Online Appendix for "Understanding Partnership Formation and Repeated Contributions in Federated Learning: An Analytical Investigation"

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Appendix A: Open-Ended Interviews with FL Researchers and Practitioners

In order to understand the status-quo and challenges of FL deployment in practice, we have conducted open-ended interviews with four FL researchers and practitioners, including (1) a professor who is leading a healthcare FL partnership across several universities; (2) a senior researcher at a multinational technology company who heads a team in developing FL software and infrastructure; (3) an entrepreneur who is building a start-up company to provide FL as a service; and (4) a professor and statistical consultant who is coordinating a university-industry collaboration team to investigate the algorithmic properties of FL. Their identities will remain undisclosed to protect the anonymity of the review process. Through extensive conversations with them, we have identified the following key challenges of deploying FL in practice:

1. Finding and recruiting partners to form a FL partnership mostly rely on pre-existing connections. "Strangers" are generally unwilling to participate in a FL partnership.

2. Coordination cost is high, especially when there is a large number of participants. One of the key design objectives of FL software and infrastructure is to automate part of the coordination (e.g., transmitting local models from and to each participant and accommodating each participant's special needs).

3. There lacks a systematic approach to enforce contribution (other than the goodwill of participants). Participants may only contribute part of their information, and it is not uncommon that participants drop out of the partnership entirely.

4. It is difficult to foster long-term engagement. Most FL partnerships dissolve after a global model is trained, without repeated contributions to continuously improve the model.

Appendix B: Illustration of Federated Learning with a Linear Regression Example

Suppose we want to estimate the linear association between variables X and Y, and collect a sample of size n, namely, $\{(x_i, y_i)\}_{i=1}^n$. We minimize the squared loss $L(\beta) = \frac{1}{n}\sum_{i=1}^n (y_i - x_i\beta)^2$ via a gradient descent algorithm,[19] which computes the estimated β at the $(k+1)$-th iteration as

$$\beta_{k+1} \leftarrow \beta_k - \nu L'(\beta_k),$$

where ν is a pre-specified learning rate, and $L'(\beta_k) = -\frac{2}{n}\sum_{i=1}^{n} x_i(y_i - x_i\beta_k)$ is the gradient of $L(\cdot)$ at β_k.

In a FL context, suppose we have two clients A and B who have collected n_A and n_B data instances, denoted as $\{(x_i, y_i)\}_{i=1}^{n_A}$ and $\{(x_i, y_i)\}_{i=n_A+1}^{n_A+n_B}$, respectively. During the $(k+1)$-th iteration, each of them computes a gradient using their local data, i.e.,

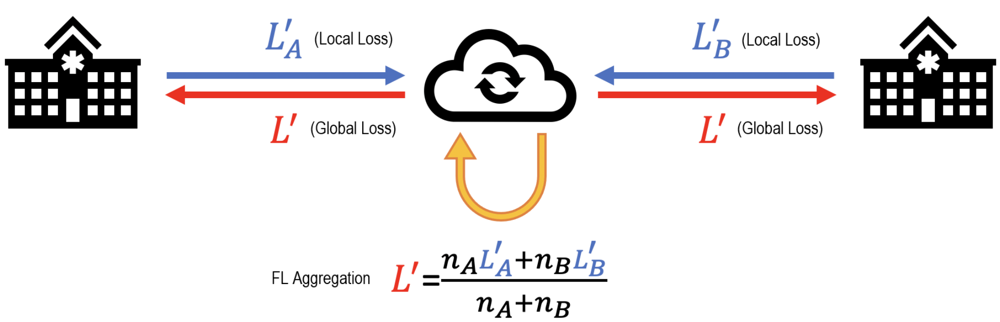
$$L'_A(\beta_k) = -\frac{2}{n_A}\sum_{i=1}^{n_A} x_i(y_i - x_i\beta_k), \text{ and } L'_B(\beta_k) = -\frac{2}{n_B}\sum_{i=n_A+1}^{n_A+n_B} x_i(y_i - x_i\beta_k),$$

Next, they upload $L'_A(\beta_k)$ and $L'_B(\beta_k)$ to the central server, which aggregates the locally-computed gradients by taking a weighted average

$$L'(\beta_k) = \frac{1}{n_A + n_B}\left\{n_A L'_A(\beta_k) + n_B L'_B(\beta_k)\right\}.$$

The central server then sends the aggregated gradient $L'(\beta_k)$ back to both clients, who individually perform parameter update to obtain β_{k+1}. This gradient aggregation process is the key idea behind "FedAvg" (McMahan et al. 2017), one of the leading and most popular FL algorithms (Li et al. 2019b). Figure 5 visualizes this process.

Figure 5 Flowchart illustrating the federated learning process



On the one hand, we can see that the aggregated gradient is computed solely based on local gradients L'_A and L'_B without the need to access clients' raw data, thereby protecting their privacy. On the other hand, note that $\frac{1}{n_A+n_B}\left\{n_A L'_A(\beta_k) + n_B L'_B(\beta_k)\right\} = -\frac{2}{n_A+n_B}\sum_{i=1}^{n_A+n_B} x_i(y_i - x_i\beta_k)$. In other words, the aggregated

[19] Although the estimation of a linear regression has explicit global solutions, we present this illustration using a gradient descent algorithm, because many other machine learning models (e.g., neural networks) are trained via gradient descent.

gradient is effectively computed as if samples from A and B were directly merged, which leads to a larger sample size $n_A + n_B$ and provides more accurate $L'(\beta_k)$ at each iteration and faster convergence than if the two clients only rely on their own data for model training.

Appendix C: Basic Results in a Standard IPD

If two players cooperate for the entire game, the total discounted payoff across all rounds would be

$$\pi^C(A) = \pi^C(B) = \sum_{t=1}^{\infty} \gamma^{t-1} R = \frac{R}{1-\gamma},$$

Suppose both players agree to adopt a "grim trigger" strategy. Without loss of generality, suppose player A defects in the first round (while B cooperates), and both players defect for the rest of the game. The total discounted payoff for player A is

$$\pi^D(A) = T + \sum_{t=2}^{\infty} \gamma^{t-1} P = T + \frac{\gamma P}{1-\gamma}$$

Therefore, the boundary condition for cooperation to arise is given by

$$\pi^C(A) > \pi^D(A) \Rightarrow \gamma > \frac{T-R}{T-P}$$

In other words, cooperation will emerge in this IPD game as long as the discount rate is not too low, i.e., when players are not too myopic and care about future rewards of mutual cooperation more than the one-time benefit of defection.

If the players instead agree to adopt a "tit-for-tat" strategy, suppose player A defects in the first round and then cooperates afterwards. Player B would cooperate in the first round, defect in the second round as a result of A's previous defection, and then cooperate afterwards. The total discounted payoff for player A in this case is

$$\pi^D(A) = T + \gamma S + \sum_{t=3}^{\infty} \gamma^{t-1} R = T + \gamma S + \frac{\gamma^2 R}{1-\gamma}$$

and the boundary condition for cooperation is

$$\pi^C(A) > \pi^D(A) \Rightarrow \gamma > \frac{T-R}{R-S}$$

Note that this is a feasible lower bound for γ because $\frac{T-R}{R-S} < 1 \Leftrightarrow 2R > T + S$ is guaranteed. Meanwhile, under tit-for-tat, another representative behavior of the two players is to alternate between cooperation and defection. Specifically, suppose A plays the sequence of actions $D \to C \to D \to C \to \ldots$, and B accordingly plays the sequence $C \to D \to C \to D \to \ldots$. The total discounted payoff for player A is

$$\pi^D(A) = T + \gamma S + \gamma^2 T + \gamma^3 S + \ldots = \frac{T + \gamma S}{1-\gamma^2}$$

and the boundary condition turns out to be the same as before

$$\pi^C(A) > \pi^D(A) \Rightarrow \gamma > \frac{T-R}{R-S}$$

Finally, a more general case of tit-for-tat is that player A defects for k consecutive rounds (starting the first round) and then cooperate afterwards, and player B cooperates on the first round, defects from round 2 to round $k+1$, and then cooperates afterwards. We show here that it is never in player A's best interest to pick $1 < k < +\infty$. Note that the total discounted payoff for player A is

$$\pi^D(A) = T + \left(\gamma P + \ldots + \gamma^{k-1} P\right) + \gamma^k S + \sum_{t=k+2}^{+\infty} \gamma^{t-1} R$$

As player A's number of defecting rounds changes from k to $k+1$, the second term above *increases* by $\gamma^k P$, the third term *decreases* by $(\gamma^k - \gamma^{k+1})S$, and the fourth term *decreases* by $\gamma^{k+1} R$. Further note that

$$\gamma^k P - (\gamma^k - \gamma^{k+1})S - \gamma^{k+1} R = \gamma^k [(P-S) - \gamma(R-S)]$$

Therefore, defecting for more rounds is beneficial for player A if $(P-S) - \gamma(R-S) > 0$, and not beneficial otherwise. As a result, the rational choice of k would either be 1 or $+\infty$, depending on the value of γ. In other words, a rational player would either defect for only one round and bear the one-time punishment ($k=1$), or simply defects for all rounds ($k = +\infty$, essentially the same as the grim trigger strategy).

Appendix D: Proofs of Theoretical Results

D.1. Proof of Theorem 1

The result can be immediately obtained by letting $\pi^C(p) > \pi^{NP}(p)$, where $\pi^C(p)$ and $\pi^{NP}(p)$ are defined in (1) and (2), respectively.

D.2. Proof of Corollary 1

For part (i), we first show that

$$\frac{U(2tx) - U(tx)}{tx} > \frac{U(2sx) - U(sx)}{sx},$$

where $s = t+1$. This can be seen by letting $g_1(\Delta) = \frac{U(a+\Delta) - U(a)}{\Delta}$, where its derivative $g_1'(\Delta) < 0$ for $\Delta > 0$ due to the strict concavity of $U(\cdot)$, and hence $g_1(\cdot)$ is a decreasing function of Δ. Similarly, we can show that $g_2(\Delta) = \frac{U(a) - U(a-\Delta)}{\Delta}$ is an increasing function of Δ. Therefore, for $x > 0$,

$$\frac{U(2tx) - U(tx)}{tx} > \frac{U(2(t+1)x) - U(tx)}{(t+2)x} > \frac{U(2(t+1)x) - U((t+1)x)}{(t+1)x} = \frac{U(2sx) - U(sx)}{sx}.$$

Then following Theorem 1, long-term FL partnership formation implies

$$\begin{aligned}
C(x) = \alpha x \; &< \; (1-\gamma)^2 \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{ U(2tx) - U(tx) \big\} \\
&= \; (1-\gamma)^2 \sum_{t=1}^{+\infty} \gamma^{t-1} \cdot tx \cdot \frac{U(2tx) - U(tx)}{tx} \\
&< \; (1-\gamma)^2 \sum_{t=1}^{+\infty} \gamma^{t-1} \cdot tx \cdot \frac{U(2x) - U(x)}{x} \\
&= U(2x) - U(x).
\end{aligned}$$

In other words, when

$$\alpha \geq \sup_{x>0} \frac{U(2x) - U(x)}{x},$$

players would not form a long-term FL partnership.

For part (ii), since $U(x)$ has continuous second derivative, $\frac{U(2x)-U(x)}{x}$ is continuously differentiable at $x > 0$. By the mean value theorem, for each x, there exists $\Delta > 0$, such that $\frac{U(2x)-U(x)}{x} = U'(x+\Delta)$. Since $U(x)$ is a strictly concave function, we have $U'(x+\Delta) \leq U'(x)$. Hence, $0 \leq \lim_{x\to\infty} \frac{U(2x)-U(x)}{x} \leq \lim_{x\to\infty} U'(x) = 0$. In other words, for any $\alpha \in (0, \sup_{x>0} \frac{U(2x)-U(x)}{x})$, there exists some finite $x_1, \ldots, x_K > 0$ for some integer $K \geq 1$, such that $\frac{U(2x_k)-U(x_k)}{x_k} = \alpha$ for any $k \in \{1, \ldots, K\}$.

Let $x^+ = \max_{k\in\{1,\ldots,K\}} x_k$, now we show that

$$\alpha x > U(2x) - U(x)$$

for any $x > x^+$. To see this, let $V(x) = \alpha x - [U(2x) - U(x)]$. Since x^+ is the largest root of $V(x) = 0$ and that $V(x)$ is continuous on $(0, +\infty)$, we must have either $V(x) > 0$ for all $x > x^+$, or $V(x) < 0$ for all $x > x^+$. Meanwhile, $V'(x) = \alpha - [2U'(2x) - U'(x)]$. Since $U(x)$ is a concave function, we have $\lim_{x\to\infty}[2U'(2x) - U'(x)] = 0$. Therefore, for any $\epsilon \in (0, \frac{1}{2}\alpha)$, there exists $\mu > 0$, such that when $x \geq \mu$, we have $2U'(2x) - U'(x) < \epsilon < \frac{1}{2}\alpha$. Let $M = U(2\mu) - U(\mu)$. Then when $x > \max\{x^+, \mu\}$, we have $U(2x) - U(x) \leq M + \epsilon(x - \mu)$. Hence,

$$\begin{aligned}
V(x) \; &\geq \; (\alpha - \epsilon)x - (M - \epsilon\mu) \\
&> \; \frac{1}{2}\alpha x - (M - \epsilon\mu).
\end{aligned}$$

Notice that α, M, ϵ, and μ are all constants. For any sufficiently large x, therefore, we have $V(x) > 0$. Hence, we have $V(x) > 0$ for all $x > x^+$. That is, $\alpha x > U(2x) - U(x)$ for all $x > x^+$. In addition, we have $\alpha x^+ = U(2x^+) - U(x^+)$. This completes the proof.

D.3. Proof of Corollary 2

Recall that $(1-\gamma)^2\sum_{t=1}^{+\infty}\gamma^{t-1}\big\{U(2tx)-U(tx)\big\}$ is convergent. Therefore, given an arbitrary ϵ-precision level, there exists $T>0$, such that for any $t>T$, we have

$$\sum_{s=T+1}^{t}\gamma^{s-1}\big\{U(2sx)-U(sx)\big\}<\epsilon$$

for a fixed γ. Now suppose that the tail terms after T are (numerically) ignorable. Let

$$f(\gamma,x) = (1-\gamma)^2\sum_{t=1}^{T}\gamma^{t-1}\big\{U(2tx)-U(tx)\big\}.$$

Since T is finite, we have

$$f(0,x)=U(2x)-U(x) \text{ and } f(1,x)=0$$

for any finite $x>0$. On the other hand, for any $\gamma\in(0,1)$, we have $f(\gamma,x)>0$. For fixed $x>0$, we notice that $f(\gamma,x)$ is a polynomial function with respect to γ.

Let

$$M(x)=\sup_{\gamma\in[0,1]}f(\gamma,x),$$

and hence $M(x)=\sup_{\gamma\in[0,1]}f(\gamma,x)\geq f(0,x)=U(2x)-U(x)>0$. Meanwhile, Theorem 1 implies that participation arises if and only if we have (γ,x) that satisfy

$$f(\gamma,x)-C(x)>0.$$

This implies that $C(x)\in(0,M(x))$. Then for any fixed x that satisfies $C(x)\in(0,M(x))$, we now investigate the boundary of γ. For the polynomial function of γ, namely $f(\gamma,x)-C(x)$, we notice that $f(\gamma,x)-C(x)>0$ when $f(\gamma,x)=M(x)$ and $f(\gamma,x)-C(x)<0$ when $\gamma=1$. Therefore, there exist $K\geq 1$ roots, namely $\gamma_1(x),\ldots,\gamma_K(x)$, such that $\gamma_k(x)\in[0,1)$ and $f(\gamma_k(x),x)-C(x)=0$ for $k\in\{1,\ldots,K\}$. Let $\gamma_u(x)=\max_{k\in\{1,\ldots,K\}}\gamma_k(x)$, where the subscript u in γ_u is an abbreviation for the word "upper" (rather than an index for any of $k\in\{1,\ldots,K\}$). Then $\gamma_u(x)<1$. And since $f(1,x)=0$, we have $f(\gamma,x)-C(x)<0$ for $\gamma\in(\gamma_u(x),1]$. In other words, for any fixed x that satisfies $C(x)\in(0,M(x))$, there always exists an upper bound $\gamma_u=\gamma_u(x)$ of γ which is strictly less than 1, such that players would not have participated in long-term FL when $\gamma\geq\gamma_u$. This completes the proof.

D.4. Proof of Theorem 2

Under the grim trigger strategy, without loss of generality, suppose B defects in the first round while A cooperates, and both players defect for all following rounds. For player B, the information stock for each round is $X_0^B = 0, X_1^B = 2x$ and $\forall t \geq 2, X_t^B = (t+1)x$. The total discounted payoff for player B is

$$\begin{aligned}
\pi^D(B) &= T(x|X_0^B) + \sum_{t=2}^{+\infty} \gamma^{t-1} P(x|X_{t-1}^B) \\
&= U(2x) - 0 + \sum_{t=2}^{+\infty} \gamma^{t-1} (U(X_{t-1}^B + x) - U(X_{t-1}^B)) \\
&= U(2x) + \gamma(U(3x) - U(2x)) + \gamma^2(U(4x) - U(3x)) + \ldots \\
&= (U(2x) + \gamma U(3x) + \gamma^2 U(4x) + \ldots) - (\gamma U(2x) + \gamma^2 U(3x) + \ldots) \\
&= (1-\gamma) \sum_{t=1}^{+\infty} \gamma^{t-1} U((t+1)x)
\end{aligned}$$

If instead both players cooperate for all rounds, the information stock for player B would be $\forall t \geq 0, X_t = 2tx$, and the total discounted payoff would be

$$\begin{aligned}
\pi^C(B) &= \sum_{t=1}^{+\infty} \gamma^{t-1} R(x|X_{t-1}^B) \\
&= \sum_{t=1}^{+\infty} \gamma^{t-1} (U(2tx) - U(2tx - 2x) - C(x)) \\
&= -\frac{C(x)}{1-\gamma} + (1-\gamma) \sum_{t=1}^{+\infty} \gamma^{t-1} U(2tx)
\end{aligned}$$

The result follows from the cooperation condition $\pi^C(B) > \pi^D(B)$.

Under tit-for-tat, suppose player B defects in the first round while A cooperates, and B cooperates in the second round while A defects, and both players cooperate for the following rounds. The information stock for player B is $X_0^B = 0, X_1^B = 2x$ and $\forall t \geq 2$, $X_t^B = (2t-1)x$. The total discounted payoff for player B is

$$\begin{aligned}
\pi^D(B) &= T(x|X_0^B) + \gamma S(x|X_1^B) + \sum_{t=3}^{+\infty} \gamma^{t-1} R(x|X_{t-1}^B) \\
&= U(2x) + \gamma(U(3x) - U(2x) - C(x)) + \gamma^2(U(5x) - U(3x) - C(x)) + \ldots \\
&= -\frac{\gamma C(x)}{1-\gamma} + U(2x) + \gamma(U(3x) - U(2x)) + \gamma^2(U(5x) - U(3x)) + \ldots \\
&= -\frac{\gamma C(x)}{1-\gamma} + U(2x) + \gamma(U(3x) - U(2x)) + \sum_{t=3}^{+\infty} \gamma^{t-1}(U(2tx - x) - U(2tx - 3x)) \\
&= -\frac{\gamma C(x)}{1-\gamma} + (1-\gamma) \left\{ U(2x) + \sum_{t=2}^{+\infty} \gamma^{t-1} U(2tx - x) \right\}
\end{aligned}$$

Same as the grim trigger strategy, we have

$$\pi^C(B) = -\frac{C(x)}{1-\gamma} + (1-\gamma)\left\{U(2x) + \sum_{t=2}^{+\infty}\gamma^{t-1}U(2tx)\right\}.$$

Therefore, the cooperation condition is given by

$$\pi^C(B) > \pi^D(B)$$
$$\Leftrightarrow C(x) < (1-\gamma)\sum_{t=2}^{+\infty}\gamma^{t-1}\left(U(2tx) - U(2tx-x)\right).$$

This completes the proof.

D.5. Proofs of Corollary 3

We first consider the tit-for-tat strategy. For a fixed x, recall that $U(\cdot)$ is a concave function, and hence

$$U(4x) - U(3x) \geq U(2tx) - U((2t-1)x) \text{ for any } t \geq 2.$$

From Theorem 2 we know that cooperation implies

$$\begin{aligned}
C(x) &< (1-\gamma)\sum_{t=2}^{+\infty}\gamma^{t-1}\big\{U(2tx) - U((2t-1)x)\big\} \\
&< (1-\gamma)\sum_{t=2}^{+\infty}\gamma^{t-1}\big\{U(4x) - U(3x)\big\} \\
&= (1-\gamma)\frac{\gamma}{1-\gamma}\big\{U(4x) - U(3x)\big\},
\end{aligned}$$

which is equivalent to

$$\gamma > \frac{C(x)}{U(4x) - U(3x)}.$$

Let $\gamma_l = \frac{C(x)}{U(4x)-U(3x)}$, which is a lower bound such that cooperation would not arise if $\gamma \leq \gamma_l$.

Next, we consider the grim trigger strategy. We first show that

$$\frac{U(2tx) - U((t+1)x)}{t-1} > \frac{U(2sx) - U((s+1)x)}{s-1},$$

where $s = t+1$. This can be seen by letting $g_1(\Delta) = \frac{U(a+\Delta)-U(a)}{\Delta}$, where its derivative $g_1'(\Delta) < 0$ for $\Delta > 0$ due to the strict concavity of $U(\cdot)$, and hence $g_1(\cdot)$ is a decreasing function of Δ. Similarly, we can show that $g_2(\Delta) = \frac{U(a)-U(a-\Delta)}{\Delta}$ is an increasing function of Δ. Therefore, for $x > 0$,

$$\frac{U(2tx) - U((t+1)x)}{(t-1)x} > \frac{U(2(t+1)x) - U((t+1)x)}{(t+1)x} > \frac{U(2(t+1)x) - U((t+2)x)}{tx} = \frac{U(2sx) - U((s+1)x)}{(s-1)x}.$$

Let

$$L_1(x) = \frac{U(2\cdot 2\cdot x) - U((2+1)x)}{2-1} = U(4x) - U(3x).$$

Then

$$L_1(x) > \frac{U(2tx) - U((t+1)x)}{t-1} \text{ for any } t \geq 3.$$

Following Theorem 2, cooperation implies

$$\begin{aligned} C(x) &< (1-\gamma)^2 \sum_{t=2}^{+\infty} \gamma^{t-1} \{U(2tx) - U((t+1)x)\} \\ &= (1-\gamma)^2 \sum_{t=2}^{+\infty} (t-1)\gamma^{t-1} \frac{U(2tx) - U((t+1)x)}{t-1} \\ &< (1-\gamma)^2 \sum_{t=2}^{+\infty} (t-1)\gamma^{t-1} L_1(x) \\ &= L_1(x)(1-\gamma)^2 \frac{\gamma}{(1-\gamma)^2}. \end{aligned}$$

That is,

$$\gamma > \frac{C(x)}{U(4x) - U(3x)}.$$

Let $\gamma_l = \frac{C(x)}{U(4x)-U(3x)}$, which is a lower bound such that cooperation would not arise if $\gamma \leq \gamma_l$. This completes the proof.

D.6. Proof of Theorem 3

Following Table 4, the total discounted payoff for player B is

$$\begin{aligned} \pi^D(B) &= U(x) - 0 + \gamma\{U(4x) - U(x) - C(x)\} + \sum_{t=3}^{+\infty} \gamma^{t-1}\{U(2tx) - U(2tx - 2x) - C(x)\} \\ &= (1-\gamma)U(x) + (1-\gamma)\sum_{t=2}^{+\infty} \gamma^{t-1} U(2tx) - \frac{\gamma}{1-\gamma} C(x) \end{aligned}$$

when B defects.

And from the proof of Theorem 2, we can see that the total discounted payoff for player B is

$$\pi^C(B) = -\frac{C(x)}{1-\gamma} + (1-\gamma)\sum_{t=1}^{+\infty} \gamma^{t-1} U(2tx)$$

when B cooperates. The result follows from the cooperation condition $\pi^C(B) > \pi^D(B)$. This completes the proof.

D.7. Proof of Theorem 4

Following Table 5, the total discounted payoff for player B is

$$\pi^D(B) = (1-\gamma)\sum_{t=1}^{+\infty} \gamma^{t-1} U(2tx - x) - \frac{\gamma}{1-\gamma} C(x)$$

when B defects.

And from the proof of Theorem 2, we can see that the total discounted payoff for player B is

$$\pi^C(B) = -\frac{C(x)}{1-\gamma} + (1-\gamma)\sum_{t=1}^{+\infty}\gamma^{t-1}U(2tx)$$

when B cooperates. The result follows from the cooperation condition $\pi^C(B) > \pi^D(B)$. This completes the proof.

D.8. Proof of Corollary 4

For grim trigger, we can see that $(\gamma_l^{(GT)}, \gamma_u^{(GT)}) \subset (0, \gamma_u)$ by observing that any γ that satisfies inequality (4) would also satisfy inequality (3) in Theorem 1. Meanwhile, it is mentioned in the text of Section 4.2.3 that $(\gamma_l^{(S+GT)}, \gamma_u^{(S+GT)}) \equiv (0, \gamma_u)$. Therefore, $(\gamma_l^{(GT)}, \gamma_u^{(GT)}) \subset (\gamma_l^{(S+GT)}, \gamma_u^{(S+GT)})$.

For tit-for-tat, we notice that any γ that satisfies inequality (5) also satisfies inequality (7). Therefore, $(\gamma_l^{(TfT)}, \gamma_u^{(TfT)}) \subset (\gamma_l^{(S+TfT)}, \gamma_u^{(S+TfT)})$. This completes the proof.

D.9. Proof of Theorem 5 (and Corollaries 1 and 2 in the presence of the fixed cost)

In the presence of the fixed cost, the discounted total payoff for participation becomes

$$\begin{aligned}\pi^C(p) &= -c_0 + \sum_{t=1}^{+\infty}\gamma^{t-1}R(x|X_{t-1}^p) \\ &= -c_0 + \sum_{t=1}^{+\infty}\gamma^{t-1}(U(2tx) - U(2tx-2x) - C(x)) = -c_0 - \frac{C(x)}{1-\gamma} + (1-\gamma)\sum_{t=1}^{+\infty}\gamma^{t-1}U(2tx).\end{aligned} \tag{9}$$

In contrast, the discounted total payoff for no participation $\pi^{NP}(p)$ in (2) does not change. Let $\pi^C(p)$ and $\pi^{NP}(p)$ be defined in (9) and (2), respectively. Then two players prefer to form a long-term FL partnership if and only if $\pi^C(p) > \pi^{NP}(p)$. The result of Theorem 5 immediately follows $\pi^C(p) > \pi^{NP}(p)$.

The results in Corollary 1 still hold. This is because players' cost sensitivity and marginal payoff gain each round are not affected by the presence of the fixed cost. Mathematically, this is reflected as

$$-(1-\gamma)c_0 + (1-\gamma)^2\sum_{t=1}^{+\infty}\gamma^{t-1}\{U(2tx) - U(tx)\} \le (1-\gamma)^2\sum_{t=1}^{+\infty}\gamma^{t-1}\{U(2tx) - U(tx)\}.$$

Then the results follow the rest of the proof of Corollary 1 naturally.

Finally, we prove the results in Corollary 2 in the presence of the fixed cost. Similar to the proof of Corollary 2, suppose that the tail terms after T are (numerically) ignorable. Let

$$f(\gamma, x) = (1-\gamma)^2\sum_{t=1}^{T}\gamma^{t-1}\{U(2tx) - U(tx)\}.$$

Since T is finite, we have

$$f(0,x)=U(2x)-U(x) \text{ and } f(1,x)=0$$

for any finite $x>0$. On the other hand, for any $\gamma\in(0,1)$, we have $f(\gamma,x)>0$. For fixed $x>0$, we notice that $f(\gamma,x)$ is a polynomial function with respect to γ. By the definition of $M(x)$ in the proof of Corollary 2, we notice that

$$M(x)=\sup_{\gamma\in[0,1]} f(\gamma,x)\geq f(0,x)=U(2x)-U(x)>0.$$

Meanwhile, Theorem 5 implies that participation arises if and only if

$$f(\gamma,x)-(1-\gamma)c_0-C(x)>0.$$

That is, $C(x)+(1-\gamma)c_0\in(0,M(x))$. Now for a fixed x that satisfies $C(x)+(1-\gamma)c_0\in(0,M(x))$, we investigate the polynomial of γ, namely $f(\gamma,x)-(1-\gamma)c_0-C(x)$, to find the boundary condition for γ. On the one hand, we have $f(\gamma,x)-(1-\gamma)c_0-C(x)>0$ for γ that satisfies $f(\gamma,x)=M(x)$. On the other hand, we have $f(\gamma,x)-(1-\gamma)c_0-C(x)<0$ when $\gamma=1$. Therefore, there exist $K\geq 1$ roots, namely $\gamma_1(x),\dots,\gamma_K(x)$, such that $\gamma_k(x)\in[0,1)$ and $f(\gamma_k(x),x)-C(x)=0$ for $k\in\{1,\dots,K\}$. Let $\gamma_u(x)=\max_{k\in\{1,\dots,K\}}\gamma_k(x)$, where the subscript u in γ_u is an abbreviation for the word "upper" (rather than an index for any of $k\in\{1,\dots,K\}$). Then $\gamma_u(x)<1$. And since $f(1,x)=0$, we have $f(\gamma,x)-C(x)<0$ for $\gamma\in(\gamma_u(x),1]$. In other words, for any fixed x that satisfies $C(x)\in(0,M(x))$, there always exists an upper bound $\gamma_u=\gamma_u(x)$ of γ which is strictly less than 1, such that players would not have participated in long-term FL when $\gamma\geq\gamma_u$.

We also note that, in the presence of the fixed cost, a positive lower bound on discount rate may also exist when the sum of the sunk cost and the cost each round, namely $c_0+C(x)$, is neither too large or too small. However, the exact boundary conditions are not straightforward and are therefore omitted.

D.10. Proof of Corollary 5

Recall that $U(x)$ is a concave function, and hence $U'(x)>U'(2x)$, or equivalently, $U'(x)>2U'(2x)-U'(x)$ for any $x>0$. Then $C'(x)\geq U'(x)$ for any $x\geq x^+$ implies $C'(x)>2U'(2x)-U'(x)$. In other words, $C'(x)$ is larger than the derivative of $U(2x)-U(x)$ for any $x\geq x^+$. Meanwhile, since $C(x^+)-(U(2x^+)-U(x^+))\geq 0$ and $C'(x)-(U(2x)-U(x))'>0$ for any $x\geq x^+$, we have $C(x)-(U(2x)-U(x))>0$ for any $x\geq x^+$. This completes the proof.

D.11. Proof of Theorem 6

We start by considering the grim trigger strategy. Without loss of generality, assume player B defects in the first round by sharing x' with player A, whereas player A cooperates and shares x. Both players defect for the rest of the game. For player B, the utility of defecting is

$$\begin{aligned}\pi^D(B) &= U(2x) - C(x') + \gamma(U(3x) - U(2x)) + \gamma^2(U(4x) - U(3x)) + \dots \\ &= -C(x') + (1-\gamma)\sum_{t=1}^{+\infty}\gamma^{t-1}U((t+1)x).\end{aligned}$$

Recall from the proof of Theorem 2 that the utility for player B under mutual cooperation is

$$\pi^C(B) = -\frac{C(x)}{1-\gamma} + (1-\gamma)\sum_{t=1}^{+\infty}\gamma^{t-1}U(2tx).$$

Therefore, the boundary condition for stable mutual cooperation can be obtained as

$$\pi^C(B) > \pi^D(B) \Leftrightarrow C(x) - (1-\gamma)C(x') < (1-\gamma)^2\sum_{t=2}^{+\infty}\gamma^{t-1}\big\{U(2tx) - U((t+1)x)\big\}.$$

Under the tit-for-tat strategy, we again assume that player B defects in the first round by sharing x' while A cooperates. Then, player A defects in the second round while B cooperates. Both players cooperate for the rest of the game. The utility for player B is

$$\begin{aligned}\pi^D(B) &= U(2x) - C(x') + \gamma(U(3x+x') - U(2x) - C(x)) + \gamma^2(U(5x+x') - U(3x+x') - C(x)) + \dots \\ &= -C(x') - \frac{\gamma C(x)}{1-\gamma} + (1-\gamma)\left\{U(2x) + \sum_{t=2}^{+\infty}\gamma^{t-1}U(2tx - x + x')\right\}.\end{aligned}$$

Therefore, the boundary condition for stable mutual cooperation can be obtained as

$$\pi^C(B) > \pi^D(B) \Leftrightarrow C(x) - C(x') < (1-\gamma)\sum_{t=2}^{+\infty}\gamma^{t-1}\left(U(2tx) - U(2tx - x + x')\right).$$

Similarly, under central server sanction, we have

$$\pi^D(B) = (1-\gamma)U(x) + (1-\gamma)\sum_{t=2}^{\infty}\gamma^{t-1}U(2tx) - C(x') - \frac{\gamma C(x)}{1-\gamma}.$$

Under sanction and grim trigger strategy, we have

$$\pi^D(B) = (1-\gamma)\sum_{t=1}^{\infty}\gamma^{t-1}U(tx) - C(x').$$

And under sanction and tit-for-tat strategy, we have

$$\pi^D(B) = (1-\gamma)U(x) + (1-\gamma)\sum_{t=2}^{\infty}\gamma^{t-1}U(2tx - x + x') - C(x') - \frac{\gamma C(x)}{1-\gamma}.$$

Then the corresponding conditions can be derived by letting $\pi^C(B) > \pi^D(B)$. This completes the proof.

D.12. Proof of Theorem 7

We notice that mathematically the proof of Theorem 6 does not require $x' > 0$. Therefore, the results in Theorem 6 hold true by replacing x' with $-\xi$. However, we also notice that $C(-\xi) = 0$. This completes the proof.

Appendix E: Special Case of An Upper Bound on the Discount Rate

Let $M_1(x) = U(2x) - U(x)$ and $M_2(x) = \frac{1}{2}\{U(4x) - U(2x)\}$. If the following condition is satisfied,

$$M_2(x) < C(x) < M_1(x),$$

then there exists

$$\gamma_u = 1 - \frac{1}{2}\sqrt{\frac{C(x) - M_2(x)}{M_1(x) - M_2(x)}} \in (0, 1)$$

such that players would not participate in long-term FL when $\gamma \geq \gamma_u$.

Now we prove the statement above. Recall from the proof of Corollary 1 that $M_2(x) < M_1(x)$, and players would participate in long-term FL only if $C(x) < M_1(x)$. Then equation (3) satisfies

$$\begin{aligned} C(x) &< (1-\gamma)^2 \left\{ U(2tx) - U(tx) + \sum_{t=2}^{+\infty} \gamma^{t-1} \{U(2tx) - U(tx)\} \right\} \\ &< (1-\gamma)^2 \left\{ U(2tx) - U(tx) + \sum_{t=2}^{+\infty} \gamma^{t-1} \cdot tx \cdot \frac{U(4x) - U(2x)}{2x} \right\} \\ &= (1-\gamma)^2 \left\{ U(2tx) - U(tx) + \left[\frac{1}{(1-\gamma)^2} - 1 \right] \cdot \frac{U(4x) - U(2x)}{2} \right\} \\ &= \{M_1(x) - M_2(x)\}(\gamma - 1)^2 + M_2(x). \end{aligned}$$

For fixed $x > 0$, let

$$f(\gamma) = \{M_1(x) - M_2(x)\}(\gamma - 1)^2 + M_2(x) - C(x).$$

and let the participation interval $PI = \{\gamma \in (0, 1) : f(\gamma) > 0\}$. Then we can see that $PI \neq \emptyset$ implies $C(x) < M_1(x)$. And when $M_2(x) < C(x)$, there exists an upper bound for γ, namely $\gamma_u = 1 - \frac{1}{2}\sqrt{\frac{C(x)-M_2(x)}{M_1(x)-M_2(x)}} \in (0, 1)$, which is the left root of $f(\gamma) = 0$, such that players would not participate in long-term FL when $\gamma \geq \gamma_u$.

Appendix F: Results on Non-Identical Players and Multiple Players

In this part, we provide detailed theoretical results in the cases of non-identical players and multiple players.

F.1. Theoretical Results on Non-Identical Players

When the two players, A and B, are no longer identical, cooperation now requires that *neither* player has an incentive to defect, or formally, $\pi^C(A) > \pi^D(A)$ *and* $\pi^C(B) > \pi^D(B)$.

Under differential cost sensitivity, following the same derivations in the proofs of Theorem 2 (Appendix D), the cooperation conditions are readily obtained by replacing $C(x)$ with $\max\{C_A(x), C_B(x)\}$. Without loss of generality, assuming that $\alpha_A < \alpha_B$, then $\max\{C_A(x), C_B(x)\} = \alpha_B x$. We summary this result in the following theorem.

THEOREM 8. *Under differential cost sensitivity ($\alpha_A < \alpha_B$), two players will cooperate if and only if player B is willing to cooperate. Specifically,*

$$C_B(x) < (1-\gamma)^2 \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{U(2tx) - U((t+1)x)\big\} \qquad \text{(under grim trigger only)}$$

$$C_B(x) < (1-\gamma) \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{U(2tx) - U((2t-1)x)\big\} \qquad \text{(under tit-for-tat only)}$$

$$\gamma < 1 - \frac{C_B(x)}{U(2x) - U(x)} \qquad \text{(under sanction only)}$$

$$C_B(x) < (1-\gamma)^2 \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{U(2tx) - U(tx)\big\} \qquad \text{(under sanction and grim trigger)}$$

$$C_B(x) < (1-\gamma) \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{U(2tx) - U((2t-1)x)\big\} \qquad \text{(under sanction and tit-for-tat)}$$

Under different value extraction capabilities, the cooperation condition can be straightforwardly derived. Take the tit-for-tat strategy as an example. Let $V_i = (1-\gamma) \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{U_i(2tx) - U_i((2t-1)x)\big\}$ where $i \in \{A, B\}$, then the two parties will cooperate as long as $C(x) < \min\{V_A, V_B\}$. Assuming $\beta_A < \beta_B$, then we have $\min\{V_A, V_B\} = V_A = (1-\gamma)\beta_A \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{U(2tx) - U((2t-1)x)\big\}$. We summarize the results in the following theorem.

THEOREM 9. *Under differential value extraction capabilities ($\beta_A < \beta_B$), two players will cooperate if and only if player A is willing to cooperation. Specifically,*

$$C(x) < (1-\gamma)^2 \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{U_A(2tx) - U_A((t+1)x)\big\} \qquad \text{(under grim trigger only)}$$

$$C(x) < (1-\gamma) \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{U_A(2tx) - U_A((2t-1)x)\big\} \qquad \text{(under tit-for-tat only)}$$

$$\gamma < 1 - \frac{C(x)}{U_A(2x) - U_A(x)} \qquad \text{(under sanction only)}$$

$$C(x) < (1-\gamma)^2 \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{ U_A(2tx) - U_A(tx) \big\} \qquad \text{(under sanction and grim trigger)}$$

$$C(x) < (1-\gamma) \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{ U_A(2tx) - U_A((2t-1)x) \big\} \qquad \text{(under sanction and tit-for-tat)}$$

Under different information acquisition capacities, without loss of generality, we assume $x_A < x_B$. Following the same calculations in the proofs of Theorem 2, we can derive the cooperation conditions, which we summarize in Theorem 10.

Theorem 10. *Under differential information acquisition capabilities ($x_A < x_B$), two players will cooperate if and only if player B is willing to cooperation. Specifically,*

$$C(x_B) < (1-\gamma)^2 \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{ U(tx_A + tx_B) - U(x_A + tx_B) \big\} \qquad \text{(under grim trigger only)}$$

$$C(x_B) < (1-\gamma) \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{ U(tx_A + tx_B) - U((t-1)x_A + tx_B) \big\} \qquad \text{(under tit-for-tat only)}$$

$$\gamma < 1 - \frac{C(x_B)}{U(x_A + x_B) - U(x_B)} \qquad \text{(under sanction only)}$$

$$C(x_B) < (1-\gamma)^2 \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{ U(tx_A + tx_B) - U(tx_B) \big\} \qquad \text{(under sanction and grim trigger)}$$

$$C(x_B) < (1-\gamma) \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{ U(tx_A + tx_B) - U((t-1)x_A + tx_B) \big\} \qquad \text{(under sanction and tit-for-tat)}$$

In a more general case, the amount of information x_A and x_B can also be time-variant across different rounds, which we denote as $x_{A,t}$ and $x_{B,t}$ for $t \geq 1$. Assuming that player B still has higher information acquisition capacity, i.e., $x_{A,t} < x_{B,t}$ for $t \geq 1$, the cooperation conditions in Theorem 10 become mathematically more complex, while the underlying dynamics remain qualitatively unchanged. For example, the cooperation conditions under the two trigger strategies are summarized as follows.

$$\sum_{t=1}^{+\infty} \gamma^{t-1} C(x_{B,t}) < (1-\gamma) \sum_{t=2}^{+\infty} \gamma^{t-1} \left\{ U\left(\sum_{s=1}^{t} (x_{A,s} + x_{B,s}) \right) - U\left(x_{A,1} + \sum_{s=1}^{t} x_{B,s} \right) \right\} \qquad \text{(under grim trigger)}$$

$$C(x_{B,1}) < (1-\gamma) \sum_{t=2}^{+\infty} \gamma^{t-1} \left\{ U\left(\sum_{s=1}^{t} (x_{A,s} + x_{B,s}) \right) - U\left(\sum_{\substack{s=1,\dots,t,\\ s\neq 2}} x_{A,s} + \sum_{s=1}^{t} x_{B,s} \right) \right\} \qquad \text{(under tit-for-tat)}$$

Finally, we show an example of two players differ in more than one aspect. If player B is more sensitive to the cost of contribution (i.e., $\alpha_A < \alpha_B$) while player A has a lower value extraction capability (i.e., $\beta_A < \beta_B$), the cooperation condition under the two trigger strategies becomes

$$C_B(x) < (1-\gamma)^2 \beta_A \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{ U(2tx) - U((t+1)x) \big\} \qquad \text{(under grim trigger)}$$

$$C_B(x) < (1-\gamma)\beta_A \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{ U(2tx) - U((2t-1)x) \big\} \qquad \text{(under tit-for-tat)}$$

which is a combination of Theorems 8 and 9.

F.2. Theoretical Results on Multiple Players

We now consider the case of multiple players. For an arbitrary player $p \in \{P_1, \dots, P_K\}$, We first derive the utility associated with stable mutual cooperation:

$$\begin{aligned}\pi^C(p) &= \sum_{t=1}^{+\infty} \gamma^{t-1} (U(Ktx) - U(Ktx - Kx) - C(x)) \\ &= -\frac{C(x)}{1-\gamma} + (1-\gamma) \sum_{t=1}^{+\infty} \gamma^{t-1} U(Ktx).\end{aligned}$$

Without loss of generality, assume that players $\{P_1, \dots, P_s\}$ are the defectors. Under the grim trigger strategy, for a defector $p \in \{P_1, \dots, P_s\}$, defecting on the first round would receive $(K-s)x$ information shared by the cooperating players, and therefore $X_1^p = (K-s+1)x$. Starting from the second round, all players defect and $\forall t \geq 2$, $X_t^p = (K-s+1)x + (t-1)x = (K-s+t)x$. The discounted total utility for defector p is:

$$\begin{aligned}\pi^D(p) &= U((K-s+1)x) - 0 + \sum_{t=2}^{+\infty} \gamma^{t-1} \big\{ U((K-s+t)x) - U((K-s+t-1)x) \big\} \\ &= (1-\gamma) \sum_{t=1}^{+\infty} \gamma^{t-1} U((K-s+t)x).\end{aligned}$$

Therefore, the boundary conditions for mutual cooperation is obtained as

$$\begin{aligned}&\pi^C(p) > \pi^D(p) \\ \Leftrightarrow\ & C(x) < (1-\gamma)^2 \sum_{t=1}^{+\infty} \gamma^{t-1} \big\{ U(Ktx) - U((K-s+t)x) \big\}.\end{aligned}$$

Under the tit-for-tat strategy, for a defector $p \in \{P_1, \dots, P_s\}$, defecting on the first round would receive $(K-s)x$ information shared by the cooperating players, and therefore $X_1^p = (K-s+1)x$. Cooperating on the second round (while the cooperators during the first round now defect) would receive an additional sx information shared among all the defectors, i.e., $X_2^p = (K-s+1)x + sx = (K+1)x$. Starting from the third round, all players cooperate and $\forall t \geq 3$, $X_t^p = (K+1)x + (t-2)Kx = (Kt-K+1)x$. The discounted total utility for defector p is:

$$\begin{aligned}\pi^D(p) =& U((K-s+1)x) - 0 + \gamma \big\{ U((K+1)x) - U((K-s+1)x) - C(x) \big\} + \\ & \sum_{t=3}^{+\infty} \gamma^{t-1} \big\{ U((Kt-K+1)x) - U((Kt-2K+1)x) - C(x) \big\} \\ =& (1-\gamma) U((K-s+1)x) + (1-\gamma) \sum_{t=2}^{+\infty} \gamma^{t-1} U((Kt-K+1)x) - \frac{\gamma C(x)}{1-\gamma}.\end{aligned}$$

Therefore, the boundary conditions for mutual cooperation is obtained as

$$\pi^C(p) > \pi^D(p)$$
$$\Leftrightarrow C(x) < (1-\gamma)\big\{U(Kx) - U((K-s+1)x)\big\} + (1-\gamma)\sum_{t=2}^{+\infty}\gamma^{t-1}\big\{U(Ktx) - U((Kt-K+1)x)\big\}.$$

The cooperation conditions under the sanction mechanism and under the combination of sanction and trigger strategies can be analogously derived, which we summarize below.

$$C(x) < (1-\gamma)(U(Kx) - U(x)) + (1-\gamma)\sum_{t=2}^{+\infty}\gamma^{t-1}\big\{U(tKx) - U((tKx-s+1)x)\big\} \qquad \text{(under sanction only)}$$

$$C(x) < (1-\gamma)^2\sum_{t=1}^{+\infty}\gamma^{t-1}\big\{U(Ktx) - U(tx)\big\} \qquad \text{(under sanction and grim trigger)}$$

$$C(x) < (1-\gamma)\sum_{t=1}^{+\infty}\gamma^{t-1}\big\{U(Ktx) - U((Kt-K+1)x)\big\} \qquad \text{(under sanction and tit-for-tat)}$$