Online Supplement to “Optimal Control of Service Systems with Heterogeneous Servers and Priority Customers”

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A. Dedicated Policy and Work-Conserving Flexible Priority Policy

Here, we provide the analysis for the dedicated and the work-conserving flexible priority policies for the 2-server-2-customer-class system, as mentioned in the introduction section.

Dedicated Policy: Under the dedicated policy, normal and VIP customers form separate queues and are served by dedicated servers. In other words, the system consists of two independent M/M/1 queues with arrival and service rates λ_1, μ_1 (for normal customers) and λ_2, μ_2 (for VIP customers), respectively (the case where server 1 is assigned to VIP customers and server 2 is assigned to normal customers can be analyzed in a similar fashion). Define $\rho_i = \frac{\lambda_i}{\mu_i}$ for $i = 1, 2$. To avoid trivial cases, we further assume that $\rho_i < 1$ for $i = 1, 2$, since otherwise the system is not stable and would incur an infinite long-run average cost. Let C_1^{D} and C_2^{D} denote the long-run average waiting costs of normal and VIP customers, respectively, under the dedicated policy. From classic results on M/M/1 queue,

$$C_1^{\mathrm{D}} = \frac{c_1 \rho_1^2}{1 - \rho_1};$$

$$C_2^{\mathrm{D}} = \frac{c_2 \rho_2^2}{1 - \rho_2}.$$

Work-Conserving Flexible Priority Policy: The derivation of the expected waiting time for normal and VIP customers, under the work-conserving flexible priority policy, makes use of a probabilistic equivalence result to the waiting time of an arbitrary customer under an M/G/1 system with multiple server's vacations. Let W_1^{F} and W_2^{F} denote the long-run average waiting time for normal and VIP customers, respectively, under the work-conserving flexible priority policy. It can be shown that the conditional expectations of W_1^{F} and W_2^{F}, given that both servers are busy, are equal to

$$E\left(W_1^{\mathrm{F}} | \text{both servers are busy}\right) = \frac{\mu_1 + \mu_2}{(\mu_1 + \mu_2 - \lambda_2)(\mu_1 + \mu_2 - \lambda_1 - \lambda_2)}; \tag{OS-A1}$$

$$E\left(W_2^{\mathrm{F}} | \text{both servers are busy}\right) = \frac{1}{\mu_1 + \mu_2 - \lambda_2},$$

respectively. The details of these results can be found in Kleinrock (1975) and Kella and Yechiali (1985).

Next, we derive the probability that both servers are busy (i.e., the *busy probability*). To do this, we construct a *heterogeneous-M/M/2 system* with arrival rate $\lambda_1 + \lambda_2$ and service rates μ_1 and μ_2. In such a system, if a customer finds an empty system with both servers available upon arrival, he/she will always enter service with the faster server. Without loss of generality, we assume $\mu_1 \geq \mu_2$. Then, it is important to notice that the only difference between this heterogeneous-M/M/2 system and the work-conserving-flexible-priority system is on the sequence of customers to be served. The M/M/2 system follows the first-come-first-served service discipline, while the priority system prioritizes VIP customers. If we ignore customer identities (normal or VIP), then the two systems are exactly the same in system statistics (e.g., queue length and service completion). By definition, a *busy period* starts from the time point when a customer arrives and finds exactly one other customer in system (under service), and ends after the time point when a customer departs and leaves exactly one other customer in system (under service). The two systems share the same busy probability, and we have the following result.

Lemma OS-A1. *Under the work-conserving flexible priority policy, the probability that both servers are busy is given by*

$$\pi = 1 - \frac{(\mu_1+\mu_2-\lambda_1-\lambda_2)\left((\lambda_1+\lambda_2)^2\mu_1 + [(\lambda_1+\lambda_2)^2 + 3(\lambda_1+\lambda_2)\mu_1 + \mu_1^2]\mu_2 + (\lambda_1+\lambda_2+\mu_1)\mu_2^2\right)}{(\lambda_1+\lambda_2)^2\mu_1^2 + [2(\lambda_1+\lambda_2)+\mu_1]\mu_1^2\mu_2 + (\lambda_1+\lambda_2+\mu_1)(\lambda_1+\lambda_2+2\mu_1)\mu_2^2 + (\lambda_1+\lambda_2+\mu_1)\mu_2^3}. \tag{OS-A2}$$

Let C_1^{F} and C_2^{F} denote the long-run average waiting costs of normal and VIP customers, respectively, under the work-conserving flexible priority policy. Since queue only exists while both servers are busy, as a result of Lemma OS-A1; Equations (OS-A1); as well as the Little's Law, we have

$$\begin{aligned} C_1^F &= \frac{c_1\pi\lambda_1(\mu_1+\mu_2)}{(\mu_1+\mu_2-\lambda_2)(\mu_1+\mu_2-\lambda_1-\lambda_2)}; \\ C_2^F &= \frac{c_2\pi\lambda_2}{\mu_1+\mu_2-\lambda_2}, \end{aligned}$$

where π is given by Equation (OS-A2).

B. Multimodularity

Here, we provide the definition and several important properties of multimodular functions.

Define $\mathbf{d}_i = \mathbf{e}_{i-1} - \mathbf{e}_i$ for $i = 2, ..., m$ such that $-\mathbf{e}_1 + \sum_{i=2}^m \mathbf{d}_i + \mathbf{e}_m = \mathbf{0}$. Let $\mathcal{F} = \{-\mathbf{e}_1, \mathbf{d}_2, ..., \mathbf{d}_m, \mathbf{e}_m\}$ and $\bar{\mathcal{F}} = \{\pm\mathbf{e}_1, ..., \pm\mathbf{e}_m, \pm\mathbf{d}_2, ..., \pm\mathbf{d}_m\}$.

Definition OS-B1. *A real-valued function $f : \mathbb{Z}^m \to \mathbb{R}$ is multimodular with respect to $\mathcal{F}$, if for all $\mathbf{x} \in \mathbb{Z}^m$; $\mathbf{a}, \mathbf{b} \in \mathcal{F}$ with $\mathbf{a} \neq \mathbf{b}$, the following property holds:*

$$f(\mathbf{x}+\mathbf{a}) + f(\mathbf{x}+\mathbf{b}) \geq f(\mathbf{x}) + f(\mathbf{x}+\mathbf{a}+\mathbf{b}).$$

For any function g defined on $\mathbb{Z}^m$, let $\Delta_{\mathbf{e}_i} g(\mathbf{x}) = g(\mathbf{x}+\mathbf{e}_i) - g(\mathbf{x})$, $\Delta_{-\mathbf{e}_i} g(\mathbf{x}) = g(\mathbf{x}-\mathbf{e}_i) - g(\mathbf{x})$, and $\Delta_{\mathbf{d}_i} g(\mathbf{x}) = \Delta_{-\mathbf{e}_i} g(\mathbf{x}) - \Delta_{-\mathbf{e}_{i-1}} g(\mathbf{x}) = g(\mathbf{x}-\mathbf{e}_{i-1}+\mathbf{d}_i) - g(\mathbf{x}-\mathbf{e}_{i-1})$. With these definitions, we can show that multimodularity can be characterized by the second order differences.

Lemma OS-B1. *A function f is multimodular if and only if*

$$\Delta_{\mathbf{a}}\Delta_{\mathbf{b}} f \leq 0$$

for all $\mathbf{a}, \mathbf{b} \in \mathcal{F}$ with $\mathbf{a} \neq \mathbf{b}$.

Lemma OS-B2. *A function f is multimodular with respect to $\mathcal{F}$ if and only if*

$$f(\mathbf{x}+\mathbf{a}_1+...+\mathbf{a}_l) + f(\mathbf{x}+\mathbf{a}_{l+1}+...+\mathbf{a}_k) \geq f(\mathbf{x}) + f(\mathbf{x}+\mathbf{a}_1+...+\mathbf{a}_k)$$

for any $\{\mathbf{a}_1, ..., \mathbf{a}_l, ..., \mathbf{a}_k\} \in \mathcal{F}, 0 < l < k$, and $\mathbf{a}_i \neq \mathbf{a}_j$ for $i \neq j$.

Essentially, compared with Lemma OS-B1, Lemma OS-B2 provides a more general characterization of multimodularity. Next, we provide some important properties of multimodular functions.

Lemma OS-B3. *For any $\mathbf{a}, \mathbf{b} \in \bar{\mathcal{F}}$ and $p, q \in \mathbb{R}$, the following properties hold:*

(a) $\Delta_{\mathbf{a}}(pf + qg) = p\Delta_{\mathbf{a}} f + q\Delta_{\mathbf{a}} g$;

(b) $\Delta_{\mathbf{a}}\Delta_{\mathbf{b}}f = \Delta_{\mathbf{b}}\Delta_{\mathbf{a}}f$.

Lemma OS-B4. *If f is multimodular with respect to $\mathcal{F}$, then the following properties hold:*

(a) $\Delta_{\mathbf{e}_i}\Delta_{\mathbf{e}_j}f \geq 0, \forall i, j$;

(b) $\Delta_{\mathbf{e}_i}\Delta_{\mathbf{e}_i}f \geq \Delta_{\mathbf{e}_j}\Delta_{\mathbf{e}_i}f, \forall i, j$;

(c) $\Delta_{\mathbf{e}_i}\Delta_{\mathbf{d}_i}f \leq 0, \forall i \neq 1$;

(d) $\Delta_{\mathbf{d}_i}\Delta_{\mathbf{d}_i}f \geq 0, \forall i \neq 1$;

(e) $\Delta_{\mathbf{e}_i}\Delta_{\mathbf{d}_j}f \geq 0$ *if* $i < j, j \neq 1$; *and* $\Delta_{\mathbf{e}_i}\Delta_{\mathbf{d}_j}f \leq 0$ *if* $i > j, j \neq 1$.

For the sake of brevity, we omit the proofs of the above four lemmas, and refer interested readers to Altman et al. (2000).

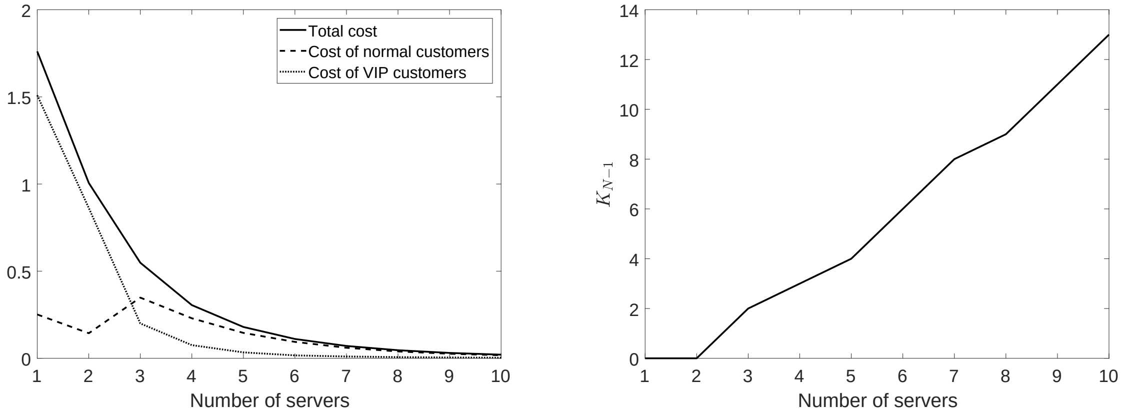
C. Additional Numerical Study

Here, we conduct numerical studies to provide further insights from our study.

Impact of Server Configuration: Our analysis has been focusing on the problem of, given the availability of multiple servers with identical or distinct service speeds, how to dynamically assign them to serve different customers. While the solution of this online optimization problem determines the performance of the system, another decision that can largely influence system performance is the configuration of servers. For example, how many servers to have? Is it better to have fewer faster servers or more slower servers (i.e., assuming fixed total service capacity and additive service rate for server collaboration, should we consider server pooling)?

Figure OS-C1 plots how things change as the system evolves from one fast server to ten slow servers. As the number of servers increases, from Figure OS-C1b, the threshold K_{N-1} (admitting a normal customer when there is one server available) increases, which means service capacity is more reserved for VIP customers. The intuition is that, as the number of servers increases, the service rate of each server decreases. Then, it takes longer for a server to complete a under-service customer and become available. In this case, if we assign a server to serve a normal customer and then a VIP customer arrives, this VIP customer will wait longer to enter service. Thus, with lower service rate, it is more important to reserve servers for future VIP arrivals. This effect, together with the fact that the system contains more servers, leads to a decreased waiting cost of VIP customers, as confirmed by Figure OS-C1a. While the change in normal customers' waiting cost is not monotone, we observe from Figure OS-C1a that the system total waiting cost decreases as the number of servers increases.

Our next numerical experiment examines the effect of server heterogeneity on customers' waiting cost. Consider a setting with two customer classes and two servers. As previous, the waiting-cost rates of normal and VIP customers are set as $\frac{c_2}{c_1} = 10$. The service rates of the two servers are set as $\mu_1 = 1 + \delta$ and $\mu_2 = 1 - \delta$, respectively, where the parameter $\delta \in \{0, 0.1, ..., 0.9\}$ captures the degree of server heterogeneity (while keeping the system total service rate $\mu_1 + \mu_2$ constant). Regarding the customer arrival rates, we



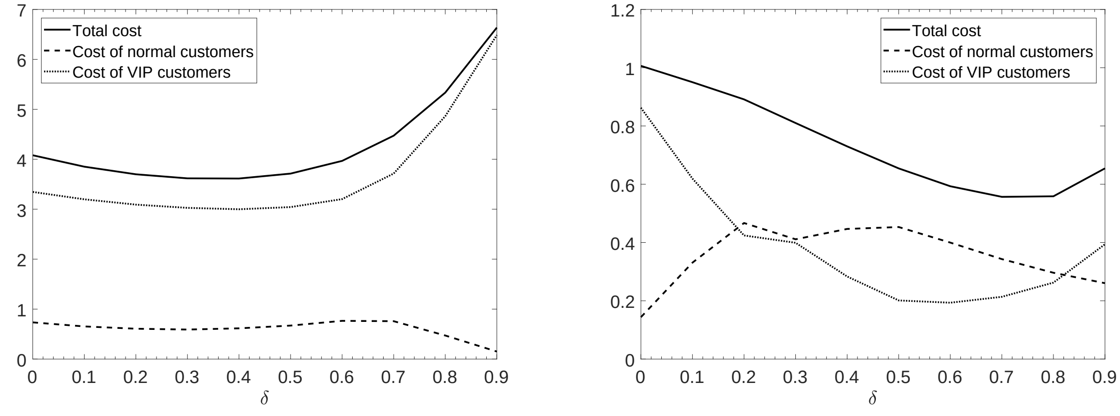
(a) Waiting Cost

(b) Threshold K_{N-1}

Figure OS-C1: Number of Servers

consider two cases. In the first case, $\lambda_1 = \lambda_2$ (i.e., the arrival rate of VIP customers equals that of normal customers); and in the second case, $\frac{\lambda_1}{\lambda_2} = 10$. For both cases, we fix the system utilization $\rho = \frac{\lambda_1+\lambda_2}{\mu_1+\mu_2} = 0.4$.

Figure OS-C2 plots how waiting costs change as δ increases (i.e., the two servers become more heterogeneous in speed). As we can see, under both cases, the system total waiting cost first decreases and then increases, indicating a better performance under a moderate level of server heterogeneity. The same pattern also holds for the waiting cost of VIP customers. However, the change in normal customers' waiting cost is again not monotone.



(a) $\lambda_1 = \lambda_2$

(b) $\lambda_1 = 10\lambda_2$

Figure OS-C2: Server Heterogeneity

Impact of Customer Segmentation: After analyzing the server-configuration effects, we next examine the impact of customer segmentation on system performance. Customer segmentation has been a popular revenue management strategy in practice and plays an important role in improving firms' bottom line (see the discussion in Talluri and van Ryzin 2004, Gallego and Topaloglu 2019 and the references therein). Here, we study how segmenting customers into multiple classes with different service-priority levels influences the system waiting cost.

In particular, we consider a system consisting of two homogeneous servers with service rates $\mu_1 = \mu_2 = 5$. There exist five groups of customers with identical arrival rates $\lambda = 0.8$ but different waiting-cost rates $\mathbf{c} = [1, 5, 10, 15, 20]$. Under the *initial setting*, the five groups of customers will not be differentiated. They are considered as a single class and served based on a first-come-first-served policy. In this case, the expected waiting-cost rate of an arbitrary arrival equals $\frac{c_1+c_2+c_3+c_4+c_5}{5} = 10.2$. Now, to segment the customers, we consider two approaches. The first one is called the *top-down* approach, where in each step, the customer group with the highest waiting-cost rate is split from the joint pool. Under this approach, we first segment a single class into two. Since Group-5 customers are associated with the highest waiting-cost rate ($c_5 = 20$), they will be split from the joint pool and become class-2 customers. The remaining four groups of customers (Groups 1-4) in the joint pool become class-1 customers. Class-2 customers (with waiting-cost rate 20) have non-preemptive priority over class-1 customers (with waiting-cost rate $\frac{1+5+10+15}{4} = 7.75$). To further segment the customers into three classes, Group-4 customers (who are associated with the highest waiting-cost rate in the current joint pool) will be split from the joint pool, and so on and so forth. The second segmentation approach is called the *bottom-up* approach, where in each step, in contrast to the top-down approach, the customer group with the lowest waiting-cost rate is split from the joint pool. Under this approach, Group-1 customers are the first to be split from the joint pool.

As shown in Figure OS-C3, under both segmentation approaches, the system total waiting cost decreases as the number of customer classes increases. Thus, it is beneficial to segment customers based on their waiting costs. However, both curves exhibit a convex-decreasing pattern implying that the marginal improvement obtained from further segmentation is diminishing. Finally, we see that the bottom-up approach outperforms the top-down approach. Intuitively, compared with the latter one, the bottom-up approach grants a larger portion of high-waiting-cost customers a higher priority and thus reduces the system waiting cost.

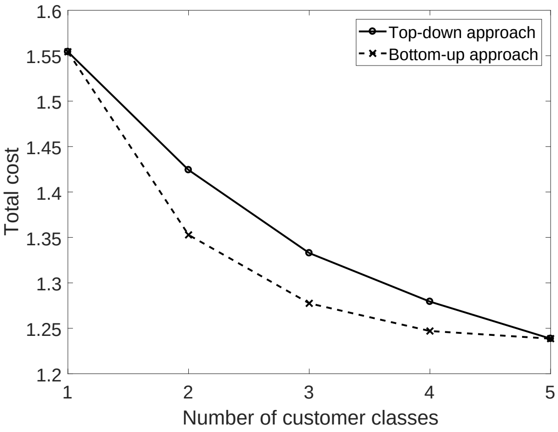


Figure OS-C3: Customer Segmentation

To gain further insights on the effect of customer segmentation, we compare the system total waiting cost under all possible segmentations with two; three; and four customer classes, and show the results in Table OS-C1. To read the table, the brackets represent how the five groups of customers are segmented. For example, (1, 4) is the case where Class 1 consists of Group-1 customers and Class 2 consists of the remaining four groups of customers (Groups 2-5). Similarly, (2, 1, 1, 1) is the case where Class 1 consists of Group-1 and Group-2 customers, Class 2 consists of Group-3 customers, Class 3 consists of Group-4 customers, and Class 4 consists of Group-5 customers. From the results, we see that (2, 3) outperforms (4, 1). Under segmentation (2, 3), the arrival rates of Class-1 and Class-2 customers equal 2λ and 3λ, respectively; while under segmentation (4, 1), the arrival rates of Class-1 and Class-2 customers equal 4λ and λ, respectively. Similarly, segmentation (1, 2, 2) outperforms segmentation (1, 1, 3). In general, we find that the system total waiting cost decreases as the customer segmentation becomes more balanced in terms of arrival rates.

Segmentation	Total Cost
(1, 4)	1.3528
(4, 1)	1.4243
(2, 3)	1.3232
(3, 2)	1.3423

(a) Two Classes

Segmentation	Total Cost
(1, 1, 3)	1.2775
(1, 3, 1)	1.2906
(3, 1, 1)	1.3328
(1, 2, 2)	1.2620
(2, 1, 2)	1.2888
(2, 2, 1)	1.2918

(b) Three Classes

Segmentation	Total Cost
(1, 1, 1, 2)	1.2470
(1, 1, 2, 1)	1.2497
(1, 2, 1, 1)	1.2536
(2, 1, 1, 1)	1.2794

(c) Four Classes

Table OS-C1: Comparison among Different Segmentation Approaches

Impact of System Utilization: As we discussed in the introduction section, a key result in the rich literature that studies parallel-server system using heavy-traffic analysis and Brownian approximations is that, the asymptotically optimal control is of a threshold-type under the so-called complete-resource-pooling condition. Intuitively speaking, the complete-resource-pooling condition requires the servers to be cross-trained so that the efforts of individual servers can be effectively combined to act as a single pool of resource. The model we consider in this paper can be represented as a complete graph, which clearly satisfies the complete-resource-pooling condition, when considered in the heavy-traffic limiting regime. Here, we investigate the pre-limit systems and examine how the system utilization influences the optimal server assignment thresholds.

In particular, we consider a system consisting of three homogeneous servers and two customer classes. For such a system, the threshold-type heuristic policy is optimal, and the optimal server assignment thresholds (K_0^*, K_1^*, K_2^*) can be obtained using the approach developed in Section 5. Now, in our numerical experiments, we fix $\mu_1 = \mu_2 = \mu_3 = 1$ and let $\lambda_1 = \lambda_2$. The system utilization $\rho = \frac{\lambda_1+\lambda_2}{\mu_1+\mu_2+\mu_3}$ is set as $\{0.8, 0.9, 0.95, 0.99\}$. Table OS-C2 summarizes how the optimal server assignment thresholds change with respect to ρ, under different values of the waiting-cost ratio $\frac{c_2}{c_1}$. We observe that for any fixed waiting-cost ratio, the optimal thresholds decrease as ρ increases, which suggests that the work-conserving flexible priority policy is asymptotically optimal in the heavy-traffic limiting regime. In addition, we find that as long as the waiting-cost ratio is large enough, the optimal threshold K_2^* (i.e., the threshold for serving a normal customer when there is only one idle server) will take a positive value (i.e., not being zero), regardless of how large ρ is.

	ρ	K_0^*	K_1^*	K_2^*
$\frac{c_2}{c_1} = 10$	0.8	0	0	1
	0.9	0	0	0
	0.95	0	0	0
	0.99	0	0	0
$\frac{c_2}{c_1} = 50$	0.8	0	0	5
	0.9	0	0	2
	0.95	0	0	1
	0.99	0	0	0
$\frac{c_2}{c_1} = 100$	0.8	0	0	8
	0.9	0	0	5
	0.95	0	0	3
	0.99	0	0	0
$\frac{c_2}{c_1} = 1000$	0.8	0	0	37
	0.9	0	0	16
	0.95	0	0	11
	0.99	0	0	5

Table OS-C2: System Utilization

D. Proof

Proof of Lemma 1: Let $v_0^*(l_1, l_2, n) = \frac{c_2}{\alpha}(l_1 + l_2 + n)$. We define w_t^* and v_t^* inductively as follows,

$$w_t^*(l_1, l_2, n) = c_1 l_1 + c_2 l_2 + \lambda_1 v_t^*(l_1 + 1, l_2, n) + \lambda_2 v_t^*(l_1, l_2 + 1, n)$$
$$+ n\mu v_t^*(l_1, l_2, n - 1) + (N - n)\mu v_t^*(l_1, l_2, n),$$

and

$$\begin{aligned} v_{t+1}^*(l_1, l_2, n) = \min_{l_1', l_2', n' \in \mathbb{N}} \quad & w_t^*(l_1', l_2', n') \\ \text{s.t.} \quad & l_1' + l_2' + n' = l_1 + l_2 + n, \\ & l_1' \leq l_1, \\ & l_2' \leq l_2, \\ & n \leq n' \leq N. \end{aligned} \tag{OS-D1}$$

Whenever the context is clear, we shall refer the above optimization problem as *problem* $v_{t+1}^*(l_1, l_2, n)$. Before we delve into the details, we first provide a high-level idea of the proof. We show that if w_t^* satisfies properties (P1)-(P8), then v_{t+1}^* and w_{t+1}^* also satisfy properties (P1)-(P8). It then immediately implies that $v^* = \lim_{t\to\infty} v_t^*$ and $w^* = \lim_{t\to\infty} w_t^*$ satisfy properties (P1)-(P8), which will complete the proof.

To facilitate the proof, we will first introduce three additional properties (P9), (P10), and (P11), which are related to the "time" dimension and will be useful for our induction analysis from time t to time $t + 1$. Properties (P1)-(P8) are also provided below for easy reference.

(P1) $0 \leq \Delta_{l_1} v^*(l_1, l_2, n) \leq \frac{c_2}{\alpha}$, $0 \leq \Delta_{l_2} v^*(l_1, l_2, n) \leq \frac{c_2}{\alpha}$, and $0 \leq \Delta_n v^*(l_1, l_2, n) \leq \frac{c_2}{\alpha}$.

(P2) $\Delta_{l_2} v^*(l_1, l_2, n) - \Delta_{l_1} v^*(l_1, l_2, n) \geq 0$.

(P3) $\Delta_{l_2} v^*(l_1, l_2, n) - \Delta_n v^*(l_1, l_2, n) \geq 0$.

(P4) Convexity:

- **(P4.1)** $\Delta_{l_1 l_1} v^*(l_1, l_2, n) \geq 0$; i.e., $v^*(l_1+2, l_2, n) - 2v^*(l_1+1, l_2, n) + v^*(l_1, l_2, n) \geq 0$.
- **(P4.2)** $\Delta_{nn} v^*(l_1, l_2, n) \geq 0$; i.e., $v^*(l_1, l_2, n+2) - 2v^*(l_1, l_2, n+1) + v^*(l_1, l_2, n) \geq 0$.
- **(P4.3)** $\Delta_{l_2 l_2} v^*(l_1, l_2, n) \geq 0$; i.e., $v^*(l_1, l_2+2, n) - 2v^*(l_1, l_2+1, n) + v^*(l_1, l_2, n) \geq 0$.

(P5) Supermodularity in $(l_1, n), (l_2, n)$ and (l_1, l_2):

- **(P5.1)** $\Delta_{l_1 n} v^*(l_1, l_2, n) \geq 0$; i.e., $v^*(l_1+1, l_2, n+1) - v^*(l_1, l_2, n+1) - v^*(l_1+1, l_2, n) + v^*(l_1, l_2, n) \geq 0$.
- **(P5.2)** $\Delta_{l_2 n} v^*(l_1, l_2, n) \geq 0$; i.e., $v^*(l_1, l_2+1, n+1) - v^*(l_1, l_2, n+1) - v^*(l_1, l_2+1, n) + v^*(l_1, l_2, n) \geq 0$.
- **(P5.3)** $\Delta_{l_1 l_2} v^*(l_1, l_2, n) \geq 0$; i.e., $v^*(l_1+1, l_2+1, n) - v^*(l_1, l_2+1, n) - v^*(l_1+1, l_2, n) + v^*(l_1, l_2, n) \geq 0$.

(P6)

- **(P6.1)** $\Delta_{l_1 l_1} v^*(l_1, l_2, n) - \Delta_{l_1 n} v^*(l_1, l_2, n) \geq 0$.
- **(P6.2)** $\Delta_{l_1 l_1} v^*(l_1, l_2, n) - \Delta_{l_1 l_2} v^*(l_1, l_2, n) \geq 0$.

(P7)

- **(P7.1)** $v^*(l_1+2, l_2, n) - 2v^*(l_1+1, l_2, n+1) + v^*(l_1, l_2, n+2) \geq 0$.
- **(P7.2)** $v^*(l_1+2, l_2, n) - 2v^*(l_1+1, l_2, n+1) + v^*(l_1, l_2+1, n+1) \geq 0$.
- **(P7.3)** $v^*(l_1+2, l_2, n) - 2v^*(l_1+1, l_2+1, n) + v^*(l_1, l_2+2, n) \geq 0$.

(P8) $(\Delta_{l_1} - \Delta_n) v^*(l_1, l_2, n)$ is single-crossing in $(-n)$. That is, if $v^*(l_1+1, 0, n+1) \geq v^*(l_1, 0, n+2)$, then we must have $v^*(l_1+1, 0, n) \geq v^*(l_1, 0, n+1)$.

(P9) Submodularity in $(l_1, t), (l_2, t), (n, t)$:

- **(P9.1)** $\Delta_{l_1 t} v^*_t(l_1, l_2, n) \leq 0$.
- **(P9.2)** $\Delta_{nt} v^*_t(l_1, l_2, n) \leq 0$.
- **(P9.3)** $\Delta_{l_2 t} v^*_t(l_1, l_2, n) \leq 0$.

(P10) $(\Delta_{l_1 t} - \Delta_{nt}) v^*_t(l_1, l_2, n) \leq 0$ and $(\Delta_{l_1 t} - \Delta_{l_2 t}) v^*_t(l_1, l_2, n) \leq 0$. That is,

- **(P10.1)** $v^*_{t+1}(l_1+1, l_2, n) - v^*_{t+1}(l_1, l_2, n+1) - v^*_t(l_1+1, l_2, n) + v^*_t(l_1, l_2, n+1) \leq 0$.
- **(P10.2)** $v^*_{t+1}(l_1+1, l_2, n) - v^*_{t+1}(l_1, l_2+1, n) - v^*_t(l_1+1, l_2, n) + v^*_t(l_1, l_2+1, n) \leq 0$.

(P11) $(\Delta_{l_1} - \Delta_n) v^*_t(l_1, 0, n)$ is single-crossing in $(-n, t)$. That is, if $v^*_t(l_1+1, 0, n+1) \geq v^*_t(l_1, 0, n+2)$, then $v^*_{t+1}(l_1+1, 0, n) \geq v^*_{t+1}(l_1, 0, n+1)$.

Our proof proceeds in the following steps. First, we show two auxiliary results (Lemmas OS-D1 and OS-D2), which provide some structural characterizations of the optimal solutions to the problem v^*_{t+1}, when w^*_t satisfies properties (P1)-(P11). Second, we show that the terminal value function $v^*_0(l_1, l_2, n) = \frac{c_2}{\alpha}(l_1+l_2+n)$ satisfies properties (P1)-(P11). Finally, we show the induction step. More specifically, we show that (i) if v^*_t satisfies properties (P1)-(P11), then w^*_t also satisfies properties (P1)-(P11); (ii) since w^*_t satisfies properties (P1)-(P11), Lemmas OS-D1 and OS-D2 hold at time t; (iii) the properties of w^*_t together with Lemmas OS-D1 and OS-D2 imply that v^*_{t+1} satisfies properties (P1)-(P11), which then implies that w^*_{t+1} also satisfies properties (P1)-(P11) (according to (i)) and hence completes the induction step. Here we remark that, for property (P11), the induction from time t to time $t+1$ is direct and does not need to go through v_{t+1}^*.

Lemma OS-D1. *Let* $(l_{1,t}^*(l_1,l_2,n), l_{2,t}^*(l_1,l_2,n), n_t^*(l_1,l_2,n))$ *be the optimal solution to problem* $v_t^*(l_1,l_2,n)$ *for any given* t *and* (l_1,l_2,n)*. If* w_t^* *and* w_{t-1}^* *satisfy property (P1)-(P11), then we have*

(a) $l_{1,t+1}^*(l_1,l_2,n) \le l_{1,t}^*(l_1+1,l_2,n) \le l_{1,t+1}^*(l_1,l_2,n)+1$*;*

(b) $l_{1,t+1}^*(l_1,l_2,n) \le l_{1,t}^*(l_1,l_2,n+1) \le l_{1,t+1}^*(l_1,l_2,n)+1$.

Lemma OS-D2. *Let* $(l_1^*(l_1,l_2,n), l_2^*(l_1,l_2,n), n^*(l_1,l_2,n))$ *be the optimal solution to problem* $v_{t+1}^*(l_1,l_2,n)$ *for any given* (l_1,l_2,n)*. If* w_t^* *satisfies property (P1)-(P11), then we have*

(a) $l_1^*(l_1,l_2,n) \le l_1^*(l_1+1,l_2,n) \le l_1^*(l_1,l_2,n)+1$*;*

(b) $l_1^*(l_1,l_2,n) \le l_1^*(l_1,l_2,n+1) \le l_1^*(l_1,l_2,n)+1$.

Next, we show that the terminal value function $v_0^*(l_1,l_2,n) = \frac{c_2}{\alpha}(l_1+l_2+n)$ satisfies properties (P1)-(P11). It is straightforward to see that $v_0^*(l_1,l_2,n)$ satisfies properties (P1)-(P8). Then by induction, w_0^* and v_1^* satisfy (P1)-(P8). We next show properties (P9)-(P11).

(P9)

$$\Delta_{l_1 t} v_0^*(l_1,l_2,n) = \Delta_{l_1} v_1^*(l_1,l_2,n) - \Delta_{l_1} v_0^*(l_1,l_2,n) = \Delta_{l_1} v_1^*(l_1,l_2,n) - \frac{c_2}{\alpha} \le 0;$$

$$\Delta_{nt} v_0^*(l_1,l_2,n) = \Delta_{n} v_1^*(l_1,l_2,n) - \Delta_{n} v_0^*(l_1,l_2,n) = \Delta_{n} v_1^*(l_1,l_2,n) - \frac{c_2}{\alpha} \le 0;$$

$$\Delta_{l_2 t} v_0^*(l_1,l_2,n) = \Delta_{l_2} v_1^*(l_1,l_2,n) - \Delta_{l_2} v_0^*(l_1,l_2,n) = \Delta_{l_2} v_1^*(l_1,l_2,n) - \frac{c_2}{\alpha} \le 0,$$

where the inequalities are due to (P1) for v_1^*.

(P10)

$$\begin{aligned}(\Delta_{l_1 t} - \Delta_{nt}) v_0^*(l_1,l_2,n) &= (\Delta_{l_1} - \Delta_n) v_1^*(l_1,l_2,n) - (\Delta_{l_1} - \Delta_n) v_0^*(l_1,l_2,n)\\ &= (\Delta_{l_1} - \Delta_n) v_1^*(l_1,l_2,n) = v_1^*(l_1+1,l_2,n) - v_1^*(l_1,l_2,n+1) \le 0;\\ (\Delta_{l_1 t} - \Delta_{l_2 t}) v_0^*(l_1,l_2,n) &= (\Delta_{l_1} - \Delta_{l_2}) v_1^*(l_1,l_2,n) - (\Delta_{l_1} - \Delta_{l_2}) v_0^*(l_1,l_2,n)\\ &= (\Delta_{l_1} - \Delta_{l_2}) v_1^*(l_1,l_2,n) \le 0,\end{aligned}$$

where $(\Delta_{l_1} - \Delta_{l_2}) v_1^*(l_1,l_2,n) \le 0$ is due to (P2) for v_1^*, and $v_1^*(l_1+1,l_2,n) \le v_1^*(l_1,l_2,n+1)$ holds because any feasible solution to $v_1^*(l_1,l_2,n+1)$ is also feasible to $v_1^*(l_1+1,l_2,n)$.

(P11) It's straightforward to see that it is optimal to assign as many customers as possible for problem $v_1^*(l_1,0,n)$. Therefore, we always have $v_1^*(l_1+1,0,n) \ge v_1^*(l_1,0,n+1)$. Hence, the single-crossing property in (P11) holds. The terminal value function, thus, satisfies all the properties.

Finally, we show the induction step from time t to $t+1$ for all properties (P1)-(P11).

(P1)

(P1.1) $0 \le \Delta_{l_1} v_{t+1}^*(l_1,l_2,n) \le \frac{c_2}{\alpha}$.

$\underline{w_t^* \Rightarrow v_{t+1}^*}$: We first show $\Delta_{l_1} v_{t+1}^*(l_1,l_2,n) \ge 0$. Let $(l_1^*(l_1+1,l_2,n), l_2^*(l_1+1,l_2,n), n^*(l_1+1,l_2,n))$ be the optimal solution to problem $v_{t+1}^*(l_1+1,l_2,n)$ (c.f., optimization problem (OS-D1)). For notational simplicity, denote $(\hat{l}_1,\hat{l}_2,\hat{n}) := (l_1^*(l_1+1,l_2,n), l_2^*(l_1+1,l_2,n), n^*(l_1+1,l_2,n))$.

Case (1). If $\hat{l}_1 \geq 1$, then it is clear that $(\hat{l}_1 - 1, \hat{l}_2, \hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n)$. It then follows that

$$\begin{aligned}\Delta_{l_1} v^*_{t+1}(l_1, l_2, n) &= v^*_{t+1}(l_1 + 1, l_2, n) - v^*_{t+1}(l_1, l_2, n) \\ &\geq w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) - w^*_t(\hat{l}_1 - 1, \hat{l}_2, \hat{n}) = \Delta_{l_1} w^*_t(\hat{l}_1 - 1, \hat{l}_2, \hat{n}) \geq 0,\end{aligned}$$

where the first inequality holds by the definition of $(\hat{l}_1, \hat{l}_2, \hat{n})$ and the feasibility of $(\hat{l}_1 - 1, \hat{l}_2, \hat{n})$ to problem $v^*_{t+1}(l_1, l_2, n)$, and the last inequality follows from the assumption that w^*_t satisfies property (P1).

Case (2). If $\hat{l}_1 = 0$, then it is clear that $(0, \hat{l}_2, \hat{n}-1)$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n)$. Therefore, we have $\Delta_{l_1} v^*_{t+1}(l_1, l_2, n) \geq \Delta_n w^*_t(0, \hat{l}_2, \hat{n} - 1) \geq 0$.

Next, we show $\Delta_{l_1} v^*_{t+1}(l_1, l_2, n) \leq \frac{c_2}{\alpha}$. Let $(\hat{l}_1, \hat{l}_2, \hat{n})$ be the optimal solution to problem $v^*_{t+1}(l_1, l_2, n)$. Then, we have $\Delta_{l_1} v^*_{t+1}(l_1, l_2, n) = v^*_{t+1}(l_1 + 1, l_2, n) - v^*_{t+1}(l_1, l_2, n) \leq w^*_t(\hat{l}_1 + 1, \hat{l}_2, \hat{n}) - w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) = \Delta_{l_1} w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) \leq \frac{c_2}{\alpha}$, where the first inequality follows from the definition of $(\hat{l}_1, \hat{l}_2, \hat{n})$ and the fact that $(\hat{l}_1 + 1, \hat{l}_2, \hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1 + 1, l_2, n)$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, $\Delta_{l_1} w^*_{t+1}(l_1, l_2, n) = c_1 + \lambda_1 \Delta_{l_1} v^*_{t+1}(l_1 + 1, l_2, n) + \lambda_2 \Delta_{l_1} v^*_{t+1}(l_1, l_2 + 1, n) + n\mu \Delta_{l_1} v^*_{t+1}(l_1, l_2, n - 1) + (N - n)\mu \Delta_{l_1} v^*_{t+1}(l_1, l_2, n) \geq 0$. Similarly, $\Delta_{l_1} w^*_{t+1}(l_1, l_2, n) = c_1 + \lambda_1 \Delta_{l_1} v^*_{t+1}(l_1 + 1, l_2, n) + \lambda_2 \Delta_{l_1} v^*_{t+1}(l_1, l_2 + 1, n) + n\mu \Delta_{l_1} v^*_{t+1}(l_1, l_2, n-1) + (N-n)\mu \Delta_{l_1} v^*_{t+1}(l_1, l_2, n) \leq c_2 + (\lambda_1 + \lambda_2 + N\mu)\frac{c_2}{\alpha} = c_2 + (1 - \alpha)\frac{c_2}{\alpha} = \frac{c_2}{\alpha}$.

(P1.2) $0 \leq \Delta_{l_2} v^*_{t+1}(l_1, l_2, n) \leq \frac{c_2}{\alpha}$.

$\underline{w^*_t \Rightarrow v^*_{t+1}}$: We first show $\Delta_{l_2} v^*_{t+1}(l_1, l_2, n) \geq 0$. Let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l^*_1(l_1, l_2 + 1, n), l^*_2(l_1, l_2 + 1, n), n^*(l_1, l_2 + 1, n))$ be the optimal solution to problem $v^*_{t+1}(l_1, l_2 + 1, n)$.

Case (1). If $\hat{l}_2 \geq 1$, then $(\hat{l}_1, \hat{l}_2 - 1, \hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n)$, which implies that $\Delta_{l_2} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{l_2} w^*_t(\hat{l}_1, \hat{l}_2 - 1, \hat{n}) \geq 0$.

Case (2). If $\hat{l}_2 = 0$, then $(\hat{l}_1, 0, \hat{n} - 1)$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n)$. Therefore, we have $\Delta_{l_2} v^*_{t+1}(l_1, l_2, n) \geq \Delta_n w^*_t(\hat{l}_1, \hat{l}_2, \hat{n} - 1) \geq 0$.

The proof of $\Delta_{l_2} v^*_{t+1}(l_1, l_2, n) \leq \frac{c_2}{\alpha}$ is similar to (P1.1), and is omitted for brevity.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: The proof is similar to (P1.1), and is omitted for brevity.

(P1.3) $0 \leq \Delta_n v^*_{t+1}(l_1, l_2, n) \leq \frac{c_2}{\alpha}$.

$\underline{w^*_t \Rightarrow v^*_{t+1}}$: We first show $\Delta_n v^*_{t+1}(l_1, l_2, n) \geq 0$. Let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l^*_1(l_1, l_2, n+1), l^*_2(l_1, l_2, n+1), n^*(l_1, l_2, n+1))$ be the optimal solution to problem $v^*_{t+1}(l_1, l_2, n + 1)$. Notice that by definition, we have $\hat{l}_1 \geq l_1, \hat{l}_2 \geq l_2, n + 1 \leq \hat{n} \leq N$, which implies $n \leq \hat{n} - 1 < N$. Therefore, $(\hat{l}_1, \hat{l}_2, \hat{n} - 1)$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n)$. It then follows that $\Delta_n v^*_{t+1}(l_1, l_2, n) \geq \Delta_n w^*_t(\hat{l}_1, \hat{l}_2, \hat{n} - 1) \geq 0$.

We next show $\Delta_n v^*_{t+1}(l_1, l_2, n) \leq \frac{c_2}{\alpha}$. Let $(\hat{l}_1, \hat{l}_2, \hat{n})$ be the optimal solution to problem $v^*_{t+1}(l_1, l_2, n)$.

Case (1). When $\hat{n} < N$, $(\hat{l}_1, \hat{l}_2, \hat{n} + 1)$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n + 1)$. Therefore, we have $\Delta_n v^*_{t+1}(l_1, l_2, n) \leq w^*_t(\hat{l}_1, \hat{l}_2, \hat{n} + 1) - w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) = \Delta_n w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) \leq \frac{c_2}{\alpha}$.

Case (2). When $\hat{n} = N$, since $n < N$, we must have either $\hat{l}_1 < l_1$ or $\hat{l}_2 < l_2$. Therefore, either $(\hat{l}_1 + 1, \hat{l}_2, \hat{n})$ or $(\hat{l}_1, \hat{l}_2 + 1, \hat{n})$ is feasible to problem $v^*_{t+1}(l_1, l_2, n + 1)$. It then follows that $\Delta_n v^*_{t+1}(l_1, l_2, n) \leq$

$w^*_t(\hat{l}_1+1,\hat{l}_2,\hat{n}) - w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) = \Delta_{l_1} w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) \le \frac{c_2}{\alpha}$, or $\Delta_n v^*_{t+1}(l_1,l_2,n) \le w^*_t(\hat{l}_1,\hat{l}_2+1,\hat{n}) - w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) = \Delta_{l_2} w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) \le \frac{c_2}{\alpha}$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $\Delta_n w^*_{t+1}(l_1,l_2,n) = \lambda_1 \Delta_n v^*_{t+1}(l_1+1,l_2,n) + \lambda_2 \Delta_n v^*_{t+1}(l_1,l_2+1,n) + n\mu \Delta_n v^*_{t+1}(l_1,l_2,n-1) + (N-n-1)\mu \Delta_n v^*_{t+1}(l_1,l_2,n) \ge 0$. Similarly, we have $\Delta_n w^*_{t+1}(l_1,l_2,n) \le (\lambda_1+\lambda_2+(N-1)\mu)\frac{c_2}{\alpha} < \frac{c_2}{\alpha}$.

(P2)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$: Note that $\Delta_{l_2} v^*_{t+1}(l_1,l_2,n) - \Delta_{l_1} v^*_{t+1}(l_1,l_2,n) \ge 0$ is equivalent to $v^*_{t+1}(l_1,l_2+1,n) - v^*_{t+1}(l_1+1,l_2,n) \ge 0$. Let $(\hat{l}_1,\hat{l}_2,\hat{n}) := (l_1^*(l_1,l_2+1,n), l_2^*(l_1,l_2+1,n), n^*(l_1,l_2+1,n))$ be the optimal solution to problem $v^*_{t+1}(l_1,l_2+1,n)$. Notice that by the feasibility constraint, we have $\hat{l}_2 \le l_2+1$.

Case (1). If $\hat{l}_2 \le l_2$, then $(\hat{l}_1,\hat{l}_2,\hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1+1,l_2,n)$, which implies that $v^*_{t+1}(l_1,l_2+1,n) - v^*_{t+1}(l_1+1,l_2,n) \ge w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) - w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) = 0$.

Case (2). If $\hat{l}_2 = l_2+1$, then $(\hat{l}_1+1,\hat{l}_2-1,\hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1+1,l_2,n)$, which implies that $v^*_{t+1}(l_1,l_2+1,n) - v^*_{t+1}(l_1+1,l_2,n) \ge w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) - w^*_t(\hat{l}_1+1,\hat{l}_2-1,\hat{n}) \ge 0$, where the last inequality holds by the assumption that w^*_t satisfies property (P2).

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $\Delta_{l_1} w^*_{t+1}(l_1,l_2,n) - \Delta_{l_2} w^*_{t+1}(l_1,l_2,n) = c_1 - c_2 + \lambda_1(\Delta_{l_1} v^*_{t+1}(l_1+1,l_2,n) - \Delta_{l_2} v^*_{t+1}(l_1+1,l_2,n)) + \lambda_2(\Delta_{l_1} v^*_{t+1}(l_1,l_2+1,n) - \Delta_{l_2} v^*_{t+1}(l_1,l_2+1,n)) + n\mu(\Delta_{l_1} v^*_{t+1}(l_1,l_2,n-1) - \Delta_{l_2} v^*_{t+1}(l_1,l_2,n-1)) + (N-n)\mu(\Delta_{l_1} v^*_{t+1}(l_1,l_2,n) - \Delta_{l_2} v^*_{t+1}(l_1,l_2,n)) \le 0$.

(P3)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$: Note that $\Delta_{l_2} v^*_{t+1}(l_1,l_2,n) - \Delta_n v^*_{t+1}(l_1,l_2,n) \ge 0$ is equivalent to $v^*_{t+1}(l_1,l_2+1,n) - v^*_{t+1}(l_1,l_2,n+1) \ge 0$, which implies that it is always optimal to serve a VIP customer if there is an available server. Let $(\hat{l}_1,\hat{l}_2,\hat{n}) := (l_1^*(l_1,l_2+1,n), l_2^*(l_1,l_2+1,n), n^*(l_1,l_2+1,n))$ be the optimal solution to problem $v^*_{t+1}(l_1,l_2+1,n)$. Notice that by the feasibility constraint, we have $\hat{l}_2 \le l_2+1$.

Case (1). If $\hat{l}_2 \le l_2$, then we must have $\hat{n} \ge n+1$, and hence $(\hat{l}_1,\hat{l}_2,\hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1,l_2,n+1)$. It then follows that $v^*_{t+1}(l_1,l_2+1,n) - v^*_{t+1}(l_1,l_2,n+1) \ge w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) - w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) = 0$.

Case (2). If $\hat{l}_2 = l_2+1$ and $\hat{n} \le N-1$, then in this case, $(\hat{l}_1,\hat{l}_2-1,\hat{n}+1)$ is a feasible solution to problem $v^*_{t+1}(l_1,l_2,n+1)$. Therefore, we have $v^*_{t+1}(l_1,l_2+1,n) - v^*_{t+1}(l_1,l_2,n+1) \ge w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) - w^*_t(\hat{l}_1,\hat{l}_2-1,\hat{n}+1) \ge 0$, where the last inequality holds by the assumption that w^*_t satisfies property (P3).

Case (3). If $\hat{l}_2 = l_2+1$ and $\hat{n} = N$, then since $n < N$, we must have $\hat{l}_1 \le l_1 - 1$. Then $(\hat{l}_1+1,\hat{l}_2-1,\hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1,l_2,n+1)$. Therefore, we have $v^*_{t+1}(l_1,l_2+1,n) - v^*_{t+1}(l_1,l_2,n+1) \ge w^*_t(\hat{l}_1,\hat{l}_2,\hat{n}) - w^*_t(\hat{l}_1+1,\hat{l}_2-1,\hat{n}) \ge 0$, where the last inequality holds by the assumption that w^*_t satisfies property (P2).

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $\Delta_{l_2} w^*_{t+1}(l_1,l_2,n) - \Delta_n w^*_{t+1}(l_1,l_2,n) = w^*_{t+1}(l_1,l_2+1,n) - w^*_{t+1}(l_1,l_2,n+1) = c_2 + \lambda_1(\Delta_{l_2} v^*_{t+1}(l_1+1,l_2,n) - \Delta_n v^*_{t+1}(l_1+1,l_2,n)) + \lambda_2(\Delta_y v^*_{t+1}(l_1,l_2+1,n) - \Delta_n v^*_{t+1}(l_1,l_2+1,n)) + n\mu(\Delta_{l_2} v^*_{t+1}(l_1,l_2,n-1) - \Delta_n v^*_{t+1}(l_1,l_2,n-1)) + (N-n)\mu(\Delta_{l_2} v^*_{t+1}(l_1,l_2,n) - \Delta_n v^*_{t+1}(l_1,l_2,n)) + \mu \Delta_n v^*_{t+1}(l_1,l_2,n) \ge 0$.

(P4) Convexity:

(P4.1)

$\underline{w_t^* \Rightarrow v_{t+1}^*}$: Let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l_1^*(l_1, l_2, n), l_2^*(l_1, l_2, n), n^*(l_1, l_2, n))$ be the optimal solution to problem $v_{t+1}^*(l_1, l_2, n)$. In addition, let $(l_1^*(l_1+2, l_2, n), l_2^*(l_1+2, l_2, n), n^*(l_1+2, l_2, n))$ be the optimal solution to problem $v_{t+1}^*(l_1+2, l_2, n)$. By Lemma OS-D2, we have $\hat{l}_1 \leq l_1^*(l_1+2, l_2, n) \leq \hat{l}_1 + 2$.

Case (1). If $l_1^*(l_1+2, l_2, n) = \hat{l}_1 + 2$, then we must have $n^*(l_1+2, l_2, n) = \hat{n}$. It then follows that $\Delta_{l_1 l_1} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{l_1 l_1} w_t^*(\hat{l}_1, \hat{l}_2, \hat{n}) \geq 0$, where the last inequality follows from the assumption that w_t^* satisfies property (P4.1).

Case (2). If $l_1^*(l_1+2, l_2, n) = \hat{l}_1 + 1$, then we must have $n^*(l_1+2, l_2, n) = \hat{n}+1$. It then follows that $\Delta_{l_1 l_1} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{l_1 n} w_t^*(\hat{l}_1, \hat{l}_2, \hat{n}) \geq 0$, where the last inequality follows from the assumption that w_t^* satisfies property (P5.2).

Case (3). If $l_1^*(l_1+2, l_2, n) = \hat{l}_1$, then we must have $n^*(l_1+2, l_2, n) = \hat{n}+2$. It then follows that $\Delta_{l_1 l_1} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{nn} w_t^*(\hat{l}_1, \hat{l}_2, \hat{n}) \geq 0$, where the last inequality follows from the assumption that w_t^* satisfies property (P4.3).

$\underline{v_{t+1}^* \Rightarrow w_{t+1}^*}$: By definition, we have $\Delta_{l_1 l_1} w_{t+1}^*(l_1, l_2, n) = \lambda_1 \Delta_{l_1 l_1} v_{t+1}^*(l_1+1, l_2, n) + \lambda_2 \Delta_{l_1 l_1} v_{t+1}^*(l_1, l_2+1, n) + n\mu \Delta_{l_1 l_1} v_{t+1}^*(l_1, l_2, n-1) + (N-n)\mu \Delta_{l_1 l_1} v_{t+1}^*(l_1, l_2, n) \geq 0$

(P4.2)

$\underline{w_t^* \Rightarrow v_{t+1}^*}$: Let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l_1^*(l_1, l_2, n), l_2^*(l_1, l_2, n), n^*(l_1, l_2, n))$ be the optimal solution to problem $v_{t+1}^*(l_1, l_2, n)$. In addition, let $(l_1^*(l_1, l_2, n+2), l_2^*(l_1, l_2, n+2), n^*(l_1, l_2, n+2))$ be the optimal solution to problem $v_{t+1}^*(l_1+2, l_2, n)$. By Lemma OS-D2, we have $\hat{l}_1 \leq l_1^*(l_1, l_2, n+2) \leq \hat{l}_1 + 2$.

Case (1). If $l_2 + n \geq N$, then $\Delta_{nn} v_{t+1}^*(l_1, l_2, n) = \Delta_{l_2 l_2} w_t^*(l_1, l_2 - N + n, N) \geq 0$.

Case (2). If $l_2 + n = N - 1$, then we have $\Delta_{nn} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{l_2 n} w_t^*(l_1, 0, N-1) \geq 0$ if $l_1^*(l_1, l_2, n) = l_1$, and $\Delta_{nn} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{l_1 l_2} w_t^*(l_1 - 1, 0, N) \geq 0$ if $l_1^*(l_1, l_2, n) = l_1 - 1$.

Case (3). If $l_2 + n \leq N - 2$, then in this case, we must have $\hat{l}_2 = 0$. We further consider three subcases.

Case (3.1). If $l_1^*(l_1, l_2, n+2) = \hat{l}_1 + 2$, then we must have $n^*(l_1, l_2, n+2) = \hat{n} \geq n+2 > n+1$. It then follows that $\Delta_{nn} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{l_1 l_1} w_t^*(l_1, l_2, n) \geq 0$.

Case (3.2). If $l_1^*(l_1, l_2, n+2) = \hat{l}_1 + 1$, then we must have $n^*(l_1, l_2, n+2) = \hat{n} + 1 \geq n+2$, which implies $\hat{n} \geq n+1$. It then follows that $\Delta_{nn} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{l_1 n} w_t^*(l_1, l_2, n) \geq 0$.

Case (3.3). If $l_1^*(l_1, l_2, n+2) = \hat{l}_1$, then we must have $n^*(l_1, l_2, n+2) = \hat{n} + 2$. It then follows that $\Delta_{nn} v_{t+1}^*(l_1, l_2, n) \geq \Delta_{nn} w_t^*(l_1, l_2, n) \geq 0$.

$\underline{v_{t+1}^* \Rightarrow w_{t+1}^*}$: By definition, we have $\Delta_{nn} w_{t+1}^*(l_1, l_2, n) = \lambda_1 \Delta_{nn} v_{t+1}^*(l_1+1, l_2, n) + \lambda_2 \Delta_{nn} v_{t+1}^*(l_1, l_2+1, n) + n\mu \Delta_{nn} v_{t+1}^*(l_1, l_2, n-1) + (N-n-2)\mu \Delta_{nn} v_{t+1}^*(l_1, l_2, n) \geq 0$.

(P4.3)

$\underline{w_t^* \Rightarrow v_{t+1}^*}$: Let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l_1^*(l_1, l_2, n), l_2^*(l_1, l_2, n), n^*(l_1, l_2, n))$ be the optimal solution to problem $v_{t+1}^*(l_1, l_2, n)$.

Case (1). If $l_2 + n \geq N$, then we have $\Delta_{l_2 l_2} v^*_{t+1}(l_1, l_2, n) = \Delta_{l_2 l_2} w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) \geq 0$.

Case (2). If $l_2 + n = N - 1$, then we have $\Delta_{l_2 l_2} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{l_2 n} w^*_t(l_1, 0, N-1) \geq 0$ when $\hat{l}_1 = l_1$, and $\Delta_{l_2 l_2} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{l_1 l_2} w^*_t(l_1 - 1, 0, N) \geq 0$ when $\hat{l}_1 = l_1 - 1$.

Case (3). If $l_2 + n \leq N - 2$, then we have $\Delta_{l_2 l_2} v^*_{t+1}(l_1, l_2, n) = \Delta_{nn} v^*_{t+1}(l_1, 0, l_2 + n) \geq 0$, where the last inequality holds because v^*_{t+1} has been shown to satisfy property (P4.2).

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: The proof is similar to (P4.1) and omitted for brevity.

(P5) Supermodularity:

(P5.1)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$:

Case (1). If $l_2 + n \geq N$, then we have $\Delta_{l_1 n} v^*_{t+1}(l_1, l_2, n) = \Delta_{l_1 l_2} w^*_t(l_1, l_2 - N + n, N) \geq 0$.

Case (2). If $l_2 + n < N$, let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l^*_1(l_1, l_2, n), l^*_2(l_1, l_2, n), n^*(l_1, l_2, n))$ be the optimal solution to problem $v^*_{t+1}(l_1, l_2, n)$. Notice that in this case, $\hat{l}_2 = 0$. We further consider the following three subcases.

Case (2.1). If $l^*_1(l_1 + 1, l_2, n + 1) = \hat{l}_1 + 2$, then we have $n^*(l_1 + 1, l_2, n + 1) = \hat{n} \geq n + 1$. Therefore, $(\hat{l}_1 + 1, \hat{l}_2, \hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n + 1)$ and problem $v^*_{t+1}(l_1 + 1, l_2, n)$. Hence, $\Delta_{l_1 n} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{l_1 l_1} w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) \geq 0$.

Case (2.2). If $l^*_1(l_1 + 1, l_2, n + 1) = \hat{l}_1 + 1$, then we have $n^*(l_1 + 1, l_2, n + 1) = \hat{n} + 1$. Therefore, $(\hat{l}_1 + 1, \hat{l}_2, \hat{n})$ is a feasible solution to problem $v^*_{t+1}(l_1 + 1, l_2, n)$, and $(\hat{l}_1, \hat{l}_2, \hat{n} + 1)$ is a feasible solution to problem $v^*_{t+1}(l_1, l_2, n + 1)$. Hence, $\Delta_{l_1 n} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{l_1 n} w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) \geq 0$.

Case (2.3). If $l^*_1(l_1 + 1, l_2, n + 1) = \hat{l}_1$, then we have $n^*(l_1 + 1, l_2, n + 1) = \hat{n} + 2$. Therefore, $(\hat{l}_1, \hat{l}_2, \hat{n} + 1)$ is a feasible solution to problem $v^*_{t+1}(l_1 + 1, l_2, n)$ and problem $v^*_{t+1}(l_1, l_2, n + 1)$. Hence, $\Delta_{l_1 n} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{nn} w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) \geq 0$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $\Delta_{l_1 n} w^*_{t+1}(l_1, l_2, n) = \lambda_1 \Delta_{l_1 n} v^*_{t+1}(l_1 + 1, l_2, n) + \lambda_2 \Delta_{l_1 n} v^*_{t+1}(l_1, l_2 + 1, n) + n\mu \Delta_{l_1 n} v^*_{t+1}(l_1, l_2, n - 1) + (N - n)\mu \Delta_{l_1 n} v^*_{t+1}(l_1, l_2, n) + \mu \Delta_n v^*_{t+1}(l_1, l_2, n) \geq 0$.

(P5.2)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$:

Case (1). If $l_2 + n \geq N$, then we have $\Delta_{l_2 n} v^*_{t+1}(l_1, l_2, n) = \Delta_{l_2 l_2} w^*_t(l_1, l_2 - N + n, N) \geq 0$.

Case (2). If $l_2 + n = N - 1$, then we have $\Delta_{l_2 n} v^*_{t+1}(l_1, l_2, n) = v^*_{t+1}(l_1, 1, N) - v^*_{t+1}(l_1, 0, N) - v^*_{t+1}(l_1, 0, N) + v^*_{t+1}(l_1, 0, N - 1)$. When $l^*_1(l_1, 0, N - 1) = l_1$, we have $\Delta_{l_2 n} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{l_2 n} w^*_t(l_1, 0, N - 1) \geq 0$. When $l^*_1(l_1, 0, N - 1) = l_1 - 1$, we have $\Delta_{l_2 n} v^*_{t+1}(l_1, l_2, n) \geq \Delta_{l_1 l_2} w^*_t(l_1 - 1, 0, N) \geq 0$.

Case (3). If $l_2 + n \leq N - 2$, then we have $l^*_2(l_1, l_2 + 1, n + 1) = 0$. It then follows that $\Delta_{l_2 n} v^*_{t+1}(l_1, l_2, n) = \Delta_{nn} v^*_{t+1}(l_1, 0, l_2 + n) \geq 0$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $\Delta_{l_2 n} w^*_{t+1}(l_1, l_2, n) = \lambda_1 \Delta_{l_2 n} v^*_{t+1}(l_1 + 1, l_2, n) + \lambda_2 \Delta_{l_2 n} v^*_{t+1}(l_1, l_2 + 1, n) + n\mu \Delta_{l_2 n} v^*_{t+1}(l_1, l_2, n - 1) + (N - n)\mu \Delta_{l_2 n} v^*_{t+1}(l_1, l_2, n) + \mu \Delta_n v^*_{t+1}(l_1, l_2, n) \geq 0$.

(P5.3)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$:

Case (1). If $l_2+n \geq N$, then we have $\Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2,n) = \Delta_{l_1 l_2} w_t^*(l_1,l_2-N+n,N) \geq 0$.

Case (2). If $l_2+n \leq N-1$, then we have $\Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2,n) = \Delta_{l_1 n} v_{t+1}^*(l_1,0,n+l_2) \geq 0$.

$\underline{v_{t+1}^* \Rightarrow w_{t+1}^*}$: By definition, we have $\Delta_{l_1 l_2} w_{t+1}^*(l_1,l_2,n) = \lambda_1 \Delta_{l_1 l_2} v_{t+1}^*(l_1+1,l_2,n) + \lambda_2 \Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2+1,n) + n\mu \Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2,n-1) + (N-n)\mu \Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2,n) \geq 0$.

(P6)

(P6.1)

$\underline{w_t^* \Rightarrow v_{t+1}^*}$: Note that $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n) \geq 0$ is equivalent to $v_{t+1}^*(l_1+2,l_2,n) - v_{t+1}^*(l_1+1,l_2,n+1) - v_{t+1}^*(l_1+1,l_2,n) + v_{t+1}^*(l_1,l_2,n+1) \geq 0$.

Case (1). If $l_2+n \geq N$, then we have $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n) = w_t^*(l_1+2,l_2-N+n,N) - w_t^*(l_1+1,l_2-N+n+1,N) - w_t^*(l_1+1,l_2-N+n,N) + w_t^*(l_1,l_2-N+n+1,N) = (\Delta_{l_1 l_1} - \Delta_{l_1 l_2}) w_t^*(l_1,l_2-N+n,N) \geq 0$.

Case (2). If $l_2+n < N$, we further consider the following two subcases.

Case (2.1). If $l_1^*(l_1+2,l_2,n) \leq l_1+1$, then we have $v_{t+1}^*(l_1+2,l_2,n) = v_{t+1}^*(l_1+1,l_2,n+1)$. Therefore, $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n) \geq v_{t+1}^*(l_1+1,l_2,n+1) - v_{t+1}^*(l_1+1,l_2,n+1) - v_{t+1}^*(l_1,l_2,n+1) + v_{t+1}^*(l_1,l_2,n+1) = 0$.

Case (2.2). If $l_1^*(l_1+2,l_2,n) = l_1+2$, we further consider the two possible scenarios under the condition $l_2+n < N$.

Case (2.2.1). When $l_2+n+1 = N$, then we have $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n) \geq w_t^*(l_1+2,0,N-1) - w_t^*(l_1+1,0,N) - w_t^*(l_1+1,0,N-1) + w_t^*(l_1,0,N) \geq 0$.

Case (2.2.2). When $l_2+n+1 < N$, since $l_1^*(l_1+2,l_2,n) = l_1+2$, we have $w_t^*(l_1+2,0,l_2+n) - w_t^*(l_1+1,0,l_2+n+1) < 0$. By property (P7), we have $w_t^*(l_1+1,0,l_2+n+1) - w_t^*(l_1,0,l_2+n+2) \leq w_t^*(l_1+2,0,l_2+n) - w_t^*(l_1+1,0,l_2+n+1) < 0$. Moreover, by property (P6), $w_t^*(l_1,0,l_2+n+1) - w_t^*(l_1-1,0,l_2+n+2) \leq w_t^*(l_1+1,0,l_2+n+1) - w_t^*(l_1,0,l_2+n+2) < 0$. Therefore, we have $l_1^*(l_1,l_2,n+1) = l_1^*(l_1,0,l_2+n+1) = l_1$. It then follows that $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n) \geq w_t^*(l_1+2,0,l_2+n) - w_t^*(l_1+1,0,l_2+n+1) - w_t^*(l_1+1,0,l_2+n) + w_t^*(l_1,0,l_2+n+1) \geq 0$.

$\underline{v_{t+1}^* \Rightarrow w_{t+1}^*}$: By definition, $\Delta_{l_1 l_1} w_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 n} w_{t+1}^*(l_1,l_2,n) = \lambda_1(\Delta_{l_1 l_1} v_{t+1}^*(l_1+1,l_2,n) - \Delta_{l_1 n} v_{t+1}^*(l_1+1,l_2,n)) + \lambda_2(\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2+1,n) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2+1,n)) + n\mu(\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n-1) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n-1)) + (N-n)\mu(\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n)) + \mu \Delta_{l_1 n} v_{t+1}^*(l_1,l_2,n) \geq 0$.

(P6.2)

$\underline{w_t^* \Rightarrow v_{t+1}^*}$: Note that $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2,n) \geq 0$ is equivalent to $v_{t+1}^*(l_1+2,l_2,n) - v_{t+1}^*(l_1+1,l_2+1,n) - v_{t+1}^*(l_1+1,l_2,n) + v_{t+1}^*(l_1,l_2+1,n) \geq 0$.

Case (1). If $l_2+n \geq N$, then we have $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2,n) = (\Delta_{l_1 l_1} - \Delta_{l_1 l_2}) w_t^*(l_1,l_2-N+n,N) \geq 0$.

Case (2). If $l_2+n < N$, then we have $\Delta_{l_1 l_1} v_{t+1}^*(l_1,l_2,n) - \Delta_{l_1 l_2} v_{t+1}^*(l_1,l_2,n) = (\Delta_{l_1 l_1} - \Delta_{l_1 n}) w_t^*(l_1,0,l_2+n) \geq 0$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, $\Delta_{l_1l_1}w^*_{t+1}(l_1,l_2,n) - \Delta_{l_1l_2}w^*_{t+1}(l_1,l_2,n) = \lambda_1(\Delta_{l_1l_1}v^*_{t+1}(l_1+1,l_2,n) - \Delta_{l_1l_2}v^*_{t+1}(l_1+1,l_2,n)) + \lambda_2(\Delta_{l_1l_1}v^*_{t+1}(l_1,l_2+1,n) - \Delta_{l_1l_2}v^*_{t+1}(l_1,l_2+1,n)) + n\mu(\Delta_{l_1l_1}v^*_{t+1}(l_1,l_2,n-1) - \Delta_{l_1l_2}v^*_{t+1}(l_1,l_2,n-1)) + (N-n)\mu(\Delta_{l_1l_1}v^*_{t+1}(l_1,l_2,n) - \Delta_{l_1l_2}v^*_{t+1}(l_1,l_2,n)) \geq 0$.

(P7)

(P7.1)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$: Denote LHS $:= v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2,n+1) + v^*_{t+1}(l_1,l_2,n+2)$.

Case (1). If $l_2+n \geq N$, then we have LHS $= w^*_t(l_1+2,l_2-N+n,N) - 2w^*_t(l_1+1,l_2-N+n+1,N) + w^*_t(l_1,l_2-N+n+2,N) \geq 0$, where the last inequality follows from the assumption that w^*_t satisfies property (P7.3).

Case (2). If $l_2+n = N-1$, then we have LHS $= v^*_{t+1}(l_1+2,0,N-1) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N)$. If $v^*_{t+1}(l_1+2,0,N-1) = w^*_t(l_1+2,0,N-1)$, then we have LHS $= w^*_t(l_1+2,0,N-1) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N) \geq 0$, by the assumption that w^*_t satisfies property (P7.2). On the other hand, if $v^*_{t+1}(l_1+2,0,N-1) = w^*_t(l_1+1,0,N)$, then we have LHS $= w^*_t(l_1+1,0,N) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N) = w^*_t(l_1,1,N) - w^*_t(l_1+1,0,N) = (\Delta_{l_2} - \Delta_{l_1})w^*_t(l_1,0,N) \geq 0$.

Case (3). If $l_2+n \leq N-2$, we further consider the following two subcases.

Case (3.1). If $l_1^*(l_1+2,l_2,n) \leq l_1+1$, then we have LHS $\geq v^*_{t+1}(l_1+1,l_2,n+1) - v^*_{t+1}(l_1+1,l_2,n+1) - v^*_{t+1}(l_1,l_2,n+2) + v^*_{t+1}(l_1,l_2,n+2) = 0$.

Case (3.2). If $l_1^*(l_1+2,l_2,n) = l_1+2$, then we have $w^*_t(l_1+2,0,l_2+n) - w^*_t(l_1+1,0,l_2+n+1) < 0$. By the assumption that w^*_t satisfies property (P7.2), it follows that $w^*_t(l_1+1,0,l_2+n+1) - w^*_t(l_1,0,l_2+n+2) \leq w^*_t(l_1+2,0,l_2+n) - w^*_t(l_1+1,0,l_2+n+1) < 0$. When $l_2+n+2 < N$, the above inequality further implies that $w^*_t(l_1,0,l_2+n+2) - w^*_t(l_1-1,0,l_2+n+3) < 0$, and hence $l_1^*(l_1,0,l_2+n+2) = l_1$. When $l_2+n+2 = N$, it is clear that $l_1^*(l_1,0,l_2+n+2) = l_1$. As a result, we have LHS $\geq w^*_t(l_1+2,0,l_2+n) - 2w^*_t(l_1+1,0,l_2+n+1) + w^*_t(l_1,0,l_2+n+2) \geq 0$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $w^*_{t+1}(l_1+2,l_2,n) - 2w^*_{t+1}(l_1+1,l_2,n+1) + w^*_{t+1}(l_1,l_2,n+2) = \lambda_1[v^*_{t+1}(l_1+3,l_2,n) - 2v^*_{t+1}(l_1+3,l_2,n+1) + v^*_{t+1}(l_1+1,l_2,n+2)] + \lambda_2[v^*_{t+1}(l_1+2,l_2+1,n) - 2v^*_{t+1}(l_1+1,l_2+1,n+1) + v^*_{t+1}(l_1,l_2+1,n+2)] + n\mu[v^*_{t+1}(l_1+2,l_2,n-1) - 2v^*_{t+1}(l_1+1,l_2,n) + v^*_{t+1}(l_1,l_2,n+1)] + (N-n-2)n\mu[v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2,n+1) + v^*_{t+1}(l_1,l_2,n+2)] + 2\mu[v^*_{t+1}(l_1+2,l_2,n) - v^*_{t+1}(l_1+1,l_2,n+1) - v^*_{t+1}(l_1+1,l_2,n) + v^*_{t+1}(l_1,l_2,n+1)] \geq 0$, where we notice that the terms in the last bracket is equal to $\Delta_{l_1l_1}v^*_{t+1}(l_1,l_2,n) - \Delta_{l_1n}v^*_{t+1}(l_1,l_2,n) \geq 0$.

(P7.2)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$: Let LHS $:= v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2,n+1) + v^*_{t+1}(l_1,l_2+1,n+1)$.

Case (1). If $l_2+n \geq N$, then we have LHS $= w^*_t(l_1+2,l_2-N+n,N) - 2w^*_t(l_1+1,l_2-N+n+1,N) + w^*_t(l_1,l_2-N+n+2,N) \geq 0$, where the last inequality holds by the assumption that w^*_t satisfies property (P7.3).

Case (2). If $l_2+n = N-1$, then we have LHS $= v^*_{t+1}(l_1+2,0,N-1) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N)$. If

$v^*_{t+1}(l_1+2,0,N-1) = w^*_t(l_1+2,0,N-1)$, then we have LHS $= w^*_t(l_1+2,0,N-1) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N) \geq 0$. On the other hand, if $v^*_{t+1}(l_1+2,0,N-1) = w^*_t(l_1+1,0,N)$, then we have LHS $= w^*_t(l_1+1,0,N) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N) = w^*_t(l_1,1,N) - w^*_t(l_1+1,0,N) = (\Delta_{l_2} - \Delta_{l_1})w^*_t(l_1,0,N) \geq 0$.

Case (3). If $l_2+n \leq N-2$, then we have LHS $= v^*_{t+1}(l_1+2,0,l_2+n) - 2v^*_{t+1}(l_1+1,0,l_2+n+1) + v^*_{t+1}(l_1,0,l_2+n+2) \geq 0$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $w^*_{t+1}(l_1+2,l_2,n) - 2w^*_{t+1}(l_1+1,l_2,n+1) + w^*_{t+1}(l_1,l_2+1,n+1) = \lambda_1[v^*_{t+1}(l_1+3,l_2,n) - 2v^*_{t+1}(l_1+2,l_2,n+1) + v^*_{t+1}(l_1+1,l_2+1,n+1)] + \lambda_2[v^*_{t+1}(l_1+2,l_2+1,n) - 2v^*_{t+1}(l_1+1,l_2+1,n+1) + v^*_{t+1}(l_1,l_2+2,n+1)] + n\mu[v^*_{t+1}(l_1+2,l_2,n-1) - 2v^*_{t+1}(l_1+1,l_2,n) + v^*_{t+1}(l_1,l_2+1,n)] + (N-n-1)\mu[v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2,n+1) + v^*_{t+1}(l_1,l_2+1,n+1)] + \mu[v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2,n) + v^*_{t+1}(l_1,l_2+1,n)] \geq 0$, since the terms in the last bracket $v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2,n) + v^*_{t+1}(l_1,l_2+1,n) = \Delta_{l_1l_1}v^*_{t+1}(l_1,l_2,n) + (\Delta_{l_2}v^*_{t+1}(l_1,l_2,n) - \Delta_{l_1}v^*_{t+1}(l_1,l_2,n)) \geq 0$.

(P7.3)

$\underline{w^*_t \Rightarrow v^*_{t+1}}$: Let LHS $:= v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2+1,n) + v^*_{t+1}(l_1,l_2+2,n)$.

Case (1). If $l_2+n \geq N$, then we have LHS $= w^*_t(l_1+2,0,N) - 2w^*_t(l_1+1,1,N) + w^*_t(l_1,2,N) \geq 0$.

Case (2). If $l_2+n = N-1$, then we have LHS $= v^*_{t+1}(l_1+2,0,N-1) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N)$. If $v^*_{t+1}(l_1+2,0,N-1) = w^*_t(l_1+2,0,N-1)$, then we have LHS $= w^*_t(l_1+2,0,N-1) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N) \geq 0$ by property (P7.2). On the other hand, if $v^*_{t+1}(l_1+2,0,N-1) = w^*_t(l_1+1,0,N)$, then we have LHS $= w^*_t(l_1+1,0,N) - 2w^*_t(l_1+1,0,N) + w^*_t(l_1,1,N) = w^*_t(l_1,1,N) - w^*_t(l_1+1,0,N) = (\Delta_{l_2} - \Delta_{l_1})w^*_t(l_1,0,N) \geq 0$.

Case (3). If $l_2+n \leq N-2$, then we have LHS $= v^*_{t+1}(l_1+2,0,l_2+n) - 2v^*_{t+1}(l_1+1,0,l_2+n+1) + v^*_{t+1}(l_1,0,l_2+n+2) \geq 0$.

$\underline{v^*_{t+1} \Rightarrow w^*_{t+1}}$: By definition, we have $w^*_{t+1}(l_1+2,l_2,n) - 2w^*_{t+1}(l_1+1,l_2+1,n) + w^*_{t+1}(l_1,l_2+2,n) = \lambda_1[v^*_{t+1}(l_1+3,l_2,n) - 2v^*_{t+1}(l_1+2,l_2+1,n) + v^*_{t+1}(l_1+1,l_2+2,n)] + \lambda_2[v^*_{t+1}(l_1+2,l_2+1,n) - 2v^*_{t+1}(l_1+1,l_2+2,n) + v^*_{t+1}(l_1,l_2+3,n)] + n\mu[v^*_{t+1}(l_1+2,l_2,n-1) - 2v^*_{t+1}(l_1+1,l_2+1,n-1) + v^*_{t+1}(l_1,l_2+2,n-1)] + (N-n)\mu[v^*_{t+1}(l_1+2,l_2,n) - 2v^*_{t+1}(l_1+1,l_2+1,n) + v^*_{t+1}(l_1,l_2+2,n)] \geq 0$.

(P8) $(\Delta_{l_1} - \Delta_n)v^*(l_1,l_2,n)$ is single-crossing in $(-n)$.

First we prove the single-crossing property for $(\Delta_{l_1} - \Delta_n)w^*_t(l_1,l_2,n)$. If $w^*_t(l_1+1,0,n+1) \geq w^*_t(l_1,0,n+2)$, then by property (P10.1), we have $w^*_{t-1}(l_1+1,0,n+1) \geq w^*_{t-1}(l_1,0,n+2)$, which in turn implies $w^*_{t-1}(l_1+1,0,n) \geq w^*_{t-1}(l_1,0,n+1)$ and $w^*_{t-1}(l_1+1,0,n-1) \geq w^*_{t-1}(l_1,0,n)$ by applying property (P8) in time period $t-1$. Moreover, the assumption $w^*_t(l_1+1,0,n+1) \geq w^*_t(l_1,0,n+2)$ further leads to $w^*_{t-1}(l_1+2,0,n) \geq w^*_{t-1}(l_1+1,0,n+1)$ by property (P7.1). Similar to the analysis in the proof of property (P11), the above four inequalities involving w^*_{t-1} respectively imply $v^*_t(l_1+1,0,n+1) = v^*_t(l_1,0,n+2)$, $v^*_t(l_1+1,0,n) = v^*_t(l_1,0,n+1)$, $v^*_t(l_1+1,0,n-1) = v^*_t(l_1,0,n)$, and $v^*_t(l_1+2,0,n) = v^*_t(l_1+1,0,n+1)$.

It then follows that $w^*_t(l_1+1,0,n) - w^*_t(l_1,0,n+1) = c_1 + \lambda_1[v^*_t(l_1+2,0,n) - v^*_t(l_1+1,0,n+1)] + \lambda_2[v^*_t(l_1+1,0,n+1) - v^*_t(l_1,0,n+2)] + n\mu[v^*_t(l_1+1,0,n-1) - v^*_t(l_1,0,n)] + (N-n-1)\mu[v^*_t(l_1+1,0,n) - v^*_t(l_1,0,n+$

$1)] + \mu \Delta_{l_1} v_t^*(l_1, 0, n) = c_1 + \Delta_{l_1} v_t^*(l_1, 0, n) > 0$. Hence $(\Delta_{l_1} - \Delta_n) w_t^*(l_1, l_2, n)$ is strictly single-crossing in $(-n)$.

Then, we prove the single-crossing property for v_{t+1}^*. If $v_{t+1}^*(l_1+1, 0, n+1) \geq v_{t+1}^*(l_1, 0, n+2)$, then we must have $w_t^*(l_1+1, 0, n+1) \geq w_t^*(l_1, 0, n+2)$, and therefore by (P8) for w_t^*, we have $w_t^*(l_1+1, 0, n) \geq w_t^*(l_1, 0, n+1)$, which leads to $v_{t+1}^*(l_1+1, 0, n) \geq v_{t+1}^*(l_1, 0, n+1)$.

(P9) Submodularity in $(l_1, t), (l_2, t), (n, t)$.

$\underline{w_t^* \Rightarrow v_{t+1}^*}$:

(P9.1)

Case (1). If $l_2 + n \geq N$, then we have $\Delta_{l_1 t} v_{t+1}^*(l_1, l_2, n) = \Delta_{l_1 t} w_t^*(l_1, l_2 - n + N, N) \leq 0$.

Case (2). If $l_2 + n < N$, let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l_{1,t+1}^*(l_1, l_2, n), l_{2,t+1}^*(l_1, l_2, n), n_{t+1}^*(l_1, l_2, n))$ be the optimal solution to problem $v_{t+1}^*(l_1, l_2, n)$. By Lemma OS-D1, $l_{1,t}^*(l_1+1, l_2, n)$ can be either $\hat{l}_1$ or $\hat{l}_1 + 1$.

Case (2.1). When $l_{1,t}^*(l_1+1, l_2, n) = \hat{l}_1$, we have $\Delta_{l_1 t} v_{t+1}^*(l_1, l_2, n) \leq \Delta_{nt} w_t^*(\hat{l}_1, 0, \hat{n}) \leq 0$.

Case (2.2). When $l_{1,t}^*(l_1+1, l_2, n) = \hat{l}_1 + 1$, we have $\Delta_{l_1 t} v_{t+1}^*(l_1, l_2, n) \leq \Delta_{l_1 t} w_t^*(\hat{l}_1, 0, \hat{n}) \leq 0$.

(P9.2)

Case (1). If $l_2 + n \geq N$, then we have $\Delta_{nt} v_{t+1}^*(l_1, l_2, n) = \Delta_{l_2 t} w_t^*(l_1, l_2 - N + n, N) \leq 0$.

Case (2). If $l_2 + n < N$, then let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l_{1,t+1}^*(l_1, l_2, n), l_{2,t+1}^*(l_1, l_2, n), n_{t+1}^*(l_1, l_2, n))$ be the optimal solution to problem $v_{t+1}^*(l_1, l_2, n)$. By Lemma OS-D1, $l_{1,t}^*(l_1, l_2, n+1)$ can be either $\hat{l}_1$ or $\hat{l}_1 + 1$.

Case (2.1). When $l_{1,t}^*(l_1, l_2, n+1) = \hat{l}_1$, we have $\Delta_{nt} v_{t+1}^*(l_1, l_2, n) = \Delta_{nt} w_t^*(\hat{l}_1, 0, \hat{n}) \leq 0$.

Case (2.2). When $l_{1,t}^*(l_1, l_2, n+1) = \hat{l}_1 + 1$, we have $\Delta_{nt} v_{t+1}^*(l_1, l_2, n) = \Delta_{l_1 t} w_t^*(\hat{l}_1, 0, \hat{n}) \leq 0$.

(P9.3)

Case (1). If $l_2 + n \geq N$, then we have $\Delta_{l_2 t} v_{t+1}^*(l_1, l_2, n) = \Delta_{l_2 t} w_t^*(l_1, l_2 - N + n, N) \leq 0$.

Case (2). If $l_2 + n < N$, then we have $\Delta_{l_2 t} v_{t+1}^*(l_1, l_2, n) = \Delta_{nt} v_{t+1}^*(l_1, 0, n + l_2) \leq 0$.

$\underline{v_{t+1}^* \Rightarrow w_{t+1}^*}$:

(P9.1) By definition, we have $\Delta_{l_1 t} w_{t+1}^*(l_1, l_2, n) = \lambda_1 \Delta_{l_1 t} v_{t+1}^*(l_1+1, l_2, n) + \lambda_2 \Delta_{l_1 t} v_{t+1}^*(l_1, l_2+1, n) + n\mu \Delta_{l_1 t} v_{t+1}^*(l_1, l_2, n-1) + (N-n)\mu \Delta_{l_1 t} v_{t+1}^*(l_1, l_2, n) \leq 0$.

(P9.2) By definition, we have $\Delta_{nt} w_{t+1}^*(l_1, l_2, n) = \lambda_1 \Delta_{nt} v_{t+1}^*(l_1+1, l_2, n) + \lambda_2 \Delta_{nt} v_{t+1}^*(l_1, l_2+1, n) + n\mu \Delta_{nt} v_{t+1}^*(l_1, l_2, n-1) + (N-n-1)\mu \Delta_{nt} v_{t+1}^*(l_1, l_2, n) \leq 0$.

(P9.3) By definition, we have $\Delta_{l_2 t} w_{t+1}^*(l_1, l_2, n) = \lambda_1 \Delta_{l_2 t} v_{t+1}^*(l_1+1, l_2, n) + \lambda_2 \Delta_{l_2 t} v_{t+1}^*(l_1, l_2+1, n) + n\mu \Delta_{l_2 t} v_{t+1}^*(l_1, l_2, n-1) + (N-n)\mu \Delta_{l_2 t} v_{t+1}^*(l_1, l_2, n) \leq 0$.

(P10) $(\Delta_{l_1 t} - \Delta_{nt}) v_t^*(l_1, l_2, n) \leq 0$ and $(\Delta_{l_1 t} - \Delta_{l_2 t}) v_t^*(l_1, l_2, n) \leq 0$.

$\underline{w_t^* \Rightarrow v_{t+1}^*}$:

(P10.1)

Case (1). If $l_{1,t}^*(l_1+1, l_2, n) < l_1 + 1$, then we have $v_t^*(l_1+1, l_2, n) = v_t^*(l_1, l_2, n+1)$. Since we always have $v_{t+1}^*(l_1+1, l_2, n) - v_{t+1}^*(l_1, l_2, n+1) \leq 0$, hence $(\Delta_{l_1 t} - \Delta_{nt}) v_t^*(l_1, l_2, n) \leq 0$.

Case (2). If $l^*_{1,t}(l_1+1,l_2,n)=l_1+1$, then by the assumption that w^*_t satisfies properties (P10.1) and (P7.1), we must have $l^*_{1,t+1}(l_1,l_2,n+1)=l_1$.

Case (2.1). If $l_2+n<N$, then $(\Delta_{l_1t}-\Delta_{nt})v^*_t(l_1,l_2,n)\le(\Delta_{l_1t}-\Delta_{nt})w^*_{t-1}(l_1,0,n+l_2)\le 0$.

Case (2.2). If $l_2+n\ge N$, then $(\Delta_{l_1t}-\Delta_{nt})v^*_t(l_1,l_2,n)=(\Delta_{l_1t}-\Delta_{l_2t})w^*_{t-1}(l_1,l_2-n+N,N)\le 0$.

(P10.2)

Case (1). If $l_2+n\ge N$, then we have $(\Delta_{l_1t}-\Delta_{l_2t})v^*_t(l_1,l_2,n)=(\Delta_{l_1t}-\Delta_{l_2t})w^*_{t-1}(l_1,l_2-N+n,N)\le 0$.

Case (2). If $l_2+n<N$, then $(\Delta_{l_1t}-\Delta_{l_2t})v^*_t(l_1,l_2,n)=(\Delta_{l_1t}-\Delta_{nt})v^*_t(l_1,0,l_2+n)\le 0$.

$\underline{v^*_{t+1}\Rightarrow w^*_{t+1}}$:

(P10.1) By definition, we have $(\Delta_{l_1t}-\Delta_{nt})w^*_{t+1}(l_1,l_2,n)=\lambda_1(\Delta_{l_1t}-\Delta_{nt})v^*_{t+1}(l_1+1,l_2,n)+\lambda_2(\Delta_{l_1t}-\Delta_{nt})v^*_{t+1}(l_1,l_2+1,n)+n\mu(\Delta_{l_1t}-\Delta_{nt})v^*_{t+1}(l_1,l_2,n-1)+(N-n-1)\mu(\Delta_{l_1t}-\Delta_{nt})v^*_{t+1}(l_1,l_2,n)+\mu\Delta_{l_1t}v^*_{t+1}(l_1,l_2,n)\le 0$.

(P10.2) Similarly, $(\Delta_{l_1t}-\Delta_{l_2t})w^*_{t+1}(l_1,l_2,n)=\lambda_1(\Delta_{l_1t}-\Delta_{l_2t})v^*_{t+1}(l_1+1,l_2,n)+\lambda_2(\Delta_{l_1t}-\Delta_{l_2t})v^*_{t+1}(l_1,l_2+1,n)+n\mu(\Delta_{l_1t}-\Delta_{l_2t})v^*_{t+1}(l_1,l_2,n-1)+(N-n)\mu(\Delta_{l_1t}-\Delta_{l_2t})v^*_{t+1}(l_1,l_2,n)\le 0$.

(P11) If $v^*_t(l_1+1,0,n+1)\ge v^*_t(l_1,0,n+2)$, then $v^*_{t+1}(l_1+1,0,n)\ge v^*_{t+1}(l_1,0,n+1)$.

First we prove the property for w^*_t. Suppose $w^*_t(l_1+1,0,n+1)\ge w^*_t(l_1,0,n+2)$. Then, we have $w^*_{t+1}(l_1+1,0,n)-w^*_{t+1}(l_1,0,n+1)=c_1+\lambda_1[v^*_{t+1}(l_1+2,0,n)-v^*_{t+1}(l_1+1,0,n+1)]+\lambda_2[v^*_{t+1}(l_1+1,0,n+1)-v^*_{t+1}(l_1,0,n+2)]+n\mu[v^*_{t+1}(l_1+1,0,n-1)-v^*_{t+1}(l_1,0,n)]+(N-n-1)\mu[v^*_{t+1}(l_1+1,0,n)-v^*_{t+1}(l_1,0,n+1)]+\mu\Delta_{l_1}v^*_{t+1}(l_1,0,n)$. It is worth noticing that, the assumption $w^*_t(l_1+1,0,n+1)\ge w^*_t(l_1,0,n+2)$ implies $v^*_{t+1}(l_1+1,0,n+1)=v^*_{t+1}(l_1,0,n+2)$. To see this, it is clear that $v^*_{t+1}(l_1+1,0,n+1)\le v^*_{t+1}(l_1,0,n+2)$, because any feasible solution to problem $v^*_{t+1}(l_1,0,n+2)$ is also feasible to problem $v^*_{t+1}(l_1+1,0,n+1)$. Moreover, $w^*_t(l_1+1,0,n+1)\ge w^*_t(l_1,0,n+2)$ implies that the optimal solution to problem $v^*_{t+1}(l_1+1,0,n+1)$ is in the feasible set of problem $v^*_{t+1}(l_1,0,n+2)$. Therefore, we must have $v^*_{t+1}(l_1+1,0,n+1)-v^*_{t+1}(l_1,0,n+2)=0$.

Following the similar argument, we have $v^*_{t+1}(l_1+2,0,n)-v^*_{t+1}(l_1+1,0,n+1)=0$, $v^*_{t+1}(l_1+1,0,n-1)-v^*_{t+1}(l_1,0,n)=0$, $v^*_{t+1}(l_1+1,0,n)-v^*_{t+1}(l_1,0,n+1)=0$, since $w^*_t(l_1+1,0,n+1)\ge w^*_t(l_1,0,n+2)$ implies $w^*_t(l_1+2,0,n)\ge w^*_t(l_1+1,0,n+1)$ (by property (P7.1)), $w^*_t(l_1+1,0,n-1)\ge w^*_t(l_1,0,n)$ (by property (P8)), and $w^*_t(l_1+1,0,n)\ge w^*_t(l_1,0,n+1)$ (by property (P8)). It then follows that $w^*_{t+1}(l_1+1,0,n)-w^*_{t+1}(l_1,0,n+1)=c_1+\mu\Delta_{l_1}v^*_{t+1}(l_1,0,n)>0$. Therefore, we must have $(\Delta_{l_1}-\Delta_n)w^*_t(l_1,0,n)$ is strictly single-crossing in $(-n,t)$.

Next, we prove the property for v^*_{t+1}. Suppose $v^*_{t+1}(l_1+1,0,n+1)\ge v^*_{t+1}(l_1,0,n+2)$, then we have $w^*_t(l_1+1,0,n+1)\ge w^*_t(l_1,0,n+2)$. By (P11) for w^*_t, we have $w^*_{t+1}(l_1+1,0,n)\ge w^*_{t+1}(l_1,0,n+1)$, which results in $v^*_{t+2}(l_1+1,0,n)\ge v^*_{t+2}(l_1,0,n+1)$. $\square$

Proof of Theorem 1: We first show that $K_0\le K_1\le K_2\le\cdots\le K_{N-1}$. Note that

$$K_i=\min\{K\in\mathbb{N}: w^*(K+1,0,i)-w^*(K,0,i+1)\ge 0\}.$$

Thus, $w^*(K_i+1,0,i)-w^*(K_i,0,i+1)\ge 0$. Since w^* satisfies property (P8), we have $w^*(K_i+1,0,i-1)-$

$w^*(K_i, 0, i) \geq 0$. Again, by the definition of K_{i-1}, we have $K_{i-1} \leq K_i$.

In the following, we show that the policy described in Theorem 1 is an optimal server assignment policy.

Note that the optimal assignment policy at state (l_1, l_2, n) is given by the solution to the following optimization problem:

$$\begin{aligned} v^*(l_1, l_2, n) = \min_{l_1', l_2', n' \in \mathbb{N}} \quad & w^*(l_1', l_2', n') \\ \text{s.t.} \quad & l_1' + l_2' + n' = l_1 + l_2 + n, \\ & l_1' \leq l_1, \\ & l_2' \leq l_2, \\ & n \leq n' \leq N. \end{aligned} \tag{OS-D2}$$

To prove Theorem 1, we need to show that at each decision epoch with state (l_1, l_2, n), the post-action state following the policy described in Theorem 1 is the optimal solution to the above optimization problem. We consider two cases: (1) $l_2 + n \geq N$ and (2) $l_2 + n < N$.

For the first case, following the policy described in Theorem 1, the post-action state is $(l_1, l_2 + n - N, N)$. Next, we show that $(l_1, l_2 + n - N, N)$ is one optimal solution to (OS-D2).

Suppose this is not true and the optimal solution is $(\hat{l}_1, \hat{l}_2, \hat{n}) \neq (l_1, l_2 + n - N, N)$. Then, we have

$$w^*(\hat{l}_1, \hat{l}_2, \hat{n}) < w^*(l_1, l_2 + n - N, N). \tag{OS-D3}$$

First, note that $\hat{l}_1 \leq l_1$ and $\hat{n} \leq N$. Let $\delta_1 = l_1 - \hat{l}_1$ and $\delta_2 = N - \hat{n}$. Then, $\hat{l}_2 - (l_2 + n - N) = \delta_1 + \delta_2$. Thus,

$$\begin{aligned} w^*(\hat{l}_1, \hat{l}_2, \hat{n}) &= w^*(l_1 - \delta_1, \hat{l}_2, N - \delta_2) \\ &\geq w^*(l_1 - \delta_1, \hat{l}_2 - \delta_2, N) \\ &\geq w^*(l_1, \hat{l}_2 - \delta_1 - \delta_2, N) \\ &= w^*(l_1, l_2 + n - N, N), \end{aligned}$$

where the first inequality holds because w^* satisfies property (P3), and the second inequality holds because w^* satisfies property (P2). This contradicts with (OS-D3).

For the second case, following the policy described in Theorem 1, it is optimal to use idle servers to serve all VIP customers. After the assignment, the state becomes $(l_1, 0, N - n - l_2)$. Then we decide whether to assign the rest of the idle servers to serve normal customers one by one. Suppose the post-action state is $(l_1^*, 0, n^*)$. We have $l_1^* + n^* = l_1 + N - n - l_2$, $l_1^* \leq K_{n^*}$, and $l_1^* + 1 > K_{n^*-1}$. Next, we show that $(l_1^*, 0, n^*)$ is one optimal solution to (OS-D2).

Suppose this is not true and the optimal solution is $(\tilde{l}_1, \tilde{l}_2, \tilde{n}) \neq (l_1^*, 0, n^*)$. Then, we have

$$w^*(\tilde{l}_1, \tilde{l}_2, \tilde{n}) < w^*(l_1^*, 0, n^*). \tag{OS-D4}$$

If $\tilde{l}_1 = l_1^*$, then $\tilde{l}_2 > 0$ and $n^* = \tilde{l}_2 + \tilde{n}$. Thus,

$$w^*(\tilde{l}_1, \tilde{l}_2, \tilde{n}) = w^*(l_1^*, \tilde{l}_2, \tilde{n}) \geq w^*(l_1^*, 0, \tilde{l}_2 + \tilde{n}) = w^*(l_1^*, 0, n^*),$$

where the inequality is because w^* satisfies property (P3). This contradicts with (OS-D4).

If $\tilde{l}_1 > l_1^*$, then $n^* > \tilde{l}_2 + \tilde{n}$. Thus, by property (P3), we have $w^*(\tilde{l}_1, \tilde{l}_2, \tilde{n}) \geq w^*(\tilde{l}_1, 0, \tilde{l}_2 + \tilde{n})$. Note that $\tilde{l}_1 \geq l_1^* + 1 > K_{n^*-1} \geq K_{\tilde{l}_2+\tilde{n}}$. By the definition of $K_{\tilde{l}_2+\tilde{n}}$, we have

$$w^*(\tilde{l}_1, 0, \tilde{l}_2 + \tilde{n}) \geq w^*(\tilde{l}_1 - 1, 0, \tilde{l}_2 + \tilde{n} + 1).$$

If $\tilde{l}_1 - 1 > l_1^*$, then $n^* > \tilde{l}_2 + \tilde{n} + 1$ and $\tilde{l}_1 - 1 \geq l_1^* + 1 > K_{n^*-1} \geq K_{\tilde{l}_2+\tilde{n}+1}$. By the definition of $K_{\tilde{l}_2+\tilde{n}+1}$, we have

$$w^*(\tilde{l}_1 - 1, 0, \tilde{l}_2 + \tilde{n} + 1) \geq w^*(\tilde{l}_1 - 2, 0, \tilde{l}_2 + \tilde{n} + 2).$$

Repeating the above process, eventually, we have

$$\begin{aligned} w^*(\tilde{l}_1, \tilde{l}_2, \tilde{n}) &\geq w^*(\tilde{l}_1, 0, \tilde{l}_2 + \tilde{n}) \\ &\geq w^*(\tilde{l}_1 - 1, 0, \tilde{l}_2 + \tilde{n} + 1) \\ &\geq w^*(\tilde{l}_1 - 2, 0, \tilde{l}_2 + \tilde{n} + 2) \\ &\geq \cdots \\ &\geq w^*(l_1^*, 0, n^*). \end{aligned}$$

This contradicts with (OS-D4).

If $\tilde{l}_1 < l_1^*$, then $\tilde{l}_2 + \tilde{n} > n^*$. Let $\bar{n} = \min\{N, \tilde{l}_2 + \tilde{n}\}$ and $\bar{l}_1 = \tilde{l}_1 + \tilde{l}_2 + \tilde{n} - \bar{n}$. Then, we have $\bar{n} \geq n^*$ and $\bar{l}_1 \leq l_1^*$.

(a) If $\bar{n} = n^*$ and $\bar{l}_1 = l_1^*$, then

$$w^*(\tilde{l}_1, \tilde{l}_2, \tilde{n}) \geq w^*(\tilde{l}_1, \tilde{l}_2 + \tilde{n} - \bar{n}, \bar{n}) \geq w^*(\tilde{l}_1 + \tilde{l}_2 + \tilde{n} - \bar{n}, 0, \bar{n}) = w^*(\bar{l}_1, 0, \bar{n}) = w^*(l_1^*, 0, n^*),$$

where the first inequality holds because w^* satisfies property (P3), and the second inequality holds because w^* satisfies property (P2). This contradicts with (OS-D4).

(b) If $\bar{n} > n^*$ and $\bar{l}_1 < l_1^*$, then similar to (a), we have $w^*(\tilde{l}_1, \tilde{l}_2, \tilde{n}) \geq w^*(\bar{l}_1, 0, \bar{n})$. Note that $\bar{l}_1 + 1 \leq l_1^* \leq K_{n^*} \leq K_{\bar{n}-1}$. By the definition of $K_{\bar{n}-1}$, we have

$$w^*(\bar{l}_1, 0, \bar{n}) \geq w^*(\bar{l}_1 + 1, 0, \bar{n} - 1).$$

If $\bar{l}_1 + 1 < l_1^*$, then $\bar{n} - 1 > n^*$ and $\bar{l}_1 + 2 \leq l_1^* \leq K_{n^*} \leq K_{\bar{n}-2}$. By the definition of $K_{\bar{n}-2}$, we have

$$w^*(\bar{l}_1 + 1, 0, \bar{n} - 1) \geq w^*(\bar{l}_1 + 2, 0, \bar{n} - 2).$$

Repeating the above process, eventually, we have

$$\begin{aligned} w^*(\tilde{l}_1, \tilde{l}_2, \tilde{n}) &\geq w^*(\bar{l}_1, 0, \bar{n}) \\ &\geq w^*(\bar{l}_1 + 1, 0, \bar{n} - 1) \\ &\geq w^*(\bar{l}_1 + 2, 0, \bar{n} - 2) \\ &\geq \cdots \\ &\geq w^*(l_1^*, 0, n^*). \end{aligned}$$

This contradicts with (OS-D4). □

Proof of Proposition 1: Consider a decision epoch $t > 0$. The state at this time point must be transited from a previous post-action state. Suppose the post-action state is (l_1, l_2, n). We consider two cases: (1)

$n = N$ and (2) $n < N$.

For the first case, at this decision epoch t, the number of busy servers is either $n-1$ or n. Clearly, it is optimal to assign at most one idle server.

For the second case, we have $l_2 = 0$ and $l_1 \leq K_n$. Since the state at time t is transited from (l_1, l_2, n), there are three possibilities for the current state: (a) $(l_1+1, 0, n)$, (b) $(l_1, 1, n)$, or (c) $(l_1, 0, n-1)$. For case (a), if $l_1 + 1 > K_n$ (i.e., $l_1 = K_n$), it is optimal to assign one idle server to a normal customer and the state becomes $(l_1, 0, n+1)$ after the assignment. Since $l_1 \leq K_n \leq K_{n+1}$, it is optimal to keep all other available servers idle. If $l_1 + 1 \leq K_n$, it is optimal to keep all available servers idle. Therefore, for case (a), it is optimal to assign at most one idle server.

For case (b) with state $(l_1, 1, n)$, it is optimal to assign one idle server to the VIP customer. The state becomes $(l_1, 0, n+1)$ after the assignment. Since $l_1 \leq K_n \leq K_{n+1}$, it is optimal to keep all other available servers idle. In this case, it is optimal to assign at most one idle server.

For case (c), if $l_1 > K_{n-1}$, it is optimal to assign one idle server to a normal customer and the state becomes $(l_1 - 1, 0, n)$ after the assignment. Since $l_1 - 1 \leq K_n$, it is optimal to keep all other available servers idle. If $l_1 \leq K_{n-1}$, it is optimal to keep all available servers idle. Therefore, for case (c), it is optimal to assign at most one idle server.

To summarize, at any decision epoch $t > 0$, it is optimal to assign at most one idle server. ☐

Proof of Lemma 2: Let $v_0^*(\mathbf{l}, n_1, n_2) = \frac{c_L}{\alpha}(l_1 + l_2 + \cdots + l_L + n_1 + n_2)$. It is straightforward to see that v_0^* satisfies the multimodularity property. We define w_t^* and v_t^* inductively as follows,

$$w_t^*(\mathbf{l}, n_1, n_2) = \sum_{i=1}^{L} c_i l_i + \sum_{i=1}^{L} \lambda_i v_t^*(\mathbf{l} + \mathbf{e}_i, n_1, n_2) + \mu_1 v_t^*(\mathbf{l}, 0, n_2) + \mu_2 v_t^*(\mathbf{l}, n_1, 0),$$

and

$$\begin{aligned} v_{t+1}^*(\mathbf{l}, n_1, n_2) = \min_{\mathbf{l}' \in \mathbb{N}^L, n_1', n_2' \in \{0,1\}} \quad & w_t^*(\mathbf{l}', n_1', n_2') \\ \text{s.t.} \quad & \sum_{i=1}^{L} l_i' + n_1' + n_2' = \sum_{i=1}^{L} l_i + n_1 + n_2, \\ & \mathbf{l}' \leq \mathbf{l}, \\ & n_1' \geq n_1, \\ & n_2' \geq n_2. \end{aligned} \qquad \text{(OS-D5)}$$

Whenever the context is clear, we shall refer the above optimization problem as *problem* $v_{t+1}^*(\mathbf{l}, n_1, n_2)$.

Similar to the proof of Lemma 1, we prove Lemma 2 by showing that if w_t^* satisfies properties (P1)-(P5), then v_{t+1}^* and w_{t+1}^* also satisfy properties (P1)-(P5). It then immediately implies that $v^* = \lim_{t\to\infty} v_t^*$ and $w^* = \lim_{t\to\infty} w_t^*$ satisfy properties (P1)-(P5), which will complete the proof. The proofs of properties (P1)-(P4) are similar to the proofs of (P1)-(P4) in Lemma 1, which are omitted for the sake of brevity. We next show that if $w_t^*(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$ and $(\mathbf{l}, n_2)$, then so are $v_{t+1}^*(\mathbf{l}, n_1, n_2)$ and $w_{t+1}^*(\mathbf{l}, n_1, n_2)$.

To prove the multimodularity property, we first show the following two auxiliary lemmas.

Lemma OS-D3. *$v^*_{t+1}(\mathbf{l}, n_1, n_2)$ is the optimal value of the following optimization problem:*

$$\begin{aligned} \min_{\mathbf{l}' \in \mathbb{N}^L, n_1', n_2' \in \{0,1\}} \quad & w^*_t(\mathbf{l}', n_1', n_2') \\ s.t. \quad & \sum_{i=1}^{L} l_i' + n_1' + n_2' = \sum_{i=1}^{L} l_i + n_1 + n_2, \\ & \sum_{i=1}^{m} l_i' \le \sum_{i=1}^{m} l_i, \quad \forall 1 \le m \le L \\ & \sum_{i=1}^{L} l_i' + n_1' \le \sum_{i=1}^{L} l_i + n_1. \end{aligned} \tag{OS-D6}$$

Lemma OS-D4. *We have $v^*_{t+1}(\mathbf{l}, n_1, 0) = \min\{\hat{w}_t(\mathbf{l}, n_1, 0), \tilde{w}_t(\mathbf{l}, n_1, 1)\}$, where*

$$\begin{aligned} \hat{w}_t(\mathbf{l}, n_1, 0) = \min_{\mathbf{l}' \in \mathbb{Z}^L_+, n_1' \in \{0,1\}} \quad & w^*_t(\mathbf{l}', n_1', 0) \\ s.t. \quad & \sum_{i=1}^{L} l_i' + n_1' = \sum_{i=1}^{L} l_i + n_1 \\ & \sum_{i=1}^{m} l_i' \le \sum_{i=1}^{m} l_i, \quad \forall 1 \le m \le L \end{aligned}$$

and

$$\begin{aligned} \tilde{w}_t(\mathbf{l}, n_1, 1) = \min_{\mathbf{l}' \in \mathbb{Z}^L_+, n_1' \in \{0,1\}} \quad & w^*_t(\mathbf{l}', n_1', 1) \\ s.t. \quad & \sum_{i=1}^{L} l_i' + n_1' + 1 = \sum_{i=1}^{L} l_i + n_1 \\ & \sum_{i=1}^{m} l_i' \le \sum_{i=1}^{m} l_i, \quad \forall 1 \le m \le L \end{aligned}$$

Moreover, if $w^*_t(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$, then $v^*_{t+1}(\mathbf{l}, n_1, 1)$, $\hat{w}_t(\mathbf{l}, n_1, 0)$, and $\tilde{w}_t(\mathbf{l}, n_1, 1)$ are all multimodular in $(\mathbf{l}, n_1)$.

We are now ready to prove property (P5). We start with the multimodularity of $v^*_{t+1}(\mathbf{l}, n_1, n_2)$ in $(\mathbf{l}, n_1)$ (i.e., the direction of $w^*_t \Rightarrow v^*_{t+1}$). Notice that in Lemma OS-D4, we have shown that if $w^*_t(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$ and $(\mathbf{l}, n_2)$, then $v^*_{t+1}(\mathbf{l}, n_1, 1)$ is multimodular in $(\mathbf{l}, n_1)$. Therefore, it remains to show that if $w^*_t(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$ and $(\mathbf{l}, n_2)$, then $v^*_{t+1}(\mathbf{l}, n_1, 0)$ is multimodular in $(\mathbf{l}, n_1)$.

We now prove the multimodularity of $v^*_{t+1}(\mathbf{l}, n_1, 0)$ in $(\mathbf{l}, n_1)$. By the definition of multimodular functions, we need to show that for any $\mathbf{a}, \mathbf{b} \in \mathcal{F} := \{-\mathbf{e}_1, \mathbf{d}_2, ..., \mathbf{d}_L, \mathbf{d}_{L+1}, \mathbf{e}_{L+1}\}$ and $\mathbf{a} \neq \mathbf{b}$, we have

$$v^*_{t+1}((\mathbf{l}, n_1, 0) + \mathbf{a}) + v^*_{t+1}((\mathbf{l}, n_1, 0) + \mathbf{b}) \ge v^*_{t+1}((\mathbf{l}, n_1, 0) + \mathbf{a} + \mathbf{b}) + v^*_{t+1}(\mathbf{l}, n_1, 0), \tag{OS-D7}$$

where $\mathbf{d}_i = \mathbf{e}_{i-1} - \mathbf{e}_i$ for each $i = 2, ..., m+1$. Denote the LHS and RHS as the left-hand-side and right-hand-side of inequality (OS-D7), respectively. To show (OS-D7), we consider the following cases.

Case (1). $\mathbf{a} = -\mathbf{e}_1$ and $\mathbf{b} \in \{\mathbf{d}_2, ..., \mathbf{d}_L\}$. In this case, consider $\mathbf{b} = \mathbf{d}_i$ for some $2 \le i \le L$. Then, we have

$$\text{LHS} - \text{RHS} = v^*_{t+1}(\mathbf{l} - \mathbf{e}_1, n_1, 0) + v^*_{t+1}(\mathbf{l} + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 0) - v^*_{t+1}(\mathbf{l} - \mathbf{e}_1 + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 0) - v^*_{t+1}(\mathbf{l}, n_1, 0)$$

Let $(\hat{\mathbf{l}}, \hat{n}_1, \hat{n}_2)$, $(\bar{\mathbf{l}}, \bar{n}_1, \bar{n}_2)$, and $(\mathbf{l}^*, n_1^*, n_2^*)$ denote the optimal solution to $v^*_{t+1}(\mathbf{l} - \mathbf{e}_1, n_1, 0)$, $v^*_{t+1}(\mathbf{l} + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 0)$, and $v^*_{t+1}(\mathbf{l}, n_1, 0)$, respectively.

Case (1.1). If $\hat{n}_2 = \bar{n}_2 = 1$, then we have

$$\text{LHS} - \text{RHS} \geq \tilde{w}_t(\mathbf{l} - \mathbf{e}_1, n_1, 1) + \tilde{w}_t(\mathbf{l} + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 1) - \tilde{w}_t(\mathbf{l} - \mathbf{e}_1 + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 1) - \tilde{w}_t(\mathbf{l}, n_1, 1) \geq 0,$$

where the last inequality follows from the multimodularity of $\tilde{w}_t(\mathbf{l}, n_1, 1)$ in $(\mathbf{l}, n_1)$.

Case (1.2). If $\hat{n}_2 = \bar{n}_2 = 0$, then we have

$$\text{LHS} - \text{RHS} \geq \hat{w}_t(\mathbf{l} - \mathbf{e}_1, n_1, 0) + \hat{w}_t(\mathbf{l} + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 0) - \hat{w}_t(\mathbf{l} - \mathbf{e}_1 + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 0) - \hat{w}_t(\mathbf{l}, n_1, 0) \geq 0,$$

where the last inequality follows from the multimodularity of $\hat{w}_t(\mathbf{l}, n_1, 0)$ in $(\mathbf{l}, n_1)$.

Case (1.3). If $\hat{n}_2 = 1$ and $\bar{n}_2 = 0$, then in this case, we have $\hat{n}_1 = 1$. Note that $\sum_{i=1}^L l_i^* \leq \sum_{i=1}^L \bar{l}_i \leq \sum_{i=1}^L l_i^* + 1$, and $n_1^* + n_2^* - 1 \leq \bar{n}_1 + \bar{n}_2 \leq n_1^* + n_2^*$. In other words, the optimal number of customers assigned to empty servers in state $(\mathbf{l} + \mathbf{e}_{i-1} - \mathbf{e}_i, n_1, 0)$ is less than that in state $(\mathbf{l}, n_1, 0)$, and the difference is less than 1. Similarly, we have $\sum_{i=1}^L l_i^* - 1 \leq \sum_{i=1}^L \hat{l}_i \leq \sum_{i=1}^L l_i^*$, and $n_1^* + n_2^* - 1 \leq \hat{n}_1 + \hat{n}_2 \leq n_1^* + n_2^*$. That is, the optimal number of customers assigned to empty servers in state $(\mathbf{l} - \mathbf{e}_1, n_1, 0)$ is less than that in state $(\mathbf{l}, n_1, 0)$, and the difference is less than 1. Therefore, $|\bar{n}_1 + \bar{n}_2 - (\hat{n}_1 + \hat{n}_2)| \leq 1$. Since $\hat{n}_1 = \hat{n}_2 = 1$ and $\bar{n}_2 = 0$, we have $\bar{n}_1 = 1$. In this case, let k be the index of the highest customer priority class that has a positive queue length, i.e., $k = \max_{1 \leq i \leq L}\{l_i > 0\}$. Then, we have

$$\begin{aligned}
&\text{LHS} - \text{RHS} \\
&\geq w_t^*(\mathbf{l} - \mathbf{e}_1 - \mathbf{e}_k, 1, 1) + w_t^*(\mathbf{l} + \mathbf{e}_{i-1} - \mathbf{e}_i, 1, 0) - w_t^*(\mathbf{l} - \mathbf{e}_1 + \mathbf{e}_{i-1} - \mathbf{e}_i, 1, 0) - w_t^*(\mathbf{l} - \mathbf{e}_k, 1, 1) \\
&= \Delta_{\mathbf{a}} w_t^*(\mathbf{l} - \mathbf{e}_k, 1, 1) - \Delta_{\mathbf{a}} w_t^*(\mathbf{l} + \mathbf{b}, 1, 0) \\
&\geq \Delta_{\mathbf{a}} w_t^*(\mathbf{l} - \mathbf{e}_k + \mathbf{b}, 1, 1) - \Delta_{\mathbf{a}} w_t^*(\mathbf{l} + \mathbf{b}, 1, 0) \\
&\geq 0,
\end{aligned}$$

where the second inequality follows from $\Delta_{\mathbf{ab}} w_t^*(\mathbf{l} - \mathbf{e}_k, 1, 1) \leq 0$, and the last inequality holds because $\Delta_{\mathbf{a},(\mathbf{e}_{L+2} - \mathbf{e}_k)} w_t^*(\mathbf{l} - \mathbf{e}_k + \mathbf{b}, 1, 0) \geq 0$, since $w_t^*(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_2)$.

Case (1.4). If $\hat{n}_2 = 0$ and $\bar{n}_2 = 1$, then similar to Case (1.3), we have $\hat{n}_1 = 1$ and $\bar{n}_1 = 1$. It then follows that

$$\begin{aligned}
&\text{LHS} - \text{RHS} \\
&\geq w_t^*(\mathbf{l} - \mathbf{e}_1, 1, 0) + w_t^*(\mathbf{l} + \mathbf{e}_{i-1} - \mathbf{e}_i - \mathbf{e}_k, 1, 1) - w_t^*(\mathbf{l} - \mathbf{e}_1 + \mathbf{e}_{i-1} - \mathbf{e}_i, 1, 0) - w_t^*(\mathbf{l} - \mathbf{e}_k, 1, 1) \\
&= \Delta_{\mathbf{b}} w_t^*(\mathbf{l} - \mathbf{e}_k, 1, 1) - \Delta_{\mathbf{b}} w_t^*(\mathbf{l} - \mathbf{e}_1, 1, 0) \\
&\geq \Delta_{\mathbf{b}} w_t^*(\mathbf{l} - \mathbf{e}_1 - \mathbf{e}_k, 1, 1) - \Delta_{\mathbf{b}} w_t^*(\mathbf{l} - \mathbf{e}_1, 1, 0) \\
&\geq 0,
\end{aligned}$$

where the second inequality follows from $\Delta_{\mathbf{ab}} w_t^*(\mathbf{l} - \mathbf{e}_k, 1, 1) \leq 0$, and the last inequality holds because $\Delta_{\mathbf{b},(\mathbf{e}_{L+2} - \mathbf{e}_k)} w_t^*(\mathbf{l} - \mathbf{e}_1, 1, 0) \geq 0$.

Case (2). $\mathbf{a} = -\mathbf{e}_1$ and $\mathbf{b} = \mathbf{d}_{L+1}$. In this case, we have $n_1 = 1$ and

$$\begin{aligned}
\text{LHS} - \text{RHS} &= v_{t+1}^*(\mathbf{l} - \mathbf{e}_1, 1, 0) + v_{t+1}^*(\mathbf{l} + \mathbf{e}_L, 0, 0) - v_{t+1}^*(\mathbf{l} - \mathbf{e}_1 + \mathbf{e}_L, 0, 0) - v_{t+1}^*(\mathbf{l}, 1, 0) \\
&= v_{t+1}^*(\mathbf{l} - \mathbf{e}_1, 1, 0) + v_{t+1}^*(\mathbf{l}, 1, 0) - v_{t+1}^*(\mathbf{l} - \mathbf{e}_1, 1, 0) - v_{t+1}^*(\mathbf{l}, 1, 0) = 0,
\end{aligned}$$

where the second equality holds because $v^*_{t+1}(\mathbf{l}+\mathbf{e}_L, 0, n_2) = v^*_{t+1}(\mathbf{l}, 1, n_2)$ for all $\mathbf{l}$ and n_2, since it is optimal to assign a highest class customer to the empty fast server.

Case (3). $\mathbf{a} = -\mathbf{e}_1$ and $\mathbf{b} = \mathbf{e}_{L+1}$. In this case, we have $n_1 = 0$ and

$$\text{LHS} - \text{RHS} = v^*_{t+1}(\mathbf{l}-\mathbf{e}_1, 0, 0) + v^*_{t+1}(\mathbf{l}, 1, 0) - v^*_{t+1}(\mathbf{l}-\mathbf{e}_1, 1, 0) - v^*_{t+1}(\mathbf{l}, 0, 0).$$

Let $(\hat{\mathbf{l}}, \hat{n}_1, \hat{n}_2)$, $(\bar{\mathbf{l}}, \bar{n}_1, \bar{n}_2)$, and $(\mathbf{l}^*, n^*_1, n^*_2)$ denote the optimal solution to $v^*_{t+1}(\mathbf{l}-\mathbf{e}_1, 0, 0)$, $v^*_{t+1}(\mathbf{l}, 1, 0)$, and $v^*_{t+1}(\mathbf{l}, 0, 0)$, respectively.

Case (3.1). If $\hat{n}_2 = 1$, then $\hat{n}_1 = 1$. In other words, it is optimal to assign a customer to server 2 in state $(\mathbf{l}-\mathbf{e}_1-\mathbf{e}_k, 1, 0)$. Thus, it is optimal to do so in state $(\mathbf{l}, 1, 0)$, which implies that $\bar{n}_2 = 1$. The rest of the proof is similar to Case (1.1) and hence omitted for brevity.

Case (3.2). If $\hat{n}_2 = 0$ and $\bar{n}_2 = 0$, then the proof is similar to Case (1.2) and hence omitted for brevity.

Case (3.3). If $\hat{n}_2 = 0$, $\bar{n}_2 = 1$, and $\hat{n}_1 = 1$, then we have

$$\begin{aligned}\text{LHS} - \text{RHS} &\geq w^*_t(\mathbf{l}-\mathbf{e}_1-\mathbf{e}_k, 1, 0) + w^*_t(\mathbf{l}-\mathbf{e}_k, 1, 1) - w^*_t(\mathbf{l}-\mathbf{e}_1-\mathbf{e}_k, 1, 1) - w^*_t(\mathbf{l}-\mathbf{e}_k, 1, 0)\\ &= \Delta_{\mathbf{e}_1\mathbf{e}_{L+2}} w^*_t(\mathbf{l}-\mathbf{e}_1-\mathbf{e}_k, 1, 0) \geq 0,\end{aligned}$$

where the last inequality follows from the multimodularity of $w^*_t(\mathbf{l}, n_1, n_2)$ in $(\mathbf{l}, n_2)$.

Case (3.4). If $\hat{n}_2 = 0$, $\bar{n}_2 = 1$, and $\hat{n}_1 = 0$, then we have

$$\begin{aligned}\text{LHS} - \text{RHS} &\geq w^*_t(\mathbf{l}-\mathbf{e}_1, 0, 0) + w^*_t(\mathbf{l}-\mathbf{e}_k, 1, 1) - w^*_t(\mathbf{l}-\mathbf{e}_1-\mathbf{e}_k, 1, 1) - w^*_t(\mathbf{l}, 0, 0)\\ &= \Delta_{-\mathbf{e}_1} w^*_t(\mathbf{l}, 0, 0) - \Delta_{-\mathbf{e}_1} w^*_t(\mathbf{l}-\mathbf{e}_k, 1, 1)\\ &\geq \Delta_{-\mathbf{e}_1} w^*_t(\mathbf{l}, 0, 1) - \Delta_{-\mathbf{e}_1} w^*_t(\mathbf{l}-\mathbf{e}_k, 1, 1)\\ &\geq 0,\end{aligned}$$

where the second inequality holds because $\Delta_{-\mathbf{e}_1, -\mathbf{e}_{L+2}} w^*_t(\mathbf{l}, 0, 1) \geq 0$, and the last inequality holds because $\Delta_{-\mathbf{e}_1, (\mathbf{e}_k-\mathbf{e}_{L+1})} w^*_t(\mathbf{l}-\mathbf{e}_k, 1, 1) \geq 0$.

Case (4). $\mathbf{a} = \mathbf{d}_{L+1}$ and $\mathbf{b} \in \{\mathbf{d}_2, ..., \mathbf{d}_L\}$. The proof of Case (4) is similar to that of Case (2), and hence is omitted for the sake of brevity.

It is worth noticing that the case of $\mathbf{a} = \mathbf{d}_{L+1}$ and $\mathbf{b} = \mathbf{e}_{L+1}$ does not exist since $(\mathbf{l}, n_1, 0) + \mathbf{d}_{L+1}$ and $(\mathbf{l}, n_1, 0) + \mathbf{e}_{L+1}$ cannot be in $\mathbb{Z}^L \times \{0, 1\} \times \{0, 1\}$ at the same time.

Case (5). $\mathbf{b} = \mathbf{e}_{L+1}$ and $\mathbf{a} \in \{\mathbf{d}_2, ..., \mathbf{d}_L\}$. In this case, we have $n_1 = 0$. The proof is similar to that of Case (3) and hence omitted for brevity.

Case (6). $\mathbf{a}, \mathbf{b} \in \{\mathbf{d}_2, ..., \mathbf{d}_L\}$. In this case, the proof is similar to that of Case (1) and hence omitted for brevity.

Combining all the above six cases, together with Lemma OS-D4, we conclude that $v^*_{t+1}(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$ for all n_2. The multimodularity of $v^*_{t+1}(\mathbf{l}, n_1, n_2)$ in $(\mathbf{l}, n_2)$ for all n_1 can be proved in a similar token, and we omit the details for the sake of brevity. This completes the proof of the $w^*_t \Rightarrow v^*_{t+1}$ direction of property (P5), i.e., if $w^*_t(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$ and $(\mathbf{l}, n_2)$, then so is $v^*_{t+1}(\mathbf{l}, n_1, n_2)$.

Finally, we show if $v^*_{t+1}(\mathbf{l}, n_1, n_2)$ multimodular in $(\mathbf{l}, n_1)$ and $(\mathbf{l}, n_2)$, then so is $w^*_{t+1}(\mathbf{l}, n_1, n_2)$, i.e., the

$v^*_{t+1} \Rightarrow w^*_{t+1}$ direction of property (P5). In what follows, we show that $w^*_{t+1}(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$. The proof of multimodularity of $w^*_{t+1}(\mathbf{l}, n_1, n_2)$ in $(\mathbf{l}, n_2)$ is similar and hence omitted for brevity.

By definition, we need to show that for any $(L+2)\times 1$ dimension vectors $\mathbf{a}, \mathbf{b} \in \mathcal{F} := \{-\mathbf{e}_1, \mathbf{d}_2, ..., \mathbf{d}_L, \mathbf{d}_{L+1}, \mathbf{e}_{L+1}\}$ and $\mathbf{a} \neq \mathbf{b}$, we have

$$w^*_{t+1}((\mathbf{l}, n_1, n_2) + \mathbf{a}) + w^*_{t+1}((\mathbf{l}, n_1, n_2) + \mathbf{b}) \geq w^*_{t+1}((\mathbf{l}, n_1, n_2) + \mathbf{a} + \mathbf{b}) + w^*_{t+1}(\mathbf{l}, n_1, n_2), \qquad \text{(OS-D8)}$$

where $\mathbf{d}_i = \mathbf{e}_{i-1} - \mathbf{e}_i$ for each $i = 2, ..., m+1$. Let LHS and RHS denote the left-hand-side and right-hand-side of inequality (OS-D8), respectively. We consider the following cases.

Case (1). $\mathbf{a}, \mathbf{b} \in \{-\mathbf{e}_1, \mathbf{d}_2, ..., \mathbf{d}_L\}$. Notice that in this case, the L^{th} and $(L+1)^{\text{th}}$ dimension of vector $\mathbf{a}, \mathbf{b}$ are equal to zero. Let $\mathbf{a}_{1:L}$ denote the first L components of vector $\mathbf{a}$. That is, $\mathbf{a}_{1:L} = (a_1, ..., a_L)$; and $\mathbf{b}_{1:L}$ is defined similarly. Then, we have

$$\begin{aligned}
&\text{LHS} - \text{RHS} \\
&= w^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L}, n_1, n_2) + w^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, n_1, n_2) - w^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L} + \mathbf{b}_{1:L}, n_1, n_2) - w^*_{t+1}(\mathbf{l}, n_1, n_2) \\
&= \sum_{i=1}^{L} \lambda_i \left[v^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L} + \mathbf{e}_i, n_1, n_2) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L} + \mathbf{e}_i, n_1, n_2) \right. \\
&\qquad\qquad \left. - v^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L} + \mathbf{b}_{1:L} + \mathbf{e}_i, n_1, n_2) - v^*_{t+1}(\mathbf{l} + \mathbf{e}_i, n_1, n_2) \right] \\
&\quad + \mu_1 \left[v^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L}, 0, n_2) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 0, n_2) - v^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L} + \mathbf{b}_{1:L}, 0, n_2) - v^*_{t+1}(\mathbf{l}, 0, n_2) \right] \\
&\quad + \mu_2 \left[v^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L}, n_1, 0) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, n_1, 0) - v^*_{t+1}(\mathbf{l} + \mathbf{a}_{1:L} + \mathbf{b}_{1:L}, n_1, 0) - v^*_{t+1}(\mathbf{l}, n_1, 0) \right] \\
&\geq 0.
\end{aligned}$$

Case (2). $\mathbf{a} = \mathbf{d}_{L+1}$ and $\mathbf{b} \in \{-\mathbf{e}_1, \mathbf{d}_2, ..., \mathbf{d}_L\}$. In this case, note that $(\mathbf{l}, n_1, n_2) + \mathbf{a} \in \mathbb{Z}^L_+ \times \{0, 1\} \times \{0, 1\}$, therefore we must have $n_1 = 1$. It then follows that

$$\begin{aligned}
&\text{LHS} - \text{RHS} \\
&= w^*_{t+1}(\mathbf{l} + \mathbf{e}_L, 0, n_2) + w^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 1, n_2) - w^*_{t+1}(\mathbf{l} + \mathbf{e}_L + \mathbf{b}_{1:L}, 0, n_2) - w^*_{t+1}(\mathbf{l}, 1, n_2) \\
&= \sum_{i=1}^{L} \lambda_i \left[v^*_{t+1}(\mathbf{l} + \mathbf{e}_L + \mathbf{e}_i, 0, n_2) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L} + \mathbf{e}_i, 1, n_2) \right. \\
&\qquad\qquad \left. - v^*_{t+1}(\mathbf{l} + \mathbf{e}_L + \mathbf{b}_{1:L} + \mathbf{e}_i, 0, n_2) - v^*_{t+1}(\mathbf{l} + \mathbf{e}_i, 1, n_2) \right] \\
&\quad + \mu_1 \left[v^*_{t+1}(\mathbf{l} + \mathbf{e}_L, 0, n_2) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 0, n_2) - v^*_{t+1}(\mathbf{l} + \mathbf{e}_L + \mathbf{b}_{1:L}, 0, n_2) - v^*_{t+1}(\mathbf{l}, 0, n_2) \right] \\
&\quad + \mu_2 \left[v^*_{t+1}(\mathbf{l} + \mathbf{e}_L, 0, 0) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 1, 0) - v^*_{t+1}(\mathbf{l} + \mathbf{e}_L + \mathbf{b}_{1:L}, 0, 0) - v^*_{t+1}(\mathbf{l}, 1, 0) \right] \\
&= -\sum_{i=1}^{L} \lambda_i \Delta_{\mathbf{ab}} v^*_{t+1}(\mathbf{l} + \mathbf{e}_i, 1, n_2) - \mu_1 \Delta_{\mathbf{e}_L \mathbf{b}} v^*_{t+1}(\mathbf{l}, 0, n_2) - \mu_2 \Delta_{\mathbf{ab}} v^*_{t+1}(\mathbf{l}, 1, 0) \\
&\geq 0,
\end{aligned}$$

where the last inequality holds because $\Delta_{\mathbf{ab}} v^*_{t+1}(\mathbf{l} + \mathbf{e}_i, 1, n_2) \leq 0$ and $\Delta_{\mathbf{ab}} v^*_{t+1}(\mathbf{l}, 1, 0) \leq 0$ (by the multimodularity of v^*_{t+1} in $(\mathbf{l}, n_1)$), as well as $\Delta_{\mathbf{e}_L \mathbf{b}} v^*_{t+1}(\mathbf{l}, 0, n_2) \leq 0$ (by Lemma OS-B4).

Case (3). $\mathbf{a} = \mathbf{e}_{L+1}$ and $\mathbf{b} \in \{-\mathbf{e}_1, \mathbf{d}_2, ..., \mathbf{d}_L\}$. In this case, we have $n_1 = 0$. Then,

$$\begin{aligned}
&\text{LHS} - \text{RHS}\\
&= w^*_{t+1}(\mathbf{l}, 1, n_2) + w^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 0, n_2) - w^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 1, n_2) - w^*_{t+1}(\mathbf{l}, 0, n_2)\\
&= \sum_{i=1}^{L} \lambda_i \left[v^*_{t+1}(\mathbf{l} + \mathbf{e}_i, 1, n_2) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L} + \mathbf{e}_i, 0, n_2)\right.\\
&\qquad\qquad \left. - v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L} + \mathbf{e}_i, 1, n_2) - v^*_{t+1}(\mathbf{l}, 0, n_2)\right]\\
&\quad + \mu_1 \left[v^*_{t+1}(\mathbf{l}, 0, n_2) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 0, n_2) - v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 0, n_2) - v^*_{t+1}(\mathbf{l}, 0, n_2)\right]\\
&\quad + \mu_2 \left[v^*_{t+1}(\mathbf{l}, 1, 0) + v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 0, 0) - v^*_{t+1}(\mathbf{l} + \mathbf{b}_{1:L}, 1, 0) - v^*_{t+1}(\mathbf{l}, 0, 0)\right]\\
&= -\sum_{i=1}^{L} \lambda_i \Delta_{\mathbf{e}_{L+1}\mathbf{b}} v^*_{t+1}(\mathbf{l} + \mathbf{e}_i, 0, n_2) - \mu_2 \Delta_{\mathbf{e}_{L+1}\mathbf{b}} v^*_{t+1}(\mathbf{l}, 0, 0)\\
&\geq 0.
\end{aligned}$$

Finally, note that the case $\mathbf{a} = \mathbf{d}_{L+1}$ and $\mathbf{b} = \mathbf{e}_{L+1}$ does not exist, since $(\mathbf{l}, n_1, 0) + \mathbf{d}_{L+1}$ and $(\mathbf{l}, n_1, 0) + \mathbf{e}_{L+1}$ cannot be in $\mathbb{Z}^L \times \{0, 1\} \times \{0, 1\}$ at the same time. This completes the proof of the $v^*_{t+1} \Rightarrow w^*_{t+1}$ direction of property (P5). ☐

Proof of Theorem 2: The proof is similar to the proof of Theorem 1 and is therefore omitted for the sake of brevity. ☐

Proof of Proposition 2: (a) Note that $K_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-1}) = K'_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-1}) + \sum_{i=1}^{i^*-1} l_i$. Thus, it is equivalent to show that $K'_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-1})$ is decreasing in l_j with a decreasing rate bounded by 1 for $j = 1, ..., i^* - 1$, and the larger j is, the higher sensitivity is. We now prove this statement for the case with $n_1 = 0$, $n_2 \in \{0, 1\}$. The proof of the case with $n_1 = 1$, $n_2 = 0$ is similar.

For simplicity, let $\mathbf{l}^K_{i^*} = (l_1, ..., l_{i^*-1}, K, 0, ..., 0)$. Then, when $n_1 = 0$ and $n_2 \in \{0, 1\}$,

$$\begin{aligned}
K'_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-1}) &= \min\{K \in \mathbb{N} : w^*(l_1, ..., l_{i^*-1}, K+1, 0, \ldots, 0, n_1, n_2) - w^*(l_1, ..., l_{i^*-1}, K, 0, \ldots, 0, n_1 + 1, n_2) \geq 0\}\\
&= \min\{K \in \mathbb{N} : \Delta_{l_{i^*}} w^*(\mathbf{l}^K_{i^*}, n_1, n_2) - \Delta_{n_1} w^*(\mathbf{l}^K_{i^*}, n_1, n_2) \geq 0\}.
\end{aligned}$$

We first show that $K'_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-1})$ is decreasing in l_j with a decreasing rate bounded by 1. Let $\bar{K}' = K'_{(n_1,n_2),i^*}(l_1, ..., l_j, \ldots, l_{i^*-1})$ and $\hat{K}' = K'_{(n_1,n_2),i^*}(l_1, ..., l_j + 1, \ldots, l_{i^*-1})$. We need to show that $\bar{K}' - 1 \leq \hat{K}' \leq \bar{K}'$.

By the definition of $K'_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-1})$, we have $\Delta_{l_{i^*}} w^*(\mathbf{l}^{\bar{K}'}_{i^*}, n_1, n_2) - \Delta_{n_1} w^*(\mathbf{l}^{\bar{K}'}_{i^*}, n_1, n_2) \geq 0$. Since w^* is multimodular in $(\mathbf{l}, n_1)$, then

$$\Delta_{l_{i^*}} w^*(\mathbf{l}^{\bar{K}'}_{i^*} + \mathbf{e}_j, n_1, n_2) - \Delta_{n_1} w^*(\mathbf{l}^{\bar{K}'}_{i^*} + \mathbf{e}_j, n_1, n_2) \geq \Delta_{l_{i^*}} w^*(\mathbf{l}^{\bar{K}'}_{i^*}, n_1, n_2) - \Delta_{n_1} w^*(\mathbf{l}^{\bar{K}'}_{i^*}, n_1, n_2) \geq 0.$$

Again, by the definition of $K'_{(n_1,n_2),i^*}(l_1, ..., l_j + 1, \ldots, l_{i^*-1})$, we have $K'_{(n_1,n_2),i^*}(l_1, ..., l_j + 1, ..., l_{i^*-1}) \leq \bar{K}'$, i.e., $\hat{K}' \leq \bar{K}'$.

By the definition of $K'_{(n_1,n_2),i^*}(l_1, ..., l_j + 1, ..., l_{i^*-1})$, we also have $\Delta_{l_{i^*}} w^*(\mathbf{l}^{\hat{K}'}_{i^*} + \mathbf{e}_j, n_1, n_2) - \Delta_{n_1} w^*(\mathbf{l}^{\hat{K}'}_{i^*} + \mathbf{e}_j, n_1, n_2) \geq 0$. Since w^* is multimodular in $(\mathbf{l}, n_1)$, then

$$\Delta_{l_{i^*}} w^*(\mathbf{l}^{\hat{K}'}_{i^*} + \mathbf{e}_{i^*}, n_1, n_2) - \Delta_{n_1} w^*(\mathbf{l}^{\hat{K}'}_{i^*} + \mathbf{e}_{i^*}, n_1, n_2) \geq \Delta_{l_{i^*}} w^*(\mathbf{l}^{\hat{K}'}_{i^*} + \mathbf{e}_j, n_1, n_2) - \Delta_{n_1} w^*(\mathbf{l}^{\hat{K}'}_{i^*} + \mathbf{e}_j, n_1, n_2) \geq 0.$$

Note that $\mathbf{l}_{i^*}^{\hat{K}'} + \mathbf{e}_{i^*} = \mathbf{l}_{i^*}^{\hat{K}'+1}$. By the definition of $K'_{(n_1,n_2),i^*}(l_1,...,l_{i^*-1})$, the above inequality implies that $K'_{(n_1,n_2),i^*}(l_1,...,l_{i^*-1}) \le \hat{K}'+1$, i.e., $\bar{K}'-1 \le \hat{K}'$.

Next, we show that the larger j is, the higher sensitivity is. Suppose $1 \le j_1 < j_2 \le i^*-1$. Let $\tilde{K}' = K'_{(n_1,n_2),i^*}(l_1,...,l_{j_1}+1,\dots,l_{j_2},\dots,l_{i^*-1})$ and $\check{K}' = K'_{(n_1,n_2),i^*}(l_1,...,l_{j_1},\dots,l_{j_2}+1,...,l_{i^*-1})$. We need to show that $\bar{K}'-\tilde{K}' \le \bar{K}'-\check{K}'$, i.e., $\tilde{K}' \ge \check{K}'$.

By the definition of $K'_{(n_1,n_2),i^*}(l_1,...,l_{j_1}+1,...,l_{j_2},...,l_{i^*-1})$, we have $\Delta_{l_{i^*}} w^*(\mathbf{l}_{i^*}^{\tilde{K}'}+\mathbf{e}_{j_1},n_1,n_2) - \Delta_{n_1} w^*(\mathbf{l}_{i^*}^{\tilde{K}'}+\mathbf{e}_{j_1},n_1,n_2) \ge 0$. Since w^* is multimodular in $(\mathbf{l},n_1)$, then
$\Delta_{l_{i^*}} w^*(\mathbf{l}_{i^*}^{\tilde{K}'}+\mathbf{e}_{j_2},n_1,n_2) - \Delta_{n_1} w^*(\mathbf{l}_{i^*}^{\tilde{K}'}+\mathbf{e}_{j_2},n_1,n_2) \ge \Delta_{l_{i^*}} w^*(\mathbf{l}_{i^*}^{\tilde{K}'}+\mathbf{e}_{j_1},n_1,n_2) - \Delta_{n_1} w^*(\mathbf{l}_{i^*}^{\tilde{K}'}+\mathbf{e}_{j_1},n_1,n_2) \ge 0$.
By the definition of $K'_{(n_1,n_2),i^*}(l_1,...,l_{j_1},...,l_{j_2}+1,...,l_{i^*-1})$, we have $K'_{(n_1,n_2),i^*}(l_1,...,l_{j_1},...,l_{j_2}+1,...,l_{i^*-1}) \le \tilde{K}'$, i.e., $\tilde{K}' \ge \check{K}'$.

(b) It is equivalent to show that $K'_{(0,0),i^*}(l_1,...,l_{i^*-1}) \le K'_{i^*,(0,1)}(l_1,...,l_{i^*-1}) \le K'_{(1,0),i^*}(l_1,...,l_{i^*-1})$. Let $K'_1 = K'_{(0,0),i^*}(l_1,...,l_{i^*-1})$, $K'_2 = K'_{i^*,(0,1)}(l_1,...,l_{i^*-1})$, and $K'_3 = K'_{(1,0),i^*}(l_1,...,l_{i^*-1})$. We need to show that $K'_1 \le K'_2 \le K'_3$.

By the definition of $K'_{(1,0),i^*}(l_1,...,l_{i^*-1})$, we have $w^*(\mathbf{l}_{i^*}^{K'_3}+\mathbf{e}_{i^*},1,0) - w^*(\mathbf{l}_{i^*}^{K'_3},1,1) \ge 0$. Since w^* satisfies property (P3), then

$$w^*(\mathbf{l}_{i^*}^{K'_3}+\mathbf{e}_{i^*},0,1) - w^*(\mathbf{l}_{i^*}^{K'_3},1,1) \ge w^*(\mathbf{l}_{i^*}^{K'_3}+\mathbf{e}_{i^*},1,0) - w^*(\mathbf{l}_{i^*}^{K'_3},1,1) \ge 0.$$

By the definition of $K'_{(0,1),i^*}(l_1,...,l_{i^*-1})$, the above inequality implies that $K'_{(0,1),i^*}(l_1,...,l_{i^*-1}) \le K'_3$, i.e., $K'_2 \le K'_3$.

By the definition of $K'_{(0,1),i^*}(l_1,...,l_{i^*-1})$, we have $w^*(\mathbf{l}_{i^*}^{K'_2}+\mathbf{e}_{i^*},0,1) - w^*(\mathbf{l}_{i^*}^{K'_2},1,1) \ge 0$. Note that w^* satisfies property (P4), i.e., if $w^*(\mathbf{l}+\mathbf{e}_k,0,1) - w^*(\mathbf{l},1,1) \ge 0$, then $w^*(\mathbf{l}+\mathbf{e}_k,0,0) - w^*(\mathbf{l},1,0) \ge 0$. This property and the above inequality together implies that

$$w^*(\mathbf{l}_{i^*}^{K'_2}+\mathbf{e}_{i^*},0,0) - w^*(\mathbf{l}_{i^*}^{K'_2},1,0) \ge 0.$$

By the definition of $K'_{(0,0),i^*}(l_1,...,l_{i^*-1})$, we have $K'_{(0,0),i^*}(l_1,...,l_{i^*-1}) \le K'_2$, i.e., $K'_1 \le K'_2$.

(c) It is equivalent to show that $K'_{(n_1,n_2),i^*-1}(l_1,...,l_{i^*-2}) \ge K'_{(n_1,n_2),i^*}(l_1,...,l_{i^*-2},0)$. We prove this for the case with $n_1=0$, $n_2\in\{0,1\}$. The proof for the case with $n_1=1$, $n_2=0$ is similar.

Let $K'_4 = K'_{(n_1,n_2),i^*-1}(l_1,...,l_{i^*-2})$ and $K'_5 = K'_{(n_1,n_2),i^*}(l_1,...,l_{i^*-2},0)$. We need to show that $K'_4 \ge K'_5$. When $n_1=0$, $n_2\in\{0,1\}$, by the definition of $K'_{(n_1,n_2),i^*-1}(l_1,...,l_{i^*-2})$, we have

$$w^*(l_1,...,l_{i^*-2},K'_4+1,0,...,0,n_1,n_2) - w^*(l_1,...,l_{i^*-2},K'_4,0,...,0,n_1+1,n_2) \ge 0. \tag{OS-D9}$$

Since w^* is multimoudular in $(\mathbf{l},n_1)$, then

$$\begin{aligned}
&w^*(l_1,...,l_{i^*-2},0,K'_4+1,...,0,n_1,n_2) - w^*(l_1,...,l_{i^*-2},0,K'_4,...,0,n_1+1,n_2)\\
\ge&w^*(l_1,...,l_{i^*-2},1,K'_4,...,0,n_1,n_2) - w^*(l_1,...,l_{i^*-2},1,K'_4-1,,...,0,n_1+1,n_2)\\
\ge&w^*(l_1,...,l_{i^*-2},2,K'_4-1,...,0,n_1,n_2) - w^*(l_1,...,l_{i^*-2},2,K'_4-2,...,0,n_1+1,n_2)\\
\ge&...\\
\ge&w^*(l_1,...,l_{i^*-2},K'_4,1,...,0,n_1,n_2) - w^*(l_1,...,l_{i^*-2},K'_4,0,...,0,n_1+1,n_2)
\end{aligned}$$

$$\geq w^*(l_1, ..., l_{i^*-2}, K_4'+1, 0, ..., 0, n_1, n_2) - w^*(l_1, ..., l_{i^*-2}, K_4', 0, ..., 0, n_1+1, n_2)$$

$$\geq 0,$$

where the last inequality follows from (OS-D9). Then, by the definition of $K'_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-2}, 0)$, we have $K'_{(n_1,n_2),i^*}(l_1, ..., l_{i^*-2}, 0) \leq K_4'$, i.e., $K_4' \geq K_5'$. □

Proof of Proposition 3: The proof is similar to the proof of Proposition 1 and is therefore omitted for the sake of brevity. □

Proof of Theorem 3: We prove this result for the case with N homogeneous servers and 2 customer classes. The proof for the case with L customer classes and 2 heterogeneous servers is similar.

The existence of optimal policy under the long-run average cost criterion can be proven via an argument involving taking the limit of $\alpha \to 0$ under the discounted total cost criterion. However, in order to apply this argument, we must show that the following two conditions hold (see Weber and Stidham 1987 and Cavazos-Cadena 1992), (1) there exists a stationary policy π that induces an irreducible positive recurrent Markov chain with finite long-run average cost C^π; (2) the number of states for which the one-stage cost $g(l_1, l_2) = c_1 l_1 + c_2 l_2 \leq C^\pi$ is finite.

In order to prove condition (1), we can choose a feasible stationary policy with parameters $K_0 = K_1 = \cdots = K_{N-1} = 0$ (i.e., the non-idling policy). It can be shown that, this stationary policy yields an irreducible positive recurrent Markov chain with a finite average cost. Condition (2) can also be easily verified, since $g(l_1, l_2) = c_1 l_1 + c_2 l_2$ is linear in (l_1, l_2) and goes to infinity when l_1 or $l_2 \to \infty$. Hence, for any positive value γ, the number of states for which $g(l_1, l_2) \leq \gamma$ is always finite. Under these conditions, Weber and Stidham (1987) show that, there exists a constant J^* and a function $f(l_1, l_2, n)$, such that

$$\begin{aligned} f(l_1, l_2, n) + J^* \geq\ & g(l_1, l_2) + \lambda_1 Tf(l_1+1, l_2, n) + \lambda_2 Tf(l_1, l_2+1, n) \\ & + n\mu Tf(l_1, l_2, (n-1)^+) + (N-n)\mu Tf(l_1, l_2, n), \end{aligned}$$

where

$$\begin{aligned} Tf(l_1, l_2, n) = \min_{l_1', l_2', n' \in \mathbb{N}} \quad & f(l_1', l_2', n') \\ \text{s.t.} \quad & l_1' + l_2' + n' = l_1 + l_2 + n, \\ & l_1' \leq l_1, \\ & l_2' \leq l_2, \\ & n \leq n' \leq N. \end{aligned}$$

Furthermore, the stationary policy that minimizes the right hand side of the above equation, for each state (l_1, l_2, n), is an optimal policy for the long-run average cost criterion, and yields a constant average cost J^*. Hence, the properties of the optimal policy under the long-run average cost criterion, determined by the function $f(l_1, l_2, n)$, are the same as those under the discounted total cost criterion, determined by the function $w^*(l_1, l_2, n)$. □

Proof of Proposition 4: To show that the optimal policy holds the three properties, we need to show that the optimal value functions $w^*(\mathbf{l},\mathbf{n})$ in Equation (1) satisfies (a) $\Delta_{l_i} w^*(\mathbf{l},\mathbf{n}) \geq \Delta_{l_j} w^*(\mathbf{l},\mathbf{n})$ for $i > j$; (b) $\Delta_{n_i} w^*(\mathbf{l},\mathbf{n}) \geq \Delta_{n_j} w^*(\mathbf{l},\mathbf{n})$ for $i > j$; and (c) $\Delta_{l_L} w^*(\mathbf{l},\mathbf{n}) \geq \Delta_{n_i} w^*(\mathbf{l},\mathbf{n})$ for all i. The rest of the proof is similar to the proof of (P1)-(P3) in Lemma 1, and we omit it for the sake of brevity. $\square$

Proof of Proposition 5: The proofs of (1) and (3) are straightforward. The proof of (2) is similar to that of Proposition 1, and is therefore omitted for the sake of brevity. $\square$

Proof of Lemma 3: Summing over the balance equations of π_s, we have

$$\begin{aligned}
\sum_{\mathbf{n}\in\{0,1\}^N} \Pi_{\mathbf{n}}(z)\cdot z^{|\mathbf{n}|} &= \sum_{\mathbf{n}\in\{0,1\}^N} \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{\mathbf{n}}} \pi_{l_1,\mathbf{n}} z^{l_1+|\mathbf{n}|} \\
&= \sum_{\mathbf{n}\in\{0,1\}^N} \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{\mathbf{n}}} \sum_{s_1} \pi_{s_1} p_{s_1,(l_1,\mathbf{n})} z^{l_1+|\mathbf{n}|} \\
&= \sum_{\mathbf{n}\neq(1,1,\cdots,1)} \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{\mathbf{n}}-1} \frac{\lambda_1 z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} \pi_{l_1,\mathbf{n}} z^{l_1+|\mathbf{n}|} \\
&\quad + \sum_{\mathbf{n}\neq(1,1,\cdots,1)} \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{\mathbf{n}}-1} \frac{\lambda_2 z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{l_1+|\mathbf{n}|} \pi_{l_1,\mathbf{n}} \\
&\quad + \sum_{\mathbf{n}\neq(1,1,\cdots,1)} \frac{(\lambda_1+\lambda_2)z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{K_{\mathbf{n}}+|\mathbf{n}|} \pi_{K_{\mathbf{n}},\mathbf{n}} \mathbf{1}_{\{K_{\mathbf{n}}\leq K_{(\mathbf{n})+1}\}} \\
&\quad + \sum_{\mathbf{n}\neq(1,1,\cdots,1)} \frac{(\lambda_1+\lambda_2)z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{K_{\mathbf{n}}+|\mathbf{n}|} \pi_{K_{\mathbf{n}},\mathbf{n}} \mathbf{1}_{\{K_{\mathbf{n}}> K_{(\mathbf{n})+1}\}} \\
&\quad + \sum_i \sum_{\substack{\mathbf{n}\neq(1,1,\cdots,1)\\|\mathbf{n}|>0}} \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{(\mathbf{n})-i}} \frac{n_i\mu_i/z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{l_1+|\mathbf{n}|} \pi_{l_1,\mathbf{n}} \\
&\quad + \sum_i \sum_{\substack{\mathbf{n}\neq(1,1,\cdots,1)\\|\mathbf{n}|>0}} \sum_{l_1=K_{(\mathbf{n})-i}+1}^{K_{\mathbf{n}}} \frac{n_i\mu_i/z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{l_1+|\mathbf{n}|} \pi_{l_1,\mathbf{n}} \\
&\quad + \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{1,1,\cdots,1}} \sum_{m=0}^{\infty} \frac{\sum_i (\mathbf{1}_{\{l_1+m\geq K_{(1,1,\cdots,1)-i}+1\}}\mu_i)}{\sum_{i=1}^N \mu_i} a_m z^{l_1+m-1+N} \pi_{l_1,1,\cdots,1} \\
&\quad + \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{1,1,\cdots,1}} \sum_{m=0}^{\infty} \frac{\sum_i \mathbf{1}_{\{l_1+m\leq K_{(1,1,\cdots,1)-i}\}}\mu_i}{\sum_{i=1}^N \mu_i} a_m z^{l_1+m-1+N} \pi_{l_1,1,\cdots,1} \\
&= \sum_{\mathbf{n}\neq(1,1,\cdots,1)} \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{\mathbf{n}}} \frac{(\lambda_1+\lambda_2)z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} \pi_{l_1,\mathbf{n}} z^{l_1+|\mathbf{n}|} \\
&\quad + \sum_i \sum_{\substack{\mathbf{n}\neq(1,1,\cdots,1)\\|\mathbf{n}|>0}} \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{\mathbf{n}}} \frac{n_i\mu_i/z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{l_1+|\mathbf{n}|} \pi_{l_1,\mathbf{n}} \\
&\quad + \sum_{l_1=\min_{\mathbf{n}'}\{K_{\mathbf{n}'}\}}^{K_{1,1,\cdots,1}} G^*(\lambda_1-\lambda_1 z)/z\cdot(z^{l_1+N}\pi_{l_1,1,\cdots,1}) \\
&= \sum_{\mathbf{n}\neq(1,1,\cdots,1)} \frac{(\lambda_1+\lambda_2)z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} \Pi_{\mathbf{n}}(z) z^{|\mathbf{n}|}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{\mathbf{n}\neq(1,1,\cdots,1)\\|\mathbf{n}|>0}} \frac{\mathbf{n}\cdot\boldsymbol{\mu}^T/z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{|\mathbf{n}|}\Pi_{\mathbf{n}}(z) + G^*(\lambda_1-\lambda_1 z)/z\cdot(z^N\Pi_{1,\cdots,1}(z))\\
=&z\Pi_{(0,\cdots,0)}(z) + \sum_{\substack{\mathbf{n}\neq(1,1,\cdots,1)\\|\mathbf{n}|>0}} \frac{(\lambda_1+\lambda_2)z+\mathbf{n}\cdot\boldsymbol{\mu}^T/z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{|\mathbf{n}|}\Pi_{\mathbf{n}}(z)\\
&+ G^*(\lambda_1-\lambda_1 z)/z\cdot(z^N\Pi_{1,\cdots,1}(z)),
\end{aligned}
$$

which implies

$$
\frac{G^*(\lambda_1-\lambda_1 z)-z}{1-z}\cdot(z^{N-1}\Pi_{1,\cdots,1}(z)) = \Pi_{(0,\cdots,0)}(z) + \sum_{\substack{\mathbf{n}\neq(1,1,\cdots,1)\\|\mathbf{n}|>0}} \frac{(\lambda_1+\lambda_2)-\mathbf{n}\cdot\boldsymbol{\mu}^T/z}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T} z^{|\mathbf{n}|}\Pi_{\mathbf{n}}(z).
$$

Notice that

$$
\lim_{z\to 1}\frac{G^*(\lambda_1-\lambda_1 z)-z}{1-z} = \frac{\sum_{i=1}^N \mu_i-\lambda_1-\lambda_2}{\sum_{i=1}^N \mu_i-\lambda_2}.
$$

Therefore, Equation OS-D1 becomes the following equation as z approaches to 1,

$$
\frac{\sum_{i=1}^N \mu_i-\lambda_1-\lambda_2}{\sum_{i=1}^N \mu_i-\lambda_2}\cdot\Pi_{1,\cdots,1}(1) = \Pi_{(0,\cdots,0)}(1) + \sum_{\substack{\mathbf{n}\neq(1,1,\cdots,1)\\|\mathbf{n}|>0}} \frac{(\lambda_1+\lambda_2)-\mathbf{n}\cdot\boldsymbol{\mu}^T}{\lambda_1+\lambda_2+\mathbf{n}\cdot\boldsymbol{\mu}^T}\Pi_{\mathbf{n}}(1).
$$

With the normalization condition $\sum_{\mathbf{n}}\Pi_{\mathbf{n}}(1)=1$, the results follow immediately. ☐

Proof of Theorem 4: The results are obtained directly from the analysis in Section 5.1.1. ☐

Proof of Proposition 6: Note that the balance equations of the CTMC for the general N-server-L-customer-class system (without queue capacity) are given by:

$$
\begin{aligned}
(\lambda+\mu)\pi_{\mathbf{l},\mathbf{n}} =& \sum_{k\in\{k|l_k>0\}}\lambda_k\pi_{\mathbf{l}-e_k,\mathbf{n}} + \sum_{j\in\{j|n_j=0\}}\mu_j(\pi_{\mathbf{l},\mathbf{n}+e_j}+\pi_{\mathbf{l},\mathbf{n}})\\
&+ \sum_{k\in\{k>i|K_{|\mathbf{n}|-1,k}\leq|\mathbf{l}|\}}\sum_{j<\hat{i}(\mathbf{n})}\lambda_k\pi_{\mathbf{l},\mathbf{n}-e_j}\mathbf{1}_{\{|\mathbf{l}|\leq K_{|\mathbf{n}|-1,i}\}}\\
&+ \sum_{(k,p)\in\{k\leq i\leq p,K_{|\mathbf{n}|-1,p}=|\mathbf{l}|,l_k>0\text{ or }p=k\}}\sum_{j<\hat{i}(\mathbf{n})}\lambda_k\pi_{\mathbf{l}-e_k+e_p,\mathbf{n}-e_j}\\
&+ \sum_{k\in\{k\geq i|K_{|\mathbf{n}|-1,k}\leq|\mathbf{l}|<K_{|\mathbf{n}|,k}\}}\sum_{p<\hat{i}(\mathbf{n})}\sum_{j\in\{p\}\cup\{j>p|n_j=0\}}\mu_j\pi_{\mathbf{l}+e_k,\mathbf{n}-e_p+e_j}, \forall(\mathbf{l},\mathbf{n})\in\mathbb{S}_\infty,
\end{aligned}
$$

and the balance equations of the CTMC for the system with capacity M are given by

$$
\begin{aligned}
(\lambda+\mu)\pi'_{\mathbf{l},\mathbf{n}} =& \sum_{k\in\{k|l_k>0\}}\lambda_k\pi'_{\mathbf{l}-e_k,\mathbf{n}} + \sum_{j\in\{j|n_j=0\}}\mu_j(\pi'_{\mathbf{l},\mathbf{n}+e_j}+\pi'_{\mathbf{l},\mathbf{n}}) + \lambda\pi'_{\mathbf{l},\mathbf{n}}\mathbf{1}_{\{|\mathbf{l}|=M\}}\\
&+ \sum_{k\in\{k>i|K_{|\mathbf{n}|-1,k}\leq|\mathbf{l}|\}}\sum_{j<\hat{i}(\mathbf{n})}\lambda_k\pi'_{\mathbf{l},\mathbf{n}-e_j}\mathbf{1}_{\{|\mathbf{l}|\leq K_{|\mathbf{n}|-1,i}\}}\\
&+ \sum_{(k,p)\in\{k\leq i\leq p,K_{|\mathbf{n}|-1,p}=|\mathbf{l}|,l_k>0\text{ or }p=k\}}\sum_{j<\hat{i}(\mathbf{n})}\lambda_k\pi'_{\mathbf{l}-e_k+e_p,\mathbf{n}-e_j}\\
&+ \sum_{k\in\{k\geq i|K_{|\mathbf{n}|-1,k}\leq|\mathbf{l}|<\min\{K_{|\mathbf{n}|,k},M\}\}}\sum_{p<\hat{i}(\mathbf{n})}\sum_{j\in\{p\}\cup\{j>p|n_j=0\}}\mu_j\pi'_{\mathbf{l}+e_k,\mathbf{n}-e_p+e_j}, \forall(\mathbf{l},\mathbf{n})\in\mathbb{S}_M.
\end{aligned}
$$

To facilitate the analysis, we define the following generation function:

$$
\Pi_{\mathbf{n}}(z) = \sum_{\mathbf{l}\in\mathbb{S}}\pi_{\mathbf{l},\mathbf{n}}z^{|\mathbf{l}|}.
$$

By summing over the balance equations, we have

$$(\lambda+\mu)\sum_{\mathbf{n}\in\{0,1\}^N}\Pi_{\mathbf{n}}(z)z^{|\mathbf{n}|}=\sum_{\mathbf{n}\in\{0,1\}^N}\sum_{i=1}^{L}\lambda_i z\Pi_{\mathbf{n}\in\{0,1\}^N}(\boldsymbol{z})z^{|\mathbf{n}|}$$

$$+\sum_{\mathbf{n}\in\{0,1\}^N}\sum_{j\in\{j|n_j=0\}}\mu_j\Pi_{\mathbf{n}}(\boldsymbol{z})z^{|\mathbf{n}|}+\sum_{\mathbf{n}\in\{0,1\}^N}\sum_{j\in\{j|n_j>0\}}\frac{\mu_j}{z}\Pi_{\mathbf{n}}(\boldsymbol{z})z^{|\mathbf{n}|}.$$

Therefore,

$$\sum_{\mathbf{n}\in\{0,1\}^N}\left[\sum_{i=1}^{L}\lambda_i(1-z)+\sum_{j\in\{j|n_j>0\}}\mu_j(1-1/z)\right]\Pi_{\mathbf{n}}(z)z^{|\mathbf{n}|}=0.$$

By canceling out $1-z$, we obtain

$$\sum_{\mathbf{n}\in\{0,1\}^N}\left[\sum_{i=1}^{L}\lambda_i-\sum_{j\in\{j|n_j>0\}}\mu_j/z\right]\Pi_{\mathbf{n}}(\boldsymbol{z})z^{|\mathbf{n}|}=0.$$

Then by letting $z=1$, and together with $\sum_{\mathbf{n}}\Pi_{\mathbf{n}}(1)=1$, we have

$$\sum_{|\mathbf{n}|<N}\sum_{j\in\{j|n_j=0\}}\mu_j\Pi_{\mathbf{n}}(1)=\mu-\lambda.$$

Given that $\sum_{j\in\{j|n_j=0\}}\mu_j\le\mu$, we have

$$1-\rho\le\sum_{|\mathbf{n}|<N}\Pi_{\mathbf{n}}(1)=1-\Pi_{\mathbf{1}}(1),$$

where $\rho:=\frac{\lambda}{\mu}$ represents the system-wide utilization. Let $\tilde{\pi}_m:=\sum_{|\mathbf{l}|=m}\pi_{\mathbf{l},\mathbf{1}}$ be the probability that all servers are busy and there are in total m customers in queue in the steady state. Note that the above inequality implies $\Pi_{\mathbf{1}}(1)\le\rho$. Next, we show that $\tilde{\pi}_{m+K_{N-1}}=\rho^m\tilde{\pi}_{K_{N-1}}<\rho^{m+1}(1-\rho)$ for any integer $m>K_{N-1}$.

For $m>K_{N-1}$, the balance equations lead to $(\lambda+\mu)\tilde{\pi}_m=\lambda\tilde{\pi}_{m-1}+\mu\tilde{\pi}_{m+1}$. Summing up equations for $m,m+1,\cdots$, we have $\tilde{\pi}_m=\rho\tilde{\pi}_{m-1}$ holds for $m>K_{N-1}$. Hence, we have $\tilde{\pi}_{m+K_{N-1}}=\rho^m\tilde{\pi}_{K_{N-1}}$. Then by $\sum_{m\ge K_{N-1}}\tilde{\pi}_{m+K_{N-1}}=\tilde{\pi}_{K_{N-1}}/(1-\rho)<\Pi_{\mathbf{1}}(1)\le\rho$ we have $\tilde{\pi}_{K_{N-1}}<\rho(1-\rho)$. Hence, $\tilde{\pi}_{m+K_{N-1}}=\rho^m\tilde{\pi}_{K_{N-1}}<\rho^{m+1}(1-\rho)$.

Using this bound for the steady state probabilities of states with a large queue length, we can derive the upper bound on the cost difference.

Let $\Delta\pi_{\mathbf{l},\mathbf{n}}=\pi_{\mathbf{l},\mathbf{n}}-\pi'_{\mathbf{l},\mathbf{n}}$. It is straightforward that $\Delta\pi_{\mathbf{l},\mathbf{n}}>0$ for $|\mathbf{l}|>M$. Notice that $\pi_{\mathbf{l},\mathbf{n}}$ and $\pi'_{\mathbf{l},\mathbf{n}}$ follow identical balance equations if $|\mathbf{l}|\le M$. By the induction on the balance equations from $|\mathbf{l}|=M$ to $|\mathbf{l}|=0$, we can show that $\Delta\pi_{\mathbf{l},\mathbf{n}}<0$ for $|\mathbf{l}|\le M$. Then for $|\mathbf{l}|>M$, we have $\pi'_{\mathbf{l},\mathbf{n}}=0$, and thus

$$0\le\sum_{|\mathbf{l}|>M}\Delta\pi_{\mathbf{l},\mathbf{n}}|\mathbf{l}|=\sum_{k>M}\tilde{\pi}_k k<\rho^{M-K_{N-1}+2}(M+\frac{1}{1-\rho}).$$

For $|\mathbf{l}|\le M$, we have $\sum_{|\mathbf{l}|\le M}\sum_{\mathbf{n}}\Delta\pi_{\mathbf{l},\mathbf{n}}=\sum_{|\mathbf{l}|\le M}\sum_{\mathbf{n}}\pi_{\mathbf{l},\mathbf{n}}-1=-\sum_{|\mathbf{l}|>M}\sum_{\mathbf{n}}\pi_{\mathbf{l},\mathbf{n}}=-\sum_{k>M}\tilde{\pi}_k>-\rho^{M-K_{N-1}+2}$. Hence, $-\rho^{M-K_{N-1}+2}M<\sum_{|\mathbf{l}|\le M}\sum_{\mathbf{n}}\Delta\pi_{\mathbf{l},\mathbf{n}}|\mathbf{l}|\le 0$. Therefore, we have

$$\begin{aligned}|C(\mathbf{K})-C'(\mathbf{K})|&=|\sum_{\mathbf{l},\mathbf{n}}\sum_{i=1}^{L}c_i l_i\pi_{\mathbf{l},\mathbf{n}}-\sum_{\mathbf{l},\mathbf{n}}\sum_{i=1}^{L}c_i l_i\pi'_{\mathbf{l},\mathbf{n}}|\\&=|\sum_{|\mathbf{l}|\le M}\sum_{\mathbf{n}}\sum_{i=1}^{L}c_i l_i\Delta\pi_{\mathbf{l},\mathbf{n}}+\sum_{|\mathbf{l}|>M}\sum_{\mathbf{n}}\sum_{i=1}^{L}c_i l_i\Delta\pi_{\mathbf{l},\mathbf{n}}|\\&\le\max\{|\sum_{|\mathbf{l}|\le M}\sum_{\mathbf{n}}\sum_{i=1}^{L}c_i l_i\Delta\pi_{\mathbf{l},\mathbf{n}}|,|\sum_{|\mathbf{l}|>M}\sum_{\mathbf{n}}\sum_{i=1}^{L}c_i l_i\Delta\pi_{\mathbf{l},\mathbf{n}}|\}\end{aligned}$$

$$
\begin{aligned}
&\le \max\{|\sum_{|\mathbf{l}|\le M}\sum_{\mathbf{n}} c_L|\mathbf{l}|\Delta\pi_{\mathbf{l},\mathbf{n}}|, |\sum_{|\mathbf{l}|> M}\sum_{\mathbf{n}} c_L|\mathbf{l}|\Delta\pi_{\mathbf{l},\mathbf{n}}|\}\\
&\le c_L \max\{\rho^{M-K_{N-1}+2}M, \rho^{M-K_{N-1}+2}(M+\frac{1}{1-\rho})\}\\
&= c_L\rho^{M-K_{N-1}+2}(M+\frac{1}{1-\rho}),
\end{aligned}
$$

where the first inequality holds because the first term is negative and the second term is positive. □

Proof of Proposition 7: We first scale λ and μ such that $\lambda+\mu=1$, and consider the following recursive equations:

$$
v_{t+1}(\mathbf{l},\mathbf{n},\mathbf{K}) = \begin{cases} w_t(\mathbf{l},\mathbf{n},\mathbf{K}), & \text{if } |\mathbf{n}|=N, \text{ or } i^*(\mathbf{l})<L \text{ and } |\mathbf{l}|\le K_{i^*(\mathbf{l}),|\mathbf{n}|},\\ v_{t+1}(\mathbf{l}-e_{i^*(\mathbf{l})},\mathbf{n}+e_{j^*(\mathbf{n})},\mathbf{K}), & \text{otherwise,} \end{cases}
$$

$$
w_t(\mathbf{l},\mathbf{n},\mathbf{K}) = \boldsymbol{c}\cdot\mathbf{l} + \sum_{i=1}^{L}\lambda_i v_t(\mathbf{l}+e_i,\mathbf{n},\mathbf{K}) + \sum_{j=1}^{N}\mu_j v_t(\mathbf{l},(\mathbf{n}-e_j)^+,\mathbf{K}),
$$

where $i^*(\mathbf{l})$ and $j^*(\mathbf{n})$ are the highest-priority class with a positive queue length, and the fastest idle server (if any), respectively. Define $i^*(\mathbf{0})=1$ and $\Delta_t v_t(\mathbf{l},\mathbf{n},\mathbf{K}) = v_{t+1}(\mathbf{l},\mathbf{n},\mathbf{K}) - v_t(\mathbf{l},\mathbf{n},\mathbf{K})$. By Theorem 8.5.6 of Puterman (2014), $\lim_{t\to\infty}\Delta_t v_t(\mathbf{l},\mathbf{n},\mathbf{K}) = C(\mathbf{K})$ for all $\mathbf{l},\mathbf{n}$.

Next, we use induction to show that, for all t

(a) $\Delta_t v_t(\mathbf{l}+e_1,\mathbf{n},\mathbf{K}+\mathbf{1}) = \Delta_t v_t(\mathbf{l},\mathbf{n},\mathbf{K}) + c_1$,

(b) $\Delta_t v_t(\mathbf{l}^{(1)},\mathbf{n}^{(1)},\mathbf{K}+e_{i_1,j_1}) + \Delta_t v_t(\mathbf{l}^{(2)},\mathbf{n}^{(2)},\mathbf{K}+e_{i_2,j_2}) - \Delta_t v_t(\mathbf{l}^{(3)},\mathbf{n}^{(3)},\mathbf{K}) - \Delta_t v_t(\mathbf{l}^{(4)},\mathbf{n}^{(4)},\mathbf{K}+e_{i_1,j_1}+e_{i_2,j_2}) \ge 0$, for $e_{i_1,j_1}\ne e_{i_2,j_2}$, $|\mathbf{l}^{(k)}|\le K_{i^*(\mathbf{l}^{(k)}),|\mathbf{n}^{(k)}|}+1$, and $|(|\mathbf{l}^{(k)}|+|\mathbf{n}^{(k)}|)-(|\mathbf{l}^{(k')}|+|\mathbf{n}^{(k')}|)|\le K_{1,N-1}+N+1$ with $k,k'=1,2,3,4$ and $k\ne k'$.

Then, by letting $t\to\infty$, we have $C(\mathbf{K}+\mathbf{1}) = C(\mathbf{K})+c_1$, and $C(\mathbf{K})$ is submodular.

With a proper-chosen end value, we can have v_0 satisfying (a) and (b). Now, suppose v_t satisfies (a) and (b). We would like to show that v_{t+1} also satisfies (a) and (b).

For (a), we first show $v_t \Rightarrow w_t$.

$$
\begin{aligned}
\Delta_t w_t(\mathbf{l}+e_1,\mathbf{n},\mathbf{K}+\mathbf{1}) - \Delta_t w_t(\mathbf{l},\mathbf{n},\mathbf{K}) =& \sum_{i=1}^{L}\lambda_i[v_t(\mathbf{l}+e_1+e_i,\mathbf{n},\mathbf{K}+\mathbf{1}) - v_t(\mathbf{l}+e_i,\mathbf{n},\mathbf{K})]\\
&+\sum_{j=1}^{N}\mu_j[v_t(\mathbf{l}+e_1,(\mathbf{n}-e_j)^+,\mathbf{K}+\mathbf{1}) - v_t(\mathbf{l},(\mathbf{n}-e_j)^+,\mathbf{K})]\\
=&(\lambda_1+\cdots+\lambda_L+\mu_1+\cdots+\mu_N)c_1 = c_1.
\end{aligned}
$$

Next, we show $w_t \Rightarrow v_{t+1}$. We consider two cases:

Case (1). If $l_L+|\mathbf{n}|\ge N$, then

$$
\begin{aligned}
&\Delta_t v_{t+1}(\mathbf{l}+e_1,\mathbf{n},\mathbf{K}+\mathbf{1}) - \Delta_t v_{t+1}(\mathbf{l},\mathbf{n},\mathbf{K}) =\\
&\qquad \Delta_t w_t(\mathbf{l}+e_1-(N-|\mathbf{n}|)e_L,\mathbf{1},\mathbf{K}+\mathbf{1}) - \Delta_t w_t(\mathbf{l}-(N-|\mathbf{n}|)e_L,\mathbf{1},\mathbf{K}) = c_1.
\end{aligned}
$$

Case (2). If $l_L+|\mathbf{n}|<N$, clearly $i^*(\mathbf{l}) = i^*(\mathbf{l}+e_1)$. Thus, $|\mathbf{l}|>K_{i^*(\mathbf{l}),|\mathbf{n}|}$ is equivalent to $|\mathbf{l}+e_1| = |\mathbf{l}|+1 > K_{i^*(\mathbf{l}+e_1),|\mathbf{n}|}+1$. If system $v_{t+1}(\mathbf{l},\mathbf{n},\mathbf{K})$ assigns a class $i^*(\mathbf{l})$ customer, then system $v_{t+1}(\mathbf{l}+e_1,\mathbf{n},\mathbf{K}+\mathbf{1})$ must also assign a class $i^*(\mathbf{l}+e_1)$ customer, and vice versa. The assignment in v_{t+2} is identical to that in period

$t{+}1$. Therefore, if we have $\hat{\mathbf{l}}, \hat{\mathbf{n}}$ such that $v_{t+1}(\mathbf{l}, \mathbf{n}, \mathbf{K}) = w_t(\hat{\mathbf{l}}, \hat{\mathbf{n}}, \mathbf{K})$, then we must have $v_{t+1}(\mathbf{l}+e_1, \mathbf{n}, \mathbf{K}+\mathbf{1}) = w_t(\hat{\mathbf{l}}+e_1, \hat{\mathbf{n}}, \mathbf{K}+\mathbf{1})$. Therefore,

$$\Delta_t v_{t+1}(\mathbf{l}+e_1, \mathbf{n}, \mathbf{K}+\mathbf{1}) - \Delta_t v_{t+1}(\mathbf{l}, \mathbf{n}, \mathbf{K}) = \Delta_t w_t(\hat{\mathbf{l}}+e_1, \hat{\mathbf{n}}, \mathbf{K}+\mathbf{1}) - \Delta_t w_t(\hat{\mathbf{l}}, \hat{\mathbf{n}}, \mathbf{K}) = c_1.$$

For (b), we first show $v_t \Rightarrow w_t$. Since $\Delta_t w_t(\mathbf{1}, \mathbf{n}, \mathbf{K}) = \sum_{i=1}^{L} \lambda_i \Delta_t v_t(\mathbf{l}+e_i, \mathbf{n}, \mathbf{K}) + \sum_{j=1}^{N} \mu_j \Delta_t v_t(\mathbf{l}, (\mathbf{n}-e_j)^+, \mathbf{K})$, we need to show the following:

(b.1) $\Delta_t v_t(\mathbf{l}^{(1)}+e_i, \mathbf{n}^{(1)}, \mathbf{K}+e_{i_1,j_1}) + \Delta_t v_t(\mathbf{l}^{(2)}+e_i, \mathbf{n}^{(2)}, \mathbf{K}+e_{i_2,j_2}) - \Delta_t v_t(\mathbf{l}^{(3)}+e_i, \mathbf{n}^{(3)}, \mathbf{K}) - \Delta_t v_t(\mathbf{l}^{(4)}+e_i, \mathbf{n}^{(4)}, \mathbf{K}+e_{i_1,j_1}+e_{i_2,j_2}) \geq 0$;

(b.2) $\Delta_t v_t(\mathbf{l}^{(1)}, (\mathbf{n}^{(1)}-e_j)^+, \mathbf{K}+e_{i_1,j_1}) + \Delta_t v_t(\mathbf{l}^{(2)}, (\mathbf{n}^{(2)}-e_j)^+, \mathbf{K}+e_{i_2,j_2}) - \Delta_t v_t(\mathbf{l}^{(3)}, (\mathbf{n}^{(3)}-e_j)^+, \mathbf{K}) - \Delta_t v_t(\mathbf{l}^{(4)}, (\mathbf{n}^{(4)}-e_j)^+, \mathbf{K}+e_{i_1,j_1}+e_{i_2,j_2}) \geq 0$.

For (b.1), notice that $|(|\mathbf{l}^{(k)}+e_i| + |\mathbf{n}^{(k)}|) - (|\mathbf{l}^{(k')}+e_i| + |\mathbf{n}^{(k')}|)| \leq K_{1,N-1}+N+1$. If $|\mathbf{l}^{(k)}+e_i| \leq K_{i^*(\mathbf{l}^{(k)}+e_i),|\mathbf{n}^{(k)}|}+1$ for all $k = 1,2,3,4$, clearly (b.1) holds. Otherwise, we must have $|\mathbf{l}^{(k)}+e_i| = K_{i^*(\mathbf{l}^{(k)}+e_i),|\mathbf{n}^{(k)}|}+2$ for some $k \in \{1,2,3,4\}$. In this case, by the definition of v_t, we have $\Delta_t v_t(\mathbf{l}^{(k)}+e_i, \mathbf{n}^{(k)}, \cdot) = \Delta_t v_t(\mathbf{l}^{(k)}+e_i-e_{i^*(\mathbf{l}^{(k)}+e_i)}, \mathbf{n}^{(k)}+e_{j^*(\mathbf{n})}, \cdot)$. Note that $v_t(\mathbf{l}^{(k)}+e_i-e_{i^*(\mathbf{l}^{(k)}+e_i)}, \mathbf{n}^{(k)}+e_{j^*(\mathbf{n})}, \cdot)$ satisfies the condition $|\mathbf{l}^{(k)}+e_i-e_{i^*(\mathbf{l}^{(k)}+e_i)}| \leq K_{i^*(\mathbf{l}^{(k)}+e_i-e_{i^*(\mathbf{l}^{(k)}+e_i)}),|\mathbf{n}^{(k)}|+1}+1$. Therefore, (b.1) holds.

For (b.2), first, similar to (b.1), we can show that the condition $|\mathbf{l}^{(k)}| \leq K_{i^*(\mathbf{l}^{(i)}),|\mathbf{n}^{(i)}|}+1$ holds. Next, we prove the condition $|(|\mathbf{l}^{(k)}| + |\mathbf{n}^{(k)}|) - (|\mathbf{l}^{(k')}| + |\mathbf{n}^{(k')}|)| \leq K_{1,N-1}+N+1$ also holds. If not, then we must have $(\mathbf{n}^{(k)}-e_j)^+ = \mathbf{n}^{(k)}$, $(\mathbf{n}^{(k')}-e_j)^+ = \mathbf{n}^{(k')}-e_j$ and $|\mathbf{l}^{(k)}| + |\mathbf{n}^{(k)}| \geq |\mathbf{l}^{(k')}| + |\mathbf{n}^{(k')}|$. Notice that in this case $n_j^{(k)} = 0$, and thus $|\mathbf{l}^{(k)}| \leq K_{i^*(\mathbf{l}^{(k)}),|\mathbf{n}^{(k)}|}+1 \leq K_{1,N-1}+1$. Then, $|\mathbf{l}^{(k')}| + |\mathbf{n}^{(k')}-e_j| \leq |\mathbf{l}^{(k)}| + |\mathbf{n}^{(k)}| \leq K_{1,N-1}+N+1$ and $|(|\mathbf{l}^{(k)}| + |\mathbf{n}^{(k)}|) - (|\mathbf{l}^{(k')}| + |\mathbf{n}^{(k')}-e_j|)| \leq K_{1,N-1}+N+1$. Therefore, (b.2) also holds and w_t satisfies (b).

Finally, we show $w_t \Rightarrow v_{t+1}$. By the definition of v_{t+1}, it is easy to verify that $\Delta_t v_{t+1}(\mathbf{l}, \mathbf{n}, \mathbf{K}) = \Delta_t w_t(\hat{\mathbf{l}}, \hat{\mathbf{n}}, \mathbf{K})$, for some $(\hat{\mathbf{l}}, \hat{\mathbf{n}})$ satisfying $|\hat{\mathbf{l}}| \leq |\mathbf{l}|$, $i^*(\hat{\mathbf{l}}) \leq i^*(\mathbf{l})$, and $|\mathbf{l}| + |\mathbf{n}| = |\hat{\mathbf{l}}| + |\hat{\mathbf{n}}|$. Therefore, we have $|\hat{\mathbf{l}}^{(k)}| \leq |\mathbf{l}^{(k)}| \leq K_{i^*(\mathbf{l}^{(k)}),|\mathbf{n}^{(k)}|}+1 \leq K_{i^*(\hat{\mathbf{l}}^{(k)}),|\hat{\mathbf{n}}^{(k)}|}+1$ and $|(|\hat{\mathbf{l}}^{(k)}| + |\hat{\mathbf{n}}^{(k)}|) - (|\hat{\mathbf{l}}^{(k')}| + |\hat{\mathbf{n}}^{(k')}|)| = |(|\mathbf{l}^{(k)}| + |\mathbf{n}^{(k)}|) - (|\mathbf{l}^{(k')}| + |\mathbf{n}^{(k')}|)| \leq K_{1,N-1}+N+1$. By the inductive assumption, $\Delta_t w_t(\hat{\mathbf{l}}^{(1)}, \hat{\mathbf{n}}^{(1)}, \mathbf{K}+e_{i_1,j_1}) + \Delta_t w_t(\hat{\mathbf{l}}^{(2)}, \hat{\mathbf{n}}^{(2)}, \mathbf{K}+e_{i_2,j_2}) - \Delta_t w_t(\hat{\mathbf{l}}^{(3)}, \hat{\mathbf{n}}^{(3)}, \mathbf{K}) - \Delta_t w_t(\hat{\mathbf{l}}^{(4)}, \hat{\mathbf{n}}^{(4)}, \mathbf{K}+e_{i_1,j_1}+e_{i_2,j_2}) \geq 0$, which implies that v_{t+1} satisfies (b). ☐

Proof of Lemma OS-A1: The corresponding heterogeneous M/M/2 system can be represented by the transition diagram in Figure OS-D1. States i, for $i = 0, 2, 3, ...$, represent the system states under which there are i customers in system, respectively. States $1A$ and $1B$ represent the system states under which Server 1 is busy whereas Server 2 is idle; and Server 1 is idle whereas Server 2 is busy, respectively.

Let π_i denote the limiting probability of state i. The set of detailed balance equations of this one-dimensional CTMC is given as follows,

$$(\lambda_1+\lambda_2)\pi_0 = \mu_1\pi_{1A} + \mu_2\pi_{1B}; \tag{OS-D10}$$

$$(\lambda_1+\lambda_2+\mu_1)\pi_{1A} = (\lambda_1+\lambda_2)\pi_0 + \mu_2\pi_2; \tag{OS-D11}$$

$$(\lambda_1+\lambda_2+\mu_2)\pi_{1B} = \mu_1\pi_2;$$

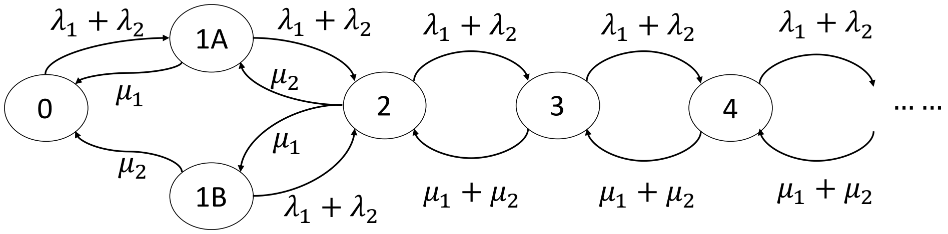


Figure OS-D1: Transition Diagram for the Heterogeneous M/M/2 System

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\pi_2 = (\lambda_1 + \lambda_2)\pi_{1A} + (\lambda_1 + \lambda_2)\pi_{1B} + (\mu_1 + \mu_2)\pi_3; \tag{OS-D12}$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\pi_i = (\lambda_1 + \lambda_2)\pi_{i-1} + (\mu_1 + \mu_2)\pi_{i+1}, \forall i = 3, 4, ... \tag{OS-D13}$$

where

$$\pi_0 + \pi_{1A} + \pi_{1B} + \sum_{i=2}^{\infty} \pi_i = 1. \tag{OS-D14}$$

Summing up Equation (OS-D13), for $i = 3, 4, ...$, gives $\pi_i = \frac{\lambda_1+\lambda_2}{\mu_1+\mu_2}\pi_{i-1}, \forall i = 3, 4, ...$. Summing up Equations (OS-D12)-(OS-D13), for $i = 3, 4, ...$, gives $\pi_2 = \frac{\lambda_1+\lambda_2}{\mu_1+\mu_2}(\pi_{1A} + \pi_{1B})$. Thus, we have $\pi_i = \left(\frac{\lambda_1+\lambda_2}{\mu_1+\mu_2}\right)^{i-1} (\pi_{1A} + \pi_{1B}), \forall i = 2, 3, ...$. Plug π_i back into Equations (OS-D11) and (OS-D14). Then, together with Equation (OS-D10), we obtain a system of linear equations with three variables π_0; π_{1A}; and π_{1B}, as follows,

$$\begin{cases} (\lambda_1 + \lambda_2)\pi_0 & = \mu_1\pi_{1A} + \mu_2\pi_{1B} \\ (\lambda_1 + \lambda_2 + \mu_1)\pi_{1A} & = (\lambda_1 + \lambda_2)\pi_0 + \mu_2\frac{\lambda_1+\lambda_2}{\mu_1+\mu_2}(\pi_{1A} + \pi_{1B}) \\ \pi_{1A} + \pi_{1B} & = (1 - \pi_0)(1 - \frac{\lambda_1+\lambda_2}{\mu_1+\mu_2}). \end{cases}$$

Solving the above system of equations results in

$$\pi_0 + \pi_{1A} + \pi_{1B} = \frac{(\mu_1 + \mu_2 - \lambda_1 - \lambda_2)\left((\lambda_1 + \lambda_2)^2\mu_1 + [(\lambda_1 + \lambda_2)^2 + 3(\lambda_1 + \lambda_2)\mu_1 + \mu_1^2]\mu_2 + (\lambda_1 + \lambda_2 + \mu_1)\mu_2^2\right)}{(\lambda_1 + \lambda_2)^2\mu_1^2 + [2(\lambda_1 + \lambda_2) + \mu_1]\mu_1^2\mu_2 + (\lambda_1 + \lambda_2 + \mu_1)(\lambda_1 + \lambda_2 + 2\mu_1)\mu_2^2 + (\lambda_1 + \lambda_2 + \mu_1)\mu_2^3}.$$

The probability that both servers are busy is equal to $1 - (\pi_0 + \pi_{1A} + \pi_{1B})$. ☐

Proof of Lemma OS-D1: (a) Let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l^*_{1,t+1}(l_1, l_2, n), l^*_{2,t+1}(l_1, l_2, n), n^*_{t+1}(l_1, l_2, n))$. We first show $l^*_{1,t}(l_1 + 1, l_2, n) \geq \hat{l}_1$. Suppose to the contrary, $l^*_{1,t}(l_1 + 1, l_2, n) = \hat{l}_1 - 1$. Then by the definition, $w^*_{t-1}(\hat{l}_1 - 1, \hat{l}_2, \hat{n} + 2) \leq w^*_{t-1}(\hat{l}_1, \hat{l}_2, \hat{n} + 1)$ (and in this case, we have $\hat{l}_2 = 0$). By property (P11), $w^*_{t-1}(\hat{l}_1, \hat{l}_2, \hat{n} + 1) \geq w^*_{t-1}(\hat{l}_1 - 1, \hat{l}_2, \hat{n} + 2)$ implies that $w^*_t(\hat{l}_1, \hat{l}_2, \hat{n}) > w^*_t(\hat{l}_1 - 1, \hat{l}_2, \hat{n} + 1)$. This contradicts with $l^*_{1,t+1}(l_1, l_2, n) = \hat{l}_1$, and hence we must have $l^*_{1,t}(l_1 + 1, l_2, n) \geq \hat{l}_1$.

We next show $l^*_{1,t}(l_1 + 1, l_2, n) \leq \hat{l}_1 + 1$. Suppose $l^*_{1,t}(l_1 + 1, l_2, n) = \hat{l}_1 + 2$. Then, we have $n^*_t(l_1 + 1, l_2, n) = \hat{n} - 1 \geq n$. By the definition, $w^*_{t-1}(\hat{l}_1 + 2, \hat{l}_2, \hat{n} - 1) < w^*_{t-1}(\hat{l}_1 + 1, \hat{l}_2, \hat{n})$ (and in this case, $\hat{l}_2 = 0$). By property (P10.1), this implies $w^*_t(\hat{l}_1 + 2, \hat{l}_2, \hat{n} - 1) < w^*_t(\hat{l}_1 + 1, \hat{l}_2, \hat{n})$. Then by property (P6.1), we have $w^*_t(\hat{l}_1 + 1, \hat{l}_2, \hat{n} - 1) < w^*_t(\hat{l}_1, \hat{l}_2, \hat{n})$. This means that $(\hat{l}_1 + 1, \hat{l}_2, \hat{n} - 1)$ is a better solution than $(\hat{l}_1, \hat{l}_2, \hat{n})$ for problem $v^*_{t+1}(l_1, l_2, n)$. This contradicts with $l^*_{1,t+1}(l_1, l_2, n) = \hat{l}_1$, and hence we must have $l^*_{1,t}(l_1 + 1, l_2, n) \leq \hat{l}_1 + 1$.

(b) The proof of part (b) is similar to that of part (a) and is therefore omitted for brevity. ☐

Proof of Lemma OS-D2: (a) Let $(\hat{l}_1, \hat{l}_2, \hat{n}) := (l_1^*(l_1, l_2, n), l_2^*(l_1, l_2, n), n^*(l_1, l_2, n))$. We first show $l_1^*(l_1 + 1, l_2, n) \geq \hat{l}_1$. Suppose to the contrary, $l_1^*(l_1 + 1, l_2, n) = \hat{l}_1 - 1$. Then by the definition, $w_t^*(\hat{l}_1 - 1, \hat{l}_2, \hat{n} + 2) \leq w_t^*(\hat{l}_1, \hat{l}_2, \hat{n} + 1)$ (and in this case, we have $\hat{l}_2 = 0$). By property (P8), $w_t^*(\hat{l}_1, \hat{l}_2, \hat{n} + 1) \geq w_t^*(\hat{l}_1 - 1, \hat{l}_2, \hat{n} + 2)$ implies that $w_t^*(\hat{l}_1, \hat{l}_2, \hat{n}) > w_t^*(\hat{l}_1 - 1, \hat{l}_2, \hat{n} + 1)$. This contradicts with $l_1^*(l_1, l_2, n) = \hat{l}_1$, and hence we must have $l_1^*(l_1 + 1, l_2, n) \geq \hat{l}_1$.

We next show $l_1^*(l_1 + 1, l_2, n) \leq \hat{l}_1 + 1$. Suppose $l_1^*(l_1 + 1, l_2, n) = \hat{l}_1 + 2$. Then, we have $n^*(l_1 + 1, l_2, n) = \hat{n} - 1 \geq n$. By the definition, $w_t^*(\hat{l}_1 + 2, \hat{l}_2, \hat{n} - 1) < w_t^*(\hat{l}_1 + 1, \hat{l}_2, \hat{n})$ (and in this case, $\hat{l}_2 = 0$). By property (P6.1), we have $w_t^*(\hat{l}_1 + 1, \hat{l}_2, \hat{n} - 1) < w_t^*(\hat{l}_1, \hat{l}_2, \hat{n})$. This means that $(\hat{l}_1 + 1, \hat{l}_2, \hat{n} - 1)$ is a better solution than $(\hat{l}_1, \hat{l}_2, \hat{n})$ for problem $v_{t+1}^*(l_1, l_2, n)$. This contradicts with $l_1^*(l_1, l_2, n) = \hat{l}_1$, and hence we must have $l_1^*(l_1 + 1, l_2, n) \leq \hat{l}_1 + 1$.

(b) The proof of part (b) is similar to that of part (a) and is therefore omitted for brevity. ☐

Proof of Lemma OS-D3: By the definition of $v_{t+1}^*(\mathbf{l}, n_1, n_2)$, it suffices to show that problems (OS-D6) and (OS-D5) are equivalent. To this end, we first show that problem (OS-D5) is equivalent to the following optimization problem:

$$\begin{aligned} \min_{\mathbf{l}' \in \mathbb{N}^L, n_1', n_2' \in \{0,1\}} \quad & w_t^*(\mathbf{l}', n_1', n_2') \\ \text{s.t.} \quad & \sum_{i=1}^{L} l_i' + n_1' + n_2' = \sum_{i=1}^{L} l_i + n_1 + n_2, \\ & \sum_{i=1}^{m} l_i' \leq \sum_{i=1}^{m} l_i, \quad \forall 1 \leq m \leq L \\ & n_1' \geq n_1, \\ & n_2' \geq n_2. \end{aligned} \tag{OS-D15}$$

Clearly, the optimal solution to problem (OS-D5) is a feasible solution to problem (OS-D15). Therefore, it remains to show that the optimal solution to problem (OS-D15) is also feasible to problem (OS-D5). We prove by contradiction. Let $(\mathbf{l}^*, n_1^*, n_2^*)$ be the optimal solution to problem (OS-D15). Suppose $(\mathbf{l}^*, n_1^*, n_2^*)$ is *not* feasible to problem (OS-D5). Then, we have $\{i : l_i^* > l_i\} \neq \emptyset$, and for any $(\tilde{\mathbf{l}}, n_1^*, n_2^*)$ with $\tilde{l}_i \leq l_i$ for each i and $\sum_{i=1}^L \tilde{l}_i = \sum_{i=1}^L l_i$, we have

$$w_t^*(\tilde{\mathbf{l}}, n_1^*, n_2^*) > w_t^*(\mathbf{l}^*, n_1^*, n_2^*). \tag{OS-D16}$$

Let $\hat{i} := \min\{i : l_i^* > l_i\}$. Since $\sum_{i=1}^{\hat{i}} l_i^* \leq \sum_{i=1}^{\hat{i}} l_i$, we can reduce $l_{\hat{i}}^*$ to $\hat{l}_{\hat{i}}$ by $l_{\hat{i}}^* - l_{\hat{i}}$ and increase l_i to $\hat{l}_i$ for each $i = 1, ..., \hat{i} - 1$ such that $\hat{l}_i \leq l_i$ for each $i = 1, ..., \hat{i}$ and $\sum_{i=1}^{\hat{i}} \hat{l}_i = \sum_{i=1}^{\hat{i}} l_i$. Since $\Delta_{l_i} w_t^*(\mathbf{l}, n_1, n_2) \geq \Delta_{l_j} w_t^*(\mathbf{l}, n_1, n_2)$ for $j < i$, we have

$$w_t^*(\hat{l}_1, .., \hat{l}_{\hat{i}}, l_{\hat{i}+1}^*, ..., l_L^*, n_1, n_2) \leq w_t^*(l_1^*, .., l_L^*, n_1, n_2)$$

We can repeat the above process and finally get a vector $(\bar{l}_1, ..., \bar{l}_L)$ with $\bar{l}_i \leq l_i$ for each $i = 1, ..., L$, and $\sum_{i=1}^L \bar{l}_i = \sum_{i=1}^L l_i$, such that

$$w_t^*(\bar{\mathbf{l}}, n_1^*, n_2^*) \leq w_t^*(\mathbf{l}^*, n_1^*, n_2^*),$$

which contradicts with (OS-D16). Therefore, the optimal solution to problem (OS-D15) must be a feasible solution to problem (OS-D5), and problem (OS-D5) is equivalent to problem (OS-D15).

In what follows, we show the equivalence between problems (OS-D15) and (OS-D6), which will then complete the proof. It is clear that the optimal solution to problem (OS-D15) is feasible to problem (OS-D6). Therefore, it remains to show that the optimal solution to problem (OS-D6) is also feasible to (OS-D15). Again, we prove by contradiction. Let $(\check{\mathbf{l}}, \check{n}_1, \check{n}_2)$ be the optimal solution to problem (OS-D6). Suppose $(\check{\mathbf{l}}, \check{n}_1, \check{n}_2)$ is not feasible to problem (OS-D15). Then we must have $\check{n}_1 < n_1$, i.e., $\check{n}_1 = 0$ and $n_1 = 1$. Note that $\sum_{i=1}^{L} \check{l}_i = \sum_{i=1}^{L} l_i$ and $\sum_{i=1}^{L} \check{l}_i + \check{n}_1 + \check{n}_2 = \sum_{i=1}^{L} l_i + n_1 + n_2$. Therefore, we have $\check{n}_2 > n_2$, i.e., $\check{n}_2 = 1$ and $n_2 = 0$. However, since it is always optimal to use the faster server first, we have

$$w_t^*(\check{\mathbf{l}}, 1, 0) \leq w_t^*(\check{\mathbf{l}}, \check{n}_1, \check{n}_2) = w_t^*(\check{\mathbf{l}}, 0, 1),$$

which implies that $(\check{l}_1, ..., \check{l}_L, 1, 0)$ is optimal for problem (OS-D6). Since it is also feasible to problem (OS-D15), this leads to a contradiction with the above assumption that the optimal solution to problem (OS-D6) is not feasible to (OS-D15). ☐

Proof of Lemma OS-D4: Note that $\tilde{w}_t(\mathbf{l}, n_1, 1)$ is the optimal cost if we assign one customer to server 2, and $\hat{w}_t(\mathbf{l}, n_1, 0)$ is the optimal cost if we do not assign any customer to server 2. Clearly, we have $v_{t+1}^*(\mathbf{l}, n_1, 0) = \min\{\hat{w}_t(\mathbf{l}, n_1, 0), \tilde{w}_t(\mathbf{l}, n_1, 1)\}$.

Now suppose $w_t^*(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$. By Lemma OS-D3, we have

$$\begin{aligned}
v_{t+1}^*(\mathbf{l}, n_1, 1) = & \min_{\mathbf{l}' \in \mathbb{N}^L, n_1' \in \{0,1\}} && w_t^*(\mathbf{l}', n_1', 1) \\
& \text{s.t.} && \sum_{i=1}^{L} l_i' + n_1' = \sum_{i=1}^{L} l_i + n_1, \\
& && \sum_{i=1}^{m} l_i' \leq \sum_{i=1}^{m} l_i, \quad \forall 1 \leq m \leq L \\
= & \min_{\mathbf{y} \in \mathbb{N}^L, z \in \{0,-1\}} && w_t^*(-\mathbf{y}, -z, 1) \\
& \text{s.t.} && \sum_{i=1}^{L} l_i + \sum_{i=1}^{L} y_i + z + n_1 = 0, \\
& && \sum_{i=1}^{m} l_i + \sum_{i=1}^{m} y_i \geq 0, \quad \forall 1 \leq m \leq L
\end{aligned}$$

Since $w_t^*(\mathbf{l}, n_1, n_2)$ is multimodular in $(\mathbf{l}, n_1)$, by Lemma 2 property (ii) in Li and Yu (2014), $w_t^*(-\mathbf{y}, -z, 1)$ is multimodular in $(\mathbf{y}, z)$ and thus multimodular in $(n_1, l_L, ..., l_1, y_1, ...y_L, z)$. Moreover, the constraint is a multimodular polyhedron defined in Li and Yu (2014). By Theorem 1 in Li and Yu (2014), $v_{t+1}^*(\mathbf{l}, n_1, 1)$ is multimodular in $(\mathbf{l}, n_1)$. The multimodularity of $\hat{w}_t(\mathbf{l}, n_1, 0)$ and $\tilde{w}_t(\mathbf{l}, n_1, 1)$ can be proven by a similar token, and we omit the details for the sake of brevity. ☐

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