user-centric allocation, implicitly compensates artist 2 for the positive externality her fans bring to the platform and to the consumption of the niche artist 1 content. Part (b) of Proposition 17 establishes that if artist 1 whose fans have a lower usage rate have a large enough majority of the users, then the platform's profits are higher with the user-centric allocation rule. This is because fans of artist 1 have a low usage rate and are a sufficiently large majority of platform users, so the positive externality of their presence on the platform is limited. Consequently, there is a limited effect of the pro-rata allocation that favors artists listened by high usage rate users. Furthermore, given low usage rate of its fans, artist 1 prefers the user-centric allocation, and since she is the artist with the better outside option (by potentially offering content through a direct channel and take away its fans from the platform) it is also in the interest of keeping all artists, that platform prefers the user-centric allocation rule. Finally, in part (c) of Proposition 17 the analysis is more nuanced and the preferred allocation rule depends on the ratio of the higher usage rate to the lower usage rate. In Subsection 3.2.2, we show that the results of Proposition 17 generalize to m artists with generic probabilities π_{ij}, for $i = 1, 2$ and $j = 1, \ldots, m$.

Figure 8 illustrates the results of Proposition 17. In this figure, $\beta^{\rm uc}$ and $\beta^{\rm pr}$ are the minimum β with which the platform can sustain both artists on it with user-centric and pro-rata allocation rules, respectively. Note that in part (a), corresponding to $q_1 < q_2$ and $\lambda_1 < \lambda_2$, the pro-rata allocation is not only preferred, but also there is a range of parameters for which the platform is profitable only with the pro-rata and is not profitable with the user-centric allocation (i.e., $\beta^{\rm pr} \leq 1 < \beta^{\rm uc}$).

Proof of Proposition 17: We use the following two lemmas in this proof.

Lemma 11. *Suppose* $\frac{\pi}{1-\pi} \geq \max\{\frac{\lambda_2}{q_1\lambda_1}, \frac{\lambda_1}{\lambda_2 q_2}\}$ *and* $\frac{\lambda_2}{\lambda_1} \leq \min\{\frac{1}{q_2}, \frac{1-\pi q_1}{\pi q_2}\}$. *In equilibrium both artists join the platform and both user types subscribe and the profit of the platform becomes* $\lambda_1 - \pi q_1 \lambda_1 - \lambda_2 \pi q_2$. *Otherwise, in any equilibrium the profit of the platform is* 0.

Proof: We first show that with the assumptions of the lemma if an artist such as i offers her products through direct channels by herself, the fans of the other artist will not buy her content and her revenue will be $q_i \lambda_i \pi$. To see this, consider artist 1 and suppose the contrary, i.e., she sets the prices such that the fans of artist 2 subscribe. This means that the subscription fee is at most $\lambda_2(1 - \pi)$. Now if this artist chooses the prices such that only her own fans subscribe, then her revenue will be at least $\lambda_1 \pi q_1$. Therefore, as long as we have $\lambda_1 \pi q_1 \geq \lambda_2(1-\pi)$ and $\lambda_2 \pi q_2 \geq \lambda_1(1-\pi)$, it is optimal for each artist to choose prices such that only her own fans join. Now consider having both artists on the platform. Because $\frac{\lambda_1}{\lambda_1} \leq \frac{1}{q_2}$, the optimal subscription fee is λ_1 and all fans subscribe. This generates revenue λ_1 for the platform. The payment that the platform need to pay artist i is the minimum payment that makes artist i indifferent between staying on the platform and offering her content by herself. As we have shows, the revenue she obtains from going solo is $\lambda_i \pi q_i$. Therefore, the overall profit of the platform is

$$\lambda_1 - \lambda_1 \pi q_1 - \lambda_2 \pi q_2.$$

If the platform wants only artist i to join it, then again as we showed only fans of artist i subscribe and the profit of platform becomes 0. Therefore, both artists joining and all users subscribing is an equilibrium if $\lambda_1 - \lambda_1 \pi q_1 - \lambda_2 \pi q_2$, which holds because of the bounds on $\frac{\lambda_2}{\lambda_1}$. ■

LEMMA 12. *Suppose* $\frac{\pi}{1-\pi} \geq \max\{\frac{\lambda_2}{q_1 \lambda_1}, \frac{\lambda_1}{\lambda_2 q_2}\}$ *and* $\frac{\lambda_2}{\lambda_1} \leq \min\{\frac{1}{q_2}, \frac{1-\pi q_1}{\pi q_2}\}$. *We have the following:*

(a) ***Pro-rata:*** *For* $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2}$ *we have* $\beta^{\text{pr}} = \pi q_1 \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_1 \pi + q_2 \frac{\lambda_2}{\lambda_1}(1-\pi)}$ *in which case the platform is paying artist* 2 *more than the revenue she would obtain if she deviates and offers her music by herself. For* $\frac{\lambda_2}{\lambda_1} \geq \frac{q_1}{q_2}$ *we have* $\beta^{\text{pr}} = \pi q_2 \frac{\lambda_2}{\lambda_1} \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_2 \pi \frac{\lambda_2}{\lambda_1} + q_1(1-\pi)}$ *in which case the platform is paying artist* 1 *more than the revenue she would obtain if she deviates and offers her music by herself.*

(b) ***User-centric:*** *For* $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2} \frac{q_2 \pi + q_1(1-\pi)}{q_1 \pi + q_2(1-\pi)}$ *we have* $\beta^{\text{uc}} = \frac{\pi q_1}{q_1 \pi + q_2(1-\pi)}$ *in which case the platform is paying artist* 2 *more than the revenue she would obtain if she deviates and offers her music by herself through a direct channel. For* $\frac{\lambda_2}{\lambda_1} \geq \frac{q_1}{q_2} \frac{q_2 \pi + q_1(1-\pi)}{q_1 \pi + q_2(1-\pi)}$ *we have* $\beta^{\text{uc}} = \frac{\pi q_2 \frac{\lambda_2}{\lambda_1}}{q_2 \pi + q_1(1-\pi)}$ *in which case the platform is paying artist* 1 *more than the revenue she would obtain if she deviates and offers her music by herself.*

Proof: For pro-rata allocation rule we must have

$$\lambda_1 \beta^{\text{pr}} \frac{\lambda_1 q_1 \pi + q_2 \lambda_2 (1-\pi)}{\lambda_1 q_1 + \lambda_2 q_2} \geq \lambda_1 \pi q_1,$$

where we used Lemma 11 to write the right-hand side, i.e., the revenue of an artist for offering her content directly. Similarly, we have

$$\lambda_1 \beta^{\text{pr}} \frac{\lambda_2 q_2 \pi + q_1 \lambda_1 (1-\pi)}{\lambda_1 q_1 + \lambda_2 q_2} \geq \lambda_2 \pi q_2.$$

Therefore, the minimum β to have both inequalities is

$$\beta^{\text{pr}} = \max\left\{ \pi q_1 \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_1 \pi + q_2 \frac{\lambda_2}{\lambda_1}(1-\pi)}, p q_2 \frac{\lambda_2}{\lambda_1} \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_2 \pi \frac{\lambda_2}{\lambda_1} + q_1(1-\pi)} \right\}.$$

By comparing the two terms of the maximum and canceling out the common terms, we see that the first term of the maximum is higher than the second term if and only if $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2}$

For user-centric allocation rule we must have

$$\lambda_1 \beta^{\text{uc}} (q_1 \pi + q_2 (1-\pi)) \geq \lambda_1 \pi q_1,$$

and

$$\lambda_1 \beta^{\text{uc}} (q_2 \pi + q_1 (1-\pi)) \geq \lambda_2 \pi q_2,$$

which leads to

$$\beta^{\mathrm{uc}} = \max\left\{\pi q_1 \frac{1}{q_1\pi + q_2(1-\pi)}, \pi q_2 \frac{\lambda_2}{\lambda_1}\frac{1}{q_2\pi + q_1(1-\pi)}\right\}.$$

By comparing the two terms of the maximum and canceling out the common terms, we see that the first term of the max is higher than the second term if and only if $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2}\frac{q_2\pi+q_1(1-\pi)}{q_1\pi+q_2(1-\pi)}$. ■

We now proceed with the proof of proposition. Given $q_1 \leq \frac{1}{2}$, using Lemma 12, we obtain

$$\beta^{\mathrm{pr}} = \pi q_2 \frac{\lambda_2}{\lambda_1}\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_2\pi\frac{\lambda_2}{\lambda_1} + q_1(1-\pi)}, \quad \beta^{\mathrm{uc}} = \pi q_2 \frac{\lambda_2}{\lambda_1}\frac{1}{q_2\pi + q_1(1-\pi)}.$$

Comparing these two expressions we find out $\beta^{\mathrm{pr}} \leq \beta^{\mathrm{uc}}$. This proves part (a). We next prove parts (b) and (c). Given, $q_1 \geq q_2$, we have the following cases:

- $\frac{\lambda_2}{\lambda_1} \in [1, \frac{q_1}{q_2}\frac{q_2\pi+q_1(1-\pi)}{q_1\pi+q_2(1-\pi)}]$: In this interval, we have

$$\beta^{\mathrm{uc}} = \frac{q_1\pi}{q_1\pi + q_2(1-\pi)} \leq q_1\pi\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_1\pi + q_2\frac{\lambda_2}{\lambda_1}(1-\pi)} = \beta^{\mathrm{pr}}.$$

- $\frac{\lambda_2}{\lambda_1} \in [\frac{q_1}{q_2}\frac{q_2\pi+q_1(1-\pi)}{q_1\pi+q_2(1-\pi)}, \frac{q_1}{q_2}]$: In this interval, we have

$$\beta^{\mathrm{uc}} = \frac{q_2\pi\frac{\lambda_2}{\lambda_1}}{q_2\pi + q_1(1-\pi)} \leq q_1\pi\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_1\pi + q_2\frac{\lambda_2}{\lambda_1}(1-\pi)} = \beta^{\mathrm{pr}},$$

if and only if $\frac{\lambda_2}{\lambda_1} \leq \bar{r}$, where $\bar{r}$ is the larger solution of $\frac{q_2\pi\frac{\lambda_2}{\lambda_1}}{q_2\pi+q_1(1-\pi)} = q_1\pi\frac{q_1+q_2\frac{\lambda_2}{\lambda_1}}{q_1\pi+q_2\frac{\lambda_2}{\lambda_1}(1-\pi)}$.

- $\frac{\lambda_2}{\lambda_1} \in [\frac{q_1}{q_2}, \infty)$: In this range, using $2p-1 \geq 0$, we have

$$\beta^{\mathrm{uc}} = \frac{q_2\pi\frac{\lambda_2}{\lambda_1}}{q_2\pi + q_1(1-\pi)} \geq q_2\pi\frac{\lambda_2}{\lambda_1}\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_2\frac{\lambda_2}{\lambda_1}\pi + q_1(1-\pi)} = \beta^{\mathrm{pr}}.$$

Therefore, for $\frac{\lambda_2}{\lambda_1} \geq \bar{r}$ we have $\beta^{\mathrm{uc}} \geq \beta^{\mathrm{pr}}$ and for $\frac{\lambda_2}{\lambda_1} \leq \bar{r}$ we have $\beta^{\mathrm{uc}} \leq \beta^{\mathrm{pr}}$. Now the question is whether pro-rata or user-centric can sustain $\frac{\lambda_2}{\lambda_1} \geq \bar{r}$ in equilibrium. This depends on whether β^{uc} is larger than 1 at $\bar{r}$ or not. There exists a $\bar{q}$ such that for $q_1 \geq \bar{q}$, β^{uc} (which is equal to β^{pr} at $\bar{r}$) is larger than 1 at $\frac{\lambda_2}{\lambda_1} = \bar{r}$. Finally, for $q_1 \leq \bar{q}$, β^{uc} and β^{pr} cross each other within the range that these two payments can sustain equilibrium. This completes the proof. ■

B.2.2. Detail of Example 1 Assuming all artists either belong to X or W, using (11), we obtain

$$\begin{aligned}\beta^{\mathrm{pr}} &= \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1}\max\left\{\max_{j\in X}\frac{q_2\lambda_2\pi_{2j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}, \max_{j\in W}\frac{\lambda_1 q_1\pi_{1j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}\right\}\\ &= \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1}\max\left\{\frac{q_2\lambda_2 x}{q_1\lambda_1 + q_2\lambda_2 x}, \frac{\lambda_1 q_1}{q_1\lambda_1 + q_2\lambda_2 w}\right\}.\end{aligned}$$

Therefore, the pivot artist is in the set X (for which $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$), if and only if we have $\frac{q_2\lambda_2 x}{q_1\lambda_1+q_2\lambda_2 x} \geq \frac{\lambda_1 q_1}{q_1\lambda_1+q_2\lambda_2 w}$. After some simplification, this results in $\frac{\lambda_2 q_2}{\lambda_1 q_1} \geq \frac{1}{xw}$ which is the same as part (a) of Example 1. Therefore, Proposition 3 establishes that the platform prefers pro-rata rule to user-centric rule.

Using (12), we obtain

$$\begin{aligned}\beta^{\mathrm{uc}} &= \frac{1}{\lambda_1}\max\left\{\max_{j\in X}\frac{q_2\lambda_2\pi_{2j}}{q_1\pi_{1j} + q_2\pi_{2j}}, \max_{j\in W}\frac{\lambda_1 q_1\pi_{1j}}{q_1\pi_{1j} + q_2\pi_{2j}}\right\}\\ &= \frac{1}{\lambda_1}\max\left\{\frac{q_2\lambda_2 x}{q_1 + q_2 x}, \frac{\lambda_1 q_1}{q_1 + q_2 w}\right\}.\end{aligned}$$

Therefore, the pivot artist is in the set W (for which $\frac{\pi_{2j}}{\pi_{1j}} \leq 1$), if and only if we have $\frac{\lambda_1 q_1}{q_1+q_2 w} \geq \frac{q_2\lambda_2 x}{q_1+q_2 x}$. After some simplification, this results in the same condition as part (b) of Example 1. Therefore, Proposition 3 establishes that the platform prefers user-centric rule to pro-rata rule.