

Internet Appendix

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A Extensions to the Economic Model

This section examines three extensions of the economic model in Section 2. Subsection A.1 shows that the results do not critically hinge on the functional form of $G(n, \theta, r)$. Subsection A.2 shows that it is possible to allow stochastic fluctuations in the excited-speculator population dynamics. Subsection A.3 shows that the main conclusions are unchanged if you allow for continuous feedback between the excited-speculator population and prices rather than letting the excited-speculator population reach steady state each trading period.

A.1 Logistic Approximation

The law of motion for excited-speculator population dynamics in Section 2 is:

$$G(n, \theta, r) = \theta \cdot r \cdot (1 - n) \times n - n \tag{A1}$$

$\Theta(n, \theta, r) = \theta \cdot r \cdot (1 - n) \times n$ is the rate at which currently apathetic speculators get excited about an asset, and $\Omega(n) = n$ is the rate at which currently excited speculators lose interest. This law of motion captures the essential behavior around at the bifurcation point of a broad range of functional forms which represent feedback trading.

Feedback trading occurs when an initial positive shock generates excess media coverage and word-of-mouth buzz, which attracts new speculators to the market, which generates even more media coverage and word-of-mouth buzz, which excites still

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more speculators, which generates still more media coverage and word-of-mouth buzz, and so on. . . This narrative incorporates four key elements:

1. First, there must be some notion of a typical size for the excited-speculator population. Without loss of generality, let us normalize this size to $n = 0$.
2. Second, the excited-speculator population dynamics should reflect the fact that their population grows due to social interactions. Traders go “mad in herds. (Mackay, 1841)” “There is a strong interpersonal component to investing, as hypothesized in epidemic models. (Shiller and Pound, 1989)”

Thus, since it is harder to excite additional speculators when there are fewer apathetic agents left to interact with, the crowd of excited speculators should grow most rapidly when it is small and then grow more and more slowly as it gets larger and larger. In short, the arrival rate should be convex in the current population size: $\partial_n \partial_n [\Theta(n, \theta, r)] < 0$.

3. Third, the probability that each excited speculator loses interest and departs the crowd should be independent of this current population size. That same Mackay (1841) epigram says that traders “recover their senses slowly and one by one”. In other words, the per capital departure rate, $\Omega(n)/n = \omega$, should be constant.

Without loss of generality, I assume that $\omega = 1$. If $n_\tau(\omega)$ and $\theta(\omega)$ represent the true values that depend on ω , then this renormalization is equivalent to re-defining $n_\tau \stackrel{\text{def}}{=} n_\tau(\omega)/\omega$ and $\theta \stackrel{\text{def}}{=} \theta(\omega)/\omega$.

4. Finally, Shiller (2000) describes how “whenever the market reaches a new high, public speakers, writers, and other prominent people suddenly appear, armed with explanations for the apparent optimism seen in the market”. He points out that “the new era thinking they promote is part of the process by which a boom may be sustained and amplified—part of the feedback mechanism that. . . can create speculative bubbles”.

Thus, the first speculators who get excited and enter the market should find it easier to attract additional friends to join them when past returns are higher: $\partial_n \partial_r [\Theta(n, \theta, r)]_{n=0} > 0$. But, to make sure we are not assuming the result, these price changes should not have any higher-order effects: $\partial_r \partial_r [\Theta(n, \theta, r)] = 0$.

The definition below converts these four elements into properties of the growth, arrival, and departure rates for the excited-speculator population.

Definition A.1 (Feedback Trading). *The population dynamics of excited speculators are governed by feedback trading if the following four conditions are satisfied:*

1. For all $r > 0$, we have that $G(0, \theta, r) = 0$.
2. For all $n \in (0, 1)$ and $r > 0$, we have that $\partial_n \partial_n [\Theta(n, \theta, r)] < 0$.
3. $\Omega(n) = n$.
4. For all $r > 0$, we have that $\partial_n \partial_r [\Theta(n, \theta, r)]_{n=0} > 0$ and $\partial_r \partial_r [\Theta(n, \theta, r)] = 0$.

An excited-speculator population governed by $\Theta(n, \theta, r) = \theta \cdot r \cdot (1 - n) \times n$ as in Equation (A1) clearly displays feedback trading. But, so does a population governed by $\tilde{\Theta}(n, \theta, r) = r \cdot (1 - e^{-\theta n})$. These functional forms look superficially different, but both satisfy the feedback-trading criteria and therefore behave identically around r_\star .

Proposition A.1 (Bubble-Generating Mechanism as a Logistic Approximation). *Suppose that a population of excited speculators obeys the law of motion $\tilde{G}(n, \theta, r)$.*

If there exists some $r_- > 0$ such that $\partial_n [\tilde{G}(n, \theta, r_-)]_{n=0} < 0$ and excited speculators engage in feedback trading (Definition A.1), the population will display a sudden qualitative change in steady-state behavior at a critical return threshold, $r_\star > r_-$.

Here is the intuition. First, if the population of excited speculators engages in feedback trading, then we know that the initial arrival rate is increasing in the price level, $\partial_n \partial_r [\Theta(n, \theta, r)]_{n=0} > 0$, for all $r > 0$. Higher returns make it easier for the first excited speculator to recruit more of his friends. We also know that there exists a return level, $r_- > 0$, such that the initial per capita growth rate of the crowd of excited speculators is negative, $\partial_n [G(n, \theta, r_-)]_{n=0} < 0$. So, via the implicit-value theorem, we know there must exist a critical return threshold, $r_\star > r_-$, such that

$$\partial_n [G(n, \theta, r)]_{n=0} \begin{cases} < 0 & \text{if } r < r_\star \\ > 0 & \text{if } r > r_\star \end{cases}$$

Put differently, if we Taylor expand a law of motion that leads to feedback trading around the point, $(0, \theta, r_\star)$, then the criteria for feedback trading imply that, to a second-order approximation, this growth rate must behave just like the logistic growth model for $n \approx 0$. The restrictions in Definition A.1 imply that there exist positive constants, $\psi, \omega > 0$, such that when $n \in [0, \epsilon)$ and $r \in (r_\star - \delta, r_\star + \delta)$ for sufficiently

small values of $\epsilon, \delta > 0$:

$$G(n, \theta, r) = \psi \cdot (r - r_\star) \times n - \omega \times n^2 + O[n^3]$$

$n = 0$ is a solution for all $r > 0$ since $G(0, \theta, r) = 0$. What's more, given the derivative at zero, $\psi \cdot (r - r_\star)$, we can see $n = 0$ will only be a stable steady state when $r < r_\star$. As soon as $r > r_\star$, this solution will switch from stable to unstable as in Figure 1. If $\psi = \omega = \theta$ and $r_\star = 1/\theta$, the growth rate in the equation above is identical to the logistic growth model. So this model is emblematic of a more general phenomenon.

Proof (Proposition A.1). The definition of feedback trading implies the following sign restrictions for the derivatives of $G(n, \theta, r)$:

1. $n = 0$ is a steady-state solution for all $r > 0$ implies that $\partial_r[G(n, \theta, r)]_{n=0} = 0$.
2. $\partial_n \partial_n[\Theta(n, \theta, r)] < 0$ and $\Omega(n) = n$ imply that $\partial_n \partial_n[G(n, \theta, r)] < 0$.
3. $\partial_n \partial_r[\Theta(n, \theta, r)]_{n=0} > 0$ and $\Omega(n) = n$ imply that $\partial_n \partial_r[G(n, \theta, r)]_{n=0} > 0$.
4. $\partial_r \partial_r[\Theta(n, \theta, r)] = 0$ and $\Omega(n) = n$ imply that $\partial_r \partial_r[G(n, \theta, r)] = 0$.

If i) there exists some $r_- > 0$ such that $\partial_n[G(n, \theta, r_-)]_{n=0} < 0$ and ii) for all $r > 0$ we have that both $\partial_n \partial_r[G(n, \theta, r)]_{n=0} > 0$ and $\partial_r \partial_r[G(n, \theta, r)]_{n=0} = 0$, then via the implicit-value theorem there must be some critical return level, $r_\star > r_-$, such that

$$\partial_r[G(n, \theta, r)]_{n=0} \begin{cases} < 0 & \text{if } r < r_\star \\ > 0 & \text{if } r > r_\star \end{cases}$$

Now, consider a Taylor expansion of $G(n, \theta, r)$ around the point $(0, \theta, r_\star)$ where $n \in [0, \epsilon]$, θ is constant, and $r \in (r_\star - \delta, r_\star + \delta)$ for sufficiently small $\epsilon, \delta > 0$:

$$\begin{aligned} G(n, \theta, r) &\approx \underbrace{G(0, \theta, r_\star)}_{=0} + \underbrace{\partial_n[G(0, \theta, r_\star)]}_{=0} \times n + \underbrace{\partial_r[G(0, \theta, r_\star)]}_{=0} \times (r - r_\star) \\ &\quad + \frac{1}{2} \cdot \underbrace{\partial_n \partial_n[G(0, \theta, r_\star)]}_{<0} \times n^2 \\ &\quad + \underbrace{\partial_n \partial_r[G(0, \theta, r_\star)]}_{>0} \times n \cdot (r - r_\star) \\ &\quad + \frac{1}{2} \cdot \underbrace{\partial_r \partial_r[G(0, \theta, r_\star)]}_{=0} \times (r - r_\star)^2 \end{aligned}$$

So for any steady-state solution, we must have $G(n, \theta, r) \approx 0 = \frac{1}{2} \cdot \partial_n \partial_n [G(0, \theta, r_\star)] \times n^2 + \partial_n \partial_r [G(0, \theta, r_\star)] \times n \cdot (r - r_\star)$. For a non-zero excited-speculator population, $n > 0$, this is only possible if $(r - r_\star) > 0$. Thus, we must have that:

$$n = -2 \cdot \frac{\partial_n \partial_r [G(0, \theta, r_\star)]}{\partial_n \partial_n [G(0, \theta, r_\star)]} \cdot (r - r_\star) > 0 \quad (\text{A1}')$$

This positive solution will only be stable if

$$0 > \partial_n G(n, \theta, r) = \partial_n \partial_n [G(0, \theta, r_\star)] \times n + \partial_n \partial_r [G(0, \theta, r_\star)] \times (r - r_\star)$$

Plugging in the functional form for n from Equation (A1') yields:

$$\partial_n \partial_n [G(0, \theta, r_\star)] \times n + \partial_r \partial_n [G(0, \theta, r_\star)] \times (r - r_\star) = -\partial_r \partial_n [G(0, \theta, r_\star)] \times (r - r_\star)$$

From the condition above, we can conclude that the solution described by Equation (A1') will always be stable for $r > r_\star$ since $\partial_r \partial_n [G(0, \theta, r_\star)] > 0$.

Thus, the logistic growth model captures the sudden qualitative change in steady-state solutions displayed by any population dynamics exhibiting feedback trading. \square

A.2 Random Fluctuations

This subsection shows that the sudden change in the steady-state behavior at $r_\star = 1/\theta$ does not disappear when noise is added to the system. To do this, consider redefining the law of motion in Equation (A1) as follows:

$$\tilde{G}(n, \theta, r) \stackrel{\text{def}}{=} G(n, \theta, r) + \sigma \times n \cdot \frac{d\varepsilon}{d\tau}$$

$\sigma > 0$ is a positive constant reflecting the instantaneous volatility of the excited-speculator population growth rate, and $\varepsilon \stackrel{\text{iid}}{\sim} \text{Normal}(0, 1)$ is a white-noise process.

This noise implies the excited-speculator population will adhere to the following stochastic law of motion for all $n_\tau \in [0, \infty)$:

$$dn_\tau = \theta \cdot (r - 1/\theta) \times n_\tau \cdot d\tau - \theta \cdot r \times n_\tau^2 \cdot d\tau + \sigma \times n_\tau \cdot d\varepsilon_\tau \quad (\text{A2})$$

I put time subscripts on n_τ and $d\varepsilon_\tau$ to highlight that these elements are now stochastic.

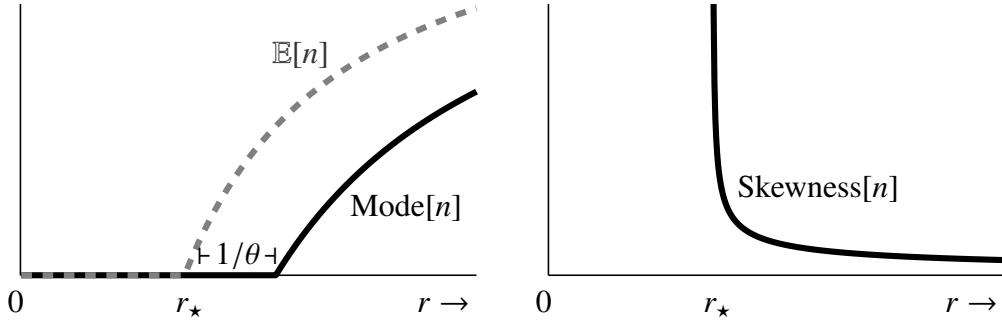


Figure A1. Stationary Distribution. The mean, mode, and skewness of the excited-speculator population (y-axis) as a function of past returns (r ; x-axis) when there are random fluctuations in population size as detailed in Proposition A.2.

Equation (A2) is just a noisy version of the law of motion described in Equation (A1) with one important difference: the range is now $n_\tau \in [0, \infty)$ rather than $n_\tau \in [0, 1)$. Because the diffusion term $\sigma \times n_\tau \cdot d\varepsilon_\tau$ contains n_τ , the noise dies away as the excited-speculator population shrinks towards zero. As a result, the population will never go negative. But it is possible for the population to exceed unity, $n_\tau > 1$.¹

The stationary distribution for the excited-speculator population in the stochastic case displays a sudden change as the asset's past return crosses a critical return threshold, $r_* = 1/\theta$. When $r < r_*$, any initial population of excited speculators almost surely goes extinct; whereas, when $r > r_*$, this is no longer the case. Adding noise does not eliminate the sudden qualitative change as shown in Figure A1.

Proposition A.2 (Bubble-Generating Mechanism with Random Fluctuations). Suppose the excited-speculator population is governed by Equation (A2) with $\sigma = \sqrt{2}$.

1. If $r > r_* = 1/\theta$, then given any initial $n_0 \in (0, \infty)$ the stationary distribution is

$$\lim_{\tau \rightarrow \infty} n_\tau = n_\infty(\theta, r) \sim \text{Gamma}(\theta \cdot r - 1, \theta \cdot r)$$

where $\text{Gamma}(a, b) \stackrel{\text{def}}{=} \frac{b^a}{\Gamma(a)} \cdot \frac{x^{a-1}}{e^{b \cdot x}}$ is the pdf for the Gamma distribution.

2. If $r < r_* = 1/\theta$, then $n_\infty(\theta, r) = 0$ almost surely.

¹One way to microfound $n_\tau > 1$ is to think about the quantity U as the typical rather than the total number of apathetic speculators in the market rather than the total number. So, whenever $n_\tau > 1$, there are more excited speculators in the market than usual (Safuan, Jovanoski, Towers, and Sidhu, 2013).

Proof (Proposition A.2). Let $n_\tau \in (0, \infty)$ denote a stochastic process

$$dn_\tau = m(n_\tau) \cdot d\tau + \sigma \cdot s(n_\tau) \cdot d\varepsilon_\tau$$

where $\tau \geq 0$, $m(n)$ denotes the drift term, $\sigma > 0$ is a positive constant, $s(n) > 0$ is the diffusion term, and $d\varepsilon_\tau$ is a standard Brownian-motion process. Assume that $s(0) = 0$ and that $m(\infty) = -\infty$. Finally, let $\text{pdf}_\tau(n_0)$ denote the probability-density function for this stochastic process at time $\tau \geq 0$ given the initial value n_0 .

The Stratonovich interpretation of the Fokker-Plank equation dictates that:

$$\partial_\tau \text{pdf}_\tau(n_0) = -\partial_n \left[\left(m(n) + \frac{1}{2} \cdot \sigma^2 \cdot \frac{ds}{dn} \cdot s(n) \right) \cdot \text{pdf}_\tau(n_0) \right] + \frac{1}{2} \cdot \sigma^2 \cdot \partial_n \partial_n \left[s(n)^2 \cdot \text{pdf}_\tau(n_0) \right]$$

A stationary distribution has the property that $\partial_\tau \text{pdf}_\tau(n_0) = 0$ for all $n_0 \in [0, \infty)$. Thus, the stationary distribution must satisfy the following condition:

$$\partial_n \left[\left(m(n) + \frac{1}{2} \cdot \sigma^2 \cdot \frac{ds}{dn} \cdot s(n) \right) \cdot \text{pdf}_\tau(n_0) \right] = \frac{1}{2} \cdot \sigma^2 \cdot \partial_n \partial_n \left[s(n)^2 \cdot \text{pdf}_\tau(n_0) \right]$$

This restriction, together with the boundary conditions that $m(\infty) = -\infty$ and $s(0) = 0$, gives us the following functional form for the stationary distribution:

$$\text{pdf}_\tau(n) = \frac{1}{\Psi \cdot s(n)} \cdot \exp\left(\frac{1}{\sigma^2/2} \cdot \int_0^n \frac{m(n')}{s(n')^2} \cdot dn' \right)$$

$\Psi \stackrel{\text{def}}{=} \int_0^\infty \frac{1}{s(n)} \cdot \exp\left(\frac{1}{\sigma^2/2} \cdot \int_0^n \frac{m(n')}{s(n')^2} \cdot dn' \right) \cdot dn < \infty$ is a renormalization factor ensuring that the pdf is well-defined—i.e., that all probabilities sum to one, $\int_0^\infty \text{pdf}_\tau(n) \cdot dn = 1$.

If we substitute in the functional form for the excited-speculator dynamics (Equation A2), we get $m(n) = \theta \cdot (r - 1/\theta) \times n - \theta \cdot r \times n^2$ and $s(n) = n$. So, when $r > r_\star = 1/\theta$, the solution dictates the density of the excited-speculator population:

$$\text{pdf}_\infty(n) = \frac{(\theta \cdot r)^{\theta \cdot r - 2}}{\Gamma(\theta \cdot r - 1)} \cdot \frac{n^{\theta \cdot r - 2}}{e^{\theta \cdot r \cdot n}}$$

This is the functional form of the Gamma distribution, $\text{Gamma}(a, b) \stackrel{\text{def}}{=} \frac{b^a}{\Gamma(a)} \cdot \frac{x^{a-1}}{e^{b \cdot x}}$, when $a \stackrel{\text{def}}{=} \theta \cdot r - 1$ and $b \stackrel{\text{def}}{=} \theta \cdot r$. This distribution is defined for all $n \in (0, \infty)$. When $r < r_\star$ this PDF is undefined, which corresponds to the solution where $n_\tau = 0$ is an absorbing boundary. See [Horsthemke and Lefever \(2006, Ch. 6.4\)](#) for details. \square

A.3 Continuous Feedback

In Section 2 speculator interactions play out on a much faster timescale than assets are priced. This subsection shows that, while modeling the continuous feedback between population dynamics and asset returns might seem more realistic, it would not eliminate the sudden qualitative change in the excited-speculator population at r_* .

Suppose that a 1% increase in the population of excited speculators increases the risky asset's return by a factor of $\epsilon \in [0, \frac{1}{\theta \cdot r})$, producing the following law of motion:

$$\tilde{G}(n, \theta, r) \stackrel{\text{def}}{=} \theta \cdot r \cdot (1 + \epsilon \cdot n) \cdot (1 - n) \times n - n \quad (\text{A3})$$

The new $(1 + \epsilon \cdot n)$ term captures the idea that an inflow of excited speculators at time τ will increase the risky asset's return, which will then make it easier for excited speculators right now to recruit their friends. If we set $\epsilon = 0$, then we get back the original law of motion in Equation (A1). By increasing ϵ , we allow transient fluctuations in the excited-speculator population to have a larger and larger effect on speculator persuasiveness via their effect on the asset's past returns.

Proposition A.3 (*Bubble-Generating Mechanism with Continuous Feedback*). *Suppose the excited-speculator population is governed by Equation (A3). Define $r_* \stackrel{\text{def}}{=} 1/\theta$.*

1. *If $r < r_*$, there is only one steady-state value for the excited-speculator population, $\mathcal{SS}(\theta, r) = \{0\}$. This lone steady state, $\bar{n} = 0$, is stable.*
2. *If $r > r_*$, there are two steady-state values, $\mathcal{SS}(\theta, r) = \{0, (1 - \epsilon)^{-1} \cdot (r - r_*)/r > 0\}$. Only the strictly positive steady state, $\bar{n} = (1 - \epsilon)^{-1} \cdot (r - r_*)/r > 0$, is stable.*

Continuous feedback does not affect the threshold return level, $r_ = 1/\theta$, at which a non-zero population of excited speculators suddenly enters the market.*

Understanding how the dynamics of the excited-speculator population interacts with an asset's past returns is very important if you want to understand how a particular bubble episode will unfold. But, it is not essential if all you want to do is understand the likelihood of a future bubble—i.e., the likelihood that r_* will be crossed.

Proof (Proposition A.3). The law of motion in Equation (A3) can be re-written as

$$\tilde{G}(n, \theta, r) = (\theta \cdot r - 1) \times n - \theta \cdot r \cdot (1 - \epsilon) \times n^2 + \text{O}[n^3]$$

Thus, if we ignore third-order terms, then we can solve for the steady-state population of excited speculators using the same logic as in the proof of Proposition 2.2:

$$\bar{n} = \begin{cases} \frac{1}{1-\epsilon} \cdot \frac{r-r_\star}{r} & \text{if } r > r_\star \\ 0 & \text{otherwise} \end{cases}$$

If we assume the strength of the continuous feedback is not too strong, $\epsilon < (\theta \cdot r)^{-1}$, then we will have that $\bar{n} < 1$.

Thus, continuous feedback does not affect the threshold, r_\star , at which speculative bubbles occur. Adding continuous feedback to the model would not alter any predictions about the likelihood of a future bubble—i.e., the likelihood that this threshold would be crossed. Because it requires excited speculators to be present, continuous feedback can only amplify the size of an existing excited-speculator population once they have already entered the market, $\frac{1}{1-\epsilon} > 1$. \square

B Additional Empirical Results

This appendix provides additional empirical results that support the findings in Sections 3 and 4. In Subsection B.1, I give evidence that the definition of a speculative bubble outlined in Subsection 3.1 is not fine-tuned. I do this by showing that the main result in Table 5a goes through when using different return cutoffs for the start date, boom size, and crash severity. In Subsection B.2, I show how to reconcile my findings with those in Greenwood, Shleifer, and You (2018). Finally, in Subsection B.3, I document that turnover sharply increases right after the start of a speculative bubble.

B.1 Definition of a Bubble Is Not Fine-Tuned

In Subsection 3.1, I define a speculative bubble as a five-year local price maximum in a particular industry with at least one boom month in the run up and a crash following the peak, and I study each bubble episode at its start date. I define a boom month as an observation with $> 100\%$ returns over the past two years (both raw and net) and $> 50\%$ raw returns over the past five years. I define a crash as a $< -40\%$ return from peak to trough. And, I define the start date as the last month prior to the peak where the industry had $< 50\%$ raw returns over the past two years.

	Start Threshold		Boom Threshold		Crash Threshold		GSY Episodes
	37.5%	62.5%	75%	125%	-30%	-50%	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
intercept	29.28** (2.29)	32.95*** (3.09)	27.89*** (2.60)	27.76** (2.23)	24.20*** (2.73)	20.04* (1.91)	18.23 (1.51)
theta	2.80** (2.20)	2.16*** (2.71)	2.98*** (3.08)	3.55** (2.49)	4.51*** (4.97)	4.70*** (4.57)	5.74*** (3.93)
Adj. R^2	0.11	0.13	0.14	0.15	0.24	0.26	0.30
#Obs	30	30	32	26	34	26	26

Table B1. Bubble Definition Is Not Fine-Tuned. Dependent variable in all regressions is `willBeBubble`, an indicator variable for whether observation precedes a bubble episode. Coefficient of +1 indicates a 1%-point increase in the likelihood of an industry-month observation being followed by a bubble. Each column reports the results of a separate regression using a matched dataset. Each pre-bubble case is matched to the most similar industry-month observation without a subsequent bubble based on `retPast2Yr (%)`, `netPast2Yr (%)`, `retPast5Yr (%)`, `bookToMkt`, and `volatility (%/year)` as of the start date of the bubble. `retPast2Yri,t (%)`: value-weighted return over past two years. `netPast2Yri,t (%)`: value-weighted return net of the market over past two years. `retPast5Yri,t (%)`: value-weighted return over past five years. `bookToMkti,t`: average book-to-market ratio in month t . `volatilityi,t (%/year)`: value-weighted daily volatility in month t . Pre-bubble and matched observations mechanically look similar along these dimensions. `theta (%)`: empirical proxy for sensitivity of speculator persuasiveness to increases in past returns. Numbers in parentheses are t -statistics clustered by industry. *, **, and ***: statistical significance at the 10%, 5%, and 1% levels.

Table B1 gives evidence that my main results in Table 5a column (1) are not fine-tuned in the sense that they do not depend on using these particular threshold values. Columns (1) and (2) show that, when I use a return threshold for my start date that is 25% higher or lower, θ still predicts the likelihood of a future bubble. For these two regressions, I use the same set of 15 bubble episodes listed in Table 1a. However, I match these cases to otherwise similar control observations earlier in each bubble's life cycle in column (1) when the industry has first realized 37.5% over the past two years. Whereas, I match later in each bubble's life cycle in column (2) when the industry has already realized 62.5% returns over the past two years.

Columns (3) and (4) in Table B1 show that, when I use a minimum return threshold for what constitutes a boom month that is 25% higher or lower, θ still predicts the likelihood of a future bubble. For these two regressions, I am changing the definition of what counts as a boom. Column (3) is more inclusive: a boom month is any observation with > 75% returns over the past two years (both raw and net) and > 50% raw returns over the past five years. This criteria adds one more bubble episode to Table 1a. Column (4) is more restrictive: a boom month is any observation with > 125% returns over the past two years (both raw and net) and > 50% raw returns over the past five years. This criteria excludes two bubble episodes Table 1a.

Columns (5) and (6) in Table B1 show that, when I use a minimum return threshold for what constitutes a crash that is 25% higher or lower, θ still predicts the likelihood of a future bubble. For these two regressions, I am changing the definition of what counts as a crash. Column (5) is more inclusive: a crash occurs so long as peak-to-trough returns are < -30%. This criteria adds two more bubble episode to Table 1a. Column (6) is more restrictive: a crash occurs whenever peak-to-trough returns are < -50%. This criteria excludes two bubble episodes Table 1a.

Finally, in column (7), I replicate the main analysis in Table 5a column (1) using the 13 bubble episodes listed in Greenwood, Shleifer, and You (2018, GSY) Table 1a contained in my sample period. I find that θ is still an economically large and statistically significant predictor for the likelihood of a future bubble in this sub-sample. Since I am using GSY's boom and crash definitions, every episodes in GSY Table 1a is contained within some episode in my Table 1a. My list of 15 bubbles also contains two additional episodes where the boom was not immediately followed by a crash.

B.2 Comparison with Greenwood et al. (2018)

I define a bubble as a local price maximum with at least one speculative boom during the run up ($\text{retPast2Yr} > 100\%$, $\text{netPast2Yr} > 100\%$, and $\text{retPast5Yr} > 50\%$) and a crash following the peak ($> 40\%$ decline). Yet, while the definitions a speculative boom and crash come from Greenwood, Shleifer, and You (2018, GSY), many of the variables that predict whether a speculative boom will be followed by a crash in GSY do not predict the ex ante likelihood of bubbles in this paper.

These discrepancies are not due to some issue with the data or some methodological conflict. Instead, they are a natural consequence of the fact that I am asking a different kind of question about bubbles. And this difference in questions leads to four key differences in our respective empirical approaches. Table B2 shows that, if I a) extend my sample period, b) exclude bubbles where the first boom month is not immediately followed by a crash, c) choose control observations based only on past returns, and d) study the first boom date rather than the start date, then I can qualitatively the main findings in GSY Table 4.

Columns (1)-(5) in Table B2 report the values provided in GSY Table 4, which reports the mean and standard deviation of various predictors for the booms with crashes (Cases; GSY Table 1a) and the booms without crashes (Controls; GSY Table 1b) respectively. Columns (6)-(10) report analogous summary statistics for the cases and control observations in a matched sample which uses monthly industry returns going back to 1926. Each bubble episode during this sample period is matched to the nearest non-bubble industry-month observation based only on past returns (retPast2Yr , netPast2Yr , and retPast5Yr rather than retPast2Yr , netPast2Yr , retPast5Yr , bookToMkt , and volatility) as of the first boom date (rather than the start date). I also exclude any would-be bubble episodes that do not immediately crash—i.e., the 19 cases consist of the bubble episodes that suffer a $< -40\%$ peak-to-trough crash within two years of the first boom date.

Table B2 shows that my analysis would look similar to GSY if I had looked at a longer sample period, if I had used GSY's more restrictive definition of a bubble, if I had matched the resulting bubble episodes to control observations based only on their past returns, and if I had done the matching as of the first boom date. Since I am matching on past returns, the cases and control observations used in columns (6)-(10)

	Values in GSY Table 4					Matched Data Using GSY Criteria				
	Cases		Controls		Diff	Cases		Controls		Diff
	Avg (1)	Sd (2)	Avg (3)	Sd (4)		Avg (6)	Sd (7)	Avg (8)	Sd (9)	
retPast2Yr	172.2	34	141.1	22	31.1*** (3.14)	158.39	28.35	151.43	22.95	6.95 (0.83)
bookToMkt	0.29	0.19	0.44	0.20	-0.15* (1.75)	0.41	0.25	0.46	0.19	-0.04 (0.62)
volatility	50.8	12	48.7	12	2.1 (0.46)	49.43	12.95	46.65	22.65	2.78 (0.47)
turnover	66.7	17	70.3	14	-3.6 (0.67)	61.07	13.08	67.57	19.07	-6.50 (1.23)
age	72.4	21	57.4	17	15.0** (2.30)	64.00	19.45	65.01	20.11	-1.01 (0.16)
ageTilt	5.3	14	-2.2	8	7.5** (2.46)	13.02	20.77	-1.16	10.94	14.18*** (2.63)
newIssuance	34.3	18	22.1	14	12.2** (2.17)	21.87	8.73	17.81	14.91	4.06 (1.02)
Δsales	28.9	18	22.9	12	6.1 (1.04)	62.09	15.47	59.12	13.09	2.97 (0.57)
CAPE	25.45	11.32	19.10	4.90	6.35* (1.87)	29.46	16.43	18.40	7.40	11.06** (2.04)
retAccel	122.8	26	90.5	33	32.3*** (2.99)	122.85	23.68	94.10	33.69	28.76*** (3.04)
#Obs	21		19			19		19		

Table B2. Case/Control Data Matched Using GSY Criteria. Columns (1)-(5): values reported in [Greenwood, Shleifer, and You \(2018, GSY\) Table 4](#). Columns (6)-(10): analogous summary statistics for a dataset of bubble episodes and matched controls constructed using GSY's criteria. Cases: mean and standard deviation for bubble episodes as of the boom date. Controls: same statistics for matched controls. Difference: difference in means across samples. Numbers in parentheses are t -statistics clustered by industry. **: statistically significant at the 5% level. **retPast2Yr** (%): value-weighted return over past two years. **bookToMkt**: average book-to-market ratio in month t . **volatility** (rank): value-weighted daily volatility in month t . **turnover** (rank): value-weighted trading volume divided by shares outstanding in month t . **age** (rank): value-weighted firm age in month t . **ageTilt** (%): difference between equal-weighted and age-weighted return over past two years. **newIssuance** (%): percent of firms issuing equity in past two years. **Δsales** (rank): value-weighted year-over-year sales growth. **CAPE**: market-wide cyclically adjusted P/E ratio in month t . **retAccel** (%): value-weighted return in months $[t - 23, t]$ minus value-weighted return in months $[t - 23, t - 12]$.

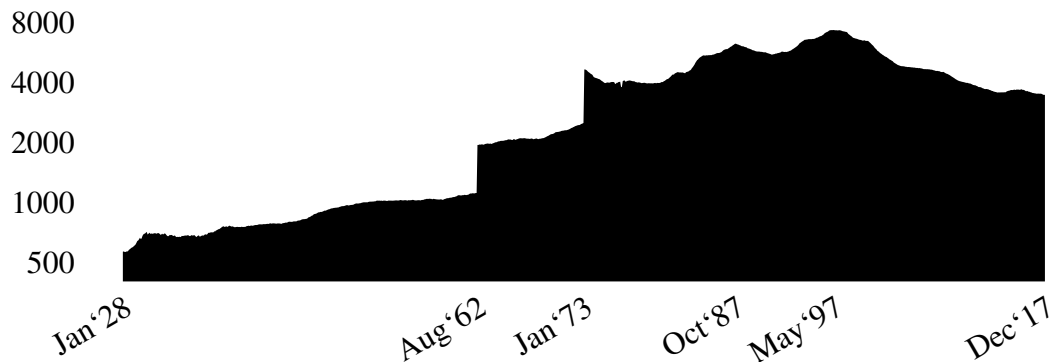


Figure B1. Number of Stocks in CRSP. Number of stocks each month from January 1928 to December 2017 with a share code of 10 or 11 in CRSP. y-axis on a log scale.

will mechanically look similar in terms of their past returns. So there is no way to fit the 31.1% difference in `retPast2Yr (%)` found in GSY's original data. However, almost every other variable looks qualitatively the same in both Greenwood, Shleifer, and You (2018, GSY)'s data and my matched data when using GSY's criteria.

`bookToMkt` is a little higher in the control observations as of the boom date. In both datasets, bubble episodes tend to have slightly more volatile returns, `volatility (rank)`, and slightly less turnover in the preceding months, `turnover (rank)`. In both datasets, bubble episodes' good returns tend to be disproportionately driven by good returns among its youngest firms, `ageTilt (%)`, and all firms in the industry are a bit more likely to have issued new equity, `newIssuance (%)`, and have slightly higher sales growth, `Δsales (rank)`. Bubble episodes tend to have booms in months where market-wide CAPE is higher in both datasets, `CAPE`. They also tend to have rapidly accelerating returns in both datasets, `retAccel (%)`.

Firm age is the one remaining difference between our results in Table B2. And this discrepancy is likely due to a quirk in how GSY define firm age as the "number of months since the firm first appeared on either Compustat or CRSP." Figure B1 shows that there are huge jumps in CRSP's data coverage in August 1962 and January 1973 that have nothing to do with economic forces. To account for this issue, I redefine firm age using Compustat's IPO date whenever this value is available. As a result, bubble episodes no longer have younger firms, `age (rank)`, but their boom returns are still driven by the strong performance of their youngest firms, `ageTilt (%)`.

Figure B2 provides another demonstration that the data themselves are not driving the difference between my results and those in Greenwood, Shleifer, and You (2018, GSY). Instead of extending the sample period and requiring booms to be immediately followed by a crash, I start with the 15 bubble episodes listed in Table 1a. I then ask, what would happen if I looked for matched controls for each of these bubble episodes based only on past returns as of the first boom date like in GSY. The dotted red vertical lines in Figure B2 depict the control observations that I would select using this alternative approach that mirrors GSY’s criteria. And the dotted blue vertical lines depict the non-bubble episodes in GSY Table 1b that occur during my sample period.

Every one of the dotted blue lines is selected as a control observation—i.e., has a dotted red line right on top of it. In other words, if I were to look for matched controls for my bubble episodes using GSY’s criteria, then I would select the control observations listed in GSY Table 1b. Figure B2 directly shows that my empirical exercise can be changed to match GSY’s findings. It just does not make sense to adopt these changes given the question I am trying to answer in this paper.

B.3 Turnover Goes Up When a Bubble Begins

If the start of a bubble episode is marked by the inflow of excited speculators, then there should be evidence of this inflow in the form of an increase in trading volume. I test for the existence of this increase in volume by regressing turnover over in the subsequent 3, 6, 9, and 12 months on the bubble-indicator variable:

$$\text{turnoverNext}\#_{i,t} = \hat{\alpha} + \hat{\beta} \times \text{willBeBubble}_{i,t} + \hat{\epsilon}_{s,q}$$

The dependent variable, `turnoverNext#`, is the value-weighted average turnover of the i th industry over the next $\#$ months. `willBeBubble` is an indicator for whether an industry is one of the 15 bubble episodes or one of the 15 matched episodes. Columns (1), (3), (5), and (7) in Table B3 show that monthly turnover increases following the start of the bubble episode.

Columns (2), (4), (6), and (8) in Table B3 report analogous results where, instead of using `willBeBubble`, I instrument for `willBeBubble` with `theta`. When I use the predicted value of `willBeBubble` from the regression in column (1) of Table 5a as the right-hand-side variable, I find a similar effect size. In short, there is strong

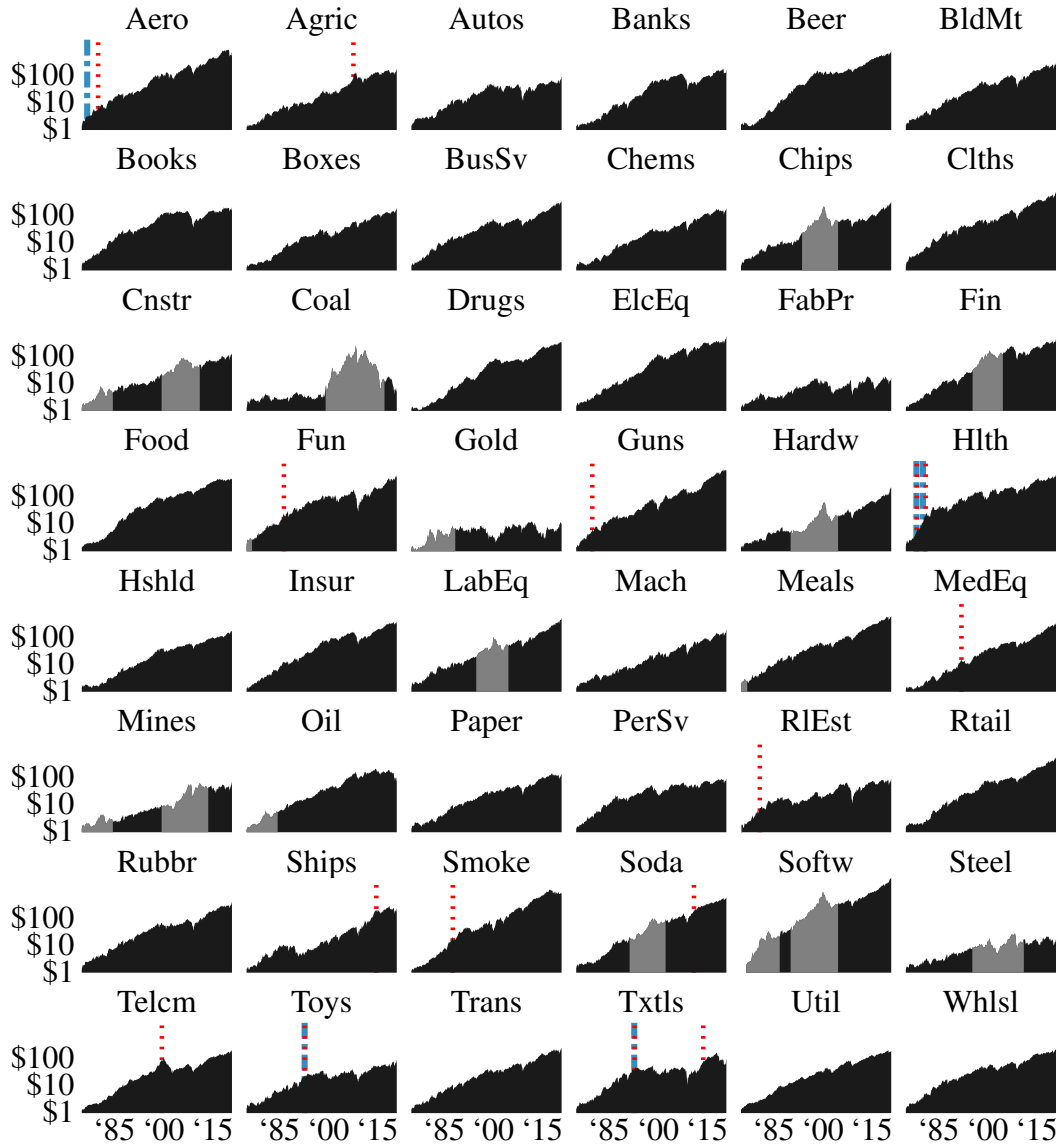


Figure B2. Matches Selected on Past Returns as of the First Boom Month. x -axis: time in months from January 1975 to December 2017. y -axis (log scale): dollar value at time t of a continuously reinvested industry-specific portfolio that started with \$1 at the opening bell on the first trading day of January 1975. Grey regions denote bubble episodes ± 2 years. Black regions indicate the normal times. Dotted red lines denote the matched control observations selected only on the basis of their similarity to a bubble episode in terms of past returns as of the first boom month. Dotted blue lines denote control observations listed in Greenwood, Shleifer, and You (2018) Table 1b which occur during normal times.

Dependent Variable:	turnoverNext3		turnoverNext6		turnoverNext9		turnoverNext12	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
intercept	25.42*** (3.43)	25.31*** (2.67)	23.42*** (3.09)	20.17* (1.68)	17.11** (2.14)	13.51 (0.95)	14.02 (1.66)	10.03 (0.66)
turnover	0.71*** (14.78)	0.71*** (12.37)	0.70*** (11.64)	0.70*** (10.70)	0.75*** (12.21)	0.75*** (10.70)	0.75*** (10.94)	0.75*** (9.39)
willBeBubble	0.25* (1.80)		0.33*** (3.31)		0.36*** (3.72)		0.38*** (3.59)	
$\hat{E}[\text{willBeBubble} \theta]$		0.24 (1.10)		0.39** (1.96)		0.42** (2.06)		0.45** (2.07)
Adj. R^2	0.81	0.79	0.86	0.83	0.88	0.86	0.87	0.84

Table B3. Turnover Goes Up When a Bubble Begins. Each column is a separate regression using the same data as in Table 2 on 15 pre-bubble observations and a matched sample of 15 observations with no subsequent bubble. The dependent variable in each regression, `turnoverNext#`, is the average value-weighted trading volume divided by shares outstanding in month over the next 3, 6, 9, or 12 months. `willBeBubble`: indicator variable for industries that will subsequently experience a bubble. Numbers in parentheses are standard errors clustered by industry. Significance: * = 10%, ** = 5%, and *** = 1%.

evidence that trading activity in an industry spikes the moment a bubble episode begins, which is consistent with the idea that bubbles are the result of a sudden inflow of excited speculators.

The key insight pertains to the ex ante likelihood that some bias-constraint pair will bind. When that happens, an excited crowd of speculators will enter the market, resulting in an increase in trading volume. How trading volume in an industry will evolve from there on out will intimately depend on the specific bias-constraint pair sustaining the bubble, though. e.g., see Figure 3 and the surrounding discussion in Section 2.3. For this reason, I only look for the initial increase in trading volume at the start of a bubble episode. It does not make sense to use the theoretical framework in this paper to resolve patterns in trading volume during bubbles (cf. DeFusco, Nathanson, and Zwick, 2017; Barberis, Greenwood, Jin, and Shleifer, 2018; Liao and Peng, 2019).

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