Appendix A: Model Discussion

Our stylized model was built in an effort to strike the right balance between representing reality, assuring tractability and capturing the relevant trade-offs. In this section, we provide further justification for the assumptions we make.

When comparing the display settings, we do not include differences in implementation costs. In practice, the labor cost for replenishment may differ across display settings; for example, under the layered display settings (`LF` and `LO`), the retailer may incur handling cost for performing inventory rotation, which is not the case under the `EA` display setting.

Our model does not include the (intangible) costs to a retailer's brand image from offering soon-to-expire products (especially at a discount). We chose to ignore these dimensions in order to focus on the variable cost dimensions but it would certainly be possible to add fixed costs to capture these effects.

Our model assumes that all consumers agree on the intrinsic quality level of the products and in particular, all prefer consuming a fresh unit to an old one if offered at the same price. As such, our model applies to most perishable grocery items with an expiration date label as well as most produce but it would not hold for products such as wine or cheese for which the consumers' WTP tends to increase with age. We also assume that consumers can easily distinguish fresh from old units. For most consumer packaged goods, the level of freshness can typically be identified via a label showing an expiration date. When old products are offered at a discount, flashy stickers (e.g., "50% discount") are often glued to the packaging in a way which makes them very visible to consumers or the products are moved to a specific area reserved for discounted products. We also assume that the percentage of passive vs active consumers (β) is an exogenous parameter which is not influenced by the display setting or the discount rate.

Further, our model assumes that the decision regarding the selling policy (`FS` vs. `SB`), the display setting, the discount age and the discount rate are strategic decisions made once by the retailer, at the beginning of the time horizon. In contrast, the replenishment quantity decision is operational and made at the beginning of every replenishment cycle, given the number of unsold units carried over from the previous period. We argue that this is reasonable for most stores since the choices made as strategic decisions have infrastructure implications for the store and cannot be easily modified.

Finally, similar to Hu et al. (2016), Chua et al. (2017) and Buisman et al. (2019), we optimize the discount rate and consider the full price p to be exogenously given. At multi-store retailers, the pricing decision on most products is centralized and calculated strategically based mostly on profit margin considerations, without consideration for waste (Chung and Li 2014) so that the base selling price is generally the same in all stores.

A.1. Technical assumptions

We make the following assumptions on the base selling price of the product: (i) $p > c + (L-1)h$ (**Assumption A4**) so that selling the product at full price is always profitable; (ii) $p < q_{R-1}$ (**Assumption A5**) so that there are always some consumers willing to purchase units from the fresh batch at full price (note that it is possible to have $p > q_n$ for $n \in \{R, ..., L-1\}$, which means that consumers might not be willing to pay full price for a product from the old batch); (iii) The base selling price of the product is such that it is not optimal to discount the fresh batch (**Assumption A6**). In the case of deterministic demand, this assumption translates to $p < \frac{1}{2}(q_{R-1} + c + (R-1)h)$. Finally, we assume that the unit holding cost satisfies the constraint $h < \min_x\{q_{x-1} - q_x\}$ (**Assumption A7**), which ensures that carrying units over to the next period is profitable because the cost of doing so is lower than the aging-caused decrease in the consumers' maximum WTP.

A.2. Sales calculations under the FS selling policy

In this section, we show how to calculate sales in each period under the FS policy. Given Assumption A1, in cycle periods $x \in \{0, ..., L-R-1\}$ only products with age x (fresh batch) and age $x+R$ (old batch) are in inventory and therefore can be sold. Then, in cycle periods $x \in \{L-R, ..., R-1\}$, only products of age x can be sold.

First, we consider the case where there are units of both batches in stock at the start of cycle period x, that is, $y_x^\tau > 0$ and $y_{x+R}^\tau > 0$, such that $x = \mod(\tau, R)$. While both batches are in stock, the units from the fresh batch are depleted at a rate of $F^x(\hat{n}, \gamma, \delta)$ and the units from the old batch are depleted at a rate of $O^x(\hat{n}, \gamma, \delta)$ until possibly, one batch stocks out. When $F^x(\hat{n}, \gamma, \delta) > 0$ and $\frac{y_x^\tau}{F^x(\hat{n},\gamma,\delta)} < \frac{y_{x+R}^\tau}{O^x(\hat{n},\gamma,\delta)}$, the fresh units runs out first, and afterward the remaining inventory of the old batch is depleted at a rate of $\left(1 - \frac{p_{x+R}}{q_{x+R}}\right)^+$. Let $N_1 = \frac{y_x^\tau}{F^x(\hat{n},\gamma,\delta)}$ and $N_2 = \left(y_{x+R}^\tau + \frac{y_x^\tau\left[\left(1-\frac{p_{x+R}}{q_{x+R}}\right)^+ - O^x(\hat{n},\gamma,\delta)\right]}{F^x(\hat{n},\gamma,\delta)}\right) / \left(1 - \frac{p_{x+R}}{q_{x+R}}\right)^+$ be the threshold store traffic values at which the inventory of fresh and old units runs out respectively. Then, the sales from the new batch in cycle period τ is $S_x^\tau = \min\{y_x^\tau, N^\tau F^x(\hat{n}, \gamma, \delta)\}$ and the sales from the old batch are given by:

$$S_{x+R}^\tau(\hat{n}, \gamma, \delta; \mathbf{y}^\tau) = \begin{cases} N^\tau O^x(\hat{n}, \gamma, \delta) & N^\tau \leq N_1 \\ N^\tau \left(1 - \frac{p_{x+R}}{q_{x+R}}\right)^+ - \frac{y_x^\tau\left[\left(1-\frac{p_{x+R}}{q_{x+R}}\right)^+ - O^x(\hat{n},\gamma,\delta)\right]}{F^x(\hat{n},\gamma,\delta)} & N_1 < N^\tau \leq N_2 \\ y_{x+R}^\tau & N^\tau > N_2 \end{cases}$$

Similarly, if both batches are in stock at the start of the period and the old batch runs out first, which implies that $O^x(\hat{n}, \gamma, \delta) > 0$ and $\frac{y_x^\tau}{F^x(\hat{n},\gamma,\delta)} \geq \frac{y_{x+R}^\tau}{O^x(\hat{n},\gamma,\delta)}$, from then on, the remaining inventory of fresh products is depleted at a rate of $\left(1 - \frac{p_x}{q_x}\right)^+$. Let $N_3 = \frac{y_{x+R}^\tau}{O^x(\hat{n},\gamma,\delta)}$ and $N_4 = \left(y_x^\tau + \frac{y_{x+R}^\tau\left[\left(1-\frac{p_x}{q_x}\right)^+ - F^x(\hat{n},\gamma,\delta)\right]}{O^x(\hat{n},\gamma,\delta)}\right) / \left(1 - \frac{p_x}{q_x}\right)^+$ be the threshold store traffic values at which the inventory of old and fresh units runs out, respectively. Then, the sales from the old batch in cycle period x are $S_{x+R}^\tau(\hat{n}, \gamma, \delta; \mathbf{y}^\tau) = \min\{y_{x+R}^\tau, N^\tau O^x(\hat{n}, \gamma, \delta)\}$ and the sales from the new batch are given by:

$$S_x^\tau(\hat{n}, \gamma, \delta; \mathbf{y}^\tau) = \begin{cases} N^\tau F^x(\hat{n}, \gamma, \delta) & N^\tau \leq N_3 \\ N^\tau \left(1 - \frac{p_x}{q_x}\right)^+ - \frac{y_{x+R}^\tau\left[\left(1-\frac{p_x}{q_x}\right)^+ - F^x(\hat{n},\gamma,\delta)\right]}{O^x(\hat{n},\gamma,\delta)} & N_3 < N^\tau \leq N_4 \\ y_x^\tau & N^\tau > N_4 \end{cases}$$

This substitution model is referred to as *stock-out based* (*dynamic*) substitution (Honhon et al. 2010) as it captures consumers' reactions when a product stocks out.

If at the start of period τ, only fresh units are in stock, then the sales are equal to $S_x^\tau = \min\left\{y_x^\tau, \left(1 - \frac{p_x}{q_x}\right) N^\tau\right\}$ for $x = \mod(\tau, R)$ and zero otherwise.

Finally, if at the start of period τ, only old units are in stock, then the sales are equal to $S_{x+R}^{\tau} = \min\left\{y_{x+R}^{\tau}, \left(1-\frac{p_{x+R}}{q_{x+R}}\right)N^{\tau}\right\}$ for $x = \mod(\tau, R)$ and zero otherwise.

The inventory transition equations are: $y_x^{\tau} = y_{x-1}^{\tau-1} - S_{x-1}^{\tau-1}$ for periods τ such that $\mod(\tau, R) = x \in \{1, ..., R-1\}$, and $y_{x+R}^{\tau} = y_{x+R-1}^{\tau-1} - S_{x+R-1}^{\tau-1}$ for periods τ such that $\mod(\tau, R) = x \in \{1, ..., L-R-1\}$.

Appendix B: Deterministic Demand

B.1. Optimal replenishment quantity

We first obtain the optimal replenishment quantity for a given fixed display setting δ, a fixed discount rate γ and a fixed discount age $\hat{n}$, under the FS and SB policies.

When store traffic is deterministic, the retailer can calculate exactly how many fresh and old units will be sold for any given replenishment quantity and therefore, can order optimally to maximize the profit. Hence, no units are ever wasted and there is always sufficient inventory to satisfy any prospective customer (i.e., one who derives a positive utility from either a fresh or an old unit). As a result, the same order quantity is optimal at every replenishment epoch. We formalize these observations in Lemma 5.

LEMMA 5. *When the store traffic is deterministic, there exists an optimal replenishment quantity to be ordered at the start of each replenishment cycle such that*

i. no unit is wasted and

ii. every prospective customers who derives a positive utility from either the fresh or the old batch buys a unit.

We define $Q^*(\hat{n}, \gamma, \delta)$ as the optimal order quantity for a given discount age $\hat{n}$, discount rate γ and display setting δ. From Lemma 5, $Q^*(\hat{n}, \gamma, \delta)$ must be equal to the demand over the replenishment cycle. In the following lemma we develop the expressions of $Q^*(\hat{n}, \gamma, \delta)$ under the FS and SB policies.

LEMMA 6. *Under deterministic store traffic, for a given discount age $\hat{n}$, discount rate γ, and display setting δ, the optimal order quantity under the FS policy is:*

$$Q_{FS}^*(\hat{n}, \gamma, \delta) = N \sum_{x=0}^{L-R-1} \left[F^x(\hat{n}, \gamma, \delta) + O^x(\hat{n}, \gamma, \delta)\right] + N \sum_{x=L-R}^{R-1} \left(1 - \frac{p_x}{q_x}\right) \tag{5}$$

and under the SB policy, it is:

$$Q_{SB}^* = N \sum_{x=0}^{R-1} \left(1 - \frac{p_x}{q_x}\right). \tag{6}$$

In (5), the first summation corresponds to the demand from consumers buying units in periods $x \in \{0, ..., L-R-1\}$ and the second summation is the demand from consumer buying from the fresh batch that is on the shelves in periods $x \in \{L-R, ..., R-1\}$. In other words, the quantity ordered by the retailer is more than what can be sold over the replenishment cycle of R periods, with the extra units being sold as old batch units in the subsequent cycle.

In (6), the demand is from consumers buying units from the fresh batch in periods $x \in \{0, ..., R-1\}$.

Given Lemma 5, under deterministic store traffic, it is enough to focus on optimizing the sum of per-period profit over the length of the replenishment cycle of R periods. Using (5) and (6), we can write the following total profit function $\Pi_{FS}(Q^*, \hat{n}, \gamma, \delta)$ over the replenishment cycle when the optimal quantity Q^* is ordered under the FS policy as:

$$\begin{aligned}
\Pi_{FS}(Q^*, \hat{n}, \gamma, \delta) &= \Bigg\{ N \sum_{x=0}^{L-R-1} \left[p_x F^x(\hat{n}, \gamma, \delta) + p_{x+R} O^x(\hat{n}, \gamma, \delta) \right] + N \sum_{x=L-R}^{R-1} p_x \left(1 - \frac{p_x}{q_x}\right) - c Q^*_{FS}(\hat{n}, \gamma, \delta) \\
&\quad - \left[N \sum_{x=1}^{L-R-1} x h F^x(\hat{n}, \gamma, \delta) + N \sum_{x=0}^{L-R-1} (x+R) h O^x(\hat{n}, \gamma, \delta) + N \sum_{x=L-R}^{R-1} x h \left(1 - \frac{p_x}{q_x}\right) \right] \Bigg\} \\
&= N \sum_{x=0}^{L-R-1} \left[(p_x - c - xh) F^x(\hat{n}, \gamma, \delta) + \left[p_{x+R} - c - (x+R)h \right] O^x(\hat{n}, \gamma, \delta) \right] \\
&\quad + N \sum_{x=L-R}^{R-1} (p_x - c - xh) \left(1 - \frac{p_x}{q_x}\right). \qquad (7)
\end{aligned}$$

Similarly, the profit over the replenishment cycle under the SB policy when the optimal quantity Q^* is ordered is:

$$\begin{aligned}
\Pi_{SB}(Q^*) &= N \sum_{x=L-R}^{R-1} p_x \left(1 - \frac{p_x}{q_x}\right) - c Q^*_{SB}(\hat{n}, \gamma) - N \sum_{x=0}^{R-1} x h \left(1 - \frac{p_x}{q_x}\right) \\
&= N \sum_{x=0}^{R-1} (p_x - c - xh) \left(1 - \frac{p_x}{q_x}\right). \qquad (8)
\end{aligned}$$

B.2. Optimal discount age

Next, we examine the optimal discount age under the SB and FS policies. First, we show that under the SB policy, it is never optimal to offer a discount.

LEMMA 7. *Under deterministic store traffic, if the retailer follows an SB policy, it is optimal not to discount the product, i.e.,* $\gamma^* = 0$ *(or equivalently* $\hat{n}^* = L - 1$*).*

In contrast, under the FS policy, it may be optimal to offer a discount. In particular, we show that under the LF display setting, which we will prove to be optimal in Theorem 1, the discount is always on units from the old batch, never on units from the fresh one.

LEMMA 8. *Under the FS policy with the LF display setting, for any discount rate* $\gamma > 0$*, the optimal discount age* $\hat{n}^* \geq R$.

One noteworthy discount age strategy (which we consider in §5) is to set $n = R$, which means that the discount is applied on the units from the old batch when the fresh batch is brought onto the store shelves. This practice may save on labor (handling) cost as the same store employee can perform the restocking and discounting (i.e., affixing discounts stickers) tasks in one trip to the product aisle.

Appendix C: Proofs

C.1. Proof of Proposition 1

When $\beta = 0$, from Table 1, we see that for all $x \in \{0, .., R-1\}$, $F^x(\hat{n}, \gamma, \delta) = P^x_F(\hat{n}, \gamma) + P^x_{FO}(\hat{n}, \gamma)$ and $O(\hat{n}, \gamma, \delta) = P^x_O(\hat{n}, \gamma) + P^x_{OF}(\hat{n}, \gamma)$ for $\delta \in \{\texttt{EA}, \texttt{LO}, \texttt{LF}\}$. Therefore, all the three display settings are equivalent.

C.2. Proof of Lemma 1

Let $P^x(\hat{n},\gamma) \equiv P^x_F(\hat{n},\gamma) + P^x_O(\hat{n},\gamma) + P^x_{FO}(\hat{n},\gamma) + P^x_{OF}(\hat{n},\gamma)$ be the probability of purchase (from either batch) in cycle period x (which is independent of the display setting). From Table 7, we can verify that, the following sets of inequalities hold for all $\hat{n}$ and γ:

$$F^x(\hat{n},\gamma,\mathtt{LO}) \leq F^x(\hat{n},\gamma,\mathtt{EA}) \leq F^x(\hat{n},\gamma,\mathtt{LF}) \leq P^x_F(\hat{n},\gamma) + P^x_{FO}(\hat{n},\gamma) + P^x_{OF}(\hat{n},\gamma) \leq P^x(\hat{n},\gamma) \tag{9}$$

$$O^x(\hat{n},\gamma,\mathtt{LF}) \leq O^x(\hat{n},\gamma,\mathtt{EA}) \leq O^x(\hat{n},\gamma,\mathtt{LO}) \leq P^x_O(\hat{n},\gamma) + P^x_{FO}(\hat{n},\gamma) + P^x_{OF}(\hat{n},\gamma) \leq P^x(\hat{n},\gamma) \tag{10}$$

$$P^x(\hat{n},\gamma) = F^x(\hat{n},\gamma,\mathtt{LO}) + O^x(\hat{n},\gamma,\mathtt{LO}) = F^x(\hat{n},\gamma,\mathtt{EA}) + O^x(\hat{n},\gamma,\mathtt{EA}) = F^x(\hat{n},\gamma,\mathtt{LF}) + O^x(\hat{n},\gamma,\mathtt{LF})$$

$$P^x(\hat{n},\gamma) \geq \max\{P^x_F(\hat{n},\gamma) + P^x_{FO}(\hat{n},\gamma) + P^x_{OF}(\hat{n},\gamma), P^x_O(\hat{n},\gamma) + P^x_{FO}(\hat{n},\gamma) + P^x_{OF}(\hat{n},\gamma)\} \tag{11}$$

If there are only units of the fresh batch in inventory in period τ i.e., $y^\tau_x > 0$ and $y^\tau_{x+R} = 0$, then $S^\tau_x(\hat{n},\gamma,\delta;\mathbf{y}^\tau) = \left(1 - \frac{p_x}{q_x}\right)^+ N^\tau$ and $S^\tau_{x+R}(\hat{n},\gamma,\delta;\mathbf{y}^\tau) = 0$ for $\delta \in \{\mathtt{EA},\mathtt{LO},\mathtt{LF}\}$. Similarly, if there are only units of the old batch in inventory in period τ i.e., $y^\tau_x = 0$ and $y^\tau_{x+R} > 0$, then $S^\tau_x(\hat{n},\gamma,\delta;\mathbf{y}^\tau) = 0$ and $S^\tau_{x+R}(\hat{n},\gamma,\delta;\mathbf{y}^\tau) = \left(1 - \frac{p_{x+R}}{q_{x+R}}\right)^+ N^\tau$ for $\delta \in \{\mathtt{EA},\mathtt{LO},\mathtt{LF}\}$. Hence, the result holds trivially when only one batch is in stock.

So now, let us assume that both batches are in stock, i.e., $y^\tau_x > 0$ and $y^\tau_{x+R} > 0$. For given $\hat{n}$ and γ, we compare the three display settings in terms of when stock-outs would occur. For each setting, the first stock-out would happen if the number of consumers is equal to $\min\{\frac{y^\tau_x}{F^x(\hat{n},\gamma,\delta)}, \frac{y^\tau_{x+R}}{O^x(\hat{n},\gamma,\delta)}\}$. Given (9) and (10), the first stock-out is either of fresh units under the LF display setting or of old units under the LO display setting. If a stock-out occurs in one setting, after that, by (11), the inventory of the other batch is depleted at a rate which is no more than the combined rate at which both fresh and old units are jointly depleted in the setting where both batches are still in stock. This means that the setting which would stock out of one batch first is the one which would stock out of everything last. Similarly, the setting which would stock out of one batch second is the one which would be second to stock out of everything. Finally, the setting which would stock out of one batch last would be the one to stock out of everything first. This observation, combined with the orderings in (9) and (10), implies that fresh units first stock out in LF, then in EA then in LO setting and old units first stock out in LO then in EA then in LF. This in turns implies the ordering of the sales.

C.3. Proof of Theorem 1

When stock traffic is deterministic the order size is equal to the demand. The proof is in three steps:

Step 1 We prove that the SB policy always yields a profit that is higher than or equal to that under the FS policy with the EA display setting.

Step 2 Under the FS policy, we prove that the EA display setting always yields a profit that is higher than or equal to that under the LO display setting.

Step 3 We prove that the exists a price threshold value above which the FS policy with LF display setting yields a higher expected profit than the SB policy.

Step 1: We first prove that, the *SB* policy yields a profit that is higher than or equal to that under FS policy with the EA setting, for any discount value $\gamma \in (0,1)$ and discount age $\hat{n} \in \{0,...,L\}$.

We do so by showing that the period profit from selling only products from the fresh batch, which is equal to $\pi^x_{SB} = N(p-c-xh)\left(1-\frac{p}{q_x}\right)$ is greater than or equal to that from selling two different batches in each period $x \in \{0,...,L-R-1\}$.

The following cases are possible:

(1) If the period index is such that $x+R<\hat{n}$ (i.e., in cycle period x no product is discounted) or $x \geq \hat{n}$ (i.e., the products from both batches are discounted), then $O^x(\hat{n},\gamma,EA)=0$ (i.e., no customer buys the old batch) and $F^x(\hat{n},\gamma,EA)=1-\frac{p}{q_x}$ so that the period profit is $N(p-c-xh)\left(1-\frac{p}{q_x}\right)$, which is the same as under SB.

(2) If the period index is such that $x \leq \hat{n} \leq x+R$ (i.e., the units from the old batch are discounted but the units from the fresh batch are not). In this case, the profit in cycle period x under FS is given by:

$$\pi^x_{FS}(\hat{n},\gamma,EA) = N\Big[(p-c-xh)F^x(\hat{n},\gamma,EA)+[p(1-\gamma)-c-(x+R)h]\,O^x(\hat{n},\gamma,EA)\Big].$$

Depending on the value γ (see Table 7 for the definition of the threshold values) the following sub-cases exist:

(3.1) If $\gamma \in [0,\gamma^x_{LM}]$, $O^x(\hat{n},\gamma,EA)=0$ (i.e., no customer buys the old batch) and $F^x(\hat{n},\gamma,EA)=1-\frac{p}{q_x}$ so that the period profit under EA is equivalent to that under SB.

(3.2) If $\gamma \in [\gamma^x_{LM},\gamma^x_{MH}]$ then the profit in cycle period x is

$$\pi^x_{FS}(\hat{n},\gamma,EA) = N\left[(p-c-xh)\left(1-\frac{\gamma p}{q_x-q_{x+R}}\right)+[p(1-\gamma)-c-(x+R)h]\left(\frac{\gamma p}{q_x-q_{x+R}}-\frac{(1-\gamma)p}{q_{x+R}}\right)\right].$$

Let $\Delta\pi^x(\gamma) \equiv \pi^x_{SB} - \pi^x_{FS}(\hat{n},\gamma,EA)$ denote the difference in profits between the SB and FS policies under EA setting in cycle period x when the discount rate is γ. At $\gamma=\gamma^x_{LM}$, we have $\pi^x_{FS}(\hat{n},\gamma^x_{LM},EA)=\pi^x_{SB}$ so that $\Delta\pi^x(\gamma_{LM})=0$.

Taking the derivative of $\Delta\pi^x(\gamma)$ with respect to γ gives:

$$\frac{\partial\Delta\pi^x(\gamma)}{\partial\gamma} = -N\left[\left(\frac{\gamma p}{q_x-q_{R+x}}-\frac{(1-\gamma)p}{q_{x+R}}\right)(-c-h(x+R)+(1-\gamma)p)+(-c+p-xh)\left(1-\frac{\gamma p}{q_x-q_{x+R}}\right)\right],$$

At $\gamma=\gamma^x_{LM}$, this derivative is equal to

$$\frac{\Delta\pi^x(\gamma)}{\partial\gamma}\Big|_{\gamma=\gamma^x_{LM}} = -\frac{p(c+xh)q_{R+x}-pq_x(c+h(R+x))}{(q_x-q_{R+x})\,q_{R+x}}>0,$$

Further, the second derivative is

$$\frac{\partial^2\Delta\pi^x(\gamma)}{\partial\gamma^2} = -\frac{2p^2q_x}{q_{R+x}\,(q_{R+x}-q_x)}>0.$$

This proves that $\Delta\pi^x(\gamma)$ is increasing (and convex), from a starting point value of zero, on the interval $[\gamma_{LM},\gamma_{MH}]$ and therefore $\Delta\pi^x(\gamma)\geq 0$ on this interval, which is equivalent to $\pi^x_{SB} \geq \pi^x_{FS}(\hat{n},\gamma,EA)$.

(3.3) If $\gamma \in [\gamma^x_{MH}, 1)$, then the profit in cycle period x is:

$$\pi^x_{FS}(\hat{n},\gamma,EA) = N\left[[p(1-\gamma)-c-(x+R)h]\left(1-\frac{(1-\gamma)p}{q_{x+R}}\right)\right].$$

Taking the derivative of $\Delta\pi^x(\gamma)$ with respect to γ gives

$$\frac{\partial\Delta\pi^x(\gamma)}{\partial\gamma} = \frac{p\left(c+h(R+x)+2(\gamma-1)p+q_{R+x}\right)}{q_{R+x}}$$

Solving $\frac{\partial\Delta\pi^x(\gamma)}{\partial\gamma}=0$ gives $\underline{\gamma}=-\frac{c+h(R+x)-2p+q_{R+x}}{2p}$. The second derivative of $\Delta\pi^x(\gamma)$ is equal to

$$\frac{\partial^2\Delta\pi^x(\gamma)}{\partial\gamma^2} = \frac{2p^2}{q_{R+x}} > 0,$$

which implies that $\Delta\pi^x(\gamma)$ is convex in γ over $[\gamma^x_{MH}, 1)$. Moreover, we have

$$\gamma^x_{MH} - \underline{\gamma} = -\frac{c+h(R+x)-2p-q_{R+x}+2q_x}{2p} > 0 \tag{12}$$

because $q_x > q_{R+x}$ and $p < \frac{1}{2}(c+h(R+x)+q_x)$ by Assumptions 2 and 3. This implies that $\Delta\pi^x(\gamma)$ is increasing over $[\gamma^x_{MH}, 1)$. Further, since $\Delta\pi^x(\gamma)|_{\gamma=\gamma^x_{MH}} = \frac{(p-q_x)\left(q_{R+x}(c+hx-p+q_x)-q_x(c+h(R+x)-p+q_x)\right)}{q_x q_{R+x}} > 0$, $\Delta\pi^x(\gamma)$ is positive over $[\gamma^x_{MH}, 1)$, which is equivalent to $\pi^x_{SB} \geq \pi^x_{FS}(\hat{n},\gamma,EA)$.

Therefore, the `SB` policy always yields a profit higher than or equal to that under the `FS` policy with the `EA` display setting.

Step 2: We next prove that, under the `FS` policy, the `LO` setting always yields a profit that is lower than or equal to that under the `EA` display setting for any discount value $\gamma \in (0,1)$ and discount age $\hat{n} \in \{0,...,L\}$.

There are three cases : (1) $\hat{n} \leq L-R-1$; (2) $L-R \leq \hat{n} \leq R-1$; and (3) $\hat{n} > R-1$. We show the proof for case 3 in detail and the same logic applies to cases 1 and 2. The profit over the replenishment period with the `FS` policy under the `LO` display setting is equal to:

$$\begin{aligned}
&\Pi_{FS}(Q^*,\hat{n},\gamma,LO) = \\
&N\sum_{x=0}^{\hat{n}-R-1}\left[(p-c-xh)\left((1-\beta)\left(1-\frac{p}{q_{x+R}}\right)+\frac{p}{q_{x+R}}-\frac{p}{q_x}\right)+(p-c-(x+R)h)\beta\left(1-\frac{p}{q_{x+R}}\right)\right] \\
&+N\sum_{x=\hat{n}-R}^{L-R-1}\Pi^x + N\sum_{t=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right) \\
= \; & N\sum_{x=0}^{\hat{n}-R-1}\left[(p-c-xh)\left(1-\frac{p}{q_x}-\beta\left(1-\frac{p}{q_{x+R}}\right)\right)+(p-c-(x+R)h)\beta\left(1-\frac{p}{q_{x+R}}\right)\right] \\
&+N\sum_{x=\hat{n}-R}^{L-R-1}\Pi^x + N\sum_{t=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right) \\
= \; & N\sum_{x=0}^{\hat{n}-R-1}\left[(p-c-xh)\left(1-\frac{p}{q_x}\right)-Rh\beta\left(1-\frac{p}{q_{x+R}}\right)\right] \\
&+N\sum_{x=\hat{n}-R}^{L-R-1}\Pi^x + N\sum_{t=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right)
\end{aligned}$$

For $x \in [\hat{n}-R, L-R-1]$, we have $\pi^x_{FS}(\hat{n},\gamma,EA)$

$$= \begin{cases} N(p-c-xh)\left(1-\dfrac{p}{q_x}\right) - NRh\beta\left(1-\dfrac{p}{q_{x+R}}\right) & \text{if } 0<\gamma<\gamma^x_{LM} \\ N(p-c-xh)(1-\beta)F^x(\hat{n},\gamma,EA) & \\ \quad + N\left[p(1-\gamma)-c-(x+R)h\right]\left(O^x(\hat{n},\gamma,EA)+\beta F^x(\hat{n},\gamma,EA)\right) & \text{if } \gamma^x_{LM}\le\gamma<\gamma^x_{MH}, \\ N\left[p(1-\gamma)-c-(x+R)h\right]\left(1-\dfrac{(1-\gamma)p}{q_{x+R}}\right) & \text{if } \gamma^x_{MH}\le\gamma<1 \end{cases}$$

where the values of γ^x_{LM} and γ^x_{MH} are defined in Table 7. Let $\Delta\pi^x(\gamma)=\pi^x_{FS}(\hat{n},\gamma,EA)-\pi^x_{FS}(\hat{n},\gamma,LO)$. For $x\in[0,\hat{n}-R-1]$, we have $\Delta\pi^x(\gamma)=Rh\beta\left(1-\frac{p}{q_{x+R}}\right)>0$; for $x\in[L-R,R-1]$, we have $\Delta\pi^x(\gamma)=0$; for $x\in[\hat{n}-R,L-R-1]$, we have $\Delta\pi^x(\gamma)$ equal to:

$$= \begin{cases} NRh\beta\left(1-\dfrac{p}{q_{x+R}}\right) & \text{if } 0<\gamma<\gamma^x_{LM} \\ N(p-c-xh)\beta F^x(\hat{n},\gamma,EA) - N\left[p(1-\gamma)-c-(x+R)h\right]\beta F^x(\hat{n},\gamma,EA) & \text{if } \gamma^x_{LM}\le\gamma<\gamma^x_{MH}, \\ 0 & \text{if } \gamma^x_{MH}\le\gamma<1 \end{cases}$$

$$= \begin{cases} NRh\beta\left(1-\dfrac{p}{q_{x+R}}\right)>0 & \text{if } 0<\gamma<\gamma^x_{LM} \\ N(\gamma p+Rh)F^x(\hat{n},\gamma,EA)>0 & \text{if } \gamma^x_{LM}\le\gamma<\gamma^x_{MH}, \\ 0 & \text{if } \gamma^x_{MH}\le\gamma<1 \end{cases}$$

In other words, for all periods $x\in\{0,...,R-1\}$, we have $\pi^x_{FS}(\hat{n},\gamma,EA)\ge\pi^x_{FS}(\hat{n},\gamma,LO)$, which implies that $\Pi_{FS}(Q^*,\hat{n},\gamma,EA)\ge\Pi_{FS}(Q^*,\hat{n},\gamma,LO)$, that is, under the `FS` policy, for any fixed γ and $\hat{n}$, the `EA` display can yield a higher profit than under the `LO` setting.

Step 3: Next, we prove that the `FS` policy under the `LF` can give a higher profit than the `SB` policy depending on the value of the selling price. We have:

$$\Pi_{FS}(Q^*,\hat{n},\gamma,LF) = N\sum_{x=0}^{\hat{n}-R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right)+N\sum_{x=\hat{n}-R}^{L-R-1}\Pi^x+N\sum_{t=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right)$$

where, for $x\in\{\hat{n}-R,...,L-R-1\}$, we have $\pi^x_{FS}(\hat{n},\gamma,LF)$

$$= \begin{cases} N(p-c-xh)\left(1-\dfrac{p}{q_x}\right) & \text{if } 0<\gamma<\gamma^x_{LM} \\ N(p-c-xh)\left[\beta\left(\dfrac{\gamma p}{q_x-q_{x+R}}-\dfrac{p}{q_x}\right)+1-\dfrac{\gamma p}{q_x-q_{x+R}}\right] & \\ \quad + N\left[p(1-\gamma)-c-(x+R)h\right]\left((1-\beta)\left(\dfrac{\gamma p}{q_x-q_{x+R}}-\dfrac{p}{q_x}\right)+\dfrac{p}{q_x}-\dfrac{(1-\gamma)p}{q_{x+R}}\right) & \text{if } \gamma^x_{LM}\le\gamma<\gamma^x_{MH}, \\ N(p-c-xh)\left[\beta\left(1-\dfrac{p}{q_x}\right)\right] & \\ \quad + N\left[p(1-\gamma)-c-(x+R)h\right]\left(\dfrac{p}{q_x}-\dfrac{(1-\gamma)p}{q_{x+R}}+(1-\beta)\left(1-\dfrac{p}{q_x}\right)\right) & \text{if } \gamma^x_{MH}\le\gamma<1 \end{cases}$$

where the values of γ^x_{LM} and γ^x_{MH} are defined in Table 7. The first and second derivative of $\pi^x_{FS}(\hat{n},\gamma,LF)$ are as follows:

$$\frac{\partial \pi^x_{FS}(\hat{n},\gamma,LF)}{\partial \gamma} = \begin{cases} 0 & \text{if } 0<\gamma<\gamma^x_{LM} \\ N\left[-(\gamma p+Rh)\dfrac{(1-\beta)p}{q_x-q_{x+R}}+(2(1-\gamma)p-c-(x+R)h)\dfrac{p}{q_{x+R}}-p\left((1-\beta)\left(\dfrac{\gamma p}{q_x-q_{x+R}}-\dfrac{p}{q_x}\right)+\dfrac{p}{q_x}\right)\right] & \text{if } \gamma^x_{LM}\le\gamma<\gamma^x_{MH} \\ N\left[-p\left(\dfrac{p}{q_x}-\dfrac{(1-\gamma)p}{q_{x+R}}+(1-\beta)\left(1-\dfrac{p}{q_x}\right)\right)+[p(1-\gamma)-c-(x+R)h]\,\dfrac{p}{q_{x+R}}\right] & \text{if } \gamma^x_{MH}\le\gamma<1 \end{cases}$$

$$\frac{\partial^2 \pi^x_{FS}(\hat{n},\gamma,LF)}{\partial \gamma^2} = \begin{cases} 0 & \text{if } 0<\gamma<\gamma^x_{LM} \\ -Np^2\left[\dfrac{(1-\beta)}{q_x-q_{x+R}}+\dfrac{1}{q_{x+R}}\right]<0 & \text{if } \gamma^x_{LM}\le\gamma<\gamma^x_{MH}, \\ -N\dfrac{2p^2}{q_{x+R}}<0 & \text{if } \gamma^x_{MH}\le\gamma<1 \end{cases}$$

This implies that, for a given discount age $\hat{n}$, the per-period profit function is concave in the discount rate γ. Since this is true for every cycle eriod x, it must be true for the profit over the replenishment period, which means that there can exist a value $\gamma^*>0$, achieving the maximum profit for a given $\hat{n}$. Then, the optimal $\hat{n}^*$ is such that $\hat{n}^*=\arg\max_{\hat{n}}\Pi_{FS}(Q^*,\hat{n},\gamma^*(\hat{n}),LF)$. Note that the profit function under the LF display setting $(\Pi_{FS}(Q^*,\hat{n}^*,\gamma^*,LF))$ is concave in p. By comparing the profit under the SB and $\Pi_{FS}(Q^*,\hat{n}^*,\gamma^*(\hat{n}^*),LF)$, we obtain the positive profit threshold $\hat{p}$ by solving $\Pi_{FS}(Q^*,\hat{n}^*,\gamma^*(\hat{n}^*),LF)=\Pi_{SB}(Q^*)$. The optimal policy is to adopt the SB policy when $p<\tilde{p}\equiv\min\{\hat{p},q_{R-1},\frac{1}{2}(q_{R-1}+c+(R-1)h)\}$ and adopt the LF policy selling both batches for $x=1,...,L-R-1$, otherwise.

C.4. Proof of Theorem 2

The above proof to Theorem 1 shows that

$$\Pi_{FS}(Q,\hat{n},\gamma,LF) = N\sum_{x=0}^{\hat{n}-R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right)+N\sum_{x=\hat{n}-R}^{L-R-1}\Pi^x+N\sum_{x=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right),$$

and the optimal γ^* satisfies the FOC

$$\frac{\partial \Pi_{FS}(Q,\hat{n},\gamma,LF)}{\partial\gamma} = N\sum_{x=\hat{n}-R}^{L-R-1}\frac{\partial \pi^x(\hat{n},\gamma,LF)}{\partial\gamma}=0.$$

Thus, γ^* is a function of β and p. The expression of $\frac{\partial\Pi^x}{\partial\gamma}$ is as follows depending on which interval γ^* falls in. $\pi^x(\hat{n},\gamma,LF)$

$$= \begin{cases} 0 & \text{if } 0<\gamma^*(\beta,p)<\gamma^x_{LM} \\ -N(\gamma^*(\beta,p)p+Rh)\dfrac{(1-\beta)p}{q_x-q_{x+R}}+N(2(1-\gamma^*(\beta,p))p-c-(x+R)h)\dfrac{p}{q_{x+R}} & \\ -Np\left((1-\beta)\left(\dfrac{\gamma^*(\beta,p)p}{q_x-q_{x+R}}-\dfrac{p}{q_x}\right)+\dfrac{p}{q_x}\right) & \text{if } \gamma^x_{LM}\le\gamma^*(\beta,p)<\gamma^x_{MH} \\ -N\left[p\left(\dfrac{p}{q_x}-\dfrac{(1-\gamma^*(\beta,p))p}{q_{x+R}}+(1-\beta)\left(1-\dfrac{p}{q_x}\right)\right)+[p(1-\gamma^*(\beta,p))-c-(x+R)h]\,\dfrac{p}{q_{x+R}}\right] & \\ & \text{if } \gamma^x_{MH}\le\gamma^*(\beta,p)<1 \end{cases}$$

To find $\frac{\partial \gamma^*}{\partial \beta}$, we take derivative of the above expression with respect to β for $x \in \{\hat{n}-R, ..., L-R-1\}$ which are shown below, sum them up, and then equate the sum to zero.

$$\begin{cases} 0 & \text{if } 0 < \gamma^*(\beta,p) < \gamma^x_{LM} \\ N\left[-\frac{\partial \gamma^*(\beta,p)}{\partial \beta}\frac{(1-\beta)p^2}{q_x - q_{x+R}} + (\gamma^*(\beta,p)p + Rh)\frac{p}{q_x - q_{x+R}} - 2\frac{p^2}{q_{x+R}}\frac{\partial \gamma^*(\beta,p)}{\partial \beta}\right] & \\ N\left[-\frac{\partial \gamma^*(\beta,p)}{\partial \beta}\frac{(1-\beta)p^2}{q_x - q_{x+R}} + \left(\frac{\gamma^*(\beta,p)p^2}{q_x - q_{x+R}} - \frac{p^2}{q_x}\right)\right] & \text{if } \gamma^x_{LM} \le \gamma^*(\beta,p) < \gamma^x_{MH} \\ N\left[-\frac{\partial \gamma^*(\beta,p)}{\partial \beta}\frac{p^2}{q_{x+R}} + p\left(1 - \frac{p}{q_x}\right) - \frac{\partial \gamma^*(\beta,p)}{\partial \gamma}\frac{p^2}{q_{x+R}}\right] & \text{if } \gamma^x_{MH} \le \gamma^*(\beta,p) < 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 < \gamma^*(\beta,p) < \gamma^x_{LM} \\ N\left[-2\frac{\partial \gamma^*(\beta,p)}{\partial \beta}p^2\left(\frac{(1-\beta)}{q_x - q_{x+R}} + \frac{1}{q_{x+R}}\right) + (2\gamma^*(\beta,p)p + Rh)\frac{p}{q_x - q_{x+R}} - \frac{p^2}{q_x}\right] & \text{if } \gamma^x_{LM} \le \gamma^*(\beta,p) < \gamma^x_{MH} \\ N\left[-2\frac{\partial \gamma^*(\beta,p)}{\partial \beta}\frac{p^2}{q_{x+R}} + p\left(1 - \frac{p}{q_x}\right)\right] & \text{if } \gamma^x_{MH} \le \gamma^*(\beta,p) < 1 \end{cases}$$

By equating the sum of the above derivatives from $x = \hat{n}^* - R$ to $L - R - 1$ to zero, we can see that $\frac{\partial \gamma^*(\beta,p)}{\partial \beta} \ge 0$ because all the coefficients before $\frac{\partial \gamma^*(\beta,p)}{\partial \beta}$ is negative and the sum of terms without $\frac{\partial \gamma^*(\beta,p)}{\partial \beta}$ is positive in the above expressions. Using the same method, we can obtain $\frac{\partial \gamma^*(\beta,p)}{\partial p} \ge 0$.

C.5. Proof of Theorem 3

Under the EA and LO display settings, the SB policy is the optimal strategy and by Lemma 5 there is no stockout. Hence, the consumer surplus is

$$CS_{SB} = N\sum_{x=0}^{R-1}\int_{\frac{p}{q_x}}^{1}(\theta q_x - p)d\theta = N\sum_{x=0}^{R-1}\frac{(q_x - p)^2}{2q_x} \tag{13}$$

Under the LF display setting, the optimal selling strategy and discount rate depends on the selling price of fresh batch. Nevertheless, compared to the SB policy, consumer surplus always (weakly) improves under the FS policy because consumers have more purchasing options in each cycle period x as detailed below.

- for $\gamma \in [0, \gamma^x_{LM})$, the fresh batch gives all consumers a higher utility and

$$CS^x_{FS} = N\int_{\frac{p}{q_x}}^{1}(\theta q_x - p)d\theta = \frac{(q_x - p)^2}{2q_x} = CS^x_{SB}$$

- for $\gamma \in [\gamma^x_{LM}, \gamma^x_{MH})$,

$$\begin{aligned} CS^x_{FS} &= N\int_{\frac{\gamma p}{q_x - q_{x+R}}}^{1}(\theta q_x - p)d\theta + N\int_{\frac{p}{q_x}}^{\frac{\gamma p}{q_x - q_{x+R}}}(\beta(\theta q_x - p)) + (1-\beta)(\theta q_{x+R} - (1-\gamma)p)d\theta \\ &+ \int_{\frac{(1-\gamma)p}{q_{x+R}}}^{\frac{p}{q_x}}(\theta q_{x+R} - p)d\theta \\ &> N\int_{\frac{\gamma p}{q_x - q_{x+R}}}^{1}(\theta q_x - p)d\theta + N\int_{\frac{p}{q_x}}^{\frac{\gamma p}{q_x - q_{x+R}}}(\beta(\theta q_x - p)) + (1-\beta)(\theta q_{x+R} - (1-\gamma)p)d\theta \\ &> N\int_{\frac{\gamma p}{q_x - q_{x+R}}}^{1}(\theta q_x - p)d\theta + N\int_{\frac{p}{q_x}}^{\frac{\gamma p}{q_x - q_{x+R}}}(\beta(\theta q_x - p)) + (1-\beta)(\theta q_x - p)d\theta \\ &> N\int_{\frac{p}{q_x}}^{1}(\theta q_x - p)d\theta = CS^x_{SB} \end{aligned}$$

because $\theta q_{x+R} - (1-\gamma)p \ge \theta q_x - p$ for $\theta \in [\frac{p}{q_x}, \frac{\gamma p}{q_x - q_{x+R}}]$

- for $\gamma \in [\gamma^x_{MH}, 1]$,

$$
\begin{aligned}
CS^x_{FS} &= N\int_{\frac{p}{q_x}}^{1} (\beta(\theta q_x - p)) + (1-\beta)(\theta q_{x+R} - (1-\gamma)p)d\theta + \int_{\frac{(1-\gamma)p}{q_{x+R}}}^{\frac{p}{q_x}} (\theta q_{x+R} - p)d\theta \\
&> N\int_{\frac{p}{q_x}}^{1} (\beta(\theta q_x - p)) + (1-\beta)(\theta q_{x+R} - (1-\gamma)p)d\theta \\
&> N\int_{\frac{p}{q_x}}^{1} (\beta(\theta q_x - p)) + (1-\beta)(\theta q_x - p)d\theta \\
&> N\int_{\frac{p}{q_x}}^{1} (\theta q_x - p)d\theta = CS^x_{SB}
\end{aligned}
$$

because $\theta q_{x+R} - (1-\gamma)p \geq \theta q_x - p$ for $\theta \in \left[\frac{(1-\gamma)p}{q_{x+R}}, 1\right]$

C.6. Proof of Lemma 2

From (7),

$$
\begin{aligned}
\Pi_{FS}(Q, L, 0, \delta) &= N\sum_{x=0}^{L-R-1}\left[(p-c-xh)F^x(\delta) + [p-c-(x+R)h]\,O^x(\delta)\right] + N\sum_{x=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right) \\
&\leq N\sum_{x=0}^{L-R-1}\left[(p-c-xh)\,(F^x(\delta)+O^x(\delta))\right] + N\sum_{x=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right) \\
&= N\sum_{x=0}^{L-R-1}\left[(p-c-xh)\left(1-\frac{p}{q_x}\right)\right] + N\sum_{x=L-R}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right) \\
&= N\sum_{x=0}^{R-1}(p-c-xh)\left(1-\frac{p}{q_x}\right) = \Pi_{SB}
\end{aligned}
$$

where the first inequality is because $h \geq 0$.

C.7. Proof of Lemma 3

Under the `SB` policy, each replenishment cycle is independent of the previous ones since all unsold products are disposed of after R periods. Hence, it is enough to optimize total expected profit over the replenishment cycle. We write the expected profit over the replenishment cycle as a function of replenishment quantity Q as follows:

$$
\begin{aligned}
\Pi_{SB}(Q;\hat{n},\gamma) &= p_0\mathbb{E}[\min\{Q, D_0\}] + p_1\mathbb{E}[\min\{(Q-D_0)^+, D_1\}] + ... + p_{R-1}\mathbb{E}[\min\{(Q-D_0-...-D_{R-2})^+, D_{R-1}\}] \\
&\quad -cQ - h\mathbb{E}[Q-D_0]^+ - ... - h\mathbb{E}[Q-D_0-...-D_{R-2}]^+ - w_R\mathbb{E}[Q\_D_0 - ... - D_{R-1}]^+ \\
&= p_0Q - p_0\mathbb{E}[Q-D_0]^+ + p_1\mathbb{E}[Q-D_0]^+ - p_1\mathbb{E}[Q-D_0-D_1]^+ ... \\
&\quad + p_{R-1}\mathbb{E}[Q-D_0-...-D_{R-2}]^+ - p_{R-1}\mathbb{E}[Q-D_0-...-D_{R-1}]^+ \\
&\quad -cQ - h\mathbb{E}[Q-D_0]^+ - ... - h\mathbb{E}[Q-D_0-...-D_{R-2}]^+ - w_R\mathbb{E}[Q\_D_0 - ... - D_{R-1}]^+ \\
&= (p_0-c)Q - (p_0-p_1+h)\mathbb{E}[Q-D_0]^+ - (p_1-p_2+h)\mathbb{E}[Q-D_0-D_1]^+ ... \\
&\quad -(p_{R-2}-p_{R-1}+h)\mathbb{E}[Q-D_0-...-D_{R-2}]^+ - (p_{R-1}+w_R)\mathbb{E}[Q-D_0-...-D_{R-1}]^+
\end{aligned}
$$

From Assumption A6, $p_0 = ... = p_{R-1} = p$ and the above expression simplifies to:

$$
\begin{aligned}
\pi_{SB}(Q) &= (p-c)Q - h\mathbb{E}[Q-D_0]^+ - h\mathbb{E}[Q-D_0-D_1]^+ ... \\
&\quad -h\mathbb{E}[Q-D_0-...-D_{R-2}]^+ - (p+w_R)\mathbb{E}[Q-D_0-...-D_{R-1}]^+
\end{aligned}
$$

The first derivative is:

$$\frac{\partial \pi_{SB}(Q)}{\partial Q} = (p-c) - hP[D_0 \le Q] - hP[D_0 + D_1 \le Q] ... - hP[D_0 + ... + D_{R-2} \le Q]$$
$$-(p+w_R)P[D_0 + ... + D_{R-1} \le Q]$$

The optimal Q^* is the value which solves this equation.

If $h=0$, it further simplifies to:

$$\pi_{SB}(Q) = (p-c)Q - (p+w_R)\mathbb{E}[Q - D_0 - ... - D_{R-1}]^+$$

which is a newsvendor model with demand equal to $D = D_0 + D_1 + ... + D_{R-1}$. So, if $h=0$, the optimal solution is $Q^* = G^{-1}\left(\frac{p-c}{p+w_R}\right)$, where G is the convolution of R random variables with mean $N\left(1 - \frac{p}{q_x}\right)$ for $x = 0, ..., R-1$.

C.8. Proof of Lemma 4

To simplify the proof we omit the superscript referring to the time period. In this proof, we compare the `LO` and `LF` display settings under a fixed sample path of consumer arrivals; the logic applies to the comparison between `LO` and `EA`. From Table 1, for any cycle period $x \in \{0, ..., L-R-1\}$, when both batches are in stock, i.e., $y_x > 0$ and $y_{x+R} > 0$, the sales rate for the old batch is weakly greater under `LO` than `LF`: $O^x(\hat{n}, \gamma, \texttt{LO}) = P_O^x(\hat{n}, \gamma) + P_{OF}^x(\hat{n}, \gamma) + \beta P_{FO}^x(\hat{n}, \gamma) \ge O^x(\hat{n}, \gamma, \texttt{LF}) = P_O^x(\hat{n}, \gamma) + (1-\beta)P_{OF}^x(\hat{n}, \gamma)$. At the same time, the total sales rate (fresh+old) is the same: $F^x(\hat{n}, \gamma, \texttt{LO}) + O^x(\hat{n}, \gamma, \texttt{LO}) = F^x(\hat{n}, \gamma, \texttt{EA}) + O^x(\hat{n}, \gamma, \texttt{EA})$.

Given this and the fact that the starting inventory of both batches is assumed to be the same under both display settings, only the following scenarios regarding the sequence of stockouts over the replenishment cycle are possible:

(1) The old batch stocks out under `LO`;

(2) No stockout of either batch under either display setting;

(3) No stockout under `LO` and the fresh batch stocks out under `LF` before the end of cycle period $L-R-1$;

(4) The fresh batch stock outs before the end of cycle period $L-R-1$ under `LO`.

In all these cases, we prove that waste, i.e., the number of leftover units of the old batch at the end of cycle period $L-R-1$, is weakly lower under `LO` than `LF`.

(1) Waste is zero under `LO`; therefore, it must be weakly lower than under `LF`;

(2) Since the sales rate for the old batch is weakly greater under `LO` than `LF` for each cycle period $x \in \{0, ..., L-R-1\}$ the number of leftover at the end of the cycle period $L-R-1$ must be weakly lower under `LO` than under `LF`.

(3) Since the sales rate for the old batch is weakly greater under `LO` than `LF`, up until the stockout epoch, more old units are sold under `LO` than under `LF`. Following the stockout epoch, the sales rate for old units is greater under `LF` (equal to $P_O^x + P_{OF}^x + P_{FO}^x$ in cycle period x) than under `LO` (where it is $P_O^x + P_{OF}^x + \beta P_{FO}^x$); however, the total sales rate (fresh + old) under `LF` is weakly lower (since type F consumers no longer buy). Hence, in every cycle period, the total sales rate is weakly higher under `LO`

than LF. Now suppose (contradiction) that, at the end of the cycle period $L-R-1$, there are fewer old units left under LF than under LO. This would imply that overall more units were sold under LF (specifically, all the fresh ones and more of the old ones), which is a contradiction to the fact that the total sales rate is always weakly higher under LO.

(4) Since the sales rate for fresh units is higher under LF than LO, it must be that the fresh batch stocked out earlier under LF. Following the stockout epoch, the total sales rate in LF is weakly lower (since type F consumers no longer buy). Hence, in every cycle period, the total sales rate is weakly higher under LO than LF. By the same contradiction argument as in (3) we conclude that the leftover inventory of old units at the end of cycle period $L-R-1$ must be weakly lower under LO.

This proves that, for a given sample path of consumer arrivals, waste under LO is lower than under LF. Since this is true for every sample path, it is also true in expectation.

C.9. Proof of Theorem 4

We drop the superscripts τ referring to the time period to simplify the notation. Since $\gamma=0$, we set $\hat{n}=L$. When $\gamma=0$, we have $P^x_O(L,0)=P^x_{OF}(L,0)=0$ from Table 7; therefore, there are only type F and FO consumers. Suppose Q is the optimal order quantity under the SB policy. Consider using the FS policy under the EA display setting and the same order quantity Q. We use y_R^{EA} to denote the old batch inventory under the EA setting at the beginning of the t-th replenishment cycle. To simplify the notation, let $D^t_F(\texttt{EA})$ and $D^t_O(\texttt{EA})$ represent the total demand for fresh and old units over the t-th replenishment cycle, respectively, under FS policy with EA display setting. And let $D^t_F(\texttt{SB})$ represent the total demand for fresh units over the t-th replenishment cycle under the SB policy.

The expected profits in the t-th replenishment cycle for the FS policy with EA display setting and SB policy for the same order quantity Q are:

$$\Pi^t_{FS}(Q,L,0,\texttt{EA}) = p\mathbb{E}[\min\{Q,D^t_F(\texttt{EA})\}]+p\mathbb{E}[\min\{y_R^{EA},D^t_O(\texttt{EA})\}]-w_L\mathbb{E}[y_R^{EA}-D^t_O(\texttt{EA})]^+-cQ, \quad (14)$$

$$\Pi^t_{SB}(Q) = p\mathbb{E}[\min\{Q,D^t_F(\texttt{SB})\}]-w_R\mathbb{E}[Q-D^t_F(\texttt{SB})]^+-cQ \quad (15)$$

From Tables 1 and 8, under the EA display setting with $\gamma=0$, we have for $x\in\{0,...,R-1\}$,

$$F^x(L,0,\texttt{EA}) = P^x_F(L,0)+P^x_{FO}(L,0)=1-\frac{p}{q_x} \quad (16)$$

$$O^x(L,0,\texttt{EA}) = 0 \quad (17)$$

that is, units from the old batch sell only after all fresh units are sold. Therefore, $D^t_F(\texttt{EA})=D^t_F(\texttt{SB})$, which implies that the first revenue term is the same in (14) and (15). In addition, we have $\mathbb{E}[\min\{y_R^{EA},D^t_O(\texttt{EA})\}]\geq 0$ and $\mathbb{E}[y_R^{EA}-D^t_O(\texttt{EA})]^+\leq\mathbb{E}[Q-D^t_F(\texttt{EA})]^+$ since $y_R^{EA}=\mathbb{E}[Q-D^t_F(\texttt{EA})]^+$. Hence, $\Pi^t_{FS}(Q,L,0,\texttt{EA})\geq\Pi^t_{SB}(Q)$. Under the optimal FS policy, the profit can be even higher by optimizing the display setting, the discount rate, the discount age, and the order quantity. Therefore, the FS policy outperforms the SB policy, that is, its optimal expected profit can only be higher under EDLP.

Next, we compare EA, LF and LO display settings under the FS policy. From Tables 1 and 8, the EA and LF display settings are equivalent when $\gamma=0$ (that is, there are only F and FO consumers). Suppose Q is the optimal order quantity under the EA setting. We use y_R^{EA} and y_R^{LO} to denote the old batch inventory under the EA and LO settings at the beginning of the t-th replenishment cycle, respectively, and let $D_F^t(\delta)$ and $D_O^t(\delta)$ for the demand of fresh and old units, respectively, under display setting $\delta \in \{\texttt{EA}, \texttt{LO}\}$ in the t-th replenishment cycle: We compare the expected profit functions when the order quantity is Q under both settings:

$$\Pi_{FS}^t(Q, L, 0, \texttt{EA}) = p\mathbb{E}[\min\{Q, D_F^t(\texttt{EA})\}] + p\mathbb{E}[\min\{y_R^{EA}, D_O^t(\texttt{EA})\}] - w_L\mathbb{E}[y_R^{EA} - D_O^t(\texttt{EA})]^+ - cQ$$
$$\Pi_{FS}^t(Q, L, 0, \texttt{LO}) = p\mathbb{E}[\min\{Q, D_F^t(\texttt{LO})\}] + p\mathbb{E}[\min\{y_R^{LO}, D_O^t(\texttt{LO})\}] - w_L\mathbb{E}[y_R^{LO} - D_O^t(\texttt{LO})]^+ - cQ$$

Under the LO display setting, for $x \in \{0, ..., R-1\}$

$$F^x(L, 0, \texttt{LO}) = P_F^x + (1-\beta)P_{FO}^x = \min\left\{\frac{p}{q_{x+R}}, 1\right\} - \frac{p}{q_x} + (1-\beta)\left(1 - \min\left\{\frac{p}{q_{x+R}}, 1\right\}\right) \tag{18}$$
$$O^x(L, 0, \texttt{LO}) = \beta P_{FO}^x = \beta\left(1 - \min\left\{\frac{p}{q_{x+R}}, 1\right\}\right) \tag{19}$$

which implies that some old units can sell before the fresh units run out.

Comparing (16) and (18), we have $F^x(L, 0, \texttt{EA}) \geq F^x(L, 0, \texttt{LO})$ and, comparing (17) and (19), we have $O^x(L, 0, \texttt{EA}) \leq O^x(L, 0, \texttt{LO})$, for $x \in \{0, ..., R-1\}$, which implies that fresh units are depleted at a faster pace than old units under EA compared to LO when both batches are in stock. Importantly, as long as both batches are in stock, the total sales rate (fresh + old units) is the same across all display settings as $F^x(L, 0, \texttt{EA}) + O^x(L, 0, \texttt{EA}) = F^x(L, 0, \texttt{LO}) + O^x(L, 0, \texttt{LO})$, for $x \in \{0, ..., R-1\}$. We next prove the following claim by induction.

Claim: At the start of every replenishment cycle, the inventory of old units is weakly higher under LO than under EA. During every replenishment cycle, the total sales (fresh + old units) are weakly higher and the waste is lower under LO than under EA.

In the first period, suppose that both display settings start with no old units. When only fresh units are in stock, the sales rate is identical under LO and EA and equal to $P_F^x + P_{FO}^x = 1 - \frac{p}{q_x}$, therefore sales in the first replenishment cycle are the same under LO and EA and there is no waste at the end of the first replenishment cycle. At the start of the second replenishment cycle, the number of old units is also the same.

Now suppose that the the inventory of old units is weakly higher under LO than under EA at the beginning of replenishment cycle $t-1$ (induction hypothesis). Since $F^x(L, 0, \texttt{LO}) \leq F^x(L, 0, \texttt{EA})$ and the fresh units inventory is the same under two settings at the start of replenishment cycle $t-1$, it is impossible for fresh units to stock out under LO before they do under EA. Further, under EA, old units begin selling only after fresh units are sold out, making it impossible for old units to stock out before fresh ones. Therefore, the only possible scenarios in terms of stockouts over replenishment cycle $t-1$ are as follows:

(0) No stockout under either EA or LO

(1) The only stockout is of fresh units under EA

(2) Fresh units stock out under EA first then old units stock out under LO

(3) Fresh units stock out under EA then fresh units stock out under LO

(4) The only stockout is of old units under LO

(5) Old units stock out under LO then fresh units stock out under EA

(6) Both batches run out under both EA and LO

We analyze each scenario separately as follows.

(0) Only fresh units are sold under EA but the total sale of fresh and old units over the replenishment cycle $t-1$ are the same under EA and LO. Waste is higher under EA (and equal to the entire inventory of old units). At the start of the next replenishment cycle t, the inventory of the old batch is weakly higher under LO (since fewer fresh units were sold), i.e, $y_R^{\texttt{EA}} \leq y_R^{\texttt{LO}}$.

(1) After fresh units stock out under EA, if the stockout cycle period $x \leq L-R-1$, only old units are sold at a rate of P_{FO}^x until expiration. If the stockout cycle period $x > L-R-1$, there is no sales of old units because of expiration. Under LO, the total sales rate (fresh+old) is weakly greater in each cycle period, so total sales over the replenishment cycle $t-1$ are weakly higher. This results in weakly more units wasted under EA than LO. At the start of the next replenishment cycle t, the inventory of the old batch is weakly higher under LO (as it is zero under EA), i.e, $y_R^{\texttt{EA}} \leq y_R^{\texttt{LO}}$.

(2) After fresh units stock out under EA, if the stockout cycle period $x \leq L-R-1$, only old units are sold at a rate of P_{FO}^x until expiration. If the stockout cycle period $x > L-R-1$, there is no sales of old units because of expiration. Under LO, the total sales rate (fresh+old) is weakly greater in each cycle period, because, even after the old units sold out, the sales rate is $P_F^x + P_{FO}^x$. Therefore, total sales over replenishment cycle $t-1$ are weakly higher. This results in weakly more units wasted under EA than LO (nothing is wasted under LO). At the start of the next replenishment cycle t, the inventory of the old batch is weakly higher under LO (as it is zero under EA), i.e, $y_R^{\texttt{EA}} \leq y_R^{\texttt{LO}}$.

(3) After fresh units stock out under EA, if the stockout cycle period $x \leq L-R-1$, only old units are sold at a rate of P_{FO}^x. If the stockout cycle period $x > L-R-1$, there is no sales of old units because of expiration. The same happens after fresh units run out in LO. Under LO, total sales over the replenishment cycle are weakly greater because LO keeps a total sale rate of $P_F^x + P_{FO}^x$ for longer. This also results in weakly more units wasted under EA than LO. At the start of replenishment cycle t, the inventory of the old batch is weakly higher under LO, i.e, $y_R^{\texttt{EA}} \leq y_R^{\texttt{LO}}$.

(4) After old units stock out under LO, the total sales rate is still equal across display settings (to $P_F^x + P_{FO}^x$). Therefore total sales over replenishment cycle $t-1$ are equal. Weakly more units are wasted under EA than LO (nothing is wasted under LO). At the start of replenishment cycle t, the inventory of the old batch is weakly higher under LO (since fewer fresh units were sold), i.e, $y_R^{\texttt{EA}} \leq y_R^{\texttt{LO}}$.

(5) After old units stock out under LO, the total sales rate is still equal across display settings (to $P_F^x + P_{FO}^x$). After the fresh units stock out under EA, only old units are sold at a rate of P_{FO}^x. Therefore, the total sale over replenishment cycle $t-1$ is weakly higher under LO. Weakly more units are wasted under EA than LO (nothing is wasted under LO). At the start of replenishment cycle t, the inventory of the old batch is weakly higher under LO (it is zero under EA), i.e, $y_R^{\texttt{EA}} \leq y_R^{\texttt{LO}}$.

(6) In this case, total sales are equal in both settings; there is no waste in replenishment cycle $t-1$ and the inventory of the old batch at the beginning of replenishment cycle t is zero in both settings, i.e., $y_R^{\texttt{EA}} = y_R^{\texttt{LO}} = 0$.

This proves the claim.

Since total sales (fresh+old) in each replenishment cycle are weakly higher under LO than under EA and there is no discount on old units (EDLP), the total revenue over each replenishment cycle is weakly higher under LO. Because the waste is weakly lower under LO than under EA in each replenishment cycle and the quantity ordered is identical, we conclude that the profit over each replenishment cycle is weakly higher under LO than under EA when all the display setting order the same Q units every period. Since the result holds for every N, it also holds in expectation. Since this result holds for a fixed Q, it also holds if the quantities are optimized for each display setting.

C.10. Proof of Theorem 5

To simplify the notation, we use $D_F^t(\delta)$ and $D_O^t(\delta)$ to represent the total demand for fresh and old units over the t-th replenishment cycle, respectively, under the FS policy with display setting $\delta \in \{\texttt{EA}, \texttt{LO}, \texttt{LF}\}$. Similarly, we use $D_F^t(\texttt{SB})$ to represent the total demand for fresh units over the t-th replenishment cycle under policy SB. Assume that Q^t is the optimal order quantity for the t-th replenishment cycle under the SB policy. Given that $h = h' = 0$, the retailer's expected profits in the t-th replenishment cycle given order quantity Q^t and inventory of the old batch $y_R^{(t-1)R}$ can be written as follows:

$$\begin{aligned}\Pi_{FS}^t(Q^t, \hat{n}, \gamma, \delta) &= p\mathbb{E}[\min\{Q^t, D_F^t(\delta)\}] + (1-\gamma)p\mathbb{E}[\min\{y_R^{(t-1)R}, D_O^t(\delta)\}] - w_L\mathbb{E}[y_R^{(t-1)R} - D_O^t(\delta)]^+ - cQ^t \\ \Pi_{SB}^t(Q^t) &= p\mathbb{E}[\min\{Q^t, D_F^t(\texttt{SB})\}] - w_R\mathbb{E}[Q^t - D_F^t(\texttt{SB})]^+ - cQ^t\end{aligned}$$

Suppose that the EA setting without discount, i.e., $\gamma = 0$, is adopted under the FS policy. Since $\gamma = 0$, we set $\hat{n} = L$. Then, we have $D_F^t(\texttt{EA}) = D_F^t(\texttt{SB})$ since, under the EA display setting with $\gamma = 0$, we have, $F^x(L, 0, \texttt{EA}) = P_F^x + P_{FO}^x$ and $O^x(L, 0, \texttt{EA}) = 0$ for $x = 0, ..., R-1$ (see Table 1 and Table 8), i.e., units from the old batch sell only after all fresh units are sold.

$$\begin{aligned}\Pi_{FS}^t(Q^t, L, 0, EA) &= p\mathbb{E}[\min\{Q^t, D_F^t(\texttt{EA})\}] + p\mathbb{E}[\min\{y_R^{(t-1)R}, D_O^t(\texttt{EA})] - w_L\mathbb{E}[y_R^{(t-1)R} - D_O^t(\texttt{EA})]^+ - cQ^t \\ \Pi_{SB}^t(Q^t) &= p\mathbb{E}[\min\{Q^t, D_F^t(\texttt{EA})\}] - w_R\mathbb{E}[Q^t - D_F^t(\texttt{EA})]^+ - cQ^t\end{aligned}$$

The first terms are the same for both policies, but there is an extra revenue term from (potentially) selling old units under the FS policy. Regarding the waste terms, we have $y_R^{(t-1)R} = Q^{t-1} - D_F^{t-1}$, implying that the expected waste at the end of the t-th replenishment cycle under the FS policy $\mathbb{E}[[Q^{t-1} - D_F^{t-1}]^+ - D_O^t]^+$, while the expected waste from the same batch under SB policy is $\mathbb{E}[Q^{t-1} - D_F^{t-1}]^+$, which is greater. Therefore, the expected waste per replenishment cycle is lower under the FS policy and therefore, the expected waste cost is lower since $w_R = w_L$. As a result, expected profit per replenishment cycle is higher under the FS policy. These results hold for any replenishment cycle. Hence, the FS policy Pareto-dominates the SB policy.

C.11. Proof of Theorem 6

To simplify the notation, we use y_R instead of $y_R^{(t-1)R}$. Under the conditions given in Theorem 6, The retailer's expected profit in the t-th replenishment cycle from 2 simplifies to:

$$\begin{aligned}\Pi_{FS}^t(Q^t, R, \gamma, \delta) &= p\mathbb{E}[\min\{Q^t, D_F\}] + (1-\gamma)p\mathbb{E}[\min\{y_R, D_O\}] - h'\mathbb{E}[Q^t - D_F] - w_L\mathbb{E}[y_R - D_O] - cQ^t \\ &= pD_F + (1-\gamma)pD_O - h'\mathbb{E}[Q^t - D_F] - w_L\mathbb{E}[y_R - D_O] - cQ^t \qquad (20)\end{aligned}$$

where D_F and D_O denote the total demand for fresh and old units during the replenishment cycle, respectively.

Writing out these expressions for each display setting, we obtain:

$$\begin{aligned}
\Pi^t_{FS}(Q^t,R,\gamma,\texttt{EA}) &= p\bar{N}\sum_{x=0}^{R-1}(P^x_F+P^x_{FO})+(1-\gamma)p\bar{N}\sum_{x=0}^{R-1}(P^x_O+P^x_{OF})\\
&\quad -h'\left[Q^t-\bar{N}\sum_{x=0}^{R-1}(P^x_F+P^x_{FO})\right]-w_L\left[y_R-\bar{N}\sum_{x=0}^{L-R-1}(P^x_O+P^x_{OF})\right]-cQ^t\\
\Pi^t_{FS}(Q^t,R,\gamma,\texttt{LO}) &= p\bar{N}\sum_{x=0}^{R-1}(P^x_F+(1-\beta)P^x_{FO})+(1-\gamma)p\bar{N}\sum_{x=0}^{R-1}(P^x_O+P^x_{OF}+\beta P^x_{FO})\\
&\quad -h'\left[Q^t-\bar{N}\sum_{x=0}^{R-1}(P^x_F+(1-\beta)P^x_{FO})\right]-w_L\left[y_R-\bar{N}\sum_{x=0}^{L-R-1}(P^x_O+P^x_{OF}+\beta P^x_{FO})\right]-cQ^t\\
\Pi^t_{FS}(Q^t,R,\gamma,\texttt{LF}) &= p\bar{N}\sum_{x=0}^{R-1}(P^x_F+P^x_{FO}+\beta P^x_{OF})+(1-\gamma)p\bar{N}\sum_{x=0}^{R-1}(P^x_O+(1-\beta)P^x_{OF})\\
&\quad -h'\left[Q^t-\bar{N}\sum_{x=0}^{R-1}(P^x_F+P^x_{FO}+\beta P^x_{OF})\right]-w_L\left[y_R-\bar{N}\sum_{x=0}^{L-R-1}(P^x_O+(1-\beta)P^x_{OF})\right]-cQ^t
\end{aligned}$$

Comparing the above three profit functions, we obtain:

$$\begin{aligned}
\Pi^t_{FS}(Q^t,R,\gamma,\texttt{LO})-\Pi^t_{FS}(Q^t,R,\gamma,\texttt{EA}) &= \beta\bar{N}\sum_{x=0}^{R-1}(-p\gamma+w_L-h')P^x_{OF}\\
\Pi^t_{FS}(Q^t,R,\gamma,\texttt{LF})-\Pi^t_{LO}(Q^t,R,\gamma,\texttt{EA}) &= \beta\bar{N}\sum_{x=0}^{R-1}(p\gamma-w_L+h')\left[P^x_{FO}+P^x_{OF}\right]
\end{aligned}$$

Therefore, LO outperforms EA, which outperforms LF if $w_L > p\gamma+h'$; otherwise, LF outperforms EA, which outperforms LO.

C.12. Proof of Corollary 1

The result follows directly from the proof of Theorem 6 since $\gamma p+h'$ is more likely to be less than w_L for a higher w_L, a lower h', or a lower p.

C.13. Proof of Lemma 5

i. Suppose (contradiction) that it is optimal to waste z units in a replenishment cycle. In this case, ordering z units fewer at the replenishment epoch would lower the purchasing and disposal costs without decreasing the revenue obtained during the replenishment cycle; therefore, profits would increase.

ii. Suppose (contradiction) that a prospective consumer leaves the store without buying a unit. Ordering one more unit for this customer to purchase would yield an additional profit for the retailer; therefore it is optimal to do so.

C.14. Proof of Lemma 6

From Lemma 5, the optimal quantity must be exactly equal to the demand from consumers over the replenishment cycle. Under the FS policy, consumer demand is equal to $N\left[F^x(\hat{n},\gamma,\delta)+O^x(\hat{n},\gamma,\delta)\right]$ in periods $x\in\{0,...,L-R-1\}$ and equal to $N\left(1-\frac{p_x}{q_x}\right)^+=N\left(1-\frac{p_x}{q_x}\right)$ in periods $x\in\{L-R,...,R-1\}$. Under the SB policy consumer demand is equal to $N\left(1-\frac{p_x}{q_x}\right)^+=N\left(1-\frac{p_x}{q_x}\right)$ in periods $x\in\{0,...,R-1\}$. In the last two equations, the positive part is dropped since $p_x\le p\le q_{R-1}\le q_x$ for $x=0,...,R-1$, where the first inequality is because p_x is either p or $(1-\gamma)p$ with $\gamma\ge 1$, the second one is by Assumption A5 and the third one is by Assumption A2.

C.15. Proof of Lemma 7

For $\hat{n}\in\{0,...,R-1\}$, the total profit over the replenishment cycle is given by (8). With no discount in any period, this is equal to $N\sum_{x=0}^{R-1}(p-c-hx)\left(1-\frac{p}{q_x}\right)$. We show that in each period $x\in\{0,...,R-1\}$, we have $(p-c-hx)\left(1-\frac{p}{q_x}\right)\ge[p(1-\gamma)-c-hx]\left(1-\frac{p(1-\gamma)}{q_x}\right)$ for any $\gamma\in[0,1]$. Simplifying this inequality, we obtain $p(1-\gamma)\le\frac{q_x+c+hx}{2}$. By Assumption A5, we have $p\le\frac{q_{R-1}+c+(R-1)h}{2}$ and by Assumption A7, we have $q_x+xh\le q_{x-1}+(x-1)h$. Combining these inequalities, we obtain $p(1-\gamma)\le p\le\frac{q_{R-1}+c+(R-1)h}{2}\le...\le\frac{q_1+c+h}{2}\le\frac{q_0+c}{2}$.

C.16. Proof of Lemma 8

To prove the result, we show, by contradiction, that the following two cases are not possible: (1) $L-R\le\hat{n}^*\le R-1$ and (2) $0\le\hat{n}^*<L-R$.

Case (1): Suppose that the optimal discount age is such that $L-R\le\hat{n}^*\le R-1$. In this case, in periods $x\in\{0,...,L-R-1\}$, both batches are offered but only the old one is discounted; in periods $x\in\{L-R,...,\hat{n}^*-1\}$, only the fresh batch is offered and without a discount; and in periods $x\in\{\hat{n}^*,...,R-1\}$, only the fresh batch is offered and is discounted. Now, consider postponing the discount age by one period, the difference in profits is:

$$\begin{aligned}\pi_{FS}(\hat{n}^*,\gamma,\mathtt{LF})-\pi_{FS}(\hat{n}^*+1,\gamma,\mathtt{LF}) &= N(p(1-\gamma)-c-\hat{n}^*h)\left(1-\frac{p(1-\gamma)}{q_{\hat{n}^*}}\right)-N(p-c-\hat{n}^*h)\left(1-\frac{p}{q_{\hat{n}^*}}\right)\\ &= -N\frac{\gamma p\left(c+q_{\hat{n}^*}+\hat{n}^*h+(\gamma-2)p\right)}{q_{\hat{n}^*}}\end{aligned}$$

which is negative because, by Assumptions A5 and A7, we have $p\le\frac{q_{R-1}+c+(R-1)h}{2}\le...\le\frac{q_{\hat{n}^*}+c+\hat{n}^*h}{2}$. Therefore, it contradicts that $\hat{n}^*$ is optimal.

Case (2): Suppose that the optimal discount age is such that $0\le\hat{n}^*<L-R$. In this case, both batches are offered, with only old one discounted in periods $x\in\{0,...,\hat{n}^*-1\}$, both batches are offered at a discount in periods $x\in\{\hat{n}^*,...,L-R-1\}$, and only the fresh batch is offered at a discount in periods $x\in\{L-R,...,R-1\}$. If the discount age is postponed by one period, the difference in profits is

$$\begin{aligned}&\pi_{FS}(\hat{n}^*,\gamma,\mathtt{LF})-\pi_{FS}(\hat{n}^*+1,\gamma,\mathtt{LF})\\ =\ &N\left[(p(1-\gamma)-c-\hat{n}^*h)F^{\hat{n}^*}(\hat{n}^*,\gamma,\mathtt{LF})+\left[p(1-\gamma)-c-(\hat{n}^*+R)h\right]O^{\hat{n}^*}(\hat{n}^*,\gamma,\mathtt{LF})\right]\\ &-N\left[(p-c-\hat{n}^*h)F^{\hat{n}^*}(\hat{n}^*+1,\gamma,\mathtt{LF})+\left[p(1-\gamma)-c-(\hat{n}^*+R)h\right]O^{\hat{n}^*}(\hat{n}^*+1,\gamma,\mathtt{LF})\right]\\ =\ &N\left[(p(1-\gamma)-c-\hat{n}^*h)F^{\hat{n}^*}(\hat{n}^*,\gamma,\mathtt{LF})\right]\end{aligned}$$

$$
\begin{aligned}
& -N\left[(p-c-\hat{n}^*h)\left[P_{FO}^{\hat{n}^*}(\hat{n}^*+1,\gamma)+\beta P_{OF}^{\hat{n}^*}(\hat{n}^*+1,\gamma)\right]+\left[p(1-\gamma)-c-(\hat{n}^*+R)h\right]\left[P_{O}^{\hat{n}^*}(\hat{n}^*+1,\gamma)+(1-\beta)P_{OF}^{\hat{n}^*}(\hat{n}^*+1,\gamma)\right]\right]\\
= & \ N\left[(p(1-\gamma)-c-\hat{n}^*h)\left(1-\frac{p(1-\gamma)}{q_{\hat{n}^*}}\right)\right]\\
& -N\left[(p-c-\hat{n}^*h)\left[\left(1-\frac{p}{q_{\hat{n}^*}}\right)-(1-\beta)P_{OF}^{\hat{n}^*}(\hat{n}^*+1,\gamma)\right]+\left[p(1-\gamma)-c-(\hat{n}^*+R)h\right]\left[P_{O}^{\hat{n}^*}(\hat{n}^*+1,\gamma)+(1-\beta)P_{OF}^{\hat{n}^*}(\hat{n}^*+1,\gamma)\right]\right]\\
= & \ N\left[(p(1-\gamma)-c-\hat{n}^*h)\left(1-\frac{p(1-\gamma)}{q_{\hat{n}^*}}\right)\right]-N\left[(p-c-\hat{n}^*h)\left[\left(1-\frac{p}{q_{\hat{n}^*}}\right)\right]\right]\\
& -N\left[\left[p(1-\gamma)-c-(\hat{n}^*+R)h-(p-c-\hat{n}^*h)\right](1-\beta)P_{OF}^{\hat{n}^*}(\hat{n}^*+1,\gamma)+\left[p(1-\gamma)-c-(\hat{n}^*+R)h\right]P_{O}^{\hat{n}^*}(\hat{n}^*+1,\gamma)\right]\\
= & \ -N\frac{\gamma p\left(c+q_{\hat{n}^*}+\hat{n}^*h+(\gamma-2)p\right)}{q_{\hat{n}^*}} \qquad \text{(from Case (1))}\\
& -N\left[(-\gamma p-Rh)(1-\beta)P_{OF}^{\hat{n}^*}(\hat{n}^*+1,\gamma)+\left[p(1-\gamma)-c-(\hat{n}^*+R)h\right]P_{O}^{\hat{n}^*}(\hat{n}^*+1,\gamma)\right]\\
= & \ -N\frac{\gamma p\left(c+q_{\hat{n}^*}+\hat{n}^*h+(\gamma-2)p\right)}{q_{\hat{n}^*}}\\
& -N\left[(-\gamma p-Rh)(1-\beta)\left[\frac{\gamma p}{q_{\hat{n}^*}-q_{\hat{n}^*+R}}-\frac{p}{q_{\hat{n}^*}}\right]+\left[p(1-\gamma)-c-(\hat{n}^*+R)h\right]\left[\frac{p}{q_{\hat{n}^*}}-\frac{(1-\gamma)p}{q_{\hat{n}^*+R}}\right]\right]\\
& = Np\left(\frac{(1-\beta)p}{q_{\hat{n}^*}-q_{\hat{n}^*+R}}+\frac{p}{q_{\hat{n}^*+R}}-\frac{p}{q_{\hat{n}^*}}\right)\gamma^2+Np\left(\frac{c}{q_{\hat{n}^*+R}}-\frac{c}{q_{\hat{n}^*}}+\frac{(\beta-1)hR}{q_{\hat{n}^*+R}-q_{\hat{n}^*}}+\frac{h(\hat{n}^*+R)}{q_{\hat{n}^*+R}}-\frac{h\hat{n}^*}{q_{\hat{n}^*}}+\frac{\beta p}{q_{\hat{n}^*}}-\frac{2p}{q_{\hat{n}^*+R}}+\frac{2p}{q_{\hat{n}^*}}-1\right)\gamma+\\
& Np\left(-\frac{c}{q_{\hat{n}^*+R}}+\frac{c}{q_{\hat{n}^*}}+\frac{\beta hR}{q_{\hat{n}^*}}-\frac{h(\hat{n}^*+R)}{q_{\hat{n}^*+R}}+\frac{h\hat{n}^*}{q_{\hat{n}^*}}+\frac{p}{q_{\hat{n}^*+R}}-\frac{p}{q_{\hat{n}^*}}\right).
\end{aligned}
$$

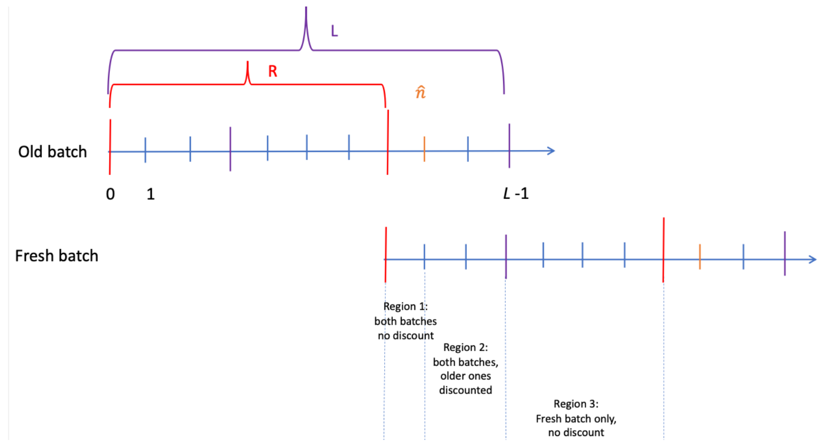
$\left(\frac{(1-\beta)p}{q_{\hat{n}^*}-q_{\hat{n}^*+R}}+\frac{p}{q_{\hat{n}^*+R}}-\frac{p}{q_{\hat{n}^*}}\right)>0$, so the profit difference function is convex in γ. At $\gamma=\frac{q_{\hat{n}^*}-q_{\hat{n}^*+R}}{q_{\hat{n}^*}}$, the difference is equal to $-\frac{p\left(q_{\hat{n}^*}-q_{\hat{n}^*+R}\right)\left(q_{\hat{n}^*}\left(c+h\hat{n}^*+q_{\hat{n}^*}-p\right)-pq_{\hat{n}^*+R}\right)}{q_{\hat{n}^*}^3}<0$ because $q_{\hat{n}^*}>q_{\hat{n}^*+R}$ and $p<\frac{c+\hat{n}^*h+q_{\hat{n}^*}}{2}$. Suppose $h<\min_{0<\hat{n}^*<L-R}\left(\frac{1}{R\left(\frac{\beta}{q_{\hat{n}^*}}+\frac{1-\beta}{q_{\hat{n}^*}-q_{\hat{n}^*+R}}\right)}-\frac{p}{R}\right)$, then the profit difference is less than 0 for $\frac{q_{\hat{n}^*}-q_{\hat{n}^*+R}}{q_{\hat{n}^*}}<\gamma<1$, and the difference is always less than 0 in this range.

Appendix D: Figures and Tables

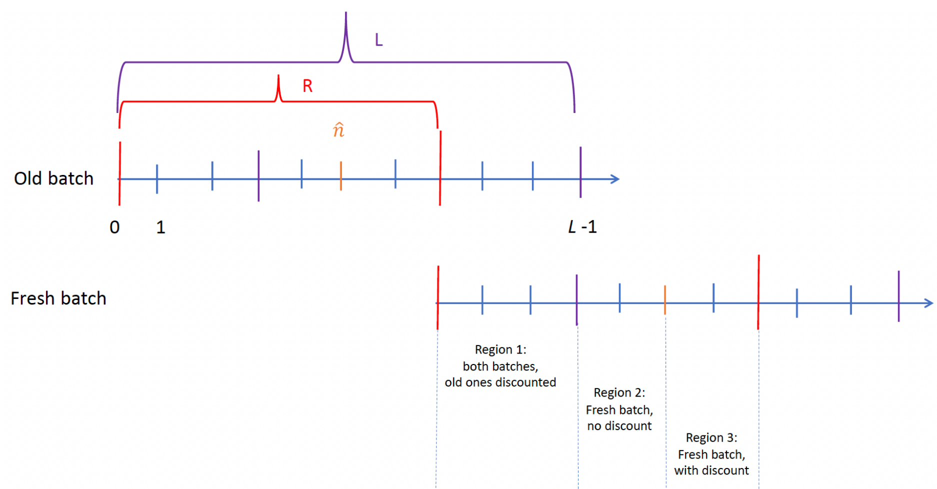
Symbol	Definition
L	Product shelf life
R	Length of replenishment cycle
$n \in \{0, ..., L-1\}$	Age of product unit
$\hat{n} \in \{0, ..., L\}$	Discount age
$x \in \{0, ..., R-1\}$	Cycle period index
$\tau \in \{0, 1, 2, ...\}$	Period index
$t \in \{1, 2, ...\}$	Replenishment cycle index
N^τ	Store traffic in period τ
$\bar{N}$	Average store traffic
q_n	Quality of product of age n
p	Base selling price
γ	Discount rate on the selling price
c	Variable unit cost
h	Per-period unit inventory holding cost
h'	Unit handling cost
w_L	Disposal cost for expired units (after L periods)
w_R	Disposal cost for units discarded after R periods
θ	Consumers' valuation of product quality
β	Proposition of passive consumers
$\delta \in \{\texttt{EA}, \texttt{LO}, \texttt{LF}\}$	Display setting
$P_i^x(\hat{n}, \gamma)$	Proportion of consumers of type $i \in \{O, F, OF, FO\}$ in cycle period x
$F^x(\hat{n}, \gamma, \delta)$	Proportion of consumers who buy fresh units in cycle period x when both batches are in stock
$O^x(\hat{n}, \gamma, \delta)$	Proportion of consumers who buy old units in cycle period x, when both batches are in stock
Q^t	Quantity of fresh units received at the start of t^{th} replenishment cycle
y_x^τ, y_{x+R}^τ	Inventory level of units of age x and $x+R$ at the start of period τ, where $x = mod(\tau, R)$
$\mathbf{y}^\tau = (y_x^\tau, y_{x+R}^\tau)$	Inventory vector at the start of period τ
$S_x^\tau(\hat{n}, \gamma, \delta; \mathbf{y}^\tau)$	Sales of products of age x (fresh units) in period τ
$S_{x+R}^\tau(\hat{n}, \gamma, \delta; \mathbf{y}^\tau)$	Sales of products of age $x+R$ (old units) in period τ
$\Pi_{FS}^t(Q^t, \hat{n}, \gamma, \delta)$	Retailer's expected profit in the t^{th} replenishment cycle under the full-shelf-life policy
$\pi_{FS}^x(\hat{n}, \gamma, \delta)$	Retailer's expected profit in the x^{th} cycle period under the full-shelf-life policy
$\Pi_{SB}^t(Q^t)$	Retailer's expected profit in the t^{th} replenishment cycle under the single-batch policy
π_{SB}^x	Retailer's expected profit in the x^{th} cycle period under the single-batch policy

Table 6 Notation

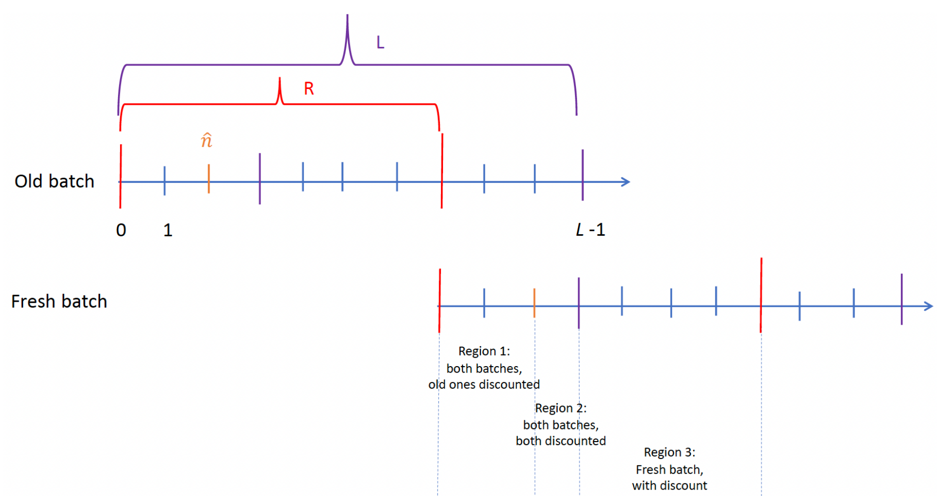
The first seven rows in Table 7 refer to periods when both fresh and old batches are on the shelves. The first row is the case where neither batch is discounted in cycle period x, rows 2-5 correspond to cases where the fresh batch is sold at the full price p and the old batch is discounted, and rows 6-7 are for cases where both batches are discounted. The last row is for cycle periods when only the fresh batch is on the shelves selling at full price.



(a) Case 1: $L = 10, R = 7, \hat{n} = 8, R - 1 < \hat{n} < L - 1$



(b) Scenario 2: $L = 10, R = 7, \hat{n} = 5, L - R - 1 < \hat{n} \leq R - 1$



(c) Scenario 3: $L = 10, R = 7, \hat{n} = 2, \hat{n} \leq L - R - 1$

Figure 5 **Batches on the store shelves in Scenarios 1 to 3**

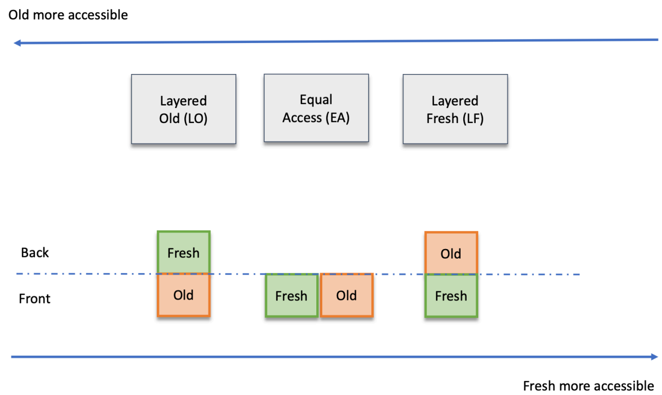
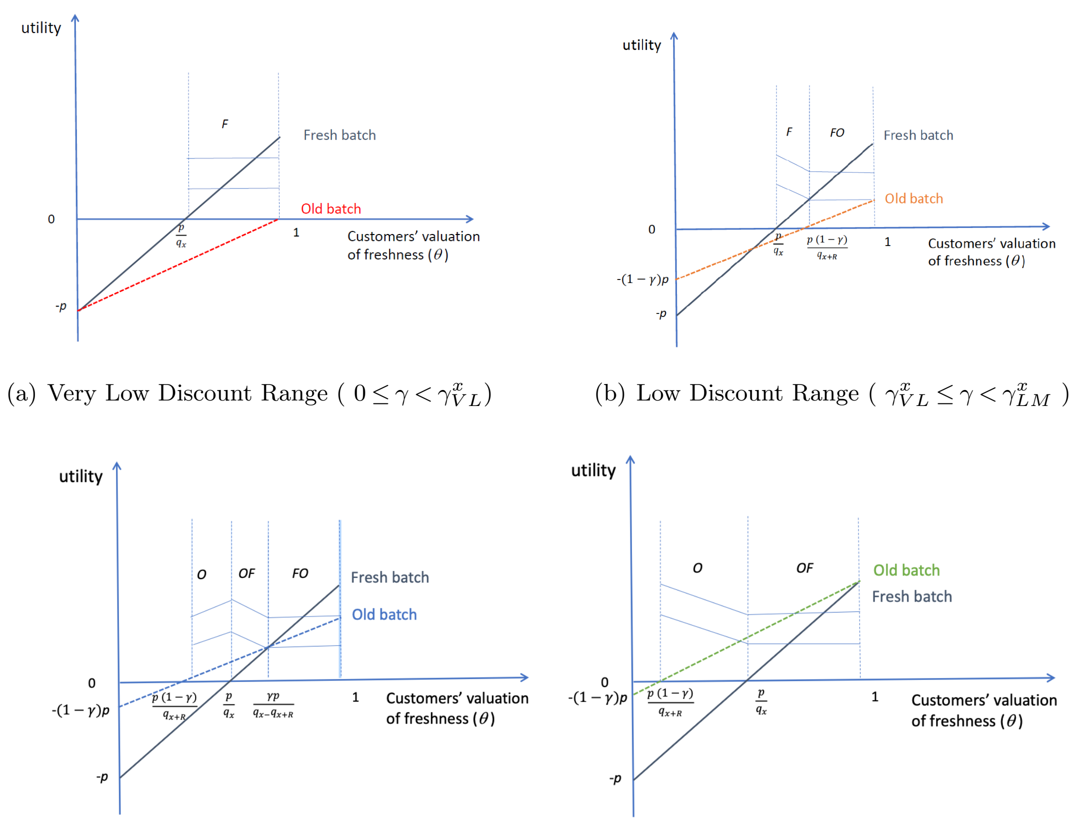


Figure 6 **Display settings**



(a) Very Low Discount Range ($0 \leq \gamma < \gamma^x_{VL}$)

(b) Low Discount Range ($\gamma^x_{VL} \leq \gamma < \gamma^x_{LM}$)

(c) Medium Discount Range ($\gamma^x_{LM} \leq \gamma < \gamma^x_{MH}$)

(d) High Discount Range ($\gamma^x_{MH} \leq \gamma < 1$)

Figure 7 **Consumer Types for Different Discount Ranges for ($x < \hat{n} \leq x + R$)**

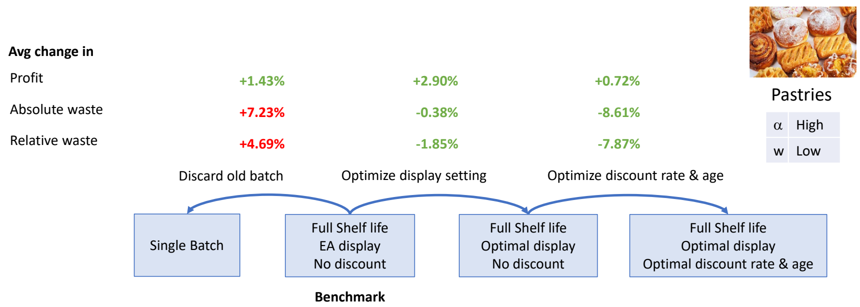


Figure 8 Average percentage change values in profit and waste for fresh pastries/pastries

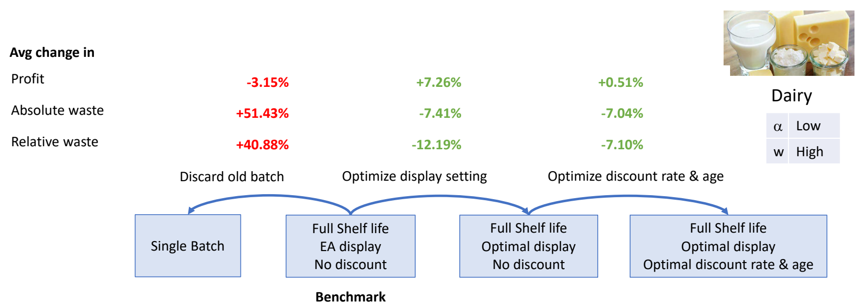


Figure 9 Average percentage change values in profit and waste for dairy products

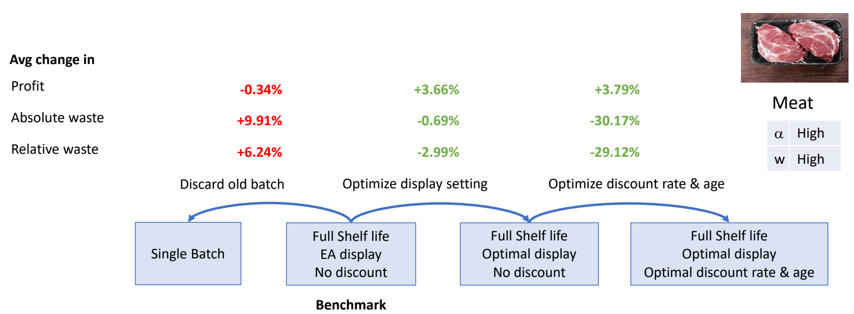


Figure 10 Average percentage change values in profit and waste for meat/fish/salads

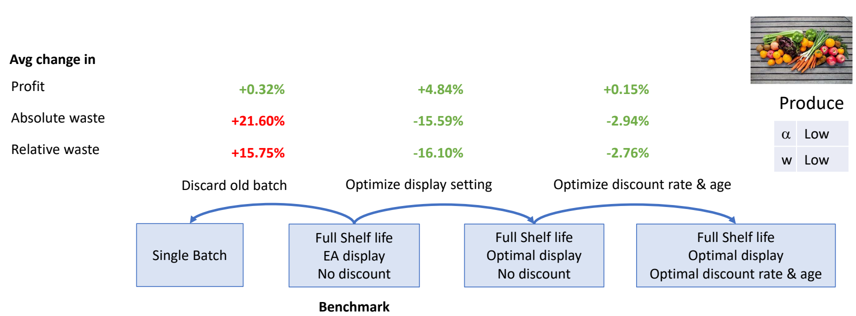


Figure 11 Average percentage change values in profit and waste for meat/fish/salads

Period	Discount age	γ range	$P_O^x(\hat{n},\gamma)$	$P_F^x(\hat{n},\gamma)$	$P_{OF}^x(\hat{n},\gamma)$	$P_{FO}^x(\hat{n},\gamma)$
$x \in \{0,...,L-R-1\}$	$\hat{n} > x+R$	NA	0	$\min\left\{\frac{p}{q_{x+R}},1\right\}-\frac{p}{q_x}$	0	$1-\min\left\{\frac{p}{q_{x+R}},1\right\}$
	$x < \hat{n} \le x+R$	$[0,\gamma_{VL}^x)$	0	$1-\frac{p}{q_x}$	0	0
	$x < \hat{n} \le x+R$	$[\gamma_{VL}^x,\gamma_{LM}^x)$	0	$\frac{p(1-\gamma)}{q_{x+R}}-\frac{p}{q_x}$	0	$1-\frac{p(1-\gamma)}{q_{x+R}}$
	$x < \hat{n} \le x+R$	$[\gamma_{LM}^x,\gamma_{MH}^x)$	$\frac{p}{q_x}-\frac{(1-\gamma)p}{q_{x+R}}$	0	$\frac{p\gamma}{q_x-q_{x+R}}-\frac{p}{q_x}$	$1-\frac{p\gamma}{q_x-q_{x+R}}$
	$x < \hat{n} \le x+R$	$[\gamma_{MH}^x,1)$	$\frac{p}{q_x}-\frac{(1-\gamma)p}{q_{x+R}}$	0	$1-\frac{p}{q_x}$	0
	$\hat{n} \le x$	$[0,\gamma_{VL}^x)$	0	$1-\frac{p}{q_x}$	0	0
	$\hat{n} \le x$	$(\gamma_{VL}^x,1]$	0	$\frac{p(1-\gamma)}{q_{x+R}}-\frac{p(1-\gamma)}{q_x}$	0	$1-\frac{p(1-\gamma)}{q_{x+R}}$
$x \in \{L-R,...,R-1\}$	$\hat{n} > x$	NA	0	$1-\frac{p}{q_x}$	0	0

$\gamma_{VL}^x = \left(1-\frac{q_{x+R}}{p}\right)^+$, $\gamma_{LM}^x = 1-\frac{q_{x+R}}{q_x}$, $\gamma_{MH}^x = \min\left\{\frac{q_x-q_{x+R}}{p},1\right\}$

Table 7 $P_O^x, P_F^x, P_{OF}^x, P_{FO}^x$ **calculations in cycle period** $x \in \{0,...,R-1\}$**.**

Display Setting	Purchasing proportions	Very low discount ($0 \le \gamma < \gamma_{VL}^x$)	Low discount ($\gamma_{VL}^x \le \gamma < \gamma_{LM}^x$)	Medium discount ($\gamma_{LM}^x \le \gamma < \gamma_{MH}^x$)	High discount ($\gamma_{MH}^x \le \gamma < 1$)
EA	Fresh with both batches in stock	P_F	P_F+P_{FO}	P_{FO}	0
	Old with both batches in stock	0	0	P_O+P_{OF}	P_O+P_{OF}
	Fresh with only fresh batch in stock	P_F	P_F+P_{FO}	$P_{FO}+P_{OF}$	P_{OF}
	Old with only old batch in stock	0	P_{FO}	$P_O+P_{FO}+P_{OF}$	P_O+P_{OF}
LO	Fresh with both batches in stock	P_F	$P_F+(1-\beta)P_{FO}$	$(1-\beta)P_{FO}$	0
	Old with both batches in stock	0	βP_{FO}	$P_O+\beta P_{FO}+P_{OF}$	P_O+P_{OF}
	Fresh with only fresh batches in stock	P_F	P_F+P_{FO}	$P_{FO}+P_{OF}$	P_{OF}
	Old with only old batches in stock	0	P_{FO}	$P_O+P_{FO}+P_{OF}$	P_O+P_{OF}
LF	Fresh with both batches in stock	P_F	P_F+P_{FO}	$P_{FO}+\beta P_{OF}$	βP_{OF}
	Old with both batches in stock	0	0	$P_O+(1-\beta)P_{OF}$	$P_O+(1-\beta)P_{OF}$
	Fresh with only fresh batches in stock	P_F	P_F+P_{FO}	$P_{FO}+P_{OF}$	P_{OF}
	Old with only old batches in stock	0	P_{FO}	$P_O+P_{FO}+P_{OF}$	P_O+P_{OF}

Table 8 Purchasing proportions for all display setting and discount combinations