Appendix A: Upper Bound Results

A.1. Proof of Proposition 1

PROPOSITION 1. *There is no non-constant function $\mathbb{E}_{Y,P}[L(\cdot,Y,P)|V=v]$, left continuous in v and with $L(\cdot,Y,P)$ convex with respect to its first argument, such that for any distribution $(Y,P,V)\sim D$ on $\{0,1\}\times\mathbb{R}_+\times\mathbb{R}_+$ satisfying Equation 1, there exists a non-negative optimal price $p^*\in\arg\max_p\mathcal{R}(p)$, satisfying $\min_p\mathbb{E}_{Y,P}[L(p,Y,P)]=\mathbb{E}_{Y,P}[L(p^*,Y,P)]$.*

Proof of Proposition 1: If $L(p,(Y,P))$ is convex with respect to its first argument, then so is $\mathbb{E}_{Y,P}[L(p,(Y,P))|V]$ (Boyd and Vandenberghe (2004), Section 3.2.1). Therefore $\mathbb{E}_{Y,P}[L(p,(Y,P))|V]$ satisfies the conditions for $L_c(p,V)$ from Mohri and Medina (2014). $\square$

A.2. Details of Bounding Algorithm

We upper-bound the worst-case revenue ratio by iteratively (i) constructing distributions against which the current loss function performs poorly and (ii) re-solving formulation (4) to obtain an improved loss function. We describe two procedures for generating adversarial distributions: one based on random sampling of Beta distributions, and one using MINLP optimization.

Sampling Beta distributions: All Beta distributions with parameters $\alpha,\beta\geq 1$ are log-concave. We use a shifted and scaled Beta CDF defined as:

$$F_{\text{Beta}}(t;\alpha,\beta,\text{loc},\text{scale})=\int_0^t\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\cdot\text{scale}}\left(\frac{x-\text{loc}}{\text{scale}}\right)^{\alpha-1}\left(1-\left(\frac{x-\text{loc}}{\text{scale}}\right)\right)^{\beta-1}dx.$$

We draw `num_dist` sets of parameters $(\alpha_k,\beta_k,\text{loc}_k,\text{scale}_k)$ uniformly from $[1,10]\times[1,10]\times[0,p_m]\times[0,p_m]$ and evaluate the performance of the current loss function L_t under the resulting distribution. The `worst_W` distributions that yield the lowest revenue ratios are added to the current set $\mathcal{F}_t$ to produce $\mathcal{F}_{t+1}$.

Function Sample_adversarial_distributions(L_t, $\mathcal{F}_t$, `num_dist`, `worst_W`)

Draw $\alpha_k,\beta_k\sim U(1,10)$ and $\text{loc}_k,\text{scale}_k\sim U(0,p_m)$ for $k=1$ to `num_dist` ;
for $k=1$ *to num_dist* **do**
  Compute $\bar{F}_i^k=1-F_{\text{Beta}}(p_i;\alpha_k,\beta_k,\text{loc}_k,\text{scale}_k)$ for all $i\in[m]$;
  Let $s^k=\arg\min_{i\in[m]}\sum_{j=1}^m L_{i,j}^1\bar{F}_j^k+L_{i,j}^0(1-\bar{F}_j^k)$;
  Let $s^*=\arg\max_{i\in[m]}p_i\bar{F}_i^k$;
  Compute $Z_k=\frac{p_{s^k}\bar{F}_{s^k}^k}{p_{s^*}\bar{F}_{s^*}^k}$;
end
Add the `worst_W` distributions with smallest Z_k to $\mathcal{F}_t$ to get $\mathcal{F}_{t+1}$;
Solve formulation (4) with $\mathcal{F}_{t+1}$ to get L_{t+1} and Z_{t+1}^{UB} ;
return L_{t+1}, $\mathcal{F}_{t+1}$, Z_{t+1}^{UB} ;

Generating adversarial distributions via MINLP: If random sampling no longer produces low-revenue distributions, we solve the MINLP formulation (5) for each pair of price indices (s,s^*), where s is the index selected by the loss function and s^* is the revenue-maximizing price. If a feasible distribution is found, we record the resulting revenue ratio.

Function Optimize_adversarial_distributions(L_t, $\mathcal{F}_t$, `worst_W`)

for $s \in [m]$ **do**
  for $s^* \in [m]$ **do**
    Solve formulation (5) with s as minimizer and s^* as maximizer ;
    If feasible, record Z^{LB}_{s,s^*} and distribution $\bar{F}_{s,s^*}$;
  end
end
Add the `worst_W` distributions with smallest Z^{LB}_{s,s^*} to $\mathcal{F}_t$ to get $\mathcal{F}_{t+1}$;
Solve formulation (4) with $\mathcal{F}_{t+1}$ to obtain L_{t+1} and Z^{UB}_{t+1} ;
return L_{t+1}, $\mathcal{F}_{t+1}$, Z^{UB}_{t+1}, Z^{LB}_{t+1} ;

Numerical results: We use a discrete price grid $\{0, 1, \ldots, 10\}$ with $m = 11$ points and $\epsilon = 10^{-4}$. The procedure is:

- Run 120 iterations of `Sample_adversarial_distributions` with `num_dist` $= 1000$ and `worst_W` $= 5$.
- Then run 12 iterations of `Optimize_adversarial_distributions` with `worst_W` $= 5$.

The final upper bound was $Z^{UB} = 0.8633$ and the final lower bound was $Z^{LB} = 0.7074$.

Appendix B: Proof of Lemma 1

Lemma 1. *(Hinge pricing loss optimality condition): Under assumptions 1 and 2:*

$$p_h = \arg\min_{\pi(X)} \mathbb{E}_{Y,P}\left[\frac{1}{\phi(P|X)}\left(cY(P-\pi(X))^+ + (1-cY)(\pi(X)-P)^+\right)\Big| X\right] = c\mathbb{E}[V|X]$$

Proof of Lemma 1:

$$\begin{aligned}
\mathbb{E}_{Y,P}[L^h_c(p_h, Y, p)|X] &= \mathbb{E}_P[\mathbb{E}_{Y|P=p}[L^h_c(p_h, Y, P)|X, P=p]|X] \\
&= \int_0^{\bar{p}} \Big(\frac{c\mathbb{E}_{Y|P=p}[Y|X, P=p](p-p_h)^+}{\phi(p|X)} + \frac{(1-c\mathbb{E}_{Y|P=p}[Y|X, P=p])(p_h-p)^+}{\phi(p|X)}\Big)\phi(p|X)dp \\
&= \int_0^{\bar{p}} \left(\frac{c\bar{F}_V(p)}{\phi(p|X)}(p-p_h)^+ + \frac{1-c\bar{F}_V(p)}{\phi(p|X)}(p_h-p)^+\right)\phi(p|X)dp \\
&= \int_{p_h}^{\bar{p}} c\bar{F}_V(p)(p-p_h)dp + \int_0^{p_h}(1-c\bar{F}_V(p))(p_h-p)dp
\end{aligned}$$

Applying the Leibniz rule to find the first-order optimality conditions:

$$\begin{aligned}
\frac{\partial}{\partial p_h}\mathbb{E}_{Y,P}[L^h_c(p_h, Y, P)|X] &= \int_{p_h}^{\bar{p}} -c\bar{F}_V(p)dp + \int_0^{p_h} 1 - c\bar{F}_V(p)dp \\
&= \int_0^{p_h} 1dp - c\int_0^{\bar{p}} \bar{F}_V(p)dp \\
&= p_h - c\mathbb{E}[V|X] = 0
\end{aligned}$$

The third equality follows from assumption 2 and the definition from Equation 1. In particular, $\mathbb{E}_{Y|P=p}[Y|X, P=p] = \mathbb{E}[Y(p,V)|X]\ \forall p \in [0, \bar{v}]$ due to unconfoundedness (i.e., the potential outcome $Y(p,V)$, the purchase outcome associated with a customer being given any price p, isn't dependent on the pricing policy generating P), and $\mathbb{E}[Y(p,V)|X] = \mathbb{P}(Y(p,V)=1|X) = \mathbb{P}(V > p|X) = \bar{F}_V(p)$ from the definition of Y. The overlap assumption is used in the second equality, where the integration is taken over the support of the distribution of P. If this does not align with the support of V, then the identity that $\int_0^{\bar{p}} \bar{F}_V(p) = \int_0^{\bar{v}} \bar{F}_V(p)dp = \mathbb{E}[V|X]$ cannot be used in the last line. Furthermore, $\phi(p|X)$ isn't well defined outside its support. $\square$

Appendix C: Proof of Theorem 1 for the Hinge Loss Function

THEOREM 1. *Let* $p^* = \arg\max_p \mathcal{R}(p)$ *and*

$$p_h = \arg\min_{\pi(X)} \mathbb{E}_{Y,P}\left[\frac{1}{\phi(P|X)}\left(cY(P-\pi(X))^+ + (1-cY)(\pi(X)-P)^+\right)\Big|X\right].$$

Under assumptions 1, 2, and 3, for $0 < c \leq 1$*, the expected worst-case revenue from pricing using the hinge loss function is bounded by*

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min\left\{\min_{z\leq -2c}\frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c},\ \min_{0<z<1}\frac{c(z-1)e^{c(z-1)}}{z\ln(z)}\right\}.$$

Furthermore, this bound is tight.

Proof of Theorem 1: We need to examine two cases: the case where the prescribed price is below the optimal price for the adversarial distribution ($p_h < p^*$), and the case where it is above ($p_h > p^*$). We prove lemmas that give revenue bounds in each case, and prove they are tight. We start with the case where $p_h < p^*$.

LEMMA 3. *Consider a pricing policy that prescribes a price* p_h *such that* $p_h \geq c\int_0^\infty \bar{F}_V(p)dp$ *and* $p_h < p^*$*, then:*

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min_{0<z<1}\frac{c(z-1)e^{c(z-1)}}{z\ln(z)}.$$

The proof can be found in Appendix C.1 and follows from manipulations of log-concave distributions. We note that the condition in Lemma 3, $p_h \geq c\int_0^\infty \bar{F}_V(p)dp$, is satisfied by the optimality conditions for the hinge pricing loss function with equality. We prove an expected revenue bound for the case where the prescribed price is greater than the optimal price $p_h \geq p^*$:

LEMMA 4. *If there exists a pricing rule* p_h *satisfying optimality conditions from Lemma 1 and* $p_h > p^*$*, then:*

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min\left\{\min_{z\leq -2c}\frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}, (c+1)e^{-c}\right\} \tag{9}$$

This is proved in Appendix C.3. Theorem 1 follows from applying the optimality conditions in Lemma 1 and combining the worst-case bounds in Lemma 3 and 4. These expressions are difficult to evaluate, but it is possible to do so using simulation. It can be shown that for the range $c > 0.792$, $\min_{z\leq-2c}\frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c} \leq (c+1)e^{-c}$. Furthermore, for $c < 0.823$, $\min_{0<z<1}\frac{c(z-1)e^{c(z-1)}}{z\ln(z)} \leq \min_{z\leq-2c}\frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}$ and $\min_{0<z<1}\frac{c(z-1)e^{c(z-1)}}{z\ln(z)} \leq (c+1)e^{-c}$. As a result, the bound $(c+1)e^{-c}$ is never the minimum for any value of c, leading to its exclusion from the bound presented in Theorem 1.

Discussion: While the bound from Lemma 3 is relatively opaque, it simplifies significantly for lower values of c. In this case, the hinge pricing policy receives a constant fraction c of the revenue.

COROLLARY 5. $\dfrac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min_{0<z<1}\dfrac{c(z-1)e^{c(z-1)}}{z\ln(z)} = c$ *for* $0 \leq c \leq 0.5$

The proof is in Appendix C.4. In this case, the adversarial distribution is simple and puts a point mass at p^*, so $\mathbb{E}[V|X] = p^*$ and the hinge price gets a fraction c of the revenue. This distribution is shown in Figure 8a. The worst-case distribution is more involved for $c > 0.5$.

An important special case of the bound from Lemma 4 occurs when $c = 1$ when the hinge pricing function prices at the conditional expected valuation $p_h = \mathbb{E}[V|X]$. In this case, the bound from Lemma 4 simplifies significantly.

COROLLARY 6. $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \min_{z \leq -2c} \frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c} = e^{-1}$ *for* $c = 1$.

We also note that for $c < 1$, a closed-form solution can be found for the minimizer of the bound.

PROPOSITION 6. *For* $0 < c < 1$,

$$\arg\min_{z \leq -2c} \frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c} = -\frac{c}{2}\left(\sqrt{\frac{c-5}{c-1}} + 1\right)$$

This is proved in Appendix C.5.

C.1. Proof of Lemma 3

LEMMA 3. *Consider a pricing policy that prescribes a price* p_h *such that* $p_h \geq c\int_0^\infty \bar{F}_V(p)dp$ *and* $p_h < p^*$, *then:*

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min_{0<z<1} \frac{c(z-1)e^{c(z-1)}}{z\ln(z)}.$$

Proof of Lemma 3:

We begin by providing a lower bound on p_h resulting from the optimality condition $p_h \geq c\int_0^\infty \bar{F}_V(p)dp$. This will lead to a revenue bound on $p_h\bar{F}_V(p)$.

$$\int_0^\infty \bar{F}_V(p)dp \geq \int_0^{p^*} \bar{F}_V(p)dp \geq \int_0^{p^*} \bar{F}_V(p^*)^{\frac{p}{p^*}} dp = \left[\frac{p^*\bar{F}_V(p^*)^{\frac{p}{p^*}}}{\ln(\bar{F}_V(p^*))}\right]_0^{p^*} = \frac{p^*\left(\bar{F}_V(p^*)-1\right)}{\ln(\bar{F}_V(p^*))} \tag{10}$$

Here, the second inequality follows due to log-concavity, $\bar{F}_V(p) \geq \bar{F}_V(0)^\theta \bar{F}_V(p^*)^{1-\theta}$ for $0 < \theta < 1$, and since $\bar{F}_V(0) = 1$. Furthermore, by substituting (10) into condition $p_h \geq c\int_0^\infty \bar{F}_V(p)dp$,

$$\frac{p_h}{p^*} \geq \frac{c}{p^*}\int_0^\infty \bar{F}_V(p)dp \geq \frac{c\left(\bar{F}_V(p^*)-1\right)}{\ln(\bar{F}_V(p^*))} \tag{11}$$

Therefore:

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h\bar{F}_V(p_h)}{p^*\bar{F}_V(p^*)} \geq \frac{p_h\bar{F}_V(p^*)^{\frac{p_h}{p^*}}}{p^*\bar{F}_V(p^*)} \tag{12}$$

The inequality follows from log-concavity. Furthermore, since $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \leq 1$, $\frac{p_h\bar{F}_V(p^*)^{\frac{p_h}{p^*}}}{p^*\bar{F}_V(p^*)} \leq 1$. We also have that $0 \leq \bar{F}_V(p^*) \leq 1$ and $0 \leq \frac{p_h}{p^*} < 1$. Therefore, according to Lemma 5, which can be found in Appendix C.2, $\frac{p_h\bar{F}_V(p^*)^{\frac{p_h}{p^*}}}{p^*\bar{F}_V(p^*)}$ is a monotone increasing function within this range, which validates the substitution of (11) into (12):

$$\begin{aligned}\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \frac{p_h\bar{F}_V(p^*)^{\frac{p_h}{p^*}}}{p^*\bar{F}_V(p^*)} &\geq \frac{c\left(\bar{F}_V(p^*)-1\right)}{\ln(\bar{F}_V(p^*))}\bar{F}_V(p^*)^{\frac{c(\bar{F}_V(p^*)-1)}{\ln(\bar{F}_V(p^*))}-1} \\ &\geq \frac{c\left(\bar{F}_V(p^*)-1\right)}{\bar{F}_V(p^*)\ln(\bar{F}_V(p^*))}e^{c(\bar{F}_V(p^*)-1)}\end{aligned}$$

Minimizing $\bar{F}_V(p^*)$ to find the worst-case distribution completes the proof. $\square$

C.2. Proof of Lemma 5

LEMMA 5. *In the domain $0 \leq x \leq 1, 0 \leq b \leq 1$ and $xb^{x-1} \leq 1$, xb^{x-1} is monotone increasing.*

Proof of Lemma 5:

$$\frac{\partial}{\partial x} xb^{x-1} = \frac{\partial}{\partial x} \frac{x}{b} e^{x\ln(b)} = \frac{e^{x\ln(b)}}{b} + \frac{x\ln(b)}{b} e^{x\ln(b)} = \frac{e^{x\ln(b)}}{b} \left[1 + x\ln(b)\right]$$

$\frac{e^{x\ln(b)}}{b} \geq 0$ for $b \geq 0$. Therefore the derivative is positive if $1 + x\ln(b) > 0$, or equivalently, $x < \frac{-1}{\ln(b)}$, and negative for $x > \frac{-1}{\ln(b)}$, with the maximum occurring at $x = \frac{-1}{\ln(b)}$. Therefore, the maximum value xb^{x-1} can take is $\frac{-e^{-1}}{b\ln(b)}$.

It can be verified that $\frac{-e^{-1}}{b\ln(b)} \geq 1$; for example, by taking the derivative, the maximum occurs when $b = e^{-1}$. Finally, $xb^{x-1} = 1$ at $x = 1$. Therefore, xb^{x-1} is monotone increasing in the range $0 \leq x < \frac{-1}{\ln(b)}$, and for $\frac{-1}{\ln(b)} \leq x \leq 1$, $xb^{x-1} \geq 1$. $\square$

C.3. Proof of Lemma 4

LEMMA 4. *If there exists a pricing rule p_h satisfying optimality conditions from Lemma 1 and $p_h > p^*$, then:*

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min\left\{\min_{z\leq -2c} \frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}, (c+1)e^{-c}\right\} \tag{9}$$

Proof of Lemma 4:

We start by proving some lower bounds on $\bar{F}_V(p_h)$, related to the gradient and position of p_h. This lower bound also acts as a lower bound on the revenue, as for the regime where $p^* \leq p_h$, we have that $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \bar{F}_V(p_h)$. We will show that:

$$\bar{F}_V(p_h) \geq e^{-sp_h(\frac{1}{c}-1)-1} \tag{13}$$

Here s is a gradient of the log of the complementary CDF at p_h:

$$s = \frac{\partial \ln(\bar{F}_V(p))}{\partial p}\Big|_{p=p_h} = \frac{\bar{F}'_V(p_h)}{\bar{F}_V(p_h)} < 0.$$

This also implies that $\bar{F}_V(p_h) \geq e^{-c}$. We begin by upper-bounding the complementary CDF. The function $\ln(\bar{F}_V(p))$ is concave, so it lies below its tangent at p_h:

$$\ln(\bar{F}_V(p)) \leq \ln(\bar{F}_V(p_h)) + s(p - p_h) \implies \bar{F}_V(p) \leq \bar{F}_V(p_h)e^{s(p-p_h)}$$

Furthermore, because $\bar{F}_V(p) \geq \bar{F}_V(p_h)^{\frac{p}{p_h}}$ for $0 \leq p \leq p_h$ and $\bar{F}_V(p) \leq \bar{F}_V(p_h)^{\frac{p}{p_h}}$ for $p_h \leq p < \infty$, it follows that $\bar{F}'_V(p_h) \leq \frac{\partial \bar{F}_V(p_h)^{\frac{p}{p_h}}}{\partial p}\Big|_{p=p_h} = \bar{F}_V(p_h)\frac{\ln(\bar{F}_V(p_h))}{p_h}$, which implies $s \leq \frac{\ln(\bar{F}_V(p_h))}{p_h}$. Furthermore, since $\bar{F}_V(p) \leq 1$, a refined upper bound is

$$\bar{F}_V(p) \leq \bar{F}_{UB}(p) := \begin{cases} 1 & \text{if } \ p \leq p_h - \frac{\ln(\bar{F}_V(p_h))}{s} \\ \bar{F}_V(p_h)e^{s(p-p_h)} & \text{if } \ p \geq p_h - \frac{\ln(\bar{F}_V(p_h))}{s} \end{cases} \tag{14}$$

Here $p_h - \frac{\ln(\bar{F}_V(p_h))}{s}$ is where the upper bounds intersect. From the optimality condition in Lemma 1:

$$\begin{aligned} p_h = c\int_0^\infty \bar{F}_V(p)dp &\leq c\left(\int_0^{p_h - \frac{\ln(\bar{F}_V(p_h))}{s}} 1dp + \int_{p_h - \frac{\ln(\bar{F}_V(p_h))}{s}}^\infty \bar{F}_V(p_h)e^{s(p-p_h)}dp\right) \\ &= c\left(p_h - \frac{\ln(\bar{F}_V(p_h))}{s} + \left[\frac{\bar{F}_V(p_h)}{s}e^{s(p-p_h)}\right]_{p_h - \frac{\ln(\bar{F}_V(p_h))}{s}}^\infty\right) \\ &= c\left(p_h - \frac{\ln(\bar{F}_V(p_h))}{s} + \frac{\bar{F}_V(p_h)}{s}e^{-\ln(\bar{F}_V(p_h))}\right) \end{aligned}$$

With some rearranging, this implies:

$$\bar{F}_V(p_h) \geq e^{-sp_h(\frac{1}{c}-1)-1} \tag{15}$$

Since $s \leq \frac{\ln(\bar{F}_V(p_h))}{p_h}$ for $0 < c \leq 1$, this implies:

$$\bar{F}_V(p_h) \geq e^{-c} \tag{16}$$

This also implies $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq e^{-1}$ when $c = 1$. We now turn to two specific revenue bounds depending on the size of c. An upper bound on the optimal revenue can be found by optimizing the upper bound on the revenue function corresponding to (14), $p^*\bar{F}_V(p^*) \leq \max_p p\bar{F}_{UB}(p)$, where $p^*_{UB} := \arg\max_p p\bar{F}_{UB}(p)$.

For certain values of c, $p^*_{UB} \geq p_h - \frac{\ln(\bar{F}_V(p_h))}{s}$, corresponding to p^*_{UB} lying on the second section of (14). It follows that

$$\max_p p\bar{F}_{UB}(p) = \max_p p\bar{F}_V(p_h)e^{s(p-p_h)}$$

From the optimality conditions, this has an maximum at $p^*_{UB} = \frac{-1}{s}$, so that $p^*_{UB}\bar{F}_{UB}(p^*_{UB}) = \dfrac{-\bar{F}_V(p_h)}{s}e^{-1-sp_h}$. Furthermore, due to the position of p^*_{UB}, we can derive some additional useful bounds. Because $p^*_{UB} \leq p_h$, this implies that:

$$sp_h \leq -1 \tag{17}$$

Furthermore, because $p^*_{UB} \geq p_h - \frac{\ln(\bar{F}_V(p_h))}{s}$, this implies that:

$$-1 \leq sp_h - \ln(\bar{F}_V(p_h)) \tag{18}$$

If we return to the expected revenue bounds and substitute $p^*_{UB}\bar{F}_{UB}(p^*_{UB})$, we have

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h\bar{F}_V(p_h)}{p^*\bar{F}_V(p^*)} \geq \frac{p_h\bar{F}_V(p_h)}{\frac{-\bar{F}_V(p_h)}{s}e^{-1-sp_h}} = -sp_he^{1+sp_h}$$

It can be shown that $-sp_he^{1+sp_h}$ is monotone increasing in sp_h for $sp_h \leq -1$, the upper bound from (17). Therefore, by substituting the lower bound, $sp_h \geq \ln(\bar{F}_V(p_h)) - 1$ from (18), we have that

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq -sp_he^{1+sp_h} \geq -(\ln(\bar{F}_V(p_h)) - 1)\bar{F}_V(p_h) \geq (c+1)e^{-c} \tag{19}$$

The second inequality follows from substituting the bound $\bar{F}_V(p_h) \geq e^{-c}$ from (16) and since $-(\ln(\bar{F}_V(p_h)) - 1)\bar{F}_V(p_h)$ is monotone increasing in the range $0 \leq \bar{F}_V(p_h) \leq 1$.

Next we address the case when p^* lies on the upper bound where $\bar{F}_{UB}(p^*) = 1$, such that $p^*_{UB} \leq p_h - \frac{\ln(\bar{F}_V(p_h))}{s}$. This means that the inverse of the statement in (18) is true, which through rearranging implies

$\bar{F}_V(p_h) \geq e^{sp_h+1}$. Furthermore we have another lower bound on $\bar{F}_V(p_h)$ from (15). Taking the maximum of these bounds gives us:

$$\bar{F}_V(p_h) \geq \begin{cases} e^{sp_h+1} & \text{for} \quad sp_h \geq -2c \\ e^{-sp_h(\frac{1}{c}-1)-1} & \text{for} \quad sp_h \leq -2c \end{cases} \tag{20}$$

It also immediate that for this case, $\max_p p\bar{F}_{UB}(p) = p_h - \frac{\ln(\bar{F}_V(p_h))}{s}$. Substituting this expression into the expected revenue ratio:

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h\bar{F}_V(p_h)}{p^*\bar{F}_V(p^*)} \geq \frac{sp_h\bar{F}_V(p_h)}{sp_h - \ln(\bar{F}_V(p_h))}.$$

Furthermore, $\dfrac{sp_h\bar{F}_V(p_h)}{sp_h - \ln(\bar{F}_V(p_h))}$ is monotone increasing in $\bar{F}_V(p_h)$ for $\bar{F}_V(p_h) \geq e^{sp_h+1}$. Therefore, we can substitute for $\bar{F}_V(p_h)$:

$$\frac{sp_h\bar{F}_V(p_h)}{sp_h - \ln(\bar{F}_V(p_h))} \geq \frac{sp_h e^{sp_h+1}}{sp_h - (sp_h+1)} = -sp_h e^{sp_h+1} \quad \forall sp_h \geq -2c$$

As mentioned previously, $-sp_h e^{sp_h+1}$ is monotone increasing for $sp_h \leq -1$, so it is minimized by $sp_h = -2c$ in the range $-2c \leq sp_h \leq -1$. The upper bound of -1 on sp_h is from (17), which also holds in this case. This leads to a bound:

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq 2ce^{-2c+1} \quad \text{if } -2c \leq sp_h \leq -1 \tag{21}$$

For $sp_h \leq -2c$, by substituting the alternative bound for $\bar{F}_V(p_h)$ from (20):

$$\frac{sp_h\bar{F}_V(p_h)}{sp_h - \ln(\bar{F}_V(p_h))} \geq \frac{sp_h e^{-sp_h(\frac{1}{c}-1)-1}}{sp_h + sp_h(\frac{1}{c}-1)+1} = \frac{csp_h e^{-sp_h(\frac{1}{c}-1)-1}}{sp_h + c} \quad \forall sp_h \leq -2c$$

Therefore, by substituting for $z = sp_h$ and taking the minimum:

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min_{z \leq -2c} \frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c} \quad \text{if } sp_h \leq -2c \tag{22}$$

Furthermore, it can be shown using simulation that $\min_{z\leq -2c} \frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c} \leq 2ce^{-2c+1}$ for all $0 < c < 1$, so the adversary will use the regime where $sp_h \leq -2c$ and the corresponding bound from (22). However, there are values of c for which $(c+1)e^{-c} \leq \min_{z\leq -2c} \dfrac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}$, leading to the bound:

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min\left\{ \min_{z\leq -2c} \frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}, (c+1)e^{-c} \right\} \quad \square$$

C.4. Proof of Corollary 5

Corollary 5. $\dfrac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min_{0<z<1} \dfrac{c(z-1)e^{c(z-1)}}{z\ln(z)} = c$ *for* $0 \leq c \leq 0.5$

Proof of Corollary 5: We achieve this by showing that the gradient of $\frac{c(z-1)e^{c(z-1)}}{z\ln(z)}$ is negative for all $0 < z < 1$ as long as $0 \leq c < 0.5$, so the solution lies on the boundary as $z \to 1$.

$$\frac{\partial}{\partial z} \frac{c(z-1)e^{c(z-1)}}{z\ln(z)} = \frac{ce^{cz-c}\left(\ln(z)(cz(z-1)+1) + (1-z)\right)}{z^2\ln(z)^2}$$

Clearly $\dfrac{ce^{cx-c}}{z^2\ln(z)^2} \geq 0$, so we will focus on showing $\ln(z)(cz(z-1)+1) + (1-z) \leq 0$. To achieve this, we will use the following bound: $\ln(z) \leq (z-1) - \frac{1}{2}(z-1)^2$ for $0 < z < 1$. This follows from a Taylor expansion of $\ln(z)$.

The second-order Taylor expansion of $\ln(z)$ around 1 is $(z-1)-\frac{1}{2}(z-1)^2$. The remainder is $R_k(\xi,z)=\dfrac{2}{\xi^3 3!}(z-1)^3$. Clearly for all $0<z<1$ and $z\le\xi\le 1$, $R_k(\xi,x)\le 0$, so $\ln(z)\le(z-1)-\frac{1}{2}(z-1)^2$ in this range. Furthermore, since $cz(z-1)+1\ge -0.25c+1\ge 0$ for $0<z<1$ and $0\le c\le 1$, we have that:

$$\ln(z)(cz(z-1)+1)+(1-x)\le((z-1)-\frac{1}{2}(z-1)^2)(cz(z-1)+1)+(1-z)=\frac{1}{2}(z-1)^2((3-z)cz-1)$$

Because $(3-z)z$ is maximized at $z^*=1$ if $0<z\le 1$ and $(3-z^*)z^*=2$, it follows that for $c\le 0.5$, $0<z<1$, $((3-z)cz-1)<0$. As a result, the gradient is negative for $0<z<1$, and the minimum occurs on the boundary $z\to 1$. Furthermore,

$$\lim_{z\to 1}\frac{c(z-1)e^{c(z-1)}}{z\ln(z)}=c$$

This follows from applying L'Hopital's rule, which states that $\lim_{f\to a}\frac{z(x)}{g(x)}=\lim_{z\to a}\frac{z'(x)}{g'(x)}$. As a result, $\lim_{z\to 1}\frac{z-1}{\ln(z)}=1$. Furthermore, clearly $\lim_{z\to 1}\frac{e^{c(z-1)}}{z}=1$, which proves the result. $\square$

C.5. Proof of Proposition 6

PROPOSITION 6. *For* $0<c<1$,

$$\underset{z\le -2c}{\arg\min}\frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}=-\frac{c}{2}\left(\sqrt{\frac{c-5}{c-1}}+1\right)$$

Proof of Proposition 6:

We begin by examining the first-order optimality conditions for $\dfrac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}$:

$$\begin{aligned}\frac{\partial}{\partial z}\frac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}&=\frac{ce^{-z(\frac{1}{c}-1)-1}}{z+c}-\frac{ce^{-z(\frac{1}{c}-1)-1}}{(z+c)^2}-\frac{cz(\frac{1}{c}-1)e^{-z(\frac{1}{c}-1)-1}}{z+c}\\&=\frac{ce^{-z(\frac{1}{c}-1)-1}}{(z+c)^2}\left(c^2z+c^2+cz^2-cz-z^2\right)\end{aligned}$$

Therefore, a root lies where $(c-1)z^2+c(c-1)z+c^2=0$. Applying the quadratic formula to find this root

$$\begin{aligned}\frac{-c(c-1)\pm\sqrt{c^2(c-1)^2-4(c-1)c^2}}{2(c-1)}&=\frac{-c(1-c)\pm\sqrt{c^2(c-1)(c-5)}}{2(c-1)}\\&=\frac{c\left(1-c\pm\sqrt{(c-1)(c-5)}\right)}{2(c-1)}\\&=c\left(-\frac{1}{2}\pm\frac{\sqrt{(c-1)(c-5)}}{2(c-1)}\right)\end{aligned}$$

We also know that $z\le -2c$. Since $0<c<1$, $\sqrt{(c-1)(c-5)}\ge 0$ and $c-1\le 0$, this constraint is only satisfied for

$$c\left(-\frac{1}{2}+\frac{\sqrt{(c-1)(c-5)}}{2(c-1)}\right)$$

This can be further simplified to:

$$-\frac{c}{2}\left(\sqrt{\frac{c-5}{c-1}}+1\right)\quad\square$$

Appendix D: Tightness of Bounds

In this section, we examine the tightness of the bounds from Theorems 1 and 2. This is achieved by finding valuation distributions that obtain the revenue given by these bounds. The worst-case distribution is dependent on c and τ. Lemma 6 shows worst-case distributions for particular parameter choices, some of which are illustrated in Figures 4 and 8.

LEMMA 6. 1. *For the hinge loss function, when $0 < c \le 0.5$, if V has a distribution*

$$\bar{F}_V(p) = \begin{cases} 1 & \text{for } 0 \le p \le 1 \\ 0 & \text{for } p > 1 \end{cases}$$

then $\dfrac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \min_{0<z<1} \dfrac{c(z-1)e^{c(z-1)}}{z\ln(z)} = c$. *This is equal to the bound from Theorem 1 for this range of c.*

2. *For the hinge loss function, when $0.5 \le c \le 0.8234$, if V has a distribution*

$$\bar{F}_V(p) = \begin{cases} g(c)^p & \text{for } 0 \le p \le 1 \\ 0 & \text{for } p > 1 \end{cases}$$

where $g(c) = \arg\min_{0<z<1} \dfrac{c(z-1)e^{c(z-1)}}{z\ln(z)}$, *then* $\dfrac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \min_{0<z<1} \dfrac{c(z-1)e^{c(z-1)}}{z\ln(z)}$. *This is equal to the bound from Theorem 1 for this range of c.*

3. *For the hinge loss function, and $0.8234 \le c \le 1$, if V has a distribution*

$$\bar{F}_V(p) = \begin{cases} 1 & \text{for } 0 \le p \le t(c) \\ e^{t(c)-p} & \text{for } p > t(c) \end{cases}$$

where $t(c) = \dfrac{1}{2}\left(\sqrt{\dfrac{c-5}{c-1}} - 1\right)$, *then* $\dfrac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \min_{z \le -2c} \dfrac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}$. *This is equal to the bound from Theorem 1 for this range of c.*

4. *For the quantile loss function, if V has a distribution*

$$\bar{F}_V(p) = \begin{cases} 1 & \text{for } 0 \le p \le 1 \\ 0 & \text{for } p > 1 \end{cases}$$

then, $\dfrac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} = 1 - \tau$. *For $0.5 \le \tau < 1$,* $\min_{0<z<1} \dfrac{(1-\tau)(z-1)e^{(1-\tau)(z-1)}}{z\ln(z)} = 1 - \tau$. *This is equal to the bound from Theorem 2 for this range of τ.*

5. *For the quantile loss function, when $0 < \tau \le 0.209$, if V has a distribution*

$$\bar{F}_V(p) = \begin{cases} 1 & \text{for } 0 \le p \le 1 \\ e^{(p-1)(1-\frac{z(\tau)}{\tau})} & \text{for } p > 1 \end{cases}$$

where $z(\tau) = \arg\min_{\tau \le z \le 1} \dfrac{z\tau(\ln(z)+1) - z^2}{\tau - z}$, *then,* $\dfrac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \min_{\tau \le z \le 1} \dfrac{z\tau(\ln(z)+1) - z^2}{\tau - z}$. *This is equal to the bound from Theorem 2 for this range of τ.*

Proof of Lemma 6:

1. Clearly $p_h = c, \bar{F}_V(p_h) = 1, p^* = 1, \bar{F}_V(p^*) = 1$. Therefore $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = c$. By Corollary 5, $\min_{0<z<1} \frac{c(z-1)e^{c(z-1)}}{z\ln(z)} = c$ for $0 < c \le 0.5$.

2. First, we will show that $p^* = 1$. The gradient of the expected revenue is positive

$$\frac{\partial \mathcal{R}(p)}{\partial p} = \frac{\partial}{\partial p} p\bar{F}_V(p) = e^{pg(c)}(1 + p\ln(g(c))) \geq 0$$

for $p\ln(g(c)) \geq -1$, or equivalently, $g(c)^p \geq e^{-1}$. Furthermore, since $g(c)^p$ is decreasing, the minimum occurs at $p = 1$. It can be numerically verified that for $c > 0.5$, $g(c) \geq e^{-1}$, so the maximum occurs at $p = 1$.

Next we will show that $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \min_{0<z<1} \frac{c(z-1)e^{c(z-1)}}{z\ln(z)}$. Due to the optimality condition of hinge loss function,

$$p_h = c\int_0^\infty \bar{F}_V(p)dp = c\int_0^\infty e^{p\ln(g(c))}dp = \frac{c\,(g(c)-1)}{\ln(g(c))}.$$

Furthermore, $\bar{F}_V(p_h) = e^{p_h \ln(g(c))}$ and $\bar{F}_V(p^*) = g(c)$. If the derived expressions are substituted back into the revenue expression:

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h\bar{F}_V(p_h)}{p^*\bar{F}_V(p^*)} = \frac{c(g(c)-1)e^{c(g(c)-1)}}{g(c)\ln(g(c))}.$$

Since $g(c) = \arg\min_{0<z<1} \frac{c(z-1)e^{c(z-1)}}{z\ln(z)}$, this proves the result. We can numerically show this is the minimizer for the bound from Theorem 1 for $c < 0.8234$.

3. First, we will show that $p^* = t(c)$ and therefore that $\bar{F}_V(p^*) = 1$ for the range of parameter values we consider. To achieve this, we will show that the revenue for the region $p \geq t(c)$ is less than $t(c)$. The gradient of the expected revenue for $\bar{F}_V(p) = e^{t(c)-p}$ at $t(c)$ is

$$\frac{\partial \mathcal{R}(p)}{\partial p}\Big|_{p=t(c)} = \frac{\partial}{\partial p} p\bar{F}_V(p)\big|_{p=t(c)} = e^{t(c)-p}(1-p)\big|_{p=t(c)} = 1 - t(c).$$

It can be shown that $t(c) \geq 1$ for $0.5 \leq c < 1$. To observe this, note that $t(0.5) = 1$ and that the gradient of $t(c)$ is positive in the range $0.5 \leq c < 1$:

$$\frac{\partial t(c)}{\partial c} = \frac{\partial}{\partial c}\frac{1}{2}\left(\sqrt{\frac{c-5}{c-1}} - 1\right) = \frac{1}{2}\left(\frac{1}{c-1} - \frac{c-5}{(c-1)^2}\right)\sqrt{\frac{c-1}{c-5}} = \frac{1}{2}\frac{4}{(c-1)^2}\sqrt{\frac{c-1}{c-5}} > 0$$

It follows that the gradient of $\mathcal{R}(p)$ evaluated at $t(c)$ is less than or equal to zero. Since $pe^{t(c)-p}$ is unimodal, the maximum revenue is achieved at $p = t(c)$.

Due to the optimality condition of hinge loss function,

$$p_h = c\int_0^\infty \bar{F}_V(p)dp = c\int_0^{t(c)} 1dp + c\int_{t(c)}^\infty e^{t(c)-p}dp = c(t(c)+1)$$

Therefore, $\bar{F}_V(p_h) = e^{t(c)(1-c)-c}$. Finally, the expected revenue ratio is:

$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h\bar{F}_V(p_h)}{p^*\bar{F}_V(p^*)} = \frac{c(t(c)+1)e^{t(c)(1-c)-c}}{t(c)} = \frac{cz^*e^{-z^*(\frac{1}{c}-1)-1}}{z^*+c}$$

Where we have made the substitution that $t(c) = \frac{-z^*}{c} - 1$ in the last equality, and the result follows from the fact that $z^* = \arg\min_{z\leq -2c} \dfrac{cze^{-z(\frac{1}{c}-1)-1}}{z+c}$. We can numerically show this is the minimizer for the bound from Theorem 1 for $c > 0.8234$.

4. It is clear that the expected revenue bound of pricing with the quantile loss function with this valuation distribution is $1-\tau$. Furthermore, from Lemma 5, $\min_{0<z<1} \frac{c(z-1)e^{c(z-1)}}{z\ln(z)} = c$ for $0 \le c \le 0.5$. Using $1-\tau = c$ completes the result. We can numerically show this is the minimizer for the bound from Theorem 2 for $0.5 < \tau < 1$.

5. For this valuation distribution, we will show that $p^* = 1$ and therefore that $\bar{F}_V(p^*) = 1$. The gradient of $p\bar{F}_V(p_h) = e^{(p-1)(1-\frac{z(\tau)}{\tau})}$ at $p=1$ is
$$\frac{\partial \mathcal{R}(p)}{\partial p}\Big|_{p=1} = \frac{\partial}{\partial p} p\bar{F}_V(p)\Big|_{p=1} = e^{(p-1)(1-\frac{z(\tau)}{\tau})}\left(1+p(1-\frac{z(\tau)}{\tau})\right)\Big|_{p=1} = 2-\frac{z(\tau)}{\tau}$$
Due to the unimodality of $pe^{(p-1)(1-\frac{z(\tau)}{\tau})}$, the maximum will occur at $p \le 1$ if the gradient is negative, which occurs if $z(\tau) \ge 2\tau$. It can be verified that $z(\tau) \ge 2\tau$ for $0 < \tau \le 0.5$ (we used numerical simulation). It follows that $p^* = 1$.

   Due to the optimality condition of the quantile loss function, we have that
$$\begin{aligned}
\tau\left(\int_0^\infty \bar{F}_V(p)dp\right) = \int_{p_q}^\infty \bar{F}_V(p)dp &\Longrightarrow \tau\left(\int_0^1 1dp + \int_1^\infty e^{(p-1)(1-\frac{z(\tau)}{\tau})}dp\right) = \int_{p_q}^\infty e^{(p-1)(1-\frac{z(\tau)}{\tau})}dp \\
&\Longrightarrow \tau\left(1+\left[\frac{e^{(p-1)(1-\frac{z(\tau)}{\tau})}}{1-\frac{z(\tau)}{\tau}}\right]_1^\infty\right) = \left[\frac{e^{(p-1)(1-\frac{z(\tau)}{\tau})}}{1-\frac{z(\tau)}{\tau}}\right]_{p_q}^\infty \\
&\Longrightarrow \tau\left(1-\frac{1}{1-\frac{z(\tau)}{\tau}}\right) = \frac{e^{(p_q-1)(1-\frac{z(\tau)}{\tau})}}{1-\frac{z(\tau)}{\tau}} \\
&\Longrightarrow z(\tau) = e^{(p_q-1)(1-\frac{z(\tau)}{\tau})} \\
&\Longrightarrow p_q = \frac{\ln(z(\tau))}{1-\frac{z(\tau)}{\tau}} + 1
\end{aligned}$$
Furthermore, by substitution of p_q into $\bar{F}_V(p)$, it follows that $\bar{F}_V(p_q) = z(\tau)$. Finally, the expected revenue is:
$$\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h\bar{F}_V(p_h)}{p^*\bar{F}_V(p^*)} = z(\tau)\left(\frac{\ln(z(\tau))}{1-\frac{z(\tau)}{\tau}}+1\right) = \frac{z(\tau)\tau(\ln(z(\tau))+1)-z(\tau)^2}{\tau - z(\tau)}$$
Since $z(\tau) = \arg\min_{\tau \le z \le 1} \dfrac{z\tau(\ln(z)+1)-z^2}{\tau - z}$, the result follows. We can numerically show this is the minimizer for the bound from Theorem 2 for $0 < \tau < 0.209$. □

Appendix E: Proof of Proposition 4

Proposition 4. *For the following customer valuation distributions, a tighter revenue ratio can be found when minimizing the expected hinge pricing loss function:*

(a) *For $V \sim$ Uniform$(0,\bar{v})$, $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = c(2-c)$. For $c^* = 0.8234$, $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = 0.9688$.*

(b) *For $V \sim$ Exponential(λ), $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = ce^{1-c}$. For $c^* = 0.8234$, $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = 0.9824$.*

Proof of Proposition 4:

For $V \sim$ Uniform$(0,\bar{v})$, $p^* = \frac{\bar{v}}{2} = \mathbb{E}[V]$. Therefore, pricing at $p = c\mathbb{E}[V]$ gives us

$$\frac{\mathcal{R}(p)}{\mathcal{R}(p^*)} = \frac{p\bar{F}(p)}{p^*\bar{F}(p^*)} = \frac{p(1-p/\bar{v})}{p^*(1-p^*/\bar{v})} = \frac{c\frac{\bar{v}}{2}(1-c\frac{\bar{v}}{2}/\bar{v})}{\frac{\bar{v}}{2}(1-\frac{\bar{v}}{2}/\bar{v})} = c(2-c).$$

For $V \sim \exp(\lambda)$, from the first order optimality conditions, we have that the optimal price is the inverse hazard rate, i.e., $p^* = \frac{\bar{F}(p)}{f(p)}$. For the exponential distribution $\bar{F}(p) = e^{-\lambda p}$ and $f(p) = \frac{1}{\lambda}e^{-\lambda p}$, so $p^* = \frac{1}{\lambda} = \mathbb{E}[V]$. By pricing at $p = c\mathbb{E}[V]$, we have:

$$\frac{\mathcal{R}(p)}{\mathcal{R}(p^*)} = \frac{p\bar{F}(p)}{p^*\bar{F}(p^*)} = \frac{pe^{-\lambda p}}{p^*e^{-\lambda p^*}} = \frac{\frac{c}{\lambda}e^{-\lambda\frac{c}{\lambda}}}{\frac{1}{\lambda}e^{-\lambda\frac{1}{\lambda}}} = ce^{1-c}. \quad \square$$

Appendix F: Proof of Lemma 2

LEMMA 2. *(Quantile optimality condition): Under assumptions 1 and 2, when only purchases are observed,*

$$\frac{\int_0^{p_q} \bar{F}_V(p)dp}{\int_0^{\infty} \bar{F}_V(p)dp} = 1 - \tau$$

where $p_q = \arg\min_{\pi(X)} \mathbb{E}_{Y,P}\left[\frac{Y}{\phi(P|X)}\left((1-\tau)(P-\pi(X))^+ + \tau(\pi(X)-P)^+\right)\Big|X\right]$.

Proof of Lemma 2:

$$\begin{aligned}
\mathbb{E}_{P|Y=1}[L^q_\tau(p_q,1,P)|X] &= \int_0^{\bar{p}} \left[(1-\tau)(p-p_q)^+ + \tau(p_q-p)^+\right] \frac{f_{P|Y=1,X}(p|Y=1,X)}{\phi(p|X)}dp \\
&= \int_0^{\bar{p}} \left[(1-\tau)(p-p_q)^+ + \tau(p_q-p)^+\right] \frac{\mathbb{P}(Y=1|P=p,X)\phi(p|X)}{\mathbb{P}(Y=1|X)\phi(p|X)}dp \\
&= \frac{1}{\mathbb{P}(Y=1|X)}\left[\int_{p_q}^{\bar{p}}(1-\tau)(p-p_q)\bar{F}_V(p)dp + \int_0^{p_q}\tau(p_q-p)\bar{F}_V(p)dp\right]
\end{aligned}$$

The second equality follows from Bayes theorem, while the third equality follows based on $\mathbb{E}[Y|P = p,X] = \mathbb{E}[Y(p,V)|X] = \mathbb{P}(V > p|X)\ \forall p \in [0,\bar{v}]$ from Equation 1 and the ignorability assumption. Applying the first-order optimality condition, and the Leibniz integral rule:

$$\begin{aligned}
\frac{\partial}{\partial p_q}\mathbb{E}_{Y,P}[L^q_\tau(p_q,Y,P)|X] &= \frac{1}{\mathbb{P}(Y=1|X)}\left[\int_0^{p_q}\tau\bar{F}_V(p)dp - \int_{p_q}^{\bar{p}}(1-\tau)\bar{F}_V(p)dp\right] = 0 \\
&\implies \tau\int_0^{p_q}\bar{F}_V(p)dp = (1-\tau)\left(\int_0^{\bar{p}}\bar{F}_V(p)dp - \int_0^{p_q}\bar{F}_V(p)dp\right) \\
&\implies \int_0^{p_q}\bar{F}_V(p)dp = (1-\tau)\int_0^{\bar{p}}\bar{F}_V(p)dp
\end{aligned}$$

Finally, the overlap implies $\int_0^{\bar{p}}\bar{F}_V(p) = \int_0^{\infty}\bar{F}_V(p)$, since $\bar{p} \geq \bar{v}$. $\square$

Appendix G: Proof of Theorem 2

THEOREM 2. *Let* $p^* = \arg\max_p \mathcal{R}(p)$ *and*

$$p_q = \underset{\pi(X)}{\arg\min}\,\mathbb{E}_{Y,P}\left[\frac{Y}{\phi(P|X)}\left((1-\tau)(P-\pi(X))^+ + \tau(\pi(X)-P)^+\right)\Big|X\right]$$

Under assumptions 1, 2, and 3, for $0 < \tau \leq 1$*, the expected worst-case revenue from pricing using the quantile loss function can be bounded:*

$$\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \min\left\{\min_{\tau\leq z\leq 1}\frac{z\tau(\ln(z)+1)-z^2}{\tau-z}, \min_{0<z<1}\frac{(1-\tau)(z-1)e^{(1-\tau)(z-1)}}{z\ln(z)}\right\}$$

Proof of Theorem 2: Similar to the hinge pricing loss function, the proof for Theorem 2 requires exploring two cases, one where the price chosen by minimizing the quantile pricing loss is above the optimal price and one where it is below. We start by bounding the case where the price obtained by minimizing the quantile pricing loss function is greater than the optimal price.

LEMMA 7. *If there exists a pricing rule p_q satisfying optimality conditions from Lemma 2 and $p_q > p^*$, then:*

$$\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \min_{\tau \leq z \leq 1} \frac{z\tau(\ln(z)+1) - z^2}{\tau - z}.$$

This again follows nontrivially from the properties of log-concavity and is proven in Appendix G.1.

For the case where $p_q < p^*$, we can apply the bound from Lemma 3 (see proof of Theorem 1 and Appendix C.1). Lemma 3 shows that a constant fraction of revenue can be achieved under a condition where the price prescribed is at least a fixed constant of the revenue achievable by a clairvoyant, $p_h \geq c\int_0^\infty \bar{F}_V(p)dp$. We show that the optimality condition for the quantile loss function implies this condition.

COROLLARY 7. *If there exists a pricing rule p_q satisfying optimality conditions from Lemma 2 and $p_q \leq p^*$*

$$\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \min_{0<z<1} \frac{(1-\tau)(z-1)e^{(1-\tau)(z-1)}}{z\ln(z)}$$

Proof of Corollary 7: From the quantile pricing optimality condition in Lemma 2,

$$\int_0^{p_q} \bar{F}_V(p)dp = (1-\tau)\int_0^\infty \bar{F}_V(p)dp$$

Since $p_q \geq \int_0^{p_q} \bar{F}_V(p)dp$, the conditions for Lemma 3 are satisfied for the constant $1-\tau$. $\square$

Theorem 2 follows from combining the bounds in Lemma 7 and Corollary 7.

Discussion: To help parse the bound from Lemma 7, it can be lower bounded by the following function with only a minor loss of accuracy.

PROPOSITION 7.

$$\min_{\tau \leq z \leq 1} \frac{z\tau(\ln(z)+1) - z^2}{\tau - z} \geq 4(1-\tau)\tau$$

This bound helps clarify the relationship with the parameter τ and clearly shows that increasing τ from 0 toward 0.5 improves the bound, as proved in Appendix G.3. Furthermore, it can be observed from Figure 7, which visualizes the revenue bound, that this bound is well approximated by a quadratic.

A version of Corollary 5 (see proof of Theorem 1 and Appendix C.4) also holds for the quantile pricing loss function, whereby the revenue ratio can be very simply lower bounded by $1-\tau$ for the range $0.5 \leq \tau < 1$. As shown in Appendix D, this bound is actually tight.

COROLLARY 8. $\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \min_{0<z<1} \frac{(1-\tau)(z-1)e^{(1-\tau)(z-1)}}{z\ln(z)} = 1-\tau$ *for* $0.5 \leq \tau < 1$

G.1. Proof of Lemma 7

LEMMA 7. *If there exists a pricing rule p_q satisfying optimality conditions from Lemma 2 and $p_q > p^*$, then:*

$$\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \min_{\tau \leq z \leq 1} \frac{z\tau(\ln(z)+1)-z^2}{\tau - z}.$$

Proof of Lemma 7:

From the optimality condition for Lemma 2, we have that $\int_0^{p_q} \tau \bar{F}_V(p)dp = \int_{p_q}^{\infty}(1-\tau)\bar{F}_V(p)dp$. By lower bounding $\int_0^{p_q} \tau \bar{F}_V(p)dp$ and upper bounding $\int_{p_q}^{\infty}(1-\tau)\bar{F}_V(p)dp$ and doing some algebraic manipulation, we are able to bound $\frac{p_q}{p^*}$, which leads to bounds on the revenue ratio.

$$\begin{aligned}\int_0^{p_q} \bar{F}_V(p)dp = \int_0^{p^*} \bar{F}_V(p)dp + \int_{p^*}^{p_q} \bar{F}_V(p)dp &\geq p^*\bar{F}_V(p^*) + \int_{p^*}^{p_q} \bar{F}_V(p^*)^{\frac{p_q-p}{p_q-p^*}} \bar{F}_V(p_q)^{\frac{p-p^*}{p_q-p^*}} dp \\ &= p^*\bar{F}_V(p^*) + \frac{(\bar{F}_V(p_q)-\bar{F}_V(p^*))(p_q-p^*)}{\ln(\bar{F}_V(p_q))-\ln(\bar{F}_V(p^*))}\end{aligned}$$

On the other hand,

$$\int_{p_q}^{\infty} \bar{F}_V(p)dp \leq \int_{p_q}^{\infty} \bar{F}_V(p^*)^{\frac{p_q-p}{p_q-p^*}} \bar{F}_V(p_q)^{\frac{p-p^*}{p_q-p^*}} dp = \frac{-\bar{F}_V(p_q)(p_q-p^*)}{\ln(\bar{F}_V(p_q))-\ln(\bar{F}_V(p^*))}$$

Therefore, from Lemma 2 we have that:

$$\begin{aligned} &\int_0^{p_q} \bar{F}_V(p)dp = \frac{(1-\tau)}{\tau}\int_{p_q}^{\infty} \bar{F}_V(p)dp \\ \implies\ & p^*\bar{F}_V(p^*) + \frac{(\bar{F}_V(p_q)-\bar{F}_V(p^*))(p_q-p^*)}{\ln(\bar{F}_V(p_q))-\ln(\bar{F}_V(p^*))} \leq \frac{(1-\tau)}{\tau}\frac{-\bar{F}_V(p_q)(p_q-p^*)}{\ln(\bar{F}_V(p_q))-\ln(\bar{F}_V(p^*))} \\ \implies\ & \frac{\left[\bar{F}_V(p^*)\left(\ln\left(\frac{\bar{F}_V(p_q)}{\bar{F}_V(p^*)}\right)+1\right) - \frac{1}{\tau}\bar{F}_V(p_q)\right]}{(\bar{F}_V(p^*)-\frac{1}{\tau}\bar{F}_V(p_q))} \leq \frac{p_q}{p^*} \qquad (23) \\ \implies\ & \frac{\bar{F}_V(p_q)\left[\bar{F}_V(p^*)\left(\ln\left(\frac{\bar{F}_V(p_q)}{\bar{F}_V(p^*)}\right)+1\right) - \frac{1}{\tau}\bar{F}_V(p_q)\right]}{\bar{F}_V(p^*)(\bar{F}_V(p^*)-\frac{1}{\tau}\bar{F}_V(p_q))} \leq \frac{p_q\bar{F}_V(p_q)}{p^*\bar{F}_V(p^*)} = \frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)}\end{aligned}$$

Where (23) follows from $\ln(\bar{F}_V(p_q)) - \ln(\bar{F}_V(p^*)) < 0$ since $p_q > p^*$, and requires $\bar{F}_V(p_q) \geq \tau$, proven in Lemma 8, which can be found in Appendix G.2. Substituting $z = \frac{\bar{F}_V(p_q)}{\bar{F}_V(p^*)}$,

$$\begin{aligned}\frac{\bar{F}_V(p_q)\left[\bar{F}_V(p^*)\left(\ln\left(\frac{\bar{F}_V(p_q)}{\bar{F}_V(p^*)}\right)+1\right) - \frac{1}{\tau}\bar{F}_V(p_q)\right]}{\bar{F}_V(p^*)(\bar{F}_V(p^*)-\frac{1}{\tau}\bar{F}_V(p_q))} &= \frac{z\bar{F}_V(p^*)\left[\bar{F}_V(p^*)(\ln(z)+1) - \frac{1}{\tau}\bar{F}_V(p^*)z\right]}{\bar{F}_V(p^*)(\bar{F}_V(p^*)-\frac{z}{\tau}\bar{F}_V(p^*))} \\ &= \frac{z\left[\tau(\ln(z)+1)-z\right]}{(\tau-z)} \leq \frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)}\end{aligned}$$

Where the requirement $\tau \leq z \leq 1$ follows from $\bar{F}_V(p^*) \geq \bar{F}_V(p_q)$, since $p_q > p^*$. $\square$

G.2. Proof of Lemma 8

LEMMA 8. *If $p^* \leq p_q$, then $\bar{F}_V(p_q) \geq \tau$.*

Proof of Lemma 8:

Due to log-concavity:

$$\bar{F}_V(p) \geq \bar{F}_V(p_q)^{\frac{p}{p_q}} \quad \forall p \leq p_q \qquad \text{and} \qquad \bar{F}_V(p_q) \geq \bar{F}_V(p)^{\frac{p_q}{p}} \quad \forall p \geq p_q$$

Therefore,

$$\int_0^{p_q} \bar{F}_V(p)dp \geq \int_0^{p_q} \bar{F}_V(p_q)^{\frac{p}{p_q}} = \left[\frac{p_q \bar{F}_V(p_q)^{\frac{p}{p_q}}}{\ln(\bar{F}_V(p_q))}\right]_0^{p_q} = \frac{p_q(\bar{F}_V(p_q)-1)}{\ln(\bar{F}_V(p_q))}$$

$$\int_{p_q}^{\infty} \bar{F}_V(p)dp \leq \int_0^{p_q} \bar{F}_V(p_q)^{\frac{p}{p_q}} = \left[\frac{p_q \bar{F}_V(p_q)^{\frac{p}{p_q}}}{\ln(\bar{F}_V(p_q))}\right]_{p_q}^{\infty} = \frac{-p_q \bar{F}_V(p_q)}{\ln(\bar{F}_V(p_q))}$$

From optimality condition in Lemma 1:

$$\begin{aligned}\int_0^{p_q} \bar{F}_V(p)dp \ = \ \frac{1-\tau}{\tau}\int_{p_q}^{\infty} \bar{F}_V(p)dp \implies & \quad \frac{p_q(\bar{F}_V(p_q)-1)}{\ln(\bar{F}_V(p_q))} \ \leq \ -\frac{1-\tau}{\tau}\frac{p_q\bar{F}_V(p_q)}{\ln(\bar{F}_V(p_q))} \\ \implies & \quad \frac{p_q(\frac{1}{\tau}\bar{F}_V(p_q)-1)}{\ln(\bar{F}_V(p_q))} \leq 0\end{aligned}$$

Therefore $\bar{F}_V(p_q) \geq \tau$, since $\ln(\bar{F}_V(p_q)) < 0$. □

G.3. Proof of Proposition 7

PROPOSITION 7.

$$\min_{\tau \leq z \leq 1} \frac{z\tau(\ln(z)+1)-z^2}{\tau - z} \geq 4(1-\tau)\tau$$

Proof of Proposition 7: We will use the bound $\ln(x) \leq x-1$, which follows from a Taylor approximation. Since we have $\tau \leq z$, then it follows that $\frac{z\tau}{\tau - z} \leq 0$. Therefore, making this substitution,

$$\frac{z\tau\left(\ln(z)+1\right)-z^2}{(\tau - z)} \geq \frac{z^2(\tau-1)}{\tau - z}$$

Furthermore, we can find the minimizer of this lower bound.

$$\begin{aligned}\frac{\partial}{\partial z}\frac{z^2(\tau-1)}{\tau-z} = & \frac{2z(\tau-1)}{\tau-z} + \frac{2z^2(\tau-1)}{(\tau-z)^2} = 0 \\ \implies & 2z(\tau-z)+z=0 \\ \implies & z = 2c\end{aligned}$$

Substituting this back into the bound,

$$\min_z \frac{z^2(\tau-1)}{\tau-z} = \frac{(2c)^2(\tau-1)}{\tau-(2c)} = 4\tau(1-\tau) \quad \square$$

Appendix H: Proof of Proposition 5

PROPOSITION 5. *For the following customer valuation distributions, a tighter revenue ratio can be found when minimizing the expected quantile loss function:*

1. *For $V \sim$ Uniform$(0, \bar{v})$, $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = 4(\sqrt{\tau} - \tau)$. For $\tau^* = 0.209$, $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = 0.9927$.*
2. *For $V \sim$ Exponential(λ), $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = -e^1 \tau \ln(\tau)$. For $\tau^* = 0.209$, $\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = 0.8893$.*

Proof of Proposition 5:

For $V \sim$ Uniform$(0, \bar{v})$, let $\bar{v} = 1$, without loss of generality. Therefore $p^* = \frac{1}{2}$. To find p_q, we need to rearrange the optimality condition $\frac{\int_0^{p_q} \bar{F}_V(p)dp}{\int_0^\infty \bar{F}_V(p)dp} = 1 - \tau$. $\int_0^{p_q} \bar{F}_V(p)dp = \int_0^{p_q} 1 - pdp = p_q - \frac{1}{2}p_q^2$. Also, $\int_0^1 \bar{F}_V(p)dp = \frac{1}{2}$. It follows that $\frac{\int_0^{p_q} \bar{F}_V(p)dp}{\int_0^\infty \bar{F}_V(p)dp} = 2p_q - p_q^2 = 1 - \tau$. By rearranging and using the quadratic formula, we have $p_q = \frac{1}{2}(2 \pm \sqrt{4 - 4(1-\tau)}) = 1 - \sqrt{\tau}$ since $0 \leq p_q \leq 1$. Therefore,

$$\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} = \frac{p_q \bar{F}(p_q)}{p^* \bar{F}(p^*)} = \frac{p_q(1-p_q)}{p^*(1-p^*)} = \frac{(1-\sqrt{\tau})(1-(1-\sqrt{\tau})))}{\frac{1}{4}} = 4(\sqrt{\tau} - \tau).$$

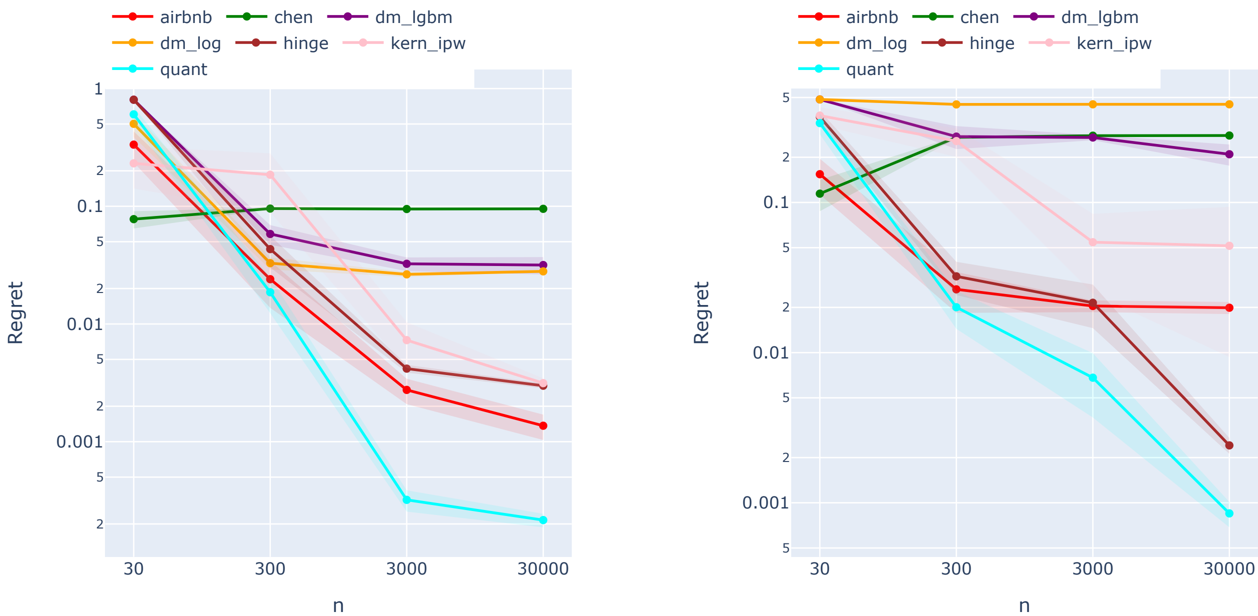
For $V \sim \exp(\lambda)$, we have $p^* = \frac{1}{\lambda}$ (see proof of Proposition 4 for a derivation). To find p_q, we need to rearrange the optimality condition $\frac{\int_0^{p_q} \bar{F}_V(p)dp}{\int_0^\infty \bar{F}_V(p)dp} = 1 - \tau$. $\int_0^{p_q} \bar{F}_V(p)dp = \int_0^{p_q} e^{-\lambda p}dp = \frac{1}{\lambda}(1 - e^{-\lambda p_q})$. Also, $\int_0^1 \bar{F}_V(p)dp = \frac{1}{\lambda}$. It follows that $\frac{\int_0^{p_q} \bar{F}_V(p)dp}{\int_0^\infty \bar{F}_V(p)dp} = 1 - e^{-\lambda p_q} = 1 - \tau$. By rearranging we have $p_q = -\frac{1}{\lambda}\ln(\tau)$. Therefore,

$$\frac{\mathcal{R}(p)}{\mathcal{R}(p^*)} = \frac{p\bar{F}(p)}{p^*\bar{F}(p^*)} = \frac{pe^{-\lambda p}}{p^* e^{-\lambda p^*}} = \frac{-\frac{1}{\lambda}\ln(\tau)e^{\lambda\frac{1}{\lambda}\ln(\tau)}}{\frac{1}{\lambda}e^{-\lambda\frac{1}{\lambda}}} = -e^1 \tau \ln(\tau). \quad \square$$

Appendix I: Experiments with Overlap Violated

In this section, we adapt the experiments from Section 5 to the setting where the historical pricing policy violates the overlap assumption. This setting is challenging, particularly from a theoretical standpoint, since the revenue obtained from unseen prices could potentially be very different from the prices we observe. Impossibility results related to counterfactual evaluation without overlap are studied in Langford et al. (2008), and bounds that explore the suboptimality of offline policy optimization without overlap under boundedness assumptions are explored in Sachdeva et al. (2020) (see bounds from Theorem 1). Despite these difficulties, Sachdeva et al. (2020) proposes practical heuristics for policy optimization without overlap, based upon the idea of avoiding (or limiting) actions that have no overlap. We implement the approach where the new policy is prevented from assigning prices with no overlap, and, in a setting where the optimal price still has coverage, we experimentally show that our loss functions still perform well.

We keep the same experimental setup as in Section 5, except that we restrict the support of the historical pricing policy to be half that of the support of the valuation distribution. For the case when valuations are linear in X, $V_i \sim$ Uniform$(X_i, X_i + 2)$, the pricing policy is set so $P_i \sim$ Uniform$(X_i + 0.5, X_i + 1.5)$. We note that for this case, the optimal (conditional) price still lies within the range of prices we observe. For the case where the valuations follow a step function, from Section 5, $V_i \sim$ Uniform$(3, 5)$ for $X_i > 1.75$, and $V_i \sim$ Uniform$(0, 2)$ for $X_i \leq 1.75$. The historical pricing policy we use sets $P_i \sim$ Uniform$(3.5, 4.5)$ for $X_i > 1.75$, and $P_i \sim$ Uniform$(0.5, 1.5)$ for $X_i \leq 1.75$. Following Sachdeva et al. (2020), we put constraints requiring $\underline{p}(X_i) \leq \pi(X_i) \leq \bar{p}(X_i)$, where $\underline{p}(X_i)$ and $\bar{p}(X_i)$ are minimum and maximum prices with support. Since the



(a) linear function (b) step function

Figure 13 Regret from a scenario with uniform historical pricing policy

historical pricing policies do not have support at 0, we need to make the adjustments to the pricing policy/ loss function as proposed in Corollaries 4 and 10.

We observe the results in Figure 13. The results are similar to the experiments with overlap, highlighting that the approach from Sachdeva et al. (2020) appears to work well in the case where the optimal price still has coverage.

We note that significantly more can be done in the case where there is *some* overlap, even if some prices are very uncommon under the historical pricing policy. This results in the distinct challenge of approaches that use inverse propensity weighting having high variance estimates. Approaches to reduce variance and improve performance include normalization via re-weighting (Lunceford and Davidian 2004, Austin and Stuart 2015), trimming of the weights to reduce the variance of the estimates (Elliott 2008, Ionides 2008), penalizing actions for which the loss function has high variance (Swaminathan and Joachims 2015, Joachims et al. 2018, Bertsimas and McCord 2018), and rebalancing to focus on evaluation at covariates for which there is better overlap (Kallus 2020). These approaches could also be applied to our setting when some propensity scores are small, which we highlight in Section 1.1.

Appendix J: Learning from Adaptive Historical Pricing Policies

We consider the case where the historical pricing policy has a dependence on previous transactions, which can occur if the policy adaptively learns from data as it is revealed. We study a panel data scenario where the seller has access to multiple independent trajectories $\mathcal{T}_j = \{(Y_{jt}, P_{jt}, X_{jt})\}_{t=1}^{T_j}$ detailing how the pricing policy has evolved for different products $j = 1, ..., m$. In this scenario, at time t, for product j, the learned pricing policy can be a function of previous observed data $S_{j,t-1} = \{(Y_{jt}, P_{jt}, X_{jt})\}_{j=1}^{t-1}$, with price density given by $\phi(P_{jt}|X_{jt}, S_{j,t-1})$.

We assume $Y_{jt}(p, V_{jt}) \perp S_{j,t-1}$ and $X_{jt} \perp S_{j,t-1}$ for all $j \in [m], t \in [T_j]$. This means that the customer's purchase decision and what type of customer is observed next is independent of previous transactions (this was also implied by the i.i.d. assumption in the main text). While this setting captures the scenario where the historical pricing policy learns over time, it does not necessarily capture the scenario in which the customer behavior changes over time, such as in response to previous prices. We make the same overlap and ignorability assumptions as previously.

Assumption 4 **(Overlap)**. $\phi\big(p \mid X_{jt}, S_{j,t-1}\big) > 0, \quad \forall p \in \mathbb{R}_+, \forall t \le T_j, \forall j \in [m]$.

Assumption 5 **(Ignorability)**. $Y_{jt}(p, V_{jt}) \perp P_{jt} \,\big|\, X_{jt}, S_{j,t-1}, \quad \forall p \in \mathbb{R}_+,\, t \le T_j,\, j \in [m]$.

We again condition on X_T and show that the hinge pricing loss still prices at the conditional expected customer valuation.

Lemma 9. *Fix a trajectory $j \in \{1, \dots, m\}$ of length T_j. Let*

$$p_{h,T_j} = \arg\min_{p_{h,T_j}} \mathbb{E}_{S_{j,T_j}|X_{j,T_j}} \left[\frac{1}{T_j} \sum_{t=1}^{T_j} \frac{cY_{jt}(P_{jt} - p_{h,t})^+ + (1 - cY_{jt})(p_{h,t} - P_{jt})^+}{\phi(P_{jt} \mid X_{jt}, S_{j,t-1})} \,\middle|\, X_{j,T_j} \right].$$

Then $p_{h,T_j} = \mathbb{E}[V_{j,T_j} \mid X_{j,T_j}]$.

Proof of Lemma 9.

$$\begin{aligned}
&\frac{\partial}{\partial p_{h,T_j}} \mathbb{E}_{S_{j,T_j}|X_{j,T_j}} \Big[\tfrac{1}{T_j} \sum_{t=1}^{T_j} \tfrac{cY_{jt}(P_{jt} - p_{h,t})^+ + (1-cY_{jt})(p_{h,t} - P_{jt})^+}{\phi(P_{jt}|X_{jt}, S_{j,t-1})} \,\Big|\, X_{j,T_j} \Big] \\
&= \frac{\partial}{\partial p_{h,T_j}} \mathbb{E}_{Y_{j,T_j}, P_{j,T_j}, S_{j,T_j-1}|X_{j,T_j}} \Big[\tfrac{1}{T_j} \tfrac{cY_{j,T_j}(P_{j,T_j} - p_{h,T_j})^+ + (1-cY_{j,T_j})(p_{h,T_j} - P_{j,T_j})^+}{\phi(P_{j,T_j}|X_{j,T_j}, S_{j,T_j-1})} \Big] \\
&= \frac{\partial}{\partial p_{h,T_j}} \mathbb{E}_{S_{j,T_j-1}|X_{j,T_j}} \Big[\mathbb{E}_{P_{j,T_j}|X_{j,T_j}, S_{j,T_j-1}} \Big[\mathbb{E}_{Y_{j,T_j}|P_{j,T_j}, X_{j,T_j}, S_{j,T_j-1}} \Big[\tfrac{1}{T_j} \tfrac{cY_{j,T_j}(P_{j,T_j} - p_{h,T_j})^+ + (1-cY_{j,T_j})(p_{h,T_j} - P_{j,T_j})^+}{\phi(P_{j,T_j}|X_{j,T_j}, S_{j,T_j-1})} \Big]\Big]\Big] \\
&= \frac{\partial}{\partial p_{h,T_j}} \mathbb{E}_{S_{j,T_j-1}|X_{j,T_j}} \Big[\int_0^\infty \frac{c\,\mathbb{E}[Y_{j,T_j} \mid X_{j,T_j}, P_{j,T_j} = p](p - p_{h,T_j})^+}{T_j} \, dp \Big] + \dots \\
&\qquad\qquad \mathbb{E}_{S_{j,T_j-1}|X_{j,T_j}} \Big[\int_0^\infty \frac{\big(1 - c\mathbb{E}[Y_{j,T_j} \mid X_{j,T_j}, P_{j,T_j} = p]\big)\,(p_{h,T_j} - p)^+}{T_j} \, dp \Big] \\
&= \frac{\partial}{\partial p_{h,T_j}} \frac{1}{T_j} \int_0^\infty \big[c\,\bar{F}_{V|X_{j,T_j}}(p)(p - p_{h,T_j})^+ + (1 - c\,\bar{F}_{V|X_{j,T_j}}(p))(p_{h,T_j} - p)^+ \big] dp \,=\, 0.
\end{aligned}$$

In the first equality, only the $t = T_j$ summand depends on p_{h,T_j}. Earlier $\{p_{h,t}\}_{t<T_j}$ are fixed, and are therefore zero when the derivative is taken. Ignorability and $Y = \mathbf{1}\{V \ge P\}$ give $\mathbb{E}[Y_{jT_j} \mid X_{j,T_j}, P_{jT_j} = p, S_{j,T_j-1}] = \bar{F}_{V|X_{j,T_j}}(p)$, removing the dependence on S_{j,T_j-1}. The remaining integral matches the i.i.d. case, resulting in the stated minimum. □

It is not practical to learn the full density $\phi(P_{jt} \mid X_{jt}, S_{j,t-1})$ when only a few trajectories are available. In extreme cases, there may be only a single trajectory, such as when the pricing policy learns from all products simultaneously and product features are incorporated into X. In this low–data regime, recent work (e.g.

Zhan et al. 2024) assumes the propensity scores are known for observed actions. This is reasonable when the historical policy is fully algorithmic and its source code or logs are accessible.

When many independent trajectories are observed, it is possible to estimate $\phi(P_{jt}|X_{jt}, S_{j,t-1})$. Practically, it is necessary to encode $S_{j,t-1}$ as a fixed-dimension embedding $\hat{S}_{j,t-1}$. This can be achieved by either using a sliding window to record the k most recent price/outcome/feature tuples in the trajectory (similar to Jagerman et al. (2019)), or by using a learned sequence embedding obtained from an LSTM (Sun et al. 2019) or causal-marked transformer (Sun et al. 2019), if a neural network model is used. Some options for learning $\phi(P_t|X_t, \hat{S}_{t-1})$ include kernel density estimation (Smola and Schölkopf 1998) or mixture-density networks (Bishop 1994, Graves 2013), where a neural head outputs the parameters of a Gaussian mixture. To avoid overfitting, which is a particular concern given that the embedding might still be high-dimensional, cross-validation should be used when fitting the model. Another common issue encountered when estimating propensity scores is high variance estimates once the reciprocal is taken. Normalization via reweighting (Lunceford and Davidian 2004) or trimming of the weights (Elliott 2008) might be necessary if some of the estimated densities are too small.

Appendix K: Learning from Historical Pricing Policies Without Low Prices

The case where the historical policy is bounded away from 0, where $p \in [\underline{p}, \bar{p}]$, requires special treatment even when the support of the historical pricing policy covers the support of the valuation distribution, i.e., $\underline{p} \leq \underline{v} \leq \bar{v} \leq \bar{p}$. In particular, we need to add a correction term to the respective loss functions to achieve the same pricing rules. First, we show how to correct the hinge pricing loss by adding a corrective factor.

K.1. Hinge Loss Correction

Corollary 9. *Suppose $\phi(p|X) > 0$ only for $p \in [\underline{p}, \bar{p}]$, and $\underline{p} \leq \underline{v} \leq \bar{v} \leq \bar{p}$. let $p_h = \arg\min_p \mathbb{E}[L_c^h(p, Y, P)|X]$. Define a shifted pricing policy as $p_h' = p_h - \underline{p}(1-c)$. Then $p_h' = c\mathbb{E}[V|X]$.*

Proof of Corollary 9:

$$\begin{aligned}
&\mathbb{E}\Big[\frac{1}{\phi(P|X)}[cY(P-p_h)^+ + (1-cY)(p_h-P)^+]\Big|X_T\Big] \\
&= \int_{\underline{p}}^{\bar{p}} \Big(\frac{c\mathbb{E}_{Y|P=p}[Y|X, P=p](p-p_h)^+}{\phi(p|X)} \\
&\qquad + \frac{(1-c\mathbb{E}_{Y|P=p}[Y|X, P=p])(p_h-p)^+}{\phi(p|X)}\Big)\phi(p|X)dp \\
&= \int_{p_h}^{\bar{p}} c\bar{F}_V(p)(p-p_h)dp + \int_{\underline{p}}^{p_h}(1-c\bar{F}_V(p))(p_h-p)dp
\end{aligned}$$

Applying the Leibniz rule to find the optimality conditions:

$$\begin{aligned}
&\frac{\partial}{\partial p_h}\mathbb{E}\Big[\frac{1}{\phi(P|X)}[cY(P-p_h)^+ + (1-cY)(p_h-P)^+]\Big|X_T\Big] \\
&= \int_{p_h}^{\bar{p}} -c\bar{F}_V(p)dp + \int_{\underline{p}}^{p_h} 1 - c\bar{F}_V(p)dp \\
&= \int_{\underline{p}}^{p_h} 1dp - c\int_{\underline{p}}^{\bar{p}} \bar{F}_V(p)dp \\
&= p_h - \underline{p} - c\left(\int_0^\infty \bar{F}_V(p)dp - \int_0^{\underline{p}} \bar{F}_V(p)dp\right) \\
&= p_h - \underline{p} - c\mathbb{E}[V|X] + c\underline{p} = 0
\end{aligned}$$

Therefore $p_h = c\mathbb{E}[V|X] + (1-c)\underline{p}$. If follows that $p'_h = p_h - \underline{p}(1-c) = \mathbb{E}[V|X]$ $\square$

As we can see, the pricing rule remains the same. Next, we show how to correct the quantile loss by adding an additional term to the loss function and also show that it remains the same.

K.2. Quantile Loss Correction

COROLLARY 10. *Suppose $\phi(p|X) > 0$ only for $p \in [\underline{p}, \bar{p}]$, and $\underline{p} \leq \underline{v} \leq \bar{v} \leq \bar{p}$. Let*

$$p_q = \arg\min_{p_q} \mathbb{E}\left[\frac{Y}{\phi(P|X)}[(1-\tau)(P-p_q)^+ + \tau(p_q-P)^+ + p_q\tau\underline{p}]\right]$$

Then

$$\frac{\int_0^{p_q} \bar{F}_V(p)dp}{\int_0^{\infty} \bar{F}_V(p)dp} = 1-\tau$$

Proof of Corollary 10:

$$\begin{aligned}
\mathbb{E}_{P|Y=1}[L^q_\tau(p_q,1,P)|X] &= \int_{\underline{p}}^{\bar{p}} \left[(1-\tau)(p-p_q)^+ + \tau(p_q-p)^+ + p_q\tau\underline{p}\right] \frac{f_{P|Y=1,X}(p|Y=1,X)}{\phi(p|X)}dp \\
&= \int_{\underline{p}}^{\bar{p}} \left[(1-\tau)(p-p_q)^+ + \tau(p_q-p)^+ + p_q\tau\underline{p}\right] \frac{\mathbb{P}(Y=1|P=p,X)\phi(p|X)}{\mathbb{P}(Y=1|X)\phi(p|X)}dp \\
&= \frac{1}{\mathbb{P}(Y=1|X)}\left[\int_{p_q}^{\bar{p}}(1-\tau)(p-p_q)\bar{F}_V(p)dp + \int_{\underline{p}}^{p_q}\tau(p_q-p)\bar{F}_V(p)dp + p_q\tau\underline{p}\right]
\end{aligned}$$

Applying the first-order optimality condition and the Leibniz integral rule:

$$\begin{aligned}
\frac{\partial}{\partial p_q}\mathbb{E}_{Y,P}[L^q_\tau(p_q,Y,P)|X] = &\frac{1}{\mathbb{P}(Y=1|X)}\left[\int_{\underline{p}}^{p_q}\tau\bar{F}_V(p)dp - \int_{p_q}^{\bar{p}}(1-\tau)\bar{F}_V(p)dp + \tau\underline{p}\right] = 0 \\
&\implies \tau\int_0^{p_q}\bar{F}_V(p)dp = (1-\tau)\left(\int_0^{\bar{p}}\bar{F}_V(p)dp - \int_0^{p_q}\bar{F}_V(p)dp\right) \\
&\implies \int_0^{p_q}\bar{F}_V(p)dp = (1-\tau)\int_0^{\infty}\bar{F}_V(p)dp \quad \square
\end{aligned}$$

Appendix L: Extension to Multiple Products

In this section, we consider an extension where multiple products are sold concurrently. We consider an assortment pricing scenario similar to Chen et al. (2023), where each customer buys at most one product from a fixed set of S products offered. Each customer has a stochastic valuation $V_1,...,V_S \in [\underline{v},\bar{v}]^S$ for each product, and was previously offered a price $P_1,...,P_S \in [0,\bar{p}]^S$ for each product, according to a distribution $\phi(P_1,...,P_S|X)$, where typically the prices are differentiated so $P_1 \neq P_2 \neq ...P_S$. We observe a binary purchase outcome $Y_1,...,Y_S$ for each product, defined as follows:

$$Y_k = \begin{cases} 1 & \text{if} \quad V_k \geq P_k \cap V_k - P_k \geq V_j - P_j \quad \forall j \in [S]\backslash\{k\} \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

As in the single product case, for a customer to purchase a product, their valuation for the product has to be greater than the price they are offered. In addition, the customer's utility, assumed to be $V_k - P_k$, must be greater than the utility from all other products. We note that under this definition, at most one product will be purchased by each customer. The expected revenue $\mathcal{R}(\pi_1(X),...,\pi_S(X))$, achieved by a policy that prescribes a price $\pi_k(X)$ for each product $k \in [S]$, is:

$$\mathbb{E}_V \left[\sum_{k \in [S]} \pi_k(X) \mathbb{1}\left\{ V_k \geq \pi_k(X) \cap V_k - \pi_k(X) \geq V_j - \pi_j(X) \ \forall j \in [S] \backslash \{k\} \right\} \middle| X \right]$$

As in the single product case, we assume overlap and ignorability, which have natural multiproduct generalizations.

Assumption 6. *(Multiproduct overlap)* $\phi(p_1, ..., p_S | X) > 0, \quad \forall p_1, ..., p_S \in [0, \bar{v}], X \in \mathcal{X}$

Assumption 7. *(Multiproduct ignorability)*

$$Y_1(p_1, ..., p_S), ..., Y_S(p_1, ..., p_S) \perp P_1, ..., P_S | X, \quad \forall p_1, ..., p_S \ [0, \bar{v}], X \in \mathcal{X}$$

To extend the results from the single item setting, we study a class of uniform pricing policies, where the price given is the same for all products $\pi_k(X) = \bar{\pi}(X), \ \forall k \in [S]$. We note that policies of this form are also studied in Chen et al. (2023). Conceptually, this approach focuses on getting as much revenue from the highest valuation product as possible and avoiding cannibalization from other products with lower earning potential. We consider an extension of the hinge pricing loss function that results in a pricing rule similar to Section 2:

$$\begin{aligned} &L^m(\bar{\pi}(X), ..., \bar{\pi}(X), Y_1, ... Y_S, P_1, ..., P_S) = \\ &\qquad \frac{1}{S} \sum_{k=1}^{S} \frac{1}{\bar{p}^{S-1} \phi(P_1, ..., P_S, X)} \left(Y_k (P_k - \pi(X))^+ + (1 - Y_k)(\pi(X) - P_k)^+ \right) \end{aligned}$$

Lemma 10. *(Multiproduct optimality condition): Let*

$$p_m = \arg\min_p \mathbb{E}_{Y,P} \left[L^m(p, ..., p, Y_1, ..., Y_S, P_1, ..., P_S) | X \right]$$

be the minimizer of the expected loss function, and $k^* = \arg\max_k \mathbb{E}_V [V_k | X]$ *be the product with the highest expected valuation. Then*

$$\frac{1}{S} \mathbb{E}[V_{k^*} | X] \leq p_m \leq \mathbb{E}[V_{k^*} | X]$$

Proof of Lemma 10

$$\begin{aligned} \mathbb{E}_{Y,P}[L^m(p_m, Y, P) | X] &= \mathbb{E}_{P,Y} \left[\frac{1}{\bar{p}^{S-1} \phi(P_1, ..., P_S, X)} \frac{1}{S} \sum_{k=1}^{S} \left(Y_k (P_k - p_m)^+ + (1 - Y_k)(p_m - P_k)^+ \right) \middle| X \right] \\ &= \mathbb{E}_P \left[\frac{\phi(P_1, ..., P_S, X)}{\bar{p}^{S-1} \phi(P_1, ..., P_S, X)} \frac{1}{S} \sum_{k=1}^{S} \Big(\mathbb{E}_{Y|P}[Y_k | X, P](P_k - p_m)^+ + (1 - \mathbb{E}_{Y|P}[Y_k | X, P])(p_m - P_k)^+ \Big) \middle| X \right] \\ &= \int_0^{\bar{p}} ... \int_0^{\bar{p}} \frac{1}{\bar{p}^{S-1} S} \sum_{k=1}^{S} \Big(\mathbb{E}_{Y|P_1,...,P_s}[Y_k | X, P_1, ..., P_s](P_k - p_m)^+ + \\ &\qquad (1 - \mathbb{E}_{Y|P_1,...,P_s}[Y_k | X, P_1, ..., P_s])(p_m - P_k)^+ \Big) dP_1 ... dP_S \end{aligned}$$

Applying the Leibniz rule to find the first-order optimality conditions:

$$\begin{aligned}\frac{\partial}{\partial p_m}\mathbb{E}_{Y,P}[L^m(p_m,Y,P)|X] = \int_0^{\bar{p}}\dots\int_0^{\bar{p}}\frac{1}{\bar{p}^{S-1}S}\sum_{k=1}^{S}\Big(&-\mathbb{E}_{Y|P_1,\dots,P_s}[Y_k|X,P_1,\dots,P_s]+\\ &(1-\mathbb{E}_{Y|P_1,\dots,P_s}[Y_k|X,P_1,\dots,P_s])\Big)dP_1\dots dP_S\\ = p_m - \int_0^{\bar{p}}\dots\int_0^{\bar{p}}\frac{1}{\bar{p}^{S-1}S}&\sum_{k=1}^{S}\mathbb{E}_{Y|P_1,\dots,P_s}[Y_k|X,P_1,\dots,P_s]dP_1\dots dP_S = 0\end{aligned}$$

Therefore

$$\begin{aligned}p_m &= \int_0^{\bar{p}}\dots\int_0^{\bar{p}}\frac{1}{\bar{p}^{S-1}S}\sum_{k=1}^{S}\mathbb{E}_{Y|P_1,\dots,P_s}[Y_k|X,P_1,\dots,P_s]dP_1\dots dP_S\\ &= \int_0^{\bar{p}}\dots\int_0^{\bar{p}}\frac{1}{\bar{p}^{S-1}S}\sum_{k=1}^{S}\mathbb{E}_{V|P_1,\dots,P_s}[\mathbb{1}\{V_k\geq P_k\cap V_k-P_k\geq V_j-P_j\quad\forall j\in[S]\backslash\{k\}\}|X,P_1,\dots,P_s]dP_1\dots dP_S\\ &\leq \int_0^{\bar{p}}\dots\int_0^{\bar{p}}\frac{1}{\bar{p}^{S-1}S}\sum_{k=1}^{S}\mathbb{E}_{V|P_k}[\mathbb{1}\{V_k\geq P_k\}|X,P_k]dP_1\dots dP_S\\ &= \sum_{k=1}^{S}\int_0^{\bar{p}}\frac{\bar{p}^{S-1}}{\bar{p}^{S-1}S}\mathbb{P}(V_k\geq P_k|X,P_k)dP_k\\ &= \frac{1}{S}\sum_{k=1}^{S}\mathbb{E}[V_k|X]\\ &\leq \max_k \mathbb{E}_V[V_k|X]\end{aligned}$$

Where the first inequality holds since $V_k\geq P_k\cap V_k-P_k\geq V_j-P_j \quad \forall j\in[S]\backslash\{k\}$ implies $V_k>P_k$. Next, we prove the lower bound. We also have :

$$\begin{aligned}p_m &= \int_0^{\bar{p}}\dots\int_0^{\bar{p}}\frac{1}{\bar{p}^{S-1}S}\mathbb{E}_{Y|P_1,\dots,P_s}\left[\sum_{k=1}^{S}Y_k\Big|X,P_1,\dots,P_s\right]dP_1\dots dP_S\\ &\geq \int_0^{\bar{p}}\dots\int_0^{\bar{p}}\frac{1}{\bar{p}^{S-1}S}\mathbb{E}_{V|P_k}[\mathbb{1}\{V_k\geq P_k\}|X,P_k]dP_1\dots dP_S \qquad \forall k\in[S]\\ &= \int_0^{\bar{p}}\frac{\bar{p}^{S-1}}{\bar{p}^{S-1}S}\mathbb{P}(V_k\geq P_k|X,P_k)dP_k \qquad \forall k\in[S]\\ &= \frac{1}{S}\mathbb{E}[V_k|X] \qquad \forall k\in[S]\end{aligned}$$

The first inequality holds since for any k,$V_k>P_k$ implies $\sum_{k=1}^{S}Y_k=1$, that is, a product will sell. Equivalently, $\mathbb{P}(V_k\geq P_k|X,P_1,..,P_S)\leq\sum_{k=1}^{S}\mathbb{P}(V_k\geq P_k\cap V_k-P_k\geq V_j-P_j\quad\forall j\in[S]\backslash\{k\}|X,P_1,..,P_S)$. Since this inequality holds for all $k\in[S]$, it also holds for $k^*=\arg\max_k\mathbb{E}_V[V_k|X]$. □

The significance of this result is that the optimal uniform multiproduct price, p_m, is within a constant fraction (between $\frac{1}{S}$ and 1) of the conditional expected customer valuation $\mathbb{E}[V_{k^*}|X]$ of the most valuable product. We highlight the connection with Section 2, where the price is set to be some known fraction of the expected customer valuation in the single product setting. Under an additional assumption, this connection can be used to extend the revenue bounds to the multiproduct setting.

The condition we require is that there is a correspondence between the expected valuation of the products and the maximum revenue that can be obtained from each. Specifically, we require the product with the highest expected valuation to also have the highest potential revenue.

ASSUMPTION 8. *The product with the highest expected valuation also has the highest potential revenue:*

$$\arg\max_k \mathbb{E}_V[V_k|X] = \arg\max_k \{\max_p \mathbb{E}_V[p\mathbb{1}\{p \le V_k\}|X]\}$$

This is a natural assumption satisfied by many practical joint product-valuation distributions that have a clear hierarchy of high-value to low-value products. Using this assumption, we can prove multiproduct revenue bounds.

LEMMA 11. *(Multiproduct revenue bound): Let*

$$p_m = \arg\min_p \mathbb{E}_{Y,P}\left[L^m(p,...,p,Y_1,...,Y_S,P_1,...,P_S)|X\right]$$

then,

$$\frac{\mathcal{R}(p_m,...,p_m)}{\max_{p_1,..,p_k}\mathcal{R}(p_1,..,p_k)} \ge \frac{1}{S}\min\left\{\min_{0<f<1}\frac{1}{S}\frac{(f-1)e^{\frac{1}{S}(f-1)}}{f\ln(f)}, e^{-1}\right\}$$

Proof of Lemma 11: From Lemma 10, have that $\frac{1}{S}\mathbb{E}[V_{k^*}|X] \le p_m \le \mathbb{E}[V_{k^*}|X]$, where $k^* = \arg\max_k \mathbb{E}_V[V_k|X]$. From assumption 8, we also have $k^* = \arg\max_k\{\max_p \mathbb{E}_V[p\mathbb{1}\{p \le V_k\}|X]\}$.

$$\begin{aligned}\mathcal{R}(p_m) &= \mathbb{E}_V\left[\sum_{k\in[S]} p_m\mathbb{1}\{V_k \ge p_m \cap V_k - p_m \ge V_j - p_m \ \forall j \in [S]\backslash\{k\}\}\middle| X\right]\\ &\ge \mathbb{E}_V\left[p_m\mathbb{1}\{V_k \ge p_m\}|X\right] = p_m\mathbb{P}(V_k \ge p_m|X) \qquad \forall k \in [S]\end{aligned}$$

Where the inequality holds since for any $k \in [S]$, $V_k > p_m$ implies that there will be a sale ($\sum_{k\in[S]}\mathbb{1}\{V_k \ge p_m \cap V_k - p_m \ge V_j - p_m \ \forall j \in [S]\backslash\{k\}\} = 1$). Furthermore, since this holds for all $k \in [S]$, we have $\mathcal{R}(p_m) \ge p_m\mathbb{P}(V_{k^*} \ge p_m|X)$.

$$\begin{aligned}\max_{p_1,..,p_k}\mathcal{R}(p_1,..,p_k) &= \max_{p_1,..,p_k}\mathbb{E}_V\left[\sum_{k\in[S]} p_k\mathbb{1}\{V_k \ge p_k \cap V_k - p_k \ge V_j - p_k \ \forall j \in [S]\backslash\{k\}\}\middle| X\right]\\ &\le \max_{p_1,..,p_k}\mathbb{E}_V\left[\sum_{k\in[S]} p_k\mathbb{1}\{V_k \ge p_k\}\middle| X\right]\\ &= \max_{p_1,..,p_k}\sum_{k=1}^{S} p_k\mathbb{P}(V_k \ge p_k|X)\\ &\le S\max_k\left\{\max_p p\mathbb{P}(V_k \ge p|X)\right\}\\ &= S\max_p p\mathbb{P}(V_{k^*} \ge p|X)\end{aligned}$$

From Figure **??**, we observe that the lower bound from Theorem 1 (where the pricing rule sets the price to $c\mathbb{E}[V|X]$) is concave and is therefore minimized at the extreme points. Let us take the lower bound, where $p_m \ge \frac{1}{S}\mathbb{E}[V_{k^*}|X]$. Using Lemma 3, and setting $c = \frac{1}{S}$, we have:

$$\frac{\mathcal{R}(p_m)}{\max_{p_1,..,p_k}\mathcal{R}(p_1,..,p_k)} \geq \frac{p_m\mathbb{P}(V_{k^*}\geq p_m|X)}{S\max_p p\mathbb{P}(V_{k^*}\geq p|X)} \geq \frac{1}{S}\min_{0<z<1}\frac{1}{S}\frac{(f-1)e^{\frac{1}{S}(f-1)}}{f\ln(f)}$$

Let us take the upper bound, where $p_m \leq \mathbb{E}[V_{k^*}|X]$. Setting $c=1$, and using Corollary 6, we have:

$$\frac{\mathcal{R}(p_m)}{\max_{p_1,..,p_k}\mathcal{R}(p_1,..,p_k)} \geq \frac{p_m\mathbb{P}(V_{k^*}\geq p_m|X)}{S\max_p p\mathbb{P}(V_{k^*}\geq p|X)} \geq \frac{1}{S}e^{-1}$$

We combine these two lower bounds to get the result. $\square$

Intuitively, this result focuses on getting as much revenue as possible from the highest expected valuation product. Since p_m prices at a fraction between $\frac{1}{S}$ and 1 of the highest expected valuation, the proof explores the revenue in either case and relies on concavity to argue that the worst case is at one of these extremes. This combines Lemma 3 for when $p_m = \frac{1}{S}$ and Corollary 6 for when $p_m = 1$ from Appendix C, together with bounds on the loss from focusing on the highest valuation product, which introduces a penalty of $\frac{1}{S}$ relative to the single product case. As such, these bounds are sensitive to the number of products sold concurrently and, therefore, perform better when there are fewer products. An alternative view is that this approach is likely to perform relatively better when there is one dominant product.

The results in this section focus on extending the results from the previous section to the multiproduct setting. We note this bound can be compared to the bound in Chen et al. (2023). Chen et al. (2023) studies a similar multiproduct setting, but with two main differences: 1) the historical pricing policy gives the same price for all products for a given customer, although this price could change for different customers, and 2) the attractiveness of each product is symmetric, meaning that each product has the same valuation in their MNL model. In this setting, they are able to obtain a 0.5 bound. Our result gives insight into what is achievable when the aforementioned conditions are relaxed. We believe that alternative pricing policies that focus on the multiproduct setting are a promising area for future study.

Appendix M: Finite-sample Bounds

The bounds introduced in previous sections apply when optimizing the expected loss function, which is often not possible in practice. It is of interest to study how different the loss will be when using a finite sample of data. This is the motivation behind the following bounds we present. We first study a linear policy class before extending to kernel-based policies that can capture a more general nonlinear policy class.

M.1. Linear Policies

An important class of pricing policies is the class of linear functions, where $\pi_\theta(X) = \langle\theta, X\rangle$. This class of functions is valued for its simplicity and interpretability. Also, when using a linear policy class, the empirical risk minimization problem of finding the best policy by optimizing the hinge and quantile pricing loss functions is convex. This is not necessarily true for nonlinear policy classes.

To achieve these bounds, we use regularized loss minimization, in which we jointly minimize the empirical risk, and a regularization function that penalizes large values of θ.

$$\hat{\theta}_\lambda = \underset{||\theta||_2\leq B}{\arg\min}\frac{1}{n}\sum_{i=1}^{n}L(\langle\theta, X_i\rangle, Y_i, P_i) + \lambda||\theta||_2^2$$

Generalization bounds are well studied for regularized convex loss functions that are ρ-Lipshitz (e.g., Shalev-Shwartz and Ben-David 2014). In Lemma 12 we show that the quantile and hinge pricing loss functions are ρ-Lipshitz. For these results, we will assume that the price range is bounded, $p \in [a,b]$, so that the historical probability of assigning a price $\phi(p|X)$ can be bounded away from 0 by a constant.

LEMMA 12. *Assume there exists $d \leq \phi(p|X)$ for all $X \in \mathcal{X}$, $p \in [a,b]$. Then $L^q_\tau(\cdot, Y, P)$ and $L^h_\tau(\cdot, Y, P)$ are ρ-Lipshitz, with $\rho = \max\{\tau, 1-\tau\}/d$.*

This simple result follows from taking the maximum absolute gradient of each loss function. This allows us to apply known generalization bounds to this setting. First, denote $\theta^* = \arg\min_\theta \mathbb{E}[L(\langle\theta, X\rangle, Y, P)]$.

LEMMA 13. *Assume there exists $d \leq \phi(p|X)$ for all $X \in \mathcal{X}$, $p \in [a,b]$. For either $L = L^h_\tau(\cdot, Y, P)$ or $L = L^q_\tau(\cdot, Y, P)$, let $\lambda = \sqrt{\frac{2\max\{\tau,1-\tau\}^2}{d^2 B^2 n}}$. For any training set of size n,*

$$\mathbb{E}[L(\langle\hat{\theta}_\lambda, X\rangle, Y, P)] \leq \mathbb{E}[L(\langle\theta^*, X\rangle, Y, P)] + \frac{B\max\{\tau, 1-\tau\}}{d}\sqrt{\frac{8}{n}}.$$

In particular, if $\epsilon > 0$, and $n \geq \frac{8(\max\{\tau,1-\tau\})^2 B^2}{d^2\epsilon^2}$, then $\mathbb{E}[L(\langle\hat{\theta}_\lambda, X\rangle, Y, P)] \leq \mathbb{E}[L(\langle\theta^*, X\rangle, Y, P)] + \epsilon$.

The result follows from Corollary 13.9 in Shalev-Shwartz and Ben-David (2014). An advantage of this bound is that it doesn't explicitly depend on the dimension of the covariate space $\mathcal{X}$, although there is a dependence on B since we require $||\theta_\lambda||_2 \leq B$. Therefore, it is still possible to get good bounds when the covariate space is high-dimensional yet sparse. This bound suggests that a large sample size n is needed for small d, which occurs when there are some prices that have a very low probability of being assigned $\phi(p|X)$. This happens when the pricing policy is unbalanced. This issue is well known in the off-policy learning community and affects inverse propensity weighting methods since, to account for observations at uncommon prices, large weights must be used, leading to high variance estimates. Approaches including normalization via re-weighting (Lunceford and Davidian 2004, Austin and Stuart 2015) and trimming of the weights to reduce the variance of the estimates (Elliott 2008, Ionides 2008) have been shown to help mitigate this problem.

We highlight that these bounds show the discrepancy between the finite sample and expected loss, whereas the pricing practitioner might be more concerned about this difference for revenue. While one might reasonably expect similar behavior, formally proving this is a challenge we leave for future work.

M.2. Kernel-Based Policies

In this section, we derive bounds for a class of kernel-based policies. These have the flexibility to model a range of policies that are nonlinear in the feature space, potentially leading to higher revenue policies. However, one drawback is that they may result in a non-convex revenue optimization problem, which may be challenging to solve compared to the linear policy class. It is also known that they may suffer from the curse of dimensionality, and have poor performance when the covariates are high-dimensional (Smola and Schölkopf 1998).

We focus on policies of the form $\pi(x) = \langle w, \Phi(x)\rangle$, where $\Phi(x)$ is a mapping that transforms the input x into a higher-dimensional space. In this transformed space, linear models can more effectively capture relationships that are nonlinear in the original space. Instead of working directly with the mapping $\Phi(x)$, we use a kernel function $K(x,x') = \langle \Phi(x), \Phi(x') \rangle$. This kernel function measures the similarity between two instances, x and x', and allows us to implicitly work in the higher-dimensional space without explicitly computing the transformation $\Phi(x)$. It can be shown that a kernel function is valid, in that it represents an inner product in some Hilbert space, if and only if it is positive semidefinite (Mohri et al. 2018). There is a broad range of suitable kernels, including polynomial, Gaussian, and sigmoid kernels, that make this a rich class for modeling the policy; see Smola and Schölkopf (1998) for a thorough treatment.

Under the assumptions that the kernel's self-similarity $K(x,x)$ is bounded by r^2, the policy π is constrained within a bounded range $[a,b]$ for all inputs, and that historical pricing probabilities are bounded away from 0, we can derive bounds on the empirical relative to the expected loss function relative in our setting.

Lemma 14. *Let $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a PDS kernel, let $\Phi : \mathcal{X} \to \Pi$ be a feature mapping associated with K, and let $\Pi = \{x \mapsto \langle w, \Phi(x) \rangle : ||w||_\Pi \leq \Lambda\}$. Assume that there exists $r > 0$ such that $K(x,x) \leq r^2$ for all $\mathcal{X} \times \mathcal{X}$ and that there exists $d \leq \phi(p|x)$ and $\pi(x) \in [a,b]$ for all $x \in \mathcal{X}$. Then, for either $L = L_\tau^h(\cdot, Y, P)$ or $L = L_\tau^q(\cdot, Y, P)$, for any $\delta > 0$, with probability at least $1-\delta$, the following inequality holds for all $\pi \in \Pi$:*

$$\mathbb{E}[L(\pi(X), Y, P)] \leq \frac{1}{n} \sum_{i=1}^n L(\pi(X_i), Y_i, P_i) + \frac{\max\{\tau, 1-\tau\}}{d} \left(2\sqrt{\frac{r^2\Lambda^2}{n}} + b\sqrt{\frac{\log \frac{1}{\delta}}{2n}} \right)$$

This result is adapted from established bounds on Rademacher complexity (Mohri et al. 2018), which is a measure of the richness of a class of functions in terms of how well they can fit random noise. The proof follows from combining Theorem 11.3 (Rademacher complexity regression bounds) and Theorem 6.12 (bound on empirical Rademacher complexity of kernel-based hypothesis) in Mohri et al. (2018). All that is required is to show that $L(h(x), y, p) \leq M$ for all $x, y, p \in \mathcal{X} \times 0, 1 \times [a,b]$, where here $M = b\frac{\max\{\tau,1-\tau\}}{d}$. Since L is ρ-Lipshitz with $\rho = \frac{\max\{\tau,1-\tau\}}{d}$ (from Lemma 12) and $h(x), p$ in $[a,b]$, the result follows. By leveraging the properties of the kernel and the boundedness of the hypothesis space, we can obtain high probability bounds on the true risk of the policy.

Many of the same insights carry over from the linear policy class, such as the $\frac{1}{\sqrt{n}}$ rate of convergence. The number of samples required is large when some prices have very low probabilities (small d). Furthermore, more samples are needed when the complexity of the hypothesis space is greater, represented by Λ and r.