Online Appendix to

"Managing Liquidity" in *Management Science*

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PROOF OF PROPOSITIONS AND LEMMAS

Proof of Lemma 1:

Substituting (2) into (1) and (4) into (3), we have

$$W_j(z_t) = z_t - \tau_t + \max_{m_{jt},a_{jt},k_{jt}} \{g(k_{jt}) - q_t k_{jt} - m_{jt} - a_{jt} + V_j(m_{jt}, a_{jt}, k_{jt})\} \tag{A1}$$

$$W(z_t) = z_t - \tau_t + \max_{m_t,a_t,k_t} \{g(k_t) - q_t k_t - m_t - a_t + \rho V_0(m_t, a_t, k_t) + (1-\rho) V_1(m_t, a_t, k_t)\} \tag{A2}$$

The first-order conditions for k_{jt} in (A1) and the first-order condition for k_t in (A2) are respectively

$$\begin{aligned} g'(k_{jt}) - q_t + \frac{\partial}{\partial k_j} V_j(m_{jt}, a_{jt}, k_{jt}) &\leq 0, \quad = 0 \text{ if } k_{jt} > 0, && \text{(A3)} \\ g'(k_t) - q_t + \rho \frac{\partial}{\partial k} V_0(m_t, a_t, k_t) + (1-\rho) \frac{\partial}{\partial k} V_1(m_t, a_t, k_t) &\leq 0, \quad = 0 \text{ if } k_t > 0. && \text{(A4)} \end{aligned}$$

Applying the envelope theorem to (5), (A1), and (A2) yields

$$\begin{aligned} \frac{\partial}{\partial k_j} V_j(m_{jt}, a_{jt}, k_{jt}) &= \beta \left\{ \theta \left[\rho \frac{\partial}{\partial k_j} W_0(z_{j,t+1}) + (1-\rho) \frac{\partial}{\partial k_j} W_1(z_{j,t+1}) \right] + (1-\theta) \frac{\partial}{\partial k_j} W(z_{j,t+1}) \right\} \\ &= \beta q_{t+1}, \end{aligned}$$

and

$$\frac{\partial}{\partial k} V_j(m_t, a_t, k_t) = \beta q_{t+1}.$$

Inada conditions on $g(\cdot)$ guarantee $k_{jt}, k_t > 0$. Hence both (A3) and (A4) hold with equality, implying $k_{1t} = k_{2t} = k_t$. Therefore, (14) obtains for buyers either with or without perfect information. Below, we analyze the agents' consumption and liquidity-holding decisions. We omit the time subscripts to avoid the cluttering of notations.

Perfect-information buyers

The first-order conditions for m_j and a_j in (A1) are respectively

$$-1+\frac{\partial}{\partial m_j}V_j(m_j,a_j,k_j) \leq 0, \quad =0 \text{ if } m_j>0, \tag{A5}$$

$$-1+\frac{\partial}{\partial a_j}V_j(m_j,a_j,k_j) \leq 0, \quad =0 \text{ if } a_j>0. \tag{A6}$$

Recall that if liquidity is plentiful, i.e., $\beta(m_j/\pi+jra_j)\geq x^*$, then $x_j=x^*$; If liquidity is scarce, i.e., $\beta(m_j/\pi+jra_j)<x^*$, then $x_j=\beta(m_j/\pi+jra_j)$. Applying the envelope theorem to (5) and (A1) gives

$$\frac{\partial}{\partial m_j}V_j(m_j,a_j,k_j)=\frac{\beta}{\pi}, \qquad \frac{\partial}{\partial a_j}V_j(m_j,a_j,k_j)=\beta r$$

for the plentiful-liquidity case, and

$$\frac{\partial}{\partial m_j}V_j(m_j,a_j,k_j)=u'(x_j)\frac{\beta}{\pi}, \qquad \frac{\partial}{\partial a_j}V_j(m_j,a_j,k_j)=u'(x_j)j\beta r+(1-j)\beta r$$

for the scarce-liquidity case. Combining the above results yields

$$\begin{aligned}\frac{\partial}{\partial m_j}V_j(m_j,a_j,k_j) &= \frac{\beta}{\pi}\max\left\langle u'(x_j),1\right\rangle, \\ \frac{\partial}{\partial a_j}V_j(m_j,a_j,k_j) &= \beta r\max\left\langle ju'(x_j)+1-j,1\right\rangle.\end{aligned}$$

Plugging these back to (A5) and (A6) and rearranging, we arrive at (12) and (13). For the arbitrage condition (15), see footnote 16.

Imperfect information buyers

The first-order conditions for m and a in (A2) are

$$-1+\rho\frac{\partial}{\partial m}V_0(m,a,k)+(1-\rho)\frac{\partial}{\partial m}V_1(m,a,k) \leq 0, \quad =0 \text{ if } m>0, \tag{A7}$$

$$-1+\rho\frac{\partial}{\partial a}V_0(m,a,k)+(1-\rho)\frac{\partial}{\partial a}V_1(m,a,k) \leq 0, \quad =0 \text{ if } a>0. \tag{A8}$$

Applying the envelope theorem, we have

$$\begin{aligned}\frac{\partial}{\partial m}V_j(m,a,k) &= \frac{\beta}{\pi}\max\left\langle u'(\tilde{x}_j),1\right\rangle \\ \frac{\partial}{\partial a}V_j(m,a,k) &= \beta r\max\left\langle ju'(\tilde{x}_j)+1-j,1\right\rangle\end{aligned}$$

where $\tilde{x}_j$ denotes DM consumption of an imperfect-information buyer in type-j meeting. Plugging the above back to (A7) and (A8) and rearranging, we obtain

$$\rho\left[u'(\tilde{x}_{0t})-1\right]+(1-\rho)\left[u'(\tilde{x}_{1t})-1\right]\leq\frac{\pi_t}{\beta}-1, \ = \ \text{if } m_t>0 \tag{16}$$

$$(1-\rho)\left[u'(\tilde{x}_{1t})-1\right] \le \frac{1}{\beta r_t}-1, \;=\; \text{if } a_t>0 \tag{17}$$

Inada conditions on $u(\cdot)$ imply that (16) holds with equality. Since $\tilde{x}_0=\tilde{x}_1=x_0$ in the case of single-liquidity holding, the equality version of (16) coincides with the equality version of (12) with $j=0$, which is simply equation (19). Mixed-liquidity holding requires (17) to hold with equality as well. Thus, $\tilde{x}_0$ and $\tilde{x}_1$ satisfy

$$u'(\tilde{x}_1)-1 \;=\; \frac{1/r-\beta}{(1-\rho)\beta}, \tag{A9}$$

$$u'(\tilde{x}_0)-1 \;=\; \frac{\pi-1/r}{\rho\beta}. \tag{A10}$$

One can easily show that the condition $\tilde{x}_1>\tilde{x}_0>x^*$ implies

$$r_t>\frac{1}{\rho\beta+(1-\rho)\pi_t}. \tag{18}$$

Q.E.D.

Proof of Lemma 2:

Let $W^e\left(k^e_{t-1},l^e_{t-1}\right)$ be the entrepreneur's value function for period t, where k^e_{t-1} is the existing capital stock and l^e_{t-1} is the previous-period loan. Using the flow-of-funds constraint (21) to substitute out c^e_t, the entrepreneur's Bellman equation is then

$$W^e\left(k^e_{t-1},l^e_{t-1}\right)=\max_{k^e_t,l^e_t}\left\{f\left(k^e_t\right)+q_tk^e_{t-1}+l^e_t-q_tk^e_t-r_{t-1}l^e_{t-1}+\eta W^e\left(k^e_t,l^e_t\right)\right\}$$

subject to the debt constraint (22). Let λ_t be the Lagrangian multiplier for (22). The first-order conditions with respect to l^e_t and k^e_t are

$$1+\eta\frac{\partial}{\partial l^e}W^e\left(k^e_t,l^e_t\right)-\lambda_t r_t=0,$$

$$f'\left(k^e_t\right)-q_t+\eta\frac{\partial}{\partial k^e}W^e\left(k^e_t,l^e_t\right)+\lambda_t\sigma^e q_{t+1}=0.$$

Substitution of the envelope conditions, $\partial W^e\left(k^e_{t-1},l^e_{t-1}\right)/\partial l^e=-r_{t-1}$ and $\partial W^e\left(k^e_{t-1},l^e_{t-1}\right)/\partial k^e=q_t$, into the above two conditions yield

$$1=\eta r_t+\lambda_t r_t, \tag{A11}$$

$$q_t=f'\left(k^e_t\right)+\zeta_t q_{t+1}, \tag{23}$$

where

$$\zeta_t\equiv\eta+\sigma^e\lambda_t.$$

Inserting $\lambda_t = 1/r_t - \eta$, as implied by (A11), into the expression for ζ_t gives (24). Given r_t that satisfies (15), we have $\lambda_t \geq \beta - \eta > 0$. Hence constraint (22) binds. *Q.E.D.*

Proof of Proposition 1:

Asset-market clearing implies $l^e + \bar{b} = \theta(1-\rho)a_1 + (1-\theta)a$. So the response of $l^e + \bar{b}$ to r can be seen from the responses of a_1 and a. Perfect-information buyers hold no liquid debt in type-0 meetings as long as r is strictly less than $1/\beta$. In type-1 meetings, a_1 and x_1 satisfy the equality version of (13) with $j = 1$ and $a_1 = x_1/(\beta r)$. It follows that

$$\frac{dx_1}{d(1/r)} = \frac{1}{\beta}\frac{1}{u''(x_1)} < 0,$$

$$\frac{da_1}{d(1/r)} = \frac{1}{\beta}\left[x_1 + \frac{u'(x_1)}{u''(x_1)}\right] = \frac{1}{\beta}\frac{u'(x_1)}{u''(x_1)}\left[x_1\frac{u''(x_1)}{u'(x_1)} + 1\right] < 0,$$

where the last inequality obtains due to the assumption that $-xu''(x)/u'(x) < 1$. Imperfect-information buyers demand positive amounts of liquid debt only in the mixed holding regime, where $\tilde{x}_0$ and $\tilde{x}_1$ satisfy (A9) and (A10), and $a = (\tilde{x}_1 - \tilde{x}_0)/(\beta r)$. Hence,

$$\frac{d\tilde{x}_0}{d(1/r)} = \frac{1}{\rho\beta}\frac{-1}{u''(\tilde{x}_0)} > 0,$$
$$\frac{d\tilde{x}_1}{d(1/r)} = \frac{1}{(1-\rho)\beta}\frac{1}{u''(\tilde{x}_1)} < 0,$$

and

$$\beta\frac{da}{d(1/r)} = \frac{u'(\tilde{x}_1)}{u''(\tilde{x}_1)}\left[\tilde{x}_1\frac{u''(\tilde{x}_1)}{u'(\tilde{x}_1)} + 1\right] + \frac{\rho}{1-\rho}\frac{1}{u''(\tilde{x}_1)} - \left[\tilde{x}_0 + \frac{1}{r}\frac{d\tilde{x}_0}{d(1/r)}\right] < 0.$$

Therefore,

$$\frac{d\left(l^e + \bar{b}\right)}{d(1/r)} = \frac{d\left[\theta(1-\rho)a_1 + (1-\theta)a\right]}{d(1/r)} < 0.$$

To prove $dl^e/d(1/r) > 0$, we first differentiate condition (26) to obtain

$$\frac{dk^e}{d(1/r)} = -\frac{\sigma^e g'(K-k^e)}{(1-\beta)f''(k^e) + (1-\zeta)g''(K-k^e)} > 0.$$

Then, we use (31) and (26) to rewrite the real quantity of private debt as

$$l^e = \frac{\sigma^e}{1-\beta}k^e g'(K-k^e)\frac{1}{r},$$

and obtain

$$\frac{dl^e}{d(1/r)} = \frac{\sigma^e}{1-\beta}\left\{k^e g'(K-k^e) + \frac{1}{r}\left[g'(K-k^e) + k^e g''(K-k^e)(-1)\right]\frac{dk^e}{d(1/r)}\right\} > 0$$

Finally, differentiating the asset-market clearing condition (30) yields

$$\frac{d\bar{b}}{d(1/r)} = \theta(1-\rho)\frac{da_1}{d(1/r)} + (1-\theta)\frac{da}{d(1/r)} - \frac{dl^e}{d(1/r)} < 0.$$

Q.E.D.

Proof of Proposition 2:

We have shown in the proof of Proposition 1 that $dk^e/dr < 0$. Then,

$$\frac{dS^{CM}}{dr} = \left[f'(k^e) - g'(K-k^e)\right]\frac{dk^e}{dr} = \frac{\beta-\zeta}{1-\beta}g'(K-k^e)\frac{dk^e}{dr}.$$

Obviously, $dS^{CM}/dr < 0$ when $\beta > \zeta = (1-\sigma^e)\eta + \sigma^e/r$, or when $r > r^{CM}$, where

$$r^{CM} \equiv \frac{\sigma^e}{\beta - (1-\sigma^e)\eta}.$$

Also, $dS^{CM}/dr > 0$ when $r < r^{CM}$. One can also show that $1/\beta - r^{CM} > 0$, and hence r^{CM} satisfies the arbitrage condition (15). *Q.E.D.*

Proof of Proposition 3:

Let (π^*, r^*) be a solution to the problem under investigation. We first show that $r^* < 1/\beta$. It would be more convenient to write (15) as $\beta \leq 1/r \leq \pi$. Let λ_r and λ_π be the Lagrangian multipliers associated with the first and second inequalities, respectively. The Kuhn-Tucker condition with respect to π is

$$\begin{aligned}\lambda_\pi &= -\theta\rho\left[u'(x_0)-1\right]\frac{\partial x_0}{\partial \pi} - \theta(1-\rho)\left[u'(x_1)-1\right]\frac{\partial x_1}{\partial \pi} \\ &\quad -(1-\theta)\rho\left[u'(\tilde{x}_0)-1\right]\frac{\partial \tilde{x}_0}{\partial \pi} - (1-\theta)(1-\rho)\left[u'(\tilde{x}_1)-1\right]\frac{\partial \tilde{x}_1}{\partial \pi}\end{aligned} \tag{A12}$$

with the complementary slackness condition:

$$\lambda_\pi\left(\pi - \frac{1}{r}\right) = 0.$$

The Kuhn-Tucker condition with respect to $1/r$ is

$$\begin{aligned}0 &= \left[f'(k^e) - g'(k)\right]\frac{dk^e}{d(1/r)} \\ &\quad +\theta\rho\left[u'(x_0)-1\right]\frac{\partial x_0}{\partial(1/r)} + \theta(1-\rho)\left[u'(x_1)-1\right]\frac{\partial x_1}{\partial(1/r)} \\ &\quad +(1-\theta)\rho\left[u'(\tilde{x}_0)-1\right]\frac{\partial \tilde{x}_0}{\partial(1/r)} + (1-\theta)(1-\rho)\left[u'(\tilde{x}_1)-1\right]\frac{\partial \tilde{x}_1}{\partial(1/r)} \\ &\quad +\lambda_r - \lambda_\pi\end{aligned} \tag{A13}$$

with the complementary slackness condition:

$$\lambda_r \left(\frac{1}{r} - \beta \right) = 0.$$

There are two cases to consider.

Case one. $\lambda_\pi = 0$. In this case the right-hand side of condition (A13) is strictly positive when evaluated at $r = 1/\beta$, since $f'(k^e) - g'(k) > 0$ and $u'(x_1) - 1 = u'(\tilde{x}_1) - 1 = 0$ when $r = 1/\beta$. Furthermore, $dk^e/d(1/r) > 0$ from (26), $u'(x_0) - 1, u'(\tilde{x}_0) - 1 \geq 0$, $\partial x_0/\partial(1/r) = 0$, $\partial \tilde{x}_0/\partial(1/r) \geq 0$, and $\lambda_r \geq 0$. These imply $\lambda_\pi > 0$: a contradiction! Hence $r = 1/\beta$ violates condition (A13). It must be true that $r^* < 1/\beta$.

Case two. $\lambda_\pi > 0$. By the complementary slackness condition, we have $1/r = \pi$. If $r = 1/\beta$, then $\pi = \beta$. But this (π, r) pair implies $u'(x_j) - 1 = u'(\tilde{x}_j) - 1 = 0$, $j \in \{0, 1\}$. Condition (A12) in turn implies $\lambda_\pi = 0$: again a contradiction! Hence in this case, it must also be true that $r^* < 1/\beta$.

We obtain stated result, $\pi^* > \beta$, by observing that if $\pi^* = \beta$ then $r^* = 1/\beta$ from (15). It follows immediately that the gross nominal illiquid rate $\pi^*/\beta > 1$. But, it should be noted that the nominal interest rate on liquid debt is zero under the optimal policy. To see this, observe that $\pi^* > \beta$ implies $u'(x_0) - 1 > 0$ and hence $\lambda_\pi > 0$ in the light of (A12). The complementary slackness condition then implies $\pi^* = 1/r^*$, hence $i^* \equiv r^*\pi^* = 1$. *Q.E.D.*