Online Appendix

Performance Evaluation, Managerial Hedging, and Contract Termination

A Several special cases

A.1 A static model without endogenous termination

Consider a static model with dates 0 and 1. There is a capital market which consists of a risk-free bond and a market portfolio. At date 1, return of the risk-free bond is a constant r and return of the market portfolio is a random variable R with a normal distribution $\mathcal{N}(m, \sigma^2)$. At date 0, a principal contracts with an agent to run a firm. If the agent exerts effort $e \in [0, \bar{e}]$, the cash flow at date 1 generated by the production capital K is a random variable X which follows a normal distribution $\mathcal{N}(eK, \psi^2 K^2)$. The correlation coefficient between R and X is ρ. The monetary cost to the agent for his effort e is beK. The principal only observes one realization of X, from which the principal cannot infer the agent's hidden effort.

The agent has a CARA utility $\bar{u}(\bar{W}_1) = -\frac{1}{\bar{\gamma}} e^{-\bar{\gamma}\bar{W}_1}$ over his wealth $\bar{W}_1$ at date 1. Let us first consider the case where the agent does not work for the principal. Given an initial wealth $\bar{W}_0$, the agent invests $\bar{\pi}$ (in monetary value) into the market portfolio and the remaining $\bar{W}_0 - \bar{\pi}$ into the risk-free bond. Then his wealth at date 1 satisfies $\bar{W}_1 = \bar{W}_0(1+r) + \bar{\pi}(R - r)$. The agent chooses $\bar{\pi}$ to solve

$$\underline{\mathcal{U}} = \max_{\bar{\pi}} \mathbb{E}[\bar{u}(\bar{W}_1)]. \tag{A.1}$$

An easy calculation shows that the agent's optimal strategy and the value function are

$$\bar{\pi}^* = \frac{m-r}{\bar{\gamma}\sigma^2} \quad \text{and} \quad \underline{\mathcal{U}} = \bar{u}\left(\bar{W}_0(1+r) + \frac{(m-r)^2}{2\bar{\gamma}\sigma^2}\right). \tag{A.2}$$

The agent uses $\underline{\mathcal{U}}$ to determine his participation constraint.

At date 0, the principal offers a take-it-or-leave-it contract to the agent with the compensation Y at date 1

$$Y = H + ZX^{\perp} + \zeta\sigma B, \tag{A.3}$$

where Z is the contract sensitivity to the relative cash flow, ζ is the sensitivity to the market shocks, and H will be determined later by the agent's participation constraint. Given Y, the agent's optimization problem is

$$\begin{aligned}\mathcal{U} &= \max_{e,\bar{\pi}} \mathbb{E}\big[\bar{u}(\bar{W}_1 + Y - beK)\big] \\ &= \max_{A,\bar{\pi}} \bar{u}\Big(\bar{W}_0(1+r) + \bar{\pi}(m-r) + H + ZeK - beK \\ &\qquad - \tfrac{1}{2}\bar{\gamma}\Big(\big(\bar{\pi}+\zeta\big)^2\sigma^2 + Z^2(1-\rho^2)\psi^2K^2\Big)\Big).\end{aligned} \tag{A.4}$$

The agent's optimal effort is $e^* = \bar{e}$ when $Z \geq b$ or $e^* = 0$ when $Z < b$; the agent's optimal private investment is

$$\bar{\pi}^* = \frac{m-r}{\bar{\gamma}\sigma^2} - \zeta. \tag{A.5}$$

The term H is determined by the agent's binding participation constraint $\mathcal{U} = \underline{\mathcal{U}}$. Calculation shows that

$$Y = \zeta(R-r) + Z(X^{\perp} - e^*K) + be^*K + \tfrac{1}{2}\bar{\gamma}Z^2(1-\rho^2)\psi^2K^2. \tag{A.6}$$

This is the static version of (22). On the right-hand side of (A.6), the first term $\zeta(R-r)$ is caused by the feedback mechanism of managerial hedging and the second term is the incentive with respect to the idiosyncratic shock.

The firm is started with an initial size S_0, among which, K is invested in the production capital, π (in monetary unit) in the market portfolio, and remaining in the risk-free bond. The firm size at date 1 is $S_1 = (1+r)(S_0 - K) + \pi(R-r) + X - Y$. The firm is liquidated at date 1 and there is no liquidation cost. The principal has a CARA utility $u(S_1) = -\frac{1}{\gamma}e^{-\gamma S_1}$. The principal's optimization problem is

$$
\begin{aligned}
\mathcal{V} &= \max_{\pi,Z,\zeta} \mathbb{E}[u(S_1)] \\
&= \max_{\pi,Z,\zeta} u\Big((1+r)(S_0-K) + \big(\pi-\zeta\big)(m-r) + (e^*-be^*)K \\
&\quad -\tfrac{1}{2}\bar{\gamma}Z^2(1-\rho^2)\psi^2K^2 - \tfrac{1}{2}\gamma\Big(\big(\pi+\tfrac{\rho\psi}{\sigma}K-\zeta\big)^2\sigma^2 + (1-Z)^2(1-\rho^2)\psi^2K^2\Big)\Big).
\end{aligned}
\tag{A.7}
$$

The optimal π and ζ satisfies

$$
\pi^* + \frac{\rho\psi}{\sigma}K - \zeta^* = \frac{m-r}{\gamma\sigma^2}, \tag{A.8}
$$

but π^* and ζ^* are not uniquely determined.

A.2 Firm without risk management and constant liquidation cost

We will derive the HJB equation satisfied by principal's value function. Recall that principal's indirect utility at firm termination is $u\big((r-\lambda)L(S_\tau)-\ell_b\big)$, where $L(s)=s-C$ and $\ell_b = \frac{1}{\gamma}(1-\frac{\delta}{r-\lambda}-\ln(\frac{r-\lambda}{\delta}))$. Let $\mathcal{V}$ as principal's optimal value and define principal's relative firm value V via

$$
\mathcal{V}(s,y) = -\frac{1}{\gamma}e^{-\gamma((r-\lambda)s-\ell_b+(r-\lambda)V(y))}. \tag{A.9}
$$

Plugging (A.9) into (33), we obtain after calculation

$$
\begin{aligned}
(r-\lambda)V = \sup_{Z,\zeta}\Big\{ &-(r-\lambda)K - \tfrac{\gamma(r-\lambda)}{2}\rho^2\psi^2K^2 + Ke^*(Z) \\
&+ \Big[ry + \tfrac{r\bar{\gamma}}{2}Z^2(1-\rho^2)\psi^2K^2 + h(e^*(Z))\Big]V_y \\
&- \tfrac{\gamma(r-\lambda)}{2}(1-\rho^2)\psi^2K^2(1+ZV_y)^2 + \tfrac{1}{2}Z^2(1-\rho^2)\psi^2K^2V_{yy} \\
&+ \zeta V_y\big[m-r-(r-\lambda)\gamma\rho\psi K\sigma\big] + \tfrac{1}{2}\zeta^2\sigma^2\big[V_{yy}-(r-\lambda)\gamma V_y^2\big]\Big\}.
\end{aligned}
\tag{A.10}
$$

Boundary conditions (31), (32), and (34) are transformed into

$$V_y + 1 = 0, \quad V_{yy} = 0, \quad \text{and} \quad V(0) = -C.^{31} \tag{A.11}$$

A.3 Firm risk management and constant liquidation cost

Plugging (39) into (33), we obtain after calculation that

$$\begin{aligned}
0 = \sup_{c,\pi,Z,\zeta} \Big\{ & \tfrac{\delta u(c)}{re^{-\gamma((r-\lambda)(s+V)-\ell)}} + \tfrac{\delta}{\gamma(r-\lambda)} + (r-\lambda)(s-K) - \tfrac{\rho\psi}{\sigma}(m-r) + e^*(Z)K - c \\
& + \Big[ry + \tfrac{r\tilde{\gamma}}{2} Z^2(1-\rho^2)\psi^2 K^2 + h(e^*(Z))\Big] V_y \\
& + \big(\pi + \tfrac{\rho\psi}{\sigma}K + \zeta V_y\big)(m-r) - \tfrac{\gamma(r-\lambda)}{2}\sigma^2\big(\pi + \tfrac{\rho\psi}{\sigma}K + \zeta V_y\big)^2 + \tfrac{1}{2}\zeta^2\sigma^2 V'_{yy} \\
& - \tfrac{\gamma(r-\lambda)}{2}(1-\rho^2)\psi^2 K^2 (1 + ZV_y)^2 + \tfrac{1}{2} Z^2 (1-\rho^2)\psi^2 K^2 V_{yy} \Big\}, \quad y \in (0, \bar{y}),
\end{aligned} \tag{A.12}$$

and the boundary conditions

$$V_y(\bar{y}) = -1, \quad V_{yy}(\bar{y}) = 0, \quad \text{and} \quad V(0) = -C. \tag{A.13}$$

The first order condition of π in (A.12) yields

$$\pi^* + \tfrac{\rho\psi}{\sigma}K + \zeta^* V_y = \frac{m-r}{(r-\lambda)\gamma\sigma^2}. \tag{A.14}$$

In (A.12), the principal chooses ζ to maximize $\frac{1}{2}\zeta^2\sigma^2 V_{yy}$. Therefore, the concavity of V implies that $\zeta^* = 0$, i.e., an RPE contract. Substitute optimal π^* and ζ^* into (A.12), we obtain the

[31] Equation (32) is reduced to $-\gamma(r-\lambda))(V_y + V_y^2) + V_{yy} = 0$. Combining the previous equation with $V_y = -1$, we obtain $V_{yy} = 0$.

following HJB equation for V:

$$
\begin{aligned}
(r-\delta)V = &\sup_{Z}\Big\{-(r-\lambda)K + Ke^*(Z) - \tfrac{\rho\psi}{\sigma}K(m-r) + \big[ry + \tfrac{\bar{\gamma}(r-\lambda)}{2}Z^2(1-\rho^2)\psi^2K^2 + h(e^*(Z))\big]V_y \\
&- \tfrac{\gamma(r-\lambda)}{2}(1-\rho^2)\psi^2K^2(1+ZV_y)^2 + \tfrac{1}{2}Z^2(1-\rho^2)\psi^2K^2V_{yy}\Big\}, \quad y\in(0,\bar{y}).
\end{aligned} \tag{A.15}
$$

B Proofs

B.1 The quiet life contract

In the quiet life contract, the agent receives no compensation, nor does he exerts any effort. He is indifferent between continuing as an incumbent and retiring. For a quiet life contract, we assume that the agent stays on the project and the principal receives the project cash flow in (1) with $e\equiv 0$. Then the principal's indirect utility $\mathcal{V}^Q$ solves

$$
\begin{aligned}
\delta\mathcal{V}^Q(S) = &\sup_{c,\pi}\Big\{\delta u(c) + \big((r-\lambda)(S-K) + \pi(m-r) - c\big)(\mathcal{V}^Q)'(S) \\
&+ \tfrac{1}{2}\Big[(\pi\sigma+\rho\psi K)^2 + (1-\rho^2)\psi^2K^2\Big](\mathcal{V}^Q)''(S)\Big\}.
\end{aligned} \tag{B.1}
$$

Calculation shows that

$$
\mathcal{V}^Q(S) = -\frac{1}{\gamma}e^{-\gamma\left((r-\lambda)(S-K)-\ell-\frac{\rho\psi}{\sigma}K(m-r)-\frac{(r-\lambda)\gamma}{2}(1-\rho^2)\psi^2K^2\right)}, \tag{B.2}
$$

where the terms $-\frac{\rho\psi}{\sigma}K(m-r)-\frac{(r-\lambda)\gamma}{2}(1-\rho^2)\psi^2K^2$ in the exponent represent the reduction due to unhedgable idiosyncratic risk of the project. We call

$$
L^Q \equiv -\frac{1}{r-\lambda}\left((r-\lambda)K + \frac{\rho\psi}{\sigma}K(m-r) + \frac{(r-\lambda)\gamma}{2}(1-\rho^2)\psi^2K^2\right)
$$

the quiet life value of the contract. The quiet life contract is never optimal as long as condition (13) holds.

B.2 Proof of Lemma 3.1

Assume that the strategy $(e^*, \bar{c}^*, \bar{\pi}^*)$ solves the agent's problem (14). Let $\mathbb{P}^{e^*}$ be the equivalent measure induced by A^* under which

$$B_t = \frac{R_t - R_0 - mt}{\sigma} \quad \text{and} \quad B_t^{\perp}(e^*) = \frac{X_t^{\perp} - X_0^{\perp} - \int_0^t e_s^* K \mathrm{d}s}{\psi K \sqrt{1-\rho^2}} \tag{B.3}$$

are two independent Brownian motions. Given $\mathcal{U}$ in (14), the process $\tilde{\mathcal{U}}^{e^*,\bar{c}^*,\bar{\pi}^*}$ defined via

$$\tilde{\mathcal{U}}_t^{e^*,\bar{c}^*,\bar{\pi}^*} = \int_0^t e^{-\bar{\delta}s} \bar{u}(\bar{c}_s^*) \mathrm{d}s + e^{-\bar{\delta}t} \mathcal{U}_t \tag{B.4}$$

is a $\mathbb{P}^{e^*}$-martingale before the contract termination time τ. By the martingale representation theorem, there exist $\mathbb{F}^{B,X^{\perp}}$-adapted processes $\widehat{\mathbb{Z}}$ and $\widehat{\zeta}$ such that

$$\mathrm{d}\tilde{\mathcal{U}}_t^{A^*,\bar{c}^*,\bar{\pi}^*} = e^{-\bar{\delta}t} \left[\widehat{\mathbb{Z}}_t \psi \sqrt{1-\rho^2} \mathrm{d}B_t^{\perp}(e_t^*) + \widehat{\zeta}_t \sigma \mathrm{d}B_t \right], \quad t < \tau, \tag{B.5}$$

where the factors $e^{-\bar{\delta}t}, \psi\sqrt{1-\rho^2}$, and σ are for convenient rescaling. Substitute (B.3) into (B.5), and we obtain that there exists some finite-variation process $\widehat{\mathbb{H}}$ such that

$$\mathrm{d}\tilde{\mathcal{U}}_t^{A^*,\bar{c}^*,\bar{\pi}^*} = e^{-\bar{\delta}t} (\mathrm{d}\widehat{\mathbb{H}}_t + \widehat{\mathbb{Z}}_t \mathrm{d}X_t^{\perp} + \widehat{\zeta}_t \sigma \mathrm{d}B_t), \quad t < \tau. \tag{B.6}$$

Given Y defined in (15), applying Itô's formula on both sides of (B.6), we show that Y necessarily follows (17) before τ.

B.3 Calculation of H and Y

For arbitrary strategies $(e, \bar{c}, \bar{\pi})$, we define

$$\mathcal{U}_t^{e,\bar{c},\bar{\pi},Z,\zeta} = -\frac{1}{\bar{\bar{\gamma}}} e^{-\bar{\gamma}\left(r\bar{W}_t - \bar{\ell} + rY_t\right)}, \tag{B.7}$$

where $\bar{W}$ satisfies (4) and Y follows

$$\mathrm{d}Y_t = \mathrm{d}H_t + Z_t \mathrm{d}X_t^{\perp} + \zeta_t \sigma \mathrm{d}B_t, \quad t \leq \tau. \tag{B.8}$$

Note that $\bar{W}_t$ depends on $(e, \bar{c}, \bar{\pi})$ before t, while Y_t, defined by the optimal value $\mathcal{U}_t$ via (15), is determined by $(e^*, \bar{c}^*, \bar{\pi}^*)$ from t onward. Therefore the value $\mathcal{U}_t^{A,\bar{c},\bar{\pi},Z,\zeta}$ represents the agent's utility at time t if he employs the strategy $(e, \bar{c}, \bar{\pi})$ up to time t and then follows the optimal strategy $(e^*, \bar{c}^*, \bar{\pi}^*)$ from time t until τ. Taking into account the utility of consumption before t, we define the process $\widetilde{\mathcal{U}}_t = e^{-\bar{\delta}t} \mathcal{U}_t^{e,\bar{c},\bar{\pi},Z,\zeta} + \bar{\delta} \int_0^t e^{-\bar{\delta}s} u(\bar{c}_s) \mathrm{d}s$. Dynamic programming principle implies that $\widetilde{\mathcal{U}}$ is a supermartingale for an arbitrary strategy $(e, \bar{c}, \bar{\pi})$, and is a martingale until time τ for the optimal strategy $(e^*, \bar{c}^*, \bar{\pi}^*)$. This means that the drift of $\widetilde{\mathcal{U}}$ is non-positive and its maximum value 0 is attained when the optimal strategy $(e^*, \bar{c}^*, \bar{\pi}^*)$ is employed. Combining (4) and (B.8), and using Itô's formula, we calculate the drift of $\widetilde{\mathcal{U}}$, divided throughout by $(-r\bar{\gamma}) e^{-\bar{\delta}t} \mathcal{U}_t^{e,\bar{c},\bar{\pi},Z,\zeta}$, to obtain

$$\begin{aligned} &\Big\{ \tfrac{\bar{\delta}}{r\bar{\gamma}} + r\bar{W}_t + Z_t e_t K - h(e_t) + \bar{\pi}_t(m - r) - \tfrac{r\bar{\gamma}}{2} \Big[Z_t^2 (1 - \rho^2) \psi^2 K^2 + (\zeta_t + \bar{\pi}_t)^2 \sigma^2 \Big] \\ &- \tfrac{\bar{\delta}\bar{u}(\bar{c}_t)}{r\bar{\gamma}\mathcal{U}_t} - \bar{c}_t \Big\} dt + dI_t + dH_t. \end{aligned} \tag{B.9}$$

The agent choose $(e, \bar{c}, \bar{\pi})$ to maximize the previous expression, leading to the maximization problem in (18). The maximizers $(e^*, \bar{c}^*, \bar{\pi}^*)$ are obtained in (19), (20), and (21). The maximum value of (B.9) is zero which is attained at $(e^*, \bar{c}^*, \bar{\pi}^*)$. Plugging (19), (20), and (21) into (B.9), we obtain

$$\mathrm{d}H_t = \left[rY_t + \zeta_t(m-r) + \tfrac{r\bar{\gamma}}{2} Z_t^2(1-\rho^2)\psi^2 K^2 + h(e_t^*) - Z_t e_t^* K\right]\mathrm{d}t - \mathrm{d}I_t. \tag{B.10}$$

Hence, (B.10), combined with (B.8), yields (22).

B.4 Verification of the agent's optimal choices

We first define the admissible class of the agent's strategies. Given the payment process I, a strategy $(e, \bar{c}, \bar{\pi})$ is *admissible* to the agent if there exist Z and ζ, adapted to $\mathbb{F}^{B,X^\perp}$, such that $\mathcal{U}^{e,\bar{c},\bar{\pi},Z,\zeta}$ defined in (B.7) satisfies the transversality condition

$$\lim_{\tilde{\tau}\to\infty} \mathbb{E}\left[e^{-\bar{\delta}\tilde{\tau}}\mathcal{U}_{\tilde{\tau}}^{e,\bar{c},\bar{\pi},Z,\zeta} 1_{\{\tilde{\tau}\le\tau\}}\right] = 0, \quad \text{for any stopping time } \tilde{\tau}. \tag{B.11}$$

We verify the optimality of $(e^*, \bar{c}^*, \bar{\pi}^*)$ in this section. For any admissible strategy $(e, \bar{c}, \bar{\pi})$, consider the process $\tilde{\mathcal{U}}$ in Appendix B.3. The drift of $\tilde{\mathcal{U}}$ is non-positive, hence $\tilde{\mathcal{U}}$ is a local supermartingale. Let $\{\tau_n : n = 1, 2, 3, \dots\}$ be a localization sequence for its local martingale part such that $\lim_{n\to\infty}\tau_n = \infty$. We have

$$\mathbb{E}\left[\bar{\delta}\int_0^{\tau_n\wedge\tau} e^{-\bar{\delta}s}\bar{u}(\bar{c}_s)\mathrm{d}s\right] + \mathbb{E}\left[e^{-\bar{\delta}\tau_n\wedge\tau}\mathcal{U}_{\tau_n\wedge\tau}^{e,\bar{c},\bar{\pi},Z,\zeta}\right] \le \tilde{\mathcal{U}}_0 = \mathcal{U}_0^{e,\bar{c},\bar{\pi},Z,\zeta} = -\frac{1}{\bar{\gamma}} e^{-\bar{\gamma}\left(r\bar{W}_0 - \bar{\ell} + rY_0\right)}. \tag{B.12}$$

The second term on the left-hand side can be decomposed into

$$\mathbb{E}\left[e^{-\bar{\delta}\tau_n}\mathcal{U}_{\tau_n}^{e,\bar{c},\bar{\pi},Z,\zeta} 1_{\{\tau_n\le\tau\}}\right] + \mathbb{E}\left[e^{-\bar{\delta}\tau}\mathcal{U}_{\tau}^{e,\bar{c},\bar{\pi},Z,\zeta} 1_{\{\tau<\tau_n\}}\right]. \tag{B.13}$$

Sending $n\to\infty$, and using monotone convergence theorem together with (B.11) and the fact that $\mathcal{U}_\tau^{e,\bar{c},\bar{\pi},Z,\zeta} = \bar{u}(r\bar{W}_\tau - \bar{\ell})$ when $\tau < \infty$, we obtain

$$\mathbb{E}\left[\bar{\delta}\int_0^\tau e^{-\bar{\delta}s}\bar{u}(\bar{c}_s)\mathrm{d}s + e^{-\bar{\delta}\tau}\bar{u}(r\bar{W}_\tau - \bar{\ell})\right] \le -\frac{1}{\bar{\gamma}}e^{-\bar{\gamma}\left(r\bar{W}_0-\ell+rY_0\right)}. \tag{B.14}$$

For strategy (e^*, c^*, π^*), $\tilde{\mathcal{U}}$ is a local martingale. Then the inequality in (B.12) is an equality. Sending $n \to \infty$ and using (B.11), the optimality of (e^*, c^*, π^*) is confirmed.

Let us now check the admissibility of the optimal strategy $(e^*, \bar{c}^*, \bar{\pi}^*)$. To this end, denote $\mathcal{U}_t = \mathcal{U}_t^{e^*,\bar{c}^*,\bar{\pi}^*,Z,\zeta}$ for simplicity of notation. Using (20), (21), and (24), we calculate

$$\mathrm{d}\left(e^{-\bar{\delta}t}\mathcal{U}_t\right) = -re^{-\bar{\delta}t}\mathcal{U}_{t-}\mathrm{d}t + e^{-\bar{\delta}t}\mathcal{U}_{t-}\Big[-\tfrac{m-r}{\sigma}\mathrm{d}B_t - r\bar{\gamma}Z_t\sqrt{1-\rho^2}\psi K\mathrm{d}B_t^\perp\Big]. \tag{B.15}$$

Therefore, for any stopping time $\tilde{\tau}$,

$$e^{-\bar{\delta}\tilde{\tau}}\mathcal{U}_{\tilde{\tau}} = \mathcal{U}_0\, e^{-r\tilde{\tau}}\mathcal{E}\left(\int_0^\cdot -\tfrac{m-r}{\sigma}\mathrm{d}B_t - r\bar{\gamma}Z_t\sqrt{1-\rho^2}\psi K\mathrm{d}B_t^\perp\right)_{\tilde{\tau}}, \tag{B.16}$$

where $\mathcal{E}(\cdot)$ is a stochastic exponential which is a supermartingale with its expectation less than 1. Therefore, due to $r > 0$, we have

$$\tfrac{1}{\mathcal{U}_0}\mathbb{E}\big[e^{-\bar{\delta}\tilde{\tau}}\mathcal{U}_{\tilde{\tau}}\big] \le \mathbb{E}[e^{-r\tilde{\tau}}] \to 0 \quad \text{as} \quad \tilde{\tau} \to \infty. \tag{B.17}$$

This verifies the transversality condition (B.11), hence the admissibility of the optimal strategy $(e^*, \bar{c}^*, \bar{\pi}^*)$.

B.5 Proofs of Propositions 4.1, 4.2, and 4.3

We will prove Proposition 4.1 here. Proposition 4.2 and 4.3 are special cases. Let us first define the principal's admissible strategy. The strategy (c, π, I, Z, ζ) is *admissible* if the following transversality condition is satisfied:

$$\lim_{\tilde{\tau}\to\infty} \mathbb{E}\Big[e^{-\delta\tilde{\tau}}\mathcal{V}(S_{\tilde{\tau}}, Y_{\tilde{\tau}})\Big] = 0, \qquad \text{for any stopping time } \tilde{\tau}, \tag{B.18}$$

where $\mathcal{V}$ is the solution of (33) with boundary conditions (31), (32), (34). S follows (4), and Y follows (22) for given (c, π, I, Z, ζ).

For $(c^*, \pi^*, I^*, Z^*, \zeta^*)$ in the statement of Proposition 4.1, suppose that it is admissible, which we will verify at the end of this section, let us first verify its optimality among all admissible strategies. To this end, for any admissible strategy (c, π, I, Z, ζ), let S and Y be the associated firm size and the relative contract value of the agent. It follows from (33) that $e^{-\delta\cdot\wedge\tau}\mathcal{V}(S_{\cdot\wedge\tau}, Y_{\cdot\wedge\tau}) + \int_0^{\cdot\wedge\tau} e^{-\delta s}u(c_s)\mathrm{d}s$ is a local supermartingale. Let $\{\tau_n : n = 1, 2, 3, \cdots\}$ be a localization sequence for its local martingale part such that $\lim_{n\to\infty}\tau_n = \infty$. We have

$$\begin{aligned}\mathcal{V}(S_0, Y_0) \geq& \mathbb{E}\left[e^{-\delta(\tau_n\wedge\tau)}\mathcal{V}(S_{\tau_n\wedge\tau}, Y_{\tau_n\wedge\tau}) + \int_0^{\tau_n\wedge\tau} e^{-\delta s}u(c_s)\mathrm{d}s\right]\\ =&\mathbb{E}\Big[e^{-\delta\tau}u((r-\lambda)L(S_\tau) - \ell)1_{\{\tau\leq\tau_n\}}\Big] + \mathbb{E}\Big[e^{-\delta\tau_n}\mathcal{V}(S_{\tau_n}, Y_{\tau_n})1_{\{\tau_n<\tau\}}\Big]\\ &+\mathbb{E}\left[\int_0^{\tau_n\wedge\tau} e^{-\delta s}u(c_s)\mathrm{d}s\right].\end{aligned} \tag{B.19}$$

Sending $n\to\infty$ on the right-hand side of (B.19) and applying (B.18) to the second term, we obtain

$$\mathcal{V}(S_0, Y_0) \geq \mathbb{E}\left[e^{-\delta\tau}u((r-\lambda)L(S_\tau) - \ell) + \int_0^{\tau} e^{-\delta s}u(c_s)\mathrm{d}s\right]. \tag{B.20}$$

When the principal chooses the strategy $(c^*, \pi^*, I^*, Z^*, \zeta^*)$, due to the choice of $(c^*, \pi^*, I^*, Z^*, \zeta^*)$ in the statement of Proposition 4.1 and (33), the inequality in (B.19) is an equality. Hence (B.20) is an equality as well, which verifies the optimality of $(c^*, \pi^*, I^*, Z^*, \zeta^*)$.

To prove the admissibility of $(c^*, \pi^*, I^*, Z^*, \zeta^*)$, combining (13) and the comparison theorem for HJB equation (33), we obtain $\mathcal{V}(S, Y) \geq \mathcal{V}^Q(S)$, where $\mathcal{V}^Q$ is the principal's value given in Section B.1 when the principal uses the quiet life contract. Under the quiet life contract, the optimal investment strategy is $\pi^* = \frac{m-r}{r\gamma\sigma^2} - \frac{\rho\psi}{\sigma}K$ and the optimal consumption rate is $c^* = (r-\lambda)(S+\Phi^Q) - \ell - \frac{1}{\gamma}\ln\left(\frac{r-\lambda}{\delta}\right)$. Then calculation shows that the firm size follows

$$\mathrm{d}S_t = \left[\frac{1}{\gamma}\left(1-\frac{\delta}{r-\lambda}\right) + \frac{(m-r)^2}{2(r-\lambda)\gamma\sigma^2} + \frac{\gamma(r-\lambda)}{2}(1-\rho^2)\psi^2K^2\right]\mathrm{d}t + \frac{m-r}{(r-\lambda)\gamma\sigma}\mathrm{d}B_t + \sqrt{1-\rho^2}\psi K\mathrm{d}B_t^{\perp}. \tag{B.21}$$

Combining the previous equation with (B.2), we obtain

$$e^{-\delta\tilde{\tau}}\mathcal{V}^Q(S_{\tilde{\tau}}) = -\frac{1}{\gamma}e^{-(r-\lambda)\tilde{\tau}}e^{\gamma(\ell-(r-\lambda)L^Q)}\mathcal{E}\left(\int_0^{\cdot} -\frac{m-r}{\sigma}\mathrm{d}B_t - \gamma(r-\lambda))\sqrt{1-\rho^2}\psi K\mathrm{d}B_t^{\perp}\right)_{\tilde{\tau}}, \tag{B.22}$$

where $\mathcal{E}(\cdot)$ is a supermartingale with its expectation less than 1. Therefore, $r > \lambda$ implies

$$\lim_{\tilde{\tau}\to\infty}\mathbb{E}[e^{-\delta\tilde{\tau}}\mathcal{V}^Q(S_{\tilde{\tau}})] = 0, \tag{B.23}$$

which confirms (B.18) because $\mathcal{V}^Q(S) \leq \mathcal{V}(S,Y) \leq 0$.

B.6 Proof of Proposition 5.1

Given the capital structure in Proposition 5.1, the budget constraint

$$\frac{b}{1+b}\Big((1+b)\mathrm{d}X_t - \mathrm{d}L_t\Big) + x\mathrm{d}t + (r-\lambda)(Y_t - U_t^*)\mathrm{d}t + U_t^*(\mathrm{d}R_t - \lambda\mathrm{d}t) = \mathrm{d}Y_t \tag{B.24}$$

holds in the implementation. Calculation shows that $\mathrm{d}Y$ evolves according to (22) with $Z \equiv b$ and $\zeta = U^* + \frac{\rho\psi}{\sigma}Kb$. The risk management account receives

$$\frac{1-b}{1+b}\,\mathrm{d}L + \frac{1}{1+b}\big((1+b)\,\mathrm{d}X - \mathrm{d}L\big) = \mathrm{d}X - \mathrm{d}I, \tag{B.25}$$

invests π^* in the market portfolio and remaining value $S - K - \pi^*$ into the risk-free bond, and pays out to (or receives from) the general partner at a continuous rate c^*. The value of the firm, which is the sum of the production capital K and the value of the risk management account, follows the dynamics (4).

B.7 Proof of Proposition 6.1

For an agent's strategy $(e, \bar{c}, \bar{\pi}, \bar{\tau})$, define

$$\mathcal{U}_t^{e,\bar{c},\bar{\pi},\bar{\tau}} = -\frac{1}{\bar{\gamma}} e^{-\bar{\gamma}\left(r\bar{W}_t - \bar{\ell} + rY_t\right)}, \tag{B.26}$$

where $\bar{W}$ satisfies (4) and Y follows (22) with the optimal contract sensitivities Z^* and ζ^*. We call the strategy $(e, \bar{c}, \bar{\pi}, \bar{\tau})$ admissible if $\mathcal{U}^{e,\bar{c},\bar{\pi},\bar{\tau}}$ satisfies the following transversality condition

$$\lim_{\nu \to \infty} \mathbb{E}\left[e^{-\bar{\delta}\nu} \mathcal{U}_\nu^{e,\bar{c},\bar{\pi},\bar{\tau}} 1_{\{\nu \le \bar{\tau}\}}\right] = 0, \quad \text{for any stopping time } \nu. \tag{B.27}$$

Consider the process

$$\tilde{\mathcal{U}}_t = \int_0^t e^{-\bar{\delta}s} \bar{u}(\bar{c}_s) \mathrm{d}s + e^{-\bar{\delta}t} \mathcal{U}_t^{e,\bar{c},\bar{\pi},\bar{\tau}}. \tag{B.28}$$

The definition of the dynamics for Y ensures that the drift of $\tilde{\mathcal{U}}$ is non-positive. Hence $\tilde{\mathcal{U}}$ is a local supermartingale. Let $\{\tau_n' : n = 1, 2, 3, \dots\}$ be a localization sequence for its local martingale part such that $\lim_{n \to \infty} \tau_n' = \infty$ a.s.. We have

$$\mathbb{E}\left[\int_0^{\tau_n' \wedge \bar{\tau}} e^{-\bar{\delta}s} \bar{u}(\bar{c}_s) \mathrm{d}s\right] + \mathbb{E}\left[e^{-\bar{\delta}\tau_n' \wedge \bar{\tau}} \mathcal{U}_{\tau_n' \wedge \bar{\tau}}^{e,\bar{c},\bar{\pi},\bar{\tau}}\right] \le \tilde{\mathcal{U}}_0 = \mathcal{U}_0^{e,\bar{c},\bar{\pi},\bar{\tau}} = -\frac{1}{\bar{\gamma}} e^{-\bar{\gamma}\left(r\bar{W}_0 - \bar{\ell} + rY_0\right)}. \tag{B.29}$$

The second term on the left-hand side above can be decomposed into

$$\mathbb{E}\left[e^{-\bar{\delta}\tau_n'} \mathcal{U}_{\tau_n'}^{e,\bar{c},\bar{\pi},\bar{\tau}} 1_{\{\tau_n' \le \bar{\tau}\}}\right] + \mathbb{E}\left[e^{-\bar{\delta}\bar{\tau}} \mathcal{U}_{\bar{\tau}}^{e,\bar{c},\bar{\pi},\bar{\tau}} 1_{\{\bar{\tau} < \tau_n'\}}\right]. \tag{B.30}$$

Because $\bar{\tau} \le \tau$, $Y_{\bar{\tau}} \ge 0$, therefore

$$\mathcal{U}_{\bar{\tau}}^{e,\bar{c},\bar{\pi},\bar{\tau}} \ge -\frac{1}{\bar{\gamma}} e^{-\bar{\gamma}(r\bar{W}_{\bar{\tau}}-\bar{\ell})} = \bar{u}(r\bar{W}_{\bar{\tau}} - \bar{\ell}). \tag{B.31}$$

Sending $n \to \infty$ and using monotone convergence theorem, together with (B.27) and (B.31), we obtain

$$\mathbb{E}\left[\int_0^{\bar{\tau}} e^{-\bar{\delta}s}\bar{u}(\bar{c}_s)\mathrm{d}s + e^{-\bar{\delta}\bar{\tau}}\bar{u}(r\bar{W}_{\bar{\tau}} - \bar{\ell})\right] \le -\frac{1}{\bar{\gamma}} e^{-\bar{\gamma}\left(r\bar{W}_0-\bar{\ell}+rY_0\right)}. \tag{B.32}$$

For the strategy $(e^*, \bar{c}^*, \bar{\pi}^*, \tau)$, where τ is defined in (16), both inequalities (B.31) and (B.32) become equalities when $\bar{\tau}$ is replaced by τ. Therefore τ is the optimal retirement time for the agent.

B.8 A no-saving and no-investment implementation

The revelation principle is widely used in the literature to deal with the agent's private saving.[32] The following result presents a contract implementation where the agent delegates the principal to manage his private wealth until his contract termination, consumes all compensation immediately after deducting effort cost, and does not save nor invest privately. Comparing with the compensation in Proposition 4.1, compensation in this no-saving and no-investment implementation can be negative.

Proposition B.1 *Any contract (Y, Z, ζ, I), under which the agent may save and invest privately, can be implemented by a no-saving and no-investment contract. Under such a contract, the agent truthfully transfers to the principal all his private wealth initially, the principal saves and invests the agent's wealth $\bar{W}$ according to $\bar{\pi}^*$ in (20), deposits I into $\bar{W}$, draws compensation flows $\tilde{c}^* = \bar{c}^* + h(e^*)$ from $\bar{W}$ to pay the agent, and finally transfers $\bar{W}_\tau$ back to the agent at the contract termination time τ. Both the agent and the principal obtain the same value under these two types*

[32] See Lemma 1 in DeMarzo and Sannikov (2006), Lemma 2 in He (2011), Lemma 1 in He (2012), and Lemma 19 in Di Tella and Sannikov (2020).

of contracts. Under the no-saving and no-investment contract, the agent does not save or invest and consumes at the rate of $\bar{c}^* = \tilde{c}^* - h(e^*)$.

Proof. Given a contract (Y, Z, ζ, I), suppose that the agent initially transfers $\bar{W}_0$ to the principal and let the principal saves and invests on his behalf. After deposing I and paying the agent at the rate of $\tilde{c} = \bar{c}^* + h(e^*)$, the value of the account follows (23). Define agent's promised utility by

$$\mathcal{U}_t = -\frac{1}{\bar{\gamma}} e^{-\bar{\gamma}\left(r\bar{W}_t - \bar{\ell} + rY_t\right)}, \tag{B.33}$$

where Y follows (22). Then calculation shows that

$$\mathrm{d}\mathcal{U}_t = -(r - \bar{\delta})\mathcal{U}_t \mathrm{d}t - \frac{m-r}{\sigma}\mathcal{U}_t \mathrm{d}B_t - r\bar{\gamma} Z_t \sqrt{1-\rho^2}\psi K \mathcal{U}_t \mathrm{d}B_t^{\perp}. \tag{B.34}$$

Before contract termination, suppose that the agent deviates from the principal's recommendation, exerts effort e' and maintains a private savings account $\bar{W}'$. The agent saves the compensation net effort cost $\tilde{c} - h(e')$ into his private account, invests privately $\bar{\pi}'$ in the market portfolio, puts aside the balance in the risk-free bond, and draws compensation flows $\bar{c}'$ from it. Then $\bar{W}'$ follows

$$\mathrm{d}\bar{W}_t' = r\bar{W}_t' \mathrm{d}t + \bar{\pi}_t'(m-r)\mathrm{d}t + \bar{\pi}_t'\sigma \mathrm{d}B_t + \tilde{c}_t \mathrm{d}t - h(e_t')\mathrm{d}t - \bar{c}_t' \mathrm{d}t, \quad 0 \le t < \tau. \tag{B.35}$$

Under $\mathbb{P}^{e'}$ induced by the agent's effort e', $\mathcal{U}$ must follow

$$\mathrm{d}\mathcal{U}_t = -\left[r - \bar{\delta} + r\bar{\gamma}(e_t' - e_t^*)Z_t K\right]\mathcal{U}_t \mathrm{d}t - \tfrac{m-r}{\sigma}\mathcal{U}_t \mathrm{d}B_t - r\bar{\gamma} Z_t \sqrt{1-\rho^2}\psi K \mathcal{U}_t \mathrm{d}B_t^{\perp,e'}, \tag{B.36}$$

where $B^{\perp,e'} = B^{\perp} - \frac{1}{\psi\sqrt{1-\rho^2}}\int_0^{\cdot}(e_t' - e_t^*)\mathrm{d}t$ is a standard Brownian motion under $\mathbb{P}^{e'}$.

Using $\bar{W}'$ and $\mathcal{U}$ as state variables, the agent's optimization problem is

$$\mathcal{U}'(\bar{W}'_t,\mathcal{U}_t) = \sup_{A',\bar{c}',\bar{\pi}'} \mathbb{E}_t \left[\bar{\delta}\int_t^\tau e^{-\bar{\delta}(s-t)}\bar{u}(\bar{c}'_s)\mathrm{d}s + e^{-\bar{\delta}(\tau-t)}\bar{u}(r(\bar{W}_\tau + \bar{W}'_\tau) - \bar{\ell})\right], \tag{B.37}$$

where $\bar{W}_\tau$ is transferred back to the agent at the termination time. The HJB equation associated with this problem is

$$\bar{\delta}\mathcal{U}'(\bar{W}',\mathcal{U}) = \sup_{e',\bar{c}',\bar{\pi}'} \Big\{\bar{\delta}\bar{u}(\bar{c}') + [r\bar{W}' + \bar{\pi}'(m-r) + \tilde{c} - h(e') - \bar{c}'] \tfrac{\partial \mathcal{U}'}{\partial \bar{W}'} + \left[\bar{\delta} - r - r\bar{\gamma}(e'-e^*)ZK\right]\mathcal{U}\tfrac{\partial \mathcal{U}'}{\partial \mathcal{U}}$$
$$+ \tfrac{1}{2}\bar{\pi}'^2\sigma^2\tfrac{\partial^2\mathcal{U}'}{\partial \bar{W}'^2} - \bar{\pi}'(m-r)\mathcal{U}\tfrac{\partial^2\mathcal{U}'}{\partial \bar{W}'\partial\mathcal{U}} + \tfrac{1}{2}\left[\tfrac{(m-r)^2}{\sigma^2} + r^2\bar{\gamma}^2Z^2(1-\rho^2)\psi^2K^2\right]\mathcal{U}^2\tfrac{\partial^2\mathcal{U}'}{\partial\mathcal{U}^2}\Big\}. \tag{B.38}$$

The solution to the above HJB equation takes the form of

$$\mathcal{U}'(\bar{W}',\mathcal{U}) = \mathcal{U}e^{-r\bar{\gamma}\bar{W}'}. \tag{B.39}$$

Moreover the optimal strategy for the agent is $(e',\bar{c}',\bar{\pi}') = (e^*, \tilde{c} - h(e^*) + r\bar{W}', 0)$. Replace $\bar{\pi}' = 0$ and $\bar{c}' = \tilde{c} - h(e^*) + r\bar{W}'$ into (B.35), and we show that $\mathrm{d}\bar{W}' \equiv 0$ and hence $\bar{W}' \equiv \bar{W}'_0 = 0$. Combining (B.33) and (B.39), we obtain $\mathcal{U}'(\bar{W}'_t,\mathcal{U}_t) = \frac{1}{\bar{\gamma}}e^{-\bar{\gamma}\left(r\bar{W}_t - \bar{\ell} + rY_t\right)}$, which is exactly the agent's promised continuation value.

Therefore, under the truth-telling contract, the agent has no incentive to deviate from the principal's recommendation and saves and invests in a private account. He achieves the same value as in the OPE contact. The principal's optimization problem is also the same under the no-saving and no-investment contract. Hence the principal's value is the same.

Similar to Proposition 6.1, if the agent has the discretion to choose his own retirement time, $\tau = \inf\{t \geq 0 : Y_t \leq 0\}$ is the optimal time for the agent to recall his private wealth $\bar{W}$ from the principal and terminate the no-saving and no-investment contract. Therefore, agent's relative contract value Y, rather than agent's total certainty equivalent, is still a state variable for the principal under the no-saving and no-investment implementation. Moreover, due to agent's CARA utility, the transfer $\tilde{c}$ to the agent may be negative under the no-saving and no-investment implementation, while the transfer I in the optimal contract of Section 4 is always nonnegative due to agent's limited liability protection.

C Four extensions

C.1 Capital investment

Instead of constant capital in the baseline model, the firm is allowed to invest in the capital in this section. With an investment rate i, the production capital follows the dynamics

$$\mathrm{d}K_t = (i_t - \delta K_t)\mathrm{d}t, \tag{C.1}$$

where δ is the capital depreciation rate. The firm size follows

$$\mathrm{d}S_t = (r-\lambda)\big(S_t - K_t - \pi_t\big)\mathrm{d}t + \pi_t(m-\lambda)\mathrm{d}t + \pi_t\sigma\mathrm{d}B_t + K_t\mathrm{d}A_t - \psi(i_t)\mathrm{d}t - \mathrm{d}I_t - c_t\mathrm{d}t, \tag{C.2}$$

where $\psi(i)$ is the adjustment cost measured in monetary unit. The capital K, hence the investment i, is observable to the principal. Principal's problem is still described by (11) with i an additional control variable for the principal. For a given contract, agent's optimal strategies are the same as in Section 3. The principal takes K as a state variable, in addition to S and Y. The HJB equation (33) is transformed to

$$\delta\mathcal{V}(k,s,y) = \sup_{c,\pi,i,Z,\zeta}\big\{\delta u(c) + \mathcal{L}\,\mathcal{V}(k,s,y)\big\}, \tag{C.3}$$

where the infinitesimal generator $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}\mathcal{V}(k,s,y) =& \Big[(r-\lambda)(s-K) - \tfrac{\rho\psi}{\sigma}K(m-r) + Ke^*(Z) - c - \psi(i)\Big]\mathcal{V}_s + \Big[i - \delta k\Big]\mathcal{V}_k \\ &+ \Big[ry + \tfrac{r\bar{\gamma}}{2}Z^2(1-\rho^2)\psi^2K^2 + h(e^*(Z))\Big]\mathcal{V}_y \\ &+ \tfrac{1}{2}(1-\rho^2)\psi^2K^2\left(1 + Z\tfrac{\mathcal{V}_{sy}}{\mathcal{V}_{ss}}\right)^2\mathcal{V}_{ss} + \tfrac{1}{2}Z^2(1-\rho^2)\psi^2K^2\left(\mathcal{V}_{yy} - \tfrac{\mathcal{V}_{sy}^2}{\mathcal{V}_{ss}}\right) \\ &+ \left(\pi + \tfrac{\rho\psi}{\sigma}K + \zeta\tfrac{\mathcal{V}_{sy}}{\mathcal{V}_{ss}}\right)(m-r)\mathcal{V}_s + \tfrac{1}{2}\left(\pi + \tfrac{\rho\psi}{\sigma}K + \zeta\tfrac{\mathcal{V}_{sy}}{\mathcal{V}_{ss}}\right)^2\sigma^2\mathcal{V}_{ss} \\ &+ \zeta(m-r)\mathcal{V}_y - \zeta(m-r)\tfrac{\mathcal{V}_{sy}}{\mathcal{V}_{ss}}\mathcal{V}_s + \tfrac{1}{2}\zeta^2\sigma^2\Big(\mathcal{V}_{yy} - \tfrac{\mathcal{V}_{sy}^2}{\mathcal{V}_{ss}}\Big).\end{aligned} \tag{C.4}$$

The optimal i is given by the first order condition $\mathcal{V}_k = \psi'(i)\mathcal{V}_s$. But the maximization problems in π and ζ remain the same as (28) and (29). Therefore the associated trade-off between contract market exposure and firm risk management remains the same qualitatively.

C.2 Leveraged firm and termination cost to creditors

Right before the firm termination time τ, the firm possesses capital K, market portfolio π_τ, and bond $S_\tau - K - \pi_\tau$. In the state of the world where $S_\tau - K - \pi_\tau < 0$, the principal needs to sell capital and the market portfolio to repay the bond creditors. Suppose that the recovery rate of capital fire sell is $q \in (0,1)$ and selling the market portfolio can obtain its full market value, then after repaying bond creditors with liquidation proceeds, the principal's net position in bond is $S_\tau - (1-q)K$. When $S_\tau - (1-q)K \geq 0$, the principal's liquidation value is $S_\tau - (1-q)K$. When $S_\tau - (1-q)K < 0$, there is still outstanding debt with bond creditors. Suppose that the principal negotiates with creditors and obtain a reduction of debt so that only $h \in (0,1)$ of outstanding debt is paid back to creditors.[33] Then principal's liquidation value in this scenario

[33] If the creditors demands a coupon rate which is higher than the risk-free rate for the risky debt, the principal is exposed to a market risk premium which is lower than what the agent has access to. Therefore, the principal has more incentive to substitute firm risk management with the contract market exposure. A similar, but more extreme case, is in Proposition 4.2 where the principal does not invest the market portfolio for firm risk management.

is $h\big(S_\tau - (1-q)K\big)$. In summary, principal's liquidation value is

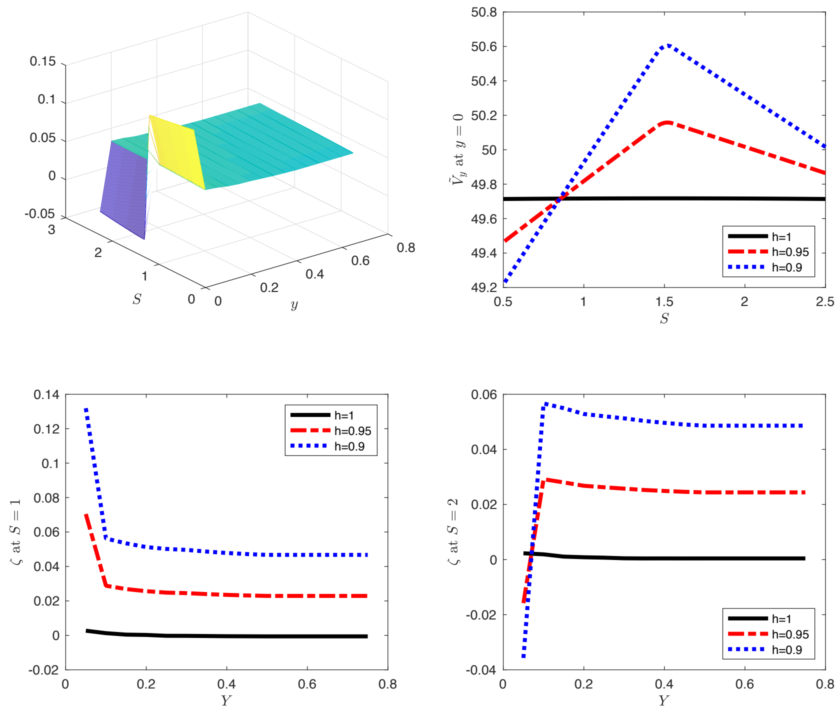
$$L(S) = \begin{cases} S - (1-q)K, & S - (1-q)K \geq 0 \\ h\big(S - (1-q)K\big), & S - (1-q)K < 0 \end{cases} \tag{C.5}$$

$$= \max\big\{S - (1-q)K, 0\big\} + h\min\big\{S - (1-q)K, 0\big\}.$$

When $h = 0$, the principal is protected by limited liability; when $h = 1$, the principal has unlimited liability. When $h < 1$ and $S-(1-q)K < 0$, the liquidation cost is $(1-h)S+h(1-q)K$ whose marginal value with respect to firm size is $1-h > 0$. Therefore each additional monetary value in firm size loses $1-h$ at liquidation and this proportional loss is larger when h is smaller. This motivates the principal to load market exposure in the contract.

Figure A1 presents numeric examples where $K = 3$, $q = 0.5$, and $h \in \{0.9, 0.95, 1\}$. When $S < 1.5$ at firm termination, the principal needs to pay $h(1.5 - S)$ to the bond creditors. Top left panel of Figure A1 shows the optimal contract market exposure ζ^* when $h = 0.9$. When Y is close to zero and $S < 1.5$, marginal liquidation cost with respect to firm size is positive. To mitigate the adverse impact on firm growth, the principal loads the contract positively on market exposure. In most part of the state space, ζ^* is positive, except when Y is extremely close to zero and $S > 1.5$. In this case, termination is imminent, rather than risk the firm size dropping below 1.5 where marginal liquidation cost is positive, the principal strategically exposes the contract negatively on market risk. The resulting negative drift $\zeta^*(m - r)$ in the dynamics of Y leads the termination more likely to happen at $S > 1.5$ where the termination cost is constant rather than at $S < 1.5$ where the termination cost is size-dependent. The choice of ζ^* close to the termination boundary is consistent with the sign of V_{sy} at $y = 0$. Top right panel shows that $V_{sy} > 0$ when $S < 1.5$ and $V_{sy} > 0$ when $S > 1.5$ for $h < 1$ cases, and $V_{sy} \equiv 0$ for $h = 1$ (constant liquidation cost). The bottom panels show ζ^* at $S = 1$ and $S = 2$. The smaller h is, the more inefficient liquidation is at $S = 1$, hence the larger ζ^* is. The bottom right panel also shows the principal's strategic termination behavior becomes more significant when h is smaller and Y is close to zero.

Figure A1



In this figure, $K = 3$, $q = 0.5$, $h = 1, 0.95$, or 0.9. All the other parameters are the same as in Figure 2.

C.3 Quadratic effort cost

In this section, we consider the model where the agent's effort cost is quadratic, i.e., $h(e) = \left(\frac{\kappa}{2}Ae^2 + be\right)K$ with $\kappa > 0$ and $b \geq 0$. Facing a quadratic effort cost, the agent's optimal effort, determined by minimizing $h(e) - ZeK$, is

$$e^*(Z) = \frac{Z - b}{\kappa} \tag{C.6}$$

and the effort cost is $h(A^*(Z)) = \frac{Z^2-b^2}{2\kappa}K$. The statement of Proposition 4.1 still holds. The only difference is that the principal chooses Z^* to maximize

$$\tfrac{Z-b}{\kappa}K\mathcal{V}_s + \tfrac{Z^2-b^2}{2\kappa}K\mathcal{V}_y + (1-\rho^2)\psi^2K^2\left(\tfrac{r\bar{\gamma}}{2}Z^2\mathcal{V}_y + Z\mathcal{V}_{sy} + \tfrac{1}{2}Z^2\mathcal{V}_{yy}\right). \tag{C.7}$$

The principal's optimal consumption rate c^*, investment position π^*, and optimal contract sensitivity ζ^* are still given by (35), (36), and (37), respectively.

We present a numerical solution to the model with a quadratic effort cost in Figure A2. When the effort cost is quadratic, Panel C shows that the principal decreases the contract sensitivity to firm cash flow near the termination boundary. This tunes down the agent's effort while still maintaining the incentive compatible constraint. More importantly, this reduces the volatility of agents relative contract value, hence decreases the probability of termination. Panel D shows that the optimal market exposure is positive when the firm liquidation cost is proportional to its size. Therefore, our results are robust under difference choice of effort cost.

C.4 Contractible managerial investment

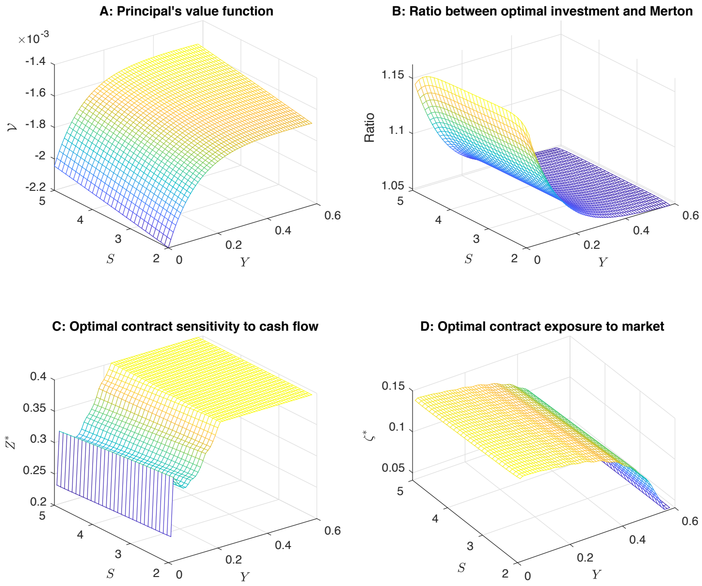
If the agent's investment in market portfolio $\bar{\pi}$ is contractible, then the principal can prescribe the agent's portfolio choice without satisfying (20). Assume that the agent can still trade the market portfolio freely after retirement, her optimization problem is

$$-\frac{1}{\bar{\gamma}}e^{-\bar{\gamma}\left(r\bar{W}_t-\bar{\ell}+rY_t\right)} = \mathcal{U}_t = \sup_{e,\bar{c}}\mathbb{E}_t\left[\bar{\delta}\int_t^\tau e^{-\bar{\delta}(s-t)}\bar{u}(\bar{c}_s)\mathrm{d}s + e^{-\bar{\delta}(\tau-t)}\bar{u}(r\bar{W}_\tau-\bar{\ell})\right], \tag{C.8}$$

subject to (4). The agent's investment $\bar{\pi}$ in the market portfolio becomes another control variable for principal's problem (11).

Addressing the agent's problem by the same methodology as in Section 3, we show that the agent's optimal effort and consumption are still given by (19) and (21), respectively. The

Figure A2



In this figure, $h(e) = e^2$ where $e \in [0, 20\%]$. All the other parameters are the same as in Figure 2.

agent's relative contract value follows

$$dY_t = \Big[rY_t + \tfrac{r\bar{\gamma}}{2} Z_t^2 (1-\rho^2)\psi^2 K^2 + h(e_t^*) + \zeta_t(m-r) + \tfrac{r\bar{\gamma}}{2}\big(\bar{\pi}_t - \bar{\pi}^*(\zeta_t)\big)^2 \sigma^2\Big] \mathrm{d}t \\ - dI_t + \zeta_t \sigma \mathrm{d}B_t + Z_t\sqrt{1-\rho^2}\psi \mathrm{d}B_t^{\perp}, \tag{C.9}$$

where $\bar{\pi}^*(\zeta) = \frac{m-r}{r\bar{\gamma}\sigma^2} - \zeta$. The contractibility of $\bar{\pi}$ gives rise to another term $\frac{r\bar{\gamma}}{2}(\bar{\pi}_t - \bar{\pi}^*(\zeta_t))^2$ that the principal can control in the dynamics of agent's relative contract value Y. A positive $\frac{r\bar{\gamma}}{2}(\bar{\pi}_t - \bar{\pi}^*(\zeta_t))^2$ increases the growth of Y comparing to that in the unobservable managerial hedging case. This positive term further helps to offset negative shocks to reduce the likelihood of Y reaching the termination boundary.

The principal's problem can be solved similarly as in Section 4.1. The difference is that, instead of (29), the principal chooses $\bar{\pi}$ and ζ to maximize

$$\zeta(m-r)\Big(\mathcal{V}_y - \tfrac{\mathcal{V}_{sy}}{\mathcal{V}_{ss}}\mathcal{V}_s\Big) + \tfrac{r\bar{\gamma}}{2}\big(\bar{\pi} - \bar{\pi}^*(\zeta)\big)^2\sigma^2\mathcal{V}_y + \tfrac{1}{2}\zeta^2\sigma^2\Big(\mathcal{V}_{yy} - \tfrac{\mathcal{V}_{sy}^2}{\mathcal{V}_{ss}}\Big). \tag{C.10}$$

The term $\frac{r\bar{\gamma}}{2}\big(\bar{\pi}-\bar{\pi}^*(\zeta)\big)^2\sigma^2\mathcal{V}_y$ is the benefit of imposing on the agent an investment strategy $\bar{\pi}$ which is different from his optimal investment strategy in (20). If $\mathcal{V}_y \le 0$, the agent's investment incentive is aligned with the interest of the principal. The principal optimally choose $\bar{\pi} = \bar{\pi}^*(\zeta)$ to maximize the objective in (C.10). However, when $\mathcal{V}_y > 0$, deviating from $\bar{\pi}^*(\zeta)$ makes the principal better off. The resulting optimal contract introduces higher value for the principal than the optimal value in the case where the agent's investment is hidden and not contractible.

D Details of the numerical method

We use the finite difference method to solve the nonlinear PDE like (33). First, we choose a grid on the state space (s, y). We need the range of y to be sufficiently large to include the free boundary. We treat the free boundary condition $\mathcal{V}_s + \mathcal{V}_y = 0$ by introducing a penalty term to the right-hand side of (33). The penalty term $p(\cdot)$ is chosen such that $p(\mathcal{V}_s + \mathcal{V}_y) = 0$ when $\mathcal{V}_s + \mathcal{V}_y \ge 0$ and $p(\mathcal{V}_s + \mathcal{V}_y) = \frac{(\mathcal{V}_s+\mathcal{V}_y)^2}{\epsilon}$ when $\mathcal{V}_s + \mathcal{V}_y < 0$, for a sufficiently small ϵ.

Starting from an initial guess of $\mathcal{V}$, we first calculate (c^*, π^*, ζ^*) using their associated first order conditions and $\mathcal{V}$. Then we update $\mathcal{V}$ by solving the elliptic PDE (33). As for the boundary conditions, we have $\mathcal{V}(s, 0) = u\big((r-\lambda)L(s)-\ell\big)$ at the termination boundary. On the upper and lower boundaries of s, we assume that each of them is approximated by the constant liquidation cost case in Online Appendix A.3. We solve the ODE (A.12) to obtain V by bisection-shooting method where the cost is $C = s - L(s)$ given s at the upper and lower boundaries, and obtain $\mathcal{V}(s, y) = -\frac{1}{\gamma}e^{-\gamma((r-\lambda)(s+V(y))-\ell)}$. Once $\mathcal{V}$ is updated, we update (c^*, π^*, ζ^*) again. Repeat the policy iteration until the difference between the new and old values of $\mathcal{V}$ is sufficiently small.