ONLINE APPENDIX

A.1 Discussion: Simultaneous Entry of Competing Firms

In the main model, we demonstrated that the desire to avoid a negative advertising war can cause an entrant to choose a product positioning that is similar to the existing offering in the market. Next, we consider the case where two competing firms enter a new market and choose their positions simultaneously.

Definition A Perfect Bayesian Equilibrium (PBE) in a simultaneous entry game consists of positioning policies $x^1, x^2 \in \{L, R\}$, advertising policies $a^2(x_1, x_2, \theta) \in \{P_2, N_1, \varnothing\}$, $a^1(x_1, x_2, \theta) \in \{P_1, N_2, \varnothing\}$ of firms, beliefs of consumers over firms types $\mathscr{F} : \theta_1 \times \theta_2 \to [0, 1]$, and purchase policies of consumers $\{g^j(x_1, x_2, a_1, a_2)\}_j \in \{1, 2, \varnothing\}$ such that

1. Consumers' choices are sequentially rational, i.e., $\{g^j(.)\}_j$ maximizes $E[U_{ij}|\mathscr{F}]$.
2. $a^2(.)$ and $a^1(.)$ constitute a Nash equilibrium of the advertising sub-game, given $\{g^j(.)\}_j$.
3. The location choices $x^1(.), x^2(.) \in \{L, R\}$ constitute a Nash equilibrium given $a^2(.)$, $a^1(.)$, and $\{g^j(.)\}_j$.
4. The consumers' beliefs $\mathscr{F}$ are updated based on $a^2(.)$ and $a^1(.)$ according to the Bayes' Rule.

When two firms simultaneously enter the market, the second stage of the game where firms determine their advertising is identical to the one in the benchmark model. Therefore, the lemmas in Section 3.2.1 still apply in the backward induction solution. What is different is the first-stage decisions about positioning. Yet, it turns out that the equilibria of the simultaneous entry game are similar to the equilibria of the entrant-incumbent game, with the slight modification that firms can choose to locate both on the right and on the left end of consumer heterogeneity line:

Proposition A.1. (Positioning under Simultaneous Entry)
(i) For each PBE of the entrant-incumbent game under co-location, there are two equivalent PBE in the simultaneous entry game, where both firms choose to locate at either R or L.
(ii) For each PBE of the entrant-incumbent game with location differentiation, there are two equivalent PBE in the simultaneous entry game, where one firm locates at R and other one firm locates at L.

Proposition A.1 demonstrates that the key insights of the benchmark model are robust to simultaneous entry of firms in a market. The equilibrium outcomes under a simultaneous game map to those under a sequential game.

A.2 Proofs of Propositions, Lemmas, and Corollaries

Proof of Proposition 1. The PBE given in Proposition 1 is defined by the following strategies and beliefs:

1. $x^2 = R$
2. $a^i(x_i, \theta) = \begin{cases} P_i, \text{if } P_i = \Pi \\ \varnothing, \text{if } P_i = 0 \end{cases}$
3. $\mathscr{F}$ puts probability 1 on $P_i = \Pi$ if $a_i = P_i$, 0 otherwise.
4. Let $B_{ij} = A_i - |\chi_j - x_i|$. If $\gamma_j < \max_i E[B_{ij}]$ then $g^j(x_2, a_1, a_2) = \varnothing$. If $\gamma_j \geq \max_i E[B_{ij}]$ and $\arg\max_i E[B_{ij}]$ is unique, then $g^j(x_2, a_1, a_2) = \arg\max_i E[B_{ij}]$. Otherwise,

$$g^j(x_2 = L, a_1, a_2) = \begin{cases} 1, \text{w.p. } 0.5 \\ 2, \text{w.p. } 0.5 \end{cases}$$

   and

$$g^j(x_2 = R, a_1, a_2) = \begin{cases} 1, \text{ if } \chi_j = L \\ 2, \text{ if } \chi_j = R \end{cases} .$$

We show that $\{x^2, a^1, a^2, \mathscr{F}, g^j\}$ is the unique PBE.

First, by definition, there exists a single g^j that maximizes $E[U_{ij}]$ for a given $\mathscr{F}$.

Second, note that the $\mathscr{F}$ is consistent with the advertising strategies of the firm types where $\mathscr{F}$, $E[A_i|a_i = P_i, a_{-i}, x_2] = (1 - \sigma_\Pi)\Pi$ and $E[A_i|a_i = \varnothing, a_{-i}, x_2] = -\sigma_\Pi \Pi$. See that the equilibrium is a separating one, i.e., each type follows a different advertising strategy. For a pooling equilibrium to exist, both types would have to announce $\varnothing$ because advertising has to be truthful. For there be no profitable deviations for the type with $P_i = \Pi$, consumer beliefs would have to put a probability less than 1 for $P_i = \Pi$ when P_i is announced which would fail the truthfulness assumption. Hence, $\mathscr{F}$ describes the unique beliefs in any PBE.

Third, in the advertising sub-game, given $\mathscr{F}$, announcing $a^i = P_i$ if $P_i = \Pi$ is strictly dominant, i.e., returns to running positive advertising is strictly larger than running no advertising:

$$E[A_i|a_i = P_i, a_{-i}, x_2] = (1 - \sigma_\Pi)\Pi > -\sigma_\Pi \Pi = E[A_i|a_i = \varnothing, a_{-i}, x_2], \quad \forall a_{-i}, x_2$$

Hence, a^1, a^2 is the unique Nash Equilibrium of the advertising sub-game given $\mathscr{F}$ and g^j.

Last, to prove the optimality of x^2, consider the potential outcomes following each location choice, summarized in Table A3.[32] The table restricts attention to the realizations of P_i. The realizations of N_i are irrelevant for firms' payoff because they cannot be advertised.

[32] Table A3 assumes $\Pi > \delta$. The proof is similar for $\Pi < \delta$.

θ	Prob.	a_1	a_2	$E[A_1]$	$E[A_2]$	D_2
$P_1 = P_2 = \Pi$	$\sigma_\Pi^2 + \rho\sigma_\Pi(1-\sigma_\Pi)$	P_1	P_2	$(1-\sigma_\Pi)\Pi$	$(1-\sigma_\Pi)\Pi$	$0.5\big[2-\Gamma(-(1-\sigma_\Pi)\Pi)-\Gamma(\delta-(1-\sigma_\Pi)\Pi)\big]$
$P_1 = \Pi, P_2 = 0$	$\sigma_\Pi(1-\sigma_\Pi)(1-\rho)$	P_1	$\varnothing$	$(1-\sigma_\Pi)\Pi$	$-\sigma_\Pi\Pi$	0
$P_1 = 0, P_2 = \Pi$	$\sigma_\Pi(1-\sigma_\Pi)(1-\rho)$	$\varnothing$	P_2	$-\sigma_\Pi\Pi$	$(1-\sigma_\Pi)\Pi$	$2-\Gamma(-(1-\sigma_\Pi)\Pi)-\Gamma(\delta-(1-\sigma_\Pi)\Pi)$
$P_1 = P_2 = 0$	$(1-\sigma_\Pi)(1-\sigma_\Pi+\rho\sigma_\Pi)$	$\varnothing$	$\varnothing$	$-\sigma_\Pi\Pi$	$-\sigma_\Pi\Pi$	$0.5\big[2-\Gamma(\sigma_\Pi\Pi)-\Gamma(\delta+\sigma_\Pi\Pi)\big]$

Table A1: Co-Location

θ	Prob.	a_1	a_2	$E[A_1]$	$E[A_2]$	D_2
$P_1 = P_2 = \Pi$	σ_Π^2	P_1	P_2	$(1-\sigma_\Pi)\Pi$	$(1-\sigma_\Pi)\Pi$	$1-\Gamma(-(1-\sigma_\Pi)\Pi)$
$P_1 = \Pi, P_2 = 0$	$\sigma_\Pi(1-\sigma_\Pi)$	P_1	$\varnothing$	$(1-\sigma_\Pi)\Pi$	$-\sigma_\Pi\Pi$	0
$P_1 = 0, P_2 = \Pi$	$\sigma_\Pi(1-\sigma_\Pi)$	$\varnothing$	P_2	$-\sigma_\Pi\Pi$	$(1-\sigma_\Pi)\Pi$	$2-\Gamma(-(1-\sigma_\Pi)\Pi)-\Gamma(\delta-(1-\sigma_\Pi)\Pi)$
$P_1 = P_2 = 0$	$(1-\sigma_\Pi)^2$	$\varnothing$	$\varnothing$	$-\sigma_\Pi\Pi$	$-\sigma_\Pi\Pi$	$1-\Gamma(\sigma_\Pi\Pi)$

Table A2: Locating Apart

Table A3: Demand for the Entrant with Negative Advertising Forbidden

We can show the expected payoff under locating apart exceeds the expected payoff under co-location in two steps. Assume the probabilities of realizations were identical across location choices. The payoff is identical for the asymmetric realizations of P_i. For the symmetric realizations, the payoff would have been larger under locating apart since

$$2-2\Gamma(-(1-\sigma_\Pi)\Pi) > 2-\Gamma(-(1-\sigma_\Pi)\Pi)-\Gamma(\delta-(1-\sigma_\Pi)\Pi)$$

and

$$2-2\Gamma(\sigma_\Pi\Pi) > 2-\Gamma(\sigma_\Pi\Pi)-\Gamma(\delta+\sigma_\Pi\Pi).$$

The extra term co-location has due to the difference in probabilities is

$$\frac{\rho\sigma_\Pi(1-\sigma_\Pi)}{2}\Big[\Gamma(-(1-\sigma_\Pi)\Pi)+\Gamma(\delta-(1-\sigma_\Pi)\Pi)-\Gamma(\sigma_\Pi\Pi)-\Gamma(\delta+\sigma_\Pi\Pi)\Big]$$

which is always negative. Hence, the expected payoff is necessarily higher under locating apart.

□

Proofs of Proposition 2, Lemmas 1, 2, and 4. The PBE is defined by

1. $x^2 = L$

2.

$$a^i(x_i = R, \theta) = \begin{cases} P_i, \text{ if } N_i = 0, P_{-i} = 0, P_i = \Pi \text{ (weak opponent)} \\ N_{-i}, \text{if above condition fails and } N_{-i} = -\beta \\ P_i, \text{ if above conditions fail and } P_i = \Pi \\ \varnothing, \text{otherwise} \end{cases}$$

and

$$a^i(x_i = L, \theta) = \begin{cases} P_i, \text{if } P_i = \Pi \\ N_{-i}, \text{if above condition fails and } N_{-i} = -\beta \\ \varnothing, \text{otherwise} \end{cases}$$

3. $\mathscr{F}$ can be constructed from Table 1.

4. Let $B_{ij} = A_i - |\chi_j - x_i|$

- if $\gamma_j < \max_i E[B_{ij}]$ then $g^j(x_2, a_1, a_2) = \varnothing$
- if above condition fails, $\arg\max_i E[B_{ij}]$ is unique, then $g^j(x_2, a_1, a_2) = \arg\max_i E[B_{ij}]$
- otherwise, $g^j(x_2 = L, a_1, a_2) = \begin{cases} 1, \text{w.p. } 0.5 \\ 2, \text{w.p. } 0.5 \end{cases}$ and $g^j(x_2 = R, a_1, a_2) = \begin{cases} 1, \text{ if } \chi_j = L \\ 2, \text{ if } \chi_j = R \end{cases}$

We prove that $\{x^2, a^1, a^2, \mathscr{F}, g^j\}$ is a PBE when

(i) $\beta > (1 - \sigma_\Pi)\Pi + \delta$ (A1a)

(ii) $\bar{\gamma} + \sigma_\beta \beta \geq (1 - \sigma_\Pi)\Pi + \delta$ (A1b)

(iii) $\Pi > (1 - \rho)(1 - \sigma_\beta)\beta$ (A1c)

(iv) $\sigma_\beta(1 - \sigma_\beta)(1 - \sigma_\Pi)(1 - \rho)(1 - \rho + \rho\sigma_\Pi)\beta + (1 - \sigma_\Pi)\sigma_\Pi(1 - \rho)\Pi - \frac{\delta}{2} \geq \sigma_\beta(1 - \sigma_\beta)\beta$
$+ (1 - \sigma_\Pi)\sigma_\Pi(1 - \sigma_\beta)(1 - 3\sigma_\beta)\Pi - (1 - \sigma_\beta)(\sigma_\beta + (1 - \sigma_\beta)\sigma_\Pi(1 - \sigma_\Pi))\delta$ (A1d)

First, g^j, by definition, maximizes $E[U_{ij}]$ given $\mathscr{F}$.

Second, see that the $\mathscr{F}$ is consistent with the advertising of the firms. Importantly, when $P_i = \Pi$ and $N_{-i} = -\beta$, consumers expect firm i to use negative advertising against strong opponents under locating apart.

Third, we discuss the optimality of $a^i(.)$, and its uniqueness given $\mathscr{F}$. Please refer to Table 1 for the posterior beliefs following each advertising outcome.

Locating Apart: If $N_i = 0$ and $P_{-i} = 0$, i.e., the opponent is weak, then $E[A_i] > E[A_{-i}]$ regardless of whether firm i does positive or negative advertising. Then positive advertising would always lead to more demand except for the scenario where only negative advertising allows stealing consumers, that is,

$$E[A_i - A_{-i}|a_i = N_i, a_{-i} = \varnothing] > \delta \text{ and } E[A_i - A_{-i}|a_i = P_i, a_{-i} = \varnothing] \leq \delta,$$

which is ruled out by the other parameter inequalities and Assumption 1. Hence, prioritizing positive advertising against 'weak opponents' is strictly dominant under locating apart (Lemma 1). If either $N_i = -\beta$ or $P_{-i} = \Pi$, i.e., the opponent is strong, then, for negative advertising to be prioritized in the advertising equilibrium, negative advertising should be effective enough to steal consumers if the opponent runs positive advertising:

$$\begin{aligned} &E[A_i - A_{-i}|a_i = N_i, a_{-i} = P_i] > \delta \\ &\qquad \Leftrightarrow \beta - (1 - \sigma_\Pi)\Pi > \delta, \end{aligned} \tag{A2}$$

which is equivalent to condition (A1a) above. Second, for negative advertising to be prioritized against strong opponents in the unique advertising equilibrium, the number of stolen consumers should be sufficient to make up for lost demand in own location:

$$\begin{aligned} D_i(a_i = N_i, a_{-i} = P_i) &\geq D_i(a_i = P_i, a_{-i} = P_i) \\ \Leftrightarrow 2 - \Gamma(-\sigma_\beta\beta) - \Gamma(\delta - \sigma_\beta\beta) &\geq 1 - \Gamma(-\sigma_\beta\beta - (1 - \sigma_\Pi)\Pi) \\ \Leftrightarrow \bar{\gamma} + \sigma_\beta\beta &\geq (1 - \sigma_\Pi)\Pi + \delta, \end{aligned} \tag{A3}$$

which is equivalent to condition (A1b) above. Since the game is symmetric, conditions (A1a) and (A1b) together imply that prioritizing negative advertising against strong opponents is strictly dominant (Lemma 2).

Co-Locating: If the opponent is weak, then $E[A_i] > E[A_{-i}]$ regardless of whether firm i does positive or negative advertising. Hence, prioritizing positive advertising against 'weak opponents' is strictly dominant under co-locating (Lemma 3).

If the opponent is strong, for positive advertising to be prioritized in the unique advertising equilibrium, it must be effective enough to steal consumers if the opponent runs negative advertising:

$$\begin{aligned} &E[A_i - A_{-i}|a_i = P_i, a_{-i} = N_i] > 0 \\ &\qquad \Leftrightarrow \Pi > (1 - \rho)(1 - \sigma_\beta)\beta, \end{aligned} \tag{A4}$$

which is equivalent to condition (A1b). Since the game is symmetric, this condition by itself implies that prioritizing positive advertising against strong opponents is strictly dominant under co-location (Lemma 4).

Lastly, to prove the optimality of x^2, consider the potential outcomes following each location choice, summarized in Table A6. Let $\xi_1 \equiv \sigma_\Pi(1 - \sigma_\Pi)(1 - \rho)$ and $\xi_2 \equiv (1 - \sigma_\Pi)(1 - \sigma_\Pi + \rho\sigma_\Pi)$ for convenience.

θ	Probability	a_1	a_2	D_2
$\{\Pi,\Pi,,\}$	$\sigma_\Pi^2+\rho\sigma_\Pi(1-\sigma_\Pi)$	P_1	P_2	$0.5\big[2-\Gamma(-(1-\sigma_\Pi)\Pi)-\Gamma(\delta-(1-\sigma_\Pi)\Pi)\big]$
$\{\Pi,0,-\beta,\}$	$\xi_1\sigma_\beta$	P_1	N_1	0
$\{0,\Pi,,-\beta\}$	$\xi_1\sigma_\beta$	N_2	P_2	$2-\Gamma((1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi)-\Gamma(\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi)$
$\{\Pi,0,0,\}$	$\xi_1(1-\sigma_\beta)$	P_1	$\varnothing$	0
$\{0,\Pi,,0\}$	$\xi_1(1-\sigma_\beta)$	$\varnothing$	P_2	$2-\Gamma(-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi)-\Gamma(\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi)$
$\{0,0,-\beta,-\beta\}$	$\xi_2\big(\sigma_\beta^2+\rho\sigma_\beta(1-\sigma_\beta)\big)$	N_2	N_1	$0.5\big[2-\Gamma(\sigma_\Pi\Pi+(1-\sigma_\beta)\beta)-\Gamma(\delta+\sigma_\Pi\Pi+(1-\sigma_\beta)\beta)\big]$
$\{0,0,0,-\beta\}$	$\xi_2\big(\sigma_\beta(1-\sigma_\beta)(1-\rho)\big)$	N_2	$\varnothing$	0
$\{0,0,-\beta,0\}$	$\xi_2\big(\sigma_\beta(1-\sigma_\beta)(1-\rho)\big)$	$\varnothing$	N_1	$2-\Gamma(\sigma_\Pi\Pi-\sigma_\beta\beta)-\Gamma(\delta+\sigma_\Pi\Pi-\sigma_\beta\beta)$
$\{0,0,0,0\}$	$\xi_2(1-\sigma_\beta)(1-\sigma_\beta+\rho\sigma_\beta)$	$\varnothing$	$\varnothing$	$0.5\big[2-\Gamma(\sigma_\Pi\Pi-\sigma_\beta\beta)-\Gamma(\delta+\sigma_\Pi\Pi-\sigma_\beta\beta)\big]$

Table A4: Co-Location

θ	Probability	a_1	a_2	D_2
$\{,,-\beta,-\beta\}$	σ_β^2	N_2	N_1	$1-\Gamma(1-\sigma_\beta)\beta)$
$\{,\Pi,0,-\beta\}$	$\sigma_\beta(1-\sigma_\beta)\sigma_\Pi$	N_2	P_2	0
$\{\Pi,,-\beta,0\}$	$\sigma_\beta(1-\sigma_\beta)\sigma_\Pi$	P_1	N_1	$2-\Gamma(-\sigma_\beta\beta)-\Gamma(\delta-\sigma_\beta\beta)$
$\{,0,0,-\beta\}$	$\sigma_\beta(1-\sigma_\beta)(1-\sigma_\Pi)$	N_2	$\varnothing$	0
$\{0,,-\beta,0\}$	$\sigma_\beta(1-\sigma_\beta)(1-\sigma_\Pi)$	$\varnothing$	N_1	$2-\Gamma(\sigma_\Pi\Pi-\sigma_\beta\beta)-\Gamma(\delta+\sigma_\Pi\Pi-\sigma_\beta\beta)$
$\{\Pi,\Pi,0,0\}$	$(1-\sigma_\beta)^2\sigma_\Pi^2$	P_1	P_2	$1-\Gamma(-(1-\sigma_\Pi)\Pi-\sigma_\beta\beta)$
$\{\Pi,0,0,0\}$	$(1-\sigma_\beta)^2\sigma_\Pi(1-\sigma_\Pi)$	P_1	$\varnothing$	0
$\{0,\Pi,0,0\}$	$(1-\sigma_\beta)^2\sigma_\Pi(1-\sigma_\Pi)$	$\varnothing$	P_2	$2-\Gamma(-(1-\sigma_\Pi)\Pi-\sigma_\beta\beta)-\Gamma(\delta-(1-\sigma_\Pi)\Pi-\sigma_\beta\beta)$
$\{0,0,0,0\}$	$(1-\sigma_\beta)^2(1-\sigma_\Pi)^2$	$\varnothing$	$\varnothing$	$1-\Gamma(\sigma_\Pi\Pi-\sigma_\beta\beta)$

Table A5: Locating Apart

Table A6: Demand for the Entrant with Negative Advertising Allowed Note. D_2 denotes the demand for the entrant. See Table 1 for $E[A_1]$ and $E[A_2]$ associated with each outcome. $\theta=\{P_1,P_2,N_1,N_2\}$. If the associated entry in θ is unspecified, that means a_1, a_2, and D_2 do not depend on the value of that entry.

With some algebra, we can simplify to

$$E[D_2|x_2=L]=\frac{1}{\bar{\gamma}-\underline{\gamma}}\bigg[\bar{\gamma}+\sigma_\beta(1-\sigma_\beta)(1-\sigma_\Pi)(1-\rho)(1-\rho+\rho\sigma_\Pi)\beta + (1-\sigma_\Pi)\sigma_\Pi(1-\rho)\Pi-\frac{\delta}{2}\bigg],\ \text{and} \tag{A5}$$

$$E[D_2|x_2=R]=\frac{1}{\bar{\gamma}-\underline{\gamma}}\bigg[\bar{\gamma}+\sigma_\beta(1-\sigma_\beta)\beta+(1-\sigma_\Pi)\sigma_\Pi(1-\sigma_\beta)(1-3\sigma_\beta)\Pi - (1-\sigma_\beta)(\sigma_\beta+(1-\sigma_\beta)\sigma_\Pi(1-\sigma_\Pi))\delta\bigg]. \tag{A6}$$

Hence, $E[D_2|x_2=L]\geq E[D_2|x_2=R]$ becomes equivalent to condition (A1d) above.

To sum up, once conditions (A1a)-(A1d) are satisfied, there exists a PBE as defined in $1-4$, which is unique given the beliefs specified by $\mathscr{F}$. □

Proof of Corollary 1. To prove the corollary, first, notice that the only advertising equilibria that can sustain co-location are where co-location leads to positive and locating apart leads to negative advertising being prioritized. Second, see that conditions (A1a) and (A1b), which are necessary for the entrant to co-locate given $\mathscr{F}$ as proven in Proposition 2, are only satisfied when δ is small enough. Let δ_1 and δ_2 be the values of δ which make the lefthand sides equal to the righthand sides of (A1a) and (A1b), respectively. Second, the condition in (A1d) is necessary for the entrant to co-locate given $\mathscr{F}$. The term multiplying δ on the lefthand side is 0.5 while it is $(1-\sigma_\beta)(\sigma_\beta+(1-\sigma_\beta)\sigma_\Pi(1-\sigma_\Pi))$ on the righthand side. Since the maximum of $x(1-x)$ is 0.25 for $x<1$, the latter term is bounded above by 0.5. Hence, as δ grows, the term with δ on the lefthand side dominates the other terms and (A1d) fails. Let δ_3 be the value of δ which makes the lefthand side equal to the righthand side. Then, for $\bar{\delta}=\min\{\delta_1,\delta_2,\delta_3\}$, the corollary follows. □

Proof of Corollary 2. First, see that (A1c) is only satisfied when ρ is large enough. Let $\underline{\rho}$ be the value of ρ where the lefthand side equals righthand side. Second, in (A1d), see that the terms with β and Π on the lefthand side disappear as ρ approaches 1. Because the coefficient of δ on the righthand side is always smaller in magnitude relative to the coefficient of δ on the lefthand side (see the Proof of Corollary 1), (A1d) fails as ρ approaches 1. Let $\bar{\rho}$ be the largest value of ρ where the lefthand side equals righthand side.[33] The corollary follows. □

Proof of Corollary 3. Let the value of σ_β where the lefthand side equals righthand side in (A1c) be $\underline{\sigma_\beta}$. Similarly, let the values of σ_Π where the lefthand side equals righthand side in (A1a) and (A1b) be $\sigma_{\Pi,1}$ and $\sigma_{\Pi,2}$ respectively, conditional on $\sigma_\beta=\underline{\sigma_\beta}$. Then, let $\underline{\sigma_\Pi}=\max\{\sigma_{\Pi,1},\sigma_{\Pi,2}\}$. The corollary follows. The reader should be careful in constructing the proof; among σ_Π, σ_β, Π, and β, there are only three free parameters because of the normalization $\sigma_\Pi\Pi=\sigma_\beta\beta$. □

Proof of Proposition 3. Here, we prove that for any set of parameters where the equilibrium in Proposition 2 (denoted by CO_{PN}) exists, there exists another equilibrium (denoted by LA_{NN}) where prioritize negative advertising against strong opponents following each location choice, and the entrant locates apart. To that end, first, we characterize the parameter set where LA_{NN} exists. Second, we show that the parameter set where the CO_{PN} exists is a strict subset of the parameter set where LA_{NN} exists.

The beliefs that are consistent with the advertising strategies in LA_{NN} are as described in Table A7.

Given the beliefs described in Table A7, (1) the necessary and sufficient conditions for negative advertising to be prioritized against strong opponents under locating apart, and (2) the necessary and sufficient conditions for positive advertising to be prioritized against weak opponents are identical to those for CO_{PN} (i.e., (A1a) and (A1b)). This is because consumer beliefs regarding prioritized advertising are identical under these scenarios. The necessary and sufficient conditions for negative advertising to be prioritized against strong opponents under co-location are:[34]

[33] See that neither $\underline{\rho}$ nor $\bar{\rho}$ are necessarily between 0 and 1.

[34] These conditions follow from the same reasoning given for (A1a) and (A1b) in the Proof of Proposition 2.

				Locating Apart		
		Prioritized Advertising				
a_i	a_{-i}	i	$-i$	$\tilde{\mathscr{F}}(P_i, N_i, P_{-i}, N_{-i})$	$E[A_i]$	$E[A_{-i}]$
P_i	P_{-i}	N_{-i}	N_i	$\{1,0,1,0\}$	$(1-\sigma_\Pi)\Pi+\sigma_\beta\beta$	$(1-\sigma_\Pi)\Pi+\sigma_\beta\beta$
P_i	N_i	N_{-i}	N_i	$\{1,1,\sigma_\Pi,0\}$	$(1-\sigma_\Pi)\Pi-(1-\sigma_\beta)\beta$	$\sigma_\beta\beta$
N_{-i}	N_i	N_{-i}	N_i	$\{\sigma_\Pi,1,\sigma_\Pi,1\}$	$-(1-\sigma_\beta)\beta$	$-(1-\sigma_\beta)\beta$
P_i	$\varnothing$	P_i	N_i	$\{1,0,0,\sigma_\beta\}$	$(1-\sigma_\Pi)\Pi+\sigma_\beta\beta$	$-\sigma_\Pi\Pi$
N_{-i}	$\varnothing$	P_i	N_i	$\{0,0,0,1\}$	$\sigma_\beta\beta-\sigma_\Pi\Pi$	$-\sigma_\Pi\Pi-(1-\sigma_\beta)\beta$
$\varnothing$	$\varnothing$	P_i	P_{-i}	$\{0,0,0,0\}$	$-\sigma_\Pi\Pi+\sigma_\beta\beta$	$-\sigma_\Pi\Pi+\sigma_\beta\beta$

				Co-location		
		Prioritized Advertising				
a_i	a_{-i}	i	$-i$	$\tilde{\mathscr{F}}(P_i, N_i, P_{-i}, N_{-i})$	$E[A_i]$	$E[A_{-i}]$
P_i	P_{-i}	N_{-i}	N_i	$\{1,0,1,0\}$	$(1-\sigma_\Pi)\Pi+\sigma_\beta\beta$	$(1-\sigma_\Pi)\Pi+\sigma_\beta\beta$
P_i	N_i	N_{-i}	N_i	$\{1,1,\sigma_\Pi,0\}$	$(1-\sigma_\Pi)\Pi-(1-\sigma_\beta)\beta$	$\sigma_\beta\beta+\rho(1-\sigma_\Pi)\Pi$
N_{-i}	N_i	N_{-i}	N_i	$\{\sigma_\Pi,1,\sigma_\Pi,1\}$	$-(1-\sigma_\beta)\beta$	$-(1-\sigma_\beta)\beta$
P_i	$\varnothing$	P_i	N_i	$\{1,0,0,\sigma_\beta\}$	$(1-\sigma_\Pi)\Pi+\sigma_\beta\beta$	$-\sigma_\Pi\Pi+\sigma_\beta\beta$
N_{-i}	$\varnothing$	P_i	N_i	$\{0,0,0,1\}$	$\sigma_\beta\beta-\sigma_\Pi\Pi$	$-\sigma_\Pi\Pi-(1-\sigma_\beta)\beta$
$\varnothing$	$\varnothing$	P_i	P_{-i}	$\{0,0,0,0\}$	$-\sigma_\Pi\Pi+\sigma_\beta\beta$	$-\sigma_\Pi\Pi+\sigma_\beta\beta$

Table A7: Posterior Beliefs Given Advertising Outcomes Under LA_{NN}. Note: Terms in purple indicate the "direct effect" of advertising, in black refer to the "inference effect," and in blue refer to the "spillover effect."

$$\beta > (1-\rho)(1-\sigma_\Pi)\Pi \tag{A7}$$

$$\bar{\gamma} + \sigma_\beta\beta > \frac{\delta}{2} - (2\rho-1)(1-\sigma_\Pi)\Pi \tag{A8}$$

Notice any set of parameters that satisfy (A7) will also satisfy (A1a) and any set of parameters that satisfy (A8) will also satisfy (A1b). In other words, when consumers believe negative advertising is prioritized, whenever firms find prioritizing negative advertising to be profitable under locating apart, they will also find it profitable under co-location. Furthermore, the entrant will locate apart whenever the advertising outcomes are identical following each location choice. So, for LA_{NN} to exist, (A1a) and (A1b) are sufficient conditions. For CO_{PN} to exist, however, (A1c) and (A1d) also need to be satisfied. Because (A1c) and (A1d) are distinct from (A1a) and (A1b), the set of parameters where CO_{PN} exists is a strict subset of the set of parameters where LA_{NN} exists. Therefore, there are no set of parameters where CO_{PN} is the unique equilibrium.

□

Proof of Lemma 6. We start by deriving a simple expression for welfare comparisons. Since the outside option provides 0 utility, the expected total consumer surplus can be written as

$$CS = \sum_{\theta} P(\theta) \Big[\int_{\gamma_L^*(\theta)}^{\bar{\gamma}} (\gamma - \gamma_L^{true}(\theta)) d\Gamma(\gamma) + \int_{\gamma_R^*(\theta)}^{\bar{\gamma}} (\gamma - \gamma_R^{true}(\theta)) d\Gamma(\gamma) \Big], \tag{A9}$$

where θ refers to the vector of values for product attributes, and $\gamma^*(\theta)$ and $\gamma^{true}(\theta)$ denote the reservation values for the consumers who are indifferent between buying the superior product or the outside option ex-ante and ex-post, respectively. The two values can differ because advertising does not always reveal all attributes of the product. If $\gamma^* > \gamma^{true}$, there are some consumers who don't buy a product, but would have enjoyed a positive utility and when $\gamma^* < \gamma^{true}$, there are some consumers who buy a product, but would have been better off with the outside option. When Γ is the uniform *cdf*, the expression becomes

$$\begin{aligned} CS &= \sum_{\theta} P(\theta) \Big[\int_{\gamma_L^*(\theta)}^{\bar{\gamma}} \frac{\gamma - \gamma_L^{true}(\theta)}{\bar{\gamma} - \underline{\gamma}} d\gamma \int_{\gamma_R^*(\theta)}^{\bar{\gamma}} \frac{\gamma - \gamma_R^{true}(\theta)}{\bar{\gamma} - \underline{\gamma}} d\gamma \Big] \\ &= \sum_{\theta} P(\theta) \Big[\frac{(\bar{\gamma} - \gamma_L^*(\theta))(\bar{\gamma} + \gamma_L^*(\theta) - 2\gamma_L^{true}(\theta))}{2(\bar{\gamma} - \underline{\gamma})} + \frac{(\bar{\gamma} - \gamma_R^*(\theta))(\bar{\gamma} + \gamma_R^*(\theta) - 2\gamma_R^{true}(\theta))}{2(\bar{\gamma} - \underline{\gamma})} \Big] \\ &= \sum_{\theta} P(\theta) \Big[\Phi + \gamma_L^*(\theta)(2\gamma_L^{true}(\theta) - \gamma_L^*(\theta)) - 2\bar{\gamma}\gamma_L^{true}(\theta) \\ &\qquad + \gamma_R^*(\theta)(2\gamma_R^{true}(\theta) - \gamma_R^*(\theta)) - 2\bar{\gamma}\gamma_R^{true}(\theta) \Big], \end{aligned} \tag{A10}$$

where Φ is only a function of $\bar{\gamma}$ and $\underline{\gamma}$, hence, it is invariant to advertising policy. Using the equilibrium strategies of firms and consumers in Propositions 1 and 2, we can characterize the values of γ^* and γ^{true} for each realization of product attributes (See Table A10).

θ	a_1	a_2	W	γ_L^*	γ_R^*	γ_L^{true}	γ_R^{true}
$\{\Pi,\Pi,\beta,\beta\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,\Pi,\beta,0\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,\Pi,0,\beta\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,\Pi,0,0\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,\beta,\beta\}$	P	0	1	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,\beta,0\}$	P	0	1	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,0,\beta\}$	P	0	1	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,0,0\}$	P	0	1	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,\beta,\beta\}$	0	P	2	$\delta-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,\beta,0\}$	0	P	2	$\delta-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,0,\beta\}$	0	P	2	$\delta-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,0,0\}$	0	P	2	$\delta-(1-\sigma_\Pi)\Pi$	$-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{0,0,\beta,\beta\}$	0	0	-	$\sigma_\Pi\Pi$	$\sigma_\Pi\Pi$	$(1-\sigma_\beta)\beta+\sigma_\Pi\Pi$	$(1-\sigma_\beta)\beta+\sigma_\Pi\Pi$
$\{0,0,\beta,0\}$	0	0	-	$\sigma_\Pi\Pi$	$\sigma_\Pi\Pi$	$(1-\sigma_\beta)\beta+\sigma_\Pi\Pi$	$\sigma_\Pi\Pi-\sigma_\beta\beta$
$\{0,0,0,\beta\}$	0	0	-	$\sigma_\Pi\Pi$	$\sigma_\Pi\Pi$	$\sigma_\Pi\Pi-\sigma_\beta\beta$	$(1-\sigma_\beta)\beta+\sigma_\Pi\Pi$
$\{0,0,0,0\}$	0	0	-	$\sigma_\Pi\Pi$	$\sigma_\Pi\Pi$	$\sigma_\Pi\ \Pi-\sigma_\beta\beta$	$\sigma_\Pi\Pi-\sigma_\beta\beta$

Table A8: Negative Advertising Banned, Firms are Located Apart

θ	a_1	a_2	W	γ_L^*	γ_R^*	γ_L^{true}	γ_R^{true}
$\{\Pi,\Pi,\beta,\beta\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,\Pi,\beta,0\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$(0.5-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(0.5-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,\Pi,0,\beta\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$(0.5-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(0.5-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,\Pi,0,0\}$	P	P	-	$-(1-\sigma_\Pi)\Pi$	$\delta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,\beta,\beta\}$	P	N	1	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,\beta,0\}$	P	N	1	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,0,\beta\}$	P	0	1	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{\Pi,0,0,0\}$	P	0	1	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,\beta,\beta\}$	N	P	2	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,\beta,0\}$	0	P	2	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,0,\beta\}$	N	P	2	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$	$\delta+(1-\sigma_\beta)\beta-(1-\sigma_\Pi)\Pi$
$\{0,\Pi,0,0\}$	0	P	2	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$	$\delta-\sigma_\beta\beta-(1-\sigma_\Pi)\Pi$
$\{0,0,\beta,\beta\}$	N	N	-	$\sigma_\Pi\Pi+(1-\sigma_\beta)\beta$	$\delta+\sigma_\Pi\Pi+(1-\sigma_\beta)\beta$	$\sigma_\Pi\Pi+(1-\sigma_\beta)\beta$	$\delta+\sigma_\Pi\Pi+(1-\sigma_\beta)\beta$
$\{0,0,\beta,0\}$	0	N	2	$\sigma_\Pi\Pi-\sigma_\beta\beta$	$\delta+\sigma_\Pi\Pi-\sigma_\beta\beta$	$\sigma_\Pi\Pi-\sigma_\beta\beta$	$\delta+\sigma_\Pi\Pi-\sigma_\beta\beta$
$\{0,0,0,\beta\}$	N	0	1	$\sigma_\Pi\Pi-\sigma_\beta\beta$	$\delta+\sigma_\Pi\Pi-\sigma_\beta\beta$	$\sigma_\Pi\Pi-\sigma_\beta\beta$	$\delta+\sigma_\Pi\Pi-\sigma_\beta\beta$
$\{0,0,0,0\}$	0	0	-	$\sigma_\Pi\Pi-\sigma_\beta\beta$	$\delta+\sigma_\Pi\Pi-\sigma_\beta\beta$	$\sigma_\Pi\Pi-\sigma_\beta\beta$	$\delta+\sigma_\Pi\Pi-\sigma_\beta\beta$

Table A9: Negative Advertising Allowed, Firms are Co-located

Table A10: The γ values for which Ex-ante and Ex-post Utilities are 0, See Table 1 for $E[A_1]$ and $E[A_2]$ associated with each outcome. $\theta=\{P_1,P_2,N_1,N_2\}$. a_1 and a_2 are the advertising outcomes that arise in the equilibria described in Propositions 1 and 2. W refers to which firm's product (if any) serves the whole market.

Given the values for γ^* and γ^{true}, we can characterize the consumer surplus associated with each realization of product attributes. In Table A11, we tabulate the associated Consumer Surplus separately for consumers located in L and R. We omit the Φ term in (A10), since it does not vary across different scenarios. Let $\xi_3 = (1 - \sigma_\Pi)\Pi$ for convenience.

	Negative Advertising Banned	Negative Advertising Allowed	
θ	Consumer Surplus (L)	Consumer Surplus (L)	Consumer Surplus (R)
$\{\Pi,\Pi,\beta,\beta\}$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma})-2\bar{\gamma}(1-\sigma_\beta)\beta$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma})-2\bar{\gamma}(1-\sigma_\beta)\beta$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{\Pi,\Pi,\beta,0\}$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma})-2\bar{\gamma}(1-\sigma_\beta)\beta$	$\xi_3(\xi_3-2(0.5-\sigma_\beta)\beta+2\bar{\gamma})-2\bar{\gamma}(0.5-\sigma_\beta)\beta$	$*-\delta(2\xi_3-2(0.5-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{\Pi,\Pi,0,\beta\}$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma})+2\bar{\gamma}\sigma_\beta\beta$	$\xi_3(\xi_3-2(0.5-\sigma_\beta)\beta+2\bar{\gamma})-2\bar{\gamma}(0.5-\sigma_\beta)\beta$	$*-\delta(2\xi_3-2(0.5-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{\Pi,\Pi,0,0\}$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma})+2\bar{\gamma}\sigma_\beta\beta$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma})+2\bar{\gamma}\sigma_\beta\beta$	$*-\delta(2\xi_3+2\sigma_\beta\beta+2\bar{\gamma}-\delta)$
$\{\Pi,0,\beta,\beta\}$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma})-2\bar{\gamma}(1-\sigma_\beta)\beta$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}))+(1-\sigma_\beta)\beta((1-\sigma_\beta)\beta-2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{\Pi,0,\beta,0\}$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma})-2\bar{\gamma}(1-\sigma_\beta)\beta$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}))+(1-\sigma_\beta)\beta((1-\sigma_\beta)\beta-2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{\Pi,0,0,\beta\}$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma})+2\bar{\gamma}\sigma_\beta\beta$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma}))+\sigma_\beta\beta(-\sigma_\beta\beta+2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{\Pi,0,0,0\}$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma})+2\bar{\gamma}\sigma_\beta\beta$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma}))+\sigma_\beta\beta(-\sigma_\beta\beta+2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{0,\Pi,\beta,\beta\}$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-2\delta)+2(1-\sigma_\beta)\beta(\delta-\bar{\sigma})-2\bar{\gamma}\delta+\delta^2$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}))+(1-\sigma_\beta)\beta((1-\sigma_\beta)\beta-2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{0,\Pi,\beta,0\}$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma}-2\delta)+2\sigma_\beta\beta(\bar{\sigma}-\delta)-2\bar{\gamma}\delta+\delta^2$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma}))+\sigma_\beta\beta(-\sigma_\beta\beta+2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{0,\Pi,0,\beta\}$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-2\delta)+2(1-\sigma_\beta)\beta(\delta-\bar{\sigma})-2\bar{\gamma}\delta+\delta^2$	$\xi_3(\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}))+(1-\sigma_\beta)\beta((1-\sigma_\beta)\beta-2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{0,\Pi,0,0\}$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma}-2\delta)+2\sigma_\beta\beta(\bar{\sigma}-\delta)-2\bar{\gamma}\delta+\delta^2$	$\xi_3(\xi_3+2\sigma_\beta\beta+2\bar{\gamma}))+\sigma_\beta\beta(-\sigma_\beta\beta+2\bar{\gamma})$	$*-\delta(2\xi_3-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{0,0,\beta,\beta\}$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi+2(1-\sigma_\beta)\beta-2\bar{\gamma})-2\bar{\gamma}(1-\sigma_\beta)\beta$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi+2(1-\sigma_\beta)\beta-2\bar{\gamma})+(1-\sigma_\beta)\beta((1-\sigma_\beta)\beta-2\bar{\gamma})$	$*\text{-}\delta(-2\sigma_\Pi\Pi-2(1-\sigma_\beta)\beta+2\bar{\gamma}-\delta)$
$\{0,0,\beta,0\}$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi+2(1-\sigma_\beta)\beta-2\bar{\gamma})-2\bar{\gamma}(1-\sigma_\beta)\beta$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi-2\sigma_\beta\beta-2\bar{\gamma})+\sigma_\beta\beta(\sigma_\beta\beta+2\bar{\gamma})$	$*\text{-}\delta(-2\sigma_\Pi\Pi+2\sigma_\beta\beta+2\bar{\gamma}-\delta)$
$\{0,0,0,\beta\}$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi-2\sigma_\beta\beta-2\bar{\gamma})+2\bar{\gamma}\sigma_\beta\beta$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi-2\sigma_\beta\beta-2\bar{\gamma})+\sigma_\beta\beta(\sigma_\beta\beta+2\bar{\gamma})$	$*\text{-}\delta(-2\sigma_\Pi\Pi+2\sigma_\beta\beta+2\bar{\gamma}-\delta)$
$\{0,0,0,0\}$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi-2\sigma_\beta\beta-2\bar{\gamma})+2\bar{\gamma}\sigma_\beta\beta$	$\sigma_\Pi\Pi(\sigma_\Pi\Pi-2\sigma_\beta\beta-2\bar{\gamma})+\sigma_\beta\beta(\sigma_\beta\beta+2\bar{\gamma})$	$*\text{-}\delta(-2\sigma_\Pi\Pi+2\sigma_\beta\beta+2\bar{\gamma}-\delta)$

Table A11: The consumer surplus following the equilibrium described in Proposition 2 **Notes:** (L) and (R) refer to the consumer surplus associated with consumers in locations L and R, respectively. The values are suppressed for consumers in R for when negative advertising is not permitted, as they are symmetrical to those for consumers in L. $*$ refers to the Consumer Surplus value for the consumers in location L in the same row, so the expression following $*$ gives the additional surplus consumers located at R enjoy relative to consumers located at L when negative advertising is allowed.

When both products have the positive attribute, both firms utilize positive advertising in equilibrium regardless of whether negative advertising is allowed and regardless of whether the products have the negative attribute or not. Hence, consumers can conclude the positive attribute is present in both products with probability 1, yet gain no additional information about the presence of the negative attribute. This is reflected in the first four rows of Table A11 which show that when $P_1 = P_2 = \Pi$, the consumer welfare becomes identical between the two scenarios as $\delta \longrightarrow 0$.

If one or more products lacks the positive attribute, the firm(s) that lacks the positive attribute (say firm i) will not be able to run positive advertising. Because positive advertising is prioritized, consumers can conclude that the positive attribute is missing in firm i's product with probability 1, regardless of whether negative advertising is allowed. If negative advertising is forbidden, consumers do not learn anything else, because firm i has to run no advertising regardless of the presence of negative attributes. If negative advertising is allowed, however, consumers also learn whether firm $-i$ has the negative attribute. If firm i runs negative advertising, consumers can conclude the negative attribute is present in firm $-i$'s product with probability 1. If firm i runs no advertising, consumers can conclude that the negative attribute is present in firm $-i$'s product with probability 0. This is reflected in the additional β terms in rows 5-16 when negative advertising is permitted in Table A11.

To tease out the change in surplus through changes in the available information, we take two steps. First, we compute the change in expected surplus when negative advertising is permitted as if the probability distribution of θ is the same for both and equal to the one under locating apart (i.e. $\rho = 0$). Second, we take $\delta \longrightarrow 0$ to suppress the change in consumer surplus due to reduced product diversity. The resulting term can be simplified as

$$4\sigma_\beta(1-\sigma_\beta)(1-\sigma_\Pi)\beta\big(2\sigma_\Pi\beta + (1-\sigma_\Pi)\bar{\gamma}\big), \tag{A11}$$

which is always positive. Hence, we conclude that the consumers have welfare gains due to additional information when negative advertising is permitted.

Lastly, the variance of β equals $\sigma_\beta(1-\sigma_\beta)$ as it has a Bernoulli distribution. See that both this variance and (A11) approach 0 as σ_β approaches 0 or 1. Hence, the welfare gains due to additional information disappear as the variance of β goes to 0. □

Proof of Lemma 7. The welfare loss due to differentiation is reflected in the additional δ terms when negative advertising is permitted in Table A11. When one firm serves the whole market, there are no additional δ terms, because consumers in one of the locations end up purchasing from a firm in the other location, regardless of whether negative advertising is permitted or not (rows 5-12). In the other outcomes (rows 1-4 and 13-16), however, there are additional terms with δ when negative advertising is permitted. The presence of such terms indicate that some consumers who could buy a product that exactly matches their preference when negative advertising is not allowed can only buy from a firm that doesn't match their preference when negative advertising is permitted. This is because when negative advertising is permitted, the firms co-locate at L, hence, consumers located at R will necessarily buy from the other location.

To tease out the change in surplus through changes in the product differentiation, we take two steps. First, we compute the change in expected surplus when negative advertising is permitted as if the probability distribution of θ is the same for both and equal to the one under locating apart (i.e. $\rho = 0$). Second, we isolate the terms with δ.[35] The resulting term can be simplified as:

$$\delta\Big[2\sigma_\Pi(1-\sigma_\Pi)(1-2\sigma_\Pi)\Pi - 2\sigma_\beta(1-\sigma_\Pi)^2(1-\sigma_\beta)\beta - (\sigma_\Pi^2 + (1-\sigma_\Pi)^2)(2\bar{\gamma}-\delta)\Big]. \tag{A12}$$

Next, we prove the term in (A12) is always negative, i.e., there is some welfare *loss* associated with the change in product differentiation. Using the normalization made earlier, plugging in $\sigma_\Pi\Pi = \sigma_\beta\beta$ yields:

$$\delta\Big[2\sigma_\Pi(1-\sigma_\Pi)(\sigma_\beta - \sigma_\Pi - \sigma_\beta\sigma_\Pi)\Pi - (\sigma_\Pi^2 + (1-\sigma_\Pi)^2)(2\bar{\gamma}-\delta)\Big]. \tag{A13}$$

Replacing $\bar{\gamma}$ in (A13) with $\delta + \sigma_\Pi\Pi + \sigma_\beta\beta$ yields:

$$\delta\Big[2\sigma_\Pi(1-\sigma_\Pi)(\sigma_\beta - \sigma_\Pi - \sigma_\beta\sigma_\Pi)\Pi - (\sigma_\Pi^2 + (1-\sigma_\Pi)^2)(2\sigma_\Pi\Pi + 2\sigma_\beta\beta + \delta)\Big]. \tag{A14}$$

From Assumption 1, $\bar{\gamma} > \delta + \sigma_\Pi\Pi + \sigma_\beta\beta$. Hence, proving (A14) is negative is sufficient to prove (A13) is negative. Next, notice that the term on the left, which is the only positive term in the expression, is maximized when $\sigma_\beta = 1$. Setting $\sigma_\beta = 1$ and collecting the terms with Π gives

$$\delta\Big[-2\sigma_\Pi^2\Pi - (\sigma_\Pi^2 + (1-\sigma_\Pi)^2)(2\sigma_\beta\beta + \delta)\Big], \tag{A15}$$

which is necessarily negative. Then (A12) has to be negative. Hence, consumers have welfare losses due to reduced product differentiation when negative advertising is permitted.

Lastly, to prove that welfare losses increase as δ increases, we take the derivative of (A12) with respect to δ, which yields

$$(2\delta - 2\bar{\gamma})(\sigma_\Pi^2 + (1-\sigma_\Pi)^2) + 2\sigma_\Pi(1-\sigma_\Pi)(\sigma_\beta(1-\sigma_\Pi) - \sigma_\Pi)\Pi. \tag{A16}$$

Notice that $\sigma_\Pi^2 + (1-\sigma_\Pi)^2 > (1-\sigma_\Pi)(\sigma_\beta(1-\sigma_\Pi) - \sigma_\Pi)$ for any value of σ_Π and σ_β. Since $\bar{\gamma} > \delta + \sigma_\Pi\Pi + \sigma_\beta\beta$ by Assumption 1, the derivative term is necessarily negative. Therefore, the welfare loss due to reduced product differentiation increases with higher δ. □

[35] In other words, we remove the terms that are associated with additional information, given in the Proof of Lemma 6.

Proof of Proposition 4. Using Table A11, we can write down the change in expected consumer surplus when negative advertising is permitted:

$$\begin{aligned}\Delta CS = & \underbrace{4\sigma_\beta(1-\sigma_\beta)(1-\sigma_\Pi)\beta\big(2\sigma_\Pi\beta+(1-\sigma_\Pi)\bar{\gamma}\big)}_{\text{Information Gains}} \\ & + \underbrace{\delta\Big[2\sigma_\Pi(1-\sigma_\Pi)(1-2\sigma_\Pi)\Pi - 2\sigma_\beta(1-\sigma_\Pi)^2(1-\sigma_\beta)\beta - (\sigma_\Pi^2+(1-\sigma_\Pi)^2)(2\bar{\gamma}-\delta)\Big]}_{\text{Product Differentiation Losses}} \\ & + 2\rho(1-\sigma_\Pi)\Big[\sigma_\Pi(2\sigma_\Pi-1)\Pi(\Pi+\delta) - 2\sigma_\Pi\Pi\bar{\gamma} - \sigma_\Pi\sigma_\beta(1-\sigma_\beta)\beta^2 \\ & \qquad + \sigma_\beta(1-\sigma_\beta)(2\sigma_\Pi-\rho\sigma_\Pi-1)\beta(2\bar{\gamma}-\beta-\delta)\Big].\end{aligned} \tag{A17}$$

The first and second terms denote the welfare change due to information and reduced product differentiation that were derived in Lemmas 6 and 7, respectively. These terms were derived keeping the distribution of θ fixed when negative advertising was permitted. The third term is the 'correction' term, which reflects the change in welfare due to the change in the distribution of θ and disappears as $\rho \longrightarrow 0$.

Collecting the terms, and using the normalization $\sigma_\Pi\Pi = \sigma_\beta\beta$, we can re-write (A17) as the summation of two terms:

$$\begin{aligned}\Delta CS = & 2\sigma_\Pi(1-\sigma_\Pi)\Pi\Big[4\sigma_\Pi(1-\sigma_\beta)\beta + 2(1-\sigma_\Pi)(1-\sigma_\beta)\bar{\gamma} + (\sigma_\beta-\sigma_\Pi-\rho\sigma_\Pi)\delta \\ & \quad + \rho(2\sigma_\Pi-1)\Pi + \rho(2\sigma_\Pi-1)\delta - 2\rho\bar{\gamma} - \rho\sigma_\Pi(1-\sigma_\beta)\beta \\ & \quad + \rho(1-\sigma_\beta)(2\sigma_\Pi-1-\rho\sigma_\Pi)(2\bar{\gamma}-\beta-\delta)\Big] \\ & \quad - \delta(2\bar{\gamma}-\delta)\Big[\sigma_\Pi^2+(1-\sigma_\Pi)^2\Big] \\ = & 2\sigma_\Pi(1-\sigma_\Pi)\Pi\Big[(1-\sigma_\beta)\big(\sigma_\Pi(4-3\rho+\rho^2)+\rho\big)\beta \\ & \quad + 2\big((1-\sigma_\beta)(1-\sigma_\Pi+\rho(2-\rho)\sigma_\Pi-\rho)-\rho\big)\bar{\gamma} \\ & \quad + \big(\sigma_\beta(1-\sigma_\Pi)-\sigma_\Pi(1-\rho^2)+\sigma_\beta\sigma_\Pi(2\rho-rho^2-1)\big)\delta \\ & \quad + \rho(2\sigma_\Pi-1)\Pi\Big] \\ & \quad - \delta(2\bar{\gamma}-\delta)\Big[\sigma_\Pi^2+(1-\sigma_\Pi)^2\Big].\end{aligned} \tag{A18}$$

Notice that ΔCS is continuous in all parameters. The second term is necessarily negative due to Assumption 1. The first term in (A18), whose sign cannot be established with certainty

- disappears if σ_Π approaches 0 or 1, or σ_β approaches 0, and
- becomes negative as σ_β approaches 1.

It is straightforward how the first term in (A18) disappears as σ_Π approaches 0 or 1. Notice that, σ_β approaching 0 is equivalent to σ_Π approaching 0, given the normalization $\sigma_\beta\beta = \sigma_\Pi\Pi$.[36] Lastly, to show that the term becomes negative as σ_β approaches 1, we plug $\sigma_\beta = 1$ in (A18), which yields

$$\Delta CS = 2\sigma_\Pi(1-\sigma_\Pi)\Pi\Big[0\beta - 2\rho\bar{\gamma} + (1-3\sigma_\Pi + 2\rho\sigma_\Pi)\delta + \rho(2\sigma_\Pi - 1)\Pi\Big] \\ -\delta(2\bar{\gamma}-\delta)\Big[\sigma_\Pi^2 + (1-\sigma_\Pi)^2\Big]. \tag{A19}$$

The term in the brackets is strictly negative because (1) $1 - 3\sigma_\Pi + 2\rho\sigma_\Pi$ is bounded above by 1, (2) $\rho(2\sigma_\Pi - 1)$ is bounded above by σ_Π and (3) $\bar{\gamma} > \delta + \sigma_\Pi\Pi + \sigma_\beta\beta$ by Assumption 1. Moreover, ΔCS will be negative for any $\sigma_\beta > \underline{\sigma_\beta}$ for some $\underline{\sigma_\beta} < 1$ because ΔCS is continuous in $\sigma_\beta\beta$.

Lastly, we prove that the change in welfare decreases with δ. Notice that the first term in (A17) does not change with δ while the second term decreases as shown in Lemma 7. Taking the derivative of (A17) with respect to δ and using $\sigma_\Pi\Pi = \sigma_\beta\beta$ yields

$$2(\delta-\bar{\gamma})(\sigma_\Pi^2 + (1-\sigma_\Pi)^2) + 2\sigma_\Pi(1-\sigma_\Pi)\Pi\Big[(\sigma_\beta(1-\sigma_\Pi) - \sigma_\Pi) + \rho\Big((2\sigma_\Pi - 1) - (1-\sigma_\beta)(2\sigma_\Pi - \rho\sigma_\Pi - 1)\Big)\Big]. \tag{A20}$$

By Assumption 1, $\bar{\gamma} > \delta + \sigma_\Pi\Pi + \sigma_\beta\beta$, so a sufficient condition for the term to be negative is:

$$(1-\sigma_\Pi)\Big[\sigma_\beta(1-\sigma_\Pi) - \sigma_\Pi + \rho\Big((2\sigma_\Pi - 1) - (1-\sigma_\beta)(2\sigma_\Pi - \rho\sigma_\Pi - 1)\Big)\Big] \\ < (\sigma_\Pi^2 + (1-\sigma_\Pi)^2). \tag{A21}$$

See that the term in brackets is bounded above by $1 - \sigma_\Pi$:

$$\begin{aligned}
&\sigma_\beta(1-\sigma_\Pi) - \sigma_\Pi + \rho\Big((2\sigma_\Pi - 1) - (1-\sigma_\beta)(2\sigma_\Pi - \rho\sigma_\Pi - 1)\Big) \\
&= \sigma_\beta(1-\sigma_\Pi) - \sigma_\Pi + \rho\Big((1-\sigma_\beta)\rho\sigma_\Pi + \sigma_\beta\sigma_\Pi - \sigma_\beta(1-\sigma_\Pi))\Big) \\
&= \sigma_\beta(1-\rho)(1-\sigma_\Pi) - \sigma_\Pi + \rho\Big((1-\sigma_\beta)\rho\sigma_\Pi + \sigma_\beta\sigma_\Pi)\Big) \\
&< \sigma_\beta(1-\rho)(1-\sigma_\Pi) - \sigma_\Pi + \sigma_\Pi \\
&< 1 - \sigma_\Pi.
\end{aligned}$$

Thus, the condition is satisfied for $\sigma_\Pi > 0$ and the total change in consumer surplus given in (A17) decreases with δ. □

Proof of Lemma 8. We postulate equilibrium strategies for the pricing search sub-games after

[36] A similar argument cannot be made for Π and β because δ is bounded above by the two in the equilibrium parameter set, hence, the second term disappears together with the first term.

each location and advertising choices, and prove that the postulated strategies indeed constitute a Nash equilibrium.

If $x_2 = R$ and $A_1 = A_2$: We postulate that, in this sub-game, consumers do not search and both firms charge $p^i = \frac{E[A_i]+\bar{\gamma}}{2}$. First, since both firms charge the same price, and the consumers receive the first quote from the firm that matches their taste, consumers have no incentive to search for a second quote. In other words, there is no profitable deviation for the consumers. Second, because the consumers do not search for a second quote, neither firm can steal consumers from the other through reduced prices. Then, the pricing problem of firm i boils down to:

$$\max_{p_i>0} p_i \left(1 + \frac{A_i - p_i + \underline{\gamma}}{\bar{\gamma} - \underline{\gamma}}\right) \tag{A22}$$

Hence, changing prices does not increase profits, since $p^i = \frac{E[A_i]+\bar{\gamma}}{2}$ is already the price that equates marginal revenue to marginal cost for firm i. Thus, the postulated strategies constitute a Nash Equilibrium of the pricing-search sub-game.

If $x_2 = L$ and $A_1 = A_2$: We postulate that, in this sub-game, consumers do not search and both firms charge $p^i = \frac{E[A_i]+\bar{\gamma}}{2} - \frac{\delta}{4}$ for δ small enough and $p^i = \frac{E[A_i]+\bar{\gamma}}{2}$ otherwise. First, since both firms charge the same price, and $A_1 = A_2$, consumers have no incentive to search for a second quote. Second, because the consumers do not search for a second quote, neither firm can steal consumers from the other through reduced prices. Then, the pricing problem of a monopolist boils down to:

$$\max_{p_i>0} \frac{1}{2} p_i \left(\max\{1 + \frac{A_i - p_i + \underline{\gamma}}{\bar{\gamma} - \underline{\gamma}}, 0\} + \max\{\frac{A_i - p_i + \underline{\gamma} - \delta}{\bar{\gamma} - \underline{\gamma}}, 0\}\right) \tag{A23}$$

The pricing problem looks different from (A22) because now, if the price is sufficiently small, firm i can serve consumers whose tastes do not exactly match firm i's product, i.e. consumers located in R.[37]

The problem may be non-convex around the solution, due to the presence of two separate markets: it may be optimal for the firm to set a price where the demand from location R equals 0. This becomes more likely as δ grows. First, assume that the firm serves both markets in the optimal solution. Then, the price that solves

$$\max_{p_i>0} \frac{1}{2} p_i \left(1 + \frac{A_i - p_i + \underline{\gamma}}{\bar{\gamma} - \underline{\gamma}} + \frac{A_i - p_i + \underline{\gamma} - \delta}{\bar{\gamma} - \underline{\gamma}}\right), \tag{A24}$$

or, $p_i^* = \frac{E[A_i]+\bar{\gamma}}{2} - \frac{\delta}{4}$, also solves (A23). If, on the other hand, $A_i - p_i^* + \underline{\gamma} - \delta < 0$, then the firm will charge $p_i^{**} = \frac{E[A_i]+\bar{\gamma}}{2}$ and only serve consumers located at L. Hence, one of these prices equates

[37] The $\frac{1}{2}$ term in the beginning signifies the fact that only half of the consumers in either location receive their quote from firm i.

marginal revenue to marginal cost for firm i. Thus, the postulated strategies constitute a Nash Equilibrium of the pricing-search sub-game.

If $A_1 \neq A_2$: We postulate that, in this sub-game, consumers do not search, and both firms charge $p^i = \frac{E[A_i]+\bar{\gamma}}{2} - \frac{\delta}{4}$ for δ small enough and $p^i = \frac{E[A_i]+\bar{\gamma}}{2}$ otherwise. The strategies here are identical to the previous sub-game, and proving that they constitute a Nash Equilibrium follows the same steps. The only difference in this case is that consumer j buys from the firm that has a larger $E[A_i] - |x_i - \chi_j|$.[38] □

Proof of Proposition 5. We restrict attention to the case where monopolists serve both markets, i.e., where δ is sufficiently small. This is also the interesting case where co-location leads to reduced prices for consumers.

The PBE is defined by:

1. $x^2 = L$

2. $$a^i(x_i = R, \theta) = \begin{cases} P_i, \text{ if } N_i = 0, P_{-i} = 0, P_i = \Pi \\ N_{-i}, \text{if above condition fails and } N_{-i} = -\beta \\ P_i, \text{ if above conditions fail and } P_i = \Pi \\ \varnothing, \text{otherwise} \end{cases}$$
   and
   $$a^i(x_i = L, \theta) = \begin{cases} P_i, \text{if } P_i = \Pi \\ N_{-i}, \text{if above condition fails and } N_{-i} = -\beta \\ \varnothing, \text{otherwise} \end{cases}$$

3. $\mathscr{F}$ is as described in Table 1.

4. $s^j(.) = $not, $\forall \theta, a_1, a_2, x_2$

5. $$p^i(x_i = R, .) = \begin{cases} \frac{E[A_i]+\bar{\gamma}}{2}, \text{if } A_1 = A_2 \\ \frac{E[A_i]+\bar{\gamma}}{2} - \frac{\delta}{4}, \text{otherwise} \end{cases}$$
   and
   $$p^i(x_i = L, .) = \left\{ \frac{E[A_i]+\bar{\gamma}}{2} - \frac{\delta}{4} \right.$$

[38] The assumption that consumers receive the first quote from the firm with larger $E[A_i] - |x_i - \chi_j|$ greatly simplifies characterizing the equilibrium. Otherwise, some consumers would be better off searching for a second quote even under identical prices, which would create incentives for firms to undercut each other's prices.

6. Let $B_{ij} = A_i - |\chi_j - x_i| - p_i$

- if $\gamma_j < \max_{i \in \mathscr{S}_j} E[B_{ij}]$ then $g^j(x_2, a_1, a_2) = \varnothing$
- if above condition fails and $\arg\max_{i \in \mathscr{S}_j} E[B_{ij}]$ is unique, then $g^j(x_2, a_1, a_2) = \arg\max_i E[B_{ij}]$
- otherwise $g^j(x_2 = L, a_1, a_2) = \begin{cases} 1, \text{w.p. } 0.5 \\ 2, \text{w.p. } 0.5 \end{cases}$ and $g^j(x_2 = R, a_1, a_2) = \begin{cases} 1, \text{ if } \chi_j = L \\ 2, \text{ if } \chi_j = R \end{cases}$

where $\mathscr{S}_j = \{1, 2\}$ if s^j=search and $\mathscr{S}_j = \{i\}$ otherwise, where i denotes the product whose free quote is received by consumer j.

We prove that $\{x^2, a^1, a^2, \mathscr{F}, g^j, s^j, p^1, p^2\}$ is a PBE when

(i) $\beta > (1 - \sigma_\Pi)\Pi + \delta$ (A25a)

(ii) $(\bar{\gamma} + \sigma_\beta \beta)(\bar{\gamma} + \sigma_\beta \beta - 2(1 - \sigma_\Pi)\Pi - 2\delta) \geq ((1 - \sigma_\Pi)\Pi - 0.5\delta)((1 - \sigma_\Pi)\Pi + \delta)$ (A25b)

(iii) $\Pi > (1 - \rho)(1 - \sigma_\beta)\beta$ (A25c)

(iv)
$$
\begin{aligned}
&\big(\sigma_\Pi^2 + \rho\sigma_\Pi(1 - \sigma_\Pi)\big)(\bar{\gamma} + (1 - \sigma_\Pi)\Pi)^2 \\
&+ 2\rho(1 - \sigma_\Pi)(1 - \rho)\sigma_\beta(\bar{\gamma} + (1 - \sigma_\Pi)\Pi - (1 - \sigma_\beta)\beta)^2 \\
&+ 2\sigma_\Pi(1 - \sigma_\Pi)(1 - \rho)(1 - \sigma_\beta)(\bar{\gamma} + (1 - \sigma_\Pi)\Pi + \sigma_\beta\beta)^2 \\
&+ (1 - \sigma_\Pi)(1 - \sigma_\Pi + \rho\sigma_\Pi)(\sigma_\beta^2 + \rho\sigma_\beta(1 - \sigma_\beta))(\bar{\gamma} - \sigma_\Pi\Pi - (1 - \sigma_\beta)\beta)^2 \\
&+ (1 - \sigma_\Pi)(1 - \sigma_\beta)(1 - \sigma_\Pi + \rho\sigma_\Pi)(1 + \sigma_\beta - \rho\sigma_\beta)(\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta)^2 - 0.5\delta^2 \\
&\geq \big(2\sigma_\beta(1 - \sigma_\beta)\sigma_\Pi + (1 - \sigma_\beta)^2(1 - \sigma_\Pi)^2\big)(\bar{\gamma} + \sigma_\beta\beta)^2 \\
&+ (1 - \sigma_\beta)^2\sigma_\Pi(2 - \sigma_\Pi)(\bar{\gamma} + \sigma_\beta\beta + (1 - \sigma_\Pi)\Pi)^2 + \sigma_\beta^2(\bar{\gamma} - (1 - \sigma_\beta)\beta)^2 \\
&+ (1 - \sigma_\beta)(1 - \sigma_\Pi)(1 + \sigma_\beta - \sigma_\Pi + \sigma_\Pi\sigma_\beta)(\bar{\gamma} + \sigma_\beta\beta - \sigma_\Pi\Pi)^2 \\
&- 0.5(1 - \sigma_\beta)(\sigma_\beta + (1 - \sigma_\beta)\sigma_\Pi(1 - \sigma_\Pi))\delta^2
\end{aligned}
\tag{A25d}
$$

First, g^j, by definition, maximizes $E[U_{ij}]$ given $\mathscr{F}$ and s_j. Second, $\mathscr{F}$ is consistent with the advertising strategies of the firm types followed in equilibrium.

Third, we discuss the optimality of $a^i(.)$ given $\mathscr{F}$. Notice that positive advertising would be prioritized against 'weak opponents' after both location choices for the same reasons as described in the Proof of Proposition 2.

Locating Apart: If the opponent is strong, then, for negative advertising to be prioritized in the advertising equilibrium, negative advertising should be effective enough to steal consumers if the opponent runs positive advertising. Since consumers don't search in equilibrium and prices are symmetric, firm i would steal consumers by guaranteeing $E[A_i - A_{-i}] > \delta$:

$$\begin{aligned} E[A_i - A_{-i}|a_i = N_i, a_{-i} = P_i] > \delta \\ \Leftrightarrow \beta - (1-\sigma_\Pi)\Pi > \delta, \end{aligned} \tag{A26}$$

which is equivalent to condition (A25a) above. Second, for negative advertising to be prioritized against strong opponents in the unique advertising equilibrium, the revenues from stolen consumers should be sufficient to make up for the lost revenues in own location ($R_i(a_i = N_i, a_{-i} = P_i) \geq R_i(a_i = P_i, a_{-i} = P_i)$):

$$\left(\frac{\sigma_\beta\beta + \overline{\gamma}}{2} - \frac{\delta}{4}\right)\left(2 - \Gamma\left(\frac{\sigma_\beta\beta + \overline{\gamma}}{2} - \frac{\delta}{4} - \sigma_\beta\beta\right) - \Gamma\left(\frac{\sigma_\beta\beta + \overline{\gamma}}{2} - \frac{\delta}{4} + \delta - \sigma_\beta\beta\right)\right) \geq$$
$$\left(\frac{(1-\sigma_\Pi)\Pi + \sigma_\beta\beta + \overline{\gamma}}{2}\right)\left(1 - \Gamma\left(\frac{(1-\sigma_\Pi)\Pi + \sigma_\beta\beta + \overline{\gamma}}{2} - \sigma_\beta\beta - (1-\sigma_\Pi)\Pi\right)\right),$$

which can be simplified as

$$(\bar{\gamma} + \sigma_\beta\beta)(\bar{\gamma} + \sigma_\beta\beta - 2(1-\sigma_\Pi)\Pi - 2\delta) \geq ((1-\sigma_\Pi)\Pi - 0.5\delta)((1-\sigma_\Pi)\Pi + \delta),$$

which is equivalent to condition (A25b) above. Since the game is symmetric, conditions (A25a) and (A25b) together imply prioritizing negative advertising against strong opponents is strictly dominant. See that (A25b) is harder to satisfy than (A1b) because

$$\begin{aligned} &(\bar{\gamma} + \sigma_\beta\beta - 2(1-\sigma_\Pi)\Pi - 2\delta) > ((1-\sigma_\Pi)\Pi - 0.5\delta) \\ \Leftrightarrow\ &\bar{\gamma} + \sigma_\beta\beta > 3(1-\sigma_\Pi)\sigma_\Pi + \frac{3\delta}{2}, \end{aligned}$$

is harder to satisfy than

$$\bar{\gamma} + \sigma_\beta\beta > (1-\sigma_\Pi)\Pi + \delta.$$

Hence, negative advertising under locating apart is less likely when pricing is introduced to the model. Because this is a necessary condition for the entrant to co-locate, co-location is also less likely when pricing is introduced.[39]

Co-Locating: If the opponent is strong, for positive advertising to be prioritized in the unique advertising equilibrium, positive advertising should be effective enough to steal consumers if the opponent runs negative advertising. Since consumers don't search in equilibrium and prices are

[39]Condition (A25d) also changes once pricing is introduced. However, condition (A25b) has precedence over condition (A25d) because (A25d) assumes (A25b) is satisfied.

symmetric, firm i would steal consumers by guaranteeing $E[A_i - A_{-i}] > 0$:

$$\begin{aligned} E[A_i - A_{-i} | a_i = P_i, a_{-i} = N_i] > 0 \\ \Pi > (1-\rho)(1-\sigma_\beta)\beta, \end{aligned} \tag{A27}$$

which is equivalent to condition (A25c) above.

Fourth, to prove the optimality of x^2, consider the potential outcomes following each location choice, summarized in Table A14.

θ	a_1	a_2	p_2 $(\frac{1}{2}*)$	R_2 $(\frac{1}{4(\bar{\gamma}-\underline{\gamma})}*)$
$\{\Pi, \Pi, , \}$	P_1	P_2	$\bar{\gamma} + (1-\sigma_\Pi)\Pi - \frac{\delta}{2}$	$(\bar{\gamma} + (1-\sigma_\Pi)\Pi)^2 - \frac{\delta^2}{4}$
$\{\Pi, 0, -\beta, \}$	P_1	N_1	$\bar{\gamma} + (1-\sigma_\Pi)\Pi - (1-\sigma_\beta)\beta - \frac{\delta}{2}$	0
$\{0, \Pi, , -\beta\}$	N_2	P_2	$\bar{\gamma} + (1-\sigma_\Pi)\Pi - (1-\sigma_\beta)\beta - \frac{\delta}{2}$	$2\big((\bar{\gamma} + (1-\sigma_\Pi)\Pi - (1-\sigma_\beta)\beta)^2 - \frac{\delta^2}{4}\big)$
$\{\Pi, 0, 0, \}$	P_1	$\varnothing$	$\bar{\gamma} + (1-\sigma_\Pi)\Pi + \sigma_\beta\beta - \frac{\delta}{2}$	0
$\{0, \Pi, , 0\}$	$\varnothing$	P_2	$\bar{\gamma} + (1-\sigma_\Pi)\Pi + \sigma_\beta\beta - \frac{\delta}{2}$	$2\big((\bar{\gamma} + (1-\sigma_\Pi)\Pi + \sigma_\beta\beta)^2 - \frac{\delta^2}{4}\big)$
$\{0, 0, -\beta, -\beta\}$	N_2	N_1	$\bar{\gamma} - \sigma_\Pi\Pi - (1-\sigma_\beta)\beta - \frac{\delta}{2}$	$(\bar{\gamma} - \sigma_\Pi\Pi - (1-\sigma_\beta)\beta)^2 - \frac{\delta^2}{4}$
$\{0, 0, 0, -\beta\}$	N_2	$\varnothing$	$\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta - \frac{\delta}{2}$	0
$\{0, 0, -\beta, 0\}$	$\varnothing$	N_1	$\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta - \frac{\delta}{2}$	$2\big((\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta)^2 - \frac{\delta^2}{4}\big)$
$\{0, 0, 0, 0\}$	$\varnothing$	$\varnothing$	$\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta - \frac{\delta}{2}$	$(\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta)^2 - \frac{\delta^2}{4}$

Table A12: Co-Location

θ	a_1	a_2	p_2 $(\frac{1}{2}*)$	R_2 $(\frac{1}{4(\bar{\gamma}-\underline{\gamma})}*)$
$\{, , -\beta, -\beta\}$	N_2	N_1	$\bar{\gamma} - (1-\sigma_\beta)\beta$	$(\bar{\gamma} - (1-\sigma_\beta\beta)^2$
$\{, \Pi, 0, -\beta\}$	N_2	P_2	$\bar{\gamma} + \sigma_\beta\beta - \frac{\delta}{2}$	0
$\{\Pi, , -\beta, 0\}$	P_1	N_1	$\bar{\gamma} + \sigma_\beta\beta - \frac{\delta}{2}$	$2\big((\bar{\gamma} + \sigma_\beta\beta)^2 - \frac{\delta^2}{4}\big)$
$\{, 0, 0, -\beta\}$	N_2	$\varnothing$	$\bar{\gamma} + \sigma_\beta\beta - \sigma_\Pi\Pi - \frac{\delta}{2}$	0
$\{0, , -\beta, 0\}$	$\varnothing$	N_1	$\bar{\gamma} + \sigma_\beta\beta - \sigma_\Pi\Pi - \frac{\delta}{2}$	$2\big(\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta)^2 - \frac{\delta^2}{4}\big)$
$\{\Pi, \Pi, 0, 0\}$	P_1	P_2	$\bar{\gamma} + \sigma_\beta\beta + (1-\sigma_\Pi)\Pi$	$(\bar{\gamma} + (1-\sigma_\Pi)\Pi + \sigma_\beta\beta)^2$
$\{\Pi, 0, 0, 0\}$	P_1	$\varnothing$	$\bar{\gamma} + \sigma_\beta\beta - \sigma_\Pi\Pi - \frac{\delta}{2}$	0
$\{0, \Pi, 0, 0\}$	$\varnothing$	P_2	$\bar{\gamma} + \sigma_\beta\beta + (1-\sigma_\Pi)\Pi - \frac{\delta}{2}$	$2\big((\bar{\gamma} + (1-\sigma_\Pi)\Pi + \sigma_\beta\beta)^2 - \frac{\delta^2}{4}\big)$
$\{0, 0, 0, 0\}$	$\varnothing$	$\varnothing$	$\bar{\gamma} + \sigma_\beta\beta - \sigma_\Pi\Pi - \frac{\delta}{2}$	$(\bar{\gamma} - \sigma_\Pi\Pi + \sigma_\beta\beta)^2$

Table A13: Locating Apart

Table A14: Revenues for the Entrant with Negative Advertising Allowed, p_2 and R_2 refer to the price and revenues for the entrant for each outcome. The terms in the paranthesis are common multipliers for all values in the columns. $\theta = \{P_1, P_2, N_1, N_2\}$ denotes the realizations for attributes. If the associated entry in θ is unspecified, that means a_i, p_i, and D_i do not depend on the value of that entry. See Table 1 for $E[A_1]$ and $E[A_2]$ associated with each outcome. Refer to Table A6 for the associated probabilities for each scenario.

With some algebra, we can simplify the expected payoff after each location choice to

$$\begin{aligned} E[R_2 | x_2 = L] = \frac{1}{4(\bar{\gamma}-\underline{\gamma})} \Big[& \big(\sigma_\Pi^2 + \rho\sigma_\Pi(1-\sigma_\Pi)\big)(\bar{\gamma} + (1-\sigma_\Pi)\Pi)^2 \\ & + 2\rho(1-\sigma_\Pi)(1-\rho)\sigma_\beta(\bar{\gamma} + (1-\sigma_\Pi)\Pi - (1-\sigma_\beta)\beta)^2 \\ & + 2\sigma_\Pi(1-\sigma_\Pi)(1-\rho)(1-\sigma_\beta)(\bar{\gamma} + (1-\sigma_\Pi)\Pi + \sigma_\beta\beta)^2 \\ & + (1-\sigma_\Pi)(1-\sigma_\Pi + \rho\sigma_\Pi)(\sigma_\beta^2 + \rho\sigma_\beta(1-\sigma_\beta))(\bar{\gamma} - \sigma_\Pi\Pi - (1-\sigma_\beta)\beta)^2 \end{aligned} \tag{A28}$$

$$+ (1-\sigma_\Pi)(1-\sigma_\beta)(1-\sigma_\Pi+\rho\sigma_\Pi)(1+\sigma_\beta-\rho\sigma_\beta)(\bar{\gamma}-\sigma_\Pi\Pi+\sigma_\beta\beta)^2 - 0.5\delta^2\Big]$$

$$E[R_2|x_2=R] = \frac{1}{4(\bar{\gamma}-\underline{\gamma})}\Big[\big(2\sigma_\beta(1-\sigma_\beta)\sigma_\Pi + (1-\sigma_\beta)^2(1-\sigma_\Pi)^2\big)(\bar{\gamma}+\sigma_\beta\beta)^2 \tag{A29}$$
$$+ (1-\sigma_\beta)^2\sigma_\Pi(2-\sigma_\Pi)(\bar{\gamma}+\sigma_\beta\beta+(1-\sigma_\Pi)\Pi)^2 + \sigma_\beta^2(\bar{\gamma}-(1-\sigma_\beta)\beta)^2$$
$$+ (1-\sigma_\beta)(1-\sigma_\Pi)(1+\sigma_\beta-\sigma_\Pi+\sigma_\Pi\sigma_\beta)(\bar{\gamma}+\sigma_\beta\beta-\sigma_\Pi\Pi)^2$$
$$- 0.5(1-\sigma_\beta)(\sigma_\beta+(1-\sigma_\beta)\sigma_\Pi(1-\sigma_\Pi))\delta^2\Big].$$

Hence, $E[R_2|x_2 = L] \geq E[R_2|x_2 = R]$ becomes equivalent to condition (A25d) above.

Last, the fact that consumers' search decisions and firms' pricing decisions constitute a Nash equilibrium of the pricing-search subgame has been established in Lemma 8.

To sum up, once conditions (A25a)-(A25d) are satisfied, there exists a PBE as defined in $1-6$. □

Proof of Proposition 6. First, consider incentives to run negative advertising against weak opponents where the entrant locates apart. In product competition, the necessary condition for negative advertising to be prioritized was $E[A_i - A_{-i}|a_i = N] > \delta \geq E[A_i - A_{-i}|a_i = P]$, i.e., only negative advertising allows stealing consumers. In political competition, negative advertising may be utilized even when it doesn't lead to stolen consumers. As long as the opponent loses sufficiently many consumers, negative advertising can be utilized. In other words, the necessary condition is $E[A_i - A_{-i}|a_i = N] \geq E[A_i - A_{-i}|a_i = P]$, which is weaker than the condition above. In the scenario where the entrant co-locates, there would be no change in the incentives and positive advertising would be prioritized against weak opponents in both political and product competition.

Second, consider incentives to run negative advertising against strong opponents. In product competition, there are two necessary conditions for negative advertising to be prioritized: (1) negative advertising allows stealing consumers when the competitor runs positive advertising and (2) the number of stolen consumers is sufficiently large. In political competition, the first condition is sufficient by itself, because a decline in total number of votes is not problematic as long as the opponent loses more votes. Hence, negative advertising is more likely against strong opponents under both co-location and locating apart.

To sum up, under any parameter set where negative advertising is prioritized in product competition, negative advertising is also prioritized in political competition. □

Proof of Proposition 7. (i) In the benchmark where $\eta = 0$, the game is symmetric between the firms. Hence, the expected winning probability is 0.5 following both co-location and locating apart under political competition. This result is independent of which equilibrium is played in the advertising subgames, because all advertising equilibria are symmetric.

(ii) In political competition, the presence of $\eta > 0$ implies that co-location leads to a strictly smaller winning probability for the entrant. This is because the winning probability is discontinuous in vote difference, hence it is discontinuous at $\eta = 0$. When a mass $\eta > 0$ prefers the incumbent, then any symmetric advertising choice leads to incumbent winning the race with probability 1 (instead of 0.5), reducing the ex-ante winning probability of the entrant following co-location strictly below 0.5.

If the entrant candidate locates apart, however, there exists a small enough $\eta > 0$ such that the winning probability of the entrant is still 0.5. Under locating apart, η only matters when the advertising outcomes favor the entrant more than the incumbent to an extent where entrant can steal the regular voters but not the voters which have the incumbency bias (measure $1-\eta$). In that case, the entrant would win with probability 1 as long as $\eta < 0.5$. For other advertising outcomes, voters in L would vote for the incumbent independent of the incumbency advantage. Hence, for $\eta < 0.5$, the winning probability of the entrant is still 0.5 following locating apart. In short, for $\eta > 0$ small enough, the entrant candidate would locate apart.

The same reasoning does not work in product competition because profits/demand are continuous in η. The presence of η can only reduce the expected demand for the entrant firm by an amount proportional to η. Hence, when η is small enough, its impact on the expected payoffs is negligible. For any situation where the entrant strictly prefers to co-locate, there exists an η small enough so that the entrant still strictly prefers to co-locate. □

Proof of Proposition A.1. Part (i) The proof follows the steps of the of Proof for Proposition 2 almost line by line. The steps for the optimality of purchase and advertising decisions, and how consumer beliefs $\mathscr{F}$ satisfy the Bayes rule given the advertising decisions is identical to the proof for Proposition 2. The only difference now is that location choice is not made in isolation by the entrant, but decided simultaneously within a game for two firms. Hence, the updated equilibrium condition would state that location decisions x^1 and x^2 constitute a Nash equilibrium. For the parameter set where the entrant decides to co-locate, the expected payoff following co-location should be larger than the expected payoff following locating apart, conditional on the location of the incumbent. Then, the Nash equilibria of the location decisions game (conditional on g^j, a^1 and a^2) would be $\{x^1, x^2\} = \{L, L\}$ and $\{x^1, x^2\} = \{R, R\}$. If $\{x^1, x^2\} = \{L, R\}$ or $\{x^1, x^2\} = \{R, L\}$, then both firms would have a profitable deviation to the other location. The observable equilibrium outcomes are identical to those of the entrant-incumbent game, up to a symmetric change in where firms are located.

Part (ii) Similarly, for the parameter set where the entrant decides to locate apart, the expected payoff following locating apart should be larger than the expected payoff following co-location, conditional on the location of the incumbent. Then, the Nash equilibria of the location decision game (conditional on g^j, a^1 and a^2) would be $\{x^1, x^2\} = \{L, R\}$ and $\{x^1, x^2\} = \{R, L\}$. If $\{x^1, x^2\} = \{L, L\}$ or $\{x^1, x^2\} = \{R, R\}$, then both firms would have a profitable deviation to the other location. The observable equilibrium outcomes are identical to those of the entrant-incumbent game, up to a symmetric change in where firms are located. $\square$