Online Appendix for "Asymptotic Optimality of Semi-Open-Loop Policies in Inventory Models with Stochastic Lead Times"

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EC.1. Auxiliary Lemmas

This appendix provides known results and conditions used in the proofs of the paper.

Theorem EC.1 **(Chen and Shao (2005))**. *Suppose that* $F:\mathbb{R}\to\mathbb{R}$ *is any Lipschitz-continuous function with Lipschitz constant at most unity, i.e., for all* $x,y\in\mathbb{R}$, $|F(x)-F(y)|\le|x-y|$. *Suppose that* $\{X_i\}_{i\ge1}$ *is any sequence of i.i.d. random variables such that* $\mathbb{E}[X_1]=0,\mathbb{E}[X_1^2]=1$, *and* $\mathbb{E}[|X_1|^3]<\infty$. *Then, for all* $n\ge1$,

$$\left|\mathbb{E}\left[F\left(n^{-1/2}\sum_{i=1}^{n}X_i\right)\right]-\mathbb{E}[F(\tilde{N})]\right|\le 3n^{-1/2}\mathbb{E}[|X_1|^3],$$

where $\tilde{N}$ *denotes a standard normal random variable.*

Theorem EC.2 **(Feinberg et al. (2014), Theorem 1.1)**. *Let* $\mathbb{S}$ *be an arbitrary metric space,* $\{\mu_n\}_{n\ge1}\subset\mathbb{P}(\mathbb{S})$ *converge weakly to* $\mu\in\mathbb{P}(\mathbb{S})$, *and* $\{f_n\}_{n\ge1}$ *be a sequence of measurable non-negative* $\mathbb{R}$*-valued functions on* $\mathbb{S}$. *Then*

$$\int_{\mathbb{S}}\liminf_{n\to\infty,s'\to s}f_n(s')\mu(ds)\le\liminf_{n\to\infty}\int_{\mathbb{S}}f_n(s)\mu_n(ds).$$

Lemma EC.1 **(Rockafellar (1970), Theorem 10.8)**. *Let* X *be an open convex subset of* $\mathbb{R}^n$ *and* $f,f_n:X\to\mathbb{R}$ *be finite-valued functions. Suppose that* f_n *is convex for each* n *and* f_n *converges to* f *pointwise on* X. *Then,* f_n *converges to* f *uniformly on any compact subset of* X.

Lemma EC.2 **(Shapiro et al. (2014), Proposition 5.1)**. *Let* X *be a nonempty closed subset of* $\mathbb{R}^n$, $f,f_n:X\to\mathbb{R}$ *be read valued functions. Then, the following two properties are equivalent: (i) for any* $x\in X$ *and any sequence* $\{x_n\}\subseteq X$ *converging to* x, *it follows that* $f_n(x_n)$ *converges to* $f(x)$, *(ii) the function* f *is continuous on* X, *and* f_n *converges to* f *uniformly on any compact subset of* X.

Lemma EC.3 **(Chen et al. (2024), Lemma A.3)**. *Let* $\{y_n\}_{n\ge1}$ *be a sequence of numbers that converges to* y_∞ *(which may be infinity). Then* $\lim_{\alpha\to1}(1-\alpha)\sum_{k=1}^{\infty}\alpha^{k-1}y_k=y_\infty$.

Assumption EC.1. **(Assumption W* in Feinberg (2016)).** *(i) The transition probability of the underlying Markov process is weakly continuous.*
(ii) The single-period cost function is $\boldsymbol{K}$*-inf-compact, where a function* $c(\cdot,\cdot):X\times A\to\mathbb{R}$ *is called* $\boldsymbol{K}$*-inf-compact if for any* λ *and any nonempty compact subset* $K\subseteq X$, *the set* $\{(x,a)\in K\times A: c(x,a)\le\lambda\}$ *is compact.*

ASSUMPTION EC.2. **(Assumption B in Feinberg (2016)).** *(i)* $v^* = \inf_{s,w} v(s,w) < \infty$. *(ii)* $\sup_{\alpha\in[0,1)} \hat{V}^{\alpha}_{\infty}(s,w) < \infty$ *for all* (s,w).

We next provide useful results related to bracket policies. Readers can refer to Hajek (1985) and Altman et al. (2003) for more about multimodularity and bracket policies.

Given $z \in \mathbb{R}^t$ and $\theta \in \mathbb{R}$, the vector $u^z(\theta) \in \mathbb{Z}^t$ is defined by

$$u^z_i(\theta) = \lfloor \theta + z_1 + \cdots + z_i \rfloor - \lfloor \theta + z_1 + \cdots + z_{i-1} \rfloor, \ \forall 1 \le i \le t.$$

The fact that $\int_0^1 \lfloor \theta + x \rfloor d\theta = x$ for any $x \in \mathbb{R}$ implies

$$\int_0^1 u^z(\theta) d\theta = z.$$

It is clear that $u^z(\theta) = u^z(\theta+1)$ and $u^z(\theta)$ is piecewise constant in θ with at most $t+1$ jumps per period. We then have the following lemma (see also Hajek 1985).

LEMMA EC.4. *Let* $z^1, ..., z^l$ *be the different vectors of* $\{u^z(\theta) \colon \theta \in [0,1)\}$. *Then* z *is a convex combination of* $z^1, ..., z^l$.

The following are useful results on multimodular functions.

THEOREM EC.3 **(Hajek (1985), Theorem 5.2)**. *Let* J *be a multimodular function on* $\mathbb{Z}^m$ *and let* $\overline{J}$ *denote its lower convex envelope. If* z *is any integer sequence with asymptotic mean* r, *then*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} J(z_k, ..., z_{k+m-1}) \ge \overline{J}(r\mathbf{e}).$$

THEOREM EC.4 **(Altman et al. (2003), Lemma 7)**. *Let* $J_t : \mathbb{Z}^{t+f(L)} \to \mathbb{R}_+$, $t = 1, 2, ...$ *be a sequence of functions, where* $f(L) \in \mathbb{Z}_+$. *If* J_t *satisfies the conditions in Lemma 6(i) and (ii) of this paper, then for any* $\theta \in [0,1)$ *and* $r \ge 0$,

$$\limsup_{n\to\infty} \frac{1}{T} \sum_{t=1}^{T} J_t(b^r_1(\theta), ..., b^r_{t+f(L)}(\theta)) \le \lim_{T\to\infty} \overline{J}_T(r\boldsymbol{e}).$$

EC.2. Missing proofs to the results in Section 2

Proof of Lemma 1. Note that at the beginning of period t, $x_{t-L-f(L)+l}$, $l = 0, ..., f(L)$, if still outstanding, will be received in period t if $t + l \le t + K^L_t$; it is not received in periods $j = t - f(L), ..., t-1$ if $K^L_j + j + 1 \le t + l$. Therefore, the summation starts from $\max_{j=t-f(L),...,t-1} \{j + K^L_j + 1\}$ and ends at $t + K^L_t$, and $q_t = \sum_{n=\max_{j=t-f(L),...,t-1}\{j+K^L_j+1\}}^{t+K^L_t} x_{n-L-f(L)}$. $\square$

Proof of Lemma 2. First, recall that the regular orders received in period t are those that are still outstanding at the beginning of period t and were placed in period $t-L-f(L)+K_t^L$ or earlier, and $o_{t,i}$ is the regular order placed $L+f(L)-i+1$ periods ago if that order has not yet arrived. Therefore, the latest placed order that is received in period t, if any, would be o_{t,l_t} according to the definition of l_t. If we receive any order in period t, we must have $e_t \le l_t$, and thus the earliest placed order that is received in period $t+1$, if any, would be o_{t+1,l_t} according to the evolution of the pipeline inventory. On the other hand, if $e_t > l_t$, there is no arriving order in period t, and the earliest placed order that is received in period $t+1$, if any, would be o_{t+1,e_t-1} according to the evolution of the pipeline inventory. Therefore, $e_{t+1} = \max\{e_t-1, l_t\}$ denotes the index (in $\boldsymbol{o}_{t+1}$) of the earliest placed order that is received in period $t+1$. This concludes that the received order quantity q_t is controlled by the Markov chain.

It is obvious that e_t, l_t both take values in $\{1, ..., f(L)+1\}$, and this Markov chain has a finite number of possible states. From its evolution, this Markov chain is irreducible, aperiodic, and thus ergodic. Its evolution only depends on the realizations of the random variables $\{K_t^L\}$, and therefore this Markov chain is exogenous. The proof is completed. □

EC.3. Proof of Theorem 1

We only prove the result for the backlogging case, as the lost-sales case can be done similarly. In addition, as all results in this section will be stated for a fixed $L>0$, we suppress some associated notational dependencies.

We denote $U \triangleq \min_{r\in[0,\mathbb{E}[D]]} C(\pi_r)$, where π_r is a constant-order type policy introduced in Section 4. Then U is a natural upper bound of OPT(L). For $\epsilon > 0$, let $\pi^{L,\epsilon}$ denote some policy such that OPT$(L) > C(\pi^{L,\epsilon}) - \frac{\epsilon}{2}$. Denote

$$\begin{aligned} U_{L,\epsilon} &\triangleq \frac{2U(L+f(L))}{\epsilon \min(p,h)} + (L+f(L)) \cdot (\frac{4U(L+f(L))}{\epsilon\min(p,h)} + \mathbb{E}[D])/p^L_{f(L)}, \\ U'_{L,\epsilon} &\triangleq (2L+2f(L)+U_{L,\epsilon}/\mathbb{E}[D]+2)(g(0)+(h+p)U_{L,\epsilon}). \end{aligned}$$

It follows from the definition of limsup that there exists some large $T_\epsilon > \frac{U'_{L,\epsilon}}{\epsilon(1-\epsilon)} + \frac{L+f(L)}{1-\epsilon}$ such that $C(\pi^{L,\epsilon}) > T^{-1}\sum_{t=1}^T \mathbb{E}[c_t^{\pi^{L,\epsilon}}] - \frac{\epsilon}{2}$ for all $T \ge (1-\epsilon)T_\epsilon - L - f(L)$. The next lemma shows that there exists some $T_{1,\epsilon}$ close to T_ϵ such that the expectations of the inventory level and pipeline inventory vector around time $T_{1,\epsilon}$ under policy $\pi^{L,\epsilon}$ are relatively small.

Lemma EC.5. *For all $\epsilon \in (0, \min(U,1))$, there exists $T_{1,\epsilon} \in [(1-\epsilon)T_\epsilon - L - f(L), T_\epsilon]$ such that*

$$C(\pi^{L,\epsilon}) > T_{1,\epsilon}^{-1}\sum_{t=1}^{T_{1,\epsilon}} \mathbb{E}[c_t^{\pi^{L,\epsilon}}] - \frac{\epsilon}{2},$$

and for all $t \in [T_{1,\epsilon}, T_{1,\epsilon}+L+f(L)-1]$,

$$\mathbb{E}[|s_t^{\pi^{L,\epsilon}}|] \le \frac{2U(L+f(L))}{\epsilon\min(p,h)}, \text{ and } \mathbb{E}[x_{t-L-f(L)}^{\pi^{L,\epsilon}}] \le \left(\frac{4U(L+f(L))}{\epsilon\min(p,h)} + \mathbb{E}[D]\right)/p^L_{f(L)}. \tag{EC.1}$$

Proof of Lemma EC.5. Note that we may partition the time interval $[(1-\epsilon)T_\epsilon - L - f(L), T_\epsilon]$ into $\lceil \frac{\epsilon T_\epsilon}{L+f(L)} \rceil$ disjoint intervals each of length $L+f(L)$, plus an additional disjoint time interval of length no more than $L+f(L)$. Suppose for contradiction that there does not exist a single interval I such that $\forall t \in I$,

$$\mathbb{E}[|s_t^{\pi^{L,\epsilon}}|] \leq \frac{2U(L+f(L))}{\epsilon \min(p,h)}, \text{ and } \mathbb{E}[x_{t-L-f(L)}^{\pi^{L,\epsilon}}] \leq \left(\frac{4U(L+f(L))}{\epsilon \min(p,h)} + \mathbb{E}[D]\right)/p_{f(L)}^L.$$

Then each of these $\lceil \frac{\epsilon T_\epsilon}{L+f(L)} \rceil$ intervals contains at least one time period t such that $\mathbb{E}[|s_t^{\pi^{L,\epsilon}}|] > \frac{2U(L+f(L))}{\epsilon \min(p,h)}$, or $\mathbb{E}[x_{t-L-f(L)}^{\pi^{L,\epsilon}}] > (\frac{4U(L+f(L))}{\epsilon \min(p,h)} + \mathbb{E}[D])/p_{f(L)}^L$. On the one hand, if $\mathbb{E}[|s_t^{\pi^{L,\epsilon}}|] > \frac{2U(L+f(L))}{\epsilon \min(p,h)}$, the cost of period $t-1$ is at least $\frac{2U(L+f(L))}{\epsilon}$. On the other hand, note that according to stochastic lead time assumption, we would receive the order quantity $x_{t-L-f(L)}$ in period t with a positive probability $p_{f(L)}^L$. If $\mathbb{E}[x_{t-L-f(L)}^{\pi^{L,\epsilon}}] > (\frac{4U(L+f(L))}{\epsilon \min(p,h)} + \mathbb{E}[D])/p_{f(L)}^L$, then it follows from the inventory update dynamics that the expected cost of either period $t-1$ or period t is at least $\frac{2U(L+f(L))}{\epsilon}$. Hence by the definition of T_ϵ, we can conclude that

$$\begin{aligned} C(\pi^{L,\epsilon}) &> \frac{\sum_{t=\lceil(1-\epsilon)T_\epsilon\rceil - L - f(L) - 1}^{T_\epsilon} \mathbb{E}[c_t^{\pi^{L,\epsilon}}]}{T_\epsilon} - \frac{\epsilon}{2} \\ &> \frac{\frac{\epsilon T_\epsilon}{L+f(L)} \times \frac{2U(L+f(L))}{\epsilon}}{T_\epsilon} - \frac{\epsilon}{2} = 2U - \frac{\epsilon}{2} > \frac{3U}{2}, \end{aligned}$$

and thus $\mathrm{OPT}(L) > \frac{3U}{2} - \frac{\epsilon}{2} > U$, which is a contradiction. This completes the proof. □

We next prove an approximate version of Theorem 1.

Lemma EC.6. *For any given $L > 0$, for all $\epsilon \in (0, \min(U,1))$, one may construct an L-dimensional vector $\boldsymbol{X}^{L,\epsilon} = (X_1^{L,\epsilon}, X_2^{L,\epsilon}, ..., X_L^{L,\epsilon})$, an $(L-f(L))$-dimensional vector $\boldsymbol{Y}^{L,\epsilon} = (Y_1^{L,\epsilon}, Y_2^{L,\epsilon}, ..., Y_{L-f(L)}^{L,\epsilon})$ and a random variable $I^{L,\epsilon}$ on a common probability space with $\{D_t\}$ and $\{K_t^L\}$ such that the following properties hold:*

(i) With probability 1, *$(\boldsymbol{X}^{L,\epsilon}, \boldsymbol{Y}^{L,\epsilon})$ is nonnegative, and $Y_i^{L,\epsilon} \leq \bar{y}$ for $i \in [1, L-f(L)]$. Also, $(\boldsymbol{X}^{L,\epsilon}, I^{L,\epsilon})$ is independent of $\{D_t\}$, $\{K_t^L\}$, and $Y_i^{L,\epsilon}$ is independent of $\{D_t\}_{t \geq i}$, $\{K_t^L\}_{t>i}$ for $i \in [1, L-f(L)]$;*

(ii) For any $1 \leq t \leq L - f(L)$,

$$I^{L,\epsilon} \sim I^{L,\epsilon} + \sum_{i=1}^{t} \sum_{n=\max\limits_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} X_n^{L,\epsilon} + \sum_{i=1}^{t} Y_i^{L,\epsilon} - \sum_{i=1}^{t} D_i;$$

(iii) $X_i^{L,\epsilon} \sim X_1^{L,\epsilon}$ for all $i = 2, ..., L$, and $Y_i^{L,\epsilon} \sim Y_1^{L,\epsilon}$ for all $i = 2, ..., L-f(L)$;

(iv) $\mathbb{E}[X_1^{L,\epsilon}] + \mathbb{E}[Y_1^{L,\epsilon}] = \mathbb{E}[D]$;

(v) $\mathrm{OPT}(L) \geq \mathbb{E}[g(Y_1^{L,\epsilon})] + h\mathbb{E}[(I^{L,\epsilon})^+] + p\mathbb{E}[(I^{L,\epsilon})^-] - 2\epsilon$.

Proof of Lemma EC.6. We assume that the initial inventory level is $\bar{\mathcal{S}} = -\sum_{i=1}^{\hat{G}+L+f(L)+1} D_{-i}$, where $\hat{G}$ is a geometrically distributed random variable with $\mathbb{P}(\hat{G}=n)=2^{-n}$ for $n \geq 1$, and the initial pipeline inventory is empty. We first claim that when analyzing the optimal long-run average cost, starting from such an initial state $(\bar{\mathcal{S}}, \mathbf{0})$ is equivalent to starting from the original initial state $(0, \mathbf{0})$, i.e., zero on-hand inventory and empty initial pipeline inventory. To see this, on the one hand, if we start from the original initial state, we run the inventory system by setting $x_t = y_t = 0$ for $\hat{G}+L+f(L)+1$ periods, then the whole system arrives at the state $(\bar{\mathcal{S}}, \mathbf{0})$. This process only generates a finite cost, and after that the two systems are coupled and controlled optimally. Therefore, the optimal long-run average cost of the system starting from $(0, \mathbf{0})$ is at most that of the system starting from $(\bar{\mathcal{S}}, \mathbf{0})$. On the other hand, if we start from the state $(\bar{\mathcal{S}}, \mathbf{0})$, we run the policy which follows the optimal policy of the system starting from $(\bar{\mathcal{S}}, \mathbf{0})$, except that we place an additional random regular order $\sum_{i=1}^{\hat{G}+L+f(L)+1} D_{-i}$ in the first period. Then after at most $L+f(L)$ periods, the system would be the same as that starting from $(0, \mathbf{0})$ and controlled optimally, and the expected cost difference remains as a constant. Therefore, the optimal long-run average cost of the system starting from $(\bar{\mathcal{S}}, \mathbf{0})$ is at most that of the system starting from the original initial state $(0, \mathbf{0})$. Combining the above, we prove our claim.

The remaining proof of this lemma involves the construction of a Markov chain which repeatedly mimics the behavior of the original Markov chain induced by the policy $\pi^{L,\epsilon}$ from period 1 to period $T_{1,\epsilon}$, and then drives the inventory process back to the state with initial inventory level $\bar{\mathcal{S}}$ and empty pipeline inventory, which takes a random time whose expectation is bounded. Finally, one can apply the theory of regenerative processes to show the existence of relevant stationary measure.

Since we need to deal with state distributions, we sometimes use "=" to refer to "equal in distribution". Let $\{B_t\}_{t\geq 1}$ denote an i.i.d. sequence of Bernoulli random variables, each of which equals 1 with probability (w.p.) $\frac{1}{2}$ and 0 w.p. $\frac{1}{2}$. We construct a discrete-time Markov process $\{Z_t^\epsilon\}_{t\geq 1} = \{(\mathcal{X}_{t,1}^\epsilon, ..., \mathcal{X}_{t,L+f(L)}^\epsilon, \mathcal{Y}_t^\epsilon, \mathcal{S}_t^\epsilon, \tau_t^\epsilon)\}_{t\geq 1}$, where $\mathcal{X}_{t,l}^\epsilon$ represents the regular order placed $L+f(L)+1-l$ periods ago, $\forall l \in [1, L+f(L)]$, $\mathcal{Y}_t^\epsilon$ represents the immediate decision, $\mathcal{S}_t^\epsilon$ represents the current inventory level, and τ_t^ϵ is a random variable. According to the stochastic lead time assumption, the received regular order quantity in period t depends on the random variables $K_{t-f(L)}^L, ..., K_t^L$, so we denote the regular ordering decision under policy $\pi^{L,\epsilon}$ in period t by $f_{x,t}^{\pi^{L,\epsilon}}(\mathcal{S}_t^\epsilon, \mathcal{X}_{t,1}^\epsilon, ..., \mathcal{X}_{t,L+f(L)}^\epsilon, K_{t-f(L)}^L, ..., K_t^L)$, and the immediate decision under policy $\pi^{L,\epsilon}$ in period t by $f_{y,t}^{\pi^{L,\epsilon}}(\mathcal{S}_t^\epsilon, \mathcal{X}_{t,1}^\epsilon, ..., \mathcal{X}_{t,L+f(L)}^\epsilon, K_{t-f(L)}^L, ..., K_t^L)$. Then $\{Z_t^\epsilon\}_{t\geq 1}$ evolves as follows.

- $\mathcal{X}_{1,l}^\epsilon = 0$, $l \in [1, L+f(L)]$, $\mathcal{S}_1^\epsilon = \bar{\mathcal{S}}, \tau_1^\epsilon = 1$.

For $t \geq 1$, the dynamics are as follows.

- $\mathcal{X}^{\epsilon}_{t+1,l} = \mathcal{X}^{\epsilon}_{t,l+1}$ for $l \in [1, L+f(L)-1]$, $\mathcal{S}^{\epsilon}_{t+1} = \mathcal{S}^{\epsilon}_{t} + \sum_{n=\max_{j=t-f(L),...,t-1}\{j-t+K^L_j+1\}}^{K^L_t} \mathcal{X}^{\epsilon}_{t,n+1} + \mathcal{Y}^{\epsilon}_{t} - D_t$.
- If $\tau^{\epsilon}_t \in [1, T_{1,\epsilon})$:
  $\mathcal{X}^{\epsilon}_{t+1,L+f(L)} = f^{\pi^{L,\epsilon}}_{x,t}(\mathcal{S}^{\epsilon}_t, \mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}, K^L_{t-f(L)}, ..., K^L_t)$,
  $\mathcal{Y}^{\epsilon}_t = f^{\pi^{L,\epsilon}}_{y,t}(\mathcal{S}^{\epsilon}_t, \mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}, K^L_{t-f(L)}, ..., K^L_t)$, $\tau^{\epsilon}_{t+1} = \tau^{\epsilon}_t + 1$.
- If $\tau^{\epsilon}_t = T_{1,\epsilon}$ and either $(\mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}) \neq \mathbf{0}$ or $\mathcal{S}^{\epsilon}_t > 0$:
  $\mathcal{X}^{\epsilon}_{t+1,L+f(L)} = \mathcal{Y}^{\epsilon}_t = 0$, $\tau^{\epsilon}_{t+1} = T_{1,\epsilon}$.
- If $\tau^{\epsilon}_t = T_{1,\epsilon}$ and $(\mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}) = \mathbf{0}$ and $\mathcal{S}^{\epsilon}_t \leq 0$:
  $\mathcal{X}^{\epsilon}_{t+1,L+f(L)} = -\mathcal{S}^{\epsilon}_t$, $\mathcal{Y}^{\epsilon}_t = 0$, $\tau^{\epsilon}_{t+1} = 0$.
- If $\tau^{\epsilon}_t = 0$ and $(\mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}) \neq \mathbf{0}$:
  $\mathcal{X}^{\epsilon}_{t+1,L+f(L)} = \mathcal{Y}^{\epsilon}_t = 0$, $\tau^{\epsilon}_{t+1} = 0$.
- If $\tau^{\epsilon}_t = 0$ and $(\mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}) = \mathbf{0}$:
  $\mathcal{X}^{\epsilon}_{t+1,L+f(L)} = \mathcal{Y}^{\epsilon}_t = 0$, $\tau^{\epsilon}_{t+1} = B_t$.

Let $z(s,x,y) \triangleq \mathbb{E}[h(s+x+y-D)^+ + p(s+x+y-D)^- + g(y)]$, and $\hat{T}_{\epsilon}$ denote a random variable distributed as the time between the Markov chain's first and second visit to a state such that $\tau^{\epsilon}_t = 1$. One may easily verify the following properties of $\{Z^{\epsilon}_t\}_{t\geq 1}$.

- It follows directly from the Markov chain dynamics that for all $t \geq 1$, $\mathcal{X}^{\epsilon}_{t,l+1} \sim \mathcal{X}^{\epsilon}_{t+1,l}$ for $l \in [1, L+f(L)-1]$.
- Conditional on the event $\{\tau^{\epsilon}_t = T_{1,\epsilon}\}$, it follows from Lemma EC.5 that the expected number of time steps until $\tau^{\epsilon}_t = 0$ and $(\mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}) = \mathbf{0}$ is at most $2L + 2f(L) + U_{L,\epsilon}/\mathbb{E}[D]$.
- Conditional on the event $\{\tau^{\epsilon}_t = 0, (\mathcal{X}^{\epsilon}_{t,1}, ..., \mathcal{X}^{\epsilon}_{t,L+f(L)}) = \mathbf{0}\}$, the number of time steps until $\tau^{\epsilon}_t = 1$ is distributed as $\hat{G}$.
- Conditional on the event $\{\tau^{\epsilon}_t = 1\}$, the joint distribution of $\{Z^{\epsilon}_i, i \in [t, t+T_{1,\epsilon}-1]\}$ is identical to that of $\{(\mathcal{X}^{\epsilon}_{i,1}, ..., \mathcal{X}^{\epsilon}_{i,L+f(L)}, \mathcal{Y}^{\epsilon}_i, \mathcal{S}^{\epsilon}_i, i), i \in [1, T_{1,\epsilon}]\}$.
- With probability one (w.p.1) $\hat{T}_{\epsilon} \geq T_{1,\epsilon}$, and $\mathbb{E}[\hat{T}_{\epsilon}] - T_{1,\epsilon} \leq 2L + 2f(L) + U_{L,\epsilon}/\mathbb{E}[D] + 2$.
- $0 \leq \mathbb{E}[\sum_{t=T_{1,\epsilon}+1}^{\hat{T}_{\epsilon}} z(\mathcal{S}^{\epsilon}_t, \sum_{n=\max_{j=t-f(L),...,t-1}\{j-t+K^L_j+1\}}^{K^L_t} \mathcal{X}^{\epsilon}_{t,n+1}, \mathcal{Y}^{\epsilon}_t)] \leq U'_{L,\epsilon}$.

Combining with the basic definitions associated with the theory of regenerative processes (e.g., Asmussen (2008)), we conclude that $\{Z^{\epsilon}_t\}_{t\geq 1}$ is a discrete-time aperiodic regenerative process, with regeneration points coinciding with visits to states such that $\tau^{\epsilon}_t = 1$. Then we may conclude the following from standard results in the theory of regenerative processes (e.g., Asmussen (2008)).

(i) $\{Z^{\epsilon}_t\}_{t\geq 1}$ converges weakly to a random vector $Z^{\epsilon}_{\infty} = (\mathcal{X}^{\epsilon}_{\infty,1}, ..., \mathcal{X}^{\epsilon}_{\infty,L+f(L)}, \mathcal{Y}^{\epsilon}_{\infty}, \mathcal{S}^{\epsilon}_{\infty}, \tau^{\epsilon}_{\infty})$ as $t \to \infty$.

(ii) Initializing the relevant Markov chain with initial conditions distributed as Z^{ϵ}_{∞}, and the stochastic lead time process with its steady state (note that we have shown that the supply system can be modeled as an ergodic Markov chain in Section 2.1), yields a stationary

Markov process $\{\bar{Z}^\epsilon_t\}_{t\geq 1}=\{(\bar{\mathcal{X}}^\epsilon_{t,1},...,\bar{\mathcal{X}}^\epsilon_{t,L+f(L)},\bar{\mathcal{Y}}^\epsilon_t,\bar{\mathcal{S}}^\epsilon_t,\bar{\tau}^\epsilon_t)\}_{t\geq 1}$. Furthermore, it follows directly from the relevant Markov chain dynamics that we may construct $\{\bar{Z}^\epsilon_t\}_{t\geq 1}$ and $\{D_i\},\{K^L_i\}$ on an appropriate probability space such that letting $I^{L,\epsilon}=\bar{\mathcal{S}}^\epsilon_1$, $(X^{L,\epsilon}_1,...,X^{L,\epsilon}_L)=(\bar{\mathcal{X}}^\epsilon_{1,1},...,\bar{\mathcal{X}}^\epsilon_{1,L})$, and $Y^{L,\epsilon}_i=\bar{\mathcal{Y}}^\epsilon_i$ for $i\in[1,L-f(L)]$ yields random vectors satisfying conditions (i) - (iv) of Lemma EC.6.

(iii) The long-run average cost satisfies

$$\mathbb{E}\Big[z\Big(\mathcal{S}^\epsilon_\infty,\sum_{n=\max_{j=-f(L),...,-1}\{j+K'^L_j+1\}}^{K'^L_0}\mathcal{X}^\epsilon_{\infty,n+1},\mathcal{Y}^\epsilon_\infty\Big)\Big]=\frac{\mathbb{E}\Big[\sum_{t=1}^{\hat{T}_\epsilon}z\Big(\mathcal{S}^\epsilon_t,\sum_{n=\max_{j=t-f(L),...,t-1}\{j-t+K^L_j+1\}}^{K^L_t}\mathcal{X}^\epsilon_{t,n+1},\mathcal{Y}^\epsilon_t\Big)\Big]}{\mathbb{E}[\hat{T}_\epsilon]},$$

where $\{K'^L_t,t\in(-\infty,\infty)\},\{K^L_t,t\in(-\infty,\infty)\}$ are mutually independent sequences of i.i.d. realizations, distributed as the random variable K^L.

Further combining (iii) with the previous bounds for $\mathbb{E}[\hat{T}_\epsilon]$, $\mathbb{E}\Big[\sum_{t=1}^{\hat{T}_\epsilon}z\Big(\mathcal{S}^\epsilon_t,\sum_{n=\max_{j=t-f(L),...,t-1}\{j-t+K^L_j+1\}}^{K^L_t}\mathcal{X}^\epsilon_{t,n+1},\mathcal{Y}^\epsilon_t\Big)\Big]$, and the definition of $T_{1,\epsilon}$, we have

$$\begin{aligned}
\mathrm{OPT}(L)&>T^{-1}_{1,\epsilon}\sum_{t=1}^{T_{1,\epsilon}}\mathbb{E}\Big[z\Big(\mathcal{S}^\epsilon_t,\sum_{n=\max_{j=t-f(L),...,t-1}\{j-t+K^L_j+1\}}^{K^L_t}\mathcal{X}^\epsilon_{t,n+1},\mathcal{Y}^\epsilon_t\Big)\Big]-\epsilon\\
&>\frac{\mathbb{E}\Big[\sum_{t=1}^{\hat{T}_\epsilon}z\Big(\mathcal{S}^\epsilon_t,\sum_{n=\max_{j=t-f(L),...,t-1}\{j-t+K^L_j+1\}}^{K^L_t}\mathcal{X}^\epsilon_{t,n+1},\mathcal{Y}^\epsilon_t\Big)\Big]-U'_{L,\epsilon}}{T_{1,\epsilon}}-\epsilon\\
&>\frac{\mathbb{E}\Big[\sum_{t=1}^{\hat{T}_\epsilon}z\Big(\mathcal{S}^\epsilon_t,\sum_{n=\max_{j=t-f(L),...,t-1}\{j-t+K^L_j+1\}}^{K^L_t}\mathcal{X}^\epsilon_{t,n+1},\mathcal{Y}^\epsilon_t\Big)\Big]}{\mathbb{E}[\hat{T}_\epsilon]}-\frac{U'_{L,\epsilon}}{(1-\epsilon)T_\epsilon-L-f(L)}-\epsilon\\
&>\mathbb{E}\Big[z\Big(\mathcal{S}^\epsilon_\infty,\sum_{n=\max_{j=-f(L),...,-1}\{j+K'^L_j+1\}}^{K'^L_0}\mathcal{X}^\epsilon_{\infty,n+1},\mathcal{Y}^\epsilon_\infty\Big)\Big]-2\epsilon,
\end{aligned}$$

which further implies that condition (v) of Lemma EC.6 is also satisfied. The proof of this lemma is completed. □

Proof of Theorem 1. To complete the proof of Theorem 1, we now prove that the sequence of random vectors $\{(X^{L,\frac{1}{n}}_1,...,X^{L,\frac{1}{n}}_L,Y^{L,\frac{1}{n}}_1,...,Y^{L,\frac{1}{n}}_{L-f(L)},S^{L,\frac{1}{n}}),n\geq\frac{1}{U}\}$ is tight. It follows from Lemma EC.6 (v) and the fact $\mathrm{OPT}(L)\leq U$ that for all $n\geq\frac{1}{U}$,

$$\mathbb{E}[|I^{L,\frac{1}{n}}|]\leq\frac{3U}{\min(p,h)}. \qquad \text{(EC.2)}$$

Combining with Lemma EC.6 (iii)–(iv) and nonnegativity, we can conclude the desired tightness, and hence existence of a subsequence $\{(X_1^{L,\frac{1}{n_i}},...,X_L^{L,\frac{1}{n_i}},Y_1^{L,\frac{1}{n_i}},...,Y_{L-f(L)}^{L,\frac{1}{n_i}},I^{L,\frac{1}{n_i}})\}_i$ converging weakly to $(X_1^L,...,X_L^L,Y_1^L,...,Y_{L-f(L)}^L,I^L)$ due to Prokhorov's theorem (Billingsley (2013)). Then Theorem 1 (i)-(iv) follow from the definition of weak convergence. In addition, because the absolute value function is continuous, it follows from Fatou's lemma for weakly converging probability (see Theorem EC.2) that

$$\mathbb{E}[|I^L|] \le \liminf_{i\to\infty} \mathbb{E}[|I^{L,\frac{1}{n_i}}|].$$

Combining with (EC.2) and the already proven Theorem 1 (iii)-(iv) completes the proof of Theorem 1 (v). A similar argument can be applied to prove (vi). The proof of Theorem 1 is completed. □

EC.4. Missing proofs to the results in Section 4

In this appendix, we provide the missing proofs to the results in Section 4.

Proof of Lemma 4. It follows from the definition of $I_L^L(r)$ that for any L such that $L-f(L)\ge m_1$,

$$\begin{aligned}\mathbb{E}[I_L^L(r)] =&\mathbb{E}\left[\sup_{l=0,1,...,L-f(L)}\left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),...,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r-\sum_{t=1}^{l}D_t\right\}\right]\\ \ge&\mathbb{E}\left[\sup_{l=0,1,...,L-f(L)}\left\{\sum_{t=1}^{l}r-\sum_{t=1}^{l}D_t\right\}\right]\\ \ge&\mathbb{E}[\max(0,m_1r-\sum_{t=1}^{m_1}D_t)]\\ =&\sigma m_1^{1/2}\mathbb{E}\left[\max\left(0,\frac{\sum_{t=1}^{m_1}(\mathbb{E}[D]-D_t)}{\sigma m_1^{1/2}}+\frac{m_1r-m_1\mathbb{E}[D]}{\sigma m_1^{1/2}}\right)\right],\end{aligned} \tag{EC.3}$$

where the first inequality holds because $\{\sum_{n=\max\limits_{j=t-f(L),...,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r\}_{t\ge1}$ are a sequence of random variables with expected value equal to r (one can also see this from the Markov chain representation presented in Section 2.1), and we can apply Jensen's inequality. Let $\tilde{N}$ denote a standard normal random variable. It follows from Theorem EC.1 and (EC.3) that,

$$\mathbb{E}[I_L^L(r)] \ge \sigma m_1^{1/2}\left(\mathbb{E}\left[\max\left(0,\tilde{N}+\frac{m_1r-m_1\mathbb{E}[D]}{\sigma m_1^{1/2}}\right)\right]-3m_1^{-1/2}\zeta\right). \tag{EC.4}$$

Note that $C_L^L(0)\ge C_L^L(r_L^L)$. Therefore,

$$p\mathbb{E}[D]\ge h\mathbb{E}[I_L^L(r_L^L)]+p\mathbb{E}[D]-pr_L^L. \tag{EC.5}$$

Combining (EC.4), (EC.5) and the fact that $0\le r_L^L\le \mathbb{E}[D]$, we have

$$p\mathbb{E}[D]\ge pr_L^L\ge h\mathbb{E}[I_L^L(r_L^L)]\ge h\sigma m_1^{1/2}\left(\mathbb{E}\left[\max\left(0,\tilde{N}+\frac{m_1r_L^L-m_1\mathbb{E}[D]}{\sigma m_1^{1/2}}\right)\right]-3m_1^{-1/2}\zeta\right),$$

which further implies

$$m_1^{-1/2}[(h\sigma)^{-1}p\mathbb{E}[D]+3\zeta] \geq \mathbb{E}\left[\max\left(0, \tilde{N}+\frac{m_1 r_L^L - m_1\mathbb{E}[D]}{\sigma m_1^{1/2}}\right)\right]. \tag{EC.6}$$

It is easily verified that $(\mathbb{E}[\max(0,\tilde{N}-1)])^{-1} \leq 13$, and thus by definition,

$$m_1 \geq ((3\zeta + (h\sigma)^{-1}p\mathbb{E}[D])(\mathbb{E}[\max(0,\tilde{N}-1)])^{-1})^2.$$

Therefore, we can conclude that the left-hand side of (EC.6) is at most $\mathbb{E}[\max(0,\tilde{N}-1)]$. It follows from the monotonicity of the function $x \mapsto \mathbb{E}[\max(0,\tilde{N}+x)]$ that

$$-1 \geq \frac{m_1 r_L^L - m_1\mathbb{E}[D]}{\sigma m_1^{1/2}},$$

namely,

$$r_L^L \leq \mathbb{E}[D] - \sigma m_1^{-1/2} = \mathbb{E}[D] - \eta.$$

The proof is completed. □

Proof of Lemma 5. We first prove that $\mathbb{P}(i_L^l(r)=l) \geq \mathbb{P}(i_\infty^L(r)=l)$ for any $0 \leq l \leq L-f(L)$. By definition, we have $\mathbb{P}(i_L^L(r)=l)=\mathbb{P}(E_1)$ and $\mathbb{P}(i_\infty^L(r)=l)=\mathbb{P}(E_1\cap E_2)$, where

$$E_1 \triangleq \left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\ldots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r - \sum_{t=1}^{l} D_t \geq \sum_{t=1}^{k}\sum_{n=\max\limits_{j=t-f(L),\ldots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r - \sum_{t=1}^{k} D_t,\ \forall\ 0 \leq k \leq L-f(L)\right\},$$

$$E_2 \triangleq \left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\ldots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r - \sum_{t=1}^{l} D_t \geq \sum_{t=1}^{k}\sum_{n=\max\limits_{j=t-f(L),\ldots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r - \sum_{t=1}^{k} D_t,\ \forall\ k \geq L-f(L)+1\right\}.$$

The desired inequality follows immediately.

Then we prove that under Assumption 2, $\mathbb{P}(i_\infty^L(r)=l) \leq \gamma^l$ for any $r \in [0, \mathbb{E}[D]-\eta]$ and $l > L-f(L)$. By definition,

$$\mathbb{P}(i_\infty^L(r)=l) \leq \mathbb{P}\left(\sum_{i=1}^{l+f(L)} r - \sum_{i=1}^{l} D_i \geq 0\right), \tag{EC.7}$$

where the inequality holds because in l consecutive periods, at most $(l+f(L))$ orders will be received. Recall that under Assumption 2, $f(L) < \eta_2 L \leq \eta_0 L/\mathbb{E}[D]$. Then for any $l > L-f(L)$, we have

$$\begin{aligned}\frac{l+f(L)}{l} r &\leq \frac{L}{L-f(L)}(\mathbb{E}[D]-\eta)\\ &< \frac{\mathbb{E}[D]}{\mathbb{E}[D]-\eta_0}(\mathbb{E}[D]-\eta)\\ &= \eta_1 E[D] < \mathbb{E}[D].\end{aligned}$$

Finally, applying a Chernoff bound, we have

$$\begin{aligned}\mathbb{P}(i^L_\infty(r)=l) \leq & \mathbb{P}\Big(\sum_{i=1}^{l+f(L)} r - \sum_{i=1}^{l} D_i \geq 0\Big)\\ \leq & \mathbb{P}\Big(\sum_{i=1}^{l} (\eta_1 \mathbb{E}[D] - D_i) \geq 0\Big)\\ \leq & \inf_{\beta>0} \mathbb{E}^l[exp(\beta(\eta_1\mathbb{E}[D]-D))] = \gamma^l,\end{aligned}$$

The proof is completed. □

Proof of Theorem 2. According to Lemma 3, it suffices to show that under Assumptions 1 and 2, for any $L \geq m_1/(1-\eta_2)$,

$$h(\mathbb{E}[I^L_\infty(r^L_L)] - \mathbb{E}[I^L_L(r^L_L)]) \leq \frac{h\gamma^{(1-\eta_2)L}}{(1-\gamma)^2}(L+\gamma-L\gamma)(\mathbb{E}[D]-\eta).$$

Let's couple the random variables $I^L_\infty(r^L_L)$ and $I^L_L(r^L_L)$, and compare them on each sample path. According to Lemma 5, we know that $\mathbb{P}(i^L_L(r^L_L)=l) \geq \mathbb{P}(i^L_\infty(r^L_L)=l)$ for any $0 \leq l \leq L-f(L)$, and the values of $I^L_\infty(r^L_L)$ and $I^L_L(r^L_L)$ are the same if $i^L_L(r^L_L) = i^L_\infty(r^L_L) = l \leq L-f(L)$. If $i^L_\infty(r^L_L) = l > L-f(L)$, the difference between $I^L_\infty(r^L_L)$ and $I^L_L(r^L_L)$ is at most $(l+f(L))r^L_L$. Therefore,

$$\begin{aligned}h(\mathbb{E}[I^L_\infty(r^L_L)] - \mathbb{E}[I^L_L(r^L_L)]) \leq & h\sum_{l=L-f(L)+1}^{\infty} \mathbb{P}(i^L_\infty(r^L_L)=l)(l+f(L))r^L_L\\ \leq & h\sum_{l=L-f(L)+1}^{\infty} \gamma^l(l+f(L))(\mathbb{E}[D]-\eta)\\ \leq & \frac{h\gamma^{(1-\eta_2)L}}{(1-\gamma)^2}(L+\gamma-L\gamma)(\mathbb{E}[D]-\eta),\end{aligned}$$

where the second inequality follows from Lemma 4 and Lemma 5, and the third inequality follows from Assumption 2 and some basic algebra. The proof is completed. □

Proof of Lemma 6. (i) Consider

$$\begin{aligned}f(y_0,y_1,...,y_{t+f(L)}) =& J_t(y_1-y_0, y_2-y_1, ..., y_{t+f(L)}-y_{t+f(L)-1})\\ =& \mathbb{E}\Big[\max_{k=0,1,...,t}\Big\{\sum_{i=t-k+1}^{t} \sum_{n=\max_{j=i-f(L),...,i-1}\{j+K^L_j+1\}}^{i+K^L_i} (y_n - y_{n-1}) - \sum_{i=t-k+1}^{t} D_i\Big\}\Big].\end{aligned}$$

For any sample path of demands and lead times, $y_1-y_0, y_2-y_1, ..., y_{t+f(L)}-y_{t+f(L)-1}$ are $t+f(L)$ orders that are possibly received in t consecutive periods. Assume $y_m - y_{m-1}$ is the last received order, then $t \leq m \leq t+f(L)$. Because the orders do not cross in time, then for any $k=1,...,t$, the term inside the curly bracket can be expressed as $y_m - y_{n_{t-k+1}} - \sum_{i=t-k+1}^{t} D_i$ for some $0 \leq n_{t-k+1} \leq m$, and thus the term inside the square bracket can be expressed as $y_m + \max\{-y_m, -y_{n_t} - D_t, -y_{n_{t-1}} - D_t - D_{t-1}, ..., -y_{n_1} - \sum_{i=1}^{t} D_i\}$, where $0 \le n_1 \le n_2 \le ... \le n_{t-1} \le n_t \le m$. This function is submodular in $(y_0, y_1, ..., y_{t+f(L)})$ (see, e.g., Example 2.6.2.(f) in Topkis 1998). Hence, $f(y_0, y_1, ..., y_{t+f(L)})$ is submodular in $(y_0, y_1, ..., y_{t+f(L)})$ and $J_t(z_1, ..., z_{t+f(L)})$ is multimodular.

(ii) For any $t \ge m$ and $(z_1, ..., z_{t+f(L)}) \in \mathbb{Z}_+^{t+f(L)}$, $J_t(z_1, \ldots, z_{t+f(L)}) \ge J_t(0, \ldots, 0, z_{t-m+1}, \ldots, z_{t+f(L)})$ holds because J_t is non-decreasing. In addition, note that by definition

$$J_m(z_{t-m+1}, ..., z_{t+f(L)}) = \mathbb{E}\left[\max_{k=0,1,...,m}\left\{\sum_{i=m-k+1}^{m}\sum_{n=\max\limits_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_{n-m+t} - \sum_{i=m-k+1}^{m} D_i\right\}\right],$$

and

$$\begin{aligned}
&J_{m+1}(z_{t-m}, z_{t-m+1}, \ldots, z_{t+f(L)})\\
=&\mathbb{E}\Bigg[\max\Bigg(\max_{k=0,1,...,m}\Bigg\{\sum_{i=m-k+2}^{m+1}\sum_{n=\max\limits_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_{n-m-1+t} - \sum_{i=m-k+2}^{m+1} D_i\Bigg\},\\
&\sum_{i=1}^{m+1}\sum_{n=\max\limits_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_{n-m-1+t} - \sum_{i=1}^{m+1} D_i\Bigg)\Bigg]\\
=&\mathbb{E}\Bigg[\max\Bigg(\max_{k=0,1,...,m}\Bigg\{\sum_{i=m-k+1}^{m}\sum_{n=\max\limits_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_{n-m+t} - \sum_{i=m-k+1}^{m} D_i\Bigg\},\\
&\sum_{i=1}^{m+1}\sum_{n=\max\limits_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_{n-m-1+t} - \sum_{i=1}^{m+1} D_i\Bigg)\Bigg],
\end{aligned}$$

where the second equality holds because $\{K_t^L\}$ and $\{D_t\}$ are i.i.d. and mutually independent. Therefore, when $z_{t-m} = 0$, it is clear that

$$J_{m+1}(0, z_{t-m+1}, \ldots, z_{t+f(L)}) \ge J_m(z_{t-m+1}, ..., z_{t+f(L)}),$$

and we can conclude that $J_t(0, \ldots, 0, z_{t-m+1}, \ldots, z_{t+f(L)}) \ge J_m(z_{t-m+1}, ..., z_{t+f(L)})$ for any $t \ge m$ by induction.

(iii) The statement is straightforward. □

Proof of Lemma 7. We first prove that

$$\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} J_t(b_1^r(\theta), ..., b_{t+f(L)}^r(\theta)) \ge \lim_{t\to\infty} \overline{J}_t(r\mathbf{e}).$$

Let $\{z_t\}_{t\geq 1}$ be any nonnegative integer sequence with an asymptotic mean r, i.e., $\lim_{T\to\infty}\frac{\sum_{t=1}^{T} z_t}{T} = r$, and m be a positive integer. By Lemma 6(ii),

$$\frac{1}{T}\sum_{t=m}^{T} J_t(z_1,\dots,z_{t+f(L)}) \geq \frac{1}{T}\sum_{t=m}^{T} J_m(z_{t-m+1},\dots,z_{t+f(L)}).$$

Thus,

$$\begin{aligned}\liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} J_t(z_1,\dots,z_{t+f(L)}) &\geq \liminf_{T\to\infty}\frac{1}{T}\sum_{t=m}^{T} J_t(z_1,\dots,z_{t+f(L)})\\ &\geq \liminf_{T\to\infty}\frac{1}{T}\sum_{t=m}^{T} J_m(z_{t-m+1},\dots,z_{t+f(L)})\\ &\geq \overline{J}_m(r\mathbf{e}),\end{aligned}$$

where the last inequality follows from Theorem EC.3. Letting m go to infinity, we have

$$\liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} J_t(z_1,\dots,z_{t+f(L)}) \geq \lim_{m\to\infty}\overline{J}_m(r\mathbf{e}).$$

In particular, this holds for the bracket policy. By Theorem EC.4,

$$\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} J_t(b_1^r(\theta),\dots,b_{t+f(L)}^r(\theta)) \leq \lim_{t\to\infty}\overline{J}_t(r\mathbf{e}).$$

Therefore, the statement holds. □

REMARK EC.1. It is worth mentioning that our proof of (8) for any $r\geq 0$ is slightly different from that of Theorem 6.2 in Hajek (1985), where r is in $(0,1)$ and the proof relies on queueing theory. Here we apply results from Altman et al. (2003) and show that the same result also holds for $r\geq 1$.

Proof of Lemma 8. Fix $t\geq 1$. Note that the right-hand side of (9) is convex and agrees with J_t on integer points. Then, using the definition of lower convex envelope, it suffices to prove $\overline{J}_t(z_1,\dots,z_{t+f(L)}) \leq \mathbb{E}\left[\max_{k=0,1,\dots,t}\left\{\sum_{i=t-k+1}^{t}\sum_{n=\max_{j=i-f(L),\dots,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_n - \sum_{i=t-k+1}^{t} D_i\right\}\right]$ for any $z\in\mathbb{R}^{t+f(L)}$.

Fix $z\in\mathbb{R}^{t+f(L)}$. For any realization of $\{D_t\}$ and $\{K_t^L\}$, we can always express the term inside the square bracket of $\mathbb{E}\left[\max_{k=0,1,\dots,t}\left\{\sum_{i=t-k+1}^{t}\sum_{n=\max_{j=i-f(L),\dots,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_n - \sum_{i=t-k+1}^{t} D_i\right\}\right]$ as $\max_{k=0,\dots,t}\left\{\sum_{i=n_{t-k+1}}^{m} z_i - \sum_{i=t-k+1}^{t} D_i\right\}$ for some $1\leq n_1\leq n_2\leq\dots\leq n_{t-1}\leq n_t\leq m< n_{t+1}$, where $t\leq m\leq t+f(L)$. Pick any $k_D\in\arg\max_{k=0,\dots,t}\left\{\sum_{i=n_{t-k+1}}^{m} z_i - \sum_{i=t-k+1}^{t} D_i\right\}$, i.e., k_D satisfies

$$\sum_{i=n_{t-k_D+1}}^{m} z_i - \sum_{i=t-k_D+1}^{t} D_i \geq \sum_{i=n_{t-k+1}}^{m} z_i - \sum_{i=t-k+1}^{t} D_i,\ \forall k=0,1,\dots,t.$$

It follows that for any $\theta \in [0,1)$,

$$\theta + \sum_{i=1}^{m} z_i - \sum_{i=n_{t-k_D+1}}^{m} z_i + \sum_{i=t-k_D+1}^{t} D_i \le \theta + \sum_{i=1}^{m} z_i - \sum_{i=n_{t-k+1}}^{m} z_i + \sum_{i=t-k+1}^{t} D_i, \ \forall k = 0,1,...,t,$$

and thus

$$\lfloor \theta + \sum_{i=1}^{n_{t-k_D+1}-1} z_i \rfloor + \sum_{i=t-k_D+1}^{t} D_i \le \lfloor \theta + \sum_{i=1}^{n_{t-k+1}-1} z_i \rfloor + \sum_{i=t-k+1}^{t} D_i, \ \forall k = 0,1,...,t,$$

because demands are all integers.

Let the vector $u^z(\theta) \in \mathbb{Z}^{t+f(L)}$ be defined by

$$u_i^z(\theta) = \lfloor \theta + z_1 + \cdots + z_i \rfloor - \lfloor \theta + z_1 + \cdots + z_{i-1} \rfloor, \ \forall 1 \le i \le t + f(L).$$

We then have

$$\sum_{i=n_{t-k_D+1}}^{m} u_i^z(\theta) - \sum_{i=t-k_D+1}^{t} D_i \ge \sum_{i=n_{t-k+1}}^{m} u_i^z(\theta) - \sum_{i=t-k+1}^{t} D_i, \ \forall 0 \le k \le t,$$

i.e.,

$$k_D \in \arg\max_{k=0,...,t} \left\{ \sum_{i=n_{t-k+1}}^{m} u_i^z(\theta) - \sum_{i=t-k+1}^{t} D_i \right\}, \ \forall \theta \in [0,1). \tag{EC.8}$$

Let $z^1, ..., z^l$ be the different integral vectors of $\{u^z(\theta) : \theta \in [0,1)\}$ (see Appendix EC.1). By Lemma EC.4, there exist nonnegative numbers $\lambda_1, ..., \lambda_l$ with $\lambda_1 + ... + \lambda_l = 1$ such that $z = \sum_{j=1}^{l} \lambda_j z^j$. Then we have

$$\begin{aligned}\max_{k=0,...,t} \left\{ \sum_{i=n_{t-k+1}}^{m} z_i - \sum_{i=t-k+1}^{t} D_i \right\} &= \sum_{i=n_{t-k_D+1}}^{m} z_i - \sum_{i=t-k_D+1}^{t} D_i \\ &= \sum_{j=1}^{l} \lambda_j \left(\sum_{i=n_{t-k_D+1}}^{m} z_i^j - \sum_{i=t-k_D+1}^{t} D_i \right) \\ &= \sum_{j=1}^{l} \lambda_j \max_{k=0,...,t} \left\{ \sum_{i=n_{t-k+1}}^{m} z_i^j - \sum_{i=t-k+1}^{t} D_i \right\},\end{aligned}$$

where the first equality follows from the definition of k_D and the third equality follows from (EC.8). Since this result holds for any sample path, we can conclude that

$$\begin{aligned}&\mathbb{E}\left[\max_{k=0,1,...,t} \left\{ \sum_{i=t-k+1}^{t} \sum_{n=\max_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_n - \sum_{i=t-k+1}^{t} D_i \right\} \right] \\ &= \sum_{j=1}^{l} \lambda_j \mathbb{E}\left[\max_{k=0,1,...,t} \left\{ \sum_{i=t-k+1}^{t} \sum_{n=\max_{j=i-f(L),...,i-1}\{j+K_j^L+1\}}^{i+K_i^L} z_n^j - \sum_{i=t-k+1}^{t} D_i \right\} \right] \\ &= \sum_{j=1}^{l} \lambda_j J_t(z^j) \ge \overline{J}_t(z),\end{aligned}$$

where the last inequality is from convexity of $\overline{J}_t$. □

Proof of Theorem 3. Because $\{D_t\}, \{K_t^L\}$ are i.i.d. and mutually independent, Lemma 8 implies

$$\lim_{t\to\infty} \overline{J}_t(r\mathbf{e}) = \mathbb{E}\left[\sup_{l=0,1,\dots}\left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r - \sum_{t=1}^{l} D_t\right\}\right].$$

Thus, by Lemma 7, the long-run average holding cost under the bracket policy $\pi_b^{r,\theta}$ is

$$\begin{aligned}\lim_{T\to\infty}\frac{h}{T}\sum_{t=1}^{T}\mathbb{E}[s_{t+1}] = &\lim_{T\to\infty}\frac{h}{T}\sum_{t=1}^{T} J_t(b_1^r(\theta),\dots,b_{t+f(L)}^r(\theta))\\ =& h\mathbb{E}\left[\sup_{l=0,1,\dots}\left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r - \sum_{t=1}^{l} D_t\right\}\right],\end{aligned}$$

which is the same as the long-run average holding cost under the constant-order policy π_r as is shown in (3).

The long-run average lost sales under a bracket policy $\pi_b^{r,\theta}$ with $r < \mathbb{E}[D]$ can be expressed as

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[(s_t + \sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} b_n^r(\theta) - D_t)^-]\\ =&\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left(\mathbb{E}[s_{t+1}] - (\mathbb{E}[s_t] + \mathbb{E}[\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} b_n^r(\theta)] - \mathbb{E}[D_t])\right) \qquad \text{(EC.9)}\\ =&-\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} b_t^r(\theta) + \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[D_t]\\ =&-r + \mathbb{E}[D],\end{aligned}$$

which is the same as that under a constant-order policy π_r. The second equality in (EC.9) holds because (i) the long-run average order quantity is $r < \mathbb{E}[D]$ according to (7), and thus $\limsup_{T\to\infty} E[s_{T+1}]/T = 0$; (ii) $\lim_{T\to\infty}\sum_{t=1}^{T} b_t^r(\theta)/T$ is the long-run average order quantity that is received under a bracket policy $\pi_b^{r,\theta}$. Combining the above, we can conclude that the long-run average cost under $\pi_b^{r,\theta}$ is the same as that under π_r. □

Proof of Lemma 9. We first obtain the closed-form expression of $C(\pi_r)$ and r_∞. When the lead time is deterministic and demand is geometrically distributed, it is well known (cf. Asmussen (2008)) that for $r \in [0, \mathbb{E}[D])$, the expected long-run average inventory level under a constant-order policy π_r equals the expected steady-state waiting time in a $Geo/D/1$ queue, and equals $\frac{\lambda r(r-1)}{2(1-\lambda r)}$ (cf. Gravey et al. (1990)). It follows from (3) that for all $r \in [0, \lambda^{-1})$,

$$C(\pi_r) = h\frac{\lambda r(r-1)}{2(1-\lambda r)} + p\lambda^{-1} - pr,$$

and $\frac{d}{dr}C(\pi_r)=\frac{h}{2}((1-\lambda)(1-\lambda r)^{-2}-1)-p$ strictly increases from $-\frac{\lambda h}{2}-p$ to ∞ on $[0,\lambda^{-1})$. It follows that r_∞, the optimal constant order, must be the unique solution to the equation $\frac{d}{dr}C(\pi_r)=0$ on $[0,\lambda^{-1})$. By a straightforward calculation we can get

$$r_\infty=\frac{h+2p-\sqrt{h^2-h^2\lambda+2hp-2h\lambda p}}{h\lambda+2\lambda p}. \tag{EC.10}$$

Given the value of r_∞, let's consider the randomized rounding policy $\pi^r_{r_\infty}$. Note that $r_u=\lceil r_\infty\rceil$, $r_l=\lfloor r_\infty\rfloor$ and $p_u=r_\infty-r_l$. At time t, under this randomized rounding policy $\pi^r_{r_\infty}$, the received order quantity is r_u with probability p_u, and r_l with probability $1-p_u$. It is clear that this is an integral ordering policy and the order quantity follows a two-point distribution D^2 with mean r_∞. In this case, the expected long-run average inventory level equals the expected steady-state waiting time in a $Geo/D^2/1$ queue. The generating function of the number of customers N_c in a $Geo/G/1$ queue is obtained in Kobayashi and Konheim (1977), and for this $Geo/D^2/1$ queue, this expression becomes

$$N(z)=(1-\lambda r_\infty)\frac{(z-1)[p_u(1-\lambda+\lambda z)^{r_u}+(1-p_u)(1-\lambda+\lambda z)^{r_l}]}{z-[p_u(1-\lambda+\lambda z)^{r_u}+(1-p_u)(1-\lambda+\lambda z)^{r_l}]}.$$

Using the fact that $r_u=r_\infty+1-p_u$ and $r_l=r_\infty-p_u$, the mean number of customers in the system given by $N'(1)$ is:

$$\mathbb{E}[N_c]=N'(1)=\frac{\lambda[2r_\infty-\lambda((-1+p_u)p_u+r_\infty+r_\infty^2)]}{2(1-\lambda r_\infty)}.$$

Using Little's law, we can obtain the expected steady-state waiting time $\mathbb{E}[W_c]$ in this $Geo/D^2/1$ queue:

$$\mathbb{E}[W_c]=\frac{\lambda(r_\infty-p_u)(r_\infty+p_u-1)}{2(1-\lambda r_\infty)}.$$

The expected long-run average cost $C(\pi^r_{r_\infty})$ is given by

$$C(\pi^r_{r_\infty})=h\mathbb{E}[W_c]+p(\lambda^{-1}-r_\infty).$$

Finally, the difference between $C(\pi_{r_\infty})$ and $C(\pi^r_{r_\infty})$ is given by

$$\begin{aligned}C(\pi_{r_\infty})-C(\pi^r_{r_\infty})&=\frac{h\lambda}{2(1-\lambda r_\infty)}[r_\infty(r_\infty-1)-(r_\infty-p_u)(r_\infty+p_u-1)]\\&=\frac{h\lambda}{2(1-\lambda r_\infty)}p_u(p_u-1)\le 0.\end{aligned} \tag{EC.11}$$

This shows that the randomized rounding policy $\pi^r_{r_\infty}$ performs worse than the optimal fractional constant-order policy π_{r_∞}. The inequality in (EC.11) becomes an equality only when p_u equals 0 or 1, i.e., r_∞ is an integer and $\pi^r_{r_\infty}$ is the same as π_{r_∞}. $\square$

Proof of Proposition 1. Consider two inventory systems, called system Π^1 and Π^2, where Π^1 is associated with a constant order quantity r^1 from the regular source, and Π^2 is associated with a constant order quantity r^2 from the regular source. Let $\{y_t^1\}_{t\geq 1}$ and $\{y_t^2\}_{t\geq 1}$ be the optimal ordering decisions from the emergency source respectively. Consider a new inventory system Π^λ with a constant order quantity $r^\lambda = \lambda r^1 + (1-\lambda)r^2$ from the regular source. Now we construct a policy for Π^λ, such that $y_t^\lambda = \lambda y_t^1 + (1-\lambda)y_t^2$ for each t. Under such a policy, for each sample path and in each period, the inventory level of Π^λ is always equal to λ times the inventory level of Π^1 plus $(1-\lambda)$ times the inventory level of Π^2, i.e., $s_t^\lambda = \lambda s_t^1 + (1-\lambda)s_t^2$. Combining the convexity of the inventory cost function and the ordering cost function $g(\cdot)$, we can conclude that that the cost incurred in system Π^λ is at most λ times the cost incurred in system Π^1 plus $(1-\lambda)$ times the cost incurred in system Π^2, which further implies the convexity of $C(\pi_r)$ in r. □

Proof of Lemma 11. According to Theorem 1 (vi), we have $\mathrm{OPT}(L) \geq \min(h,p)|\mathbb{E}[I^L]|$. Note that $U_0 = C(\pi_0) \geq \mathrm{OPT}(L)$ for any $L > 0$. Letting $\bar{S} \triangleq \frac{U_0}{\min(h,p)}$ yields the desired result. □

Proof of Lemma 12. The first inequality in (14) directly follows from (13), and it should hold for any subsequence. To establish the convergence result, note that $\{(r^L, s^L)\}_{L\geq 1}$ are uniformly bounded in L, it follows from Bolzano-Weierstrass theorem (cf. Rudin et al. (1964)) that there exists a subsequence $\{L_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty}(r^{L_n}, s^{L_n}) = (r^\infty, s^\infty) \in \Omega \triangleq [0, \mathbb{E}[D]] \times [-\bar{S}, \bar{S}]. \tag{EC.12}$$

We next fix a sufficiently small $\epsilon > 0$ and let $\Omega^\epsilon \triangleq [-\epsilon, \mathbb{E}[D]+\epsilon] \times [-\bar{S}-\epsilon, \bar{S}+\epsilon]$. Note that one can follow the proof of Proposition 1 to show that $V_L^\alpha(r, s, w_\infty)$ is jointly convex in (r,s) on Ω^ϵ. Then combining the joint convexity and the fact that $V_L^\alpha(r,s,w_\infty)$ converges to $V_\infty^\alpha(r,s,w_\infty)$ pointwise (cf. Feinberg (2016)), from Lemma EC.1, we conclude that $V_L^\alpha(r,s,w_\infty)$ converges to $V_\infty^\alpha(r,s,w_\infty)$ uniformly on Ω. Finally, combining the above uniform convergence with (EC.12) and Lemma EC.2 implies that

$$(1-\alpha)\liminf_{n\to\infty} V_{L_n - L'}^\alpha(r^{L_n}, s^{L_n},, w_\infty) = (1-\alpha)V_\infty^\alpha(r^\infty, s^\infty, w_\infty).$$

The proof is completed. □

Proof of Theorem 4. Under Assumption 3, we can follow Lemma 3 in Bai et al. (2023) to show that $\mathrm{OPT}(L)$ is non-decreasing in L. Combining this with Lemmas 12 and 13, we have

$$\begin{aligned}\lim_{L\to\infty}\mathrm{OPT}(L) &\geq \liminf_{n\to\infty}\mathrm{OPT}(L_n)\\ &\geq \lim_{\alpha\to 1}(1-\alpha)V_\infty^\alpha(s^\infty, w_\infty)\\ &= v^*\\ &= C(\pi_{r^\infty})\\ &\geq \min_{0\leq r\leq \mathbb{E}[D]} C(\pi_r) = U.\end{aligned}$$

On the other hand, and $U \geq \mathrm{OPT}(L)$ for any $L > 0$. Combining the above inequalities yields the statement. □

EC.5. Generalizations of Theorem 4

In this appendix, we explore generalizations of Theorem 4. To be specific, we discuss the difficulty in establishing the convergence rate of the optimality gap, and show how to extend the asymptotic results in dual-sourcing backlog inventory models in which the support of the random component of lead time increases as the deterministic component increases.

EC.5.1. Difficulty in establishing the convergence rate of optimality gap

Xin and Goldberg (2018) showed that in the dual-sourcing inventory model with deterministic lead times, the optimality gap of the tailored base-surge policy decreases at an inverse-polynomial rate as the lead time of the regular supplier increases. They begin by deriving a lower bound of the optimal cost (refer to their equation (9), which resembles our Lemma 10), expressed as the discounted total cost of an infinite-horizon inventory problem, where the optimal policy is a base-stock policy. Then they carefully analyze the Markov process representing the inventory position process under such an optimal stationary base-stock policy, and mention that the overshoot (the amount by which the inventory level exceeds the base stock level) is analogous to the waiting time in a $GI/GI/1$ queue. Finally, they utilize well-known results for single-server queues (see their Lemma 5) to establish the inverse-polynomial convergence rate.

In contrast, our problem presents a key challenge: in the infinite-horizon inventory problem, the optimal policy is a state-dependent modified base-stock policy, where the order-up-to level varies based on the current underlying state w (with w indicating the most recent period in which the placed regular constant order has been received, as per Lemma 2). Consequently, when analyzing the overshoot, we must take into account the fluctuations in the underlying base-stock levels over time. To the best of our knowledge, there is currently no established convergence rate for such a complex Markov process to reach its steady state. Therefore, to establish an explicit bound on the convergence rate in the dual-sourcing inventory model with stochastic lead times, new mathematical methodologies are necessary, and we leave this as a future research direction.

EC.5.2. General stochastic lead times in the dual-sourcing backlog inventory model

In this subsection, we analyze the case where the support of random lead time part goes to infinity as the deterministic lead time part increases. We will make the following mild assumption regarding the stochastic lead time process throughout this subsection.

Assumption EC.3. *There exists some* $\eta_4 \in (0,1)$ *such that* $f(L) : \mathbb{Z}_+ \to \mathbb{Z}_+$ *satisfies* $f(L) \leq \eta_4 L$ *for any* $L > 0$*, and* $K^L = \lceil \xi f(L) \rceil$*, where* ξ *is a continuous random variable in* $[0,1]$*, whose distribution has continuous density, equal to* $p_c \in (0,\infty)$ *at point* 1*.*

Assumption EC.3 basically says that the support of the random variable K^L can increase (even linearly) in L and, as $L \to \infty$, the probability of K^L taking a value j such that $f(L) - j = o(f(L))$ is approximately $p_c/f(L)$. We note that this assumption is mild and allows for a sequence of lead time distributions such that the coefficient of variation of the lead time does not converge to zero as L increases, e.g., the uniform distribution. Then we can show that constant-order type policies are asymptotically optimal in the dual-sourcing backlog inventory model under Assumption EC.3.

THEOREM EC.5. *In the dual-sourcing backlog inventory model with divisible products and no order crossover, under Assumption EC.3, we have* $\lim_{L\to\infty} \mathrm{OPT}(L) = C(\pi_0)$*, where* $C(\pi_0)$ *implements a modified base-stock policy from the emergency supplier.*

In the following, we provide a proof sketch of Theorem EC.5. Recall notation: if f, g are two functions from $\mathbb{Z}_+$ to $\mathbb{R}_+$, then $f = O(g)$ if there exists a positive constant $c > 0$ such that $f(n) \le c \cdot g(n)$ for every sufficiently large n, $f = \Theta(g)$ if there exist positive constants $c_1, c_2 > 0$ such that $c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n)$ for every sufficiently large n.

First, we give an intuition for the structure of the arrival process of the regular orders, placed at constant rate $r > 0$. Let W denote the current value of the random part of the lead time; namely, at a given time t, all orders placed at time $t - L - W$ has arrived. Denote $w = \mathbb{E}W$. Then, the average size b of a non-zero arrival (when such occurs) should be $b = \Theta(w)$. Given Assumption EC.3, the frequency on non-zero arrivals $P(L)$ should be $P(L) = \Theta(w/f(L)) = \Theta(b/f(L))$. But, the average rate of arrivals must be r, so that $P(L)b = \Theta(1)$, implying $b^2/f(L) = \Theta(1)$. We conclude that the asymptotics $b = \Theta(\sqrt{f(L)})$ and $P(L) = \Theta(1/\sqrt{f(L)})$ should hold.

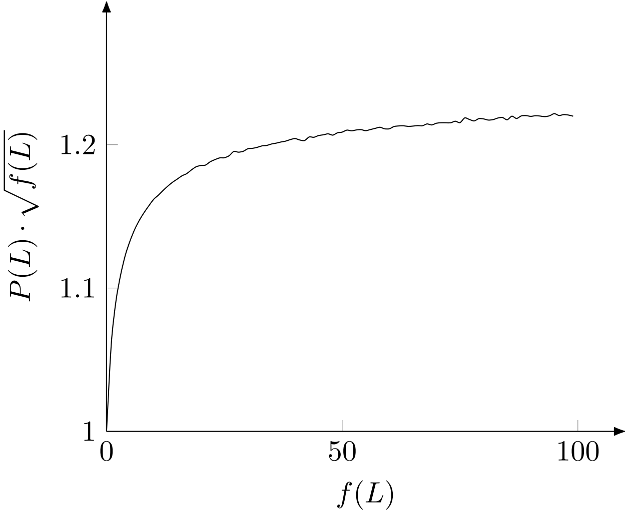
This argument can be carried out rigorously, we do not do it to save space. We also provide an example to demonstrate the relationship between $P(L)$ and $f(L)$. Assume that K^L is uniformly distributed over $\{0, 1, ..., f(L)\}$, we compute the value of $P(L)$ as $f(L)$ increases from 0 to 100 through simulation, and report the result in Figure EC.1. One can see that in this example, $P(L)$ converges to 0 at the rate of $1/\sqrt{f(L)}$ as $f(L)$ increases, which numerically illustrates the claimed property.

Next, we rewrite the lower bound obtained in Lemma 10 under Assumption EC.3 instead of Assumption 3. We have that

$$\mathrm{OPT}(L) \ge (1-\alpha) V^{\alpha}_{L-f(L)}(r^L, s^L, w^L_{\infty}),$$

where the superscript "L" in w^L_∞ means that in the $(L - f(L))$-period single-sourcing backlog inventory problem with Markov-modulated demands, the underlying Markov chain varies as L increases. We then claim that, under Assumption EC.3, $r^L = O(\frac{1}{\sqrt{f(L)}})$, which converges to 0 as L increases. To see this, because $P(L) = \Theta(\frac{1}{\sqrt{f(L)}})$, we know that under a constant-order type

Figure EC.1 **Relationship between $P(L)$ and $f(L)$ when K^L follows a uniform distribution**

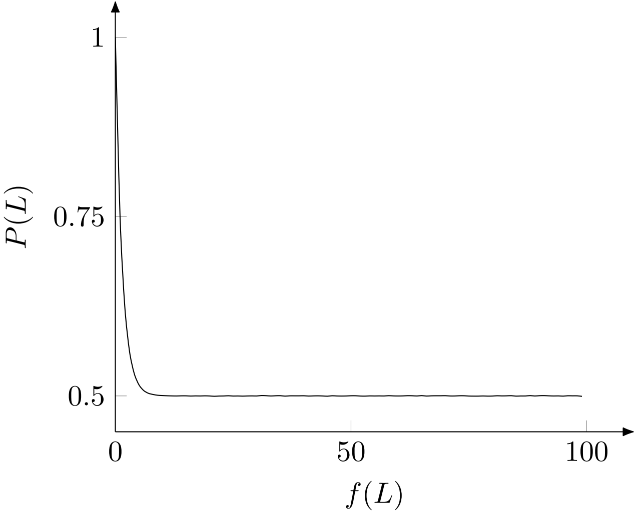


policy π_{r^L}, the expected size of an arriving regular order is of order $\Theta(r^L\sqrt{f(L)})$. In each period, the inventory level can at most decrease by $\mathbb{E}[D]$ in expectation, then it will take $\Theta(r^L\sqrt{f(L)})$ periods to consume the new arriving order by Wald's equation, and therefore the expected average inventory cost incurred during these periods is at least $\Theta(r^L\sqrt{f(L)})$. Note that under Assumption EC.3, $\lim_{L\to\infty}\frac{\sqrt{f(L)}}{L-f(L)}=0$, then there are infinitely many new arriving orders in the infinite-horizon problem, and the (undiscounted) average cost in each period is at least $\Theta(r^L\sqrt{f(L)})$. It then follows from Lemma EC.3 that we must have $r^L=O(\frac{1}{\sqrt{f(L)}})$, otherwise the lower bound becomes infinity, which leads to a contradiction.

Note that when the constant order quantity is zero, we receive nothing from the regular supplier in the original dual-sourcing backlog inventory model, and thus the change of the stochastic lead time will not affect the inventory model. Therefore, similar to Lemma 12, we can obtain that in the limit, the lower bound of $\mathrm{OPT}(L)$ equals $(1-\alpha)$ times the optimal total discounted cost in an infinite-horizon single-sourcing backlog inventory problem with i.i.d. demands D_t in each period, i.e., there is only one state in the corresponding Markov chain. In its long-run average counterpart, the modified base-stock policy is known to be optimal. Finally, following the proof of Theorem 4, we can conclude the asymptotic optimality of a semi-open-loop policy, which orders nothing from the regular supplier, and implements a modified base-stock policy from the emergency supplier.

We make some final comments about Assumption EC.3 used in the dual-sourcing backlog inventory model and Assumption 2 in the lost-sales inventory model. First, both assumptions allow the support of the stochastic lead time part to grow linearly in L. Second, Assumption 2 allows the lead time distribution to change arbitrarily as L increases, and we can leverage the special structure of inventory dynamics under a constant-order policy to derive explicit convergence rate in the lost-sales inventory models. In the dual-sourcing inventory backlog model, Assumption EC.3

Figure EC.2 **Relationship between $P(L)$ and $f(L)$ when K^L follows a two-point distribution**



basically leads to the situation in which the optimal regular constant order quantity converges to zero. It is straightforward to verify that this property also holds for the lost-sales inventory model. Specifically, under Assumption EC.3, a constant-order policy π_{r^L} incurs an average holding cost $\Theta(r^L\sqrt{f(L)})$. Consequently, we must have $r^L = O(\frac{1}{\sqrt{f(L)}})$, and then we can implement the vanishing discount factor approach to establish the asymptotic optimality. However, such a phenomenon does not necessarily exist without Assumption EC.3. For instance, assume that K^L follows a two-point distribution with $\mathbb{P}(K^L = 0) = \frac{1}{2}$ and $\mathbb{P}(K^L = f(L)) = \frac{1}{2}$. We compute the value of $P(L)$ as $f(L)$ increases from 0 to 100 through simulation, and report the relationship between $P(L)$ and $f(L)$ in Figure EC.2. One can see that in this example, $P(L)$ converges to $\frac{1}{2}$ instead of 0. Therefore, a constant regular order results in $\Theta(1)$ frequency of non-zero arrivals, with $\Theta(1)$ average size of a non-zero arrival. Using this fact, it can be further shown that the optimal constant regular order may be non-zero. To extend our theoretical results to dual-sourcing backlog inventory models with more general stochastic lead times, the major difficulty lies in the analysis of the corresponding inventory dynamics when implementing a state-dependent modified base-stock policy, and we leave this as a future research direction.

EC.6. Proof details to the results in Section 5

EC.6.1. The model with order crossover

To carry out the asymptotic analysis for the models with order crossover, we first show that there also exists a finite-state, exogenous, ergodic Markov chain which controls the received order quantity in each period, and we suppress any associated notational dependency on L as this dependency is clear from the context. In particular, we define a Markov chain $\{\boldsymbol{w}_t = (w_{t,0}, ..., w_{t,f(L)})\}_{t\geq 1}$, where for any $0 \leq i \leq f(L)$, $w_{t,i}$ is a random variable representing whether $x_{t-f(L)-L+i}$, the regular order placed $L+f(L)-i$ periods ago, is received before period t, in period t, or after period t. If this order is received before period t, $w_{t,i}=-1$; if it is received in period t, $w_{t,i}=0$; if it will arrive after period t, $w_{t,i}=1$. According to this definition, we can see that the random order quantity to be received in period t is $q_t=\sum_{i=0}^{f(L)}(1-|w_{t,i}|)\cdot x_{t-f(L)-L+i}$. We further define $\{B_i\}_{0\le i\le f(L)}$ as Bernoulli random variables such that $\mathbb{P}(B_i=0)=\frac{p^L_{f(L)-i}}{\sum_{l=0}^{i}p^L_{f(L)-l}}$, and use these random variables to update the conditional probability that a placed order is received in the current period if it has not been received before. Then the evolution of this Markov chain is as follows:

- For any $1\le i\le f(L)$, if $w_{t,i}$ is realized to be 1, i.e., the regular order placed $L+f(L)-i$ periods ago will arrive in the future, then $w_{t+1,i-1}$ follows the same distribution of a copy of the random variable B_{i-1}; otherwise, $w_{t+1,i-1}=-1$, i.e., the regular order placed in period $t-L-f(L)+i$ arrived in period t $(w_{t,i}=0)$ or even earlier $(w_{t,i}=-1)$.
- $w_{t+1,f(L)}$ follows the same distribution of a copy of the random variable $B_{f(L)}$.

Through the above construction, we can use the Markov chain $\{\boldsymbol{w}_t\}_{t\ge1}$ to update the conditional probability that the corresponding order is received in the current period. It is easy to conclude that $\{\boldsymbol{w}_t\}_{t\ge1}$ is a finite-state, exogenous, ergodic Markov chain in the sense that its evolution is independent of all other events, including the demand process.

We next discuss how to establish the asymptotic results for the models with order crossover and divisible products. Since the proof is almost identical to that in the model without order crossover, we only provide a proof sketch and explain why it works.

Lower bound on the optimal cost: Similar to Theorem 1, we provide a lower bound on the optimal long-run average cost for the inventory model. Note that we can use the ergodic Markov chain $\{\boldsymbol{w}_t\}_{t\ge1}$ to express the regular order quantity received in each period, and thus we can follow the proof of Theorem 1 to construct the random vector representing the steady state of the Markov chain induced by an optimal policy. Within the proof of Theorem 1 (see Section EC.3), the bounds and estimates only depend on the support of the lead time, making them applicable to both the order crossover and no order crossover cases.

Single-sourcing lost-sales inventory model with divisible products: The proofs for both order crossover and no order crossover cases are almost the same (with minor differences in the definitions of several terms). For instance, those key steps such as the probability bound in (EC.7) only depend on the support of the lead time and are applicable to both cases. Therefore, we can conclude that under Assumptions 1 and 2, constant-order policies are asymptotically optimal and the optimality gap decays exponentially fast to zero as L increases.

Dual-sourcing backlog inventory model with divisible products: Because the supply system can be interpreted as the ergodic Markov chain $\{\boldsymbol{w}_t\}_{t\ge1}$ which controls the regular order quantity received in each period, we can follow the same procedure to establish a lower bound of the optimal cost, expressed as the discounted cost in a single-sourcing backlog inventory problem with Markov-modulated demands, as shown in Lemma 10. We can further implement the vanishing discount factor approach to establish the convergence of the discounted problem to the long-run average counterpart, as outlined in Section 4.3. Consequently, we can draw the same conclusion for the model with order crossover, i.e., under Assumption 3, the policy which places a constant order from the regular supplier and implements a state-dependent modified base-stock policy from the emergency supplier is asymptotically optimal. Finally, following the mapping from the joint pricing and inventory model to a dual-sourcing inventory model in Section 4.4, we can also establish the asymptotic optimality of constant-order dynamic pricing policies under the same condition.

EC.6.2. The model with Markov-modulated demands

Lower bound on the optimal cost: We first establish a lower bound on the optimal long-run average cost for the inventory model, which is similar to Theorem 1. Since the Markov chain $\{\bar{w}_t\}_{t\geq 1}$ regulating the demand process is ergodic, there exists a unique stationary distribution $\bar{w}_\infty$. Therefore, we can incorporate the state of this Markov chain into the system state, just as we have done for the Markov chain $\{w_t\}_{t\geq 1}$ which regulates the supply system. We can then follow the proof of Theorem 1 to construct the random vector representing the steady state of the Markov chain induced by an optimal policy, where the nature is also in its steady state $\bar{w}_\infty$. This steady-state random vector further provides us a lower bound on the optimal cost.

Single-sourcing lost-sales inventory model with divisible products: The proof of establishing the asymptotic optimality of constant-order policies is almost identical to the proof presented in Section 4.1. To be specific, we first use the steady-state random vector and apply Jensen's inequality to obtain a lower bound similar to that in (5). The only difference is that the demand process is in its stationary version, i.e., the demand variables follow the compound probability distribution that results from the steady state of the nature. Then we can follow the remaining proof steps to conclude the asymptotic optimality of constant-order policies.

Dual-sourcing backlog inventory model with divisible products: In the dual-sourcing backlog inventory model with Markov-modulated demands, by utilizing the steady-state random vector, we can further apply Jensen's inequality to derive a similar lower bound to that in (13), expressed as the discounted total cost of a single-sourcing backlog inventory problem with Markov-modulated demands. The only difference is that here the Markov state includes not only $\{w_t\}_{t\geq 1}$ which controls the regular order quantity received in each period, but also the nature state $\{\bar{w}_t\}_{t\geq 1}$ regulating the original demand process. Finally, following the analysis in Section 4.3, we can draw the same conclusion as in Theorem 4 that a tailored state-dependent base-surge policy, which always places a constant order from the regular supplier and implements a state-dependent modified base-stock policy from the emergency supplier, is asymptotically optimal. Similar results can be established in the joint pricing and inventory model by mapping it to a dual-sourcing inventory model.

EC.6.3. The model with random supply function

We consider the inventory model with divisible products and random supply functions. In this model, although the order placed in period t is x_t, we can only receive $s(x_t, Z_t)$ units, where $s(x,z): \mathbb{R}_+^2 \to \mathbb{R}_+$ is a deterministic function and $\{Z_t\}$ is a sequence of i.i.d. nonnegative random variables having the same distribution as a random variable Z with $0 < \mathbb{E}[Z] < \infty$. We assume the supply randomness is realized when the order is received, i.e., the realization of Z_t is observed only when we receive the order placed in period t. We assume that $s(x,z) = 0$ when $x = 0$, it is continuous and increasing in $x \geq 0$ for all $z \geq 0$, and $\{Z_t\}$ are independent of $\{D_t\}$ and $\{K_t^L\}$. Following Lemma 1, the supply that is received in period t is $\sum_{n=\max_{j=t-f(L),\ldots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} s(x_{n-L-f(L)}, Z_{n-L-f(L)})$.

For any $x \geq 0$, define $\mu(x) = \mathbb{E}[s(x,Z)]$ as the mean received quantity when the order quantity is x. We also define $\bar{\mu} = \sup_{x \geq 0} \mu(x)$ and

$$\bar{x} = \sup\{x \geq 0 : \mu(x) < \bar{\mu}\}$$

if the set $\{x \geq 0 : \mu(x) < \bar{\mu}\}$ is nonempty and $\bar{x} = 0$ otherwise. For each $0 \leq \mu \leq \bar{\mu}$, define

$$\rho(\mu) = \inf\{0 \leq x \leq \bar{x} : \mathbb{E}[s(x,Z)] \geq \mu\} \text{ and } \hat{s}(\mu, Z) = s(\rho(\mu), Z).$$

Thus, $\{\hat{s}(\mu, Z) : 0 \leq \mu \leq \bar{\mu}\}$ is a random supply function with respect to μ. Next we introduce the concept of stochastic linearity in midpoint, and readers are referred to Feng and Shanthikumar (2018) for more discussions on this concept and its properties.

Definition EC.1 (Feng and Shanthikumar 2018). A stochastic function $\{Y(x) : x \in \mathcal{X}\}$ for some convex set $\mathcal{X}$ is *stochastically linear in midpoint*, written as $\{Y(x) : x \in \mathcal{X}\} \in \mathrm{SL}(mp)$, if for any $x_1, x_2 \in \mathcal{X}$, there exist $\hat{Y}(x_1)$ and $\hat{Y}(x_2)$ defined on a common probability space such that

1. $\hat{Y}(x_i) =^d Y(x_i)$, $i = 1, 2$, and
2. $(\hat{Y}(x_1) + \hat{Y}(x_2))/2 \leq_{cv} Y((x_1 + x_2)/2)$, where $Y_1 \leq_{cv} Y_2$ represents Y_1 is smaller than Y_2 in the concave order; that is, $\mathbb{E}[f(Y_1)] \leq \mathbb{E}[f(Y_2)]$ for any concave function $f : \mathbb{R} \to \mathbb{R}$.

Throughout this subsection, we consider the same assumption imposed in Bu et al. (2020).

Assumption EC.4. *(a) The function $\mu(x)$ is strictly increasing in $x \in [0, \bar{x}]$; (b) the transformed random supply function $\{\hat{s}(\mu, Z) : 0 \leq \mu \leq \bar{\mu}\} \in \mathrm{SL}(mp)$.*

With this additional assumption regarding the random supply function, we can establish the asymptotic results for both the single-sourcing lost-sales and dual-sourcing backlog inventory model with divisible products. The key to its analysis is that, instead of dealing with the placed order quantities $\{x_t\}$, which can destroy convexity of the cost function, we deal with the mean received quantities $\{\mu_t\}$ with $\mu_t \triangleq \mu(x_t)$. This approach helps preserve the convexity of the cost function under Assumption EC.4, thus leading to the asymptotic optimality of constant-order type policies.

Single-sourcing lost-sales inventory model with divisible products: Let's discuss how to establish the asymptotic optimality of constant-order policies in the single-sourcing lost-sales inventory model with random supply function and divisible products. First, we can follow Theorem 1 and Lemma EC.1 in Bu et al. (2020) to construct a lower bound of the optimal cost as follows:

$$\mathrm{OPT}(L) \geq h\mathbb{E}\left[\sup_{l=0,1,\dots,L-f(L)}\left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} s(X_n^L, Z_n) - \sum_{t=1}^{l} D_t\right\}\right] + p\mathbb{E}[D] - p\mathbb{E}[s(X_1^L, Z)], \tag{EC.13}$$

where X_i^L corresponds to the regular order placed in period $T+i-L-f(L)-1$ (the same as those defined in Theorem 1), and $\{Z_t\}$ are random variables in the supply function. However, the right-hand side of (EC.13) is not convex in $\boldsymbol{X}^L$ and we cannot apply Jensen's inequality directly. To overcome this difficulty, we will deal with the mean received order quantity. Define $M_i^L \triangleq \mathbb{E}[s(X_i^L, Z_i)|X_i^L]$, and by definition we have

$$\mathrm{OPT}(L) \geq h\mathbb{E}\left[\sup_{l=0,1,\dots,L-f(L)}\left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} \hat{s}(M_n^L, Z_n) - \sum_{t=1}^{l} D_t\right\}\right] + p\mathbb{E}[D] - p\mathbb{E}[\hat{s}(M_1^L, Z)].$$

It then follows from Proposition 1 in Chen et al. (2025) that, under Assumption EC.4, we have

$$\mathrm{OPT}(L) \geq h\mathbb{E}\left[\sup_{l=0,1,\dots,L-f(L)}\left\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} \mathbb{E}[\hat{s}(M_n^L, Z_n)] - \sum_{t=1}^{l} D_t\right\}\right] + p\mathbb{E}[D] - p\mathbb{E}[\hat{s}(M_1^L, Z)].$$

Because $X_i^L \sim X_1^L$ for all $i = 2, \dots, L$, we have $\mathbb{E}[\hat{s}(M_i^L, Z_i)] = \mathbb{E}[\hat{s}(M_1^L, Z)]$ for all $i = 2, \dots, L$. Therefore, instead of dealing with the mean placed order quantity $\mathbb{E}[X_1^L]$ in the lower bound (5), we deal with the mean received order quantity in the model with random supply function. It suffices to follow the subsequent analysis in Section 4.2 and the proof of Theorem 1 in Bu et al. (2020) to establish the asymptotic optimality of constant-order policies in the single-sourcing lost-sales with random supply function and divisible products.

Dual-sourcing backlog inventory model with divisible products: We briefly discuss how to establish the asymptotic optimality results in the dual-sourcing backlog inventory model with random supply function (imposed on the regular source) and divisible products. By employing the same variable transformation technique described above, we can derive a similar lower bound to that in (12), expressed as the discounted total cost of a single-sourcing backlog inventory problem with Markov-modulated demands. Note that the random supply function, which only affects the received order quantity from the regular source, is now incorporated into the demand process in the corresponding single-sourcing backlog inventory problem. Therefore, following the analysis in Section 4.3, we can draw the same conclusion as in Theorem 4, with the additional Assumption EC.4 applied to the dual-sourcing inventory model with random supply function. Similar results can be established in the joint pricing and inventory model by mapping it to a dual-sourcing inventory model.

EC.7. General lead time model

Song and Zipkin (1996) introduced a Markovian model of the supply system to model the no order crossover phenomenon. Specifically, the supply system is modeled as a discrete Markov process $\{i_t\}$, and it is exogenous in the sense that the evolution of this Markov process is independent of all other events, including orders and demands. The transition of i_t is driven by the realization of a random variable a_t, where $\{a_t\}_{t\geq 1}$ are i.i.d. exogenous inputs. Given the current supply system state $i_t = i$ and the current input $a_t = a$, we have $i_{t+1} = Q(i,a)$, where $Q(\cdot,\cdot)$ is a deterministic function.

Imagine the supply system as a string of positions, which might represent successive production stages or geographic locations, with $K+1$ positions in total. Each order has a position $k \in [K+1]$. When an order is placed, it enters the supply system at position 1 and subsequently passes through some or all of the other positions. A higher numbered position indicates more progress through the supply system, and position K indicates the completion of processing, i.e., the delivery of the order. The dummy position $K+1$ implies that this order has arrived in an earlier period. We use a vector $\boldsymbol{w}_t = (w_{t,1}, ..., w_{t,L+f(L)})$ to record the positions of pipeline orders in period t, where $w_{t,n}$ represents the position of the order placed in period $t-L-f(L)+n-1$ (some orders may have arrived by period t, and we record their positions as $K+1$ for completeness). In period t, given state $i_t = i$ and input $a_t = a$, orders currently at position k moves to position $M(k|i,a)$ (with $M(K|i,a) = K+1$ for any (i,a) by definition). Therefore, the order received in period t is $q_t = \sum_{\{n: M(w_{t,n}|i_t,a_t)=K\}} x_{t-f(L)-L+n-1}$.

Let $M^l(k|i)$ denote the l-step movement random variable, where $M^1 = M$. Thus, given $i_t = i$, the order at position k moves to position $M^l(k|i)$ at time $t+l$, depending on the realizations of $a_t, a_{t+1}, ..., a_{t+l-1}$. The lead time for an order placed in period t with $i_t = i$, is the random variable

$$L(i,t) = \min\{l \geq 0 : M^{l+1}(1|i) = K\}.$$

We further make the following assumptions:

1. $M(k|i,a)$ is nondecreasing in k for any (i,a), ensuring that orders never cross in time.
2. $L \le L(i,t) \le L + f(L)$ for any state i.
3. $\{(\boldsymbol{w}_t, i_t)\}_{t\ge 1}$ forms a finite-state, exogenous, ergodic Markov chain.

For this general lead time model, we can incorporate the Markov chain $\{(\boldsymbol{w}_t, i_t)\}_{t\ge 1}$ into the system state. One can then follow Section 5.1 to extend the results to the model with general lead times, and we omit the details.

EC.8. Proof of Theorem 6

In this appendix, we provide the proof of Theorem 6. Within this proof, we use $C^L_\infty(\pi_{S,r})$ to denote the long-run average cost of the capped base-stock policy $\pi_{S,r}$. The key is to establish the convergence of $C^L_\infty(\pi_{S,r})$ to $C^L_\infty(\pi_r)$ as $S \to \infty$ (see Theorem EC.6). To establish this convergence, we first derive an upper bound of $C^L_\infty(\pi_{S,r})$ (see Lemma EC.8), and then use large deviations to bound the gap between the upper bound and $C^L_\infty(\pi_r)$ (see Lemma EC.9).

For any $L > 0$, under each capped base-stock policy $\pi_{S,r}$, there exists a Markov chain describing the inventory dynamics, and $C^L_\infty(\pi_{S,r})$ depends on the long-run behavior and steady-state distribution of the induced Markov chain. The following lemma establishes the existence of such a steady-state distribution. Because the proof is quite similar to the argument used in Theorem 1 to prove the existence of the steady-state distribution under an optimal policy for the inventory problem, we omit the details here.

Lemma EC.7. *For any $L > 0$, under the Markov chain induced by each policy $\pi_{S,r}$, there exists a steady-state distribution $(I^{L,c}, q_1^{L,c}, \ldots, q_{L+f(L)}^{L,c})$ where $I^{I,c}$ represents the on-hand inventory at the beginning of the period, and $q_i^{L,c}$ represents the order placed $L + f(L) + 1 - i$ periods ago.*

Recall that $I^L_\infty(r)$, defined in (2), is the steady-state on-hand inventory under the constant-order policy π_r. The next lemma obtains an upper bound of $C^L_\infty(\pi_{S,r})$.

Lemma EC.8. *For any $L > 0$, under each policy $\pi_{S,r}$,*

$$C^L_\infty(\pi_{S,r}) \le h\mathbb{E}[I^L_\infty(r)] + p(\mathbb{E}[D] - r) + p\mathbb{E}[((I^L_\infty(r) + (L + f(L) + 1)r - S) \wedge r)^+].$$

Proof. Because the order placed by $\pi_{S,r}$ in each period is always no greater than the order capacity r, the steady-state on-hand inventory under policy $\pi_{S,r}$ is bounded from above by $I^L_\infty(r)$. Therefore, w.p.1 we have

$$I^{L,c} \le I^L_\infty(r), \text{ and } 0 \le q_i^{L,c} \le r, \forall 1 \le i \le L + f(L). \tag{EC.14}$$

Let $pipeline^{L,c}$ denote the current total pipeline inventory (before new order arrivals), then according to the stochastic lead time assumption, we have

$$\sum_{i=1}^{L} q_{i+f(L)}^{L,c} \leq pipline^{L,c} \leq \sum_{i=1}^{L+f(L)} q_i^{L,c}. \tag{EC.15}$$

Let $q_{L+f(L)+1}^{L,c}$ be the order placed based on the current inventory state, then according to the definition of capped base-stock policy, we have

$$q_{L+f(L)+1}^{L,c} = \min\left\{\left(S - I^{L,c} - pipeline^{L,c}\right)^{+}, r\right\},$$

which further implies w.p.1 that

$$\begin{aligned} -q_{L+f(L)+1}^{L,c} &= \max\left\{\left(I^{L,c} + pipeline^{L,c} - S\right) \wedge 0, -r\right\} \\ &\leq \max\left\{\left(I^{L,c} + (L+f(L))r - S\right) \wedge 0, -r\right\} \end{aligned} \tag{EC.16}$$

where the inequality follows from (EC.14) and (EC.15). Combining the stationarity with (EC.14), (EC.16) and the fact that $\mathbb{E}[D - q_{L+f(L)+1}^{L,c}]$ captures the amount of long-run average lost sales, we have

$$\begin{aligned} C_\infty^L(\pi_{S,r}) &= h\mathbb{E}[I^{L,c}] + p\mathbb{E}[D - q_{L+f(L)+1}^{L,c}] \\ &= h\mathbb{E}[I^{L,c}] + p(\mathbb{E}[D] - r) + p\mathbb{E}[r - q_{L+f(L)+1}^{L,c}] \\ &\leq h\mathbb{E}[I_\infty^L(r)] + p(\mathbb{E}[D] - r) + p\mathbb{E}[((I_\infty^L(r) + (L+f(L)+1)r - S) \wedge r)^+], \end{aligned}$$

and the proof is completed. □

According to Lemma EC.8, we have

$$\begin{aligned} C_\infty^L(\pi_{S,r}) - C_\infty^L(\pi_r) &\leq p\mathbb{E}[[(I_\infty^L(r) + (L+f(L)+1)r - S) \wedge r]^+] \\ &\leq rp\mathbb{P}(I_\infty^L(r) + (L+f(L)+1)r - S \geq 0). \end{aligned} \tag{EC.17}$$

The next lemma helps us to bound the probability in (EC.17).

LEMMA EC.9. *For any $r \in [0, \mathbb{E}[D])$, $x > f(L)r$, and $\beta > 0$ such that $\phi_r(\beta) \triangleq \mathbb{E}[e^{\beta(r-D)}] < 1$, we have*

$$\mathbb{P}(I_\infty^L(r) \geq x) \leq e^{-\beta(x - f(L)r)} \frac{\phi_r(\beta)}{1 - \phi_r(\beta)}.$$

Proof. Note that

$$\begin{aligned}
\mathbb{P}(I^L_\infty(r)\ge x) &= \mathbb{P}\Bigg(\sup_{l=0,1,\dots}\Bigg\{\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r-\sum_{t=1}^{l}D_t\Bigg\}\ge x\Bigg)\\
&\le \sum_{l\ge 1}\mathbb{P}\Bigg(\Bigg(\sum_{t=1}^{l}\sum_{n=\max\limits_{j=t-f(L),\dots,t-1}\{j+K_j^L+1\}}^{t+K_t^L} r-\sum_{t=1}^{l}D_t\Bigg)\ge x\Bigg)\\
&\le \sum_{l\ge 1}\mathbb{P}\Bigg(\Bigg((l+f(L))r-\sum_{t=1}^{l}D_t\Bigg)\ge x\Bigg)\\
&= \sum_{l\ge 1}\mathbb{P}(e^{\beta(\sum_{t=1}^{l}(r-D_t))}\ge e^{\beta(x-f(L)r)})\\
&\le e^{-\beta(x-f(L)r)}\sum_{l\ge 1}(\phi_r(\beta))^l\\
&= e^{-\beta(x-f(L)r)}\frac{\phi_r(\beta)}{1-\phi_r(\beta)}
\end{aligned}$$

where the second inequality holds because in l consecutive periods, at most $(l+f(L))$ orders will be received, and the third inequality comes from Markov's inequality. The proof is completed. □

Combining (EC.17) and Lemma EC.9, we can derive the following theorem which states that $C^L_\infty(\pi_{S,r})$ converges to $C^L_\infty(\pi_r)$ as S increases.

THEOREM EC.6. *For any $r\in[0,\mathbb{E}[D])$, $S>(L+2f(L)+1)r$, and $\beta>0$ such that $\phi_r(\beta)<1$, we have*

$$C^L_\infty(\pi_{S,r})-C^L_\infty(\pi_r)\le rpe^{-\beta(S-(L+2f(L)+1)r)}\frac{\phi_r(\beta)}{1-\phi_r(\beta)}.$$

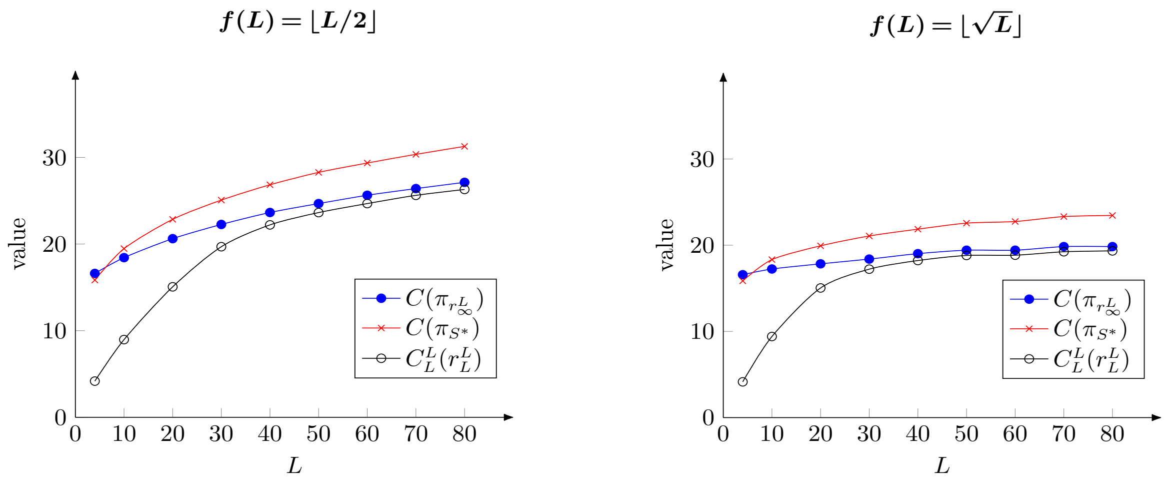
Finally, combining Theorem EC.6 and Theorem 2, we can conclude that capped base-stock policies are asymptotically optimal and complete the proof of Theorem 6.

EC.9. Numerical Results in Section 6

Table EC.1 Comparison between constant-order policy and base-stock policy

		Poisson Demand							
		$L=1$		$L=4$		$L=10$		$L=30$	
$\mathbb{E}[D]=2$	$p=1$	1.36	-4.81%	1.45	10.56%	1.52	13.15%	1.59	14.54%
	$p=4$	3.43	-18.70%	3.68	3.20%	3.90	12.27%	4.24	17.75%
	$p=9$	5.48	-30.21%	5.82	-7.84%	6.13	5.26%	6.65	15.30%
	$p=39$	11.99	-53.90%	12.50	-34.11%	12.92	-18.25%	13.62	0.69%
$\mathbb{E}[D]=5$	$p=1$	2.66	-5.07%	3.04	13.67%	3.28	18.73%	3.58	20.27%
	$p=4$	6.19	-15.36%	7.16	3.58%	7.96	12.02%	9.21	17.46%
	$p=9$	9.50	-28.18%	10.76	-6.62%	11.85	4.79%	13.75	13.11%
	$p=39$	19.84	-52.62%	21.50	-32.09%	22.98	-17.36%	25.38	0.04%
$\mathbb{E}[D]=10$	$p=1$	4.67	-4.34%	5.59	15.19%	6.17	20.61%	6.87	22.18%
	$p=4$	10.08	-12.31%	12.57	3.34%	14.52	10.85%	17.35	16.85%
	$p=9$	14.82	-25.33%	18.00	-5.66%	20.69	3.96%	25.02	11.44%
	$p=39$	29.67	-50.89%	33.39	-29.28%	36.89	-15.46%	42.82	-0.81%
$\mathbb{E}[D]=30$	$p=1$	12.68	-6.12%	15.84	15.05%	17.73	22.02%	20.01	23.67%
	$p=4$	23.67	-8.39%	33.48	2.77%	40.39	9.11%	49.86	15.87%
	$p=9$	32.08	-20.72%	44.24	-3.25%	54.33	2.87%	69.49	9.39%
	$p=39$	57.92	-46.13%	71.89	-23.99%	84.57	-11.53%	106.51	-1.09%
		Geometric Demand							
		$L=1$		$L=4$		$L=10$		$L=30$	
$\mathbb{E}[D]=2$	$p=1$	1.17	-2.19%	1.29	12.05%	1.40	14.70%	1.51	15.99%
	$p=4$	3.15	-11.81%	3.45	3.48%	3.71	9.95%	4.10	15.42%
	$p=9$	5.15	-22.25%	5.53	-4.02%	5.87	5.60%	6.47	13.10%
	$p=39$	11.70	-44.93%	12.09	-27.41%	12.49	-13.90%	13.43	0.76%
$\mathbb{E}[D]=5$	$p=1$	3.55	-2.16%	3.73	8.15%	3.88	10.97%	4.05	12.40%
	$p=4$	9.59	-12.29%	10.13	3.79%	10.62	10.47%	11.35	15.46%
	$p=9$	15.82	-22.63%	16.58	-4.42%	17.28	5.79%	18.44	14.05%
	$p=39$	36.30	-45.88%	37.10	-28.15%	38.17	-14.78%	39.92	1.80%
$\mathbb{E}[D]=10$	$p=1$	7.41	-2.00%	7.75	7.21%	7.98	10.13%	8.29	11.06%
	$p=4$	20.15	-12.30%	21.14	3.55%	22.00	10.87%	23.41	15.28%
	$p=9$	33.41	-23.02%	34.79	-4.54%	35.99	6.16%	38.25	14.04%
	$p=39$	77.16	-46.36%	79.42	-29.30%	80.69	-15.20%	83.84	1.61%
$\mathbb{E}[D]=30$	$p=1$	22.88	-2.24%	23.81	6.71%	24.35	9.61%	25.20	10.54%
	$p=4$	62.29	-12.29%	65.10	3.77%	67.70	10.31%	71.50	15.09%
	$p=9$	103.50	-23.11%	107.59	-4.58%	111.11	5.91%	116.97	14.82%
	$p=39$	238.88	-46.27%	244.35	-29.03%	248.50	-14.85%	258.17	1.75%

Figure EC.3 Comparison between constant-order policy and the lower bound


$f(L)=\lfloor L/2\rfloor$

$f(L)=\lfloor\sqrt{L}\rfloor$

Table EC.2 Comparison between policies in small lead time regime

	Poisson Demand				Geometric Demand			
Policy	$L=1$	$L=2$	$L=3$	$L=4$	$L=1$	$L=2$	$L=3$	$L=4$
	$p=1$							
Capped base-stock	2.51	3.02	3.31	3.48	3.46	3.71	3.87	4.00
Base-stock	2.53	3.22	3.59	3.84	3.47	3.88	4.10	4.26
Constant-order	2.67	3.05	3.34	3.48	3.55	3.73	3.88	4.01
	$p=4$							
Capped base-stock	5.22	6.77	7.83	8.56	8.41	9.67	10.39	10.93
Base-stock	5.23	6.89	8.03	8.93	8.42	9.85	10.76	11.47
Constant-order	6.26	7.16	7.96	8.68	9.99	10.42	10.77	11.10
	$p=9$							
Capped base-stock	6.81	9.19	10.83	12.14	12.16	14.45	15.90	16.91
Base-stock	6.83	9.23	10.93	12.30	12.19	14.55	16.14	17.34
Constant-order	11.25	12.17	12.96	13.68	15.88	16.64	17.28	17.96
	$p=39$							
Capped base-stock	9.37	13.07	15.81	18.04	19.59	23.78	26.69	29.05
Base-stock	9.37	13.10	15.85	18.08	19.63	23.83	26.75	29.15
Constant-order	41.21	42.15	43.00	43.68	46.00	46.83	47.52	47.84

Table EC.3 **Comparison between open-loop policies**

		$\pi_b^{r_\infty,\theta}$	$\pi_{r_\infty}^r$	$\pi_{r_{int}}$
$\mathbb{E}[D]=100$	$p=1$	72.84	0.00%	0.01%
	$p=4$	199.00	0.00%	0.00%
	$p=9$	334.20	0.00%	0.00%
	$p=19$	521.87	0.00%	0.00%
	$p=39$	784.86	0.00%	0.02%
$\mathbb{E}[D]=10$	$p=1$	6.93	0.33%	0.99%
	$p=4$	18.96	0.11%	0.21%
	$p=9$	31.85	0.10%	0.46%
	$p=19$	49.75	0.17%	4.53%
	$p=39$	74.82	0.04%	0.24%
$\mathbb{E}[D]=5$	$p=1$	3.25	1.45%	2.69%
	$p=4$	8.92	0.94%	6.55%
	$p=9$	14.99	0.08%	0.04%
	$p=19$	23.43	0.61%	6.71%
	$p=39$	35.25	0.70%	27.66%
$\mathbb{E}[D]=2$	$p=1$	0.95	9.66%	5.32%
	$p=4$	2.74	9.64%	45.84%
	$p=9$	4.66	7.24%	92.95%
	$p=19$	7.33	5.28%	159.15%
	$p=39$	11.07	3.80%	252.31%

Table EC.4 **Comparison between integral policies with order crossover**

Policy	$L=1$	$L=4$	$L=10$	$L=30$	$L=1$	$L=4$	$L=10$	$L=30$
	$p=1$, $\mathbb{E}[D]=2$				$p=1$, $\mathbb{E}[D]=5$			
Base-stock	1.23	1.41	1.51	1.59	3.68	4.00	4.17	4.30
Constant-order	1.16	1.24	1.27	1.32	3.55	3.64	3.70	3.77
Bracket	1.16	1.24	1.27	1.32	3.55	3.64	3.70	3.77
Randomized rounding	1.18	1.24	1.27	1.32	3.56	3.65	3.70	3.77
	$p=4$, $\mathbb{E}[D]=2$				$p=4$, $\mathbb{E}[D]=5$			
Base-stock	3.03	3.53	3.86	4.22	9.17	10.50	11.33	12.21
Constant-order	4.16	4.24	4.27	4.32	9.98	10.22	10.39	10.61
Bracket	3.15	3.35	3.48	3.66	9.57	9.92	10.14	10.43
Randomized rounding	3.35	3.51	3.62	3.78	9.64	9.97	10.19	10.46
	$p=9$, $\mathbb{E}[D]=2$				$p=9$, $\mathbb{E}[D]=5$			
Base-stock	4.38	5.29	5.93	6.66	13.47	16.01	17.85	19.81
Constant-order	9.16	9.24	9.27	9.32	15.82	16.30	16.64	17.12
Bracket	5.15	5.40	5.59	5.85	15.80	16.30	16.61	17.08
Randomized rounding	5.43	5.65	5.82	6.04	15.87	16.34	16.62	17.08
	$p=39$, $\mathbb{E}[D]=2$				$p=39$, $\mathbb{E}[D]=5$			
Base-stock	7.13	8.94	10.58	12.86	21.92	27.37	32.34	39.06
Constant-order	39.16	39.24	39.27	39.32	45.82	46.30	46.64	47.12
Bracket	11.66	11.91	12.19	12.56	36.35	36.82	37.37	38.10
Randomized rounding	11.97	12.33	12.53	12.88	36.48	37.06	37.70	38.31

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