Online Companion for "Buy-One-Get-One Promotions in a Two-Echelon Supply Chain"

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MS-OPM-20-01139

Proofs of Statements

This companion contains five sections. In Section EC.1, we provide the proof of EDLP policy. In Section EC.2, we consider the two-period models without stockpiling for BOGO and PR. In Section EC.3, we take consumers' stockpiling behavior into consideration. In Section EC.4, we consider multi-period and demand uncertainty for BOGO and PR. Finally, in Section EC.5, we relax our assumptions by considering consumers' time-inconsistency, heterogeneous marginal utility, heterogeneous holding cost, and manufacturer-initiated BOGO.

Appendix EC.1: Everyday Low Price (EDLP)

Proof of EDLP policy. Under EDLP, the retailer optimizes a single price for selling the product throughout all periods. A consumer buys the product as long as her net utility is non-negative. In a two-period decentralized supply chain, if the retailer does not implement any promotions, the expected profit function of the retailer is

$$\pi_{nr}(p_n) = D_n(p_n - w_n) = 2(1-p_n)(p_n - w_n).$$

Taking the first and second derivatives of $\pi_{nr}(p_n)$, we obtain the following results:

$$\frac{d\pi_{nr}}{dp_n} = 2(1-2p_n+w_n) \quad \text{and} \quad \frac{d^2\pi_{nr}}{dp_n^2} = -4.$$

Since $\frac{d^2\pi_{nr}}{dp_n^2} = -4 < 0$, the retailer's profit function is concave in p_n. From the first-order-condition, we get $p_n^* = \frac{1+w_n}{2}$. Then, the optimal expected demand of the product is $D_n^* = 1 - w_n$, and the optimal expected profit of the retailer is $\pi_{nr}^* = \frac{(1-w_n)^2}{2}$. The manufacturer's problem is to maximize

$$\pi_{nm}(w_n) = (1-w_n)(w_n - c), \quad \text{where } 0 < w_n \leq 1.$$

Taking the first and second derivatives of $\pi_{nm}(w_n)$, we obtain the following results:

$$\frac{d\pi_{nm}}{dw_n} = (1-2w_n+c) \quad \text{and} \quad \frac{d^2\pi_{nm}}{dw_n^2} = -2.$$

Since $\frac{d^2\pi_{nm}}{dp_n^2} = -2 < 0$, the manufacturer's profit function is concave in w_n. From the first-order-condition, we get $w_n^* = \frac{1+c}{2}$. Then, the optimal results are $p_n^* = \frac{3+c}{4}$, $D_n^* = \frac{1-c}{2}$, $\pi_{nr}^* = \frac{(1-c)^2}{8}$, and $\pi_{nm}^* = \frac{(1-c)^2}{4}$. Similarly, in a multi-period model ($T+1$ periods ($T \in \mathbb{N}^+$)), the optimal prices are $w_n^* = \frac{1+c}{2}$ and $p_n^* = \frac{3+c}{4}$. The optimal expected profits for the retailer and manufacturer are $\pi_{nr}^* = \frac{(1+T)(1-c)^2}{16}$ and $\pi_{nm}^* = \frac{(1+T)(1-c)^2}{8}$, respectively. $\square$

Appendix EC.2: Models Without Stockpiling (NSP)

Proof of Lemma 1. In Case I, the retailer's expected profit function can be rewritten as

$$\pi_{br}(p_b,p_g) = 2(1-p_b)(p_b-w_b) + (1-\frac{p_g}{\theta})(p_g-w_b).$$

Taking the first and second derivatives of $\pi_{br}(p_b,p_g)$, we obtain the following results:

$$\frac{\partial \pi_{br}}{\partial p_b} = 2-4(p_b+2w_b), \quad \frac{\partial^2 \pi_{br}}{\partial p_b^2} = -4, \quad \frac{\partial \pi_{br}}{\partial p_g} = \frac{\theta+w_b-2p_g}{\theta}, \quad \text{and} \quad \frac{\partial^2 \pi_{br}}{\partial p_g^2} = -\frac{2}{\theta}.$$

Therefore, the Hessian matrix of the second derivatives of retailer's profit function is

$$H = \begin{pmatrix} -4 & 0 \\ 0 & -\frac{2}{\theta} \end{pmatrix}.$$

Hence, $|H_1| = -4 < 0$, and $|H_2| = \frac{8}{\theta} > 0$. It means that the retailer's profit function under this scenario is jointly concave in p_b and p_g. From the first-order-condition, we can get $p_b^* = \frac{1+w_b}{2}$, $p_g^* = \frac{\theta+w_b}{2}$, and $D_b^* = \frac{3\theta-w_b-2\theta w_b}{2\theta}$. Since $0 \le p_g \le p_b \le 1$, $D_b \ge 0$, and $p_g < \theta \le \frac{p_g}{p_b}$[8], we get $0 < w_b \le \theta$. Comparing the retailer's profit with that of the EDLP model, we have, for any given $0 < w = w_b = w_n \le \theta$, $\pi_{br} - \pi_{nr} = \frac{\theta(2+\theta-6w)+w^2(1+2\theta)}{4\theta} - \frac{(1-w)^2}{2} = \frac{(\theta-w)^2}{4\theta} \ge 0$. Therefore, it is optimal for the retailer to offer a BOGO when $0 < w_b \le \theta$.

In Case II, the retailer's expected profit function is

$$\pi_{br}(p_b,\ p_g) = \left(2-p_b-\frac{p_b+p_g}{1+\theta}\right)(p_b-w_b) + \left(1-\frac{p_b+p_g}{1+\theta}\right)(p_g-w_b).$$

Taking the first and second derivatives of $\pi_{br}(p_b,p_g)$, we obtain the following results:

$$\frac{\partial \pi_{br}}{\partial p_b} = \frac{2(1-p_g)+3w_b+\theta(2+w_b)-2p_b(2+\theta)}{1+\theta}, \quad \frac{\partial^2 \pi_{br}}{\partial p_b^2} = -\frac{2(2+\theta)}{1+\theta},$$

$$\frac{\partial \pi_{br}}{\partial p_g} = \frac{1+\theta-2(p_g+p_b-w_b)}{1+\theta}, \quad \text{and} \quad \frac{\partial^2 \pi_{br}}{\partial p_g^2} = -\frac{2}{1+\theta}.$$

Therefore, the Hessian matrix of the second derivatives of retailer's profit function is

$$H = \begin{pmatrix} -\frac{2(2+\theta)}{1+\theta} & -\frac{2}{1+\theta} \\ -\frac{2}{1+\theta} & -\frac{2}{1+\theta} \end{pmatrix}.$$

Hence, $|H_1| = -\frac{2(2+\theta)}{1+\theta} < 0$, and $|H_2| = \frac{4}{1+\theta} > 0$. It means that the retailer's profit function under this scenario is jointly concave in p_b and p_g. From the first-order-condition, we have $p_b^* = \frac{1+w_b}{2}$, $p_g^* = \frac{\theta+w_b}{2}$, and $D_b^* = \frac{3+3\theta-w_b(5+\theta)}{2(1+\theta)}$. For reasonableness, $0 \le p_g \le p_b \le 1$, $D_b \ge 0$, and $\frac{p_g}{p_b} < \theta \le 1$, it shows that $(\theta-1)w_b > 0$, which is not possible for $0 < \theta \le 1$. Therefore, in Case II, the retailer will not offer a BOGO. Combining the results in Case I, Lemma 1 is proved.

The optimal results for BOGO-NSP and PR-NSP are presented in the following table.

[8] If $p_g \ge \theta$, it is the case of EDLP.

	EDLP	BOGO	PR
Regular price	$p_n = \frac{1+w}{2}$	$p_b = \frac{1+w}{2}$	$p_r = \frac{1+w}{2}$
Promotional price	N/A	$p_g = \frac{\theta+w}{2}$	$p_d = \frac{2\theta+w+\theta w}{2(1+\theta)}$
Product demand	$D_n = 1-w$	$D_g = \frac{3\theta-w-2\theta w}{2\theta}$	$D_d = \frac{3\theta-w-2\theta w}{2\theta}$
Retailer's profit	$\pi_{nr} = \frac{(1-w)^2}{2}$	$\pi_{br} = \frac{\theta(2+\theta-6w)+w^2(1+2\theta)}{4\theta}$	$\pi_{dr} = \frac{\theta(1+5\theta)-6\theta w(1+\theta)+w^2\left(1+3\theta+2\theta^2\right)}{4\theta(1+\theta)}$

Table EC.1 Retailer's Optimal Policies in Two Periods with NSP, $w_b = w_d = w$

Note that consumer choices in Table 1 do not consider behavioral aspects of BOGO (i.e., consumers think they are getting a better deal with BOGO than an equivalent PR). These aspects can be incorporated with minor changes. Consumers perceive the average price in BOGO, given by $\bar{p}_{bogo} = \frac{p_b+p_g}{2}$, to be less than $\bar{p}_{bogo}$. Thus, while the average price is $\frac{p_b+p_g}{2}$ consumers perceive it as $\frac{p_b+\hat{p}_g}{2} < \frac{p_b+p_g}{2}$. Let the parameter k represents the degree of consumers' BOGO bias, then $\frac{p_b+\hat{p}_g}{2} = k\frac{p_b+p_g}{2}$, where $k<1$. For BOGO prices of p_b and p_g, consumers use $\hat{p}_g = kp_g - (1-k)p_b < p_g$ instead of p_g in making their choices. Therefore, this aspect of consumer behavior can be incorporated by replacing p_g in Table 1 by $kp_g - (1-k)p_b < p_g$ while still using p_g to compute profit. □

Proof of Lemma 2. In PR-NSP, the retailer's expected profit function is

$$\pi_{dr}\left(p_r,\, p_d\right) = (1-p_r)(p_r - w_d) + \left(2 - p_d - \frac{p_d}{\theta}\right)(p_d - w_d).$$

Taking the first and second derivatives of $\pi_{dr}(p_r,\, p_d)$, we obtain the following results:

$$\frac{\partial \pi_{dr}}{\partial p_r} = 1 - 2p_r + w_d, \quad \frac{\partial^2 \pi_{dr}}{\partial p_r^2} = -2,$$

$$\frac{\partial \pi_{dr}}{\partial p_d} = \frac{2\theta + w_d + \theta w_d - 2p_d(1+\theta)}{\theta}, \quad \text{and} \quad \frac{\partial^2 \pi_d r}{\partial p_r^2} = -\frac{2(1+\theta)}{\theta}.$$

Therefore, the Hessian matrix of the second derivatives of retailer's profit function is

$$H = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{2(1+\theta)}{\theta} \end{pmatrix}.$$

Hence, $|H_1| = -2 < 0$, and $|H_2| = \frac{4(1+\theta)}{\theta} > 0$. It means that the retailer's profit function is jointly concave in p_r and p_d. From the first-order-condition, we get $p_r = \frac{1+w_d}{2}$, $p_d = \frac{2\theta+w_d+\theta w_d}{2(1+\theta)}$, and $D_d = \frac{3\theta-w_d-2\theta w_d}{2\theta}$. In Case I, since $0 \le p_d \le p_r \le 1$, $D_d \ge 0$, and $p_d < \theta \le \frac{p_d}{p_r}$[9], we have $0 < w_d \le \frac{2\theta^2}{1+\theta}$ for any positive $0 < \theta \le 1$. Case II requires $0 \le p_d \le p_r \le 1$, $D_d \ge 0$, and $\frac{p_d}{p_r} < \theta \le 1$, which is infeasible. Therefore, under PR, we have, when $0 < w_d \le \frac{2\theta^2}{1+\theta}$, $p_r^* = \frac{(1+w_d)}{2}$, $p_d^* = \frac{2\theta+w_d+\theta w_d}{2(1+\theta)}$, $D_d^* = \frac{3\theta-w_d-2\theta w_d}{2\theta}$, and $\pi_{dr}^* = \frac{\theta(1+5\theta)-6\theta w_d(1+\theta)+w_d^2\left(1+3\theta+2\theta^2\right)}{4\theta(1+\theta)}$; when $\frac{2\theta^2}{1+\theta} < w_d \le 1$, $p_d = p_r = \frac{1+w_b}{2}$, and the

[9] If $p_d \ge \theta$, it is the case of EDLP.

retailer will not offer a price reduction. Comparing the retailer's profit with that of EDLP, we get, for any given $0 < w = w_d = w_n \leq \frac{2\theta^2}{1+\theta}$, when $\frac{1}{3} < \theta < 1$ and $0 < w_d \leq \theta - (1-\theta)\sqrt{\frac{\theta}{1+\theta}}$, $\pi_{dr} = \frac{\theta(1+5\theta)-6\theta w(1+\theta)+w^2\left(1+3\theta+2\theta^2\right)}{4\theta(1+\theta)} > \frac{(1-w)^2}{2}$. Otherwise, it is suboptimal for the retailer to offer a price reduction. The optimal solutions for the retailer are provided in Table EC.1. □

Proof of Corollary 1. Given a wholesale price $w = w_d = w_b$, since $\pi^*_{br} - \pi^*_{dr} = \frac{(1-\theta)^2}{4(1+\theta)} \geq 0$, the retailer will offer a BOGO or an EDLP. In addition, since $\frac{d(\pi^*_{br}-\pi^*_{dr})}{d\theta} = \frac{(\theta-1)(\theta+3)}{4(1+\theta)^2} \leq 0$, the profit difference between BOGO and PR for the retailer is decreasing in θ. □

Proof of Proposition 1. Based on Lemma 1, the manufacturer's problem is to maximize

$$\pi_{bm}(w_b) = \begin{cases} \frac{(3\theta-w_b-2\theta w_b)(w_b-c)}{2\theta}, & 0 < w_b \leq \theta \\ (1-w_b)(w_b-c), & \theta < w_b \leq 1 \end{cases}.$$

When $0 < w_b \leq \theta$, taking the first and second derivatives of $\pi_{bm}(w_b)$, we obtain the following results:

$$\frac{d\pi_{bm}}{dw_b} = \frac{c - 2w_b + 3\theta + 2c\theta - 4w_b\theta}{2\theta}, \quad \text{and} \quad \frac{d^2\pi_{bm}}{dw_b^2} = -\frac{2\theta+1}{\theta}.$$

Hence, $\frac{d^2\pi_{bm}}{dw_b^2} = -\frac{2\theta+1}{\theta} < 0$, the manufacturer's profit function is concave in w_b. From the first-order-condition, we get $w_b = \frac{3\theta+c+2c\theta}{2(1+2\theta)}$. Also, when $\theta \leq w_b \leq 1$, we get $w_b = \frac{1+c}{2}$. Hence, there are three cases as follows:

- Case 1: $\frac{3\theta+c+2c\theta}{2(1+2\theta)} < \frac{1+c}{2} < \theta$.
  Since $\frac{(3\theta-c-2c\theta)^2}{8\theta(1+2\theta)} > (1-\theta)(\theta-c)$, BOGO dominates EDLP.
- Case 2: $\frac{3\theta+c+2c\theta}{2(1+2\theta)} < \theta < \frac{1+c}{2}$.
  If $\frac{\left(1+c-c^2\right)+(1-c)\sqrt{1+4c+c^2}}{5-4c} < \theta \leq \frac{1+c}{2}$ and $\frac{(3\theta-c-2c\theta)^2}{8\theta(1+2\theta)} > \frac{(1-c)^2}{4}$, BOGO dominates EDLP. If $\frac{3\theta+c+2c\theta}{2(1+2\theta)} < \theta < \frac{\left(1+c-c^2\right)+(1-c)\sqrt{1+4c+c^2}}{5-4c}$ and $\frac{(3\theta-c-2c\theta)^2}{8\theta(1+2\theta)} < \frac{(1-c)^2}{4}$, EDLP dominates BOGO.
- Case 3: $\theta < \frac{3\theta+c+2c\theta}{2(1+2\theta)} < \frac{1+c}{2}$.
  Since $(1-\theta)(\theta-c) < \frac{(1-c)^2}{4}$, EDLP dominates BOGO.

Combining the cases above, we obtain the results in Proposition 1. □

Proof of Corollary 2. The optimal results for the supply chain with no stockpiling are presented in Table EC.2.

Based on the results in Table EC.2, we can obtain the following results:

- $\frac{dw_b}{d\theta} = \frac{3}{2(1+2\theta)^2} > 0$, $\frac{dp_b}{d\theta} = \frac{3}{4(1+2\theta)^2} > 0$, and $\frac{dp_g}{d\theta} = \frac{8\theta^2+8\theta+5}{4(1+2\theta)^2} > 0$.
- $\frac{dw_b}{dc} = \frac{1}{2} > 0$ and $\frac{dp_b}{dc} = \frac{dp_g}{dc} = \frac{1}{4} > 0$.
- $\frac{dD_b}{d\theta} = \frac{c}{4\theta^2} > 0$, $\frac{dD_b}{dc} = -\frac{1+2\theta}{4\theta} < 0$, $\frac{d\pi_{bm}}{d\theta} = \frac{9\theta^2-c^2(1+2\theta)^2}{8\theta^2(1+2\theta)^2} > 0$, and $\frac{d\pi_{bm}}{dc} = \frac{c-3\theta+2c\theta}{4\theta} < 0$.
- $\frac{d\pi_{br}}{dc} = \frac{c-3\theta+2c\theta}{8\theta} < 0$. In addition, if $\frac{2}{5} < \theta < \frac{3\sqrt{3}-2}{4}$ and $0 < c < \tilde{c}(\theta)$, $\frac{d\pi_{br}}{d\theta} = \frac{\theta^2(16\theta^2+16\theta-23)-(c+2c\theta)^2}{16\theta^2(1+2\theta)^2} < 0$; if $\frac{3\sqrt{3}-2}{4} < \theta < 1$ and $0 < c < \tilde{c}(\theta)$, $\frac{d\pi_{br}}{d\theta} = \frac{\theta^2(16\theta^2+16\theta-23)-(c+2c\theta)^2}{16\theta^2(1+2\theta)^2} > 0$.

	EDLP	BOGO	PR
Wholesale price	$w_n = \frac{1+c}{2}$	$w_b = \frac{3\theta+c+2c\theta}{2(1+2\theta)}$	$w_d = \frac{3\theta+c+2c\theta}{2(1+2\theta)}$
Regular price	$p_n = \frac{3+c}{4}$	$p_b = \frac{2+c+\theta(7+2c)}{4(1+2\theta)}$	$p_r = \frac{2+c+\theta(7+2c)}{4(1+2\theta)}$
Promotional price	N/A	$p_g = \frac{c(1+2\theta)+\theta(5+4\theta)}{4(1+2\theta)}$	$p_d = \frac{c+\theta(7+3c)+\theta^2(11+2c)}{4(1+2\theta)(1+\theta)}$
Product demand	$D_n = \frac{1-c}{2}$	$D_g = \frac{3\theta-c-2c\theta}{4\theta}$	$D_d = \frac{3\theta-c-2c\theta}{4\theta}$
Retailer's profit	$\pi_{nr} = \frac{(1-c)^2}{8}$	$\pi_{br} = \frac{\theta(8-7\theta+8\theta^2)+(1+2\theta)(c^2(1+2\theta)-6c\theta)}{16\theta(1+2\theta)}$	$\pi_{dr} = \frac{\theta(4+\theta+13\theta^2)+(1+2\theta)(1+\theta)(c^2(1+2\theta)-6c\theta)}{16\theta(1+2\theta)(1+\theta)}$
Manufacturer's profit	$\pi_{nm} = \frac{(1-c)^2}{4}$	$\pi_{bm} = \frac{(3\theta-c-2c\theta)^2}{8\theta(1+2\theta)}$	$\pi_{dm} = \frac{(3\theta-c-2c\theta)^2}{8\theta(1+2\theta)}$

Table EC.2 Manufacturer-Retailer Optimal Policies in Two Periods with NSP

Corollary 2 follows. □

Proof of Corollary 3. We omit the proof of Corollary 3(a). Please see the results in Table EC.2. For the rest of Corollary 3, we have that $\pi^*_{br} - \pi^*_{dr} = \pi^*_{bsc} - \pi^*_{dsc} = \frac{(1-\theta)^2}{4(1+\theta)} \geq 0$, and $\frac{d(\pi^*_{br}-\pi^*_{dr})}{d\theta} = \frac{(\theta-1)(\theta+3)}{4(1+\theta)^2} \leq 0$. If $0 \leq c < \tilde{c}(\theta)$,

$$\frac{\pi_{br}{}^*}{\pi_{br}{}^* + \pi_{bm}{}^*} = \frac{-6c\theta(1+2\theta) + (c+2c\theta)^2 + \theta\,(8-7\theta+8\theta^2)}{-18c\theta(1+\theta) + 3(c+2c\theta)^2 + \theta\,(8+11\theta+8\theta^2)} > \frac{1}{3}.$$

If $0 \leq c < \bar{c}(\theta)$,

$$\frac{\pi_{dr}{}^*}{\pi_{dr}{}^* + \pi_{dm}{}^*} = \frac{(1+\theta)(c+2c\theta)^2 - 6c\theta\,(1+3\theta+2\theta^2) + \theta\,(4+\theta+13\theta^2)}{3(1+\theta)(c+2c\theta)^2 - 18c\theta\,(1+3\theta+2\theta^2) + \theta\,(4+19\theta+31\theta^2)} > \frac{1}{3}.$$

Hence, in the region of $0 \leq c < \bar{c}(\theta)$, we can obtain

$$\frac{\pi_{br}{}^*}{\pi_{br}{}^* + \pi_{bm}{}^*} > \frac{\pi_{dr}{}^*}{\pi_{dr}{}^* + \pi_{dm}{}^*} > \frac{\pi_{nr}{}^*}{\pi_{nr}{}^* + \pi_{nm}{}^*} = \frac{1}{3}.$$

Under BOGO, the surplus of individual consumer located at v is given by

$$S\,(v) = \begin{cases} (2+\theta)\,v - 2p_b - p_g, & \text{if } \frac{p_g}{\theta} \leq v \leq 1 \\ 2v - 2p_b, & \text{if } p_b \leq v < \frac{p_g}{\theta} \end{cases}.$$

We integrate over individual consumer's surplus to find the total consumer surplus

$$CS_b = \frac{-6c\theta(1+2\theta) + (c+2c\theta)^2 + \theta\,(8+\theta\,(-7+8\theta))}{32\theta(1+2\theta)}.$$

Combining the profits of the retailer and manufacturer, the social welfare is

$$SW_b = \frac{7(c+2c\theta)^2 - 42c\theta(1+2\theta) + 3\theta(8+5\theta+8\theta^2)}{32\theta(1+2\theta)}.$$

Under PR, the surplus of individual consumer located at v is given by

$$S\,(v) = \begin{cases} (2+\theta)\,v - 2p_d - p_r, & \text{if } \frac{p_d}{\theta} \leq v \leq 1 \\ 2v - p_d - p_r, & \text{if } p_r \leq v < \frac{p_d}{\theta} \\ v - p_d, & \text{if } p_d \leq v < p_r \end{cases}.$$

We integrate over individual consumer's surplus to find the total consumer surplus

$$CS_d = \frac{(1+\theta)\,(c+2c\theta)^2 - 6c\theta(1+3\theta+2\theta^2) + \theta\,(20+\theta-35\theta^2+32\theta^3)}{32\theta(1+\theta)(1+2\theta)}.$$

Combining the profits of the retailer and manufacturer, the social welfare is

$$SW_d = \frac{7(1+\theta)(c+2c\theta)^2 - 42c\theta(1+3\theta+2\theta^2) + \theta(28+39\theta+27\theta^2+32\theta^3)}{32\theta(1+\theta)(1+2\theta)}.$$

Similarly, the consumer surplus under EDLP is $CS_n = 2\int_{p_n}^{1}(x-p_n)\,dx = \frac{(1-c)^2}{16}$. Since, $CS_d - CS_b = \frac{3(1-\theta)^2}{8(1+\theta)} > 0$, $CS_b - CS_n = \frac{c^2(1+2\theta)-2c\theta(1+2\theta)+\theta\left(6-11\theta+8\theta^2\right)}{32\theta(1+2\theta)} > 0$, $SW_d - SW_b = \frac{(1-\theta)^2}{8(1+\theta)} > 0$, we get $CS_d > CS_b > CS_n$ and $SW_d > SW_b > SW_n$. Corollary 3 follows. □

Appendix EC.3: Models with Stockpiling (SP)

Proof of Lemma 3. We have proved the case of $p_b - p_g < h \le 1$ in Lemma 1. When $0 < h \le p_b - p_g$, we can rewrite Table 2 by taking the lower bound of each case into account. Then, we get Tables EC.3, EC.4, and EC.5.

h	p_g	Consumer Choice on v
$0 < h \le \frac{p_b(1-\theta)}{1+\theta}$	$0 < p_g < p_b - h$	$(2,1): \frac{p_b-h}{\theta} \le v \le 1$ $(2,0)s: \frac{p_b+p_g+h}{2} \le v < \frac{p_b-h}{\theta}$ $(0,0): 0 \le v < \frac{p_b+p_g+h}{2}$
	$p_b - h \le p_g < p_b$	$(2,1): \frac{p_g}{\theta} \le v \le 1$ $(1,1): p_b \le v < \frac{p_g}{\theta}$ $(0,0): 0 \le v < p_b$
$\frac{p_b(1-\theta)}{1+\theta} < h \le p_b(1-\theta)$	$0 < p_g < \frac{h(1+\theta)-p_b(1-\theta)}{1-\theta}$	$(2,1): \frac{p_b-h}{\theta} \le v \le 1$ $(2,0)s: \frac{h}{1-\theta} \le v < \frac{p_b-h}{\theta}$ $(2,0): \frac{p_b+p_g}{1+\theta} \le v < \frac{h}{1-\theta}$ $(0,0): 0 \le v < \frac{p_b+p_g}{1+\theta}$
	$\frac{h(1+\theta)-p_b(1-\theta)}{1-\theta} \le p_g \le p_b - h$	$(2,1): \frac{p_b-h}{\theta} \le v \le 1$ $(2,0)s: \frac{p_b+p_g+h}{2} \le v < \frac{p_b-h}{\theta}$ $(0,0): 0 \le v < \frac{p_b+p_g+h}{2}$
	$p_b - h < p_g \le p_b$	$(2,1): \frac{p_g}{\theta} \le v \le 1$ $(1,1): p_b \le v < \frac{p_g}{\theta}$ $(0,0): 0 \le v < p_b$

Table EC.3 Consumer Choices Under BOGO with Stockpiling, $p_b < \theta$

In Tables EC.3, EC.4, and EC.5, there are six strategies depending on consumer choices for the retailer. Strategy 1: $\{(2,1),(1,1),(0,0)\}$; Strategy 2: $\{(2,1),(2,0)s,(0,0)\}$; Strategy 3: $\{(2,1),(2,0)s,(2,0),(0,0)\}$; Strategy 4: $\{(2,0)s,(0,0)\}$; Strategy 5: $\{(2,0)s,(2,0),(0,0)\}$; Strategy

h	p_g	Consumer Choice on v
$0 < h \le p_b - \theta$	$0 < p_g < p_b - h$	$(2,0)s: \frac{p_b+p_g+h}{2} \le v < \frac{p_b-h}{\theta}$ $(0,0): 0 \le v < \frac{p_b+p_g+h}{2}$
	$p_b - h \le p_g < p_b$	$(1,1): p_b \le v < 1$ $(0,0): 0 \le v < p_b$
$p_b - \theta < h \le \frac{p_b(1-\theta)}{1+\theta}$	$0 < p_g < p_b - h$	$(2,1): \frac{p_b-h}{\theta} \le v \le 1$ $(2,0)s: \frac{p_b+p_g+h}{2} \le v < \frac{p_b-h}{\theta}$ $(0,0): 0 \le v < \frac{p_b+p_g+h}{2}$
	$p_b - h \le p_g \le \theta$	$(2,1): \frac{p_g}{\theta} \le v \le 1$ $(1,1): p_b \le v < \frac{p_g}{\theta}$ $(0,0): 0 \le v < p_b$
	$\theta < p_g \le p_b$	$(1,1): p_b \le v < 1$ $(0,0): 0 \le v < p_b$
$\frac{p_b(1-\theta)}{1+\theta} < h \le p_b(1-\theta)$	$0 < p_g < \frac{h(1+\theta)-p_b(1-\theta)}{1-\theta}$	$(2,1): \frac{p_b-h}{\theta} \le v \le 1$ $(2,0)s: \frac{h}{1-\theta} \le v < \frac{p_b-h}{\theta}$ $(2,0): \frac{p_b+p_g}{1+\theta} \le v < \frac{h}{1-\theta}$ $(0,0): 0 \le v < \frac{p_b+p_g}{1+\theta}$
	$\frac{h(1+\theta)-p_b(1-\theta)}{1-\theta} \le p_g \le p_b - h$	$(2,1): \frac{p_b-h}{\theta} \le v \le 1$ $(2,0)s: \frac{p_b+p_g+h}{2} \le v < \frac{p_b-h}{\theta}$ $(0,0): 0 \le v < \frac{p_b+p_g+h}{2}$
	$p_b - h < p_g \le \theta$	$(2,1): \frac{p_g}{\theta} \le v \le 1$ $(1,1): p_b \le v < \frac{p_g}{\theta}$ $(0,0): 0 \le v < p_b$
	$\theta < p_g \le p_b$	$(1,1): p_b \le v < 1$ $(0,0): 0 \le v < p_b$

Table EC.4 Consumer Choices Under BOGO with Stockpiling, $\theta < p_b < \frac{1+\theta}{2}$

6: $\{(1,1),(0,0)\}$. It is straightforward to see that Strategy 6 is the case of EDLP. We first investigate Strategy 4. Then, the retailer's profit function is

$$\pi_{br}(p_b,\, p_g) = \left(1 - \frac{p_b + p_g + h}{2}\right)(p_b - w_b) + \left(1 - \frac{p_b + p_g + h}{2}\right)(p_g - w_b).$$

Taking the first and second derivatives of $\pi_{br}(p_b,\, p_g)$, we obtain the following results:

$$\frac{\partial \pi_{br}}{\partial p_b} = 1 - \frac{h}{2} - p_b - p_g + w_b, \quad \frac{\partial^2 \pi_{br}}{\partial p_b^2} = -1,$$

$$\frac{\partial \pi_{br}}{\partial p_g} = 1 - \frac{h}{2} - p_b - p_g + w_b, \quad \text{and} \quad \frac{\partial^2 \pi_{br}}{\partial p_g^2} = -1.$$

Hence, the Hessian matrix of the second derivatives of retailer's profit function is

$$H = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

We have that $|H_1| = -1 < 0$ and $|H_2| = 0$. It means that the retailer's profit function under this scenario is jointly quasi-concave in p_b and p_g. From the first-order-condition, we get $p_b + p_g =$

h	p_g	Consumer Choice on v
$0<h\leq\frac{p_b(1-\theta)}{1+\theta}$	$0<p_g<p_b-h$	$(2,0)s:\frac{p_b+p_g+h}{2}\leq v<1$ $(0,0):0\leq v<\frac{p_b+p_g+h}{2}$
	$p_b-h\leq p_g<p_b$	$(1,1):p_b\leq v<1$ $(0,0):0\leq v<p_b$
$\frac{p_b(1-\theta)}{1+\theta}<h\leq p_b-\theta$	$0<p_g<\frac{h(1+\theta)-p_b(1-\theta)}{1-\theta}$	$(2,0)s:\frac{h}{1-\theta}\leq v<1$ $(2,0):\frac{p_b+p_g}{1+\theta}\leq v<\frac{h}{1-\theta}$ $(0,0):0\leq v<\frac{p_b+p_g}{1+\theta}$
	$\frac{h(1+\theta)-p_b(1-\theta)}{1-\theta}\leq p_g\leq p_b-h$	$(2,0)s:\frac{p_b+p_g+h}{2}\leq v<1$ $(0,0):0\leq v<\frac{p_b+p_g+h}{2}$
	$p_b-h<p_g\leq p_b$	$(1,1):p_b\leq v<1$ $(0,0):0\leq v<p_b$
$p_b-\theta<h\leq p_b(1-\theta)$	$0<p_g<\frac{h(1+\theta)-p_b(1-\theta)}{1-\theta}$	$(2,1):\frac{p_b-h}{\theta}\leq v\leq 1$ $(2,0)s:\frac{h}{1-\theta}\leq v<\frac{p_b-h}{\theta}$ $(2,0):\frac{p_b+p_g}{1+\theta}\leq v<\frac{h}{1-\theta}$ $(0,0):0\leq v<\frac{p_b+p_g}{1+\theta}$
	$\frac{h(1+\theta)-p_b(1-\theta)}{1-\theta}\leq p_g\leq p_b-h$	$(2,1):\frac{p_b-h}{\theta}\leq v\leq 1$ $(2,0)s:\frac{p_b+p_g+h}{2}\leq v<\frac{p_b-h}{\theta}$ $(0,0):0\leq v<\frac{p_b+p_g+h}{2}$
	$p_b-h<p_g\leq\theta$	$(2,1):\frac{p_g}{\theta}\leq v\leq 1$ $(1,1):p_b\leq v<\frac{p_g}{\theta}$ $(0,0):0\leq v<p_b$
	$\theta<p_g\leq p_b$	$(1,1):p_b\leq v<1$ $(0,0):0\leq v<p_b$

Table EC.5 Consumer Choices Under BOGO with Stockpiling, $\frac{1+\theta}{2}<p_b<1$

$\frac{2-h+2w_b}{2}$ and $\pi_{br}=\frac{(h-2+2w_b)^2}{8}$. When $0<h<p_b-\theta$ and $0<p_g<p_b-h$, we see that $\pi_{br}<\frac{(1-w_b)^2}{2}$. Hence, it is suboptimal for the retailer to offer a BOGO under this strategy. Similarly, we can get that $\pi_{br}<\frac{(1-w_b)^2}{2}$ under Strategy 5. Thus, we only need to investigate Strategies 1, 2, and 3. Note that, when $p_g=p_b-h$, consumers are indifferent between $(2,0)s$ and $(1,1)$. The retailer can set a slightly higher $\hat{p}_g=p_g+\varepsilon$ to make $\hat{p}_g+h>p_b$, which will induce consumers to choose $(1,1)$ option for a higher net utility. We rewrite the tables to focus on Strategies 1, 2, and 3, and get three solutions as follows:

- Solution 1: $(p_b,p_g)=\left(\frac{h+w_b+\theta}{2},\frac{h(1+3\theta)-(1-\theta)(\theta+w_b)}{2(1-\theta)}\right)$, if $0<\theta\leq 1$, $\frac{\theta-\theta^2}{1+3\theta}<h\leq\frac{\theta-\theta^2}{1+\theta}$, and $0<w_b<\frac{-h+\theta-3h\theta-\theta^2}{-1+\theta}$; or $0<\theta\leq 1$, $\frac{\theta-\theta^2}{1+\theta}<h<\frac{1-\theta}{2}$, and $\frac{-h+\theta-h\theta-\theta^2}{-1+\theta}<w_b<\frac{-h+\theta-3h\theta-\theta^2}{-1+\theta}$. The expected profit of the retailer is $\pi_{br1}=\frac{h^2\left(1-6\theta-3\theta^2\right)+(-1+\theta)^2\left({w_b}^2-10w_b\theta+\theta^2\right)-2h(-1+\theta)(\theta(3+\theta)+w_b(-1+5\theta))}{4(-1+\theta)^2\theta}$.
- Solution 2: $(p_b,p_g)=\left(\frac{2(\theta-1)(w_b+2\theta+w_b\theta)+h(\theta^2+3\theta-2)}{2(\theta^2+\theta-2)},\frac{\theta h}{1-\theta}\right)$, if $0<\theta\leq 1$, $0<h\leq\frac{4\theta-4\theta^2}{2+5\theta+\theta^2}$, and $0<w_b<\frac{-2h-h\theta-2\theta^2+h\theta^2+2\theta^3}{-2+2\theta^2}$; or $0<\theta\leq 1$, $\frac{4\theta-4\theta^2}{2+5\theta+\theta^2}<h<\frac{1-\theta}{2}$, and $\frac{-2h+4\theta-5h\theta-4\theta^2-h\theta^2}{-2+2\theta^2}<w_b<\frac{-2h-h\theta-2\theta^2+h\theta^2+2\theta^3}{-2+2\theta^2}$. The expected profit of the retailer is $\pi_{br2}=\frac{h^2(4-12\theta-3\theta^2+2\theta^3+\theta^4)+4(-1+\theta)^2(4\theta^2+{w_b}^2(1+\theta)^2-2w_b\theta(4+\theta))+4h(2-3\theta+\theta^2)(2\theta+w_b(-1+2\theta+\theta^2))}{8(-1+\theta)^2\theta(2+\theta)}$.

- Solution 3: $(p_b,p_g)=\left(\frac{3\theta+2h+(1+2\theta)w_b}{2(1+2\theta)},\frac{\theta(3-4h)+(1+2\theta)w_b}{2(1+2\theta)}\right)$, if $0<\theta\le\frac{1}{4}$, $\frac{1}{4}(1-4\theta)<h<\frac{1-\theta}{2}$, and $0<w_b<\frac{4h\theta+4\theta^2-\theta}{1+2\theta}$; or $\frac{1}{4}<\theta<1$, $0<h\le\frac{1-\theta}{2}$, and $0<w_b<\frac{4h\theta+4\theta^2-\theta}{1+2\theta}$. The expected profit of the retailer is $\pi_{br3}=\frac{-6w_b\theta(1+2\theta)+\theta(-8h^2-8h(-1+\theta)+9\theta)+(w_b+2w_b\theta)^2}{4\theta(1+2\theta)}$.

In the comparison of the solutions above, we see that Solution 3 dominates Solutions 1 and 2 for the retailer. Comparing Solution 3 with the results under EDLP, we get that, if $0<\theta\le\frac{2}{5}$, $\frac{2(1-\theta)-\sqrt{2\theta(1+2\theta)}}{4}<h<\frac{1-\theta}{2}$, and $0<w_b<\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}$; or $\frac{2}{5}<\theta<1$, $0<h<\frac{1-\theta}{2}$, and $0<w_b<\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}$, BOGO dominates EDLP. Similarly, we identify the other solutions as follows:

- Solution 1: $(p_b,p_g)=\left(\frac{1+w_b}{2},\frac{\theta+w_b}{2}\right)$, if $0<\theta\le1$, $\frac{1-\theta}{2}<h<1-\theta$, and $0<w_b<\theta$. The expected profit of the retailer is $\pi_{br1}=\frac{\theta(2+\theta-6w_b)+{w_b}^2(1+2\theta)}{4\theta}$.
- Solution 2: $(p_b,p_g)=\left(\frac{1+w_b}{2},\frac{h\theta}{1-\theta}\right)$, if $0<\theta\le\frac{1}{2}$, $\frac{1-\theta}{2}<h<1-\theta$, and $0<w_b<\frac{1-2h-\theta}{\theta-1}$; or $\frac{1}{2}<\theta\le1$, $\frac{1-\theta}{2}<h<\frac{1-\theta}{2\theta}$, and $0<w_b<\frac{1-2h-\theta}{\theta-1}$; or $\frac{1}{2}<\theta\le1$, $\frac{1-\theta}{2\theta}<h<1-\theta$, and $\frac{1-\theta-2h\theta}{\theta-1}<w_b<\frac{1-2h-\theta}{\theta-1}$. The expected profit of the retailer is $\pi_{br2}=\frac{1+{w_b}^2(-1+\theta)^2-2(1-h+h^2)\theta+(1-2h)\theta^2-2w_b(-1+\theta)(-2+h+2\theta)}{2(-1+\theta)^2}$.
- Solution 3: $(p_b,p_g)=\left(\frac{h(5\theta^2-1)+(\theta-1)(w_b(1+3\theta)+2\theta(1+2\theta))}{2(\theta-1)(1+2\theta)},\frac{\theta(2h+\theta-1)}{1-\theta}\right)$, if $0<\theta\le\frac{1}{\sqrt{3}}$, $\frac{1-\theta}{2}<h<1-\theta$, and $\frac{2\theta-h-4h\theta+2\theta^2-5h\theta^2-4\theta^3}{3\theta^2-2\theta-1}<w_b<\frac{-h-2h\theta-h\theta^2}{3\theta^2-2\theta-1}$; or $\frac{1}{\sqrt{3}}<\theta<1$, $\frac{1-\theta}{2}<h<\frac{2\theta+2\theta^2-4\theta^3}{1+4\theta+5\theta^2}$, and $0<w_b<\frac{-h-2h\theta-h\theta^2}{3\theta^2-2\theta-1}$; or $\frac{1}{\sqrt{3}}<\theta<1$, $\frac{2\theta+2\theta^2-4\theta^3}{1+4\theta+5\theta^2}<h<1-\theta$, and $\frac{2\theta-h-4h\theta+2\theta^2-5h\theta^2-4\theta^3}{3\theta^2-2\theta-1}<w_b<\frac{-h-2h\theta-h\theta^2}{3\theta^2-2\theta-1}$. The expected profit of the retailer is $\pi_{br3}=-h-2w_b-\frac{4h^2}{3(-1+\theta)^2}-\frac{2h(9+5h+3w_b)}{9(-1+\theta)}+\frac{(h-w_b)^2}{4\theta}+\frac{{w_b}^2}{1+\theta}-\frac{(h+3w_b)^2}{36+72\theta}$.
- Solution 4: $(p_b,p_g)=\left(\frac{h}{1-\theta},\frac{\theta(2h+\theta-1)}{1-\theta}\right)$, if $0<\theta\le1$, $\frac{1-\theta}{2}<h<1-\theta$, and $0<w_b\le1$. The expected profit of the retailer is $\pi_{br4}=\frac{(1-h-\theta)\left((-1+\theta)\left(3w_b+\theta+5w_b\theta+2\theta^2\right)+h(2+\theta(5+4\theta))\right)}{(-1+\theta)^2(1+\theta)}$.
- Solution 5: $(p_b,p_g)=\left(\frac{h}{1-\theta},\frac{2h+(\theta-1)(1+\theta+2w_b)}{2(\theta-1)}\right)$, if $0<\theta\le1$, $\frac{1-\theta}{2}<h<\frac{1-\theta^2}{2}$, and $0<w_b<\frac{1-2h-2h\theta-\theta^2}{2(\theta-1)}$; or $0<\theta\le1$, $\frac{1-\theta^2}{2}<h<1-\theta$, and $\frac{1-2h-\theta^2}{2(\theta-1)}<w_b<\frac{1-2h-2h\theta-\theta^2}{2(\theta-1)}$. The expected profit of the retailer is $\pi_{br5}=\frac{-4h^2(1+\theta)-4h(1+w_b)(-1+\theta^2)+(-1+\theta)^2\left(4{w_b}^2-8w_b(1+\theta)+(1+\theta)^2\right)}{4(-1+\theta)^2(1+\theta)}$.
- Solution 6: $(p_b,p_g)=\left(\frac{h+\theta+w_b}{2},\frac{h(1+3\theta)-(1-\theta)(\theta+w_b)}{2(1-\theta)}\right)$, if $0<\theta\le1$, $\frac{1-\theta}{2}<h<1-\theta$, and $\frac{\theta-h-h\theta-\theta^2}{\theta-1}<w_b<h+\theta$. The expected profit of the retailer is $\pi_{br6}=\frac{h^2\left(1-6\theta-3\theta^2\right)+(1-\theta)^2\left({w_b}^2-10w_b\theta+\theta^2\right)-2h(-1+\theta)(\theta(3+\theta)+w_b(-1+5\theta))}{4(1-\theta)^2\theta}$.
- Solution 7: $(p_b,p_g)=\left(\frac{2(\theta-1)(w_b+2\theta+\theta w_b)+h(\theta^2+3\theta-2)}{2(\theta^2+\theta-2)},\frac{h\theta}{1-\theta}\right)$, if $0<\theta\le1$, $\frac{1-\theta}{2}<h<1-\theta$, and $\frac{4\theta-2h-5h\theta-4\theta^2+h\theta^2}{2\theta^2-2}<w_b<\frac{h\theta^2-2\theta^2-2h-h\theta+2\theta^3}{2\theta^2-2}$. The expected profit of the retailer is $\pi_{br7}=\frac{h^2(4-12\theta-3\theta^2+2\theta^3+\theta^4)+4(-1+\theta)^2(4\theta^2+{w_b}^2(1+\theta)^2-2w_b\theta(4+\theta))+4h(2-3\theta+\theta^2)(2\theta+w_b(-1+2\theta+\theta^2))}{8(-1+\theta)^2\theta(2+\theta)}$.

We show that Solution i $(i=2,3,...,7)$ is dominated by Solution 1 or the EDLP solution. Lemma 3 is proved. □

Proof of Lemma 4. Consumer choices in PR-SP are presented in Table EC.6. Similar to the proof of Lemma 3, in model PR-SP, there are six solutions as follows:

- Solution 1: $(p_r,p_d)=\left(\frac{h}{1-\theta},\frac{h\theta}{1-\theta}\right)$, if $0<\theta\le1$, $0<h<1-\theta$, and $0<w_d\le1$. The expected profit of the retailer is $\pi_{dr1}=-\frac{3w_d(-1+\theta)^2+h^2(1+\theta+\theta^2)+h(-1+\theta)(1+2\theta+w_d(2+\theta))}{(-1+\theta)^2}$.

h	θ	v	Consumer Choice	Case Designation
$0<h\le p_r-p_d$	$0<\theta<\frac{p_d}{p_d+h}$	$\frac{p_r-h}{\theta}\le v\le 1$	$(2,1)$	SP1
		$p_d+h\le v<\frac{p_r-h}{\theta}$	$(2,0)s$	
		$p_d\le v<p_d+h$	$(1,0)$	
		$0\le v<p_d$	$(0,0)$	
	$\frac{p_d}{p_d+h}\le\theta<\frac{p_r-h}{p_r}$	$\frac{p_r-h}{\theta}\le v\le 1$	$(2,1)$	SP2
		$\frac{h}{1-\theta}\le v<\frac{p_r-h}{\theta}$	$(2,0)s$	
		$\frac{p_d}{\theta}\le v<\frac{h}{1-\theta}$	$(2,0)$	
		$p_d\le v<\frac{p_d}{\theta}$	$(1,0)$	
		$0\le v<p_d$	$(0,0)$	
	$\frac{p_r-h}{p_r}\le\theta\le 1$	$p_r\le v\le 1$	$(2,1)$	SP3
		$\frac{p_d}{\theta}\le v<p_r$	$(2,0)s$	
		$p_d\le v<\frac{p_d}{\theta}$	$(1,0)$	
		$0\le v<p_d$	$(0,0)$	
$p_r-p_d<h\le 1$	$0<\theta<\frac{p_d}{p_r}$	$\frac{p_d}{\theta}\le v\le 1$	$(2,1)$	SP4
		$p_r\le v<\frac{p_d}{\theta}$	$(1,1)$	
		$p_d\le v<p_r$	$(1,0)$	
		$0\le v<p_d$	$(0,0)$	
	$\frac{p_d}{p_r}\le\theta\le 1$	$p_r\le v\le 1$	$(2,1)$	SP5
		$\frac{p_d}{\theta}\le v<p_r$	$(2,0)$	
		$p_d\le v<\frac{p_d}{\theta}$	$(1,0)$	
		$0\le v<p_d$	$(0,0)$	

Table EC.6 Consumer Choices in Two Periods with PR-SP

- Solution 2: $(p_r,p_d)=\left(\frac{h}{1-\theta},\frac{2\theta+(1+\theta)w_d}{2(1+\theta)}\right)$, if $0<\theta\le 1$, $\frac{1-\theta}{1+\theta}<h<1-\theta$, and $0<w_d<\frac{2\theta(1-h-\theta-h\theta)}{\theta^2-1}$. The expected profit of the retailer is $\pi_{dr2}=\frac{-4w_d\theta(-2+h+2\theta)(-1+\theta^2)+{w_d}^2\left(-1+\theta^2\right)^2-4\theta\left(-(-1+\theta)^2\theta+h^2(1+\theta)+h\left(-1+\theta^2\right)\right)}{4(-1+\theta)^2\theta(1+\theta)}$.
- Solution 3: $(p_r,p_d)=\left(\frac{h+\theta+w_d}{2},\frac{h\theta}{1-\theta}\right)$, if $0<\theta\le 1$, $0<h<\frac{\theta(1-\theta)}{1+\theta}$, and $0<w_d<h+\theta$; or $0<\theta\le 1$, $\frac{\theta(1-\theta)}{1+\theta}<h<1-\theta$, and $\frac{h(1+\theta)-\theta(1-\theta)}{1-\theta}<w_d<h+\theta$. The expected profit of the retailer is $\pi_{dr3}=\frac{(h-w_d+\theta)^2}{4\theta}-\frac{(w_d(-1+\theta)+h\theta)(-2+h+2\theta+h\theta)}{(-1+\theta)^2}$.
- Solution 4: $(p_r,p_d)=\left(\frac{1+w_d}{2},\frac{h\theta}{1-\theta}\right)$, if $0<\theta<\frac{1}{2}$, $\frac{1-\theta}{2}<h<1-\theta$, and $0<w_d<\frac{1-2h-\theta}{\theta-1}$; or $\frac{1}{2}<\theta<1$, $\frac{1-\theta}{2}<h\le\frac{1-\theta}{2\theta}$, and $0<w_d<\frac{1-2h-\theta}{\theta-1}$; or $\frac{1}{2}<\theta<1$, $\frac{1-\theta}{2\theta}<h<1-\theta$, and $\frac{1-\theta-2h\theta}{\theta-1}<w_d<\frac{1-2h-\theta}{\theta-1}$. The expected profit of the retailer is $\pi_{dr4}=\frac{1}{4}(-1+w_d)^2-\frac{(w_d(-1+\theta)+h\theta)(-2+h+(2+h)\theta)}{(-1+\theta)^2}$.
- Solution 5: $(p_r,p_d)=\left(\frac{3\theta+2h(1+\theta)+(1+2\theta)w_d}{2(1+2\theta)},\frac{\theta(3-2h)+(1+2\theta)w_d}{2(1+2\theta)}\right)$, if $0<\theta<\frac{1}{4}$, $\frac{1-4\theta}{2}<h<\frac{3(1-\theta)}{2(2+\theta)}$, and $0<w_d<\frac{\theta(2h+4\theta-1)}{1+2\theta}$; or $0<\theta<\frac{1}{4}$, $\frac{3(1-\theta)}{2(2+\theta)}<h<1-\theta$, and $\frac{\theta(3(\theta-1)+2h(2+\theta))}{1+\theta-\theta^2}<w_d<\frac{\theta(2h+4\theta-1)}{1+2\theta}$; or $\frac{1}{4}<\theta<1$, $0<h<\frac{3(1-\theta)}{2(2+\theta)}$, and $0<w_d<\frac{\theta(2h+4\theta-1)}{1+2\theta}$; or $\frac{1}{4}<\theta<1$, $\frac{3(1-\theta)}{2(2+\theta)}<h<1-\theta$, and $\frac{\theta(3(\theta-1)+2h(2+\theta))}{1+\theta-\theta^2}<w_d<\frac{\theta(2h+4\theta-1)}{1+2\theta}$. The expected profit of the retailer is $\pi_{dr5}=\frac{-6w_d\theta(1+2\theta)+(w_d+2w_d\theta)^2+\theta\left(-4h(-1+\theta)+9\theta-4h^2(1+\theta)\right)}{4\theta(1+2\theta)}$.
- Solution 6: $(p_r,p_d)=\left(\frac{1+w_d}{2},\frac{2\theta+w_d+\theta w_d}{2(1+\theta)}\right)$, if $0<\theta\le 1$, $\frac{1-\theta}{2(1+\theta)}<h<1-\theta$, and $0<w_d<\frac{2\theta^2}{1+\theta}$. The expected profit of the retailer is $\pi_{dr6}=\frac{-6w_d\theta(1+\theta)+\theta(1+5\theta)+{w_d}^2\left(1+3\theta+2\theta^2\right)}{4\theta(1+\theta)}$.

In the comparison of the solutions above, Solutions 5 and 6 dominate the others. By comparing Solutions 5 and 6 with the results under EDLP, Lemma 4 is proved. □

Proof of Corollary 4. The retailer's optimal policies in BOGO-SP and PR-SP are shown in Table EC.7. Comparing the results in BOGO-SP and PR-SP, we first have that, if $0<\theta<\frac{1}{3}$ and $\frac{1-\theta}{2}-\frac{1}{2}\sqrt{\frac{\theta(1+2\theta)}{2}}<h\leq 1$, or $\frac{1}{3}<\theta<\frac{2}{5}$ and $0<h\leq\frac{1-\theta-\sqrt{(3\theta-1)(2\theta+1)}}{2(1+\theta)}$, BOGO dominates EDLP, and EDLP dominates PR.

Parameters	BOGO		PR	
	Interior Solution	Boundary Solution	Interior Solution	Boundary Solution
Regular price	$p_b=\frac{1+w}{2}$	$p_b=\frac{3\theta+2h+(1+2\theta)w}{2(1+2\theta)}$	$p_r=\frac{1+w}{2}$	$p_r=\frac{3\theta+2h(1+\theta)+(1+2\theta)w}{2(1+2\theta)}$
Promotional price	$p_g=\frac{\theta+w}{2}$	$p_g=\frac{\theta(3-4h)+(1+2\theta)w}{2(1+2\theta)}$	$p_d=\frac{2\theta+w+\theta w}{2(1+\theta)}$	$p_d=\frac{\theta(3-2h)+(1+2\theta)w}{2(1+2\theta)}$
Product demand	$D_b=\frac{3\theta-w-2\theta w}{2\theta}$	$D_b=\frac{3\theta-w-2\theta w}{2\theta}$	$D_d=\frac{3\theta-w-2\theta w}{2\theta}$	$D_d=\frac{3\theta-w-2\theta w}{2\theta}$
Retailer's profit	$\pi_{br}=\tau_1{}^{\ddagger}+\frac{2+\theta}{4}$	$\pi_{br}=\tau_1+\frac{9\theta-8h^2+8h(1-\theta)}{4(1+2\theta)}$	$\pi_{dr}=\tau_1+\frac{1+5\theta}{4(1+\theta)}$	$\pi_{dr}=\tau_1+\frac{9\theta-4h^2(1+\theta)+4h(1-\theta)}{4(1+2\theta)}$

‡ $\tau_1=\frac{w^2(1+2\theta)-6w\theta}{4\theta}$

Table EC.7 Retailer's Optimal Policies in Two Periods with SP, $w_b=w_d=w$

We divide the intersecting region ($\frac{1}{3}<\theta\leq 1$ and $\max\left\{0,\frac{1-\theta-\sqrt{(3\theta-1)(2\theta+1)}}{2(1+\theta)}\right\}<h<1$) into three parts and have the results as follows:

- Region 1: $\frac{1}{3}<\theta\leq 1$ and $\frac{1-\theta}{2}<h\leq 1$, where $\pi_{dr}{}^*=\frac{\theta(1+5\theta)-6\theta w_d(1+\theta)+w_d{}^2\left(1+3\theta+2\theta^2\right)}{4\theta(1+\theta)}$ and $\pi_{br}{}^*=\frac{\theta(2+\theta-6w_b)+w_b{}^2(1+2\theta)}{4\theta}$. Given $w_b=w_d=w$, $\pi_{br}{}^*-\pi_{dr}{}^*=\frac{(\theta-1)^2}{4(1+\theta)}>0$. Hence, BOGO dominates PR, and PR dominates EDLP in Region 1.
- Region 2: $\frac{1}{3}<\theta\leq 1$ and $\frac{1-\theta}{2+2\theta}<h\leq\frac{1-\theta}{2}$, where $\pi_{br}{}^*=\frac{-6w_b\theta(1+2\theta)+\theta(-8h^2-8h(-1+\theta)+9\theta)+(w_b+2w_b\theta)^2}{4\theta(1+2\theta)}$ and $\pi_{dr}{}^*=\frac{\theta(1+5\theta)-6\theta w_d(1+\theta)+w_d{}^2\left(1+3\theta+2\theta^2\right)}{4\theta(1+\theta)}$. Given $w_b=w_d=w$, $\pi_{br}{}^*-\pi_{dr}{}^*=\frac{8h\left(1-\theta^2\right)-(\theta-1)^2-8h^2(1+\theta)}{4(1+\theta)(1+2\theta)}>0$. In addition, $\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}>\theta-(1-\theta)\sqrt{\frac{\theta}{1+\theta}}$. Hence, BOGO dominates PR, and PR dominates EDLP in Region 2.
- Region 3: $\frac{1}{3}<\theta\leq 1$ and $\max\left\{0,\frac{1-\theta-\sqrt{(3\theta-1)(2\theta+1)}}{2(1+\theta)}\right\}<h<\frac{1-\theta}{2+2\theta}$, where $\pi_{br}{}^*=\frac{-6w_b\theta(1+2\theta)+\theta(-8h^2-8h(-1+\theta)+9\theta)+(w_b+2w_b\theta)^2}{4\theta(1+2\theta)}$ and $\pi_{dr}{}^*=\frac{-6w_d\theta(1+2\theta)+(w_d+2w_d\theta)^2+\theta\left(-4h(-1+\theta)+9\theta-4h^2(1+\theta)\right)}{4\theta(1+2\theta)}$. Given $w_b=w_d=w$, $\pi_{br}{}^*-\pi_{dr}{}^*=\frac{(1-h)h(1-\theta)}{1+2\theta}>0$. In addition, $\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}>\theta-\sqrt{\frac{2\theta(2h(-1+\theta)+(-1+\theta)^2+2h^2(1+\theta))}{1+2\theta}}$. Hence, BOGO dominates PR, and PR dominates EDLP in Region 3.

To sum up, Corollary 4 is proved. □

Proof of Proposition 2. Let $\hat{c}=2\theta-1-\frac{(1-2h-\theta)\sqrt{2\theta_2}}{\sqrt{\theta}}+\sqrt{\frac{2\left(4h^2-4h(1-\theta)+\frac{(1-\theta)\left(\theta_1-2\theta^2+(1-2h-\theta)\sqrt{2\theta\theta_2}\right)}{\theta_2}\right)}{\theta}}$, $\check{c}=\frac{\theta(4\theta-1)}{\theta_2}-2(1-2h-\theta)\sqrt{\frac{2\theta}{\theta_2}}$, $\Phi_1=\frac{\theta_1-2\theta^2+(1-\theta)\sqrt{2\theta\theta_2}}{4\theta_2}$, $\Phi_2=\frac{1}{8}\left(4-4\theta+(1-4\theta)\sqrt{\frac{2\theta}{\theta_2}}\right)$, and $\Phi_3=\frac{2\theta_1-4\theta^2-\sqrt{\left((4\theta-1)\sqrt{\theta(5\theta-2)}-1+\theta\left(3-4\theta+8\theta^2\right)\right)\theta_2}}{4\theta_2}$. The supply chain's optimal solutions are presented in Table EC.8.

We will discuss all the results in three regions as follows:

Parameters	BOGO $(h,\ \theta)\in\omega_1$	BOGO $(h,\ \theta)\in\omega_2$ or $(h,\ \theta)\in\omega_3$ and $0<c<\check{c}$	BOGO $(h,\ \theta)\in\omega_4$ or $(h,\ \theta)\in\omega_3$ and $\check{c}<c<\hat{c}$
w_b	$\frac{3\theta+c+2c\theta}{2(1+2\theta)}$	$\frac{3\theta+c+2c\theta}{2(1+2\theta)}$	$\rho=\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}$
p_b	$\frac{2+c+\theta(7+2c)}{4(1+2\theta)}$	$\frac{c+4h+9\theta+2c\theta}{4(1+2\theta)}$	$\frac{2h+3\theta+\rho+2\theta\rho}{2+4\theta}$
p_g	$\frac{c(1+2\theta)+\theta(5+4\theta)}{4(1+2\theta)}$	$\frac{c+9\theta+2c\theta-8h\theta}{4(1+2\theta)}$	$\frac{\rho+\theta(3-4h+2\rho)}{2+4\theta}$
D_b	$\frac{3\theta-c-2c\theta}{4\theta}$	$\frac{3\theta-c-2c\theta}{4\theta}$	$\frac{3\theta-\rho-2\theta\rho}{2\theta}$
π_{br}	$\frac{\theta(8-7\theta+8\theta^2)+\tau_3}{16\theta(1+2\theta)}$ ‡	$\frac{\theta(9\theta-32h^2+32h(1-\theta))+\tau_3}{16\theta(1+2\theta)}$	$\frac{(\rho+\theta(2\rho-3))^2-8h\theta(h+\theta-1)}{4\theta(1+2\theta)}$
π_{bm}	$\frac{(3\theta-c-2c\theta)^2}{8\theta(1+2\theta)}$	$\frac{(3\theta-c-2c\theta)^2}{8\theta(1+2\theta)}$	$\frac{(c-\rho)(\rho+\theta(2\rho-3))}{2\theta}$

‡ $\tau_3=(1+2\theta)\left(c^2(1+2\theta)-6c\theta\right)$

Table EC.8 Manufacturer-Retailer Optimal Solutions in Two Periods with SP

- Region 1: $0<\theta\le 1$ and $\frac{1-\theta}{2}<h\le 1$. The manufacturer's problem is to maximize
$$\pi_{bm}(w_b)=\begin{cases}\frac{3\theta-w_b-2\theta w_b}{2\theta}(w_b-c)\,, & 0<w_b\le\theta\\ (1-w_b)(w_b-c)\,, & \theta<w_b\le 1\end{cases}.$$
Similar to the proof of Proposition 1, there are three cases as follows:
–Case 1: $\frac{3\theta+c+2c\theta}{2(1+2\theta)}<\frac{1+c}{2}<\theta$. When $0<w_b\le\theta$, $\pi_{bm}=\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}$. When $\theta<w_b\le 1$, $\pi_{bm}=(\theta-c)(1-\theta)$. By comparing the profits, we get, when $\frac{1+c}{2}<\theta<1$, $\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}>(\theta-c)(1-\theta)$, BOGO dominates EDLP for ${w_b}^*=\frac{3\theta+c+2c\theta}{2(1+2\theta)}$.
–Case 2: $\frac{3\theta+c+2c\theta}{2(1+2\theta)}<\theta\le\frac{1+c}{2}$. When $0<w_b\le\theta$, $\pi_{bm}=\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}$. When $\theta<w_b\le 1$, $\pi_{bm}=\frac{(1-c)^2}{4}$. By comparing the profits, we get, when $\frac{\left(1+c-c^2\right)+(1-c)\sqrt{1+4c+c^2}}{5-4c}<\theta\le\frac{1+c}{2}$, $\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}>\frac{(1-c)^2}{4}$, BOGO dominates EDLP for ${w_b}^*=\frac{3\theta+c+2c\theta}{2(1+2\theta)}$.
–Case 3: $\theta<\frac{3\theta+c+2c\theta}{2(1+2\theta)}<\frac{1+c}{2}$. When $0<w_b\le\theta$, $\pi_{bm}=(\theta-c)(1-\theta)$. When $\theta<w_b\le 1$, $\pi_{bm}=\frac{(1-c)^2}{4}$. By comparing the profits, we get, $(\theta-c)(1-\theta)<\frac{(1-c)^2}{4}$, EDLP dominates BOGO for ${w_b}^*=\frac{1+c}{2}$.
- Region 2: $0<\theta<\frac{2}{5}$ and $\frac{1-\theta}{2}-\frac{1}{2}\sqrt{\frac{\theta(1+2\theta)}{2}}<h\le\frac{1-\theta}{2}$. The manufacturer's problem is to maximize
$$\pi_{bm}(w_b)=\begin{cases}\frac{3\theta-w_b-2\theta w_b}{2\theta}(w_b-c)\,, & 0<w_b\le\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}\\ (1-w_b)(w_b-c)\,, & \theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}<w_b\le 1\end{cases}.$$
Note that $\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}=\rho$, there are three cases as follows:
–Case 1: $\frac{3\theta+c+2c\theta}{2(1+2\theta)}<\frac{1+c}{2}<\rho$. When $0<w_b\le\theta$, $\pi_{bm}=\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}$; when $\theta<w_b\le 1$, $\pi_{bm}=(\rho-c)(1-\rho)$. By comparing the profits, we get, $\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}<(\rho-c)(1-\rho)$, EDLP dominates BOGO.
–Case 2: $\frac{3\theta+c+2c\theta}{2(1+2\theta)}<\rho\le\frac{1+c}{2}$. When $0<w_b\le\theta,\pi_{bm}=\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}$; when $\theta<w_b\le 1,\pi_{bm}=\frac{(1-c)^2}{4}$. By comparing the profits, we get $\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}<\frac{(1-c)^2}{4}$, EDLP dominates BOGO.
–Case 3: $\rho<\frac{3\theta+c+2c\theta}{2(1+2\theta)}<\frac{1+c}{2}$. When $0<w_b\le\theta$, $\pi_{bm}=\frac{3\theta-\rho-2\theta\rho}{2\theta}(\rho-c)$; when $\theta<w_b\le 1$, $\pi_{bm}=\frac{(1-c)^2}{4}$. By comparing the profits, we get, $\frac{3\theta-\rho-2\theta\rho}{2\theta}(\rho-c)<\frac{(1-c)^2}{4}$, EDLP dominates BOGO.
- Region 3 (Region ω_2, Region ω_3 and Region ω_4): $\frac{2}{5}<\theta<1$ and $0<h\le\frac{1-\theta}{2}$. The manufacturer's problem is to maximize
$$\pi_{bm}(w_b)=\begin{cases}\frac{3\theta-w_b-2\theta w_b}{2\theta}(w_b-c)\,, & 0<w_b\le\theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}\\ (1-w_b)(w_b-c)\,, & \theta-(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}<w_b\le 1\end{cases}.$$

In this Region, we have three cases as follows:

–Case 1: $\frac{3\theta+c+2c\theta}{2(1+2\theta)} < \frac{1+c}{2} < \rho$. When $0 < w_b \le \theta$, $\pi_{bm} = \frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}$; when $\theta < w_b \le 1$, $\pi_{bm} = (\rho - c)(1-\rho)$. By comparing the profits, we get, when $\frac{1}{2} < \theta < \frac{(5+\sqrt{13})}{12}$, $\frac{1}{8}\left(4 - 4\theta - \sqrt{2}\sqrt{-2+\frac{1}{\theta} - 4\theta + 8\theta^2}\right) < h < \frac{1-\theta}{2}$, and $0 < c < 2\theta - 1 - 2(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}$; or $\frac{5+\sqrt{13}}{12} \le \theta < 1$, $0 < h < \frac{1-\theta}{2}$, and $0 < c < 2\theta - 1 - 2(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}$, $\frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)} > (\rho - c)(1-\rho)$, BOGO dominates EDLP for ${w_b}^* = \frac{3\theta+c+2c\theta}{2(1+2\theta)}$.

–Case 2: $\frac{3\theta+c+2c\theta}{2(1+2\theta)} < \rho \le \frac{1+c}{2}$. When $0 < w_b \le \theta$, $\pi_{bm} = \frac{(c-3\theta+2c\theta)^2}{8\theta(1+2\theta)}$; when $\theta < w_b \le 1$, $\pi_{bm} = \frac{(1-c)^2}{4}$. By comparing the profits and combining the result in Case 1, we get, when $\frac{2}{5} < \theta < 1$, $\Phi_1 < h < \frac{1-\theta}{2}$ (Region ω_2), and $0 < c < \theta - (1-\theta)\sqrt{\frac{2\theta}{1+2\theta}}$; or $\frac{2}{5} < \theta < 1$, $\max\{\Phi_2, 0\} < h < \Phi_1$ (Region ω_3), and $0 < c < \frac{\theta(4\theta-1)}{1+2\theta} - 2(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}}$, BOGO dominates EDLP for ${w_b}^* = \frac{3\theta+c+2c\theta}{2(1+2\theta)}$.

–Case 3: $\rho < \frac{3\theta+c+2c\theta}{2(1+2\theta)} < \frac{1+c}{2}$. When $0 < w_b \le \theta$, $\pi_{bm} = \frac{3\theta-\rho-2\theta\rho}{2\theta}(\rho - c)$; when $\theta < w_b \le 1$, $\pi_{bm} = \frac{(1-c)^2}{4}$. By comparing the profits, we get, when $\frac{2}{5} < \theta < \frac{3\sqrt{57}-1}{32}$, $\max\{\Phi_3, 0\} < h < \theta_2$ (Region ω_4), and $0 < c < \hat{c}$; or $\frac{2}{5} < \theta < 1$, $\max\{\Phi_2, 0\} < h < \Phi_1$ (Region ω_3), and $\frac{\theta(4\theta-1)}{1+2\theta} - 2(1-2h-\theta)\sqrt{\frac{2\theta}{1+2\theta}} = \check{c} < c < \hat{c}$, $\frac{3\theta-\rho-2\theta\rho}{2\theta}(\rho - c) > \frac{(1-c)^2}{4}$, BOGO dominates EDLP for ${w_b}^* = \rho$.

To sum up, Proposition 2 is proved. □

Appendix EC.4: Multi-Period and Demand Uncertainty

Let $T_m = m + T$, $m \in \mathbb{N}^+$, and $T_\theta = 1 + T_1\theta$. Proposition EC.1 identifies the retailer's optimal policy in BOGO-MSPP.

Proposition EC.1. *With BOGO-MSPP, the retailer's best response between BOGO and EDLP is:*

a) If $(1-\theta)/2 < h \le 1$ and $0 < w_b < \theta$, offer a BOGO with IS (i.e., ${p_b}^* < {p_g}^* + h$).

b) If $\max\left\{(1-\theta)/2 - \sqrt{\theta+\theta^2 T_1}/2\sqrt{T_1} \equiv \underline{\hat{\theta}}, 0\right\} < h \le (1-\theta)/2$ and $0 < w_b < \theta - (1-2h-\theta)\sqrt{\theta T_1}/\sqrt{T_\theta}$, offer a BOGO with BS (i.e., ${p_b}^* = {p_g}^* + h$).

c) Otherwise, offer an EDLP.

Proof of Proposition EC.1. Consumer choices in MSPP without stockpiling under BOGO and PR are shown in Table EC.9. Hence, the retailer's optimal pricing policy in the multi-period model still follows Proposition 1. Without stockpiling, a consumer's consumption patterns coincide with her purchase patterns, which results in T not affecting the retailer's optimal policy.

With stockpiling, the retailer's optimal solutions are presented in Table EC.10.

In Lemma 3, we have shown that the retailer will offer a BOGO according to the interior solution ($p_b^* < p_g^* + h$) or the boundary solution ($p_b^* = p_g^* + h$). Consider a multi-period model in which there are T ($T \ge 1$, $T \in \mathbb{N}^+$) regular-price periods after a promotion period. We first investigate the scenario with $T = 2$. Similar to the proof of Lemma 3, the retailer's best response between BOGO and EDLP is as follows: If $\frac{1-\theta}{2} < h \le 1$ and $0 < w_b < \theta$, the optimal interior solution

Options	BOGO-MSPP		PR-MSPP	
	Case I: $0<\theta\le p_g/p_b$	Case II: $p_g/p_b<\theta\le 1$	Case I: $0<\theta\le p_d/p_r$	Case II: $p_d/p_r<\theta\le 1$
$(2,1,...,1)_{T+1}$	$\frac{p_g}{\theta}\le v\le 1$	$p_b\le v\le 1$	$\frac{p_d}{\theta}\le v\le 1$	$p_r\le v\le 1$
$(2,0,...,0)_{T+1}$	N/A	$\frac{p_b+p_g}{1+\theta}\le v<p_b$	N/A	$\frac{p_d}{\theta}\le v<p_r$
$(1,1,...,1)_{T+1}$	$p_b\le v<\frac{p_g}{\theta}$	N/A	$p_r\le v<\frac{p_d}{\theta}$	N/A
$(1,0,...,0)_{T+1}$	N/A	N/A	$p_d\le v<p_r$	$p_d\le v<\frac{p_d}{\theta}$
$(0,0,...,0)_{T+1}$	$0\le v<p_b$	$0\le v<\frac{p_b+p_g}{1+\theta}$	$0\le v<p_d$	$0\le v<p_d$

Table EC.9 Consumer Choices in MSPP with BOGO and PR, T $(T\ge 1,\ T\in\mathbb{N}^+)$

Parameters	BOGO	
	Interior Solution	Boundary Solution
p_b	$\frac{1+w}{2}$	$\frac{2h+w+(2+T+(1+T)w)\theta}{2(1+(1+T)\theta)}$
p_g	$\frac{\theta+w}{2}$	$\frac{w+(2+T-2h(1+T)+(1+T)w)\theta}{2(1+(1+T)\theta)}$
D_b	$\frac{(2+T)\theta-w(1+(1+T)\theta)}{2\theta}$	$\frac{(2+T)\theta-w(1+(1+T)\theta)}{2\theta}$
π_{br}	$\frac{(1+\theta+T\theta)w^2-2(2+T)\theta w+\theta(1+T+\theta)}{4\theta}$	$\frac{h(1-\theta+h(T\theta-1))}{1+(1+T)\theta}+\tau_2$‡

Parameters	PR	
	Interior Solution	Boundary Solution
p_r	$\frac{1+w}{2}$	$\frac{w+(2+T+(1+T)w)\theta+2h(1+\theta)}{2(1+(1+T)\theta)}$
p_d	$\frac{2\theta+(1+\theta)w}{2(1+\theta)}$	$\frac{w+(2+w+T(1-2h+w))\theta}{2(1+(1+T)\theta)}$
D_d	$\frac{(2+T)\theta-w(1+(1+T)\theta)}{2\theta}$	$\frac{(2+T)\theta-w(1+(1+T)\theta)}{2\theta}$
π_{dr}	$\frac{T(1-w)^2}{4}+\frac{(w-(2-w)\theta)^2}{4\theta(1+\theta)}$	τ_2

‡ $\tau_2=\frac{(1+(1+T)\theta)^2w^2-2(2+T)(1+(1+T)\theta)w\theta+\theta(4(1-h)hT+\theta(4+T(4-4h(1+h)+T)))}{4\theta(1+(1+T)\theta)}$

Table EC.10 Retailer's Optimal Policies with MSPP, $w_b=w_d=w$**,** T $(T\ge 1,\ T\in\mathbb{N}^+)$

is $(p_b^*,p_g^*)=\left(\frac{1+w_b}{2},\frac{\theta+w_b}{2}\right)$. If $\max\left\{0,\frac{1-\theta}{2}-\frac{\sqrt{3\theta(1+3\theta)}}{6}\right\}$ and $0<w_b<\theta-(1-2h-\theta)\sqrt{\frac{3\theta}{1+3\theta}}$, the optimal boundary solution is $(p_b^*,p_g^*)=\left(\frac{2h+w_b+4\theta+3w_b\theta}{1+3\theta},\frac{w_b+(4-6h+3w_b)\theta}{1+3\theta}\right)$. Otherwise, it is optimal for the retailer to offer a single price. Again, the optimal BOGO pricing policy with $T=3$ is as follows: If $\frac{1-\theta}{2}<h\le 1$ and $0<w_b<\theta$, the optimal interior solution is $(p_b^*,p_g^*)=\left(\frac{1+w_b}{2},\frac{\theta+w_b}{2}\right)$. If $\max\left\{0,\frac{1-\theta}{2}-\frac{\sqrt{\theta(1+4\theta)}}{4}\right\}$ and $0<w_b<\theta-(1-2h-\theta)\sqrt{\frac{4\theta}{1+4\theta}}$, the optimal boundary solution is $(p_b^*,p_g^*)=\left(\frac{2h+w_b+5\theta+4w_b\theta}{2(1+4\theta)},\frac{w_b+(5-8h+4w_b)\theta}{2(1+4\theta)}\right)$. We show that the retailer can counter stockpiling as long as the optimal prices meet the necessary and sufficient conditions as follows: $(p_b^*,p_g^*)\in\left\{p_b^*-p_g^*\le h, p_g^*\le\theta, p_g^*/p_b^*>\theta\right\}$. Hence, we obtain the retailer's best response between BOGO and EDLP is as follows: If $\frac{1-\theta}{2}<h\le 1$ and $0<w_b<\theta$, the optimal interior solution is $(p_b^*,p_g^*)=\left(\frac{1+w_b}{2},\frac{\theta+w_b}{2}\right)$. If $\max\left\{0,\frac{1-\theta}{2}-\frac{1}{2}\sqrt{\frac{\theta+\theta^2+T\theta^2}{1+T}}\right\}$ and $0<w_b<\theta-(1-2h-\theta)\sqrt{\frac{(1+T)\theta}{1+(1+T)\theta}}$, the optimal boundary solution is $(p_b^*,p_g^*)=\left(\frac{2h+w_b+(2+T+(1+T)w_b)\theta}{2(1+(1+T)\theta)},\frac{w_b+(2+T-2h(1+T)+(1+T)w_b)\theta}{2(1+(1+T)\theta)}\right)$. Otherwise, EDLP dominates BOGO, and the retailer will offer a single price. Proposition EC.1 follows. □

Proposition EC.2 shows the retailer's optimal response in PR-MSPP.

PROPOSITION EC.2. *With PR-MSPP, the retailer's best response between PR and EDLP is:*

a) If $1/3 < \theta \le 1$, $(1-\theta)/2\theta_1 < h \le 1$ *and* $0 < w_d < \theta - (1-\theta)\sqrt{\theta}/\sqrt{\theta_1}$, *offer a PR with IS (i.e.,* $p_r{}^* < p_d{}^* + h$*).*

b) If $1/3 < \theta \le 1$, $\max\left\{(1-\theta)/2\theta_1 - \sqrt{(3\theta-1)T_\theta}/2\theta_1\sqrt{T} \equiv \tilde{\underline{\theta}}, 0\right\} < h \le (1-\theta)/2\theta_1$ *and* $0 < w_d < \theta - \sqrt{\theta[(1-\theta)^2 + T(4h^2\theta_1 - 4h(1-\theta) + (1-\theta)^2)]}/\sqrt{T_\theta}$, *offer a PR with BS (i.e.,* $p_r{}^* = p_d{}^* + h$*).*

c) Otherwise, offer an EDLP.

Proof of Propostion EC.2. The proof is similar to that of Proposition EC.1 and is omitted. □

Proof of Corollary 5. The proof is similar to that of Corollary 4 and is omitted. □

Proof of Proposition 3. Let $\Psi_1 = \frac{1-\theta}{4} + \frac{(1-\theta)\sqrt{T_1}}{4}\sqrt{\frac{\theta}{T_\theta}}$, $\Psi_2 = \frac{1-\theta}{2} - \frac{2\theta T_1 - T}{4}\sqrt{\frac{\theta}{T_1 T_\theta}}$, $\Psi_3 = \frac{1-\theta}{2} - \frac{\sqrt{(3+TT_2)\theta - T_1 - 4TT_1\theta^2 + 4T_1{}^2\theta^3 + 2(2\theta T_1 - T)\sqrt{\theta(3\theta-1+T(2\theta-1))}}}{4\sqrt{T_1 T_\theta}}$, $\tilde{c}(T) = \theta - (1-\theta)\sqrt{\frac{T_1\theta}{T_\theta}}$, $\check{c}(T) = \frac{\theta(2\theta T_1 - T)}{T_\theta} - 2(1-2h-\theta)\sqrt{\frac{\theta T_1}{T_\theta}}$, and $\hat{c}(T) = 2\theta - 1 - \frac{2(1-2h-\theta)\sqrt{T_\theta}}{\sqrt{\theta T_1}} + 2\sqrt{\frac{4h(h+\theta-1)}{\theta T_1} + \frac{(1-\theta)\left(1+T(1-\theta)\theta+\theta^2+(1-2h-\theta)\sqrt{\theta T_1 T_\theta}\right)}{\theta T_1 T_\theta}}$. We present the optimal solutions for the supply chain in Table EC.11.

Parameters	BOGO	BOGO	BOGO
	$(h,\ \theta) \in \Omega_1$	$(h,\ \theta) \in \Omega_2$ or $(h,\ \theta) \in \Omega_3$ and $0 < c < \check{c}(T)$	$(h,\ \theta) \in \Omega_4$ or $(h,\ \theta) \in \Omega_3$ and $\check{c}(T) < c < \hat{c}(T)$
w_b	$\frac{c+(2+c+c(1+T))\theta}{2(1+(1+T)\theta)}$	$\frac{c+(2+c+c(1+T))\theta}{2(1+(1+T)\theta)}$	$\xi^{\ddagger\ddagger}$
p_b	$\frac{2+c+(4+c+(3+c)T)\theta}{4(1+\theta+T\theta)}$	$\frac{c+4h+(6+c+(3+c)T)\theta}{4(1+\theta+T\theta)}$	$\frac{2h+\xi+\theta(2+T+\xi+T\xi)}{2(1+\theta+T\theta)}$
p_g	$\frac{1}{2}\left(\theta + \frac{c+(2+c+T+cT)\theta}{2(1+\theta+T\theta)}\right)$	$\frac{c+(6+c-4h+(3+c-4h)T)\theta}{4(1+\theta+T\theta)}$	$\frac{\xi+\theta(2+T-2h(1+T)+\xi+T\xi)}{2(1+\theta+T\theta)}$
D_b	$\frac{(2+T)\theta-c(1+\theta+T\theta)}{4\theta}$	$\frac{(2+T)\theta-c(1+\theta+T\theta)}{4\theta}$	$\frac{(2+T)\theta-(1+\theta+T\theta)\xi}{2\theta}$
π_{br}	$\frac{(1+T)(1-2h-\theta)^2}{4(1+\theta+T\theta)} + \tau_4{}^{\ddagger}$	τ_4	$\tau_5{}^{\ddagger\ddagger\ddagger}$
π_{bm}	$\frac{(c+(c-2-(1-c)T)\theta)^2}{8\theta(1+\theta+T\theta)}$	$\frac{(c+(-2+c+(-1+c)T)\theta)^2}{8\theta(1+\theta+T\theta)}$	$\frac{(c-\xi)(\xi+\theta(-2+T(-1+\xi)+\xi))}{2\theta}$

‡ $\tau_4 = \frac{c^2(1+\theta+T\theta)^2 - 2c(2+T)\theta(1+\theta+T\theta) + \theta\left(16(1-h)h(1+T) + \left((2+T)^2 - 16h(1+T)\right)\theta\right)}{16\theta(1+\theta+T\theta)}$

‡‡ $\xi = \theta - (1-2h-\theta)\sqrt{\frac{(1+T)\theta}{1+(1+T)\theta}}$

‡‡‡ $\tau_5 = \frac{4h(1+T)(1-\theta)\theta - 4h^2(1+T)\theta + (\xi+\theta(-2+T(-1+\xi)+\xi))^2}{4\theta(1+\theta+T\theta)}$

Table EC.11 Manufacturer-Retailer Optimal Policies with MSPP, T $(T \ge 1,\ T \in \mathbb{N}^+)$

The rest of the proof is similar to that of Proposition 2 and is omitted. □

Options	BOGO-MCPP		PR-MCPP	
	Case I: $0<\theta\leq p_g/p_b$	Case II: $p_g/p_b<\theta\leq 1$	Case I: $0<\theta\leq p_d/p_r$	Case II: $p_d/p_r<\theta\leq 1$
$(\underbrace{2,2,...,2}_{n_i},\underbrace{1,1,...,1}_{\mathcal{T}-n_i})$	$\frac{p_g}{\theta}\leq v\leq 1$	$p_b\leq v\leq 1$	$\frac{p_d}{\theta}\leq v\leq 1$	$p_r\leq v\leq 1$
$(\underbrace{2,2,...,2}_{n_i},\underbrace{0,0,...,0}_{\mathcal{T}-n_i})$	N/A	$\frac{p_b+p_g}{1+\theta}\leq v<p_b$	N/A	$\frac{p_d}{\theta}\leq v<p_r$
$(\underbrace{1,1,...,1}_{\mathcal{T}})$	$p_b\leq v<\frac{p_g}{\theta}$	N/A	$p_r\leq v<\frac{p_d}{\theta}$	N/A
$(\underbrace{1,1,...,1}_{n_i},\underbrace{0,0,...,0}_{\mathcal{T}-n_i})$	N/A	N/A	$p_d\leq v<p_r$	$p_d\leq v<\frac{p_d}{\theta}$
$(\underbrace{0,0,...,0}_{\mathcal{T}})$	$0\leq v<p_b$	$0\leq v<\frac{p_b+p_g}{1+\theta}$	$0\leq v<p_d$	$0\leq v<p_d$

Table EC.12 Consumer Choices in MCPP with BOGO and PR, $\mathcal{T}$ $(\mathcal{T}\geq 2,\ \mathcal{T}\in\mathbb{N}^+)$

Proof of Proposition 4. Consumer choices in MCPP without stockpiling are presented in Table EC.4. The retailer's expected profit functions under Cases I and II of BOGO are

$$\pi_{br,I}(p_b,\,p_g)=\underbrace{\mathcal{T}\left[(1-p_b)(p_b-w_b)\right]}_{Regular\ price\ sales}+\underbrace{n_1\left(1-\frac{p_g}{\theta}\right)(p_g-w_b)}_{BOGO\ price\ sales}$$

and

$$\pi_{br,II}(p_b,\,p_g)=\underbrace{\left[n_1\left(p_b-\frac{p_b+p_g}{1+\theta}\right)+\mathcal{T}(1-p_b)\right](p_b-w_b)}_{Regular\ price\ sales}+\underbrace{n_1\left(1-\frac{p_b+p_g}{1+\theta}\right)(p_g-w_b)}_{BOGO\ price\ sales}$$

respectively. The retailer's expected profit function under Cases I and II of PR is

$$\pi_{dr,I}(p_r,\,p_d)=\pi_{dr,II}(p_r,\,p_d)=\underbrace{(\mathcal{T}-n_2)(1-p_r)(p_r-w_d)}_{Regular\ price\ sales}+\underbrace{n_2\left(2-p_d-\frac{p_d}{\theta}\right)(p_d-w_d)}_{Reduced\ price\ sales}.$$

The optimal solutions to Proposition 4 are provided in Table EC.13.

	BOGO	PR
Regular price	$p_b=\frac{1+w}{2}$	$p_r=\frac{1+w}{2}$
Promotional price	$p_g=\frac{\theta+w}{2}$	$p_d=\frac{2\theta+w+\theta w}{2(1+\theta)}$
Product demand	$D_b=\frac{n_1(\theta-w)+\theta\mathcal{T}(1-w)}{2\theta}$	$D_d=\frac{n_2(\theta-w)+\theta\mathcal{T}(1-w)}{2\theta}$
Retailer's profit	$\pi_{br}=\frac{n_1(\theta-w)^2+\theta\mathcal{T}(1-w)^2}{4\theta}$	$\pi_{dr}=\frac{\mathcal{T}(1-w)^2}{4}+\frac{n_2}{4}\left(\frac{3\theta-1}{\theta+1}+\frac{w^2}{\theta}-2w\right)$

Table EC.13 Retailer's Optimal Policies with MCPP, $w_b=w_d=w$, $\mathcal{T}$ $(\mathcal{T}\geq 2,\ \mathcal{T}\in\mathbb{N}^+)$

We further show the retailer's optimal policies in Figure EC.1. If $n_1\geq n_2$, as shown in Figure EC.1a, BOGO fully dominates PR with higher *profit margin* and *cumulative sales.* If $n_2>n_1$, as shown in Figures EC.1b, EC.1c, and EC.1d, PR could dominate BOGO. Also, PR is more profitable if θ is large and/or w is small, which means PR can dominate BOGO due to large cumulative sales and a high profit margin. As θ increases, the disadvantage of PR's profit margin decreases. As w decreases, the advantage of more sales with PR increases. As n_2 increases, the *cumulative sales effect* is more significant, which helps PR outperform BOGO in a larger region of w-θ space.

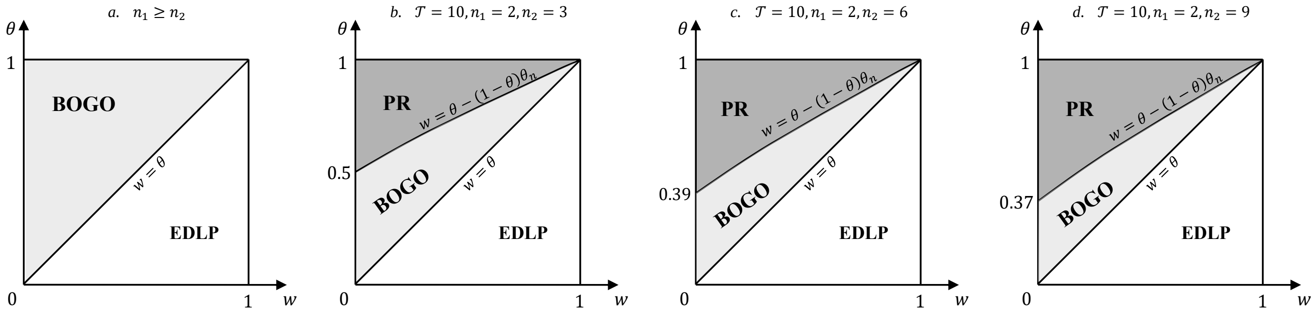

Figure EC.1 Retailer's Optimal Policies with MCPP

The rest of the proof is similar to that of Lemma 1 and is omitted. □

Proof of Proposition 5. Let $c_{m1} = \theta + 2\theta_n(\theta - 1) + \frac{(\theta-1)\theta\mathcal{T}}{n_2+\theta\mathcal{T}}$, $c_{m2} = \frac{2(\theta-1)\theta_n(n_2+\theta\mathcal{T})}{n_1+\theta\mathcal{T}} + \frac{\theta(n_1+(2\theta-1)\mathcal{T})}{n_1+\theta\mathcal{T}} - \frac{2(\theta-1)\theta}{n_1+\theta\mathcal{T}}\sqrt{\frac{\theta_n\mathcal{T}(n_2-n_1)+n_2^2}{\theta} + \frac{n_2(\mathcal{T}-n_2)}{\theta+1}}$, and $c_{m3} = \theta - (1-\theta)\theta\sqrt{\frac{\mathcal{T}}{\theta(n_1+\theta\mathcal{T})}}$, where $\theta_n = \sqrt{\frac{\theta n_2}{(\theta+1)(n_2-n_1)}}$. The optimal solutions for the supply chain are provided in Table EC.14. We further show the supply chain's equilibrium in Figure EC.2. If $n_1 \geq n_2$, as shown in Figure EC.2a, BOGO and EDLP will be the better alternatives for the manufacturer and supply chain. If $n_1 < n_2$, as shown in Figures EC.2b, EC.2c, and EC.2d, PR becomes optimal when θ is high and/or c is low. Also, the regions where PR outperforms BOGO and EDLP increase with PR's duration.

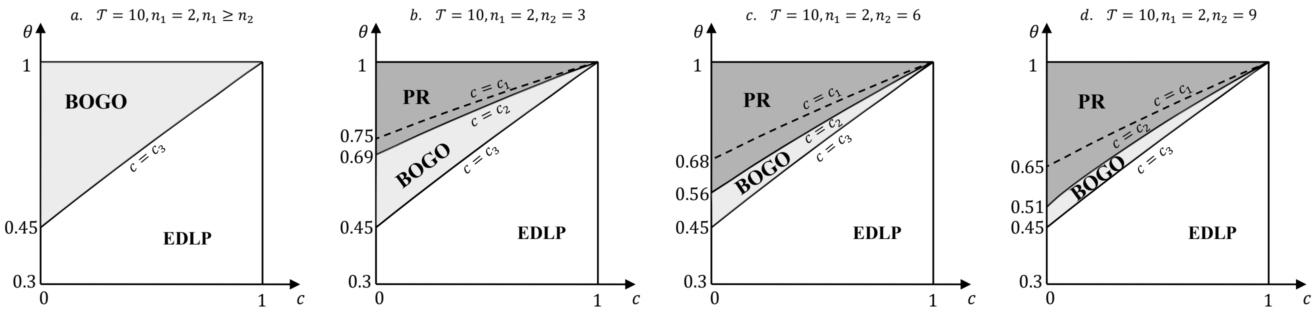

Figure EC.2 Optimal Policies for the SC in MCPP

According to Proposition 4, the manufacturer's problem is to maximize

$$\pi_m(w) = \begin{cases} \frac{(w-c)[n_2(\theta-w)-\theta\mathcal{T}(w-1)]}{2\theta}, & 0 < w \leq \theta + (\theta-1)\sqrt{\frac{\theta n_2}{(\theta+1)(n_2-n_1)}} \\ \frac{(w-c)[n_1(\theta-w)-\theta\mathcal{T}(w-1)]}{2\theta}, & \theta + (\theta-1)\sqrt{\frac{\theta n_2}{(\theta+1)(n_2-n_1)}} < w \leq \theta \\ \frac{\mathcal{T}(1-w)(w-c)}{2}, & \theta < w < 1 \end{cases} \cdot$$

Parameters	BOGO
	Interior Solution
w_b	$\frac{1}{2}\left(c+\frac{\theta(n_1+\mathcal{T})}{n_1+\theta\mathcal{T}}\right)$
p_b	$\frac{1}{4}\left(c+\frac{\theta(n_1+\mathcal{T})}{n_1+\theta\mathcal{T}}+2\right)$
p_g	$\frac{1}{2}\left(\frac{1}{2}\left(c+\frac{\theta(n_1+\mathcal{T})}{n_1+\theta\mathcal{T}}\right)+\theta\right)$
D_b	$\frac{\theta(n_1+\mathcal{T})-c(n_1+\theta\mathcal{T})}{4\theta}$
π_{br}	$\frac{1}{16}\left(c^2\left(\frac{n_1}{\theta}+\mathcal{T}\right)-2c(n_1+\mathcal{T})+\frac{\theta n_1^2+2(\theta(2\theta-3)+2)n_1\mathcal{T}+\theta\mathcal{T}^2}{n_1+\theta\mathcal{T}}\right)$
π_{bm}	$\frac{(\theta(n_1+\mathcal{T})-c(n_1+\theta\mathcal{T}))^2}{8\theta(n_1+\theta\mathcal{T})}$

Parameters	PR	
	Interior Solution	Boundary Solution
w_d	$\frac{1}{2}\left(c+\frac{\theta(n_2+\mathcal{T})}{n_2+\theta\mathcal{T}}\right)$	$\theta-(1-\theta)\sqrt{\frac{\theta n_2}{(\theta+1)(n_2-n_1)}}$
p_r	$\frac{1}{4}\left(c+\frac{\theta(n_2+\mathcal{T})}{n_2+\theta\mathcal{T}}+2\right)$	$\frac{w_d+1}{2}$
p_d	$\frac{1}{4}\left(c+\frac{\theta((\theta+5)n_2+5\theta\mathcal{T}+\mathcal{T})}{(\theta+1)(n_2+\theta\mathcal{T})}\right)$	$\frac{2\theta+\theta w_d+w_d}{2\theta+2}$
D_d	$\frac{\theta(n_2+\mathcal{T})-c(n_2+\theta\mathcal{T})}{4\theta}$	$\frac{n_2(\theta-w_d)-\theta\mathcal{T}(w_d-1)}{2\theta}$
π_{dr}	$\frac{1}{16}\left(\frac{c^2(n_2+\theta\mathcal{T})}{\theta}-2c(n_2+\mathcal{T})+\tau_6^{\ddagger}\right)$	$\frac{1}{4}\left(n_2\left(\frac{3\theta-1}{\theta+1}+\frac{w_d^2}{\theta}-2w_d\right)+\mathcal{T}(w_d-1)^2\right)$
π_{dm}	$\frac{(\theta(n_2+\mathcal{T})-c(n_2+\theta\mathcal{T}))^2}{8\theta(n_2+\theta\mathcal{T})}$	$\frac{(c-w_d)(n_2(w_d-\theta)+\theta\mathcal{T}(w_d-1))}{2\theta}$

‡ $\tau_6=\frac{\left(9\theta-3\theta^2-4\right)n_2^2+2\left(3\theta^2-3\theta+2\right)n_2\mathcal{T}+\theta(\theta+1)\mathcal{T}^2}{(\theta+1)(n_2+\theta\mathcal{T})}$

Table EC.14 Manufacturer-Retailer Optimal Policies with MCPP, $\mathcal{T}$ ($\mathcal{T}\geq 2,\ \mathcal{T}\in\mathbb{N}^+$)

The rest of the proof is similar to that of Proposition 2 and is omitted. □

Lemma EC.1 identifies the retailer's optimal policy in PR-UNSP.

LEMMA EC.1. *In a DSC with PR-UNSP, for a wholesale price w_d, the retailer's best response is:*

a) If $0<w_d\leq\min\left\{2\theta^2/\theta_1,(\theta(2\Lambda_d-1)-(1-\theta)\sqrt{\theta\Lambda_d/\theta_1})/(\theta(\Lambda_d-1)+\Lambda_d)\right\}\equiv w_d^K$, offer a PR.

b) Otherwise, offer an EDLP.

Proof of Lemma EC.1. The retailer's expected profit functions under BOGO-UNSP are

$$\pi_{br,I}(p_b,\ p_g)=\underbrace{(1+\Lambda_b)\left[(1-p_b)(p_b-w_b)\right]}_{\text{Regular price sales}}+\underbrace{\Lambda_b\left(1-\frac{p_g}{\theta}\right)(p_g-w_b)}_{\text{BOGO price sales}}$$

and

$$\pi_{br,II}(p_b,\ p_g)=\underbrace{\left[\Lambda_b\left(1-\frac{p_b+p_g}{1+\theta}\right)+(1-p_b)\right](p_b-w_b)}_{\text{Regular price sales}}+\underbrace{\Lambda_b\left(1-\frac{p_b+p_g}{1+\theta}\right)(p_g-w_b)}_{\text{BOGO price sales}}$$

respectively. The retailer's expected profit function under PR-UNSP is

$$\pi_{dr,I}(p_r,\ p_d)=\pi_{dr,II}(p_r,\ p_d)=\underbrace{(1-p_r)(p_r-w_d)}_{\text{Regular price sales}}+\underbrace{\Lambda_d\left(2-p_d-\frac{p_d}{\theta}\right)(p_d-w_d)}_{\text{Reduced price sales}}.$$

The rest of the proof is similar to that of Lemma 2 and is omitted. □

Proof of Proposition 6. Given a wholesale price $w = w_d = w_b$, we get $\pi^*_{br} = \frac{(1-w)^2\theta+(w^2+\theta-(4-w)w\theta+\theta^2)\Lambda_b}{4\theta}$ and $\pi^*_{dr} = \frac{(1-w)^2}{4} + \frac{(w-(2-w)\theta)^2\Lambda_d}{4\theta(1+\theta)}$. As long as $\Lambda_d > \Lambda_b$ and $w < w^K = \min\left\{\frac{2\theta^2}{1+\theta}, \frac{2\theta}{1+\theta} - \frac{1-\theta}{1+\theta}\sqrt{\frac{\theta\Lambda_b}{(\Lambda_d-\Lambda_b)}}\right\}$, PR dominates BOGO; otherwise, BOGO or EDLP is optimal for the retailer.

Moreover, as we mentioned in the main text, new and existing consumers behave the same under PR. For BOGO, existing consumers may buy two or one unit in period 1. New consumers may behave in one of two ways. Some consumers whose interest in the product is raised by BOGO but are uncertain about it may buy only one unit. Some new consumers, motivated by BOGO, may buy two units. We prove that assuming all new consumers in BOGO buy two units or having some buy one unit leads to the same qualitative result. When new consumers are only attracted by buying two units, we get the critical wholesale price is $w^{\bar{K}} = \min\left\{\frac{2\theta^2}{1+\theta}, \frac{2\theta(1+\theta)(\Lambda_d-\Lambda_b)-\sqrt{(1-\theta)^2\theta(4\theta+(1-\theta)^2\Lambda_b)\Lambda_d-\theta(1-\theta^2)^2\Lambda_b}}{\Lambda_d-1+\theta(2-4\Lambda_b+\theta(\Lambda_d-1)+2\Lambda_d)}\right\}$. Note that $\Lambda_d = 1+\lambda_d$ and $\Lambda_b = 1+\lambda_b$, to be consistent, we use Λ_d and Λ_b to do the mathematical transformation. Given the value of $\{\theta, \Lambda_d, \Lambda_b\}$ in the following range:

Parameters	Range
θ	$0.01 - 1.00$
Λ_d	$1.01 - 5.00$
Λ_b	$1.01 - \Lambda_d$

Table EC.15 Numerical Analysis for the Critical Wholesale Prices

Define $\Delta = \frac{\|w^{\bar{K}}-w^K\|}{w^K} \times 100\%$ as the deviation of the two critical wholesale prices, given the value in the table above, we get that $\Delta \in [0, 6.5\%]$ when $\Lambda_d \le 2$, $\Delta \in [0, 8.7\%]$ when $\Lambda_d \le 3$, $\Delta \in [0, 9.3\%]$ when $\Lambda_d \le 4$, $\Delta \in [0, 11.4\%]$ when $\Lambda_d \le 5$. We show that these two critical wholesale prices are very close, and for cases where the expected number of new consumers is as large as the number of exiting consumers, the two critical wholesale prices are within 6.5% from each other. To simplify the analysis, we assume new consumers may buy one unit in the main text. □

Proposition EC.3 characterizes the SC's equilibrium in UNSP model. We show that w^* must be one of five possible solutions: $w^*_1 = (c\Lambda_d + \theta(1+c+(2+c)\Lambda_d))/2(\theta+\theta_1\Lambda_d)$ (PR-IS), $w^*_2 = w^K$ (PR-BS), $w^*_3 = (c\Lambda_b + \theta(1+c+(2+c)\Lambda_b))/2(\theta+\theta_1\Lambda_b)$ (BOGO-IS), $w^*_4 = \theta$ (BOGO-BS), and $w^*_5 = (1+c)/2$ (EDLP).

Proposition EC.3. *In a DSC with UNSP, choice among BOGO, PR, and EDLP is as follows:*

a) If $\Lambda_d \le \Lambda_b$,

i) when $0 \le c < \min\{c_5, c_6\}$, *the SC offers a BOGO with IS,* $w^* = w_3^*$;

ii) when $\min\{c_5, c_6\} = c_6$,

i. *if* $c_6 \le c < c_7$, *the SC offers a BOGO with BS,* $w^* = w_4^*$;

ii. *if* $c_7 \le c \le 1$, *the SC offers an EDLP,* $w^* = w_5^*$;

iii) when $\min\{c_5, c_6\} = c_5$ *and* $c_5 \le c \le 1$, *the SC offers an EDLP,* $w^* = w_5^*$.

b) If $\Lambda_d > \Lambda_b$,

i) when $0 \le c < c_1$, *the SC offers a PR with IS,* $w^* = w_1^*$;

ii) when $c_1 \le c < \min\{c_2, c_3, c_4\}$, *the SC offers a PR with BS,* $w^* = w_2^*$;

iii) when $\min\{c_2, c_3, c_4\} = c_2$ *and* $c_2 \le c < \min\{c_5, c_6\}$, *the SC follows the policy in (a)-(i);*

iv) when $\min\{c_2, c_3, c_4\} = c_2$ *and* $\min\{c_5, c_6\} \le c < 1$, *the SC follows the policies in (a)-(ii) or (a)-(iii) depending on the values of* c_5 *and* c_6;

v) when $\min\{c_2, c_3, c_4\} = c_3$,

i. *if* $c_3 \le c < c_7$, *the SC offers a BOGO with BS,* $w^* = w_4^*$;

ii. *if* $c_7 \le c \le 1$, *the SC offers an EDLP,* $w^* = w_5^*$;

vi) when $\min\{c_2, c_3, c_4\} = c_4$ *and* $c_4 \le c \le 1$, *the SC offers an EDLP,* $w^* = w_5^*$.

Proof of Propostion EC.3. We define some critical production costs as follows:

$$c_1 = \begin{cases} \frac{2\theta-(1-\theta)(2\eta_1+\eta_2)}{1+\theta}, & 0<\theta<\frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)} \\ \frac{\eta_2\left(4\theta^2(1+\Lambda_d)+\theta(2\Lambda_d-1)-(2\Lambda_d+1)\right)}{1+\theta}, & \frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)}\le\theta\le 1 \end{cases},$$

$$c_2 = \begin{cases} \frac{\eta_3(3\theta+2(1+\theta)\Lambda_b-1-2\eta_1(1-\theta))}{1+\theta} - \frac{2\eta_1\eta_3(1-\theta)\Lambda_d}{\theta} + \frac{2\eta_3\eta_4(1-\theta)}{(1+\theta)\theta}, & 0<\theta<\frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)} \\ \frac{\eta_3(2\Lambda_b-1+\theta(2\Lambda_b-1)-4\Lambda_d+4\theta^2(1+\Lambda_d))+2\sqrt{2}(1-\theta)\eta_3\eta_5}{1+\theta}, & \frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)}\le\theta\le 1 \end{cases},$$

$$c_3 = \begin{cases} \frac{2\theta}{1+\theta} + \frac{{\eta_1}^2(1-\theta)}{(1+\theta)(\theta-\eta_1)} - \eta_6, & 0<\theta<\frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)} \\ \frac{\theta(1+\Lambda_b+\theta(2\Lambda_b+\theta\Lambda_b-3\theta-4(1+\theta)\Lambda_d))}{(1+\theta)(\Lambda_b+\theta(\Lambda_b-1-2\Lambda_d)-2\Lambda_d)}, & \frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)}\le\theta\le 1 \end{cases},$$

$$c_4 = \begin{cases} \frac{2\theta}{1+\theta} - \frac{\eta_1(1-\theta)}{\eta_2(1+\theta)} + \eta_7, & 0<\theta<\frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)} \\ \frac{2\theta^2(1+\Lambda_d)-2\Lambda_d+\sqrt{4(1-\theta^2)^2{\Lambda_d}^2-(1+\theta-2\theta^2)^2}}{1+\theta}, & \frac{\Lambda_b}{4(\Lambda_d-\Lambda_b)}\le\theta\le 1 \end{cases},$$

$c_5 = \frac{\theta(2\Lambda_b-1)-\sqrt{2\theta\eta_3}(1-\theta)\Lambda_b}{\theta(\Lambda_b-1)+\Lambda_b}$, $c_6 = \eta_3\left(2\theta(1+\Lambda_b)-1\right)$, and $c_7 = \theta - \Lambda_b + \theta\Lambda_b + (1-\theta)\sqrt{{\Lambda_b}^2-1}$, where $\eta_1 = \sqrt{\frac{\theta\Lambda_b}{(\Lambda_d-\Lambda_b)}}$, $\eta_2 = \frac{\theta}{\theta+\Lambda_d+\theta\Lambda_d}$, $\eta_3 = \frac{\theta}{\theta+\Lambda_b+\theta\Lambda_b}$, $\eta_4 = \sqrt{\theta(1+\theta)\left(\eta_1(\Lambda_d-\Lambda_b)+\Lambda_b(\theta+\Lambda_d+\theta\Lambda_d)\right)}$, $\eta_5 = \sqrt{(1+\theta)(\Lambda_d-\Lambda_b)(1+2\Lambda_d+2\theta(1+\Lambda_d))}$, $\eta_6 = \frac{(1-\theta)\theta\left({\eta_1}^2+\theta(1+\theta)(\theta-(1-\theta)\Lambda_b)-\eta_1\theta(2+\theta+\Lambda_b+\theta\Lambda_b)\right)}{(1+\theta)(\theta-\eta_1)(\theta(\theta+\Lambda_b+\theta\Lambda_b)-\eta_1(\theta+\Lambda_d+\theta\Lambda_d))}$, and $\eta_7 = \frac{(1-\theta)\sqrt{(1+2\eta_1-\theta)\theta\Lambda_b-(1+2\eta_1)\theta\Lambda_d+(1+\theta)^2\Lambda_b{\Lambda_d}^2}}{(1+\theta)\sqrt{\theta(\Lambda_d-\Lambda_b)}}$. The optimal solutions for the supply chain is provided in Table EC.16.

Using the results in Proposition 9, the manufacturer's expected profit function is, if $\Lambda_d \le \Lambda_b$,

$$\pi_m(w) = \begin{cases} \frac{[(1+2\Lambda_b)\theta-\theta w-\Lambda_b(1+\theta)w](w-c)}{2\theta}, & 0<w\le\theta \\ (1-w)(w-c), & \theta<w\le 1 \end{cases}.$$

Parameters	BOGO	
	Interior Solution	Boundary Solution
w_b	$\frac{c\Lambda_b+\theta(1+c+(2+c)\Lambda_b)}{2(\theta+\Lambda_b+\theta\Lambda_b)}$	θ
p_b	$\frac{1+w_b}{2}$	$\frac{1+\theta}{2}$
p_g	$\frac{\theta+w_b}{2}$	θ
D_b	$\frac{\theta(1+w_b+(2-w_b)\Lambda_b)-\Lambda_b w_b}{2\theta}$	$\frac{(1-\theta)(1+\Lambda_b)}{2}$
π_{br}	$\frac{(1-w_b)^2\theta+\Lambda_b\left(w_b{}^2+\theta-(4-w_b)w_b\theta+\theta^2\right)}{4\theta}$	$\frac{(1-\theta)^2(1+\Lambda_b)}{4}$
π_{bm}	$\frac{(\theta(1+w_b+(2-w_b)\Lambda_b)-\Lambda_b w_b)(w_b-c)}{2\theta}$	$\frac{(\theta-c)(1-\theta)(1+\Lambda_b)}{2}$

Parameters	PR	
	Interior Solution	Boundary Solution
w_d	$\frac{c\Lambda_d+\theta(1+c+(2+c)\Lambda_d)}{2(\theta+\Lambda_d+\theta\Lambda_d)}$	w^K
p_r	$\frac{1+w_d}{2}$	$\frac{1+w^K}{2}$
p_d	$\frac{2\theta+(1+\theta)w_d}{2(1+\theta)}$	$\frac{2\theta+(1+\theta)w^K}{2(1+\theta)}$
D_d	$\frac{\theta(1+w_d+(2-w_d)\Lambda_d)-\Lambda_d w_d}{2\theta}$	$\frac{\theta(1+w^K+(2-w^K)\Lambda_d)-\Lambda_d w^K}{2\theta}$
π_{dr}	$\frac{(1-w_d)^2+\frac{(w_d+(-2+w_d)\theta)^2\Lambda_d}{\theta(1+\theta)}}{4}$	$\frac{(1-w^K)^2+\frac{\left(w^K+(-2+w^K)\theta\right)^2\Lambda_d}{\theta(1+\theta)}}{4}$
π_{dm}	$\frac{(\theta(1+w_d+(2-w_d)\Lambda_d)-\Lambda_d w_d)(w_d-c)}{2\theta}$	$\frac{\left(\theta(1+w^K+(2-w^K)\Lambda_d)-\Lambda_d w^K\right)\left(w^K-c\right)}{2\theta}$

Table EC.16 Manufacturer-Retailer Optimal Policies with UNSP

If $\Lambda_d > \Lambda_b$,

$$\pi_m(w) = \begin{cases} \frac{[(1+2\Lambda_d)\theta-\theta w-\Lambda_d(1+\theta)w](w-c)}{2\theta}, & 0 < w \le w^K \\ \frac{[(1+2\Lambda_b)\theta-\theta w-\Lambda_b(1+\theta)w](w-c)}{2\theta}, & w^K < w \le \theta \\ (1-w)(w-c), & \theta < w \le 1 \end{cases}.$$

The rest of the proof is similar to that of Proposition 2 and is omitted. □

Proof of Proposition 7. For a loss-averse retailer, the penalty-adjusted expected market size is

$$\begin{aligned}
\mathbb{E}\left[s_i - r\left[s_0-s_i\right]^+\right] &= \int_{s_0}^{\infty} sg_i(s)ds + (1+r)\int_1^{s_0} sg_i(s)ds - rs_0G_i(s_0) \\
&= \int_{s_0}^{\infty} sg_i(s)ds + \int_1^{s_0} sg_i(s)ds + r\int_1^{s_0} sg_i(s)ds - rs_0G_i(s_0) \\
&= \mathbb{E}(s_i) + r\int_1^{s_0} sg_i(s)ds - rs_0G_i(s_0) \\
&= \mathbb{E}(s_i) + rs_0G_i(s_0) - r\int_1^{s_0} G_i(s)ds - rs_0G_i(s_0) \\
&= \Lambda_i - r\int_1^{s_0} G_i(s)ds \le \Lambda_i.
\end{aligned}$$

Replacing Λ_i by $\Lambda_i - r\int_1^{s_0} G_i(s)ds$, it is straightforward to see that the retailer's expected profits decrease with $r\int_1^{s_0} G_i(s)ds$. The previous analysis is applicable for any pdf of s_i, $i = b$ *or* d. More specifically, we consider a simple pdf (i.e., a two-point distribution) given by

$$s_i = \begin{cases} \mathcal{S}_i, & \text{with probability } p_{r_i} \\ 1, & \text{with probability } 1-p_{r_i} \end{cases}, \ i = b \text{ or } d,$$

where $\mathcal{S}_i$ ($\mathcal{S}_i > 1$) is the realized market size with promotions in period 1. This means the promotion-caused brand switching effect has two states. A high-state with a significant number of new consumers in the promotion period, which can be estimated from historical promotion data. A low-state with relatively fewer brand switchers which we normalize to 1. The expected market size is $\mathbb{E}[s_i] = p_{ri}\mathcal{S}_i + 1 - p_{ri} = (\mathcal{S}_i - 1)\, p_{ri} + 1 \equiv \bar{\Lambda}_i$, $i = b$ *or* d. Replacing Λ_i in the previous analysis by $\bar{\Lambda}_i$, we can see that the probability of a high-state market size p_{ri} plays a vital role in determining which promotion is best. As shown above, the retailer's profit increases in $\mathcal{S}_i$ and p_{ri}, which means PR could be preferred by the retailer if $p_{rd} > p_{rb}$ even for a smaller PR realized market size, that is, $\mathcal{S}_b > \mathcal{S}_d$.

According to Proposition 7, the penalty-adjusted expected market size in the gain-loss model is $\mathbb{E}\left[s_i - r\,[s_0 - s_i]^+\right] = p_{ri}\mathcal{S}_i + (1 - p_{ri})(1 - r(s_0 - 1))$, $i = b$ *or* d. For simplicity, we assume that $1 < s_0 < \mathcal{S}_i$, which means the realized market size in the high-state is larger than the reference level of the market size with BOGO and PR. Since $\mathbb{E}\left[s_i - r\,[s_0 - s_i]^+\right] < \mathbb{E}[s_i]$, a loss-averse retailer incurs a loss in expected profits due to the risk of overestimating market expansion. The expected profit reduction under BOGO for a given wholesale price $w_b = w$ depending on p_{rb} is

$$\Delta_{br} = \frac{(w^2 + \theta - (4 - w)w\theta + \theta^2)\,(1 - p_{rb})r(s_0 - 1)}{4\theta}.$$

The expected profit reduction under PR for a given wholesale price $w_d = w$ depending on p_{rd} is

$$\Delta_{dr} = \frac{(w - (2 - w)\theta)^2\,(1 - p_{rd})r(s_0 - 1)}{4\theta(1 + \theta)}.$$

It is straightforward to see that Δ_{ir} ($i = b$ *or* d) is a monotone decreasing function of p_{ri}, which also means that the likelihood of a large market size decreases the expected profit reductions under BOGO and PR. Since the standard deviation σ_i ($i = b$ *or* d) is a function of p_{ri}, that is, $\sigma_i = (\mathcal{S}_i - 1)\sqrt{p_{ri}(1 - p_{ri})}$. We show that Δ_{ir} is concavely decreasing in σ_i, which means that the larger the standard deviation, the faster the expected profit reductions will occur. As a result, a loss-averse retailer prefers a high market expansion with either BOGO or PR in a low uncertainty (dispersion) manner.

To sum up, the effectiveness of promotions in expanding the market size should consider both the expected expansion and its dispersion. The analysis above indicates that the variability information in the distribution function, pre-set reference level of market size, and probability of high-state market size affect the profitability of promotions and can alter the retailer's optimal choice. □

Appendix EC.5: Model Extensions

Proof of Proposition 8. The profits of the retailer under BOGO and PR are

$$\pi_{br} = \left(1 - \frac{p_g}{\theta}\right)(p_g + p_b - 2w) + \left(\frac{p_g}{\theta} - p_b\right)(p_b - w) + (1 - \mu_p)\left(1 - \frac{p_g}{\theta}\right)(p_b - w) + \left(\frac{p_g}{\theta} - p_b\right)(p_b - w)$$

and

$$\pi_{dr} = \left(1 - \frac{p_d}{\theta}\right)(2p_d - 2w) + \left(\frac{p_d}{\theta} - p_d\right)(p_d - w) + (1-\mu_p)\left(1 - \frac{p_d}{\theta}\right)(p_r - w) + \left(\frac{p_d}{\theta} - p_r\right)(p_r - w),$$

respectively. For a given w and $\theta > \frac{\sqrt{1+\mu_p^2}-1}{2} \equiv \mu_1$, the retailer's profit functions are jointly concave in the prices, and the optimal prices under BOGO and PR are

$$(p_b^*, p_g^*) = \left(w + \frac{w\mu_p + \theta(4 - 4w - \mu_p)}{8\theta - \mu_p^2}, \frac{\theta(4\theta + 2w(2-\mu_p) + (2-\mu_p)\mu_p)}{8\theta - \mu_p^2}\right)$$

and

$$(p_r^*, p_d^*) = \left(\frac{2(1+w)\theta(1+\theta) + (w + w\theta - 2\theta^2)\mu_p - w\mu_p^2}{4\theta(1+\theta) - \mu_p^2}, \frac{\left(\theta\left(4\theta + w(2+2\theta-\mu_p) + \mu_p - \mu_p^2\right)\right)}{4\theta(1+\theta) - \mu_p^2}\right),$$

respectively. The resulting optimal profits are

$$\pi_{br}^* = \frac{2\left(\theta(2+\theta-\mu_p) + w^2(1+2\theta-\mu_p) + w(\mu_p - \theta(6-\mu_p))\right)}{8\theta - \mu_p^2}$$

and

$$\pi_{dr}^* = \frac{w^2(1+\theta)(1+2\theta-\mu_p) + \theta\left(1 - \mu_p^2 + \theta(5 - (2-\mu_p)\mu_p)\right) + w\left(\mu_p + \mu_p^2 - 2\theta^2(3-\mu_p) - \theta\left(6 - \mu_p + \mu_p^2\right)\right)}{4\theta(1+\theta) - \mu_p^2},$$

respectively. Let

$$\begin{aligned}\Delta Z &= \pi_{dr}^* - \pi_{br}^* \\ &= \frac{(1-\theta)\left(8\theta^3 + w(1-\mu_p)\mu_p^2(w+\mu_p) + \theta\mu_p^2\left(3 + 2w^2 - (2-\mu_p)\mu_p + 2w(1+\mu_p)\right) - 8\theta^2(1 + \mu_p(w + \mu_p - 1))\right)}{32\theta^2(1+\theta) - 4\theta(3+\theta)\mu_p^2 + \mu_p^4}.\end{aligned}$$

It can be shown that ΔZ is convex in w and two roots for $\Delta Z = 0$ are $\underline{w} = \frac{8\theta^2 - 2\theta\mu_p - (1+2\theta-\mu_p)\mu_p^2}{2\mu_p(1+2\theta-\mu_p)} - \frac{(1-\mu_p)\sqrt{(8\theta-\mu_p^2)(4\theta^2+4\theta-\mu_p^2)}}{2\mu_p(1+2\theta-\mu_p)}$ and $\bar{w} = \frac{8\theta^2 - 2\theta\mu_p - (1+2\theta-\mu_p)\mu_p^2}{2\mu_p(1+2\theta-\mu_p)} + \frac{(1-\mu_p)\sqrt{(8\theta-\mu_p^2)(4\theta^2+4\theta-\mu_p^2)}}{2\mu_p(1+2\theta-\mu_p)}$. Thus, when $w < \underline{w}$ or $w > \bar{w}$, $\Delta Z > 0$. In addition, if $\mu_1 < \theta < \frac{1-\mu_p+\mu_p^2}{2} - \frac{(1-\mu_p)\sqrt{2+\mu_p^2}}{2\sqrt{2}} \equiv \mu_2$, $\underline{w} < \bar{w} < 0$, then $\Delta Z > 0$ for all $0 < w < 1$. If $\mu_2 < \theta < \frac{1-\mu_p+\mu_p^2}{2} + \frac{(1-\mu_p)\sqrt{2+\mu_p^2}}{2\sqrt{2}} \equiv \mu_3$, $\underline{w} < 0 < \bar{w}$, then $\Delta Z > 0$ for $w > \bar{w}$. If $\mu_3 < \theta \leq 1$, $0 < \underline{w} < \bar{w}$, then $\Delta Z > 0$ for $w < \underline{w}$ or $w > \bar{w}$. Ensuring $\bar{w} < 1$ requires $\mu_2 < \theta < \frac{\mu_p^2}{2} + \frac{\mu_p\sqrt{1+\mu_p^2}}{2\sqrt{2}} \equiv \mu_4$. The result follows. □

Proof of Proposition 9. Consumer choices with heterogeneous marginal utility are presented in Table EC.17. By solving the cases in Table EC.17, we obtain eight solutions shown as follows:

- Solution 1. $\{p_b, p_g\} = \left\{\frac{1}{2}(1+w_b), \frac{\bar{\alpha}w_b\theta_H + (\alpha w_b + \theta_H)\theta_L}{2\bar{\alpha}\theta_H + 2\alpha\theta_L}\right\}$, if $0 < \alpha < 1$, $\frac{\theta_H}{2-\theta_H} < \theta_L < \theta_H$, and $\frac{\bar{\alpha}\theta_H(\theta_H-\theta_L)}{(1-\theta_H)(\bar{\alpha}\theta_H+\alpha\theta_L)} < w_b < \frac{\theta_L((1-2\alpha)\theta_H+2\alpha\theta_L)}{\bar{\alpha}\theta_H+\alpha\theta_L}$; or if $\frac{\theta_H(\theta_H+(\theta_H-2)\theta_L)}{(\theta_H-\theta_L)(\theta_H+2(\theta_H-1)\theta_L)} < \alpha < 1$, $\frac{\theta_H}{2} < \theta_L < \frac{\theta_H}{2-\theta_H}$, and $\frac{\bar{\alpha}\theta_H(\theta_H-\theta_L)}{(1-\theta_H)(\bar{\alpha}\theta_H+\alpha\theta_L)} < w_b < \frac{\theta_L((1-2\alpha)\theta_H+2\alpha\theta_L)}{\bar{\alpha}\theta_H+\alpha\theta_L}$. This is the interior solution of Case 1.
- Solution 2. $\{p_b, p_g\} = \{\frac{\bar{\alpha}(2(1+\theta_H)+w_b(2+(2-\alpha)\theta_H)) - \alpha(1+(2-\alpha)w_b)\theta_L}{4\bar{\alpha}+2(2-\alpha)(\bar{\alpha}\theta_H+\alpha\theta_L)}, \frac{\bar{\alpha}w_b(2+(2-\alpha)\theta_H)+(2\bar{\alpha}+(2-\alpha)\alpha w_b+(2-\alpha)\theta_H)\theta_L}{4\bar{\alpha}+2(2-\alpha)(\bar{\alpha}\theta_H+\alpha\theta_L)}\}$, if $0 < \alpha < 1$, $\frac{\theta_H}{2} < \theta_L < \theta_H$, and $0 < w_b < \min\left\{\frac{\theta_L((2-(5-2\alpha)\alpha)\theta_H+2(\bar{\alpha}+(2-\alpha)\alpha\theta_L))}{\bar{\alpha}(2+(2-\alpha)\theta_H)+(2-\alpha)\alpha\theta_L}, \frac{2\bar{\alpha}(1+\theta_H)(\theta_H-\theta_L)}{(1-\theta_H)(2\bar{\alpha}+(2-\alpha)(\bar{\alpha}\theta_H+\alpha\theta_L))}\right\}$; or if $0 < \alpha < \frac{2+5\theta_H-4\theta_L}{4(\theta_H-\theta_L)} - \frac{1}{4}\sqrt{\frac{4+4\theta_H+9\theta_H^2-24\theta_H\theta_L+16\theta_L^2}{(\theta_H-\theta_L)^2}}$, $0 < \theta_L < \frac{\theta_H}{2}$, and $0 < w_b < \frac{\theta_L((2-(5-2\alpha)\alpha)\theta_H+2(\bar{\alpha}+(2-\alpha)\alpha\theta_L))}{\bar{\alpha}(2+(2-\alpha)\theta_H)+(2-\alpha)\alpha\theta_L}$. This is the interior solution of Case 2.

Options	$\theta_H > \theta_L > p_g$, BOGO to both groups			
	Case 1: $p_g < \theta_L < \theta_H < p_g/p_b$	Case 2: $p_g < \theta_L < p_g/p_b < \theta_H$		Case 3: $p_g < p_g/p_b < \theta_L < \theta_H$
	High-/Low-Type	High-Type	Low-Type	High-/Low-Type
(2, 1)	$\frac{p_g}{\theta_i} \le v \le 1$	$p_b \le v \le 1$	$\frac{p_g}{\theta_L} \le v \le 1$	$p_b \le v \le 1$
(2, 0)	N/A	$\frac{p_b+p_g}{1+\theta_H} \le v < p_b$	N/A	$\frac{p_b+p_g}{1+\theta_i} \le v < p_b$
(1, 1)	$p_b \le v < \frac{p_g}{\theta_i}$	N/A	$p_b \le v < \frac{p_g}{\theta_L}$	N/A
(0, 0)	$0 \le v < p_b$	$0 \le v < \frac{p_b+p_g}{1+\theta_H}$	$0 \le v < p_b$	$0 \le v < \frac{p_b+p_g}{1+\theta_i}$
Options	$\theta_H > p_g > \theta_L$, BOGO to the high-type group			
	Case 4: $\theta_L < p_g < \theta_H < p_g/p_b$		Case 5: $\theta_L < p_g < p_g/p_b < \theta_H$	
	High-Type	Low-Type	High-Type	Low-Type
(2, 1)	$\frac{p_g}{\theta_H} \le v \le 1$	N/A	$p_b \le v \le 1$	N/A
(2, 0)	N/A	N/A	$\frac{p_b+p_g}{1+\theta_H} \le v < p_b$	N/A
(1, 1)	$p_b \le v < \frac{p_g}{\theta_H}$	$p_b \le v \le 1$	N/A	$p_b \le v \le 1$
(0, 0)	$0 \le v < p_b$	$0 \le v < p_b$	$0 \le v < \frac{p_b+p_g}{1+\theta_H}$	$0 \le v < p_b$
Options	Case 6: $p_g > \theta_H > \theta_L$, BOGO to no one (EDLP)			
	High-/Low-Type			
(1, 1)	$p_b \le v \le 1$			
(0, 0)	$0 \le v < p_b$			

Table EC.17 Consumer Choices in BOGO-NSP with Heterogeneous Marginal Utility

- Solution 3. $\{p_b, p_g\} = \left\{\frac{1}{2}(1+w_b), \frac{1}{2}(w_b+\theta_H)\right\}$, if $0<\alpha<1$, $\frac{\theta_H}{2} < \theta_L < \theta_H$, and $2\theta_L - \theta_H < w_b < \theta_H$; or if $0<\alpha<1$, $0<\theta_L<\frac{\theta_H}{2}$, and $0<w_b<\theta_H$. This is the interior solution of Case 4.
- Solution 4. $\{p_b, p_g\} = \{\frac{-(1+\theta_H)(2+\theta_L)+w_b(-3+(-3+2\alpha)\theta_H-2\alpha\theta_L)}{2(-2+\theta_H(-2+\alpha+(-1+\alpha)\theta_L)-\theta_L(1+\alpha+\alpha\theta_L))}, \frac{\theta_L(-(1+\theta_H)(2+\theta_L)+w_b(-3+(-3+2\alpha)\theta_H-2\alpha\theta_L))}{2(-2+\theta_H(-2+\alpha+(-1+\alpha)\theta_L)-\theta_L(1+\alpha+\alpha\theta_L))}\}$, if $0<\alpha<1$, $0<\theta_L<\theta_H$, and $0<w_b<\frac{-2-\theta_L(1+2\alpha+2\alpha\theta_L)+\theta_H(2(-1+\alpha)+(-1+2\alpha)\theta_L)}{-3+(-3+2\alpha)\theta_H-2\alpha\theta_L}$. This is the boundary solution of Cases 2 and 3.
- Solution 5. $\{p_b, p_g\} = \left\{\frac{-(2+\theta_H)\theta_L+w_b((-1+\alpha)\theta_H-(2+\alpha)\theta_L)}{2(-1+\alpha)\theta_H^2-4\theta_L-2\alpha\theta_H\theta_L}, \frac{\theta_H(-(2+\theta_H)\theta_L+w_b((-1+\alpha)\theta_H-(2+\alpha)\theta_L))}{2(-1+\alpha)\theta_H^2-4\theta_L-2\alpha\theta_H\theta_L}\right\}$, if $0<\alpha<1$, $\frac{\theta_H}{2}<\theta_L<\theta_H$, and $0<w_b<\frac{\theta_L(\theta_H(2+(-1+2\alpha)\theta_H)-2(2+\alpha\theta_H)\theta_L)}{\theta_H((-1+\alpha)\theta_H-(2+\alpha)\theta_L)}$; or if $0<\alpha<\frac{2\theta_H-\theta_H^2-4\theta_L}{-2\theta_H^2+2\theta_H\theta_L}$, $\frac{1}{4}(2\theta_H-\theta_H^2)<\theta_L<\frac{\theta_H}{2}$, and $0<w_b<\frac{\theta_L(\theta_H(2+(-1+2\alpha)\theta_H)-2(2+\alpha\theta_H)\theta_L)}{\theta_H((-1+\alpha)\theta_H-(2+\alpha)\theta_L)}$. This is the boundary solution of Cases 1 and 2.
- Solution 6. $\{p_b, p_g\} = \left\{\frac{1}{2}(1+w_b), \theta_L\right\}$, if $0<\alpha<1$, $\frac{\theta_H}{2}<\theta_L<\theta_H$, and $\max[0, 2\theta_L-1] < w_b < \frac{2\theta_L-\theta_H}{\theta_H}$. This is the boundary solution of Cases 1 and 4.
- Solution 7. $\{p_b, p_g\} = \left\{\frac{2+(2+\alpha)w_b+\alpha\theta_H}{4+2\alpha\theta_H}, \frac{\theta_H(2+(2+\alpha)w_b+\alpha\theta_H)}{4+2\alpha\theta_H}\right\}$, if $0<\alpha<1$, $\frac{\theta_H}{2}<\theta_L<\theta_H$, and $2\theta_L-\theta_H<w_b<\theta_H$; or if $0<\alpha<1$, $0<\theta_L<\frac{\theta_H}{2}$, and $0<w_b<\theta_H$. This is the boundary solution of Cases 4 and 5.
- Solution 8. $\{p_b, p_g\} = \left\{\frac{1}{2}(1+w_b), \theta_H\right\}$. This is the boundary solution of Cases 4 and 6. In addition, the retailer's excepted profit with Solution 8 is the same as that in Case 6 (EDLP).

Similarly, we combine all the solutions above and show that the boundary solutions (Solution 4 to Solution 8) are dominated by the interior solutions (Solution 1 to Solution 3) or EDLP solution. Proposition 9 is proved. □

Numerical Analysis for Consumers with Heterogeneous Marginal Utility

To better understand the retailer's pricing policy for any given w_b, we examine the optimal policy for selected values of α and w_b in the (θ_L, θ_H) space which are shown in Figure EC.3. In

Figure EC.3a, if θ_L is small and θ_H is relatively large (i.e., Case 4), the retailer will price such that BOGO only attracts the high-type consumers who choose from $\{(2,1),(1,1),(0,0)\}$, and the low-type consumers who choose from $\{(1,1),(0,0)\}$. When θ_L is large, BOGO attracts both types of consumers. In this case, if θ_H is close to θ_L (i.e., Case 1), the retailer will price such that BOGO will attract both low-type and high-type consumers who choose from $\{(2,1),(1,1),(0,0)\}$. If θ_H is considerably larger than θ_L (i.e., Case 2), the retailer will price such that the low-type consumers will choose from $\{(2,1),(1,1),(0,0)\}$, however, the high-type consumers will choose from $\{(2,1),(2,0),(0,0)\}$. This is because the gap between θ_L and θ_H is so large that the retailer cannot prevent some high-type consumers from choosing $(2,0)$ which is less preferable by the retailer.

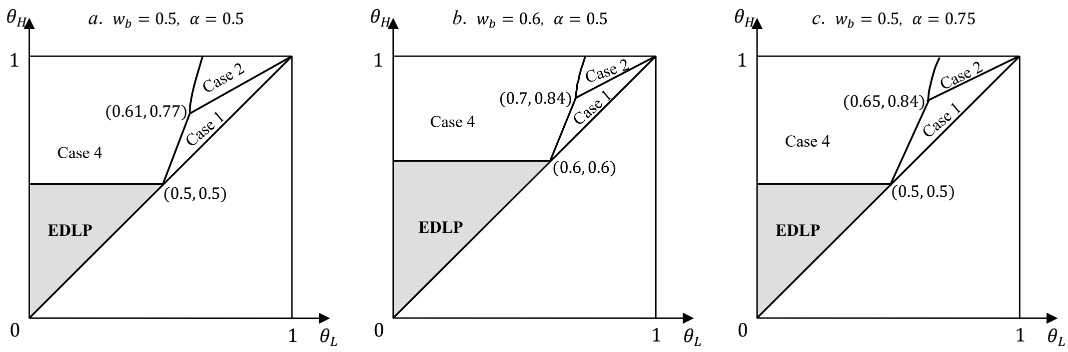

Figure EC.3 Retailer's Best Response with Heterogeneous Marginal Utility

In Figure EC.3b, with a larger wholesale price w_b, the retailer is less likely to offer BOGO because of the decreasing profit margin, thus the size of the EDLP region increases. The sizes of the regions of Cases 1, 2, and 4 decrease. For a larger proportion of high-type consumers α shown in Figure EC.3c, the size of the EDLP region does not change (due to unchanged wholesale price), the sizes of regions of Cases 1 and 4 increase and the size of the region of Case 2 decreases. These changes are because the retailer capitalizes on the larger α to increase the number of consumers choosing the most profitable options $\{(2,1),(1,1),(0,0)\}$, which are the solutions for Cases 1 and 4.

Because of the complexity of the retailer's response, the manufacturer's problem is very complex and we therefore examine it numerically. Figure EC.4 shows the solution for $\theta_H = \theta_L + 0.15$ which enables us to present the range of possible cases. As the figure shows, for $\theta_L < 0.442$ ($\theta_H < 0.592$), the SC uses EDLP. For $0.442 \le \theta_L \le 0.606$ ($0.592 \le \theta_H \le 0.756$), the prices are such that BOGO is only used by high θ consumers (Case 4 in Table EC.17). The first drop in w_b in Figure EC.4a is caused by the retailer's BOGO. When $\theta_L = 0.442$ and $\theta_H = 0.592$, θ_H is high enough for the retailer to use BOGO. Similar to the case of homogeneous consumers, the retailer will decrease the regular price in response to the lower wholesale price. As a result, the retailer's profit jumps and her share of the SC profit increases substantially as shown in Figure EC.4b. The next range is

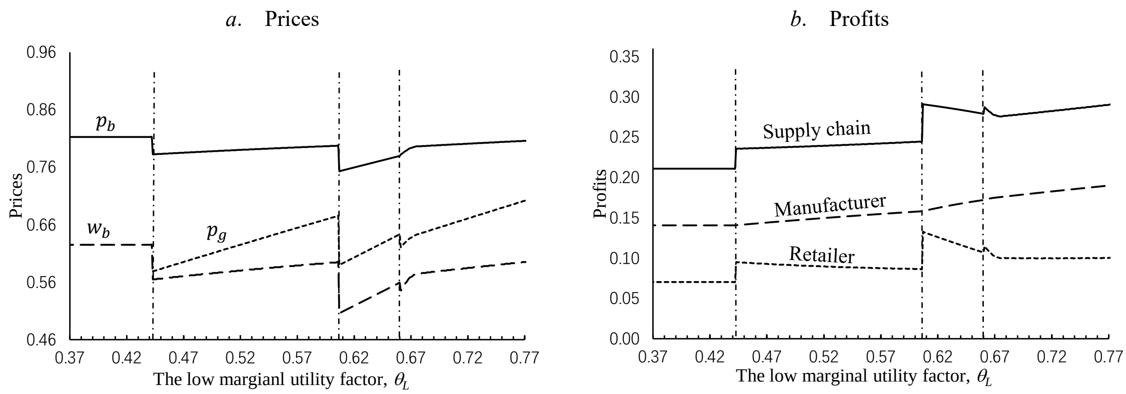

Figure EC.4 Consumers with Heterogeneous Marginal Utility, $\alpha=0.50$, $c=0.2$, $\theta_H=\theta_L+0.15$

$0.607<\theta_L\le 0.66$ $(0.757<\theta_H\le 0.81)$, in which it is optimal to offer BOGO to both groups (Case 1 in Table EC.17), the manufacturer drops the wholesale price to $w_b=0.506$, and the retailer drops the regular and BOGO prices shown in Figure EC.4a. This is because when values of θ_i $(i=L,H)$ are not very large, the retailer could price such that $p_g^*<\theta_L$ and $\theta_H<\frac{p_g^*}{p_b^*}$. As a result, all consumers choose from $\{(2,1),(1,1),(0,0)\}$ which is the most profitable for the retailer. Starting at $\theta_L=0.661$ and for $\theta_L\ge 0.661$ $(\theta_H\ge 0.811)$, the SC uses the pricing policy of Case 2 in Table EC.17. Since θ_H is considerably large, the retailer cannot prevent some high-type consumers from choosing (2,0) which is not preferred by the retailer. As shown in Figure EC.4a, the retailer decreases the price gap between p_b and p_g, and the manufacturer increases the wholesale price. Similar to the case of homogeneous consumers, the manufacturer's profit increases in θ_i, but the retailer's profit decreases in θ_i in each region. The reason is that the wholesale price in each range is increasing in $\theta_i (i=H,L)$, but the retailer's profit from regular-price sales is decreasing in θ_i. □

Proof of Proposition 10. To simplify the cases, we define some strategic options for consumers. Option 1: {(2,1), (2,0)s, (0,0)}. With this option, if $\frac{p_b-h_j}{\theta}\le v\le 1$, $(j=\{H,\ L\})$, a consumer will choose (2,1); if $\frac{p_b+p_g+h_j}{2}\le v<\frac{p_b-h_j}{\theta}$, a consumer will choose (2,0)s; and if $0\le v<\frac{p_b+p_g+h_j}{2}$, a consumer will not buy. We use subscript $\{H,L\}$ to distinguish the consumer type, e.g., $S1_H$ means strategic option 1 for the high-type consumers. Option 2: {(2,1), (2,0)s, (2,0), (0,0)}. For example, for the low-type consumers, if $\frac{p_b+p_g-h_L}{p_b+p_g+h_L}\le\theta<\frac{p_b-h_L}{p_b}$, and $h_L<h_H<p_b-p_g$, a low-type consumer will choose $S2_L$. And, Option 3: {(2,1), (2,0), (0,0)} and Option 4: {(2,1), (1,1), (0,0)}. We show consumer choices in Table EC.18.

From Table EC.18, we obtain four optimal solutions as follows:

- Solution 1. $\{p_b^*,p_g^*\}=\left\{\frac{2(h_L+w_b)\bar{\gamma}\gamma+\left(h_L+w_b-(3h_L+w_b-4)\gamma+2(h_L+2w_b)\gamma^2\right)\theta+\bar{\gamma}(1+2w_b\gamma)\theta^2}{2\theta+2\gamma\left(2-2\gamma(1-\theta)^2-\theta+2\theta^2\right)},\right.$ $\left.\frac{\theta(2\bar{\gamma}-2h_L\bar{\gamma}(1+\gamma(-1+\theta))+\theta-5\gamma\theta)+w_b(\theta+\gamma(2-2\gamma(1-\theta)^2-\theta+2\theta^2))}{2\theta+2\gamma(2-2\gamma(1-\theta)^2-\theta+2\theta^2)}\right\}$, if $0<\theta\le 1$, $\frac{1}{3}<\gamma<1$, $\max\left\{0,\frac{\theta(4\gamma^2(1-\theta)^2-\theta+\gamma\theta(1-4\theta))}{\theta+\gamma(2\bar{\gamma}+\theta+6\gamma\theta+4\bar{\gamma}\theta^2)}\right\}<h_L<\frac{2(1-3\gamma)(1-\theta)\theta}{2\gamma^2(1-\theta)^2+\theta-\gamma(2+\theta)(1+2\theta)}$, $\frac{2(3\gamma-1)(1-\theta)\theta+h_L\bar{\gamma}(2\gamma(1-\theta)^2+3\theta)}{2\theta+2\gamma(2-2\gamma(1-\theta)^2-\theta+2\theta^2)}<h_H<$

h	θ	High-Type	Low-Type
$h_L < h_H \leq \frac{2h_L p_b}{h_L+p_b+p_g} < p_b - p_g$	$0 < \theta < \frac{p_b+p_g-h_H}{p_b+p_g+h_H}$	$S1_H$	$S1_L$
	$\frac{p_b+p_g-h_H}{p_b+p_g+h_H} \leq \theta < \frac{p_b+p_g-h_L}{p_b+p_g+h_L}$	$S2_H$	$S1_L$
	$\frac{p_b+p_g-h_L}{p_b+p_g+h_L} \leq \theta < \frac{p_b-h_H}{p_b}$	$S2_H$	$S2_L$
	$\frac{p_b-h_H}{p_b} \leq \theta < \frac{p_b-h_L}{p_b}$	$S3_H$	$S2_L$
	$\frac{p_b-h_L}{p_b} \leq \theta \leq 1$	$S3_H$	$S3_L$
$h_L < \frac{2h_L p_b}{h_L+p_b+p_g} \leq h_H < p_b - p_g$	$0 < \theta < \frac{p_b+p_g-h_H}{p_b+p_g+h_H}$	$S1_H$	$S1_L$
	$\frac{p_b+p_g-h_H}{p_b+p_g+h_H} \leq \theta < \frac{p_b-h_H}{p_b}$	$S2_H$	$S1_L$
	$\frac{p_b-h_H}{p_b} \leq \theta < \frac{p_b+p_g-h_L}{p_b+p_g+h_L}$	$S3_H$	$S1_L$
	$\frac{p_b+p_g-h_L}{p_b+p_g+h_L} \leq \theta < \frac{p_b-h_L}{p_b}$	$S3_H$	$S2_L$
	$\frac{p_b-h_L}{p_b} \leq \theta \leq 1$	$S3_H$	$S3_L$
$h_L < p_b - p_g \leq h_H$	$0 < \theta < \frac{p_g}{p_b}$	$S4_H$	$S1_L$
	$\frac{p_g}{p_b} \leq \theta < \frac{p_b+p_g-h_L}{p_b+p_g+h_L}$	$S3_H$	$S1_L$
	$\frac{p_b+p_g-h_L}{p_b+p_g+h_L} \leq \theta < \frac{p_b-h_L}{p_b}$	$S3_H$	$S2_L$
	$\frac{p_b-h_L}{p_b} \leq \theta \leq 1$	$S3_H$	$S3_L$
$p_b - p_g \leq h_L < h_H$	$0 < \theta < \frac{p_g}{p_b}$	$S4_H$	$S4_L$
	$\frac{p_g}{p_b} \leq \theta \leq 1$	$S3_H$	$S3_L$

Table EC.18 Consumer Choices in BOGO-SP with Heterogeneous Holding Cost

1, and $0 < w_b < \frac{h_L(2\bar{\gamma}\gamma+(1+\gamma+6\gamma^2)\theta+4\bar{\gamma}\gamma\theta^2)+\theta(\theta-\gamma(4\gamma(1-\theta)^2-\theta+4\theta^2))}{\theta+\gamma(2-2\gamma(1-\theta)^2-\theta+2\theta^2)}$. This is an interior solution that satisfies $h_L < p_b - p_g < h_H$.

- Solution 2. $\{p_b^*, p_g^*\} = \left\{\frac{w_b+2h_H\gamma+h_L\bar{\gamma}(1-\theta)+3\theta+2h_H\theta\bar{\gamma}+2w_b\theta}{2+4\theta}, \frac{w_b+h_L\bar{\gamma}(1-\theta)+3\theta+2w_b\theta-2h_H(\bar{\gamma}+\theta(1+\gamma))}{2+4\theta}\right\}$, if $0 < \theta \leq 1$, $\frac{1}{3} < \gamma < 1$, $\max\left\{0, \frac{\theta-4\theta^2}{\bar{\gamma}+3\theta+\gamma\theta}\right\} < h_L < \frac{2(1-3\gamma)(1-\theta)\theta}{2\gamma^2(1-\theta)^2+\theta-\gamma(2+\theta)(1+2\theta)}$, $h_H < \frac{2(3\gamma-1)(1-\theta)\theta+h_L\bar{\gamma}(2\gamma(1-\theta)^2+3\theta)}{2\theta+2\gamma(2-2\gamma(1-\theta)^2-\theta+2\theta^2)}$, and $0 < w_b < \frac{-2h_H\gamma+2h_H(-1+\gamma)\theta+\theta(-1+4\theta)+h_L(1+\gamma+5\theta-\gamma\theta)}{1+2\theta}$. This is a boundary solution in which the high-type consumers are indifferent between stockpiling and no stockpiling (i.e., $p_b - p_g = h_H$).
- Solution 3. $\{p_b^*, p_g^*\} = \{\frac{1+w_b}{2}, \frac{\theta+w_b}{2}\}$, if $0 < \theta \leq 1$, $0 < \gamma < 1$, $\frac{1-\theta}{2} < h_L < h_H$, and $0 < w_b \leq \theta$. This is an interior solution that satisfies $p_b - p_g < h_L$.
- Solution 4. $\{p_b^*, p_g^*\} = \left\{\frac{2h_L+w_b+3\theta+2w_b\theta}{2+4\theta}, \frac{w_b+3\theta-4h_L\theta+2w_b\theta}{2+4\theta}\right\}$, if $0 < \theta \leq 1$, $0 < \gamma < 1$, $\max\{0, \frac{1}{4}(1-4\theta)\} < h_L < \frac{1-\theta}{2}$, $h_L < h_H < 1$, and $0 < w_b < \frac{4h_L\theta+4\theta^2-\theta}{1+2\theta}$. This is a boundary in which the low-type consumers are indifferent between stockpiling and no stockpiling (i.e., $p_b - p_g = h_L$).

To sum up, Proposition 10 is proved. □

Numerical Analysis for Consumers with Heterogeneous Holding Cost

To better understand the retailer's policy for a given w_b, we examine the retailer's optimal policy for selected values of γ, θ, and w_b in the (h_L, h_H) space, which are shown in Figure EC.5. Define $\Delta p = {p_b}^* - {p_g}^*$. If h_L is high, the retailer can price such that $\Delta p < h_L$ and no consumers stockpile. If h_L is medium high, the retailer can price such that the two prices are just far enough apart to prevent the low-type consumers from stockpiling (i.e., $\Delta p = h_L$). The high-type consumers will not stockpile in this case either. If h_L is low and h_H is high enough, the retailer can only price to prevent the high-type consumers from stockpiling (i.e., $h_L < \Delta p < h_H$). Some low-type consumers will stockpile in this case. If h_L is low and h_H is relatively low, the retailer will price such that the two prices are just apart enough to prevent the high-type consumers from stockpiling (i.e., $\Delta p = h_H$). Some low-type consumers will stockpile in this case.

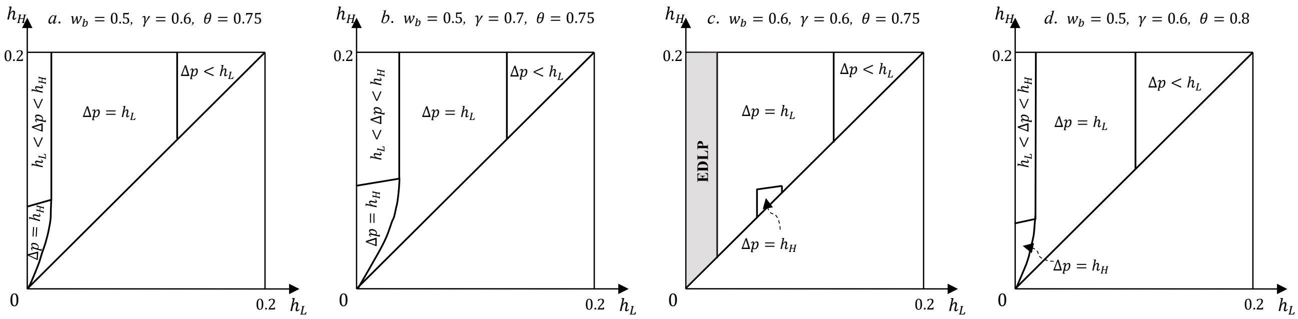

Figure EC.5 Retailer's Best Response with Heterogeneous Holding Cost

Figure EC.5b shows that as the proportion of high-type consumers γ increases, the size of the regions of $h_L < \Delta p < h_H$ and $\Delta p = h_H$ increase, the size of the region of $\Delta p < h_L$ does not change, and the size of the region of $\Delta p = h_L$ decreases. With larger γ, the retailer can extract more surplus from a large number of high-type consumers while letting the low-type consumers get a better deal by stockpiling. As w_b increases, Figure EC.5c shows that the region of $h_L < \Delta p < h_H$ disappears, the sizes of regions of $\Delta p = h_H$ and $\Delta p = h_L$ decrease, the size of the region of $\Delta p < h_L$ does not change, and the size of the region of EDLP increases. As θ increases, Figure EC.5d shows that the size of the region of $\Delta p < h_L$ increases, the sizes of the regions of $\Delta p = h_L$, $\Delta p = h_H$, and $h_L < \Delta p < h_H$ decrease. With larger θ, consumers are more likely to consume two units in period 1 and the retailer can extract more surplus from the high-type consumers because they are less likely to stockpile and therefore reduces Δp.

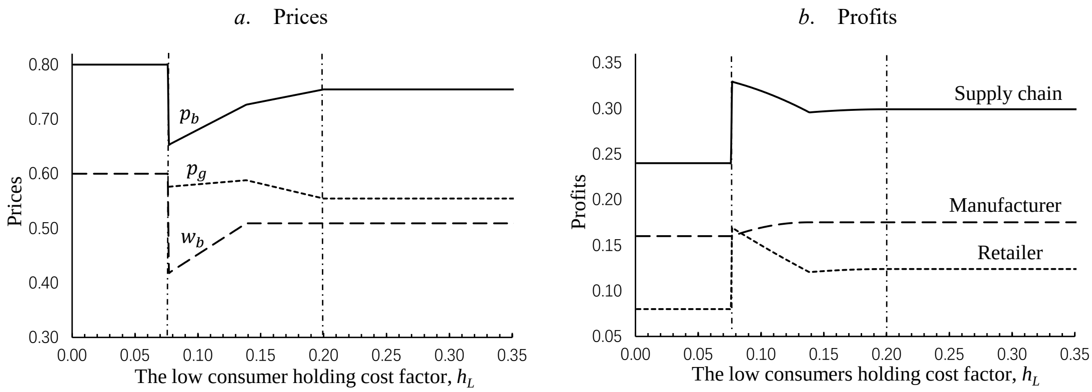

Figure EC.6 Consumers with Heterogeneous Holding Cost, $\gamma = 0.50$, $c = 0.2$, $h_H = h_L + 0.15$

We examine the manufacturer's problem numerically. Figure EC.6 shows the solutions under the relationship $h_H = h_L + 0.15$. As the figure shows, for $h_L < 0.077$ ($h_H < 0.227$), the SC uses EDLP with $w_b = 0.6$ and $p_b = 0.8$. For $0.077 \le h_L \le 0.20$ $(0.227 \le h_H \le 0.35)$, the SC prices such that low-type consumers are indifferent between stockpiling and no stockpiling and high type consumers will not stockpile (i.e., $\Delta p = h_L$). For $h_L = 0.077$ and $h_H = 0.227$ it is optimal for the retailer to use EDLP for any wholesale price above $w_b = 0.4183$. Then, it becomes optimal for the retailer to offer BOGO to both types of consumers if $w_b = 0.4183$ where the manufacturer's profit also is the same as in EDLP. This is the first drop in w_b in Figure EC.6a. Similar to the case of homogeneous consumers, the retailer's profit jumps and her share of the SC profit increases substantially. The next range is $0.20 < h_L < 0.35$, in which it is optimal to offer BOGO where both types of consumers do not stockpile with $\Delta p < h_L = 0.2$, and prices and profits no longer depend on h_L and h_H as shown in Figures EC.6a-b. $\square$

Proposition EC.4 identifies the SC's optimal policy in MBOGO.

PROPOSITION EC.4. *With MBOGO-NSP, the SC choice between MBOGO and EDLP is:*

a) If $0 < c \le (3(1+5\theta_1 - 2\theta^2) - 3\sqrt{2(1-\theta)(1+14\theta_1+\theta^2)\theta_4})/(2\theta(11+2\theta_1)-3) \equiv \acute{c}(\theta)$*, offer an MBOGO with BS (i.e.,* $p_g^* = \theta p_b^*$*),* $w^* = (9(1+c)+\theta(7+6c-\theta))/(\theta^2+14\theta_1+1)$.

b) If $\acute{c}(\theta) < c \le 2\theta - 1$*, offer an MBOGO with IS (i.e.,* $p_g^* > \theta p_b^*$*),* $w^* = (1+c)/2$.

c) Otherwise, offer an EDLP.

Proof of Proposition EC.4 We distinguish between two cases:

- Case I: $0 < \theta \le \frac{p_g}{p_b}$. The retailer's and manufacturer's expected profits are:

$$\pi_{br,I}^{MBOGO}(p_b) = \left[2(1-p_b) + \left(1 - \frac{p_g}{\theta}\right)\right](p_b - w_b), \qquad \text{and}$$

$$\pi_{bm,I}^{MBOGO}(p_g, w_b) = 2(1-p_b)(w_b - c) + \left(1 - \frac{p_g}{\theta}\right)(w_b - c + p_g - p_b).$$

- Case II: $\frac{p_g}{p_b} < \theta \le 1$. The retailer's and manufacturer's expected profits are:

$$\pi_{br,II}^{MBOGO}(p_b) = \left[\left(2 - p_b - \frac{p_b+p_g}{1+\theta}\right) + \left(1 - \frac{p_b+p_g}{1+\theta}\right)\right](p_b - w_b), \qquad \text{and}$$

$$\pi_{bm,II}^{MBOGO}(p_g, w_b) = \left(2 - p_b - \frac{p_b+p_g}{1+\theta}\right)(w_b - c) + \left(1 - \frac{p_b+p_g}{1+\theta}\right)(w_b - c + p_g - p_b).$$

In Case I, if $p_g < p_b < \frac{p_g}{\theta}$, we can obtain that the optimal retail price is

$$p_b{}^*(p_g, w_b) = \begin{cases} \frac{3\theta+2w_b\theta-p_g}{4\theta}, & \text{if } \frac{3\theta}{5} < p_g < \theta \text{ and } \max\left\{0, \frac{p_g-3\theta+4p_g\theta}{2\theta}\right\} < w_b < \frac{5p_g-3\theta}{2\theta} \\ \frac{p_g}{\theta}, & \text{if } p_g < \theta \text{ and } \max\left\{0, \frac{5p_g-3\theta}{2\theta}\right\} < w_b < 1 \\ p_g, & \text{if } \frac{1}{2} < \theta < 1, \ \frac{3\theta}{1+4\theta} < p_g < \theta, \text{ and } 0 < w_b < \frac{p_g-3\theta+4p_g\theta}{2\theta} \end{cases}.$$

In Case II, if $\frac{p_g}{\theta} \leq p_b < 1$, we can obtain the optimal retail price is

$$p_b{}^*(p_g, w_b) = \begin{cases} \frac{3-2p_g+3w_b+3\theta+w_b\theta}{6+2\theta}, & \text{if } p_g < \theta \text{ and } \max\left\{0, \frac{6p_g-3\theta+4p_g\theta-3\theta^2}{3\theta+\theta^2}\right\} < w_b < \frac{3+2p_g-\theta}{3+\theta} \\ \frac{p_g}{\theta}, & \text{if } \frac{3\theta+3\theta^2}{6+4\theta} < p_g < \theta \text{ and } 0 < w_b < \frac{6p_g-3\theta+4p_g\theta-3\theta^2}{3\theta+\theta^2} \\ 1, & \text{if } p_g < \theta \text{ and } \frac{3+2p_g-\theta}{3+\theta} < w_b < 1 \end{cases}.$$

First, the regions of $\{p_g,\, w_b\}$ are given by:

Region κ_1 : if $\frac{3\theta}{5} < p_g < \theta$ and $\max\left\{0, \frac{p_g-3\theta+4p_g\theta}{2\theta}\right\} < w_b < \frac{5p_g-3\theta}{2\theta}$;

Region κ_2 : if $0 < p_g < \theta$ and $\max\left\{0, \frac{6p_g-3\theta+4p_g\theta-3\theta^2}{3\theta+\theta^2}\right\} < w_b < \frac{3+2p_g-\theta}{3+\theta}$;

Region κ_3 : if $\frac{3\theta+3\theta^2}{6+4\theta} < p_g < \frac{3\theta}{5}$ and $0 < w_b < \frac{6p_g-3\theta+4p_g\theta-3\theta^2}{3\theta+\theta^2}$, or $\frac{3\theta}{5} < p_g < \theta$ and $\frac{5p_g-3\theta}{2\theta} < w_b < \frac{6p_g-3\theta+4p_g\theta-3\theta^2}{3\theta+\theta^2}$;

Region κ_4 : if $\frac{1}{2} < \theta < 1$, $\frac{3\theta}{1+4\theta} < p_g < \theta$, and $0 < w_b < \frac{p_g-3\theta+4p_g\theta}{2\theta}$;

Region κ_5 : if $p_g < \theta$ and $\frac{3+2p_g-\theta}{3+\theta} < w_b < 1$.

Combining the two cases above, we have the following result to show the retailer's best response for a given wholesale price w_b and the second unit price p_g,

$$p_b{}^*(p_g, w_b) = \begin{cases} \frac{3\theta+2w_b\theta-p_g}{4\theta}, & \text{if } \{p_g,\ w_b\} \in \kappa_1 \\ \frac{3-2p_g+3w_b+3\theta+w_b\theta}{6+2\theta}, & \text{if } \{p_g,\ w_b\} \in \kappa_2 \\ \frac{p_g}{\theta}, & \text{if } \{p_g,\ w_b\} \in \kappa_3 \\ p_g, & \text{if } \{p_g,\ w_b\} \in \kappa_4 \\ 1, & \text{if } \{p_g,\ w_b\} \in \kappa_5 \end{cases}.$$

The optimal product demand and retailer's expected profit are

$$D_b{}^*(p_g, w_b) = \begin{cases} \left(\frac{p_g+\theta-2w_b\theta}{2\theta}\right) + \left(1 - \frac{p_g}{\theta}\right), & \text{if } \{p_g,\ w_b\} \in \kappa_1 \\ \left(\frac{6-2p_g+7\theta+\theta^2-w_b(6+5\theta+\theta^2)}{2(1+\theta)(3+\theta)}\right) + \left(1 - \frac{3(1+\theta)+2p_g(2+\theta)+w_b(3+\theta)}{2(1+\theta)(3+\theta)}\right), & \text{if } \{p_g,\ w_b\} \in \kappa_2 \\ \left(2 - \frac{2p_g}{\theta}\right) + \left(1 - \frac{p_g}{\theta}\right), & \text{if } \{p_g,\ w_b\} \in \kappa_3 \\ 1 - w_b, & \text{if } \{p_g,\ w_b\} \in \kappa_4 \\ 0, & \text{if } \{p_g,\ w_b\} \in \kappa_5 \end{cases},$$

and

$$\pi_{br}{}^*(p_g, w_b) = \begin{cases} \frac{(p_g-(3-2w_b)\theta)^2}{8\theta^2}, & \text{if } \{p_g,\ w_b\} \in \kappa_1 \\ \frac{(2p_g-3(1+\theta)+w_b(3+\theta))^2}{4(1+\theta)(3+\theta)}, & \text{if } \{p_g,\ w_b\} \in \kappa_2 \\ \frac{3(\theta-p_g)(p_g-w_b\theta)}{\theta^2}, & \text{if } \{p_g,\ w_b\} \in \kappa_3 \\ \frac{(1-w_b)^2}{2}, & \text{if } \{p_g,\ w_b\} \in \kappa_4 \\ 0, & \text{if } \{p_g,\ w_b\} \in \kappa_5 \end{cases}.$$

Second, based on the retailer's optimal response, the manufacturer's problem is:

$$\pi_{bm}(p_g, w_b) = \begin{cases} \left(\frac{p_g+\theta-2w_b\theta}{2\theta}\right)(w_b-c) + \left(1-\frac{p_g}{\theta}\right)\left(w_b - c + \frac{p_g-3\theta+4\theta p_g-2w_b\theta}{4\theta}\right), & \text{if } \{p_g,\ w_b\} \in \kappa_1 \\ \left(\frac{6-2p_g+7\theta+\theta^2-w_b(6+5\theta+\theta^2)}{2(1+\theta)(3+\theta)}\right)(w_b-c) + & \\ \left(\frac{3+5\theta+2\theta^2-2p_g(2+\theta)-w_b(3+\theta)}{2(1+\theta)(3+\theta)}\right)\left(w_b - c - \frac{3(1+\theta)+w_b(3+\theta)-2p_g(4+\theta)}{6+2\theta}\right), & \text{if } \{p_g,\ w_b\} \in \kappa_2 \\ \left(2-\frac{2p_g}{\theta}\right)(w_b-c) + \left(1-\frac{p_g}{\theta}\right)\left(w_b - c + p_g - \frac{p_g}{\theta}\right), & \text{if } \{p_g,\ w_b\} \in \kappa_3 \\ (1-w_b)(w_b-c), & \text{if } \{p_g,\ w_b\} \in \kappa_4 \\ 0, & \text{if } \{p_g,\ w_b\} \in \kappa_5 \end{cases}.$$

By solving the problem above, we can obtain the following solutions:

- Solution 1: $\{p_g,\, w_b\} = \left\{\frac{\theta(2+c+2\theta)}{1+4\theta},\ \frac{1+c}{2}\right\}$, if $0 \le c \le 1$ and $\frac{1+c}{2} < \theta \le 1$, the manufacturer's expected profit is $\pi_{bm1} = \frac{1+c^2-6c\theta+2c^2\theta+2\theta^2}{2+8\theta}$.
- Solution 2: $\{p_g,\, w_b\} = \left\{\frac{3\theta}{1+4\theta}, \frac{1+c}{2}\right\}$, if $0 \le c \le 1$ and $\frac{1}{2} < \theta < \frac{11-c}{16+4c}$, the manufacturer's expected profit is $\pi_{bm2} = \frac{1+2c+c^2+4\theta-16c\theta+4c^2\theta}{4+16\theta}$.
- Solution 3: $\{p_g,\, w_b\} = \left\{\frac{\theta(11+3c+\theta)}{13+2\theta}, \frac{5p_g-3\theta}{2\theta}\right\} = \left\{\frac{\theta(11+3c+\theta)}{13+2\theta}, \frac{16+15c-\theta}{26+4\theta}\right\}$, if $0 \le c \le 1$ and $\max\{0, 3c-2\} < \theta < 1$, the manufacturer's expected profit is $\pi_{bm3} = \frac{(2-3c+\theta)^2}{26+4\theta}$.
- Solution 4: $\{p_g,\, w_b\} = \left\{0, \frac{9+11\theta+2\theta^2+c(3+\theta)^2}{(3+\theta)(5+2\theta)}\right\}$, if $0 < c < \frac{2}{3}$ and $0 < \theta < \frac{2-3c}{4+c}$, the manufacturer's expected profit is $\pi_{bm4} = \frac{c^2(3+\theta)^3-4(1+\theta)^2(-3+2\theta)-4c(9+18\theta+11\theta^2+2\theta^3)}{4(1+\theta)(3+\theta)(5+2\theta)}$.
- Solution 5: $\{p_g,\, w_b\} = \left\{\frac{-3+10\theta+\theta^2+2c(3+\theta)}{2(7+\theta)}, \frac{2(3+c)}{7+\theta}\right\}$, if $0 < c < \frac{1}{2}$ and $-5-c+\sqrt{28+4c+c^2} \le \theta \le 1$; or if $\frac{1}{2} \le c < 1$ and $-1+2c < \theta \le 1$, the manufacturer's expected profit is $\pi_{bm5} = \frac{(3+\theta)(1-2c+\theta)^2}{4(1+\theta)(7+\theta)}$.
- Solution 6: $\{p_g,\, w_b\} = \left\{0, \frac{3-\theta}{3+\theta}\right\}$, if $0 \le c \le 1$ and $0 < \theta \le 1$, the manufacturer's expected profit is $\pi_{bm6} = \frac{\theta(3(1-\theta)-2c(3+\theta))}{(1+\theta)(3+\theta)}$.
- Solution 7: $\{p_g,\, w_b\} = \left\{\frac{\theta(24+23\theta+\theta^2+3c(3+\theta))}{2(15+14\theta+\theta^2)}, \frac{9+9c+7\theta+6c\theta-\theta^2}{15+14\theta+\theta^2}\right\}$, if $0 \le c \le 1$ and $\max\{0, 3c-2\} < \theta \le 1$, the manufacturer's expected profit is $\pi_{bm7} = \frac{(3+\theta)(2-3c+\theta)^2}{4(15+14\theta+\theta^2)}$.

The optimal solution(s) should be higher than the EDLP solution and the others at same time within the region of $\{c, \theta\}$. By comparing the solutions above, we can obtain the optimal policy for the manufacturer in Proposition EC.4. □

Proof of Corollary 6. The optimal solutions for the supply chain are provided in Table EC.19.

Parameters	MBOGO Boundary Solution $0 < c \le \acute{c}(\theta)$	MBOGO Interior Solution $\acute{c}(\theta) < c \le 2\theta - 1$
w_b	$\frac{9+9c+7\theta+6c\theta-\theta^2}{15+14\theta+\theta^2}$	$\frac{1+c}{2}$
p_b	$\frac{24+23\theta+\theta^2+3c(3+\theta)}{2(15+14\theta+\theta^2)}$	$\frac{1+7\theta+2c\theta}{2+8\theta}$
p_g	$\frac{\theta(24+23\theta+\theta^2+3c(3+\theta))}{2(15+14\theta+\theta^2)}$	$\frac{\theta(2+c+2\theta)}{1+4\theta}$
D_b	$\frac{3(2+\theta-3c)(3+\theta)}{2(15+14\theta+\theta^2)}$	$\frac{3\theta-c-2c\theta}{1+4\theta}$
π_{br}	$\frac{9(1+\theta)(3+\theta)(2-3c+\theta)^2}{4(15+14\theta+\theta^2)^2}$	$\frac{(c-3\theta+2c\theta)^2}{2(1+4\theta)^2}$
π_{bm}	$\frac{(3+\theta)(2-3c+\theta)^2}{4(15+14\theta+\theta^2)}$	$\frac{1+c^2-6c\theta+2c^2\theta+2\theta^2}{2+8\theta}$

Table EC.19 Manufacturer-Retailer Optimal Policies with MBOGO-NSP

Similar to the proof of Corollary 2, we can prove Corollaries 6(a), 6(b), and 6(c). Similar to the proof of Corollary 3, in the model of MBOGO, we integrate over individual consumer's surplus to find the total consumer surplus, if $0 \le c \le \acute{c}(\theta)$

$$CS_b = \frac{(2+\theta)(3+\theta)^2(2-3c+\theta)^2}{8(15+\theta(14+\theta))^2}.$$

If $\acute{c}(\theta) < c \leq 2\theta - 1$, the consumer surplus is

$$CS_b = \frac{1 + \theta(4 + 2c^2 - 7\theta + 4(-3 + c)c\theta + 8\theta^2)}{4(1 + 4\theta)^2}.$$

Combining the profits of the retailer and manufacturer, if $0 \leq c \leq \acute{c}(\theta)$, the social welfare is

$$SW_b = \frac{3(3 + \theta)(2 - 3c + \theta)^2(18 + 17\theta + \theta^2)}{8(15 + 14\theta + \theta^2)^2}.$$

If $\acute{c}(\theta) < c \leq 2\theta - 1$, the total social welfare is

$$SW_b = \frac{-12c\theta(2 + 7\theta) + c^2(4 + 22\theta + 28\theta^2) + 3(1 + 4\theta + 5\theta^2 + 8\theta^3)}{4(1 + 4\theta)^2}.$$

Comparing the results above with those in Corollary 3, Corollary 6 follows. □