ONLINE APPENDIX

Strategy-proof Multi-Issue Mediation: An Application to Online Dispute Resolution

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1 Discussion and Extensions

In this section, we provide a general discussion of our main model in light of the results obtained in the paper. To this end, we elaborate on some of our essential modeling assumptions, discuss the role they play in driving the positive results of our paper, and offer directions in which they can be extended to cases not covered in the main exposition. We also consider how one can go about formulating the mediation problem in a standard Bayesian setting such as that of Myerson and Sattertwaite (1983) and offer a reconciliation of the possibility results in our setup with the impossibility result in their setting.

1.1 Modeling conflicting preferences

When describing a dispute, using diametrically opposed preferences over alternatives is intuitive because it resembles the standard bargaining problem, which is modeled as a zero-sum game. It is also unavoidable when there are just two available alternatives. However, it is conceivable that many other situations where preferences are not necessarily diametrically opposed could also depict a dispute where there are more than two alternatives. Consider, for example, a case where the set of available alternatives (other than the outside option) is $\mathbf{X} = \{x_1, x_2, x_3, x_4, x_5\}$ and the negotiators' preferences are as follows:

Negotiator 1: $x_1 \; x_2 \; x_3 \; x_4 \; x_5$

Negotiator 2: $x_3 \; x_5 \; x_4 \; x_2 \; x_1$

These preferences are not diametrically opposed, but they are certainly conflicting to some extent, as the agents cannot agree on their best alternative. Notice, however, that alternatives

x_4 and x_5 are (Pareto) dominated by x_3. So, if negotiators' preferences over bundles are monotonic and selecting an efficient outcome by the mediation protocol is desired, then the presence of these two alternatives is irrelevant to the problem and can be eliminated from the preferences. Thus, this particular dispute problem can be transformed into a reduced problem where the only available alternatives are x_1, x_2, and x_3, and the negotiators' preferences over these three are diametrically opposed. This observation easily generalizes to any set of alternatives, any preference profile, and multi-issue mediation problems (under monotonic preferences over bundles). Namely, eliminating inefficient alternatives yields a reduced dispute with diametrically opposed preferences.

In this section, we will generalize our arguments for single-issue mediation problems but for any discrete set of alternatives and preference profiles. Let $\mathbf{X}$ be a nonempty set of available alternatives and Θ be the set of all complete, transitive, and antisymmetric preference relations on $\mathbf{X}$. Define $max(\theta)$ as the maximal element of the preference ordering $\theta \in \Theta$, namely if $x^\star = max(\theta)$, then $x^\star \ \theta \ x$ for all $x \in \mathbf{X} \setminus \{x^\star\}$. Therefore, a ***two-person, single-issue dispute*** (dispute in short) problem is a list $D = (\theta_1, \theta_2, \mathbf{X})$ where $\theta_i \in \Theta$ for $i = 1, 2$ and $max(\theta_1) \neq max(\theta_2)$.

For any nonempty subset $\widetilde{\mathbf{X}} \subseteq \mathbf{X}$, let $\theta|_{\widetilde{\mathbf{X}}}$ denote the restriction of the preference ordering $\theta \in \Theta$ on $\widetilde{\mathbf{X}}$. Therefore, define $\widetilde{D} = (\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\mathbf{X}})$ to be a dispute reduced from $D = (\theta_1, \theta_2, \mathbf{X})$ whenever $\widetilde{\mathbf{X}} \subseteq \mathbf{X}$ and $\widetilde{\theta}_i = \theta_i|_{\widetilde{\mathbf{X}}}$ for $i = 1, 2$.

Lemma A.1.1. *By eliminating all the Pareto inefficient alternatives, any dispute D can be reduced to an equivalent dispute $\widetilde{D}$ where the negotiators' preferences are diametrically opposed.*

Proof. Let $\widetilde{\mathbf{X}} \subseteq \mathbf{X}$ be the set of alternatives that survive the elimination of Pareto inefficient alternatives. That is, none of the alternatives in $\widetilde{\mathbf{X}}$ is Pareto inefficient. Renumber the elements in $\widetilde{\mathbf{X}}$, and suppose, without loss of generality, that $\widetilde{\mathbf{X}} = \{x_1, ..., x_m\}$ where $m \geq 2$, and negotiator 1 ranks alternatives as $x_k \ \widetilde{\theta}_1 \ x_{k+1}$. If x_m is not the best alternative for $\widetilde{\theta}_2$ on $\widetilde{\mathbf{X}}$, then there must exist some x_k where $k < m$ such that $x_k \ \widetilde{\theta}_2 \ x_m$. But this contradicts the assumption that x_m is not Pareto inefficient. Thus, negotiator 2 must rank x_m as the top alternative. With similar reasoning, if x_{m-1} is not negotiator 2's second-best alternative, then it must be Pareto inefficient, contradicting the assumption that x_{m-1} survives after the deletion of Pareto inefficient alternatives. Iterating this logic implies that the rankings of the negotiators must be diametrically opposed. □

1.2 More than two issues

Our two-issue model with a single main issue and a single supplementary issue is without loss of generality. If there are more than two issues in the dispute, then we can regroup these issues under two types of categories depending on whether an issue has certain or uncertain gains from mediation. In particular, let category-**X** be the collection of issues that exhibit uncertain gains from mediation (i.e., non-negotiable alternatives are the negotiators' private information), and category-**Y** be the collection of issues that exhibit certain gains (i.e., it is common knowledge that all alternatives in these issues are negotiable for all types). Under this regrouping, each negotiator now faces a vector of alternatives for each category. The negotiators' preferences over these vectors (of alternatives) need not be diametrically opposed in general. However, as long as the negotiators' preferences are strict and monotonic, by applying the transformation discussed in Section 5.1, we can eliminate all inefficient vectors. This brings us back to an environment analogous to our main model, in which preferences over vectors are diametrically opposed.

When there are multiple parties involved in a dispute, as would be the case for community/public disputes, we can similarly regroup them to be represented by either negotiator, effectively treating them as clones of the two negotiators. Nevertheless, there might be cases where negotiators' preferences are extremely dispersed, and so grouping them into two "representative" agents is not feasible. Such mediation environments are both practically and theoretically more complex than the one we study here and in need of further research.

1.3 Reconciliation with the negative results in Bayesian settings

The influential work of Myerson and Sattertwaite (1983) [henceforth MS] is an important milestone in showing the difficulty of efficient trade in bargaining problems with asymmetric information. It is useful to discuss the underlying factors that are absent in the MS model, which may account for the possibility results in our model under dominant strategies. "Budget balancedness" is automatically satisfied in our setup, and so, "budget imbalance" is not the driving force for our possibility result. The MS model lacks a second issue with a rich set of efficient alternatives.

The mechanism design problem in MS concerns a bilateral trade between a buyer and a seller, who have private information about the valuations of a good. The mechanism has two components: the probability of trade, p, and the transfer, x, both of which are functions of the traders' reports. If no trade occurs, then $x = p = 0$ (the outside option), and so both traders receive zero utility. The utility functions are $u_b = v_b p - x$ for the buyer and $u_s = x - v_s p$ for the seller, where the valuations v_b, v_s are the traders' private information.

The buyer (seller) prefers lower (higher) transfers in MS, and it is a priori uncertain whether a transfer leading to a mutually beneficial trade exists. Moreover, the quasi-linear utility functions in MS also satisfy the monotonicity and the quid pro quo assumptions. Despite all these similarities, the impossibility of MS is not at odds with our results because the MS model translates as a single-issue mediation problem in our setup, where the transfer is the issue with uncertain gains. Efficiency in MS implies that the probability of efficient trade is generically either 0 or 1, depending on whether or not the buyer's valuation is higher than the seller's valuation. This means that the probability of trade cannot be considered as a second issue since we require the second issue to have at least as many alternatives as the main issue.

What is needed for a possibility is a new issue with a sufficient number of efficient alternatives, as in the case of issue $\mathbf{Y}$ in our model. To provide an illustration of the above points, in the following example, we offer a simple adaptation of the MS setup in our model and demonstrate how one can overcome the impossibility by adding an extra issue:

Example 1 (Possibility in the augmented MS framework): Suppose that the seller and the buyer now negotiate not only over the terms of trade but also over the division of a commonly known surplus of size 0.4, which is relevant only if the buyer and seller agree on the terms of the trade. We refer to the latter as issue $\mathbf{Y}$. The valuations of the good to the buyer and the seller are v_b and v_s, respectively. We assume that each negotiator knows her valuation and believes that the opponent's valuation is drawn from $[0, 1]$. The mediator privately solicits the traders' valuations and recommends a quadruple (p, x, y_s, y_b), where p denotes the probability of trade, x is the transfer, and y_s, and y_b are respectively the seller's and the buyer's share of the surplus. The preferences of the traders are as follows: $U_b = pv_b - x + y_b$ and $U_s = x - pv_s + y_s$. Suppose that each trader has only two types, $v_b, v_s \in \{0.2, 0.6\}$.

	$v_b = 0.6$		$v_b = 0.2$	
$v_s = 0.6$	$p = 1$	$y_s = 0$	*No*	$y_s = 0$
	$x = 0.6$	$y_b = 0.4$	*trade*	$y_b = 0$
$v_s = 0.2$	$p = 1$	$y_s = 0.2$	$p = 1$	$y_s = 0.4$
	$x = 0.4$	$y_b = 0.2$	$x = 0.2$	$y_b = 0$

Efficiency implies $p = 1$ and $y_b + y_s = 0.4$ if $v_b \geq v_s$, and $p = y_b = y_s = 0$ if $v_s < v_b$. Individual rationality implies that the traders' utilities are non-negative. The following mechanism is strategy-proof, efficient, and individually rational:[1]

[1]The seller of type $v_s = 0.2$ has no incentive to mimic type $v_s = 0.6$. This is true because the seller's payoff under truth-telling (which is 0.4 regardless of the buyer's type) is higher than or equal to her deviation payoffs 0.4 (if the buyer is of type $v_b = 0.6$) and 0 (if the buyer is of type $v_b = 0.2$). Similarly, the seller of type $v_s = 0.6$

2. Proofs of the Results

2.1 Sketch of the proof of Theorem 1

The proof follows four main steps: (1) establishing an injective and decreasing function $\mathbf{y}$ from $\mathbf{X}$ to $\mathbf{Y}$, and thus the set $\mathbf{B}^{\mathbf{y}}$; (2) proving that each entry of the lower half of the matrix f comes from the set $\mathbf{B}^{\mathbf{y}}$; (3) establishing the binary relation $\trianglerighteq$ over $\mathbf{X}$ that is transitive and antisymmetric; and (4) proving that each entry of the lower half of the matrix f is in fact the maximal element of a particular subset of $\mathbf{X}$ with respect to the partial order $\trianglerighteq$.

All four steps utilize the following core idea, which we call the **weak axiom of revealed precedence (WARP)**: If two distinct alternatives x, x' in issue $\mathbf{X}$ are in the zone of mutual gain at some type profile, and f suggests a bundle with x at that profile, then it cannot be the case that f suggests x' at another type profile where both x and x' are in the zone of mutual gain. Therefore, whenever the zone of mutual gain is non-empty, a strategy-proof, efficient, and individually rational mediation mechanism behaves as if it is a single-valued "choice mechanism" that satisfies the weak axiom of revealed preference (see Rubinstein 2012).

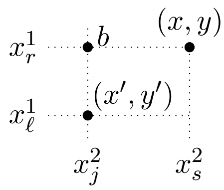


Figure 1

Suppose WARP does not hold and two distinct alternatives x and x' of issue $\mathbf{X}$ are in the zone of mutual gain at some type profiles as in Figure 1. By individual rationality of the mechanism, bundle b is acceptable for type x_r^1 of Negotiator 1. It is also acceptable for a more *accepting* type x_ℓ^1. Strategy-proofness implies that type x_r^1 weakly prefers bundle b to (x', y'), i.e., $u_1(b, x_r^1) \geq u_1(x', y', x_r^1)$ for all $u_1 \in \mathcal{D}_1$. The same inequality holds for type x_ℓ^1 as well by the invariance property. The converse of the last inequality is also true (i.e., $u_1(x', y', x_\ell^1) \geq u_1(b, x_\ell^1)$ for all $u_1 \in \mathcal{D}_1$) by strategy-proofness. Therefore, $b = (x', y')$ since u_1 is strict. Repeating the symmetric arguments for Negotiator 2 suffices to show $b = (x, y)$, implying that all these three bundles are the same, which contradicts our presumption that x and x' are distinct.

We set $\mathbf{y}\left(f_{k,k}^{\mathbf{X}}\right) = f_{k,k}^{\mathbf{Y}}$ for $k \in \mathcal{I}$, and use strategy-proofness of the mechanism and the bundles on the main and second diagonals to prove that the mapping $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ is injective and decreasing. In particular, at any entry on the second diagonal (i.e., $f_{k+1,k}$), individual rationality

has no incentive to mimic type $v_s = 0.2$ because her payoff under truth-telling and deviation is 0. Symmetric arguments apply to the buyer.

and efficiency of f, and the deal-breaker property of the utilities imply that f recommends either x_k or x_{k+1}. Suppose, without loss of generality, that $f^{\mathbf{X}}_{k+1,k} = x_k$. Because utilities are monotonic, strategy-proofness requires $f_{k+1,k} = f_{k,k}$. Finally, because Negotiator 2 prefers x_{k+1} (i.e., $f^{\mathbf{X}}_{k+1,k+1}$) to x_k (i.e., $f^{\mathbf{X}}_{k,k}$), monotonicity of preferences and strategy-proofness require that Negotiator 2 must prefer $\mathbf{y}(f^{\mathbf{X}}_{k,k})$ to $\mathbf{y}(f^{\mathbf{Y}}_{k+1,k+1})$, implying that $\mathbf{y}$ must be injective and decreasing. We denote the set of all bundles on the main diagonal by $\mathbf{B}^{\mathbf{y}}$ (Step 1).

Similar arguments and WARP imply that the lower half of the matrix f consists of the logrolling bundles, $\mathbf{B}^{\mathbf{y}}$ (Step 2). Much like the case in rationalizable choice rules (see Rubinstein 2012), WARP implies that f behaves as if it follows a binary relation (which we call a precedence order) over the set of alternatives in the main issue $\mathbf{X}$ in such a way that it always picks the alternative in issue $\mathbf{X}$ that is revealed to be "better" than any other alternative in the zone of mutual gain. Therefore, we construct the partial order as follows. Take any type profile (x^1_ℓ, x^2_j) where $j \leq \ell$, and consider the set of all alternatives in the zone of mutual gain at that profile (i.e., $[x_j, x_\ell]$). We say $f^{\mathbf{X}}_{\ell,j} \trianglerighteq x$ whenever $x \in [x_j, x_\ell]$ (Step 3).

It follows from construction and WARP that the binary relation $\trianglerighteq$ is antisymmetric and transitive, and each entry of the lower half of the matrix is indeed the maximal element of the zone of mutual gain at the type profile corresponding to that entry (Step 4). By individual rationality and the deal-breaker property of the preferences, f must choose ϕ above the diagonal when the zone of mutual gain is empty.

2.2 Formal Proof of Theorem 1

We start with establishing a series of preliminary results. For any mechanism f and type profile $t = (x^1_\ell, x^2_j) \in \mathbf{T}$, we let $f(t) = (f^{\mathbf{X}}_t, f^{\mathbf{Y}}_t) = (f^{\mathbf{X}}_{\ell,j}, f^{\mathbf{Y}}_{\ell,j}) = f_{\ell,j}$.

Lemma 1. *If mechanism f is efficient and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then for any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ with $j \leq \ell$ we have $f^{\mathbf{X}}_t \in [x_j, x_\ell]$, where $[x_j, x_\ell] = \mathbf{N}(x^1_\ell) \cap \mathbf{N}(x^2_j)$ denoting the set of all mutually negotiable alternatives at type profile t.*

Proof. Take any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ with $j \leq \ell$, and so $\mathbf{N}(x^1_\ell) = \{x_1, \ldots, x_j, \ldots, x_\ell\}$ and $\mathbf{N}(x^2_j) = \{x_j, \ldots, x_\ell, \ldots, x_m\}$. Therefore, $\mathbf{N}(x^1_\ell) \cap \mathbf{N}(x^2_j) = [x_j, x_\ell]$. Because f is efficient over $\mathcal{D}$ and negotiators' utility functions $u_1 \in \mathcal{D}_1$ and $u_2 \in \mathcal{D}_2$ satisfy deal-breaker property, $f(t) \neq \phi$. Moreover, we must have $f^{\mathbf{X}}_t \notin \{x_1, \ldots, x_{j-1}, x_{\ell+1}, \ldots, x_m\}$ because f is individually rational over $\mathcal{D}$, alternatives in $\{x_{\ell+1}, \ldots, x_m\}$ are non-negotiable for type x^1_ℓ of Negotiator 1, alternatives in $\{x_1, \ldots, x_{j-1}\}$ are non-negotiable for type x^2_j of Negotiator 2, and every $(u_1, u_2) \in \mathcal{D}$ satisfy deal-breaker property. Hence, it must be that $f^{\mathbf{X}}_t \in [x_j, x_\ell]$. $\square$

Lemma 2 (WARP). *Assume that mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$. For any $t, t' \in \mathbf{T}$ and distinct alternatives $x, x' \in \mathbf{X}$, satisfying $x, x' \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$ for $i = 1, 2$, if $f_t^{\mathbf{X}} = x$, then $f_{t'}^{\mathbf{X}} \neq x'$.*

Proof. Assume that f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$, and suppose for a contradiction that there are $t = (t_1, t_2) = (x_\ell^1, x_j^2) \in \mathbf{T}$, $t' = (t'_1, t'_2) = (x_{\ell'}^1, x_{j'}^2) \in \mathbf{T}$, and distinct alternatives $x_k, x_{k'} \in \mathbf{X}$, satisfying $x_k, x_{k'} \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$ for $i = 1, 2$, such that $f_t^{\mathbf{X}} = x_k$ and $f_{t'}^{\mathbf{X}} = x_{k'}$. Suppose, without loss of generality, that $\ell < \ell'$ (if $\ell = \ell'$, then just skip the steps involving bundle b and Negotiator 1 below). Consider the less accepting type of Negotiator 1 (i.e., x_ℓ^1) and type profile $(x_\ell^1, x_{j'}^2)$. It must be the case that $j' \leq \ell$. To prove this last claim, suppose for a contradiction that $j' > \ell$. Because $x_k \in \mathbf{N}(x_\ell^1)$ and $x_k \in \mathbf{N}(x_{j'}^2)$ we have $k \leq \ell$ and $k \geq j'$. These two inequalities imply $j' \leq k \leq \ell$, contradicting with $j' > \ell$.

Therefore, Lemma 1 requires that $f(t_1, t'_2) = b$ for some $b \in \mathbf{B} \setminus \{\phi\}$. Note that all three bundles (i.e., b, $f(t)$ and $f(t')$) are acceptable by two types of Negotiator 1. Bundles $f(t)$ and $f(t')$ are acceptable by both because $x_k, x_{k'} \in \mathbf{N}(t_1) \cap \mathbf{N}(t'_1)$. Bundle b is also acceptable by both types of Negotiator 1 because f is individually rational and suggests b at type profile (t_1, t'_2), meaning b is acceptable for type t_1, and t'_1 is more accepting than t_1.

Strategy-proofness requires $u_1(b, t_1) \geq u_1(f(t'), t_1)$ for all $u_1 \in \mathcal{D}_1$, and type-invariance of u_1 implies $u_1(b, t'_1) \geq u_1(f(t'), t'_1)$ because both b and $f(t')$ are acceptable by types t_1 and t'_1. Strategy-proofness also requires $u_1(f(t'), t'_1) \geq u_1(b, t'_1)$ for all $u_1 \in \mathcal{D}_1$. The last two inequalities imply $b = f(t')$ because u_1's are strict. Moreover, strategy-proofness requires $u_2(f(t), t_2) \geq u_2(f(t'), t_2)$ for all $u_2 \in \mathcal{D}_2$ because $f(t_1, t'_2) = b$ and $b = f(t')$, and type-invariance of u_2's implies $u_2(f(t), t'_2) \geq u_2(f(t'), t'_2)$ because $f(t)$ and $f(t')$ are acceptable by both t_2 and t'_2. (This last claim is true because $f_{t_1, t'_2}^{\mathbf{X}} = x_{k'}$ and $x_k, x_{k'} \in \mathbf{N}(t_2) \cap \mathbf{N}(t'_2)$). Strategy-proofness also requires $u_2(f(t'), t'_2) \geq u_2(f(t), t'_2)$ for all $u_2 \in \mathcal{D}_2$. Because u_2's are strict, the last two inequalities imply $f(t) = f(t')$, contradicting that x_k and $x_{k'}$ are distinct alternatives. □

Lemma 3. *If mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then there exists an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ such that $f_{k,k} = (x_k, \mathbf{y}(x_k))$ for every $k \in \mathcal{I}$.*

Proof. Assume that mechanism f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$. Lemma 1 implies $f_{k,k}^{\mathbf{X}} = x_k$ for any $k \in \mathcal{I}$ and $f_{k+1,k}^{\mathbf{X}} \in \{x_k, x_{k+1}\}$ whenever $k \neq m$. Furthermore, it must be that $f_{k+1,k} \in \{f_{k,k}, f_{k+1,k+1}\}$. To prove this suppose, for a contradiction, that $f_{k+1,k} \notin \{f_{k,k}, f_{k+1,k+1}\}$. If $f_{k+1,k}^{\mathbf{X}} = x_k$, which is equal to $f_{k,k}^{\mathbf{X}}$, then strategy-proofness and monotonicity of $u_1 \in \mathcal{D}_1$ require $f_{k+1,k}^{\mathbf{Y}} = f_{k,k}^{\mathbf{Y}}$, namely $f_{k+1,k} = f_{k,k}$.

On the other hand, if $f^{\mathbf{X}}_{k+1,k} = x_{k+1}$, which is equal to $f^{\mathbf{X}}_{k+1,k+1}$, then strategy-proofness and monotonicity of $u_2 \in \mathcal{D}_2$ require $f^{\mathbf{Y}}_{k+1,k} = f^{\mathbf{Y}}_{k+1,k+1}$, namely $f_{k+1,k} = f_{k+1,k+1}$, leading to the desired contradiction.

Next, we set $\mathbf{y}(x_k) = f^{\mathbf{Y}}_{k,k}$ for $k \in \mathcal{I}$. To prove that $\mathbf{y}$ is injective and decreasing, we prove the following: For any $k \in \mathcal{I} \setminus \{m\}$, if $\mathbf{y}(x_k) = y_\ell$ and $\mathbf{y}(x_{k+1}) = y_{\ell'}$, then $\ell' < \ell$. Suppose for a contradiction that there is some k such that $\mathbf{y}(x_k) = y_\ell$, $\mathbf{y}(x_{k+1}) = y_{\ell'}$ and $\ell' \geq \ell$. Recall that $f_{k+1,k} \in \{f_{k,k}, f_{k+1,k+1}\}$. If $f_{k+1,k} = f_{k+1,k+1}$, then strategy-proofness require that $u_1(f_{k+1,k+1}, t_1) \geq u_1(f_{k,k}, t_1)$ for all $u_1 \in \mathcal{D}_1$, where $t_1 = x^1_{k+1}$. Because $f_{k+1,k+1} = (x_{k+1}, \mathbf{y}(x_{k+1})) = (x_{k+1}, y_{\ell'})$ and $f_{k,k} = (x_k, \mathbf{y}(x_k)) = (x_k, y_\ell)$, the last inequality implies $u_1(x_{k+1}, y_{\ell'}, t_1) \geq u_1(x_k, y_\ell, t_1)$, contradicting that u_1's are decreasing and $\ell' \geq \ell$. However, if $f_{k+1,k} = f_{k,k}$, then strategy-proofness require $u_2(f_{k,k}, t_2) \geq u_2(f_{k+1,k+1}, t_2)$ for all $u_2 \in \mathcal{D}_2$. This inequality implies that $u_2(x_k, y_\ell, t_2) \geq u_2(x_{k+1}, y_{\ell'}, t_2)$, contradicting that u_2 is increasing and $\ell' \geq \ell$. Hence, it must be that $\ell' < \ell$, so $\mathbf{y}$ is injective and decreasing. □

Lemma 4. *If mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then for any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$, we have*

$$f(t) = f(x^1_\ell, x^2_j) = \begin{cases} \left(x^*_{[x_j,x_\ell]}, \mathbf{y}\left(x^*_{[x_j,x_\ell]}\right)\right) & \text{if } j \leq \ell \\ \phi & \text{otherwise;} \end{cases}$$

where $x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$ and $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ is an injective and decreasing function.

Proof. Assume that mechanism f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$. Lemma 3 ensures that there is an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ such that $f_{k,k} = (x_k, \mathbf{y}(x_k))$ for every $k \in \mathcal{I}$. Let $\mathbf{B}^{\mathbf{y}} = \{(x_k, \mathbf{y}(x_k)) | x_k \in \mathbf{X}\}$ be the set of all bundles on the main (first) diagonal. Take any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ where $\ell, j \in \mathcal{I}$. If $\ell < j$, then efficiency and individual rationality of f and the deal-breaker property of all $(u_1, u_2) \in \mathcal{D}$ require $f(t) = \phi$.

Therefore, we assume $j < \ell$ for the rest of the proof, and prove that $f_{\ell,j} = (x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})) \in \mathbf{B}^{\mathbf{y}}$ for some $x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$. To show this, we suppose, for a contradiction, that there exists such j and ℓ such that either $f_{\ell,j} \notin \mathbf{B}^{\mathbf{y}}$ or $x^*_{[x_j,x_\ell]} \notin [x_j, x_\ell]$ holds. Lemma 1 requires that $f^{\mathbf{X}}_{\ell,j} = x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$. Therefore, it must be that $x^*_{[x_j,x_\ell]} = x_k$ for some $j \leq k \leq \ell$ and $f_{\ell,j} \notin \mathbf{B}^{\mathbf{y}}$. Lemma 2 requires $f^{\mathbf{X}}_{s,r} = x_k$ for all $k \leq s \leq \ell$ and $j \leq r \leq k$ because $f^{\mathbf{X}}_{k,k} = x_k$ (by Lemma 3), $x_k \in \mathbf{N}(x^1_s) \cap \mathbf{N}(x^2_r)$ for all s and r satisfying $k \leq s$ and $r \leq k$, and $x^*_{[x_j,x_\ell]} = x_k$. Consider now the type profile (x^1_k, x^2_j). Strategy-proofness of f and monotonicity of any $u_2 \in \mathcal{D}_2$ require that $f_{k,j} = f_{k,k}$ because $f^{\mathbf{X}}_{k,j} = x_k = f^{\mathbf{X}}_{k,k}$. Similarly, strategy-proofness of f and monotonicity of any $u_1 \in \mathcal{D}_1$ require $f_{k,j} = f_{\ell,j}$ because $f^{\mathbf{X}}_{\ell,j} = x_k = f^{\mathbf{X}}_{k,j}$. Thus, it must be that $f_{\ell,j} = f_{k,k} = (x_k, \mathbf{y}(x_k))$, contradicting that $f_{\ell,j} \notin \mathbf{B}^{\mathbf{y}}$. Hence, we have $f_{\ell,j} = (x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})) \in \mathbf{B}^{\mathbf{y}}$ for some $x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$. $\square$

Lemma 5. *If mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then there exists a partial order $\trianglerighteq$ on $\mathbf{X}$ such that for any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ with $j \leq \ell$, $f^{\mathbf{X}}_{\ell,j} \trianglerighteq x$ for all $x \in [x_j, x_\ell]$.*

Proof. Assume that mechanism f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$. We construct $\trianglerighteq$ as follows: For any $x_k, x_{k'} \in \mathbf{X}$

$$x_k \ \trianglerighteq \ x_{k'} \iff f^{\mathbf{X}}_{\max\{k,k'\},\min\{k,k'\}} = x_k.$$

By construction, $\trianglerighteq$ is antisymmetric and reflexive. To prove that $\trianglerighteq$ is a partial order on $\mathbf{X}$ we need to show that it is transitive; namely for any triple $x, x', x'' \in \mathbf{X}$, if $x \trianglerighteq x'$ and $x' \trianglerighteq x''$, then it is not the case that $x'' \trianglerighteq x$. Suppose for a contradiction that there exists three distinct $x_k, x_{k'}, x_{k''} \in \mathbf{X}$ such that $x_k \trianglerighteq x_{k'}$, $x_{k'} \trianglerighteq x_{k''}$, and $x_{k''} \trianglerighteq x_k$. Suppose, without loss of generality, that $k < k' < k''$. By construction of $\trianglerighteq$, it must be that $f^{\mathbf{X}}_{k'',k'} = x_{k'}$ and $f^{\mathbf{X}}_{k'',k} = x_{k''}$. Moreover, Lemma 4 implies $f^{\mathbf{X}}_{k'',k'} \in [x_{k'}, x_{k''}]$ and $f^{\mathbf{X}}_{k'',k} \in [x_k, x_{k''}]$: Namely, f suggests $x_{k'}$ (i.e., $f^{\mathbf{X}}_{k'',k'} = x_{k'}$) while both $x_{k'}$ and $x_{k''}$ are mutually negotiable (i.e., $x_{k'}, x_{k''} \in [x_{k'}, x_{k''}]$) and suggests $x_{k''}$ (i.e., $f^{\mathbf{X}}_{k'',k} = x_{k''}$) while these two alternatives are still mutually negotiable ($x_{k'}, x_{k''} \in [x_k, x_{k''}]$ because $k < k' < k''$), contradicting (by Lemma 2) that f is strategy-proof, efficient, and individually rational over $\mathcal{D}$. Thus, $\trianglerighteq$ is transitive.

Now take any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ with $j \leq \ell$ and set $f^{\mathbf{X}}_{\ell,j} = x_s$ for some $j \leq s \leq \ell$. To show $f^{\mathbf{X}}_{\ell,j} \trianglerighteq x$ for all $x \in [x_j, x_\ell]$, suppose for a contradiction that there exists $x_{s'} \in [x_j, x_\ell]$ and it is not the case that $x_s \trianglerighteq x_{s'}$. Suppose, without loss of generality, $s' < s$. Because $x_s \trianglerighteq x_{s'}$ is not true, construction of $\trianglerighteq$ requires that $f^{\mathbf{X}}_{s,s'} \neq x_s$. Let $f^{\mathbf{X}}_{s,s'} = x_{s''}$ for some $x_{s''} \in [x_{s'}, x_s]$. With all the given information, consider the following two profiles; $t = (x^1_\ell, x^2_j)$ and $t' = (x^1_s, x^2_{s'})$. We have $x_{s''}, x_s \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$ for $i = 1, 2$, $f^{\mathbf{X}}_t = x_s$ and $f^{\mathbf{X}}_{t'} = x_{s''}$, contradicting (by Lemma 2) that f is strategy-proof, efficient and individually rational over $\mathcal{D}$. Hence, it must be that $f^{\mathbf{X}}_{\ell,j} = \underset{[x_j,x_\ell]}{\boldsymbol{max}} \ \trianglerighteq$. $\square$

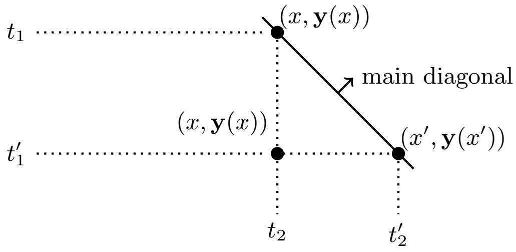
Proof of Theorem 1: Proof immediately follows from Lemmata 2-4. ■

2.3 Sketch of the Proof of Theorem 2

Consider the "if" part. Mechanism $f^{\succeq_{\mathbf{y}}}$ satisfies individual rationality because it never suggests a non-negotiable alternative or unacceptable outcome. For efficiency, consider a bundle $b = (x_b, \mathbf{y}(x_b))$ suggested by $f^{\succeq_{\mathbf{y}}}$ at some type profile $t \in \mathbf{T}$. Suppose for a contradiction that another bundle $a = (x_a, \mathbf{y}(x_a)) \in \mathbf{B}^{\mathbf{y}}$, where x_a is also in the zone of mutual gain at the type profile t, Pareto dominates b. Because $f^{\succeq_{\mathbf{y}}}$ suggests b while both x_a and x_b are available, we must have $x_b \succeq_{\mathbf{y}} x_a$. Moreover, because negotiators' utility functions satisfy quid pro quo, there is a Negotiator i such that for all $u_i \in \mathcal{D}_i$, $u_i(x_a, y, t_i) > u_i(x_b, y, t_i)$ for all $y \in Y$ while $u_i(b, t_i) > u_i(a, t_i)$, contradicting that bundle a Pareto dominates b (the strict inequalities follow from the fact that u_i's are strict and $a \neq b$). However, if $a \notin \mathbf{B}^{\mathbf{y}}$, then by condition (i.2) of Definition 1, there is no alternative $y \in \mathbf{Y}$ that can be paired with x_a so that bundle a Pareto dominates b. Hence, f must be efficient.

Regarding strategy-proofness of f, a profitable deviation is never possible, by the deal-breaker property, from or to a type profile in which $f^{\succeq_{\mathbf{y}}}$ suggests ϕ. Therefore, consider a type profile where $f^{\succeq_{\mathbf{y}}}$ suggests a bundle $b' = (x_{b'}, \mathbf{y}(x_{b'})) \neq \phi$. Any deviation of, say, Negotiator 1 to a less-accepting type to get $a' = (x_{a'}, \mathbf{y}(x_{a'})) \neq b'$ is never profitable because (1) we must have $x_{b'} \succeq_{\mathbf{y}} x_{a'}$ since the logrolling mechanism $f^{\succeq_{\mathbf{y}}}$ suggests b' when both $x_{a'}$ and $x_{b'}$ are in the zone of mutual gain; (2) bundle a' must be appearing on a lower row on the main diagonal of $f^{\succeq_{\mathbf{y}}}$ than b' does because $\succeq_{\mathbf{y}}$ is transitive; and thus (3) it must be the case that Negotiator 1 prefers alternative $x_{a'}$ over $x_{b'}$ (and Negotiator 2 prefers $x_{b'}$ over $x_{a'}$) whenever both these alternatives are negotiable for her, and so, Negotiator 1 must prefer bundle b' over a' because preferences satisfy quid pro quo and $x_{b'} \succeq_{\mathbf{y}} x_{a'}$. Similar reasoning proves that Negotiator 1 has no incentive to deviate to a more accepting type. Hence, $f^{\succeq_{\mathbf{y}}}$ is strategy-proof.

Consider now the "only if" part. By Theorem 1, strategy-proofness, efficiency, and individual rationality of f over some non-empty subset $\mathcal{D}$ of $\mathcal{U}$ imply an injective and decreasing function $\mathbf{y}$ and a partial order $\succeq_{\mathbf{y}}$ such that $f = f^{\succeq_{\mathbf{y}}}$. To prove $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$, and so $\mathcal{D}$ satisfies quid pro quo, we need to show that $\succeq_{\mathbf{y}}$ and $\mathbf{y}$ satisfy Definition 1. Condition (1.ii) is simply implied by the efficiency of f over $\mathcal{D}$. For condition (1.i) take any x, x' with $x \succeq_{\mathbf{y}} x'$. By the construction of $\succeq_{\mathbf{y}}$ in the proof of Theorem 1, we know that $x \succeq_{\mathbf{y}} x'$ implies that f must be suggesting a bundle with x at some type profile where both x and x' are in the zone of mutual gain (e.g., (t_1', t_2), see the figure below). Assuming, without loss of generality, that Negotiator 1 prefers x to x', i.e., $u_1(x, y, t_1') \geq u_1(x', y, t_1')$ for all $u_1 \in \mathcal{D}_1$ and $y \in Y$, Negotiator 2 must have $u_2(x, y, t_2) \leq u_2(x', y, t_2)$ for all $u_2 \in \mathcal{D}_2$ because her preferences over individual issues are diametrically opposed. It is easy to verify that strategy-proofness of f implies $u_2(x, \mathbf{y}(x), t_2) \geq u_2(x', \mathbf{y}(x'), t_2)$ because otherwise type t_2 would deviate to t_2'. The last inequality is what condition (1.i) requires.



Finally, the collection of sets $[x_j, x_\ell]$ where $1 \leq j \leq \ell \leq m$ constitutes the set of all non-empty zones of mutual gain, and every doubleton $\{x, x'\} \subseteq [x_j, x_\ell]$ has a least upper bound in $[x_j, x_\ell]$, which is $x^*_{[x_j, x_\ell]}$. Thus, $(S, \trianglerighteq)$ is a semilattice for any non-empty zone of mutual gain, as required by condition (2) of Definition 1.

2.4 Formal Proof of Theorem 2

Lemma 6. *If mechanism $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism for some partial order $\succeq_\mathbf{y}$ on $\mathbf{X}$, then for any distinct logrolling bundles $b, b' \in \mathbf{B}^\mathbf{y}$, where $b = (x_k, \mathbf{y}(x_k))$ and $b' = (x_{k'}, \mathbf{y}(x_{k'}))$, and $\ell, \ell', j, j' \in \mathcal{I}$, where $j \leq \ell$, $j' \leq \ell'$, $f^{\succeq_\mathbf{y}}_{\ell,j} = b$, and $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$, we have the following:*

(*i*) $b \in V(b) \equiv \left\{ f^{\succeq_\mathbf{y}}_{r,s} | r, s \in \mathcal{I} \text{ and } s \leq k \leq r \right\}$.

(*ii*) $\{b, b'\} \nsubseteq V(b) \cap V(b')$.

(*iii*) *If $\ell' < \ell$ and $j' = j$, then $k' \leq k$.*

(*iv*) *If $\ell' = \ell$ and $j' < j$, then $k' \leq k$.*

Proof. Assume that $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, namely for any $r, s \in \mathcal{I}$ with $s \leq r$, $f^{\succeq_\mathbf{y}}_{r,s} = \left(x^*_{[x_s, x_r]}, \mathbf{y}(x^*_{[x_s, x_r]}) \right)$ where $x^*_{[x_s, x_r]} \succeq_\mathbf{y} x$ for all $x \in [x_s, x_r]$. Take any two bundles $b, b' \in \mathbf{B}^\mathbf{y}$, satisfying $b = (x_k, \mathbf{y}(x_k))$, $b' = (x_{k'}, \mathbf{y}(x_{k'}))$, and $x_k \neq x_{k'}$, and any indices $\ell, \ell', j, j' \in \mathcal{I}$, satisfying $j \leq \ell$, $j' \leq \ell'$, $f^{\succeq_\mathbf{y}}_{\ell,j} = b$, and $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$.

To prove the claim of (i), suppose for a contradiction that $b \notin V(b)$. Namely, either $j \leq \ell < k$ or $\ell \geq j > k$ holds. In either case we have $x_k \notin [x_j, x_\ell]$, contradicting that $f^{\succeq_\mathbf{y}}_{\ell,j}$ is a logrolling mechanism. To prove claim of (ii), suppose for a contradiction that $\{b, b'\} \subseteq V(b) \cap V(b')$; namely, $j \leq k, k' \leq \ell$ and $j' \leq k, k' \leq \ell'$. These two inequalities imply $x_k, x_{k'} \in [x_j, x_\ell] \cap [x_{j'}, x_{\ell'}]$. Because $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism and $f^{\succeq_\mathbf{y}}_{\ell,j} = b$, it must be that $x_k = x^*_{[x_j, x_\ell]}$, and so $x_k \succeq_\mathbf{y} x_{k'}$. Similarly, because $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$, it must be that $x_{k'} \succeq_\mathbf{y} x_k$, contradicting that $\succeq_\mathbf{y}$ is antisymmetric and $x_k \neq x_{k'}$. To prove claim of (iii), assume that $\ell' < \ell$ and $j' = j$, and suppose for a contradiction that $k < k'$. Because $f^{\succeq_\mathbf{y}}_{\ell,j} = b$ and $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$, claim of (i) implies $j \leq k \leq \ell$ and $j' \leq k' \leq \ell'$. Because $\ell' < \ell$ and $k < k'$, it must be that $j' = j \leq k < k' \leq \ell' < \ell$; namely $\{b, b'\} \subseteq V(b) \cap V(b')$, contradicting the claim of (ii). Symmetric arguments suffice to prove the claim of (iv). □

Lemma 7. *If the non-empty set $\mathcal{D} \subseteq \mathcal{U}$ satisfies quid pro quo and $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, where the partial order $\succeq_\mathbf{y}$ is in $\Pi_\mathcal{D}$, then $f^{\succeq_\mathbf{y}}$ is efficient over $\mathcal{D}$.*

Proof. Assume that $\mathcal{D}$ satisfies quid pro quo and $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, where $\succeq_\mathbf{y} \in \Pi_\mathcal{D}$. Take any type profile $t = (t_1, t_2) = (x^1_\ell, x^2_j) \in \mathbf{T}$. If $j > \ell$, then $f^{\succeq_\mathbf{y}}(t) = \phi$. By deal-breaker property, no bundle in $\mathbf{B}\backslash\{\phi\}$ would make one negotiator better off without hurting the other. Suppose, for the rest of the proof, that $j \leq \ell$. Because $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, it must be that $f^{\succeq_\mathbf{y}}(t) = (x_k, \mathbf{y}(x_k)) \in \mathbf{B}^\mathbf{y}$ for some $k \in [x_j, x_\ell]$. Next, we show that any other bundle would make at least one negotiator worse off and thus conclude that $f^{\succeq_\mathbf{y}}$ is efficient. We prove this claim in two steps:

For step 1, we prove that any bundle in $\mathbf{B}^\mathbf{y}\backslash\{(x_k, \mathbf{y}(x_k))\}$ would make one of the negotiators worse off. To show this we suppose, for a contradiction, that there exists a bundle $(x_{k'}, \mathbf{y}(x_{k'})) \in \mathbf{B}^\mathbf{y}\backslash\{(x_k, \mathbf{y}(x_k))\}$, which is acceptable by both negotiators at type profile t, and that $u_i(x_{k'}, \mathbf{y}(x_{k'}), t_i) \geq u_i(x_k, \mathbf{y}(x_k), t_i)$ for all $i \in \mathbf{I}$ and all $(u_1, u_2) \in \mathcal{D}$, and the inequality is strict for at least one negotiator/utility function. Because $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, it must be that $x_k = \underset{[x_j, x_\ell]}{\boldsymbol{max}} \succeq_\mathbf{y}$, and so $x_k \succeq_\mathbf{y} x_{k'}$. Therefore, because $\mathcal{D}$ satisfies quid pro quo and $\succeq_\mathbf{y} \in \Pi_\mathcal{D}$, there must exist a Negotiator $i \in \mathbf{I}$ satisfying $u_i(x_k, \mathbf{y}(x_k), t_i) > u_i(x_{k'}, \mathbf{y}(x_{k'}), t_i)$ for all $u_i \in \mathcal{D}_i$ by condition $i.1$ of Definition 1 (strict inequality follows from the fact that u_i is strict), which yields the desired contradiction.

For step 2, we prove that any bundle in $\mathbf{B}\backslash\mathbf{B}^\mathbf{y}$ would make one of the negotiators worse off. To show this, we suppose, for a contradiction, that there exists a bundle $(x_{k''}, y_r) \in \mathbf{B}\backslash\mathbf{B}^\mathbf{y}$, which is acceptable by both negotiators at type profile t, such that $u_i(x_{k''}, y_r, t_i) \geq u_i(x_k, \mathbf{y}(x_k), t_i)$ for all $i \in \mathbf{I}$ and all $(u_1, u_2) \in \mathcal{D}$, and the inequality is strict for at least one negotiator/utility function. Let $\mathbf{y}(x_k) = y_s$ and $\mathbf{y}(x_{k''}) = y_{s''}$. If $x_{k''} \notin [x_j, x_\ell]$, then it must be non-negotiable alternative for at least one of the negotiators at type profile t, and so by deal-breaker property $u_i(x_{k''}, y_r, t_i) < u_i(x_k, \mathbf{y}(x_k), t_i))$ holds for some $i \in \mathbf{I}$. Therefore, it must be that $x_{k''} \in [x_j, x_\ell]$. Because $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, $x_k = \underset{[x_j, x_\ell]}{\boldsymbol{max}} \succeq_\mathbf{y}$, and so $x_k \succeq_\mathbf{y} x_{k''}$. Because $\mathcal{D}$ satisfies quid pro quo and $\succeq_\mathbf{y} \in \Pi_\mathcal{D}$, there must exists a Negotiator i satisfying $u_i(x_k, y_s, t_i) \geq u_i(x_{k''}, y_{s''}, t_i)$ for all $u_i \in \mathcal{D}_i$ by condition $i.1$ of Definition 1. Suppose, without loss of generality, that the last inequality is true for Negotiator 1; namely $u_1(x_k, y_s, t_1) \geq u_1(x_{k''}, y_{s''}, t_1)$ for all $u_1 \in \mathcal{D}_1$. Condition $i.1$ of Definition 1 also implies that $u_1(x_{k''}, y, t_1) \geq u_1(x_k, y, t_1)$ for all $u_1 \in \mathcal{D}_1$ and all $y \in \mathbf{Y}$, or equivalently $k'' \leq k$. Because each u_1 is a decreasing function, the last two inequalities imply $s \leq s''$. There are three exhaustive cases regarding the value of r in comparison to s and s'', and we consider each case next:

First, consider the case where $s \leq s'' \leq r$. Because $y_r \neq \mathbf{y}(x_{k''})$, it must be that $s'' < r$.

Therefore, $u_1(x_{k''}, y_{s''}, t_1) > u_1(x_{k''}, y_r, t_1)$ for all $u_1 \in \mathcal{D}_1$ since u_1's are decreasing. The last inequality and $u_1(x_k, y_s, t_1) \geq u_1(x_{k''}, y_{s''}, t_1)$ for all $u_1 \in \mathcal{D}_1$ yield $u_1(x_k, y_s, t_1) > u_1(x_{k''}, y_r, t_1)$ for all $u_1 \in \mathcal{D}_1$, contradicting that that both negotiators find bundle $(x_{k''}, y_r)$ better than (x_k, y_s). Second, consider the case where $r \leq s \leq s''$. Because the negotiators' preferences over alternatives are diametrically opposed and any $u_2 \in \mathcal{D}_2$ is increasing, it must be that for any $u_2 \in \mathcal{D}_2$, $u_2(x_k, y_s, t_2) \geq u_2(x_{k''}, y_s, t_2) \geq u_2(x_{k''}, y_r, t_2)$, one of which is strict because $(x_{k''}, y_r) \neq (x_k, y_s)$, contradicting that both negotiators find bundle $(x_{k''}, y_r)$ better than (x_k, y_s). Third, consider the case where $s < r < s''$. By condition $i.2$ of Definition 1, there is no $y_r \in \mathbf{Y}$ where $u_i(x_{k''}, y_r, t_i) \geq u_i(x_k, y_s, t_i)$ for all $i \in \mathbf{I}$ and $(u_1, u_2) \in \mathcal{D}$, contradicting again that both negotiators find bundle $(x_{k''}, y_r)$ better than (x_k, y_s). □

Lemma 8. *If the non-empty set $\mathcal{D} \subseteq \mathcal{U}$ satisfies quid pro quo and $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, where the partial order $\succeq_\mathbf{y}$ is in $\Pi_\mathcal{D}$, then $f^{\succeq_\mathbf{y}}$ is strategy-proof over $\mathcal{D}$.*

Proof. Assume that $\mathcal{D}$ satisfies quid pro quo and $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, where $\succeq_\mathbf{y} \in \Pi_\mathcal{D}$. Take any type profile $t = (t_1, t_2) = (x^1_\ell, x^2_j) \in \mathbf{T}$. We want to prove that no player has an incentive to deviate from t. For this purpose, consider, without loss of generality, deviations of type x^1_ℓ of Negotiator 1, so fix j or x^2_j. If $\ell < j$, then $f^{\succeq_\mathbf{y}}_{\ell,j} = \phi$ and Negotiator 1 can change the outcome only if she deviates to a more accepting type (i.e., $x^1_{\ell'}$ where $\ell' > j > \ell$). In this case, $x^*_{[x_j, x_{\ell'}]} \in [x_j, x_{\ell'}]$, where $[x_j, x_{\ell'}] \cap \mathbf{N}(x^1_\ell) = \varnothing$, so $f^{\succeq_\mathbf{y}}_{\ell',j}$ is an unacceptable bundle for type x^1_ℓ. Therefore, the deal-breaker property of the utility functions implies that type x^1_ℓ of Negotiator 1 has no profitable deviation whenever $\ell < j$.

If $\ell = j$, then type x^1_ℓ of Negotiator 1 can deviate to a less accepting type or a more accepting type to change the outcome. In the first case, she deviates to a type $x^1_{\ell'}$ where $\ell' < \ell = j$, implying $f^{\succeq_\mathbf{y}}(x^1_{\ell'}, x^2_j) = f^{\succeq_\mathbf{y}}_{\ell',j} = \phi$. In the second case (i.e., she deviates to a type $x^1_{\ell''}$ where $\ell = j < \ell''$) it must be that $x^*_{[x_j, x_{\ell''}]} \in [x_j, x_{\ell''}]$, where $[x_j, x_{\ell''}] \cap \mathbf{N}(x^1_\ell) = \{x_\ell\}$, and so $f^{\succeq_\mathbf{y}}(x^1_{\ell''}, x^2_j) = f^{\succeq_\mathbf{y}}_{\ell'',j}$ is an unacceptable bundle for her unless $f^{\succeq_\mathbf{y}}_{\ell'',j}$ is the same as $f^{\succeq_\mathbf{y}}_{\ell,j} = (x_\ell, \mathbf{y}(x_\ell))$. In any case, deal-breaker property of the utility functions implies that type x^1_ℓ of Negotiator 1 has no profitable deviation whenever $\ell = j$.

Suppose now that $j < \ell$. By deal-breaker property, deviating to a type $x^1_{\ell'}$ where $\ell' < j < \ell$ is not profitable for type x^1_ℓ of Negotiator 1 because $f^{\succeq_\mathbf{y}}_{\ell',j} = \phi$. If she deviates to a type $x^1_{\ell''}$ where $j \leq \ell'' < \ell$, and if $f^{\succeq_\mathbf{y}}_{\ell,j} = (x_k, \mathbf{y}(x_k))$ and $f^{\succeq_\mathbf{y}}_{\ell'',j} = (x_{k''}, \mathbf{y}(x_{k''}))$, then by condition (iii) of Lemma 6 we must have $k'' \leq k$. Because the negotiators' utilities are monotonic and $\mathcal{D}$ satisfies logrolling, condition $i.1$ of Definition 1 implies $u_1(x_k, \mathbf{y}(x_k), t_1) \geq u_1(x_{k''}, \mathbf{y}(x_{k''}), t_1)$ or equivalently $u_1(f^{\succeq_\mathbf{y}}_{\ell,j}, t_1) \geq u_1(f^{\succeq_\mathbf{y}}_{\ell'',j}, t_1)$ for all $u_1 \in \mathcal{U}_1$. Thus, deviation to $x^1_{\ell''}$ can never be profitable for type x^1_ℓ of Negotiator 1. Finally, if she deviates to a more accepting type $x^1_{\ell'''}$

where $j \leq \ell < \ell'''$ and gets something different than $f^{\succeq_{\mathbf{y}}}_{\ell,j}$, then it must be that $x^*_{[x_j,x_{\ell'''}]} \in [x_{\ell+1}, x_{\ell'''}]$: To prove the last claim, suppose for a contradiction that $x^*_{[x_j,x_{\ell'''}]} \notin [x_{\ell+1}, x_{\ell'''}]$. Because $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, it must be that $x^*_{[x_j,x_{\ell'''}]} \in [x_j, x_{\ell'''}]$. The last two conditions imply $x^*_{[x_j,x_{\ell'''}]} \in [x_j, x_\ell]$. Because $f^{\succeq_{\mathbf{y}}}_{\ell,j} = \left(x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})\right)$, it must be that $x^*_{[x_j,x_\ell]} \succeq_{\mathbf{y}} x^*_{[x_j,x_{\ell'''}]}$. Similarly, because $f^{\succeq_{\mathbf{y}}}_{\ell''',j} = \left(x^*_{[x_j,x_{\ell'''}]}, \mathbf{y}(x^*_{[x_j,x_{\ell'''}]})\right)$ and $[x_j, x_\ell] \subset [x_j, x_{\ell'''}]$, it must be that $x^*_{[x_j,x_{\ell'''}]} \succeq_{\mathbf{y}} x^*_{[x_j,x_\ell]}$, contradicting that $x^*_{[x_j,x_{\ell'''}]} \neq x^*_{[x_j,x_\ell]}$ and $\succeq_{\mathbf{y}}$ is antisymmetric. Therefore, it must be that $x^*_{[x_j,x_{\ell'''}]} \in [x_{\ell+1}, x_{\ell'''}]$, where $[x_{\ell+1}, x_{\ell'''}] \cap \mathbf{N}(x^1_\ell) = \varnothing$, and so $f^{\succeq_{\mathbf{y}}}_{\ell''',j}$ is an unacceptable bundle for type x^1_ℓ of Negotiator 1. Therefore, deviating to a more accepting type $x^1_{\ell'''}$ cannot be profitable for her by the deal-breaker property of the utility functions. Hence, type x^1_ℓ of Negotiator 1 has no profitable deviation whenever $j < \ell$.

Symmetric arguments would prove the same conclusion for Negotiator 2. Since we exhausted all possible deviations of type x^1_ℓ of Negotiator 1, we can conclude that $f^{\succeq_{\mathbf{y}}}$ is strategy-proof over $\mathcal{D}$. □

Proof of Theorem 2:

Proof of 'if': Assume that $\mathcal{D}$ satisfies quid pro quo and $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, where $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$. The set $[x_j, x_\ell]$ is a connected subset of $\mathbf{X}$ whenever $j \leq \ell$, and $\succeq_{\mathbf{y}}$ is a semilattice for all connected subsets of $\mathbf{X}$. Therefore, $\underset{[x_j,x_\ell]}{\boldsymbol{max}} \succeq_{\mathbf{y}}$ uniquely exists whenever $j \leq \ell$. $f^{\succeq_{\mathbf{y}}}$ never suggests a non-negotiable alternative, and so, it is individually rational over $\mathcal{D}$. $f^{\succeq_{\mathbf{y}}}$ is efficient over $\mathcal{D}$ by Lemma 7 and strategy-proof over $\mathcal{D}$ by Lemma 8.

***Proof of 'only if'*:** Now assume that the mediation mechanism f is strategy-proof, efficient, and individually rational over $\mathcal{D}$. Theorem 1 implies an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$, a partial order $\succeq_{\mathbf{y}}$ on $\mathbf{X}$ such that $f = f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism. We need to prove that $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$, and thus $\mathcal{D}$ satisfies quid pro quo.

To show that part $(1.i)$ of Definition 1 holds, let $x_k, x_{k'} \in \mathbf{X}$ be two distinct alternatives and $x_k \succeq_{\mathbf{y}} x_{k'}$. Suppose, without loss of generality, $k' < k$. Because any $u_1 \in \mathcal{D}_1$ is decreasing and $k' < k$, it must be that $u_1(x_{k'}, y, .) > u_1(x_k, y, .)$ for any $y \in \mathbf{Y}$. Strategy-proofness of f requires $u_1(f_{k,k'}, x^1_k) \geq u_1(f_{k',k'}, x^1_k)$ for all $u_1 \in \mathcal{D}_1$, and thus by type invariance, $u_1(f_{k,k'}, t_1) \geq u_1(f_{k',k'}, t_1)$ for all $u_1 \in \mathcal{D}_1$ and all $t_1 \in \mathbf{T}_1$ with $f^{\mathbf{X}}_{k,k'}, f^{\mathbf{X}}_{k',k'} \in \mathbf{N}(t_1)$. Recall the construction of $\succeq_{\mathbf{y}}$ in the proof of Theorem 1: $x_k \succeq_{\mathbf{y}} x_{k'}$ if and only if $f_{k,k'} = (x_k, \mathbf{y}(x_k))$. Therefore, the last inequality implies $u_1(x_k, \mathbf{y}(x_k), t_1) > u_1(x_{k'}, \mathbf{y}(x_{k'}), t_1)$ because f is a logrolling mechanism (i.e., $f_{k',k'} = (x_{k'}, \mathbf{y}(x_{k'}))$ and x_k and $x_{k'}$ are two distinct alternatives. Thus, as required by part $(1.i)$ of Definition 1, for all $u_1 \in \mathcal{D}_1$ and all $y \in \mathbf{Y}$ we have $u_1(x_k, \mathbf{y}(x_k), t_1) > u_1(x_{k'}, \mathbf{y}(x_{k'}), t_1)$ and $u_1(x_{k'}, y, t_1) > u_1(x_k, y, t_1)$.

To show that part $(1.ii)$ of Definition 1 holds, suppose for a contradiction that there is some

$t_i \in \mathbf{T}_i$ with $x_k, x_{k'} \in \mathbf{N}(t_i)$ for $i = 1, 2$, and $y \in \mathbf{Y}$ with $u_1(x_{k'}, y, t_1) > u_1(x_k, \mathbf{y}(x_k), t_1)$ and $u_2(x_{k'}, y, t_2) \geq u_2(x_k, \mathbf{y}(x_k), t_2)$ for all $(u_i, u_{-i}) \in \mathcal{D}$. Then, consider the type profile $(x_k^1, x_{k'}^2)$ where both x_k and $x_{k'}$ are negotiable by both types, and $f(t) = (x_k, \mathbf{y}(x_k))$ (as discussed in the previous paragraph). The last two inequalities, which are true for the types x_k^1 and $x_{k'}^2$ by type invariance, imply that both these types prefer the bundle $(x_{k'}, y)$ over $f(t)$ for all $(u_1, u_2) \in \mathcal{D}$, contradicting that f is efficient over $\mathcal{D}$.

To show that part (2) of Definition 1 holds, recall that all sets of the form $[x_j, x_\ell]$ where $j, \ell \in \mathcal{I}$ and $j \leq \ell$ designate all the connected subsets of $\mathbf{X}$. Because $x^*_{[x_j, x_\ell]} = \boldsymbol{max}_{[x_j, x_\ell]} \succeq_{\mathbf{y}}$, it must be that every doubleton $\{x, x'\} \subseteq [x_j, x_\ell]$ has a least upper bound in $[x_j, x_\ell]$, denoted by $x^*_{[x_j, x_\ell]}$, and thus the poset $(S, \succeq_{\mathbf{y}})$ is a semilattice for all connected subset S of $\mathbf{X}$. Hence, $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$ and $\mathcal{D}$ satisfies quid pro quo. ■

2.5 Proofs of the Remaining Results

Proof of Theorem 3: Assume that $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ be an injective and decreasing function and $\succeq_{\mathbf{y}}$ be a partial order on $\mathbf{X}$. We first show $(i) \Rightarrow (ii)$. For this purpose, assume that $f_{\ell,j} = \left(x^*_{[x_j, x_\ell]}, \mathbf{y}(x^*_{[x_j, x_\ell]})\right)$ for all $\ell, j \in \mathcal{I}$ with $j \leq \ell$. Let $r_1 \in \mathcal{I}$ is such that $x_{r_1} = \boldsymbol{max}_{\mathbf{X}} \succeq_{\mathbf{y}}$. Therefore, it must be that $f_{r_1, r_1} = (x_{r_1}, \mathbf{y}(x_{r_1})) \in \mathbf{B}^{\mathbf{y}}$. Consider the matrix of f: All the entries on row r_1 to the left of entry f_{r_1, r_1}, all the entries on column r_1 below entry f_{r_1, r_1}, and all the entries in between must fill up with bundle $(x_{r_1}, t(x_{r_1}))$ because x_{r_1} has the highest rank over $\mathbf{X}$ and f always acts like the logrolling mechanism $f^{\succeq_{\mathbf{y}}}$ at these entries. Thus, the rectangle $\square^{r_1}_{m,1}$ fills up with $(x_{r_1}, \mathbf{y}(x_{r_1}))$. We let $\square^{r_1}_{m,1}$ be the first element of the rectangular partition of $\triangle_{m,1}$. Note that, when $m \geq 3$, the so-far-unfilled $\triangle_{m,1} \backslash \square^{r_1}_{m,1}$ consists of at least one triangle (if $r_1 \in \{1, m\}$) and at most two triangles (if $r_1 \notin \{1, m\}$).

Next, take an arbitrary triangle $\triangle_{s,r} \in \triangle_{m,1} \backslash \square^{r_1}_{m,1}$. Note that either $s = r_1$ and $r = 1$, or $s = m$ and $r = r_1 + 1$. Let $f_{r_2, r_2} = (x_{r_2}, \mathbf{y}(x_{r_2})) \in \mathbf{B}^{\mathbf{y}}$ with $r_2 \neq r_1$ denote the logrolling bundle on the hypotenuse of $\triangle_{s,r}$ that satisfies $x_{r_2} = \boldsymbol{max}_{[x_s, x_r]} \succeq_{\mathbf{y}}$. Once again, starting from the hypotenuse of $\triangle_{s,r}$ all the so-far-unfilled entries on row r_2 to the left of entry f_{r_2, r_2}, all the so-far-unfilled entries on column r_2 below entry f_{r_2, r_2}, and all entries in between must fill up with bundle $(x_{r_2}, \mathbf{y}(x_{r_2}))$ because x_{r_2} has the highest rank among the alternatives in $[x_s, x_r]$. Thus, let $\square^{r_2}_{s,r}$ denote the second element of the rectangular partition of $\triangle_{m,1}$. Note that the so-far-unfilled set $\triangle_{m,1} \backslash \left\{\square^{r_1}_{m,1} \cup \square^{r_2}_{s,r}\right\}$ consists of at least one triangle. Iterate this reasoning and at each step, pick a triangle from the so-far-unfilled subset of $\triangle_{m,1}$ and fill its corresponding rectangle with the bundle whose first component has the highest precedence with respect to $\succeq_{\mathbf{y}}$. By the finiteness of the problem, the rectangular partition is obtained in m steps.

Now we show $(ii) \Rightarrow (i)$. For this reason, we assume that the triangle $\triangle_{m,1}$ has a rectangular partition (denote it by $\mathcal{P}^1$) such that f assigns a unique bundle from the set of logrolling bundles $\mathbf{B^y}$ to each rectangle in this partition. In this rectangular partition $\mathcal{P}^1$ of $\triangle_{m,1}(\equiv \triangle^1)$, let $\square^{r_1} \subset \triangle^1$ be the rectangle that includes the entry at the bottom left corner of triangle $\triangle^1$ (i.e., $f_{m,1}$). We construct the precedence order $\succeq_{\mathbf{y}}$ as follows: Let $f^{\mathbf{X}}_{m,1} = x_{r_1}$ have the higher precedence rank than any other alternative in $\mathbf{X}$; namely, $x_{r_1} \succeq_{\mathbf{y}} x$ for all $x \in \mathbf{X}$. Next consider $\triangle^1 \backslash \square^{r_1}$ which has a triangular partition $\mathcal{P}^2$ that consists of at most two triangles. Take an arbitrary triangle $\triangle^2 \in \mathcal{P}^2$ and let $\square^{r_2} \subset \triangle^2$ denote the rectangle that includes the entry at the bottom left corner of triangle $\triangle^2$, say $(x_{r_2}, \mathbf{y}(x_{r_2}))$. We let x_{r_2} have a higher precedence rank than any other alternative in $\mathbf{X}$ that appears on the hypotenuse of $\triangle^2$. Namely, if $r_2 < r_1$, then $x_{r_2} \succeq_{\mathbf{y}} f^{\mathbf{X}}_{k,k}$ for all $k \in \{1, ..., r_2-1, r_2+1, ..., r_1-1\}$, and if $r_2 > r_1$, then $x_{r_2} \succeq_{\mathbf{y}} f^{\mathbf{X}}_{k,k}$ for all $k \in \{r_1+1, ..., r_2-1, r_2+1, ..., m\}$. Iterate in this fashion by considering an arbitrary triangle from the remaining partition $\triangle^1 \backslash \{\square^{r_1}, \square^{r_2}\}$. At the end of this finite procedure (consisting of exactly m steps), we obtain a transitive, antisymmetric but possibly incomplete strict precedence order $\succeq_{\mathbf{y}}$ on $\mathbf{B^y}$. Moreover, by construction we have $f_{\ell,j} = \left(x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})\right)$ where $x^*_{[x_j,x_\ell]} = \underset{[x_j,x_\ell]}{\boldsymbol{max}} \succeq_{\mathbf{y}}$ for all $\ell, j \in \mathcal{I}$ with $j \leq \ell$. This completes the proof. ■

Proof of Theorem 4: Constrained Shortlisting mechanism clearly belongs to a logrolling mechanisms family. Fix the set of logrolling bundles $\mathbf{B^y} = \{(x, \mathbf{y}(x)) | x \in \mathbf{X}\}$ and the family of logrolling mechanisms whose range is $\mathbf{B^y} \cup \{\phi\}$. Let $b_j = (x_j, \mathbf{y}(x_j)) \in \mathbf{B^y}$ denote a logrolling bundle. To see that the rank variance of a CS mechanism is lower than any other member of the logrolling mechanism family, we simply consider two cases about the number of possible alternatives.

First, when m is odd, $var(b_k) = (m+1)^2$. For any $b_{k-j}, b_{k+j} \in \mathbf{B^y}$ with $j < k$, we have $var(b_{k-j}) = var(b_{k+j}) = 2(\frac{(m+1)}{2} - j)^2 + 2(\frac{(m+1)}{2} + j)^2 = (m+1)^2 + 4j^2$. Thus, $var(b_k) < var(b)$ for any $b \in \mathbf{B^y} \backslash \{b_k\}$. Since any member of the logrolling mechanism family must pick an element of $\mathbf{B^y}$ whenever the mutual zone of agreement is non-empty (by Theorem 1), minimization of rank variance requires that $x_k \succeq^{CS}_{\mathbf{y}} x$ for any $x \in \mathbf{X}$. Also observe that $var(b_k) < var(b_{k-1}) < \ldots < var(b_1)$ and $var(b_k) < var(b_{k+1}) < \ldots < var(b_m)$. Thus, minimization of rank variance subsequently requires that $x_{k-1} \succeq^{CS}_{\mathbf{y}} \ldots \succeq^{CS}_{\mathbf{y}} x_1$ and $x_{k+1} \succeq^{CS}_{\mathbf{y}} \ldots \succeq^{CS}_{\mathbf{y}} x_m$. Note that when m is odd, the rank variance of the unique CS mechanism is strictly less than any other member of the logrolling mechanisms family.

Second, when m is even, $var(b_{\overline{k}}) = var(b_{\underline{k}}) = \frac{1}{2}(m^2 + (m+2)^2)$. For any $b_{\underline{k}-j}, b_{\overline{k}+j} \in \mathbf{B^y}$ with $j < k$, we have $var(b_{\underline{k}-j}) = var(b_{\overline{k}+j}) = 2(\frac{m}{2} - j)^2 + 2(\frac{(m+2)}{2} + j)^2 = \frac{1}{2}(m^2 + (m+2)^2) + 4j^2$. Hence, $var(b_{\overline{k}}) = var(b_{\underline{k}}) < var(b)$ for any $b \in \mathbf{B^y} \backslash \{b_{\overline{k}}, b_{\underline{k}}\}$. Note that we also have $var(b_{\underline{k}}) =$

$var(b_{\overline{k}}) < var(b_{\underline{k}-1}) < \ldots < var(b_1)$ and $var(b_{\underline{k}}) = var(b_{\overline{k}}) < (b_{\overline{k}+1}) < \ldots < var(b_m)$. Then, minimization of rank variance subsequently requires that either $x_{\overline{k}} \succeq_{\mathbf{y}}^{CS} x_{\underline{k}}$ or $x_{\underline{k}} \succeq_{\mathbf{y}}^{CS} x_{\overline{k}}$ together with $x_{\underline{k}-1} \succeq_{\mathbf{y}}^{CS} \ldots \succeq_{\mathbf{y}}^{CS} x_1$ and $x_{\overline{k}+1} \succeq_{\mathbf{y}}^{CS} \ldots \succeq_{\mathbf{y}}^{CS} x_m$. Note that when m is even, rank variance of a CS mechanism is weakly less than any other member of the logrolling mechanisms family. ■

3. A Practical Depiction of Quid Pro Quo

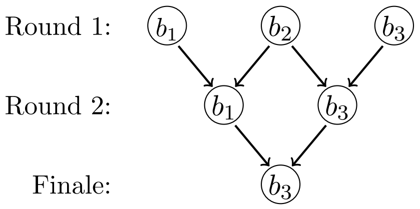
Definition 1 expresses the quid pro quo property based on an order-theoretic semilattice structure. An equivalent and arguably more intuitive description uses a recursive and algorithmic process on the set of logrolling bundles, which we present here through a simple example. This alternative structure is practically useful since it helps better understand the quid pro quo property and its role in Theorem 2. The essence of quid pro quo is that the negotiators' preferences allow an "*elimination tournament*" of the form we discuss below, where there is always a winner of each match-up at each round. Each round of this tournament effectively represents the corresponding diagonal of the strategy-proof, efficient, and individually rational mechanism to be constructed.

As an example, consider the case with three alternatives in each issue. The tournament always starts with all logrolling bundles ordered from b_1 to b_3 (see the left side of Figure 2 below), where $b_k = (x_k, y_{4-k})$.[2] Consider the most accepting types of each negotiator (i.e., types x_3^1 and x_1^2). In the first round of the tournament, each logrolling bundle matches up with its adjacent neighbor (i.e., both b_1 and b_3 match only with b_2). In the match-up between b_k and b_{k+1}, the "winner" is b_{k+1} if Negotiator 1 unambiguously ranks b_{k+1} over b_k (i.e., $u_1(b_{k+1}, x_3^1) \geq u_1(b_k, x_3^1)$ for all $u_1 \in \mathcal{D}_1$). Otherwise (if Negotiator 2 unambiguously ranks b_k over b_{k+1}), the winner of this match-up is b_k. These are the conditions implied by $(1.i)$ of Definition 1. If Negotiator 1 unambiguously ranks b_{k+1} over b_k and Negotiator 2 also unambiguously ranks b_k over b_{k+1}, then both of these bundles can be the winner. In such cases, the mediator (i.e., the partial order $\succeq_{\mathbf{y}}$ that we create) has the freedom to choose either of these two bundles to proceed to the next round.

We suppose in our example that Negotiator 2 unambiguously ranks b_1 over b_2 and Negotiator 1 unambiguously ranks b_3 over b_2. Therefore, b_1 and b_3 win over b_2 and move to the next round. In the second round, the winners of the first round (i.e., b_1 and b_3) match up (see the second row on the left side of Figure 2 below). In the second round b_3 (respectively, b_1) would be the

[2] When $m = n$, then decreasing function $\mathbf{y}$ is unique. For cases where $m < n$, the process may start with any such $\mathbf{y}$. If the logrolling bundles in $\mathbf{B}^{\mathbf{y}}$ fail to satisfy $(1.ii)$ of Definition 1, then $\mathbf{y}$ should be replaced and the entire process should be repeated with the new logrolling bundles.

winner if Negotiator 1 (respectively, Negotiator 2) unambiguously ranks b_3 over b_1 (respectively, b_1 over b_3).



	x_1^2	x_2^2	x_3^2
x_1^1	b_1		
x_2^1	b_1	b_2	
x_3^1	b_3	b_3	b_3

Figure 2: *An example for the elimination tournament and the matrix representation for the corresponding mediation rule*

If Negotiator 1 ranks b_1 over b_3 for some $u_1 \in \mathcal{D}_1$ and Negotiator 2 ranks b_3 over b_1 for some $u_2 \in \mathcal{D}_2$, then there is no winner of this match-up, and the process fails. In this case, we return to the previous round(s). If the mediator had the freedom to choose the winner of any match-ups in the earlier rounds, then we replace the winner(s) of these match-ups and reiterate the process. If the mediator had no freedom to choose the winner in the earlier rounds, or if none of these reiterations yield a match-up in the second round with a winner, then the process fails, meaning that $\mathcal{D}$ does not satisfy quid pro quo.

Suppose, for the sake of argument, Negotiator 1 unambiguously ranks b_3 over b_1 and Negotiator 2 unambiguously ranks b_1 over b_3 in our simple example. Then, either bundle can be the winner of the second round. In the illustration above, we have illustrated b_3 as the winner of the tournament.

The match-up configurations in this entire tournament are, in fact, the join semilattice structure implied by condition (2) of Definition 1. Theorem 2 says that we can use this tournament structure in creating logrolling mechanisms that are strategy-proof, efficient, and individually rational over $\mathcal{D}$. The winners of each round fill up the corresponding diagonals. For the tournament described above, the order of the logrolling bundles in the first round gives the placement order of these bundles (from the top corner to the bottom corner) along the first diagonal, and the order in the second round gives the placement order along the second diagonal, and the last winner, b_3, fills up the bottom left entry of this matrix (which is the last diagonal). The constructed mechanism corresponds to the logrolling mechanism $f^{\succeq_{\mathbf{y}}}$ with $x_3 \succeq_{\mathbf{y}} x_1 \succeq_{\mathbf{y}} x_2$. Recall that both b_1 and b_3 were winners in the second round in our example, so the negotiators' preferences (i.e., $\mathcal{D}$) also admit a second logrolling mechanism $f^{\succeq'_{\mathbf{y}}}$ with $x_1 \succeq'_{\mathbf{y}} x_3 \succeq'_{\mathbf{y}} x_2$.