Online Appendix for "Asymptotic Optimality of Base-Stock Policies for Perishable Inventory Systems"

Jinzhi Bu, Xiting Gong, Xiuli Chao

Appendix A. Proof of Lemma 1 in Section 2

We prove the lemma by considering the case when $\mathbb{P}(D \geq S/m) > 0$ and the case when $\mathbb{P}(D \geq S/m) = 0$ separately. For convenience, we denote the state space under base-stock policy π_S as

$$\mathcal{X} = \{(x_1, x_2, \ldots, x_{m-1}) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_{m-1} \leq S\}.$$

Case 1: $\mathbb{P}(D \geq S/m) > 0$. In this case, similar to the proof of Theorem 3 in Huh et al. (2009), we first prove that the Markov chain $\{X_t(S) := (x^{\pi_S}_{t,1}, x^{\pi_S}_{t,2}, \ldots, x^{\pi_S}_{t,m-1}) : t \geq 1\}$ converges to some random vector $X_\infty(S) = (x^{\pi_S}_{\infty,1}, x^{\pi_S}_{\infty,2}, \ldots, x^{\pi_S}_{\infty,m-1})$ in distribution. Applying this result, we can prove Lemma 1 as follows. Since $X_t(S)$ converges in distribution to $X_\infty(S)$, one can verify that $\big(x^{\pi_S}_{t,1}, D_t\big)$ converges in distribution to $\big(x^{\pi_S}_{\infty,1}, D\big)$, where D is independent of $X_\infty(S)$. Since the function $(x_1 - d)^+$ is continuous in $0 \leq x_1 \leq S$, $d \geq 0$, and bounded from above by S, by applying Theorem 3.2.3 in Durrett (2010), we obtain

$$\lim_{t \to \infty} \mathbb{E}[(x^{\pi_S}_{t,1} - D_t)^+] = \mathbb{E}[(x^{\pi_S}_{\infty,1} - D)^+].$$

Therefore, Lemma 1 is established by letting $O_\infty(S) =^d (x^{\pi_S}_{\infty,1} - D)^+$, where $=^d$ denotes "equal in distribution".

We now prove that the stationary distribution of the Markov chain $\{X_t(S) : t \geq 1\}$ exists, which is also the limiting distribution that $\{X_t(S) : t \geq 1\}$ converges to. By applying Theorem 16.0.2 of Meyn and Tweedie (1993), we only need to construct a measurable subset $\mathbf{U} \subseteq \mathcal{X}$, a nontrivial measure $\nu(\cdot)$ (i.e., $\nu(\mathcal{X}) > 0$), and a positive integer $t^* \geq 1$, such that

$$\mathbb{P}(X_{t^*}(S) \in B | X_1(S) = \mathbf{x}_1) \geq \nu(B) \tag{EC.1}$$

for any $\mathbf{x}_1 \in \mathbf{U}$ and any measurable subset $B \subseteq \mathcal{X}$.

Let $\mathbf{U} = \mathcal{X}$, $t^* = 2m - 1$, and the measure $\nu(\cdot)$ be defined as follows: for any measurable set $B \subseteq \mathcal{X}$,

$$\nu(B) \triangleq \mathbb{P}\bigg(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2, \Big((S - \sum_{t=m}^{2m-2} D_t)^+, (S - \sum_{t=m+1}^{2m-2} D_t)^+, \ldots, (S - D_{2m-2})^+\Big) \in B\bigg).$$

It is easy to verify that $\nu(\cdot)$ is a measure. In addition, since $\nu(\mathcal{X}) = (\mathbb{P}(D \geq S/m))^{2m-2} > 0$, it is nontrivial.

To complete the proof, it remains to verify inequality (EC.1). Note that

$$\begin{aligned}
&\mathbb{P}(X_{2m-1}(S) \in B | X_1(S) = \mathbf{x}_1) \\
&\geq \mathbb{P}\Big(X_{2m-1}(S) \in B, D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2 | X_1(S) = \mathbf{x}_1\Big) \\
&= \mathbb{P}\Big(X_{2m-1}(S) \in B | D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2, X_1(S) = \mathbf{x}_1\Big) \mathbb{P}\Big(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big) \\
&= \mathbb{P}\Big(((S - \sum_{t=m}^{2m-2} D_t)^+, \ldots, (S - D_{2m-2})^+) \in B | D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2, X_1(S) = \mathbf{x}_1\Big) \mathbb{P}\Big(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big) \\
&= \mathbb{P}\Big(((S - \sum_{t=m}^{2m-2} D_t)^+, \ldots, (S - D_{2m-2})^+) \in B | D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big) \mathbb{P}\Big(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big) \\
&= \nu(B), \qquad \text{(EC.2)}
\end{aligned}$$

where the first equality follows from the conditional probability formula and the independence between $(D_1, D_2, \ldots, D_{2m-2})$ and the initial state $X_1(S)$, the second equality follows from Corollary 2 of Cooper and Tweedie (2002), which states that if $D_t \geq S/m$ for any $1 \leq t \leq 2m-2$, then $x^{\pi_S}_{2m-1,i} = (S - \sum_{t=m+i-1}^{2m-2} D_t)^+$ for any $1 \leq i \leq m-1$ regardless of the initial state, the third equality follows from the independence between $(D_1, D_2, \ldots, D_{2m-2})$ and $X_1(S)$, and the last equality follows from the definition of $\nu(\cdot)$. The proof of Lemma 1 for Case 1 is complete.

Case 2: $\mathbb{P}(D \geq S/m) = 0$. In this case, the Markov chain $\{X_t(S) : t \geq 1\}$ may not have a stationary distribution, since one can construct a cyclic Markov chain similar to that in §3.2 of Huh et al. (2009). So, we prove $\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[(x^{\pi_S}_{t,1} - D_t)^+] = S/m - \mathbb{E}[D]$ directly by showing that

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[(x^{\pi_S}_{t,1} - D_t)^+] \geq \frac{S}{m} - \mathbb{E}[D] \geq \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[(x^{\pi_S}_{t,1} - D_t)^+]. \qquad \text{(EC.3)}$$

By letting $O_\infty(S) =^d S/m - D$, we prove Lemma 1 under Case 2.

We first prove the first inequality in (EC.3). From the recursion (19), one can verify that

$$\sum_{i=t-m+1}^{t} \mathbb{E}[o^{\pi_S}_i] \geq S - m\mathbb{E}[D], \quad \forall t \geq m. \qquad \text{(EC.4)}$$

Suppose $T = km + l$, where $k \geq 1$ and $1 \leq l < m$. Then, we have

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[o^{\pi_S}_t] \geq \frac{1}{km+l} \sum_{s=0}^{k-1} \sum_{t=sm+1}^{(s+1)m} \mathbb{E}[o^{\pi_S}_t] \geq \frac{k}{km+l}(S - m\mathbb{E}[D]),$$

where the second inequality follows from (EC.4). By taking $\liminf_{T\to\infty}$ on both sides of the above inequalities, we obtain the first inequality in (EC.3).

We next prove the second inequality in (EC.3). For any vector $\mathbf{x} = (x_1, \ldots, x_{m-1}) \in \mathcal{X}$, consider the following three systems under the same base-stock policy π_S: System 1 starts with the initial state $\mathbf{x}_1^1 = \mathbf{x}$; System 2 starts with $\mathbf{x}_1^2 = (x_{m-1}, x_{m-1}, \dots, x_{m-1})$; and System 3 starts with $\mathbf{x}_1^3 = (S/m, 2S/m, \dots, (m-1)S/m)$. For $1 \le k \le 3$, let $\mathbf{x}_t^k$ and o_t^k denote the system state and the amount of outdates in System k in period t, respectively. Then, we have

$$\sum_{t=1}^{T} o_t^1 \le \sum_{t=1}^{T} o_t^2 \le o_1^2 + \sum_{t=2}^{T} o_t^3, \quad \forall T \ge 2, \tag{EC.5}$$

under any given demand sample path. Note that $x_{1,i}^1 \le x_{1,i}^2$ for all $1 \le i \le m-1$ and $x_{t,m}^1 = x_{t,m}^2 = S$ for any $t \ge 1$. In addition, System 2 will be empty at the beginning of period 2 (i.e., $\mathbf{x}_2 = \mathbf{0}$). Then, $x_{2,i}^2 \le x_{2,i}^3$ for all $1 \le i \le m-1$ and $x_{t,m}^2 = x_{t,m}^3 = S$ for any $t \ge 2$. Then, the two inequalities in (EC.5) can be proven by using the recursion (19) and induction.

Since $D_t < S/m$ for any $t \ge 1$, one can verify that

$$\mathbf{x}_t^3 = \left(\frac{S}{m}, \frac{2S}{m}, \dots, \frac{(m-1)S}{m}\right)$$

and $o_t^3 = S/m - D_t$ for any $t \ge 1$. Applying (EC.5), we obtain

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[o_t^1] \le \limsup_{T\to\infty} \left\{ \frac{1}{T}\mathbb{E}[o_1^2] + \frac{1}{T} \sum_{t=2}^{T} \mathbb{E}[o_t^3] \right\} = \frac{S}{m} - \mathbb{E}[D].$$

Thus, the second inequality in (EC.3) is also satisfied. The proof of Lemma 1 under Case 2 is then complete. Q.E.D.

Appendix B. Proofs of Theorems 2 to 4 in Section 3

B.1. Proof of Theorem 2

For convenience, let $\mu \triangleq \mathbb{E}[\hat{D}_1]$, $\sigma \triangleq \sqrt{\mathrm{Var}[\hat{D}_1]}$ and $\sigma_X \triangleq \sqrt{\mathrm{Var}[X_1]}$ (recall the definitions of $\hat{D}_1$ and X_1 from §3.1). To highlight the dependence on n, we use the functional form $D(n)$ to represent the single-period demand. We only consider the case $\sigma_X > 0$, and similar arguments apply to the case $\sigma_X = 0$, i.e., when X_1 is deterministic. We divide the proof of Theorem 2 into three major steps.

First, we prove that for any fixed $k > 1$, there exists $n_1(k) > 0$ such that $S_n^{NP} \le k\mu n$, for any $n \ge n_1(k)$. To this end, we first prove the following inequality:

$$S_n^{NP} \le \mathbb{E}[D(n)] + \sqrt{p\mathrm{Var}[D(n)]/h}. \tag{EC.6}$$

Let $\mathcal{M}$ be the following set of distributions: $\mathcal{M} = \big\{\widetilde{F}$ is a cdf of $D:\ \mathbb{E}_{\widetilde{F}}[D] = \mathbb{E}[D(n)], \mathrm{Var}_{\widetilde{F}}[D] = \mathrm{Var}[D(n)]\big\}$, where $\mathbb{E}_{\widetilde{F}}[\cdot]$ denotes the expectation taken with respect to cdf $\widetilde{F}(\cdot)$. Then

$$\begin{aligned} h(S_n^{NP} - \mathbb{E}[D(n)]) &\le (h+p)\mathbb{E}[(D(n) - S_n^{NP})^+] + h\big(S_n^{NP} - \mathbb{E}[D(n)]\big) \\ &= \min_{S\ge 0} \Big\{ h\mathbb{E}[(S - D(n))^+] + p\mathbb{E}[(D(n) - S)^+] \Big\} \\ &\le \min_{S\ge 0} \max_{\widetilde{F}\in\mathcal{M}} \Big\{ h\mathbb{E}_{\widetilde{F}}[(S-D)^+] + p\mathbb{E}_{\widetilde{F}}[(D-S)^+] \Big\} \\ &= \sqrt{ph\mathrm{Var}[D(n)]}, \end{aligned}$$

where the last equality follows from Scarf (1958). The above inequality directly implies (EC.6).

Recall that $\hat{D}_j$'s are independent of $N(n)$. From Wald's equation and the law of total variance,

$$\mathbb{E}[D(n)] = \mu\mathbb{E}[N(n)], \quad \text{and} \quad \mathrm{Var}[D(n)] = \sigma^2\mathbb{E}[N(n)] + \mu^2\mathrm{Var}[N(n)]. \tag{EC.7}$$

Moreover, from the classic central limit theorem for the renewal process (see e.g., Theorem 4.3.2 in Gallager 2012), we have the random variable

$$\frac{N(n) - \frac{1}{\mathbb{E}[X_1/n]}}{\sqrt{\mathrm{Var}[X_1/n]} \cdot \sqrt{\frac{1}{(\mathbb{E}[X_1/n])^3}}}$$

converges in distribution to the standard normal random variable when $n \to \infty$. Since $\mathbb{E}[X_1] = 1$ and $\mathrm{Var}[X_1] = \sigma_X^2$, it follows from the convergence properties that

$$\lim_{n\to\infty} \frac{\mathbb{E}[N(n)]}{n} = 1, \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathrm{Var}[N(n)]}{n} = \sigma_X^2.$$

Then, it follows from (EC.6) and (EC.7) that $\limsup_{n\to\infty} S_n^{NP}/n \le \mu$. As a result, for any $k > 1$, there exists some $n_1(k)$ such that $S_n^{NP} \le k\mu n$ when $n \ge n_1(k)$.

Second, we prove that there exists some constant $\nu > 0$, which only depends on the distribution of X_1, such that for any fixed $0 < \delta < \nu$ and $n \ge 2(1+\delta)/\delta$,

$$\mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)} \hat{D}_j\Big)\Big] \le \exp\Big(-\frac{\delta^2 n}{8(1+\delta)^2\nu^2}\Big) + \big(\mathbb{E}[e^{-\lambda \hat{D}_1}]\big)^{\frac{n}{1+\delta}}. \tag{EC.8}$$

For any $\delta > 0$, we first note that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)} \hat{D}_j\Big)\Big] &= \mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)} \hat{D}_j\Big)\Big| N(n) < \lceil\frac{n}{1+\delta}\rceil\Big]\mathbb{P}\big(N(n) < \lceil\frac{n}{1+\delta}\rceil\big) \\
&\quad + \mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)} \hat{D}_j\Big)\Big| N(n) \ge \lceil\frac{n}{1+\delta}\rceil\Big]\mathbb{P}\big(N(n) \ge \lceil\frac{n}{1+\delta}\rceil\big) \\
&\le \mathbb{P}\Big(N(n) < \lceil\frac{n}{1+\delta}\rceil\Big) + \mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{\lceil\frac{n}{1+\delta}\rceil} \hat{D}_j\Big)\Big] \\
&\le \mathbb{P}\Big(N(n) < \frac{n}{1+\delta}\Big) + \big(\mathbb{E}[e^{-\lambda \hat{D}_1}]\big)^{\frac{n}{1+\delta}}.
\end{aligned} \tag{EC.9}$$

To show (EC.8), it suffices to bound $\mathbb{P}\big(N(n) < \frac{n}{1+\delta}\big)$ from above by $\exp(-\frac{\delta^2 n}{8(1+\delta)^2\nu^2})$. For convenience, let $k(n,\delta) = \lceil\frac{n}{1+\delta}\rceil$. Note that

$$\mathbb{P}\Big(N(n) < \frac{n}{1+\delta}\Big) = \mathbb{P}\big(N(n) < k(n,\delta)\big) = \mathbb{P}\big(\sum_{i=1}^{k(n,\delta)} X_i > n\big), \tag{EC.10}$$

where the second equality follows from the definition of $N(n)$. Since X_1 is a nonnegative r.v. and $\mathbb{E}[e^{sX_1}] < \infty$ for some $s > 0$ by our assumption, $X_1 - \mathbb{E}[X_1]$ is a sub-exponential r.v. (see

Proposition 2.7.1 of Vershynin (2018) for equivalent definitions of a sub-exponential r.v.). Then, we apply Bernstein's inequality for sub-exponential random variables (see e.g., Theorem 1.13 in Rigollet and Hütter 2015) to obtain the following result: there exists some constant $\nu > 0$ (which only depends on the distribution of X_1) such that when $\delta < \nu$ and $n \geq 2(1+\delta)/\delta$,

$$\begin{aligned}
\mathbb{P}\Big(\sum_{i=1}^{k(n,\delta)} X_i > n\Big) &= \mathbb{P}\Big(\frac{1}{k(n,\delta)}\sum_{i=1}^{k(n,\delta)}(X_i - 1) > \frac{n}{k(n,\delta)} - 1\Big) \\
&\leq \exp\Big(-\frac{k(n,\delta)}{2}\big((\frac{\frac{n}{k(n,\delta)}-1}{\nu})^2 \wedge \frac{\frac{n}{k(n,\delta)}-1}{\nu}\big)\Big) \\
&= \exp\Big(-\frac{(n-k(n,\delta))^2}{2k(n,\delta)\nu^2}\Big) \\
&\leq \exp\Big(-\frac{\delta^2 n}{8(1+\delta)^2\nu^2}\Big), \qquad \text{(EC.11)}
\end{aligned}$$

where the first inequality follows from Bernstein's inequality and $n \geq \frac{n}{1+\delta} + 1 > k(n,\delta)$, the second equality holds since $\frac{n}{k(n,\delta)} - 1 \leq 1 + \delta - 1 < \nu$ when $\delta < \nu$, and the second inequality holds since

$$\frac{(n-k(n,\delta))^2}{k(n,\delta)} > \frac{(n-(\frac{n}{1+\delta}+1))^2}{\frac{n}{1+\delta}+1} = \frac{(\delta n - (1+\delta))^2}{(1+\delta)(n+(1+\delta))} \geq \frac{\delta^2 n}{4(1+\delta)^2},$$

where the last inequality holds since $1+\delta \leq n\delta/2$. Then, (EC.8) follows from (EC.9), (EC.10) and (EC.11).

Finally, we bound the RHS of inequality (23). Combining the results established in the previous two steps, we obtain that for any $k > 1$, $0 < \delta < \nu$ and $n \geq n_1(k) \vee (2(1+\delta)/\delta)$,

$$e^{\frac{1}{m}\lambda S_n^{NP}}\mathbb{E}[e^{-\lambda\sum_{j=1}^{N(n)}\hat{D}_j}] \leq e^{f_1(\lambda)n} + e^{f_2(\lambda)n},$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are defined as

$$f_1(\lambda) \triangleq \frac{\lambda k\mu}{m} - \frac{\delta^2}{8(1+\delta)^2\nu^2}, \quad \text{and} \quad f_2(\lambda) \triangleq \frac{\lambda k\mu}{m} + \frac{1}{1+\delta}\log(\mathbb{E}[e^{-\lambda\hat{D}_1}]).$$

Note that $f_1(\lambda) < 0$ when λ is sufficiently small. When $m \geq 2$ (otherwise the RHS of (23) equals zero), $f_2'(0) = (k/m - 1/(1+\delta))\mu < 0$ when $1 < k < 2/(1+\delta)$ and $0 < \delta < 1$. Since $f_2(0) = 0$, there exists some $\lambda_0(k,\delta) > 0$ such that $f_2(\lambda) < 0$ when $0 < \lambda < \lambda_0(k,\delta)$. Thus, by choosing δ, k and λ satisfying the following three conditions: (i) $0 < \delta < 1 \wedge \nu$; (ii) $1 < k < 2/(1+\delta)$; and (iii) $0 < \lambda < \lambda_0(k,\delta) \wedge [\delta^2 m/(8k\mu(1+\delta)^2\nu^2)]$, we have

$$C_n(\widetilde{S}_n) - \mathrm{OPT}_n \leq K_1 e^{-K_2 n}, \quad \forall n \geq n_1(k) \vee 2(1+\delta)/\delta,$$

where $K_1 = 2^m(m-1)\theta/(\lambda m e) \geq 0$ and $K_2 = -\max\{f_1(\lambda), f_2(\lambda)\}m > 0$. Q.E.D.

B.2. Proof of Theorem 3

From the sketched proof of Theorem 3 in §6.2, it suffices to prove inequalities (25) and (26).

We first prove inequality (25). Note that

$$\begin{aligned} C(\widetilde{S}) - \mathrm{OPT} &\le \theta\Big(\mathbb{E}\Big[\Big(\frac{\widetilde{S}}{m} - D\Big)^+\Big] - \mathbb{E}\Big[\Big(\frac{\widetilde{S}}{m} - \frac{1}{m}\sum_{i=1}^m D_i\Big)^+\Big]\Big) \\ &= \theta\Big(\mathbb{E}\Big[\Big(D - \frac{\widetilde{S}}{m}\Big)^+\Big] - \mathbb{E}\Big[\Big(\frac{1}{m}\sum_{i=1}^m D_i - \frac{\widetilde{S}}{m}\Big)^+\Big]\Big), \end{aligned} \tag{EC.12}$$

where the inequality follows from Lemmas 1 and 2 and Proposition 2, and the equality follows from $x^+ = x + (-x)^+$ and $\mathbb{E}[D] = \mathbb{E}[\frac{1}{m}\sum_{i=1}^m D_i]$. Thus, inequality (25) holds.

We next prove inequality (26). We first provide an upper bound on $C(\widetilde{S})$. Note that

$$p\mathbb{E}[(D-\widetilde{S})^+] = p\mathbb{E}[(D-\widetilde{S})\cdot 1_{\{D>\widetilde{S}\}}] \le p\big(\bar{D}-\widetilde{S}\big)\cdot \mathbb{P}(D>\widetilde{S}) \le (h+\theta)\big(\bar{D}-\widetilde{S}\big), \tag{EC.13}$$

where the first inequality follows from $D \le \bar{D}$, a.s., and the second one follows from inequality (5). Since $\widetilde{S} \ge \underline{S}$, we have

$$\begin{aligned} C(\widetilde{S}) &\le h\mathbb{E}\big[(\widetilde{S}-D)^+\big] + \theta\mathbb{E}\big[O_\infty(\widetilde{S})\big] + (h+\theta)\big(\bar{D}-\widetilde{S}\big) \\ &\le h\mathbb{E}\big[(\underline{S}-D)^+\big] + \theta\mathbb{E}\big[O_\infty(\underline{S})\big] + (h+\theta)\big(\bar{D}-\underline{S}\big), \end{aligned} \tag{EC.14}$$

where the first inequality follows from (EC.13), and the second one follows from $a^+ - b^+ \le (a-b)^+$, and the following inequality: for any $0 \le S_1 \le S_2$,

$$\mathbb{E}[O_\infty(S_2)] \le \mathbb{E}[O_\infty(S_1)] + S_2 - S_1. \tag{EC.15}$$

The inequality (EC.15) can be proved as follows. First, by using the system dynamics, we can prove by induction that $x_{t,1}^{\pi_{S_1}} \le x_{t,1}^{\pi_{S_2}}$ for any $t \ge 1$. Second, note that for any $t \ge 1$, $x_{t+1,i}^{\pi_S} = (x_{t,i+1}^{\pi_S} - x_{t,1}^{\pi_S} \vee D_t)^+$ for $1 \le i \le m-1$ and $x_{t,m}^{\pi_S} = S$. We can prove by induction that $x_{t,i}^{\pi_{S_2}} \le x_{t,i}^{\pi_{S_1}} + S_2 - S_1$ for any $t \ge 1$ and $1 \le i \le m$. Then, by the definition of o_t^π, we obtain $o_t^{\pi_{S_2}} \le o_t^{\pi_{S_1}} + S_2 - S_1$ for any $t \ge 1$. Finally, the inequality (EC.15) follows by applying the definition of $\mathbb{E}[O_\infty(S)]$. Thus, inequality (26) holds. Q.E.D.

B.3. Proof of Theorem 4

From the sketched proof of Theorem 4 in §6.2, it suffices to prove equality (24). Recall that $\widetilde{S}_\theta$ is the minimizer of $\widetilde{C}_\theta(S)$ over $[0,\infty)$. One can easily show that $\widetilde{S}_\theta$ decreases in θ. Thus, the limit $\lim_{\theta\to\infty}\widetilde{S}_\theta$ exists and we denote it by $\widetilde{S}_\infty$. Next, we show that $\widetilde{S}_\infty \le S^0$, where $S^0 \triangleq \sup\{S \ge 0 : \mathbb{E}[(S - \sum_{i=1}^m D_i)^+] = 0\}$. One can verify that $S^0 = m\underline{D}$ (recall that $\underline{D} = \inf\{x : F(x) > 0\}$). By the definition of $\widetilde{C}_\theta(S)$ in (4), for any $\theta \ge 0$, we have

$$\begin{aligned} \widetilde{C}_\theta(\widetilde{S}_\theta) &= h\mathbb{E}[(\widetilde{S}_\theta - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\theta)^+] + \frac{\theta}{m}\mathbb{E}\Big[\Big(\widetilde{S}_\theta - \sum_{i=1}^m D_i\Big)^+\Big] \\ &\le \min_{0\le S\le m\underline{D}} \widetilde{C}_\theta(S) = \min_{0\le S\le m\underline{D}} \big\{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\big\}. \end{aligned} \tag{EC.16}$$

Note that inequality (EC.16) holds for all θ, and its RHS is a constant. This implies $\widetilde{S}_\infty \le S^0$. Thus,

$$\begin{aligned}\lim_{\theta\to\infty}\left\{h\mathbb{E}[(\widetilde{S}_\theta - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\theta)^+]\right\} =& h\mathbb{E}[(\widetilde{S}_\infty - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\infty)^+] \\ \ge& \min_{0\le S\le m\underline{D}}\left\{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\right\}. \qquad \text{(EC.17)}\end{aligned}$$

After combining inequalities (EC.16) and (EC.17), we obtain equality (24). Q.E.D.

Appendix C. Proofs of Statements in Section 4

C.1. Proof of Lemma 3

First, we prove that $C^{LIFO}(S) \ge \hat{C}_L(S)$. From the system dynamics under the LIFO issuance policy described in §4.1, we have the following recursion on the outdating process under base-stock policy π_S: under any demand sample path,

$$o^{\pi_S}_{t+m-1} = \min_{i\in\{0,\dots,m-1\}}\left\{(S-D_{t+i})^+ - \sum_{j=t+i}^{t+m-2} o^{\pi_S}_j\right\}, \quad \forall t\ge 1. \qquad \text{(EC.18)}$$

Then, it follows that

$$\sum_{j=t}^{t+m-1} o^{\pi_S}_j \ge \min_{i\in\{0,\dots,m-1\}}\{(S-D_{t+i})^+\} = (S-\max\{D_t, D_{t+1},\dots,D_{t+m-1}\})^+, \quad \forall t\ge 1.$$

After taking the expectation on both sides of the above inequality, we obtain

$$\sum_{j=t}^{t+m-1} \mathbb{E}[o^{\pi_S}_j] \ge \mathbb{E}[(S-\max\{D_1,D_2,\dots,D_m\})^+], \quad \forall t\ge 1. \qquad \text{(EC.19)}$$

By the definition of $C^{LIFO}(S)$, we have

$$\begin{aligned}C^{LIFO}(S) = \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[C^{\pi_S}_t] &= h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \limsup_{T\to\infty}\frac{\theta}{mT}\sum_{t=1}^{mT}\mathbb{E}[o^{\pi_S}_t] \\ &\ge h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \frac{\theta}{m}\mathbb{E}[(S-\max\{D_1,D_2,\dots,D_m\})^+],\end{aligned}$$

where the inequality follows from inequality (EC.19). By the definition of $\hat{C}_L(S)$, we conclude that $C^{LIFO}(S) \ge \hat{C}_L(S)$.

Next, we prove that $C^{LIFO}(S) \le \hat{C}_U(S)$. Under policy π_S, the total leftover inventory in each period t after satisfying demand D_t is $(S-D_t)^+$. Note that the outdated inventories in periods $t, t+1, \dots, t+m-1$ are part of this inventory. Then, we obtain

$$\sum_{j=t}^{t+m-1} o^{\pi_S}_j \le (S-D_t)^+, \quad \forall t\ge 1.$$

The rest of the proof is similar to that of the first part and we omit it for brevity. Q.E.D.

C.2. Proof of Theorem 5

First, we prove part (a). Since $\widetilde{S} \le S^{NP}$ and $\hat{S} \le S^{NP}$, it follows from inequality (27) that

$$C^{LIFO}(\hat{S}) - \mathrm{OPT}^{LIFO} \le \frac{3\theta}{2m}\mathbb{E}[(S^{NP} - D)^+].$$

Thus, part (a) holds.

Next, we prove part (b). From inequality (27), we also have

$$\begin{aligned} C^{LIFO}(\hat{S}) - \mathrm{OPT}^{LIFO} \le & \frac{\theta}{2m}\Big(2(m-1)\mathbb{E}[D] + \mathbb{E}[(D-\hat{S})^+] + \mathbb{E}[(D-\widetilde{S})^+] - \mathbb{E}[(\max\{D_1,\dots,D_m\} - \hat{S})^+] \\ & + \mathbb{E}[(\max\{D_1,\dots,D_m\} - \widetilde{S})^+] - 2\mathbb{E}[(\sum_{i=1}^m D_i - \widetilde{S})^+]\Big) \\ \le & \frac{(m-1)\theta}{m}\mathbb{E}[D], \end{aligned} \tag{EC.20}$$

where the two inequalities follow from $x^+ = x + (-x)^+$ for any x and that $D_1, \dots, D_m$ and D are *i.i.d.* r.v.'s. When demand D is unbounded, one can easily verify that $\lim_{p\to\infty} \widetilde{C}_p(\widetilde{S}_p) = \infty$. Since $\mathrm{OPT}^{LIFO} \ge \widetilde{C}(\widetilde{S})$ by Proposition 2, it follows that $\lim_{p\to\infty} \mathrm{OPT}_p^{LIFO} = \infty$. Combining this with inequality (EC.20), we obtain part (b).

Finally, we prove part (c). From Proposition 2 and the proof of Theorem 4, we have

$$\lim_{\theta\to\infty} \mathrm{OPT}_\theta^{LIFO} \ge \lim_{\theta\to\infty} \widetilde{C}_\theta(\widetilde{S}_\theta) = \min_{0\le S\le m\underline{D}} \{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\}. \tag{EC.21}$$

By a similar proof to that of Theorem 4, we can also prove that $\lim_{\theta\to\infty} \hat{S}_\theta = \underline{D}$ and

$$\lim_{\theta\to\infty} \theta(\mathbb{E}[(\hat{S}_\theta - D)^+] + \mathbb{E}[(\hat{S}_\theta - \max\{D_1,\dots,D_m\})^+]) = 0.$$

Since $C^{LIFO}(S) \le \hat{C}_U(S)$ for any $S \ge 0$ by Lemma 3, it follows that

$$\limsup_{\theta\to\infty} C_\theta^{LIFO}(\hat{S}_\theta) \le \limsup_{\theta\to\infty} \hat{C}_{U,\theta}(\hat{S}_\theta) = p(\mathbb{E}[D] - \underline{D}) + \frac{1}{m}\lim_{\theta\to\infty}(\theta\mathbb{E}[(\hat{S}_\theta - D)^+]) = p(\mathbb{E}[D] - \underline{D}). \tag{EC.22}$$

When $\underline{D} = 0$, after combining inequalities (EC.21) and (EC.22), we obtain $\lim_{\theta\to\infty}(C_\theta^{LIFO}(\hat{S}_\theta) - \mathrm{OPT}_\theta^{LIFO}) = 0$. Thus, part (c) holds. Q.E.D.

C.3. Proof of Proposition 1

From Lemma 3, we have $C^{LIFO}(S^{LIFO,*}) \ge \min_{S\ge 0} \hat{C}_L(S)$. In addition, by a similar proof to that of Theorem 4, we can prove that $\lim_{\theta\to\infty}\big(\min_{S\ge 0} \hat{C}_{L,\theta}(S)\big) = p(\mathbb{E}[D] - \underline{D})$. Combining these results with inequality (EC.22) and the definition of $S^{LIFO,*}$, we obtain $\lim_{\theta\to\infty} C_\theta(S_\theta^{LIFO,*}) = p(\mathbb{E}[D] - \underline{D})$.

Next, we construct an admissible policy and show that its long-run average cost is strictly less than $p(\mathbb{E}[D] - \underline{D})$ and independent of θ when $m \ge 2$ and $S^{NP} > \underline{D} > 0$. Without loss of generality, suppose that the system is initially empty. Consider the following admissible policy:

order $\min\{S^{NP}, m\underline{D}\}$ units in periods 1, $m+1$, $2m+1$, …, whereas use the base-stock policy with level $\underline{D}$ in all other periods. One can easily check that 1) there is no inventory outdating under this policy; 2) the expected holding and penalty cost in each of periods 1, $m+1$, $2m+1$, … is

$$h\mathbb{E}[(\min\{S^{NP}, m\underline{D}\} - D)^+] + p\mathbb{E}[(D - \min\{S^{NP}, m\underline{D}\})^+],$$

which is strictly less than $p(\mathbb{E}[D] - \underline{D})$ when $m \geq 2$ and $S^{NP} > \underline{D} > 0$ by the definition of S^{NP}; and 3) the expected holding and penalty cost in each of all other periods is at most $p(\mathbb{E}[D] - \underline{D})$ because by the definition of S^{NP}, we have

$$h\mathbb{E}[(S - D)^+] + p\mathbb{E}[(D - S)^+] \leq p(\mathbb{E}[D] - \underline{D}),\ \forall S \in [\underline{D}, \min\{S^{NP}, m\underline{D}\}].$$

Combining these results together, we conclude that the long-run average cost under this policy is strictly less than $p(\mathbb{E}[D] - \underline{D})$ and independent of θ.

Finally, by the definition of OPT^{LIFO}, we obtain $\lim_{\theta\to\infty} \mathrm{OPT}_\theta^{LIFO} < p(\mathbb{E}[D] - \underline{D})$. As a result, $\lim_{\theta\to\infty}(C_\theta^{LIFO}(S_\theta^{LIFO,*}) - \mathrm{OPT}_\theta^{LIFO}) > 0$. Q.E.D.

C.4. Proof of Lemma 4

Note that the long-run average holding and penalty cost under base-stock policy π_S always equals $h\mathbb{E}[(S - D)^+] + p\mathbb{E}[(D - S)^+]$, regardless of the inventory issuance policy. From the definitions of $C^{FIFO}(S)$, $C^{LIFO}(S)$ and $C^{GI}(S)$, it suffices to prove the following inequalities under any demand sample path:

$$\sum_{i=1}^{t} o_i^{FIFO,\pi_S} \leq \sum_{i=1}^{t} o_i^{GI,\pi_S} \leq \sum_{i=1}^{t} o_i^{LIFO,\pi_S}, \quad \forall t \geq 1, \tag{EC.23}$$

where o_t^{FIFO,π_S}, o_t^{GI,π_S}, and o_t^{LIFO,π_S} denote the amounts of outdates in period t under base-stock policy π_S, when the FIFO, general, and LIFO inventory issuance policies are adopted respectively.

Since the system is initially empty, $o_t^{FIFO,\pi_S} = o_t^{GI,\pi_S} = o_t^{LIFO,\pi_S} = 0$ for any $t = 1, \ldots, m-1$. Thus, the inequalities in (EC.23) hold when $1 \leq t \leq m-1$. Now consider a general period $t \geq m$. Suppose inductively that the inequalities in (EC.23) hold for each period $s = 1, 2, \ldots, t-1$. In the following, we prove that the inequalities in (EC.23) hold for period t. Then, by induction, the inequalities in (EC.23) hold for any period $t \geq 1$.

First, we prove the following inequalities under a general issuance policy: for any $t \geq m$:

$$\left(S - \sum_{i=t-m+1}^{t} D_i - \sum_{i=t-m+1}^{t-1} o_i^{GI,\pi_S}\right)^+ \leq o_t^{GI,\pi_S} \leq \min_{t-m+1 \leq i \leq t}\left\{(S - D_i)^+ - \sum_{j=i}^{t-1} o_j^{GI,\pi_S}\right\}. \tag{EC.24}$$

To see the first inequality in (EC.24), we note that the S units of total inventory at the beginning of period $t-m+1$ are either used to satisfy demands or outdated in periods $t-m+1, \ldots, t$.

Since the total demand in periods $t-m+1,\dots,t$ is $\sum_{i=t-m+1}^{t} D_i$, we have $\sum_{i=t-m+1}^{t} o_i^{GI,\pi_S} \ge (S-\sum_{i=t-m+1}^{t} D_i)^+$, leading to the first inequality in (EC.24) due to $o_t^{GI,\pi_S} \ge 0$. To see the second inequality in (EC.24), we note that for each $t-m+1 \le i \le t$, the outdated inventory in periods $i, i+1, \dots, t$ are part of the leftover inventory at the end of period i after satisfying demand in period i, whose amount equals $(S-D_i)^+$. Thus, for each $t-m+1 \le i \le t$, we have $\sum_{j=i}^{t} o_j^{GI,\pi_S} \le (S-D_i)^+$, leading to the second inequality in (EC.24).

Applying the first inequality in (EC.24), we obtain

$$\begin{aligned}\sum_{i=1}^{t} o_i^{GI,\pi_S} &\ge \max\left\{ S-\sum_{i=t-m+1}^{t} D_i + \sum_{i=1}^{t-m} o_i^{GI,\pi_S},\quad \sum_{i=1}^{t-1} o_i^{GI,\pi_S}\right\}\\ &\ge \max\left\{ S-\sum_{i=t-m+1}^{t} D_i + \sum_{i=1}^{t-m} o_i^{FIFO,\pi_S},\quad \sum_{i=1}^{t-1} o_i^{FIFO,\pi_S}\right\}\\ &= \sum_{i=1}^{t} o_i^{FIFO,\pi_S}, \end{aligned}\tag{EC.25}$$

where the second inequality follows from the inductive assumption and the identity follows from the recursion for the outdated inventory under the FIFO issuance policy in equation (19). Similarly, applying the second inequality in (EC.24), the inductive assumption and the recursion for the outdated inventory under the LIFO issuance policy in equation (EC.18), we obtain $\sum_{i=1}^{t} o_i^{GI,\pi_S} \le \sum_{i=1}^{t} o_i^{LIFO,\pi_S}$, which, together with (EC.25), leads to (EC.23). Q.E.D.

Appendix D. Proof of Lemma 5 in Section 5

We first prove inequality (13). From the system dynamics under policy π_S, one can easily verify that for $t \ge m+L$,

$$\begin{aligned} x_{t,m}^{\pi_S} &= S-\sum_{i=t-L}^{t-1} D_i - \sum_{i=t-L}^{t-1} o_i^{\pi_S},\\ o_t^{\pi_S} &= \Big(S-\sum_{i=t-m-L+1}^{t} D_i - \sum_{i=t-m-L+1}^{t-1} o_i^{\pi}\Big)^+. \end{aligned}$$

Since $o_i^{\pi_S}$ is non-negative for any period i and $\{D_t : t\ge 1\}$ are *i.i.d.* random variables, it follows from the above two identities that, for $t \ge m+L$,

$$\mathbb{E}[(x_{t,m}^{\pi_S}-D_t)^+] \le \mathbb{E}\Big[\Big(S-\sum_{i=1}^{L+1} D_i\Big)^+\Big], \tag{EC.26}$$

$$\mathbb{E}[o_t^{\pi_S}] \le \mathbb{E}\Big[\Big(S-\sum_{i=1}^{m+L} D_i\Big)^+\Big]. \tag{EC.27}$$

In addition, for $t \geq m+2L$, we have

$$\begin{aligned}\mathbb{E}[(D_t - x_{t,m}^{\pi_S})^+] &\leq \mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + \sum_{i=t-L}^{t-1}\mathbb{E}[o_i^{\pi_S}] \\ &\leq \mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + L\mathbb{E}\Big[\Big(S - \sum_{i=1}^{m+L} D_i\Big)^+\Big], \qquad \text{(EC.28)}\end{aligned}$$

where the second inequality follows from inequality (EC.27).

Recall that $C_t^{\pi_S} = h(x_{t,m}^{\pi_S} - D_t)^+ + b(D_t - x_{t,m}^{\pi_S})^+ + \theta o_t^{\pi_S}$. From the definition of $C^L(S)$, we have

$$\begin{aligned}C^L(S) &= \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[C_t^{\pi_S}] = \limsup_{T\to\infty}\frac{1}{T}\sum_{t=m+2L}^{T}\mathbb{E}[C_t^{\pi_S}] \\ &\leq h\mathbb{E}\Big[\Big(S - \sum_{i=1}^{L+1} D_i\Big)^+\Big] + b\mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + (\theta + bL)\mathbb{E}\Big[\Big(S - \sum_{i=1}^{m+L} D_i\Big)^+\Big],\end{aligned}$$

where the inequality follows from inequalities (EC.26) to (EC.28). Therefore, inequality (13) holds.

We next prove inequality (14). For any admissible policy $\pi \in \Pi$, one can easily verify that

$$x_{t+L,m}^{\pi} = x_{t,m+L}^{\pi} - \sum_{i=t}^{t+L-1} D_i - \sum_{i=t}^{t+L-1} o_i^{\pi}, \quad \forall t \geq 1, \qquad \text{(EC.29)}$$

$$\sum_{i=t}^{t+m+L-1} o_i^{\pi} \geq \big(x_{t,m+L}^{\pi} - \sum_{i=t}^{t+m+L-1} D_i\big)^+, \quad \forall t \geq 1. \qquad \text{(EC.30)}$$

For any $t \geq 1$, it follows from (EC.29) and (EC.30) that

$$\begin{aligned}&h(x_{t+L,m}^{\pi} - D_{t+L})^+ + b(D_{t+L} - x_{t+L,m}^{\pi})^+ \\ =&h\big(x_{t,m+L}^{\pi} - \sum_{i=t}^{t+L} D_i - \sum_{i=t}^{t+L-1} o_i^{\pi}\big)^+ + b\big(\sum_{i=t}^{t+L} D_i + \sum_{i=t}^{t+L-1} o_i^{\pi} - x_{t,m+L}^{\pi}\big)^+ \\ \geq&h\big(x_{t,m+L}^{\pi} - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - x_{t,m+L}^{\pi}\big)^+ - h\sum_{i=t}^{t+L-1} o_i^{\pi}, \qquad \text{(EC.31)}\end{aligned}$$

and

$$(m+L)\sum_{t=1}^{T+m+L-1} o_t^{\pi} \geq \sum_{t=1}^{T}\sum_{i=t}^{t+m+L-1} o_i^{\pi} \geq \sum_{t=1}^{T}(x_{t,m+L}^{\pi} - \sum_{i=t}^{t+m+L-1} D_i)^+. \qquad \text{(EC.32)}$$

From the definition of C_t^{π}, we further have

$$\begin{aligned}\sum_{t=1}^{T+m+L} C_t^{\pi} &\geq \sum_{t=1}^{T+m}(h(x_{t+L,m}^{\pi} - D_{t+L})^+ + b(D_{t+L} - x_{t+L,m}^{\pi})^+) + \theta\sum_{t=1}^{T+m+L} o_t^{\pi} \\ &\geq \sum_{t=1}^{T+m}\left\{h\big(x_{t,m+L}^{\pi} - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - x_{t,m+L}^{\pi}\big)^+\right\} + (\theta - hL)\sum_{t=1}^{T+m+L-1} o_t^{\pi} \\ &\geq \sum_{t=1}^{T}\left\{h\big(x_{t,m+L}^{\pi} - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - x_{t,m+L}^{\pi}\big)^+ + \frac{\theta - hL}{m+L}(x_{t,m+L}^{\pi} - \sum_{i=t}^{t+m+L-1} D_i)^+\right\}, \qquad \text{(EC.33)}\end{aligned}$$

where the first inequality is obtained from dropping the holding and backlogging costs in the first L periods, the second one follows from inequality (EC.31) and the inequality $\sum_{t=1}^{T+m}\sum_{i=t}^{t+L-1} o_i^\pi \le L\sum_{t=1}^{T+m+L-1} o_i^\pi$, and the last inequality follows from $\theta \ge hL$ and inequality (EC.32).

Since $x_{t,m+L}^{\pi_S} = S$ for each $t \ge 1$, inequality (14) follows directly after applying inequality (EC.33) to policy π_S. Q.E.D.

Appendix E. Proofs of Lemma 6 and Proposition 4 in Section 6

E.1. Proof of Lemma 6

Since the generic demand D has a finite upper support $\bar{D}$, we can define the state space as

$$\mathcal{X} = \{\mathbf{x} = (x_1, x_2, \ldots, x_{m-1}) : 0 \le x_1 \le x_2 \le \ldots \le x_{m-1} \le \bar{D}\}.$$

Based on the discussion after Lemma 6, it remains to prove that there exist an optimal policy, denoted by π^*, for our system under the average-cost criterion, and a sequence of discount factors $\{\alpha_n \in (0,1) : n \ge 1\}$ converging to one such that

$$x_{t,m}^{\pi^*}(\mathbf{x}) = \lim_{n\to\infty} x_{t,m}^{\alpha_n,*}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}, \forall t \ge 1,$$

where for any system state $\mathbf{x} \in \mathcal{X}$, $x_{t,m}^{\pi^*}(\mathbf{x})$ is the order-up-to level in period t under policy π^* and $x_{t,m}^{\alpha_n,*}(\mathbf{x})$ is the order-up-to level under the optimal policy in period t for the system under the discounted-cost criterion with discount factor $\alpha_n \in (0,1)$.

According to Theorem 3.8 in Schäl (1993), the above statement is proven once we verify a set of three conditions (i.e., a general assumption, and conditions (S) and (B) stated in Schäl (1993)) for our perishable inventory system. For brevity, we omit their detailed statements. Among the three conditions, the general assumption and condition (S) can be easily verified, and we omit the details for brevity. To verify condition (B), we need to prove the following inequality:

$$\sup_{\alpha<1}\left(J_\alpha^*(\mathbf{x}) - \inf_{\mathbf{x}'\in\mathcal{X}} J_\alpha^*(\mathbf{x}')\right) < \infty, \quad \forall \mathbf{x} \in \mathcal{X}, \tag{EC.34}$$

where $J_\alpha^*(\mathbf{x}) \triangleq \inf_\pi J_\alpha^\pi(\mathbf{x})$ and $J_\alpha^\pi(\mathbf{x})$ denotes the expected total discounted cost under an admissible policy π with the initial state $\mathbf{x}$ and discount factor $\alpha \in (0,1)$.

We now prove inequality (EC.34). For an initial state $\mathbf{x}' \in \mathcal{X}$, let $\mathbf{x}_m^{\alpha*}(\mathbf{x}')$ be the system state in period m under the optimal policy for the system under the discounted-cost criterion with discount factor $\alpha \in (0,1)$. From Theorem 1 in Nandakumar and Morton (1993), there exists a simple, compact positive ordering region (P.O.R.) of the system state $\mathbf{x}$ including $\mathbf{x} = \mathbf{0}$ such that it is optimal not to order outside the P.O.R. and once the system enters the P.O.R., it can never leave it. Since the product has a fixed lifetime of m periods, $\mathbf{x}_m^{\alpha *}(\mathbf{x}')$ must be in the P.O.R. under any demand sample path regardless of the initial state $\mathbf{x}'$. Then, we have

$$J_\alpha^*(\mathbf{x}') \geq \alpha^{m-1}\mathbb{E}[J_\alpha^*(\mathbf{x}_m^{\alpha *}(\mathbf{x}'))] \geq \alpha^{m-1} J_\alpha^*(\mathbf{0}), \tag{EC.35}$$

where the first inequality follows from the definition of $J_\alpha^*(\cdot)$, and the second one holds since $J_\alpha^*(\mathbf{x})$ is increasing in $\mathbf{x}$ in the P.O.R. by Theorem 1 in Nandakumar and Morton (1993).

For any initial state $\mathbf{x} \in \mathcal{X}$, we define a feasible policy π as follows: it does not place any order in the first $m-1$ periods, and then orders optimally under the discounted-cost criterion with the discount factor $\alpha \in (0,1)$ from period m onwards. Then, we have

$$\begin{aligned} J_\alpha^*(\mathbf{x}) - \inf_{\mathbf{x}' \in \mathcal{X}} J_\alpha^*(\mathbf{x}') &\leq J_\alpha^\pi(\mathbf{x}) - \alpha^{m-1} J_\alpha^*(\mathbf{0}) \\ &\leq (m-1) \cdot \big((h+\theta) x_{m-1} + p\mathbb{E}[D]\big), \end{aligned}$$

where the first inequality follows from the definition of $J_\alpha^*(\cdot)$ and (EC.35), and the second inequality follows from the definition of policy π and the fact that the inventory levels in the first $m-1$ periods under policy π do not exceed x_{m-1}, and $\alpha < 1$. Since $J_\alpha^*(\mathbf{x}) - \inf_{\mathbf{x}' \in \mathcal{X}} J_\alpha^*(\mathbf{x}')$ is uniformly bounded for all $\alpha \in (0,1)$, inequality (EC.34) holds. Q.E.D.

E.2. Proof of Proposition 4

Note that equation (EC.29) and inequality (EC.30) in the proof of Lemma 5 hold for any admissible policy π under any issuance policy. Then, it follows that inequality (EC.33) holds for any admissible policy π under any issuance policy. From the definition of C_t^π in (2), we have

$$\begin{aligned} \sum_{t=1}^{T+m+L} \mathbb{E}[C_t^\pi] &\geq \sum_{t=1}^{T} \Big\{ h\big(\mathbb{E}[x_{t,m+L}^\pi] - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - \mathbb{E}[x_{t,m+L}^\pi]\big)^+ + \frac{\theta - hL}{m+L}(\mathbb{E}[x_{t,m+L}^\pi] - \sum_{i=t}^{t+m+L-1} D_i)^+ \Big\} \\ &\geq T\widetilde{C}^L(\widetilde{S}^L), \end{aligned} \tag{EC.36}$$

where the first inequality follows from inequality (EC.33), the independence between $x_{t,m+L}^\pi$ and $(D_t, \ldots, D_{t+m+L-1})$ for each period t and the conditional Jensen's inequality, and the second inequality follows from the definition of $\widetilde{C}^L(\cdot)$ and $\widetilde{S}^L$. Then, it follows from the definition of C^π and inequality (EC.36) that

$$C^\pi = \limsup_{T\to\infty} \frac{1}{T+m+L} \sum_{t=1}^{T+m+L} \mathbb{E}[C_t^\pi] \geq \widetilde{C}^L(\widetilde{S}^L).$$

Since the above inequality holds for any admissible policy $\pi \in \Pi$ and under any issuance policy, we obtain $\mathrm{OPT}^L \geq \widetilde{C}^L(\widetilde{S}^L)$ for any $L \geq 1$ under any issuance policy. Q.E.D.

Appendix F. Asymptotic Convergence Rate in Large Penalty Costs

In this appendix, we characterize the convergence rate for the optimality gap of policy $\pi_{\widetilde{S}}$ in the classic system with large unit penalty cost p under four classes of demands. The results are presented in the following proposition, in which for non-negative functions $f(x)$ and $g(x)$, the notation $f(x) = \mathcal{O}(g(x))$ means that $\limsup_{x\to\infty} f(x)/g(x) < \infty$. These results show that the asymptotic rate at which the optimality gap of policy $\pi_{\widetilde{S}}$ converges to zero in the unit penalty cost depends on the specific demand distribution and differs for different demand distributions. Similar results can be established for other classes of continuous demand distributions.

Proposition EC.1. *Suppose that demand D is a continuous random variable with probability density function $f(\cdot)$. Then, the following results hold:*

(a) If D follows a Weibull distribution with shape parameter $\beta > 0$, then $C_p^{FIFO}(\widetilde{S}_p) - \mathrm{OPT}_p^{FIFO} = \mathcal{O}\big(p^{-\frac{1}{m^\beta}} \cdot (\log p)^{\frac{1}{\beta}-1}\big)$;

(b) If D follows a fat-tailed distribution with parameter $\alpha > 1$, then $C_p^{FIFO}(\widetilde{S}_p) - \mathrm{OPT}_p^{FIFO} = \mathcal{O}(p^{-(1-\frac{1}{\alpha})})$;

(c) If D is bounded and $f(\bar{D}) > 0$, then $C_p^{FIFO}(\widetilde{S}_p) - \mathrm{OPT}_p^{FIFO} = \mathcal{O}(p^{-1})$;

(d) If D follows a triangular distribution, then $C_p^{FIFO}(\widetilde{S}_p) - \mathrm{OPT}_p^{FIFO} = \mathcal{O}(p^{-\frac{1}{2}})$.

Proof of Proposition EC.1. The proofs of parts (a)-(b) are based on inequality (25) and the proofs of parts (c)-(d) are based on inequality (26). For convenience, let $\bar{F}(x) \triangleq 1 - F(x)$ for a c.d.f. $F(\cdot)$.

Proof of part (a). Since $\widetilde{S}_p \geq \underline{S}_p$ and from inequality (25), it suffices to prove

$$\mathbb{E}\Big[\big(D - \frac{1}{m}\underline{S}_p\big)^+\Big] \sim p^{-\frac{1}{m^\beta}}(\log p)^{\frac{1}{\beta}-1}, \tag{EC.37}$$

where the notation $f(p) \sim g(p)$ represents the limit $\lim_{p\to\infty} f(p)/g(p)$ exists and is positive. Since both $\mathbb{E}[(D - \frac{1}{m}\underline{S}_p)^+]$ and $p^{-\frac{1}{m^\beta}}(\log p)^{\frac{1}{\beta}-1}$ converge to zero as $p \to \infty$, by applying L'Hospital's rule, we obtain

$$\lim_{p\to\infty} \frac{\mathbb{E}[(D - \frac{1}{m}\underline{S}_p)^+]}{p^{-\frac{1}{m^\beta}}(\log p)^{\frac{1}{\beta}-1}} = \lim_{p\to\infty} \frac{\frac{1}{m}\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{p^{-\frac{1}{m^\beta}-1}(\log p)^{\frac{1}{\beta}-1}\big(\frac{1}{m^\beta} - (\frac{1}{\beta} - 1)(\log p)^{-1}\big)},$$

where $\underline{S}_p'$ denotes the derivative of $\underline{S}_p$ with respect to p. Therefore, it suffices to derive the order of $\underline{S}_p'$ and $\bar{F}(\frac{1}{m}\underline{S}_p)$.

One can verify that $\underline{S}_p = \alpha\big(\log(\frac{p+h}{h+\theta})\big)^{\frac{1}{\beta}}$ under a Weibull distribution for $\bar{F}(\cdot)$ with scale parameter α (> 0) and shape parameter β (> 0). By taking the derivative with respect to p, we obtain

$$\underline{S}_p' = \frac{\alpha}{\beta(p+h)}\left(\log\left(\frac{p+h}{h+\theta}\right)\right)^{\frac{1}{\beta}-1} \sim p^{-1}(\log p)^{\frac{1}{\beta}-1}.$$

In addition, it follows that $\bar{F}(\frac{1}{m}\underline{S}_p)=(\frac{p+h}{h+\theta})^{-\frac{1}{m^\beta}}\sim p^{-\frac{1}{m^\beta}}$. Therefore,

$$\frac{\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{p^{-\frac{1}{m^\beta}-1}(\log p)^{\frac{1}{\beta}-1}\big(\frac{1}{m^\beta}-(\frac{1}{\beta}-1)(\log p)^{-1}\big)}\sim\frac{p^{-1}(\log p)^{\frac{1}{\beta}-1}p^{-\frac{1}{m^\beta}}}{p^{-\frac{1}{m^\beta}-1}(\log p)^{\frac{1}{\beta}-1}\big(\frac{1}{m^\beta}-(\frac{1}{\beta}-1)(\log p)^{-1}\big)}\sim 1,$$

which implies (EC.37).

Proof of part (b). Since $\widetilde{S}_p\ge\underline{S}_p$ and from inequality (25), it suffices to prove

$$\mathbb{E}\Big[\Big(D-\frac{1}{m}\underline{S}_p\Big)^+\Big]\sim p^{-(1-\frac{1}{\alpha})}. \tag{EC.38}$$

Since both $\mathbb{E}\big[(D-\frac{1}{m}\underline{S}_p)^+\big]$ and $p^{-(1-\frac{1}{\alpha})}$ converge to zero as $p\to\infty$, by applying L'Hospital's rule, we obtain

$$\lim_{p\to\infty}\frac{\mathbb{E}[(D-\frac{1}{m}\underline{S}_p)^+]}{p^{-(1-\frac{1}{\alpha})}}=\lim_{p\to\infty}\frac{\frac{1}{m}\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{(1-\frac{1}{\alpha})p^{-(2-\frac{1}{\alpha})}}.$$

Hence again, we only need to derive the order of $\underline{S}_p'$ and $\bar{F}(\frac{1}{m}\underline{S}_p)$.

By definition, the tail function of a fat-tailed distribution satisfies $\bar{F}(x)\sim x^{-\alpha}$ for some $\alpha>1$. Therefore, $\bar{F}(\underline{S}_p)=(h+\theta)/(p+h)\sim\underline{S}_p^{-\alpha}$, which implies that $\underline{S}_p\sim p^{\frac{1}{\alpha}}$. From L'Hospital's rule, we obtain $\underline{S}_p'\sim p^{\frac{1}{\alpha}-1}$. In addition, $\bar{F}(\frac{1}{m}\underline{S}_p)\sim(\frac{1}{m}\underline{S}_p)^{-\alpha}\sim(\underline{S}_p)^{-\alpha}\sim p^{-1}$. Therefore,

$$\frac{\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{p^{-(2-\frac{1}{\alpha})}}\sim\frac{p^{\frac{1}{\alpha}-1}p^{-1}}{p^{-(2-\frac{1}{\alpha})}}\sim 1,$$

which implies (EC.38).

Proof of part (c). From inequality (26), it suffices to prove that $\bar{D}-\underline{S}_p\sim p^{-1}$. Since $F(\underline{S}_p)=(p-\theta)/(p+h)$, by taking the derivative with respect to p on both sides, we obtain $\underline{S}_p'\cdot f(\underline{S}_p)=\frac{h+\theta}{(p+h)^2}$. Since $f(\bar{D})>0$, we have

$$\lim_{p\to\infty}\frac{\bar{D}-\underline{S}_p}{p^{-1}}=\lim_{p\to\infty}\frac{\underline{S}_p'}{p^{-2}}=\lim_{p\to\infty}\frac{(h+\theta)}{p^{-2}(p+h)^2f(\underline{S}_p)}=\frac{h+\theta}{f(\bar{D})}\in(0,\infty),$$

which implies that $\bar{D}-\underline{S}_p\sim p^{-1}$.

Proof of part (d). From inequality (26), it suffices to prove that $\bar{D}-\underline{S}_p\sim p^{-\frac{1}{2}}$. Let $[\underline{D},\bar{D}]$ be the support of the triangular distribution and a be the mode. Then, the tail distribution function $\bar{F}(\cdot)$ on $[\underline{D},\bar{D}]$ is given by

$$\bar{F}(x)=\begin{cases}1-\frac{(x-\underline{D})^2}{(\bar{D}-\underline{D})(a-\underline{D})}, & \underline{D}\le x\le a;\\ \frac{(\bar{D}-x)^2}{(\bar{D}-\underline{D})(\bar{D}-a)}, & a\le x\le\bar{D}.\end{cases}$$

When p is sufficiently large, $(h+\theta)/(p+h)$ is very close to zero and therefore, $\underline{S}_p$ satisfies

$$\bar{F}(\underline{S}_p)=\frac{(\bar{D}-\underline{S}_p)^2}{(\bar{D}-\underline{D})(\bar{D}-a)}=\frac{h+\theta}{p+h}.$$

Therefore, $\bar{D}-\underline{S}_p\sim p^{-\frac{1}{2}}$. Q.E.D.

Appendix G. A Class of Asymptotically Optimal Base-Stock Policies

In this appendix, we consider a class of base-stock policies for the classic system under the FIFO issuance policy, denoted as $\{\pi_{\widetilde{S}^{\alpha,\beta}}: \alpha \geq 0, \beta \geq 0\}$, and extend some of our results established for base-stock policy $\pi_{\widetilde{S}}$ in §3 to this class of policies. For any $\alpha \geq 0$ and $\beta \geq 0$, we first define the approximate cost function $\widetilde{C}^{\alpha,\beta}(S)$ as

$$\widetilde{C}^{\alpha,\beta}(S) \triangleq h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \theta UL^{\alpha,\beta}(S), \quad \forall S \geq 0, \tag{EC.39}$$

where

$$UL^{\alpha,\beta}(S) \triangleq \frac{\alpha}{m}\mathbb{E}\left[\left(S - \sum_{i=1}^{m} D_i\right)^+\right] + \beta\mathbb{E}\left[\left(\frac{S}{m} - D\right)^+\right].$$

That is, we construct the approximate cost function $\widetilde{C}^{\alpha,\beta}(S)$ by approximating the term $\mathbb{E}[O_\infty(S)]$ by a non-negative linear combination of its lower bound $\frac{1}{m}\mathbb{E}[(S - \sum_{i=1}^m D_i)^+]$ and its upper bound $\mathbb{E}[(\frac{S}{m} - D)^+]$ (see Lemma 2). Then, we define $\widetilde{S}^{\alpha,\beta}$ as the minimizer of function $\widetilde{C}^{\alpha,\beta}(S)$, i.e.,

$$\widetilde{S}^{\alpha,\beta} = \inf \arg\min_{S \geq 0} \widetilde{C}^{\alpha,\beta}(S).$$

Similar to inequality (5), one can easily verify that

$$F^{-1}\left(\frac{p}{p+h+\theta(\alpha+\beta)}\right) \leq \widetilde{S}^{\alpha,\beta} \leq F^{-1}\left(\frac{p}{p+h}\right) \tag{EC.40}$$

The following proposition extends Theorem 1 to 4 to the class of base-stock policies $\{\pi_{\widetilde{S}^{\alpha,\beta}}: \alpha \geq 0, \beta \geq 0\}$ under certain conditions.

Proposition EC.2. *(a) When $\beta = 0$, the optimality gap of policy $\pi_{\widetilde{S}^{\alpha,0}}$ converges to zero exponentially fast in the lifetime m;*

(b) Under the assumption of Theorem 2, the optimality gap of policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ converges to zero exponentially fast in the demand population size n;

(c) When $\alpha + \beta = 1$ or demand D is bounded, $\lim_{p\to\infty}(C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p) = 0$;

(d) When $\alpha + \beta > 0$, $\lim_{\theta\to\infty}(C_\theta(\widetilde{S}_\theta^{\alpha,\beta}) - \mathrm{OPT}_\theta) = 0$.

Part (a) shows that Theorem 1 holds for policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ when $\beta = 0$. When $\beta > 0$, it remains unknown whether the optimality gap of $\pi_{\widetilde{S}^{\alpha,\beta}}$ converges to zero exponentially fast in the lifetime m. Part (b) extends Theorem 2 to policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ with arbitrary $\alpha \geq 0$ and $\beta \geq 0$. Part (c) reveals two cases under which $\pi_{\widetilde{S}^{\alpha,\beta}}$ is asymptotically optimal with large p. When $\alpha + \beta = 1$, the difference between $UL^{\alpha,\beta}(S)$ and $\frac{1}{m}\mathbb{E}[(S - \sum_{i=1}^m D_i)^+]$ converges to zero when $S \to \infty$. Therefore, these two approximations of $\mathbb{E}[O_\infty(S)]$ are roughly the same when p is large. On the other hand, when D is bounded, both $\widetilde{S}$ and $\widetilde{S}^{\alpha,\beta}$ converge to $\bar{D}$. Thus, in both cases, $\pi_{\widetilde{S}^{\alpha,\beta}}$ is asymptotically optimal with large p. Part (d) can be explained as follows. When either α or β is positive, base-stock level $\widetilde{S}^{\alpha,\beta}$ converges to $\widetilde{S}_\infty$ and the long-run average outdating costs under policy $\widetilde{S}^{\alpha,\beta}$ converges to zero as $\theta \to \infty$. Thus, from the discussion in §3.3, $\pi_{\widetilde{S}^{\alpha,\beta}}$ is asymptotically optimal with large unit outdating costs. Note that when $\alpha = \beta = 0$, $\pi_{S^{NP}}$ is asymptotically optimal with large unit outdating costs only when $S^{NP} \le m\underline{D}$. This is because S^{NP} is independent of θ, and when $S^{NP} > m\underline{D}$, by Lemma 2(b), the long-run average outdating cost under $\pi_{\widetilde{S}^{NP}}$ is positive and increases linearly in θ. Thus, $\pi_{\widetilde{S}^{NP}}$ is *not* asymptotically optimal with large θ when $S^{NP} > m\underline{D}$.

Similar to Proposition EC.1, we can also characterize the convergence rate of the optimality gap for base-stock policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ as p increases, and prove that $\pi_{\widetilde{S}^{\alpha,\beta}}$ satisfies Proposition EC.1 (a)-(b) when $\alpha + \beta = 1$ and Proposition EC.1 (c)-(d), established for $\pi_{\widetilde{S}}$. The details are omitted.

Proof of Proposition EC.2. First, we prove parts (a) and (b). Similar to the proofs of Theorems 1 and 2, if we can establish the following upper bound on the optimality gap of policy $\pi_{\widetilde{S}^{\alpha,\beta}}$:

$$\begin{aligned} C(\widetilde{S}^{\alpha,\beta}) - \mathrm{OPT} \le \theta\bigg(& \frac{m-\alpha}{m}\mathbb{E}\bigg[\Big(\widetilde{S}^{\alpha,\beta} - \sum_{i=1}^{m} D_i\Big)^+\bigg] - \beta\mathbb{E}\bigg[\Big(\frac{1}{m}\widetilde{S}^{\alpha,\beta} - D\Big)^+\bigg] \\ & + \frac{\alpha-1}{m}\mathbb{E}\bigg[\Big(\widetilde{S} - \sum_{i=1}^{m} D_i\Big)^+\bigg] + \beta\mathbb{E}\bigg[\Big(\frac{1}{m}\widetilde{S} - D\Big)^+\bigg]\bigg), \end{aligned} \tag{EC.41}$$

then the results in Proposition EC.2 can be proven easily using the similar arguments in the proofs of Theorems 1 and 2.

Now, we show inequality (EC.41). From Proposition 2 and the optimality of $\widetilde{S}^{\alpha,\beta}$, we obtain

$$C(\widetilde{S}^{\alpha,\beta}) - \mathrm{OPT} \le C(\widetilde{S}^{\alpha,\beta}) - \widetilde{C}^{\alpha,\beta}(\widetilde{S}^{\alpha,\beta}) + \widetilde{C}^{\alpha,\beta}(\widetilde{S}) - \widetilde{C}(\widetilde{S}). \tag{EC.42}$$

From Lemma 2(a), and the definitions of $C(\cdot)$, $\widetilde{C}^{\alpha,\beta}(\cdot)$ and $UL^{\alpha,\beta}(\cdot)$, we have

$$\begin{aligned} C(\widetilde{S}^{\alpha,\beta}) - \widetilde{C}^{\alpha,\beta}(\widetilde{S}^{\alpha,\beta}) &\le \theta\mathbb{E}\Big[\big(\widetilde{S}^{\alpha,\beta} - \sum_{i=1}^{m} D_i\big)^+\Big] - \theta UL^{\alpha,\beta}(\widetilde{S}^{\alpha,\beta}) \\ &= \theta\Big(1 - \frac{\alpha}{m}\Big)\mathbb{E}\Big[\big(\widetilde{S}^{\alpha,\beta} - \sum_{i=1}^{m} D_i\big)^+\Big] - \theta\beta\mathbb{E}\Big[\Big(\frac{\widetilde{S}^{\alpha,\beta}}{m} - D\Big)^+\Big]. \end{aligned} \tag{EC.43}$$

From the definitions of $\widetilde{C}^{\alpha,\beta}(\cdot)$ and $\widetilde{C}(\cdot)$, we also have

$$\widetilde{C}^{\alpha,\beta}(\widetilde{S}) - \widetilde{C}(\widetilde{S}) = \theta\frac{\alpha-1}{m}\mathbb{E}\Big[\big(\widetilde{S} - \sum_{i=1}^{m} D_i\big)^+\Big] + \theta\beta\mathbb{E}\Big[\Big(\frac{\widetilde{S}}{m} - D\Big)^+\Big]. \tag{EC.44}$$

Combining (EC.42)-(EC.44), we obtain inequality (EC.41).

Then, we prove part (c). First, we consider the case with $\alpha + \beta = 1$ and unbounded demand. Following the proof of inequality (EC.41) while replacing the use of the upper bound from Lemma 2(a) with that from Lemma 2(a) in inequality (EC.43), we can obtain the following inequality:

$$C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p \le \theta\mathbb{E}\Big[\Big(\frac{\widetilde{S}_p^{\alpha,\beta}}{m} - D\Big)^+\Big] - \theta UL^{\alpha,\beta}(\widetilde{S}_p^{\alpha,\beta}) + \theta UL^{\alpha,\beta}(\widetilde{S}_p) - \frac{\theta}{m}\mathbb{E}\Big[\big(\widetilde{S}_p - \sum_{i=1}^{m} D_i\big)^+\Big].$$

Then, by the definition of $UL^{\alpha,\beta}(\cdot)$ and since $\beta = 1-\alpha$ and $x^+ = x + (-x)^+$ for any $x \in \Re$, after some simple algebra we obtain that

$$\begin{aligned} C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p \le & \alpha\theta\biggl(\mathbb{E}\Bigl[\Bigl(D - \frac{\widetilde{S}_p^{\alpha,\beta}}{m}\Bigr)^+\Bigr] - \frac{1}{m}\mathbb{E}\Bigl[\bigl(\sum_{i=1}^m D_i - \widetilde{S}_p^{\alpha,\beta}\bigr)^+\Bigr]\biggr) \\ & + (1-\alpha)\theta\biggl(\mathbb{E}\Bigl[\Bigl(D - \frac{\widetilde{S}_p}{m}\Bigr)^+\Bigr] - \frac{1}{m}\mathbb{E}\Bigl[\Bigl(\sum_{i=1}^m D_i - \widetilde{S}_p\Bigr)^+\Bigr]\biggr). \end{aligned} \tag{EC.45}$$

When the demand D is unbounded, one can easily verify that $\lim_{p\to\infty} \widetilde{S}_p^{\alpha,\beta} = \lim_{p\to\infty} \widetilde{S}_p = \infty$, and thus, the RHS of inequality (EC.45) converges to zero as $p \to \infty$. Therefore, $\lim_{p\to\infty}(C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p) = 0$ when $\alpha+\beta = 1$ and demand D is unbounded.

Next, we consider the case with bounded demand (i.e., $\bar{D} < \infty$). In this case, we prove the following inequality:

$$C(\widetilde{S}^{\alpha,\beta}) - \mathrm{OPT} \le (h+\theta)(\bar{D} - \underline{S}) + (h + \theta(\alpha+\beta))(\bar{D} - \widetilde{S}^{\alpha,\beta}). \tag{EC.46}$$

Together with $\lim_{p\to\infty} \underline{S}_p = \lim_{p\to\infty} \widetilde{S}_p^{\alpha,\beta} = \bar{D}$, this directly leads to $\lim_{p\to\infty}(C(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p) = 0$.

The proof of inequality (EC.46) is analogous to that of inequality (26) for base-stock policy $\pi_{\widetilde{S}}$. The following are the differences. First, since $\widetilde{S}^{\alpha,\beta} \ge F^{-1}(\frac{p}{p+h+\theta(\alpha+\beta)})$ from (EC.40), the inequality $p\mathbb{E}[(D-\widetilde{S})^+] \le (h+\theta)\bigl(\bar{D} - \widetilde{S}\bigr)$ should be modified to $p\mathbb{E}[(D - \widetilde{S}^{\alpha,\beta})^+] \le (h+\theta(\alpha+\beta))\bigl(\bar{D} - \widetilde{S}^{\alpha,\beta}\bigr)$. Based on the above inequality, $\widetilde{S}^{\alpha,\beta} \le \bar{D}$, and the fact that $\mathbb{E}[O_\infty(S)]$ increases in S (which can be easily proven from the recursion (19)), inequality (EC.14) should be modified accordingly to

$$C(\widetilde{S}^{\alpha,\beta}) \le h(\bar{D} - \mathbb{E}[D]) + \theta\mathbb{E}\bigl[O_\infty(\bar{D})\bigr] + (h+\theta(\alpha+\beta))\bigl(\bar{D} - \widetilde{S}^{\alpha,\beta}\bigr).$$

Combining this inequality with the inequality in Proposition 3 and since $\mathbb{E}[O_\infty(S)] - S$ is decreasing in S, we obtain inequality (EC.46). The proof of part (c) is complete.

Finally, we prove part (d). Suppose $\alpha+\beta > 0$. By applying similar arguments to those in the proof of Theorem 4, we can easily show that $\lim_{\theta\to\infty} \theta UL^{\alpha,\beta}(\widetilde{S}_\theta^{\alpha,\beta}) = 0$ and

$$\lim_{\theta\to\infty} h\mathbb{E}[(\widetilde{S}_\theta^{\alpha,\beta} - D)^+] + p\mathbb{E}[(D - \widetilde{S}_\theta^{\alpha,\beta})^+] = \min_{0\le S\le m\underline{D}} \{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\}. \tag{EC.47}$$

Since $\mathbb{E}[O_\infty(S)] \le \mathbb{E}[(S - \sum_{i=1}^m D_i)^+]$ by Lemma 2(a) and $\mathbb{E}[O_\infty(S)] \le \mathbb{E}[(S/m - D)^+]$ by Lemma 2(b), it follows from $\lim_{\theta\to\infty} \theta UL^{\alpha,\beta}(\widetilde{S}_\theta^{\alpha,\beta}) = 0$ that $\lim_{\theta\to\infty} \theta\mathbb{E}[O_\infty(\widetilde{S}_\theta^{\alpha,\beta})] = 0$. Therefore,

$$\lim_{\theta\to\infty}(C_\theta(\widetilde{S}_\theta^{\alpha,\beta}) - \mathrm{OPT}_\theta) = \lim_{\theta\to\infty} \theta\mathbb{E}[O_\infty(\widetilde{S}_\theta^{\alpha,\beta})] = 0,$$

where the first equality follows from Lemma 1, (EC.47), and the proof of Theorem 4. Q.E.D.

Appendix H. Figures in Section 7

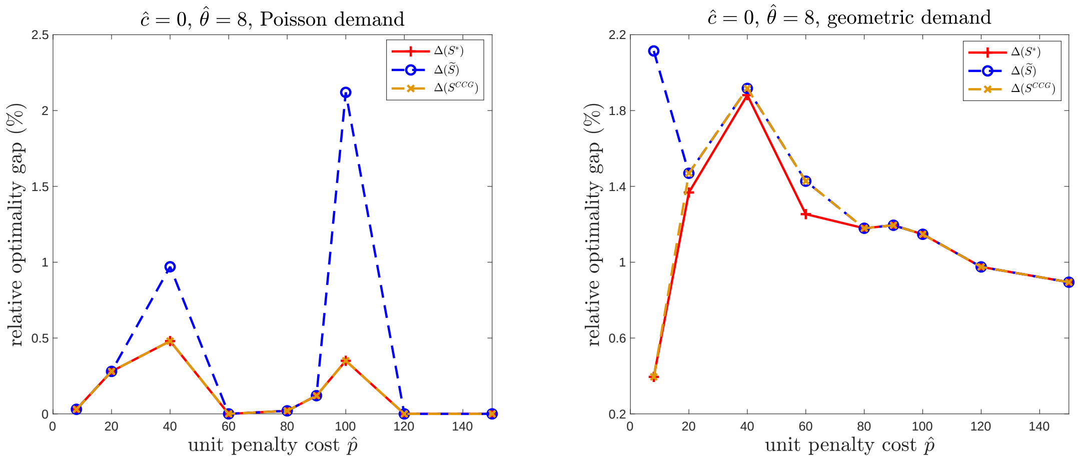

Figure EC.1 **Performances of base-stock policies in the lost-sales system under FIFO issuance policy and different unit penalty costs** $\hat{p}\in\{8,20,40,60,80,90,100,120,150\}$**, or correspondingly, under different service levels** $\hat{p}/(\hat{p}+\hat{h})\times 100\%\in\{88.89\%,95.24\%,97.56\%,98.36\%,98.77\%,98.90\%,99.01\%,99.17\%,99.34\%\}$

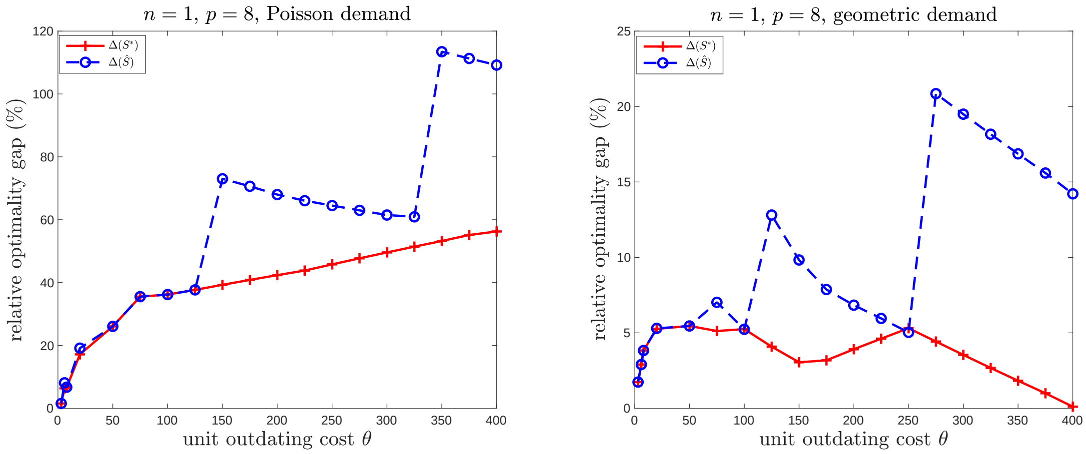

Figure EC.2 **Performances of base-stock policies in the lost-sales system under LIFO issuance policy and different unit outdating costs** $\hat{\theta}\in\{3,6,8,20,50,75,100,125,150,175,200,225,250,275,300,325,350,375,400\}$

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